1. Find disjunctive and conjunctive normal forms.

(i) $a1 = (x1 \land x2) \lor ((\neg x3 \lor x1) \land \neg x2) \lor x2$

$x1x2x3$

X1	X2	X3	$(x1 \land x2)$	$((\neg x3 \lor x1) \land \neg x2)$	$x2$	$(x1 \land x2) \lor ((\neg x3 \lor x1) \land \neg x2) \lor x2$
1	1	1	1	0	1	1
1	1	0	1	0	1	1
1	0	1	0	1	0	1
1	0	0	0	1	0	1
0	1	1	0	0	1	1
0	1	0	0	0	1	1
0	0	1	0	0	0	0
0	0	0	0	1	0	1

DNF

$E= x1x2x3 + x1x2\bar{x}3 + x1\bar{x}2x3 + x1\bar{x}2\bar{x}3 + \bar{x}1x2x3 + \bar{x}1x2\bar{x}3 + \bar{x}1\bar{x}2\bar{x}3$

CNF

$\neg E= \bar{x}1\bar{x}2x3$

$E=\neg\neg E = \overline{\bar{x}1\bar{x}2x3}$

$E= \neg\neg E = x1x2\bar{x}3$

(ii) $a2 = (x1 \lor ((x1 \land x2) \lor (\neg x1 \land \neg x2 \land \neg x3))) \land x3$

X1	X2	X3	$x1$	$x1 \land x2$	$\neg x1 \land \neg x2 \land \neg x3$	$x3$	$(x1 \lor ((x1 \land x2) \lor (\neg x1 \land \neg x2 \land \neg x3))) \land x3$
1	1	1	1	1	0	1	1
1	1	0	1	1	0	0	0
1	0	1	1	0	0	1	1
1	0	0	1	0	0	0	0
0	1	1	0	0	0	1	0
0	1	0	0	0	0	1	0
0	0	1	0	0	0	0	0
0	0	0	0	0	1	0	0

DNF
$E=x1x2x3 + x1\bar{x}2x3$

CNF
$\neg E= x1x2\bar{x}3 + x1\bar{x}2\bar{x}3+ \bar{x}1x2x3 + \bar{x}1x2\bar{x}3 + \bar{x}1\bar{x}2x3 + \bar{x}1\bar{x}2\bar{x}3$

$E= \neg\neg E =\overline{x1x2\bar{x}3 + x1\bar{x}2\bar{x}3+ \bar{x}1x2x3 + \bar{x}1x2\bar{x}3 + \bar{x}1\bar{x}2x3 + \bar{x}1\bar{x}2\bar{x}3}$
$E= \neg\neg E =(\bar{x}1+\bar{x}2+x3)(\bar{x}1+x2+x3)(x1+\bar{x}2+\bar{x}3)(x1+\bar{x}2+x3)(x1+x2+\bar{x}3)(x1+x2+x3)$

(iii) $a3 = (x1 \vee x2) \wedge \neg(x1 \wedge x2)$
$a3 = (x1 \vee x2) \wedge (\neg x1 \vee \neg x2)$

X1	X2	$x1 \vee x2$	$\neg x1 \vee \neg x2$	$(x1 \vee x2) \wedge (\neg x1 \vee \neg x2)$
1	1	1	0	0
0	0	0	1	0
1	0	1	1	1
0	1	1	1	1

DNF
$E= x1\bar{x}2 + \bar{x}1x2$

CNF
$\neg E= \bar{x}1\bar{x}2 + x1x2$

$E= \neg\neg E = \overline{\bar{x}1\bar{x}2 + x1x2}$
$E= \neg\neg E = (x1+x2)(\bar{x}1+ \bar{x}2)$

2. Prove that nor is functionally complete. That is if we let $p * q$ mean $\neg(p \vee q)$ show that the other connectives, $\wedge$, $\vee$, $\neg$ and $\rightarrow$ are expressible in terms of $*$.

$p \wedge q = \neg(\neg p \vee \neg q)$
$p \vee q = \neg(\neg(p \vee q))$
$\neg p = \neg(p \vee q)$
$p \rightarrow q = \neg(\neg p \vee q)$

3. Negate and simplify
(i) $\forall x(p \rightarrow \forall y(q \wedge By)) \wedge r$
$\neg(\forall x(p \rightarrow \forall y(q \wedge By)) \wedge r)$
$\exists x\neg(p \rightarrow \forall y(q \wedge By)) \wedge r$
$\exists x(p \wedge \neg\forall y(q \wedge By)) \wedge r$
$\exists x(p \wedge \exists y\neg(q \wedge By)) \wedge r$
$\exists x(p \wedge \exists y(\neg q \vee \neg By)) \wedge r$

(ii) $\forall x \exists y \exists z \forall t(r \vee \forall m(p \leq q))$
$= \neg(\forall x \exists y \exists z \forall t(r \vee \forall m(p \leq q)))$
$= \exists x \neg(\exists y \exists z \forall t(r \vee \forall m(p \leq q)))$
$= \exists x \forall y \neg(\exists z \forall t(r \vee \forall m(p \leq q)))$
$= \exists x \forall y \forall z \neg(\forall t(r \vee \forall m(p \leq q)))$
$= \exists x \forall y \forall z \exists t \neg(r \vee \forall m(p \leq q)))$
$= \exists x \forall y \forall z \exists t(\neg r \wedge \neg(\forall m(p \leq q))))$
$= \exists x \forall y \forall z \exists t(\neg r \wedge (\exists m \neg(p \leq q))))$
$= \exists x \forall y \forall z \exists t(\neg r \wedge (\exists m \neg(p \leq q)))) = \exists x \forall y \forall z \exists t(\neg r \wedge (\exists m(p > q)))$

3. Using the properties of logical equivalence, prove that $(P \wedge (P \rightarrow (Q \rightarrow R))) \rightarrow (Q \rightarrow R)$ is a tautology.

$(P \wedge (P \rightarrow (Q \rightarrow R))) \rightarrow (Q \rightarrow R)$
$(P \wedge (\neg P \vee (\neg Q \vee R))) \rightarrow (\neg Q \vee R)$
$\neg(P \wedge (\neg P \vee (\neg Q \vee R))) \vee (\neg Q \vee R)$
$(\neg P \vee \neg(\neg P \vee (\neg Q \vee R))) \vee (\neg Q \vee R)$
$(\neg P \vee (P \wedge \neg(\neg Q \vee R))) \vee (\neg Q \vee R)$
$(\neg P \vee (P \wedge (Q \wedge \neg R))) \vee (\neg Q \vee R)$
$(\neg P \vee (P \wedge Q) \wedge \neg R))) \vee (\neg Q \vee R)$
$(\neg P \vee P) \wedge (\neg P \vee Q) \wedge \neg R))) \vee (\neg Q \vee R)$
$1 \wedge (\neg P \vee Q) \wedge \neg R))) \vee (\neg Q \vee R)$
$(\neg P \vee Q) \wedge \neg R))) \vee (\neg Q \vee R)$
$(\neg P \wedge \neg R) \vee (Q \wedge \neg R) \vee (\neg Q \vee R)$
$(\neg P \wedge \neg R) \vee 1$
$=1$

5. Let P xy mean that “x prefers y”. Let Ox mean that “x is old” and Nx mean that “x is new”, and Qx mean that “x is a person” and Hx mean that “x is a house”. Let a be Adam and b be Betty. Using only this dictionary translate the following into predicate logic.

(i) Betty is old.
Ob
(ii) All old people prefer new houses.
$\forall x \forall y(((Qx \wedge Ox) \wedge (Hy \wedge Ny)) \rightarrow Pxy)$
(iii) Some young (i.e. not old) people prefer new houses.
$\exists x \forall y(((Qx \wedge \neg Ox) \wedge (Hy \wedge Ny)) \rightarrow Pxy)$
(iv) Every old person prefers some old house.
$\forall x \exists y(((Qx \wedge Ox) \wedge (Hy \wedge Ny)) \rightarrow Pxy)$

6. Prove by a direct argument that if x and y are odd integers, then x + y is even.
E.g Direct proof
argue from the premisses to the conclusion by a direct chain of reasoning

x=2n+1
Y=2m+1
X+Y= 2p
X+Y=(2n+1) + (2m+1)
x+y= 2n+2m+2

$x+y=2(n+m+1)$
$x+y=2p$
$p=n+m+1$

7. Let x be an integer. Prove by contrapositive that if x^2 is odd, then x is odd.
Claim x^2 is odd, then x is odd.
Contrapositive RTP: $\neg q \rightarrow \neg p$
x^2 is even
x is even
$x = 2n$
$x^2= (2n)^2$
$x^2= 4n^2$
$x^2=2(2n^2)$
$x^2=2m$
$m=2n^2$