

# 命题逻辑系统

## 作业参考答案

# 1.证明下列公式是重言式

$$(a) A \rightarrow (\neg A \rightarrow B)$$

Proof.

任取  $v \in \Omega$ , 以  $E$  记公式  $A \rightarrow (\neg A \rightarrow B)$ , 则由  $v$  的同态可知:

$$v(E) = v(A) \rightarrow (\neg v(A) \rightarrow v(B))$$

分别用  $a, b$  表示  $v(A), v(B)$ , 则上式可写成:  $v(E) = a \rightarrow (\neg a \rightarrow b)$

$\therefore$  证明  $E$  是重言式  $\Leftrightarrow$  证明  $a \rightarrow (\neg a \rightarrow b) = 1 \Leftrightarrow$  证明  $a = 1$  时,  $\neg a \rightarrow b$  不为 0.

$$\therefore \neg a \rightarrow b = 0 \rightarrow b = 1$$

$\therefore A \rightarrow (\neg A \rightarrow B)$  是重言式。



# 1.证明下列公式是重言式

$$(b) (A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$$

Proof.

任取  $v \in \Omega$ , 以  $E$  记  $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ , 则由  $v$  的同态可知:  $v(E) = (v(A) \rightarrow (v(B) \rightarrow v(C))) \rightarrow ((v(A) \rightarrow v(B)) \rightarrow (v(A) \rightarrow v(C)))$   
分别用  $a, b, c$  表示  $v(A), v(B), v(C)$ , 则上式可写成:

$$v(E) = (a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c))$$

$\therefore$  证明  $E$  是重言式  $\Leftrightarrow$  证明  $(a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$

$\Leftrightarrow$  证明  $a \rightarrow (b \rightarrow c) = 1$  时,  $(a \rightarrow b) \rightarrow (a \rightarrow c)$  不为 0.

1)  $a = 0$  时,  $(a \rightarrow b) \rightarrow (a \rightarrow c) = (0 \rightarrow b) \rightarrow (0 \rightarrow c) = 1$

2)  $a = 1$  时, 则  $b \rightarrow c = 1$

若  $b = 0$ , 则  $(a \rightarrow b) \rightarrow (a \rightarrow c) = (1 \rightarrow 0) \rightarrow (1 \rightarrow c) = 0 \rightarrow (1 \rightarrow c) = 1$

若  $b = 1$ , 则  $c = 1$ , 所以  $(a \rightarrow b) \rightarrow (a \rightarrow c) = 1$ .

$\therefore A \rightarrow (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$  是重言式。 □

# 1.证明下列公式是重言式

$$(c) (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$$

Proof.

任取  $v \in \Omega$ , 以  $E$  记公式  $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$ , 则由  $v$  的同态可知:

$$v(E) = (\neg v(A) \rightarrow \neg v(B)) \rightarrow (v(B) \rightarrow v(A))$$

分别用  $a, b$  表示  $v(A), v(B)$ , 则上式可写成:  $v(E) = (\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a)$

$\therefore$  证明  $E$  是重言式  $\Leftrightarrow$  证明  $(\neg a \rightarrow \neg b) \rightarrow (b \rightarrow a) = 1$

$\Leftrightarrow$  证明  $\neg a \rightarrow \neg b = 1$  时,  $b \rightarrow a$  不为 0.

1)  $a = 0$ , 即  $\neg a = 1$  时, 则  $\neg b = 1$ ,  $b = 0$ , 所以  $b \rightarrow a = 0 \rightarrow 0 = 1$

2)  $a = 1$ , 即  $\neg a = 0$  时,

若  $b = 1$ ,  $b \rightarrow a = 1 \rightarrow 1 = 1$

若  $b = 0$ ,  $b \rightarrow a = 0 \rightarrow 1 = 1$

$\therefore (\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$  是重言式。



## 2.证明下列各条成立

$$(a) (A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$$

Proof.

$$\begin{aligned} \text{左边} &= \neg(A \vee B) \vee C \\ &= (\neg A \wedge \neg B) \vee C \\ &= (\neg A \vee C) \wedge (\neg B \vee C) \\ &= (A \rightarrow C) \wedge (B \rightarrow C) = \text{右边} \\ \therefore (A \vee B) \rightarrow C &= (A \rightarrow C) \wedge (B \rightarrow C) \end{aligned}$$



## 2.证明下列各条成立

$$(b) (A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C)$$

Proof.

$$\begin{aligned} \text{左边} &= \neg(A \wedge B) \vee C \\ &= (\neg A \vee \neg B) \vee C \\ &= (\neg A \vee C) \vee (\neg B \vee C) \\ &= (A \rightarrow C) \vee (B \rightarrow C) = \text{右边} \\ \therefore (A \wedge B) \rightarrow C &= (A \rightarrow C) \vee (B \rightarrow C) \end{aligned}$$



## 2.证明下列各条成立

$$(c) A \rightarrow (B \rightarrow C) = B \rightarrow (A \rightarrow C)$$

Proof.

$$\begin{aligned}\text{左边} &= A \rightarrow (\neg B \vee C) \\ &= (\neg A) \vee (\neg B \vee C) \\ &= (\neg B) \vee (\neg A \vee C) \\ &= B \rightarrow (\neg A \vee C) \\ &= B \rightarrow (A \rightarrow C) = \text{右边} \\ \therefore A \rightarrow (B \rightarrow C) &= B \rightarrow (A \rightarrow C)\end{aligned}$$



### 3.求公式 $(\neg p_1 \rightarrow p_2) \rightarrow p_3$ 的析取范式和合取范式

#### Solution

$$\begin{aligned} & (\neg p_1 \rightarrow p_2) \rightarrow p_3 \\ &= (p_1 \vee p_2) \rightarrow p_3 \\ &= \neg(p_1 \vee p_2) \vee p_3 \\ &= (\neg p_1 \wedge \neg p_2) \vee p_3 \Rightarrow \text{析取范式} \\ &= (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee p_3 \\ &= (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_3) \vee (\neg p_1 \wedge p_3) \\ &= (\neg p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge \neg p_3) \vee (p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge \neg p_2 \wedge p_3) \\ &\quad \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (\neg p_1 \wedge \neg p_2 \wedge p_3) \end{aligned}$$

#### Solution

$$\begin{aligned} & (\neg p_1 \rightarrow p_2) \rightarrow p_3 \\ &= (p_1 \vee p_2) \rightarrow p_3 \\ &= \neg(p_1 \vee p_2) \vee p_3 \\ &= (\neg p_1 \wedge \neg p_2) \vee p_3 \Rightarrow \text{合取范式} \\ &= (\neg p_1 \vee p_3) \wedge (\neg p_2 \vee p_3) \\ &= (\neg p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \wedge (p_1 \vee \neg p_2 \vee p_3) \wedge (\neg p_1 \vee \neg p_2 \vee p_3) \end{aligned}$$



## 4. 计算下列逻辑公式的真度

$$(a) (p_1 \vee p_2) \rightarrow p_3$$

### Solution

给定命题变元 $p_1, p_2, p_3$ , 作真值表, 有:

| 公式                               | (0, 0, 0) | (0, 0, 1) | (0, 1, 0) | (0, 1, 1) |
|----------------------------------|-----------|-----------|-----------|-----------|
| $(p_1 \vee p_2) \rightarrow p_3$ | 1         | 1         | 0         | 1         |

| (1, 0, 0) | (1, 0, 1) | (1, 1, 0) | (1, 1, 1) |
|-----------|-----------|-----------|-----------|
| 0         | 1         | 0         | 1         |

$$\therefore \text{其真度为 } \tau((p_1 \vee p_2) \rightarrow p_3) = \frac{|T|}{2^3} = \frac{5}{8}.$$

## 4. 计算下列逻辑公式的真度

$$(b) (p_1 \rightarrow p_2) \vee (p_3 \rightarrow p_4)$$

### Solution

给定命题变元 $p_1, p_2, p_3, p_4$ , 作真值表, 有:

| 公式   | $(0, 0, 0, 0)$ | $(0, 0, 0, 1)$ | $(0, 0, 1, 0)$ | $(0, 0, 1, 1)$ |
|--|----------------|----------------|----------------|----------------|
| $(p_1 \rightarrow p_2) \vee (p_3 \rightarrow p_4)$ | 1              | 1              | 1              | 1              |

|                |                |                |                |                |
|----------------|----------------|----------------|----------------|----------------|
| $(0, 1, 0, 0)$ | $(0, 1, 0, 1)$ | $(0, 1, 1, 0)$ | $(0, 1, 1, 1)$ | $(1, 0, 0, 0)$ |
| 1              | 1              | 1              | 1              | 1              |

|                |                |                |                |                |
|----------------|----------------|----------------|----------------|----------------|
| $(1, 0, 0, 1)$ | $(1, 0, 1, 0)$ | $(1, 0, 1, 1)$ | $(1, 1, 0, 0)$ | $(1, 1, 0, 1)$ |
| 1              | 0              | 1              | 1              | 1              |

|                |                |
|----------------|----------------|
| $(1, 1, 1, 0)$ | $(1, 1, 1, 1)$ |
| 1              | 1              |

$$\therefore \text{其真度为 } \tau((p_1 \rightarrow p_2) \vee (p_3 \rightarrow p_4)) = \frac{|T|}{2^4} = \frac{15}{16}.$$

## 4. 计算下列逻辑公式的真度

$$(c) (\neg p_1 \rightarrow p_2) \rightarrow p_3$$

### Solution

给定命题变元 $p_1, p_2, p_3$ , 作真值表, 有:

| 公式   | (0, 0, 0) | (0, 0, 1) | (0, 1, 0) | (0, 1, 1) |
|--|-----------|-----------|-----------|-----------|
| $(\neg p_1 \rightarrow p_2) \rightarrow p_3$ | 1         | 1         | 0         | 1         |

| (1, 0, 0) | (1, 0, 1) | (1, 1, 0) | (1, 1, 1) |
|-----------|-----------|-----------|-----------|
| 0         | 1         | 0         | 1         |

$$\therefore \text{其真度为 } \tau((\neg p_1 \rightarrow p_2) \rightarrow p_3) = \frac{|T|}{2^3} = \frac{5}{8}.$$

## 5. 设 $A \downarrow B$ 表示 $\neg(A \vee B)$ , 证明连接符 $\{\downarrow\}$ 是命题逻辑连接符集的充足集

Proof.

$\{\neg, \vee\}$  是充足集, 所以, 对任意公式  $F$ ,  $F$  能用 ' $\neg$ ' 和 ' $\vee$ ' 表示, 用数学归纳法证明: 若  $|F| = 1$ , 即  $F$  为原子公式. 则必为一个原子命题或其否定形式, 记原子命题为  $p$ , 则  $F = p$  或  $F = \neg p$  成立, 于是有:

- 1)  $F = p = \neg\neg(p \vee p) = \neg(p \downarrow p) = \neg((p \downarrow p) \vee (p \downarrow p)) = (p \downarrow p) \downarrow (p \downarrow p)$
- 2)  $F = \neg p = \neg(p \vee p) = p \downarrow p.$

假设, 对于所有的公式  $F$ , 若  $|F| < n$  则  $F$  能用 ' $\downarrow$ ' 表示. 现考虑任意满足  $|F| = n$  的公式  $F$ :

- 1) 若  $F = \neg A$  ( $A$  为  $F$  的子公式), 由  $\downarrow$  的定义得:  
 $F = \neg A = \neg(A \vee A) = A \downarrow A.$
- 2) 若  $F = A \vee B$  ( $A, B$  为  $F$  的子公式), 由  $\downarrow$  的定义及 1) 得:  
 $F = A \vee B = \neg(\neg(A \vee B)) = \neg(A \downarrow B) = (A \downarrow B) \downarrow (A \downarrow B).$

无论以上何种情况,  $A, B$  均满足  $|A| < n, |B| < n$ . 所以,  $A, B$  必能用 ' $\downarrow$ ' 表示.  $\therefore$  由数学归纳法知所有公式  $F$  均能用  $\downarrow$  表示, 即  $\{\downarrow\}$  是充足集. □

6.研究题: 任给分数 $\frac{n}{m}(0 \leq n \leq m)$ , 是否存在一个命题公式 $A$  使得 $A$  的真度 $\tau(A) = \frac{n}{m}$ ?

Proof.

不能. 由真度的定义知, 对于任意公式 $A$ , 其真度的形式必定为:

$$\tau(A) = \frac{M}{2^N}$$

若 $|A| = n$ ,  $M, N \in \mathbb{N}^+$ . 若该命题成立, 即对于任意 $0 \leq n \leq m$ , 有:

$$\frac{n}{m} = \frac{M}{2^N}$$

$q = \frac{n}{m} \leq 1$  是小于1的任意有理数, 即所有小于1的有理数都具有 $\frac{M}{2^N}$  的形式, 这是不可能的!  $\therefore$  原命题不成立. □

# 1. 试证:

1)  $\vdash A \rightarrow (B \rightarrow (A \rightarrow B)).$

Proof.

构造推演:

- |    |   |            |
|----|---|------------|
| 1. | $B \rightarrow (A \rightarrow B)$   | $L_1$      |
| 2. | $(B \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow (B \rightarrow (A \rightarrow B)))$ | $L_1$      |
| 3. | $A \rightarrow (B \rightarrow (A \rightarrow B))$   | $MP(1, 2)$ |



# 1.试证:

2)  $\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)).$

Proof.

构造推演:

|              |   |            |
|--------------|---|------------|
| 1.           | $B \rightarrow C$   | $\Gamma$   |
| 2.           | $(B \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$                                 | $L_1$      |
| 3.           | $A \rightarrow (B \rightarrow C)$   | $MP(1, 2)$ |
| 4.           | $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ | $L_2$      |
| 5.           | $(A \rightarrow B) \rightarrow (A \rightarrow C)$   | $MP(3, 4)$ |
| $\therefore$ | $\{B \rightarrow C\} \vdash (A \rightarrow B) \rightarrow (A \rightarrow C)$                      |            |

由演绎定理得到,  $\vdash (B \rightarrow C) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$



# 1.试证:

3)  $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ .

Proof.

构造推演:

|              |   |            |
|--------------|---|------------|
| 1.           | $A$   | $\Gamma$   |
| 2.           | $A \rightarrow (A \rightarrow B)$                 | $\Gamma$   |
| 3.           | $A \rightarrow B$                                 | $MP(1, 2)$ |
| 4.           | $B$   | $MP(1, 3)$ |
| $\therefore$ | $\{A, A \rightarrow (A \rightarrow B)\} \vdash B$ |            |

由演绎定理得到,  $\{A \rightarrow (A \rightarrow B)\} \vdash A \rightarrow B$   
 $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$





## 2. 试证:

1)  $(A \rightarrow (B \rightarrow C)) \approx (B \rightarrow (A \rightarrow C))$ .

Proof.

" $\Rightarrow$ " 证  $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))$ :

|              |   |            |
|--------------|---|------------|
| 1.           | $A \rightarrow (B \rightarrow C)$   | $\Gamma$   |
| 2.           | $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$ | $L_2$      |
| 3.           | $(A \rightarrow B) \rightarrow (A \rightarrow C)$   | $MP(1, 2)$ |
| 4.           | $B$   | $\Gamma$   |
| 5.           | $B \rightarrow (A \rightarrow B)$   | $L_1$      |
| 6.           | $A \rightarrow B$   | $MP(4, 5)$ |
| 7.           | $A \rightarrow C$   | $MP(3, 6)$ |
| $\therefore$ | $\{A \rightarrow (B \rightarrow C), B\} \vdash A \rightarrow C$                                   |            |

由演绎定理得到,  $\{A \rightarrow (B \rightarrow C)\} \vdash B \rightarrow (A \rightarrow C)$   
 $\vdash ((A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C)))$

## Proof (Cont.)

" $\Leftarrow$ " 证  $\vdash (B \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow (B \rightarrow C))$ :

|              |   |            |
|--------------|---|------------|
| 1.           | $B \rightarrow (A \rightarrow C)$   | $\Gamma$   |
| 2.           | $(B \rightarrow (A \rightarrow C)) \rightarrow ((B \rightarrow A) \rightarrow (B \rightarrow C))$ | $L_2$      |
| 3.           | $(B \rightarrow A) \rightarrow (B \rightarrow C)$   | $MP(1, 2)$ |
| 4.           | $A$   | $\Gamma$   |
| 5.           | $A \rightarrow (B \rightarrow A)$   | $L_1$      |
| 6.           | $B \rightarrow A$   | $MP(4, 5)$ |
| 7.           | $B \rightarrow C$   | $MP(3, 6)$ |
| $\therefore$ | $\{B \rightarrow (A \rightarrow C), A\} \vdash B \rightarrow C$                                   |            |

由演绎定理得到,  $\{B \rightarrow (A \rightarrow C)\} \vdash A \rightarrow (B \rightarrow C)$

$\vdash (B \rightarrow (A \rightarrow C)) \rightarrow (A \rightarrow (B \rightarrow C))$

综上,  $(A \rightarrow (B \rightarrow C)) \approx (B \rightarrow (A \rightarrow C))$ .



## 2. 试证:

$$2) (A \rightarrow (A \rightarrow B)) \approx (A \rightarrow B).$$

Proof.

" $\Rightarrow$ " 证  $\vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ :

- |    |                                   |            |
|----|-----------------------------------|------------|
| 1. | $A \rightarrow (A \rightarrow B)$ | $\Gamma$   |
| 2. | $A$                               | $\Gamma$   |
| 3. | $A \rightarrow B$                 | $MP(1, 2)$ |
| 4. | $B$                               | $MP(2, 3)$ |

$$\therefore \{ \{ A \rightarrow (A \rightarrow B), A \} \vdash B$$

由演绎定理得到,  $\{ A \rightarrow (A \rightarrow B) \} \vdash A \rightarrow B, \vdash (A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$

" $\Leftarrow$ " 证  $\vdash (A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$ :

- |    |   |            |
|----|---|------------|
| 1. | $A \rightarrow B$   | $\Gamma$   |
| 2. | $(A \rightarrow B) \rightarrow (A \rightarrow (A \rightarrow B))$ | $L_1$      |
| 3. | $A \rightarrow (A \rightarrow B)$                                 | $MP(1, 2)$ |

$$\therefore \{ A \rightarrow B \} \vdash A \rightarrow (A \rightarrow B)$$

由演绎定理得到,  $\vdash A \rightarrow B \rightarrow A \rightarrow (A \rightarrow B)$

综上,  $A \rightarrow (A \rightarrow B) \approx A \rightarrow B$ .



# 1. 利用 $L$ 的完备性定理证明以下各式成立:

$$(a) \vdash (\neg A \rightarrow A) \rightarrow A$$

Proof.

$$\begin{aligned} & (\neg A \rightarrow A) \rightarrow A \\ = & (\neg(\neg A) \vee A) \rightarrow A \\ = & A \rightarrow A \\ = & \neg A \vee A \end{aligned}$$

任取 $v \in \Omega$ , 可知:  $v(\neg A \vee A) = 1 \therefore \models ((\neg A \rightarrow A) \rightarrow A)$   
由完备性定理可知,  $\vdash ((\neg A \rightarrow A) \rightarrow A)$



# 1. 利用 $L$ 的完备性定理证明以下各式成立:

$$(b) \vdash \neg(A \rightarrow B) \rightarrow (B \rightarrow A)$$

Proof.

$$\begin{aligned} & \neg(A \rightarrow B) \rightarrow (B \rightarrow A) \\ = & \neg(\neg A \vee B) \rightarrow (\neg B \vee A) \\ = & \neg(A \wedge \neg B) \vee (\neg B \vee A) \\ = & (\neg A \vee B) \vee (\neg B \vee A) \\ = & \neg A \vee A \vee \neg B \vee B \end{aligned}$$

任取 $v \in \Omega$ , 可知:  $v(\neg A \vee A \vee \neg B \vee B) = 1$

$$\therefore \models (\neg(A \rightarrow B) \rightarrow (B \rightarrow A))$$

由完备性定理可知,  $\vdash (\neg(A \rightarrow B) \rightarrow (B \rightarrow A))$



# 1.利用L的完备性定理证明以下各式成立:

$$(c) ((A \vee B) \rightarrow C) \approx (A \rightarrow C) \wedge (B \rightarrow C)$$

Proof.

$$\begin{aligned} & (A \vee B) \rightarrow C \\ = & \neg(A \vee B) \vee C \\ = & (\neg A \wedge \neg B) \vee C \\ = & (\neg A \wedge \neg B) \vee (\neg A \vee \neg B \vee C) \wedge C \\ = & (\neg A \vee C) \wedge (\neg B \vee C) \\ = & (A \rightarrow C) \wedge (B \rightarrow C) \\ \therefore & (A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C) \end{aligned}$$

由于 $(A \vee B) \rightarrow C = (A \rightarrow C) \wedge (B \rightarrow C)$

任取 $v \in \Omega$ , 则 $v((A \vee B) \rightarrow C) = v((A \rightarrow C) \wedge (B \rightarrow C))$

$\therefore \models (((A \vee B) \rightarrow C) \rightarrow (A \rightarrow C) \wedge (B \rightarrow C))$

且 $\models ((A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$

由完备性定理可知,  $\vdash (((A \vee B) \rightarrow C) \rightarrow (A \rightarrow C) \wedge (B \rightarrow C))$

且 $\vdash ((A \rightarrow C) \wedge (B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$

所以,  $((A \vee B) \rightarrow C) \approx (A \rightarrow C) \wedge (B \rightarrow C)$



# 1.利用 $L$ 的完备性定理证明以下各式成立:

$$(d) ((A \wedge B) \rightarrow C) \approx (A \rightarrow C) \vee (B \rightarrow C)$$

Proof.

$$\begin{aligned} & (A \wedge B) \rightarrow C \\ = & \neg(A \wedge B) \vee C \\ = & (\neg A \vee \neg B) \vee C \\ = & (\neg A \vee C) \vee (\neg B \vee C) \\ = & (A \rightarrow C) \vee (B \rightarrow C) \\ \therefore & (A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C) \end{aligned}$$

由于 $(A \wedge B) \rightarrow C = (A \rightarrow C) \vee (B \rightarrow C)$

任取 $v \in \Omega$ , 则 $v((A \wedge B) \rightarrow C) = v((A \rightarrow C) \vee (B \rightarrow C))$

$$\therefore \models (((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow C) \vee (B \rightarrow C))$$

$$\text{且} \models ((A \rightarrow C) \vee (B \rightarrow C) \rightarrow ((A \wedge B) \rightarrow C))$$

由完备性定理可知,  $\vdash (((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow C) \vee (B \rightarrow C))$

$$\text{且} \vdash ((A \rightarrow C) \vee (B \rightarrow C) \rightarrow ((A \wedge B) \rightarrow C))$$

所以,  $((A \wedge B) \rightarrow C) \approx (A \rightarrow C) \vee (B \rightarrow C)$



2. 设 $\Gamma \subseteq F(S)$ ,  $\Gamma$ 是有限集,  $A \in F(S)$ . 证明:  
 $\Gamma \vdash A$ 当且仅当 $\Gamma \models A$ . 其中 $\Gamma \models A$ 定义为对于任何赋值 $v$ 若对于 $\Gamma$ 中的每个成员 $B$ 只要 $v(B) = 1$ 就有 $v(A) = 1$ .

Proof.

首先, 由 $\Gamma$ 的有限性, 不妨设 $\Gamma = \{F_1, F_2, \dots, F_n\}$ .

先证明一个引理,

引理1: 设 $\Gamma \subseteq F(S)$ ,  $A, B \in F(S)$ 为任一公式, 则 $\Gamma \cup \{A\} \models B$  当且仅当 $\Gamma \models A \rightarrow B$ .

证明:

充分性, 若 $\Gamma \cup \{A\} \models B$ , 根据定义, 对于任何赋值 $v$ , 若有 $v(F_i) = 1$  ( $\forall F_i \in \Gamma$ ) 且 $v(A) = 1$ , 则 $v(B) = 1$ 成立. 对于 $v(A \rightarrow B)$ , 当 $v(A) = 0$ 时,  $v(A \rightarrow B) = 1$ , 当 $v(A) = 1$ 时, 因为 $v(F_i) = 1$  ( $\forall F_i \in \Gamma$ ) 成立, 所以 $v(B) = 1$ 成立. 所以对任何赋值 $v$ , 有若 $v(F_i) = 1$  ( $\forall F_i \in \Gamma$ )成立, 则 $v(A \rightarrow B) = 1$ 成立, 即 $\Gamma \models A \rightarrow B$ .



## Proof(Cont.)

必要性, 若 $\Gamma \models A \rightarrow B$ , 即对任何赋值 $v$ , 若 $v(F_i) = 1$  ( $\forall F_i \in \Gamma$ )成立, 则 $v(A \rightarrow B)$ 成立. 若 $v(A) = 1$ , 则必有 $v(B) = 1$ . 所以有 $\Gamma \cup \{A\} \models B$ 成立. 引理1证毕.

现证明原命题,

1)充分性:

$$\because \Gamma \vdash A, \text{ 即 } \{F_1, F_2, \dots, F_n\} \vdash A$$

$$\therefore \{F_1, F_2, \dots, F_{n-1}\} \vdash F_n \rightarrow A$$

$$\therefore \{F_1, F_2, \dots, F_{n-2}\} \vdash F_{n-1} \rightarrow (F_n \rightarrow A)$$

.....

$$\therefore \vdash F_1 \rightarrow (F_2 \rightarrow \dots \rightarrow (F_n \rightarrow A)) \dots$$

$$\text{由完备性得: } \models F_1 \rightarrow (F_2 \rightarrow \dots \rightarrow (F_n \rightarrow A)) \dots$$

$$\text{由引理1得: } \{F_1\} \models F_2 \rightarrow (F_3 \rightarrow \dots \rightarrow (F_n \rightarrow A)) \dots$$

$$\therefore \{F_1, F_2\} \models F_3 \rightarrow \dots \rightarrow (F_n \rightarrow A) \dots$$

.....

$$\therefore \Gamma \models A$$

## Proof(Cont.)

2)必要性:

$$\begin{aligned} & \therefore \Gamma \models A, \text{ 即 } \{F_1, F_2, \dots, F_n\} \models A \\ & \text{由引理1得: } \{F_1, F_2, \dots, F_{n-1}\} \models F_n \rightarrow A \\ & \therefore \{F_1, F_2, \dots, F_{n-2}\} \models F_{n-1} \rightarrow (F_n \rightarrow A) \\ & \quad \dots\dots \\ & \therefore \models F_1 \rightarrow (F_2 \rightarrow \dots \rightarrow (F_n \rightarrow A))\dots \\ & \text{由完备性得: } \vdash F_1 \rightarrow (F_2 \rightarrow \dots \rightarrow (F_n \rightarrow A))\dots \\ & \therefore \{F_1\} \vdash F_2 \rightarrow (F_3 \rightarrow \dots \rightarrow (F_n \rightarrow A))\dots \\ & \therefore \{F_1, F_2\} \vdash F_3 \rightarrow \dots \rightarrow (F_n \rightarrow A))\dots \\ & \quad \dots\dots \\ & \therefore \Gamma \vdash A \end{aligned}$$

综上, 若 $\Gamma$ 有限, 则 $\Gamma \vdash A$  当且仅当  $\Gamma \models A$ .