Predictive Modeling Homework 1 Part 1

1.

(a) Since Y_1, Y_2, Y_3 are independent and they are all normally distributed, their joint distribution is multivariate normal distribution.

$$\begin{aligned}
\sigma^2 =& 1,\ \sigma = 1 \\
f(y_1, y_2, y_3) =& \prod_{i=1}^{3} f(y_i) \\
=& \prod_{i=1}^{3} \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(y_i-\mu)^2}{2\sigma^2}) \\
=& \frac{1}{\sigma^3 (2\pi)^{3/2}} exp(-\frac{\Sigma_{i=1}^{3}(y_i-\mu)^2}{2\sigma^2}) \\
=& \frac{1}{(2\pi)^{3/2}} exp(-\frac{\Sigma_{i=1}^{3}(y_i-\mu)^2}{2})
\end{aligned}$$

(b)Likelihood function:

$$\begin{aligned}
y_1 =& 4.6, y_2 = 6.3, y_3 = 5.0 \\
L(\mu) =& \prod_{i=1}^{3} f(y_i) \\
=& \frac{1}{(2\pi)^{3/2}} exp(-\frac{\Sigma_{i=1}^{3}(y_i-\mu)^2}{2}) \\
=& \frac{1}{(2\pi)^{3/2}} exp(-\frac{(4.6-\mu)^2 + (6.3-\mu)^2 + (5.0-\mu)^2}{2})
\end{aligned}$$

Log likelihood function:

$$\begin{aligned}
l(\mu) =& log(L(\mu)) \\
=& -\frac{3}{2}log(2\pi) - \frac{1}{2}\Sigma_{i=1}^{3}(y_i-\mu)^2 \\
=& -\frac{3}{2}log(2\pi) - \frac{1}{2}(4.6-\mu)^2 - \frac{1}{2}(6.3-\mu)^2 - \frac{1}{2}(5.0-\mu)^2
\end{aligned}$$

(c)Score function:

$$\begin{aligned}
y_1 =& 4.6, y_2 = 6.3, y_3 = 5.0 \\
l'(\mu) =& \Sigma_{i=1}^{3}(y_i-\mu) = 15.9 - 3\mu
\end{aligned}$$

Observed information:

$$j(\mu) = -l''(\mu) = 3$$

Expected information:

$$i(\mu) = E[j(\mu)] = 3$$

2.

2. (a) $D(\vec{Y};\mu(\vec{X})) = -2\log f_{(\vec{Y};\mu(\vec{X}))}(\vec{y})$

$$= -2\log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(y_i-\mu)^2}{2\sigma^2}\right)\right)$$

$$= n\log(2\pi) + 2n\log(\sigma) + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2$$

(b) $D(\vec{Y};\mu(\vec{X})) = -2\log f_{(\vec{Y};\mu(\vec{X}))}(\vec{y})$

$$= -2\log\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma\, y_i} \exp\left(-\frac{(\log y_i-\mu)^2}{2\sigma^2}\right)\right)$$

$$= n\log(2\pi) + 2n\log(\sigma) + 2\sum_{i=1}^{n}\log(y_i) + \frac{1}{\sigma^2}\sum_{i=1}^{n}(\log y_i-\mu)^2$$

(c) $D(\vec{Y};\mu(\vec{X})) = -2\log f_{(\vec{Y};\mu(\vec{X}))}(\vec{y})$ $\qquad \mu(\vec{x}) = p.$

$$= -2\log\left(\prod_{i=1}^{n}\binom{y_i}{m} p^{y_i}(1-p)^{m-y_i}\right)$$

$$= -2\sum_{i=1}^{n}\log\binom{y_i}{m} - 2\log p\cdot\sum_{i=1}^{n}y_i - 2\log(1-p)\left(mn-\sum_{i=1}^{n}y_i\right)$$

$$= -2\sum_{i=1}^{n}\log\binom{y_i}{m} - 2\log\left(\frac{p}{1-p}\right)\sum_{i=1}^{n}y_i - 2mn\log(1-p)$$

(d) $D(\vec{Y};\mu(\vec{X})) = -2\log f_{(\vec{Y};\mu(\vec{X}))}(\vec{y})$ $\qquad \mu(\vec{x}) = \lambda$

$$= -2\log\left(\prod_{i=1}^{n}\frac{\lambda^{y_i}}{y_i!}e^{-\lambda}\right)$$

$$= -2\log\lambda\cdot\sum_{i=1}^{n}y_i + \cancel{2\log}\, 2\sum_{i=1}^{n}\log(y_i!) + 2n\lambda$$

3.

(a) $\log Y_i \sim N(\mu, \sigma^2)$

$$\mu_Y = E[Y_i] = \int_0^\infty y \frac{1}{y\sqrt{2\pi}\sigma} \exp\left(-\frac{(\log y - \mu)^2}{2\sigma^2}\right) dy \quad \text{let } x = \log y \quad \text{i.e } y = e^x \quad dy = e^x dx$$

$$= \int_{-\infty}^\infty \cancel{e^x} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) e^x dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + x\right) dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2 - 2\mu x - 2\sigma^2 x + \mu^2}{2\sigma^2}\right) dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{[x-(\mu+\sigma^2)]^2 - 2\mu\sigma^2 - \sigma^4}{2\sigma^2}\right) dx$$

$$= 1 \cdot \exp\left(\mu + \frac{1}{2}\sigma^2\right)$$

$$= \exp\left(\mu + \frac{\sigma^2}{2}\right)$$

$$\sigma_Y^2 = E[(Y_i - \mu_Y)^2] = E[Y_i^2] - [E(Y_i)]^2$$

$$E[Y_i^2] = \int_0^\infty y^2 \frac{1}{y\sqrt{2\pi}\sigma} \exp\left(-\frac{(\log y - \mu)^2}{2\sigma^2}\right) dy \quad \text{let } x = \log y \quad y = e^x \quad dy = e^x dx$$

$$= \int_{-\infty}^\infty e^x \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \cdot e^x dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2} + 2x\right) dx$$

$$= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-(\mu+2\sigma^2))^2 - 4\mu\sigma^2 - 4\sigma^4}{2\sigma^2}\right) dx$$

$$= \exp(2\mu + 2\sigma^2) = \mu_Y^2 \cdot \exp(\sigma^2)$$

$$\sigma_Y^2 = E[Y_i^2] - [E(Y_i)]^2$$

$$= \mu_Y^2 \cdot \exp(\sigma^2) - \mu_Y^2$$

$$= \mu_Y^2 (\exp(\sigma^2) - 1)$$

(b) For $X_k = \log Y_k$, $k=1,\cdots,n$

$X_k \sim N(\mu,\sigma^2)$

$$f(x_k) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x_k-\mu)^2}{2\sigma^2}\right).$$

$$L_X(\theta) = \prod_{k=1}^{n} f(x_k) = (2\pi)^{-\frac{n}{2}}\sigma^{-n}\exp\left(-\frac{\sum_{k=1}^{n}(x_k-\mu)^2}{2\sigma^2}\right)$$

$$l_X(\theta) = \log L_X(\theta) = -\frac{n}{2}\log(2\pi) - n\log\sigma - \frac{1}{2\sigma^2}\sum_{k=1}^{n}(x_k-\mu)^2$$

$$\frac{\partial l_X(\theta)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{k=1}^{n}(x_k-\mu) = 0$$

$$\hat{\mu}_X = \frac{1}{n}\sum_{k=1}^{n} X_k$$

$$\frac{\partial l_X(\theta)}{\partial\sigma} = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{k=1}^{n}(x_k-\mu)^2 = 0$$

$$\hat{\sigma}_X^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\hat{\mu})^2 = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \sum_{k=1}^{n}X_k\right)^2$$

According to the properties of MLE estimators, μ_Y and σ_Y^2 are functions of μ_X, σ_X^2

$\therefore$ the MLE for (μ_Y, σ_Y^2) is

$$\hat{\mu}_Y = \exp\left(\hat{\mu}_X + \frac{\hat{\sigma}_X^2}{2}\right) = \exp\left(\frac{1}{n}\sum_{i=1}^{n}X_i + \frac{1}{2}S\right)$$

$$\hat{\sigma}_Y = \hat{\mu}_Y^2\left(\exp(S) - 1\right)$$

where $S = \frac{1}{n}\sum_{i=1}^{n}\left(X_i - \frac{1}{n}\sum_{k=1}^{n}X_k\right)^2$

$X_k = \log Y_k$, $k=1,\cdots,n$

4.

4. According to definition:

$$I(\vec{\theta}) = E\left[\left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}} \right) \left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}} \right)^T \right]$$

$$= E\left[\left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}} \right) \left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}^T} \right) \Big| \vec{\theta} \right]$$

$$\frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) = \frac{\partial}{\partial \vec{\theta}} \left(\frac{\frac{\partial}{\partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})}{f_{\vec{Y}}(\vec{y};\vec{\theta})} \right)$$

$$= \frac{\frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})}{f_{\vec{Y}}(\vec{y};\vec{\theta})} - \frac{\frac{\partial}{\partial \vec{\theta}} f_{\vec{Y}}(\vec{y};\vec{\theta}) \cdot \frac{\partial}{\partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})}{f_{\vec{Y}}(\vec{y};\vec{\theta})^2}$$

$$= \frac{\frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})}{f_{\vec{Y}}(\vec{y};\vec{\theta})} - \frac{\partial}{\partial \vec{\theta}} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) \cdot \frac{\partial}{\partial \vec{\theta}^T} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) \quad ①$$

$$\because E\left(\frac{\frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})}{f_{\vec{Y}}(\vec{y};\vec{\theta})} \Big| \vec{\theta} \right) = \int \frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} f_{\vec{Y}}(\vec{y};\vec{\theta})\, d\vec{y}$$

$$= \frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} \cdot 1 = 0. \quad ②$$

$\therefore$ plug ② into ①.

$$E\left[\frac{\partial^2}{\partial \vec{\theta} \partial \vec{\theta}^T} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) \right] = -E\left[\frac{\partial}{\partial \vec{\theta}} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) \cdot \frac{\partial}{\partial \vec{\theta}^T} \log f_{\vec{Y}}(\vec{y};\vec{\theta}) \right]$$

$$\therefore I(\vec{\theta}) = E\left[\left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}} \right) \left(\frac{\partial \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}} \right)^T \right]$$

$$= -E\left[\left(\frac{\partial^2 \log f_{\vec{Y}}(\vec{y};\vec{\theta})}{\partial \vec{\theta}^2} \right) \right]$$