命题逻辑系统

作业参考答案

1.证明下列公式是重言式

(a) $A \to (\neg A \to B)$

Proof.

任取$v \in \Omega$，以E记公式$A \to (\neg A \to B)$，则由v的同态可知：

$$v(E) = v(A) \to (\neg v(A) \to v(B))$$

分别用a, b表示$v(A), v(B)$，则上式可写成：$v(E) = a \to (\neg a \to b)$

$\therefore$证明E是重言式$\Leftrightarrow$证明$a \to (\neg a \to b) = 1 \Leftrightarrow$证明$a = 1$时，$\neg a \to b$不为0.

$\because \neg a \to b = 0 \to b = 1$

$\therefore A \to (\neg A \to B)$是重言式。 $\square$

1.证明下列公式是重言式

(b) $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$

Proof.

任取$v \in \Omega$，以E记$(A \to (B \to C)) \to ((A \to B) \to (A \to C))$，则由$v$的同态可知：$v(E) = (v(A) \to (v(B) \to v(C))) \to ((v(A) \to v(B)) \to (v(A) \to v(C)))$
分别用a, b, c表示$v(A), v(B), v(C)$，则上式可写成：

$$v(E) = (a \to (b \to c)) \to ((a \to b) \to (a \to c))$$

$\therefore$证明E是重言式$\Leftrightarrow$证明$(a \to (b \to c)) \to ((a \to b) \to (a \to c)) = 1$
$\Leftrightarrow$证明$a \to (b \to c) = 1$时，$(a \to b) \to (a \to c)$ 不为0.

1) $a = 0$时，$(a \to b) \to (a \to c) = (0 \to b) \to (0 \to c) = 1$
2) $a = 1$时，则$b \to c = 1$
   若$b = 0$，则$(a \to b) \to (a \to c) = (1 \to 0) \to (1 \to c) = 0 \to (1 \to c) = 1$
   若$b = 1$，则$c = 1$，所以$(a \to b) \to (a \to c) = 1$.

$\therefore A \to (B \to C)) \to ((A \to B) \to (A \to C))$是重言式。 □

1.证明下列公式是重言式

(c) $(\neg A \to \neg B) \to (B \to A)$

Proof.

任取$v \in \Omega$，以E记公式$(\neg A \to \neg B) \to (B \to A)$，则由$v$的同态可知：

$$v(E) = (\neg v(A) \to \neg v(B)) \to (v(B) \to v(A))$$

分别用a, b表示$v(A), v(B)$，则上式可写成：$v(E) = (\neg a \to \neg b) \to (b \to a)$

$\therefore$证明E是重言式$\Leftrightarrow$证明$(\neg a \to \neg b) \to (b \to a) = 1$

$\Leftrightarrow$证明$\neg a \to \neg b = 1$ 时，$b \to a$不为0.

1) $a = 0$，即$\neg a = 1$时，则$\neg b = 1$，$b = 0$，所以$b \to a = 0 \to 0 = 1$
2) $a = 1$，即$\neg a = 0$时，
   若$b = 1$，$b \to a = 1 \to 1 = 1$
   若$b = 0$，$b \to a = 0 \to 1 = 1$

$\therefore (\neg A \to \neg B) \to (B \to A)$是重言式。 □

2.证明下列各条成立

(a) $(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$

Proof.

$$
\begin{aligned}
\text{左边} &= \neg(A \vee B) \vee C \\
&= (\neg A \wedge \neg B) \vee C \\
&= (\neg A \vee C) \wedge (\neg B \vee C) \\
&= (A \rightarrow C) \wedge (B \rightarrow C) = \text{右边}
\end{aligned}
$$

$\therefore (A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$ □

2.证明下列各条成立

(b) $(A \wedge B) \to C = (A \to C) \vee (B \to C)$

Proof.

$$
\begin{aligned}
\text{左边} &= \neg(A \wedge B) \vee C \\
&= (\neg A \vee \neg B) \vee C \\
&= (\neg A \vee C) \vee (\neg B \vee C) \\
&= (A \to C) \vee (B \to C) = \text{右边}
\end{aligned}
$$

$\therefore (A \wedge B) \to C = (A \to C) \vee (B \to C)$ $\square$

2.证明下列各条成立

(c) $A \to (B \to C) = B \to (A \to C)$

Proof.

$$
\begin{aligned}
\text{左边} &= A \to (\neg B \vee C) \\
&= (\neg A) \vee (\neg B \vee C) \\
&= (\neg B) \vee (\neg A \vee C) \\
&= B \to (\neg A \vee C) \\
&= B \to (A \to C) = \text{右边}
\end{aligned}
$$

$\therefore A \to (B \to C) = B \to (A \to C)$ $\square$

3.求公式$(\neg p_1 \to p_2) \to p_3$的析取范式和合取范式

Solution

$$
\begin{aligned}
& (\neg p_1 \to p_2) \to p_3 \\
= & (p_1 \vee p_2) \to p_3 \\
= & \neg(p_1 \vee p_2) \vee p_3 \\
= & (\neg p_1 \wedge \neg p_2) \vee p_3 \Rightarrow \text{析取范式} \\
= & (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee p_3 \\
= & (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_3) \vee (\neg p_1 \wedge p_3) \\
= & (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \\
& \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge p_3)
\end{aligned}
$$

Solution

$$
\begin{aligned}
& (\neg p_1 \to p_2) \to p_3 \\
= & (p_1 \vee p_2) \to p_3 \\
= & \neg(p_1 \vee p_2) \vee p_3 \\
= & (\neg p_1 \wedge \neg p_2) \vee p_3 \Rightarrow \text{合取范式} \\
= & (\neg p_1 \vee p_3) \wedge (\neg p_2 \vee p_3) \\
= & (\neg p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3)
\end{aligned}
$$

4.计算下列逻辑公式的真度

(a) $(p_1 \vee p_2) \to p_3$

Solution

给定命题变元p_1, p_2, p_3, 作真值表, 有:

公式	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$
$(p_1 \vee p_2) \to p_3$	*1*	*1*	*0*	*1*

$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
0	*1*	*0*	*1*

$\therefore$ 其真度为$\tau((p_1 \vee p_2) \to p_3) = \frac{|T|}{2^3} = \frac{5}{8}$.

4.计算下列逻辑公式的真度

(b) $(p_1 \to p_2) \vee (p_3 \to p_4)$

Solution

给定命题变元p_1, p_2, p_3, p_4, 作真值表, 有:

公式	$(0,0,0,0)$	$(0,0,0,1)$	$(0,0,1,0)$	$(0,0,1,1)$
$(p_1 \to p_2) \vee (p_3 \to p_4)$	*1*	*1*	*1*	*1*

$(0,1,0,0)$	$(0,1,0,1)$	$(0,1,1,0)$	$(0,1,1,1)$	$(1,0,0,0)$
1	*1*	*1*	*1*	*1*

$(1,0,0,1)$	$(1,0,1,0)$	$(1,0,1,1)$	$(1,1,0,0)$	$(1,1,0,1)$
1	*0*	*1*	*1*	*1*

$(1,1,1,0)$	$(1,1,1,1)$
1	*1*

$\therefore$ 其真度为$\tau((p_1 \to p_2) \vee (p_3 \to p_4)) = \frac{|T|}{2^4} = \frac{15}{16}$.

4.计算下列逻辑公式的真度

(c) $(\neg p_1 \to p_2) \to p_3$

Solution

给定命题变元p_1, p_2, p_3, 作真值表, 有:

公式	$(0,0,0)$	$(0,0,1)$	$(0,1,0)$	$(0,1,1)$
$(\neg p_1 \to p_2) \to p_3$	*1*	*1*	*0*	*1*

$(1,0,0)$	$(1,0,1)$	$(1,1,0)$	$(1,1,1)$
0	*1*	*0*	*1*

$\therefore$ 其真度为$\tau((\neg p_1 \to p_2) \to p_3) = \frac{|T|}{2^3} = \frac{5}{8}$.

5.设$A \downarrow B$表示$\neg(A \vee B)$，证明连接符$\{\downarrow\}$ 是命题逻辑连接符集的充足集

Proof.

$\{\neg, \vee\}$ 是充足集, 所以, 对任意公式F, F 能用'$\neg$' 和'$\vee$'表示,
用数学归纳法证明: 若$|F| = 1$, 即F为原子公式. 则必为一个原子命题或其否定形式, 记原子命题为p, 则$F = p$ 或$F = \neg p$成立, 于是有:

1) $F = p = \neg\neg(p \vee p) = \neg(p \downarrow p) = \neg((p \downarrow p) \vee (p \downarrow p)) = (p \downarrow p) \downarrow (p \downarrow p)$
2) $F = \neg p = \neg(p \vee p) = p \downarrow p$.

假设, 对于所有的公式F, 若$|F| < n$ 则F 能用'$\downarrow$'表示. 现考虑任意满足$|F| = n$的公式F:

1) 若$F = \neg A$(A 为F的子公式), 由$\downarrow$的定义得：
   $F = \neg A = \neg(A \vee A) = A \downarrow A$.
2) 若$F = A \vee B$(A, B 为F的子公式), 由$\downarrow$的定义及1)得:
   $F = A \vee B = \neg(\neg(A \vee B)) = \neg(A \downarrow B) = (A \downarrow B) \downarrow (A \downarrow B)$.

无论以上何种情况，A, B 均满足$|A| < n$, $|B| < n$. 所以, A, B必能用'$\downarrow$' 表示.
$\therefore$ 由数学归纳法知所有公式F均能用$\downarrow$ 表示, 即$\{\downarrow\}$ 是充足集. □

6.研究题：任给分数$\frac{n}{m}(0 \leq n \leq m)$，是否存在一个命题公式$A$ 使得A 的真度$\tau(A) = \frac{n}{m}$?

Proof.

不能. 由真度的定义知, 对于任意公式A, 其真度的形式必定为:

$$\tau(A) = \frac{M}{2^N}$$

若$|A| = n$, $M, N \in \mathbb{N}^+$. 若该命题成立, 即对于任意$0 \leq n \leq m$, 有:

$$\frac{n}{m} = \frac{M}{2^N}$$

$q = \frac{n}{m} \leq 1$是小于1的任意有理数, 即所有小于1的有理数都具有$\frac{M}{2^N}$ 的形式, 这是不可能的! $\therefore$ 原命题不成立. □

1.试证：

1) $\vdash A \to (B \to (A \to B))$.

Proof.

构造推演:

$$
\begin{array}{lcc}
1. & B \to (A \to B) & L_1 \\
2. & (B \to (A \to B)) \to (A \to (B \to (A \to B))) & L_1 \\
3. & A \to (B \to (A \to B)) & MP(1,2)
\end{array}
$$

□

1.试证：

2) $\vdash (B \to C) \to ((A \to B) \to (A \to C))$.

Proof.

构造推演:

$$
\begin{array}{lcc}
1. & B \to C & \Gamma \\
2. & (B \to C) \to (A \to (B \to C)) & L_1 \\
3. & A \to (B \to C) & MP(1,2) \\
4. & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) & L_2 \\
5. & (A \to B) \to (A \to C) & MP(3,4) \\
\therefore & \{B \to C\} \vdash (A \to B) \to (A \to C) &
\end{array}
$$

由演绎定理得到，$\vdash (B \to C) \to ((A \to B) \to (A \to C))$

$\square$

1.试证：

3) $\vdash (A \to (A \to B)) \to (A \to B)$.

Proof.

构造推演:

$$
\begin{array}{llc}
1. & A & \Gamma \\
2. & A \to (A \to B) & \Gamma \\
3. & A \to B & MP(1,2) \\
4. & B & MP(1,3) \\
\therefore & \{A, A \to (A \to B)\} \vdash B &
\end{array}
$$

由演绎定理得到，$\{A \to (A \to B)\} \vdash A \to B$

$\vdash (A \to (A \to B)) \to (A \to B)$

□

2. 试证:

1) $(A \to (B \to C)) \approx (B \to (A \to C))$.

Proof.

" $\Rightarrow$ " 证 $\vdash ((A \to (B \to C)) \to (B \to (A \to C)))$:

$$
\begin{array}{lcc}
1. & A \to (B \to C) & \Gamma \\
2. & (A \to (B \to C)) \to ((A \to B) \to (A \to C)) & L_2 \\
3. & (A \to B) \to (A \to C) & MP(1,2) \\
4. & B & \Gamma \\
5. & B \to (A \to B) & L_1 \\
6. & A \to B & MP(4,5) \\
7. & A \to C & MP(3,6) \\
\therefore & \{A \to (B \to C), B\} \vdash A \to C &
\end{array}
$$

由演绎定理得到，$\{A \to (B \to C)\} \vdash B \to (A \to C)$

$\vdash ((A \to (B \to C)) \to (B \to (A \to C)))$

Proof (Cont.)

"$\Leftarrow$" 证 $\vdash (B \to (A \to C)) \to (A \to (B \to C))$:

$$
\begin{array}{lcc}
1. & B \to (A \to C) & \Gamma \\
2. & (B \to (A \to C)) \to ((B \to A) \to (B \to C)) & L_2 \\
3. & (B \to A) \to (B \to C) & MP(1,2) \\
4. & A & \Gamma \\
5. & A \to (B \to A) & L_1 \\
6. & B \to A & MP(4,5) \\
7. & B \to C & MP(3,6) \\
\therefore & \{B \to (A \to C), A\} \vdash B \to C &
\end{array}
$$

由演绎定理得到，$\{B \to (A \to C)\} \vdash A \to (B \to C)$

$$\vdash (B \to (A \to C)) \to (A \to (B \to C))$$

综上，$(A \to (B \to C)) \approx (B \to (A \to C))$.

$\square$

2.试证:

2) $(A \to (A \to B)) \approx (A \to B)$.

Proof.

" $\Rightarrow$ " 证 $\vdash (A \to (A \to B)) \to (A \to B)$:

$$\begin{array}{lcc}
1. & A \to (A \to B) & \Gamma \\
2. & A & \Gamma \\
3. & A \to B & MP(1,2) \\
4. & B & MP(2,3) \\
\therefore & \{\{A \to (A \to B), A\} \vdash B &
\end{array}$$

由演绎定理得到, $\{A \to (A \to B)\} \vdash A \to B$, $\vdash (A \to (A \to B)) \to (A \to B)$

" $\Leftarrow$ " 证 $\vdash (A \to B) \to (A \to (A \to B))$:

$$\begin{array}{lcc}
1. & A \to B & \Gamma \\
2. & (A \to B) \to (A \to (A \to B)) & L_1 \\
3. & A \to (A \to B) & MP(1,2) \\
\therefore & \{A \to B\} \vdash A \to (A \to B) &
\end{array}$$

由演绎定理得到, $\vdash A \to B \to A \to (A \to B)$

综上, $A \to (A \to B) \approx A \to B$. $\square$

1.利用L的完备性定理证明以下各式成立：

(a) $\vdash (\neg A \to A) \to A$

Proof.

$$
\begin{aligned}
& (\neg A \to A) \to A \\
= \quad & (\neg(\neg A) \vee A) \to A \\
= \quad & A \to A \\
= \quad & \neg A \vee A
\end{aligned}
$$

任取$v \in \Omega$，可知：$v(\neg A \vee A) = 1 \therefore \models ((\neg A \to A) \to A)$
由完备性定理可知，$\vdash ((\neg A \to A) \to A)$

$\square$

1.利用L的完备性定理证明以下各式成立：

(b) $\vdash \neg(A \to B) \to (B \to A)$

Proof.

$$\begin{aligned}
& \neg(A \to B) \to (B \to A) \\
= \quad & \neg(\neg A \vee B) \to (\neg B \vee A) \\
= \quad & \neg(A \wedge \neg B) \vee (\neg B \vee A) \\
= \quad & (\neg A \vee B) \vee (\neg B \vee A) \\
= \quad & \neg A \vee A \vee \neg B \vee B
\end{aligned}$$

任取$v \in \Omega$，可知：$v(\neg A \vee A \vee \neg B \vee B) = 1$

$\therefore \models (\neg(A \to B) \to (B \to A))$

由完备性定理可知，$\vdash (\neg(A \to B) \to (B \to A))$

$\square$

1.利用L的完备性定理证明以下各式成立：

(c) $((A \vee B) \to C) \approx (A \to C) \wedge (B \to C)$

Proof.

$$
\begin{array}{rl}
 & (A \vee B) \to C \\
= & \neg(A \vee B) \vee C \\
= & (\neg A \wedge \neg B) \vee C \\
= & (\neg A \wedge \neg B) \vee (\neg A \vee \neg B \vee C) \wedge C \\
= & (\neg A \vee C) \wedge (\neg B \vee C) \\
= & (A \to C) \wedge (B \to C) \\
\therefore & (A \vee B) \to C = (A \to C) \wedge (B \to C)
\end{array}
$$

由于$(A \vee B) \to C) = (A \to C) \wedge (B \to C)$

任取$v \in \Omega$，则$v((A \vee B) \to C) = v((A \to C) \wedge (B \to C))$

$\therefore \models (((A \vee B) \to C) \to (A \to C) \wedge (B \to C))$

且$\models ((A \to C) \wedge (B \to C) \to ((A \vee B) \to C))$

由完备性定理可知，$\vdash (((A \vee B) \to C) \to (A \to C) \wedge (B \to C))$

且$\vdash ((A \to C) \wedge (B \to C) \to ((A \vee B) \to C))$

所以，$((A \vee B) \to C) \approx (A \to C) \wedge (B \to C)$

$\square$

1.利用L的完备性定理证明以下各式成立：

(d) $((A \wedge B) \to C) \approx (A \to C) \vee (B \to C)$

Proof.

$$
\begin{aligned}
& (A \wedge B) \to C \\
= \quad & \neg(A \wedge B) \vee C \\
= \quad & (\neg A \vee \neg B) \vee C \\
= \quad & (\neg A \vee C) \vee (\neg B \vee C) \\
= \quad & (A \to C) \vee (B \to C) \\
\therefore \quad & (A \wedge B) \to C = (A \to C) \vee (B \to C)
\end{aligned}
$$

由于$(A \wedge B) \to C) = (A \to C) \vee (B \to C)$

任取$v \in \Omega$，则$v((A \wedge B) \to C) = v((A \to C) \vee (B \to C))$

$\therefore \models (((A \wedge B) \to C) \to (A \to C) \vee (B \to C))$

且$\models ((A \to C) \vee (B \to C) \to ((A \wedge B) \to C))$

由完备性定理可知，$\vdash (((A \wedge B) \to C) \to (A \to C) \vee (B \to C))$

且$\vdash ((A \to C) \vee (B \to C) \to ((A \wedge B) \to C))$

所以，$((A \wedge B) \to C) \approx (A \to C) \vee (B \to C)$

□

2. 设$\Gamma \subseteq F(S)$, Γ是有限集，$A \in F(S)$. 证明：
$\Gamma \vdash A$当且仅当$\Gamma \models A$。其中$\Gamma \models A$定义为对于任何赋值v若对于Γ中的每个成员B只要$v(B) = 1$就有$v(A) = 1$.

Proof.

首先，由Γ的有限性，不妨设$\Gamma = \{F_1, F_2, ..., F_n\}$.

先证明一个引理，
引理1：设$\Gamma \subseteq F(S)$, $A, B \in F(S)$为任一公式，则$\Gamma \cup \{A\} \models B$ 当且仅当$\Gamma \models A \to B$.

证明:
充分性，若$\Gamma \cup \{A\} \models B$，根据定义，对于任何赋值$v$，若有$v(F_i) = 1$ $(\forall F_i \in \Gamma)$且$v(A) = 1$，则$v(B) = 1$成立. 对于$v(A \to B)$，当$v(A) = 0$时，$v(A \to B) = 1$，当$v(A) = 1$时，因为$v(F_i) = 1$ $(\forall F_i \in \Gamma)$ 成立，所以$v(B) = 1$成立. 所以对任何赋值v，有若$v(F_i) = 1$ $(\forall F_i \in \Gamma)$成立，则$v(A \to B) = 1$成立，即$\Gamma \models A \to B$.

Proof(Cont.)

必要性，若$\Gamma \models A \to B$，即对任何赋值v，若$v(F_i) = 1$ $(\forall F_i \in \Gamma)$成立，则$v(A \to B)$成立．若$v(A) = 1$，则必有$v(B) = 1$．所以有$\Gamma \cup \{A\} \models B$成立．引理1证毕．

现证明原命题，
1)充分性：

$$
\begin{gathered}
\because \Gamma \vdash A,\ \text{即}\{F_1, F_2, ..., F_n\} \vdash A \\
\therefore \{F_1, F_2, ..., F_{n-1}\} \vdash F_n \to A \\
\therefore \{F_1, F_2, ..., F_{n-2}\} \vdash F_{n-1} \to (F_n \to A) \\
...... \\
\therefore \vdash F_1 \to (F_2 \to ... \to (F_n \to A))...) \\
\text{由完备性得：} \models F_1 \to (F_2 \to ... \to (F_n \to A))...) \\
\text{由引理1得：} \{F_1\} \models F_2 \to (F_3 \to ... \to (F_n \to A))...) \\
\therefore \{F_1, F_2\} \models F_3 \to ... \to (F_n \to A))...) \\
...... \\
\therefore \Gamma \models A
\end{gathered}
$$

Proof(Cont.)

2)必要性:

$$\because \Gamma \models A,\ 即\{F_1, F_2, ..., F_n\} \models A$$
$$由引理1得:\ \{F_1, F_2, ..., F_{n-1}\} \models F_n \to A$$
$$\therefore \{F_1, F_2, ..., F_{n-2}\} \models F_{n-1} \to (F_n \to A)$$
$$......$$
$$\therefore \models F_1 \to (F_2 \to ... \to (F_n \to A))...)$$
$$由完备性得:\ \vdash F_1 \to (F_2 \to ... \to (F_n \to A))...)$$
$$\therefore \{F_1\} \vdash F_2 \to (F_3 \to ... \to (F_n \to A))...)$$
$$\therefore \{F_1, F_2\} \vdash F_3 \to ... \to (F_n \to A))...)$$
$$......$$
$$\therefore \Gamma \vdash A$$

综上, 若Γ有限, 则$\Gamma \vdash A$ 当且仅当$\Gamma \models A$.