The regularity number of a finite group

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Joint work with Tim Burness

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Remark: $b(G, \Omega)$ is the smallest k for which there exist $g_1, \ldots, g_k \in G$ such that $\bigcap_{i=1}^{k} H^{g_i} = 1$ (i.e. G has a regular orbit on $(G/H)^k$).

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The **base number** of G, denoted by $B(G)$, is the maximum base size over all transitive faithful permutation representations of G.

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Question: When does H have a regular orbit on $(G/K)^k$ for some $K \leqslant G$ core-free?

For example, if H is a stabiliser of a k-set in S_n, then what is its base size when it acts on partitions of $\{1, \ldots, n\}$?

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Remark: $b(G, \Omega) \leqslant k \iff (\underbrace{H, \ldots, H}_{k})$ is regular.

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The **regularity number** of G, denoted by $R(G)$, is the smallest k such that all core-free k-tuples of G are regular.

Remark: $B(G)$ is the smallest k such that all core-free **conjugate** k-tuples of G are regular.

If $\mathcal{S} = \{H \leqslant G \,:\, H \text{ core-free}\}$ and $\mathcal{P} \subseteq \mathcal{S}$, then we define:

- $R_{\mathcal{P}}(G) = \min\{k \,:\, \text{every tuple in } \mathcal{P}^k \text{ is regular}\}$
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$$b(G, \Omega) \leqslant B(G) \leqslant R(G)$$

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- If Ω is a set of k-spaces and $e_1, \dots, e_n$ is a basis for V, then

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- $R(G) - B(G)$ can be arbitrarily large!

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- **Vdovin:** reduction to almost simple groups
- **Burness:** $R_{\mathrm{sol\,max}}(G) \leqslant 5$ & sporadic socle
- **Baykalov:** alternating socle & current work on classical groups

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Conjecture 1: If G is almost simple, then $R_{\mathrm{ns}}(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$.

Conjecture 2: $R_{\mathrm{sol}}(G) \leqslant 5$ for every finite group G.

Results

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Theorem A (A-M & Burness | 2024+)

Let G be almost simple with socle A_n. Then

- If $G \in \{S_n, A_n\}$, then $R(G) = n - |S_n : G|$
- $R_{\mathrm{ns}}(G) \leqslant 6$, with $R_{\mathrm{ns}}(G) = 2$ if $n \geqslant 13$
- $R_{\mathrm{sol\,max}}(G) \leqslant 5$, with $R_{\mathrm{sol\,max}}(G) = 2$ if $n \geqslant 17$

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Theorem B (A-M & Burness | 2024+)

Let G be almost simple with sporadic socle. Then

- $R(G) \leqslant 7$ with equality if and only if $G = \mathrm{M}_{24}$
- $R_{\mathrm{sol}}(G) \leqslant 3$

Theorem C (A-M | 2024+)

Let G be almost simple with classical socle and natural module V.

- If $\dim V \geqslant 11$, then $R_{\mathrm{ns}}(G) \leqslant 4$
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Future goals:

1. Prove Conjecture 1 for all almost simple groups of Lie type
2. Prove that $R_{\mathrm{sol\,max}}(G) \leqslant 5$ for all almost simple groups of Lie type
3. Prove Conjecture 2 for S_n and A_n