The regularity number of a finite group

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joint work with Tim Burness

Groups in Florence IV

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Examples

- $b(S_n, \{1, \ldots, n\}) = n - 1$
- $b(\mathrm{GL}(V), V) = \dim(V)$

Note: In general, computing $b(G, \Omega)$ is **hard**.

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- **Vdovin:** reduction to almost simple groups
- **Burness:** primitive groups & sporadic socle
- **Baykalov:** alternating socle & current work on classical groups

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Remark: If G is transitive with point stabiliser H, then $b(G,\Omega) \leqslant k \iff$ the k-tuple $(H, \dots, H)$ is regular.

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If $\mathcal{S} = \{H \leqslant G \,:\, H \text{ core-free}\}$ and $\mathcal{P} \subseteq \mathcal{S}$, then we define:

$$R_{\mathcal{P}}(G) = \min\{k \,:\, \text{every tuple in } \mathcal{P}^k \text{ is regular}\}$$

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- **Conjecture 1:** If G is almost simple, then $R_{\mathrm{ns}}(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$.
- **Conjecture 2:** We have $R_{\mathrm{sol}}(G) \leqslant 5$ for every finite group.

Results

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Theorem A (A-M & Burness | 2024+)

Let G be almost simple with socle A_n. Then

- If $G \in \{S_n, A_n\}$, then $R(G) = n - |S_n : G|$
- $R_{\mathrm{ns}}(G) \leqslant 6$, with $R_{\mathrm{ns}}(G) = 2$ if $n \geqslant 13$
- $R_{\mathrm{sol\,max}}(G) \leqslant 5$, with $R_{\mathrm{sol\,max}}(G) = 2$ if $n \geqslant 17$

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Theorem B (A-M & Burness | 2024+)

Let G be almost simple with sporadic socle. Then

- $R(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$
- $R_{\mathrm{sol}}(G) \leqslant 3$

Final remarks

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Future goals:

1. Prove Conjecture 1 for all almost simple groups of Lie type
2. Prove that $R_{\mathrm{sol\,max}}(G) \leqslant 5$ for all almost simple groups of Lie type
3. Prove Conjecture 2 for S_n and A_n