Victoria University of Wellington

School of Mathematics, Statistics and Operations Research

MATH 161 Notes	1. Propositional Logic

WARNING These notes are **not** complete. For a complete set of notes you will also need the notes given in lectures.

A **statement** or **proposition** is a sentence that is either true or false.

Which of the following are statements?

Wellington is a city
Wellington is attractive
$3 + 4 = 7$
Open the gate!
Is 7 a prime number?

We often claim that one statement can be **inferred** from others. For example, from the statements.

All dogs are animals.
Fido is a dog.

We can correctly infer that Fido is an animal. This appears to be correct or **valid** reasoning. However, given the statements

All dogs are animals.
Tiddles is an animal.

It would clearly be **invalid** reasoning to deduce that Tiddles is necessarily a dog. But why?

Logic is the study of the principles of valid reasoning.

Truth Values: We write 1 for a true statement and 0 for a false one. Every statement is assigned one of these values.

A **compound statement** is build from others using connections. For example the statement "I will think about logic and I will do my assignment" is made from the **primitive statements** "I will think about logic"; "I will do my assignment". In the above compound statement "and" functions as a logical connective.

A **truth-table** tells us how the truth of a compound statement relates to the truth of the underlying primitive statements.

Negation From proposition p we get "not p". Notation $\neg p$. Many other notations, eg the text uses $\sim p$.

Truth condition: $\neg p$ is true when p is false and false when p is true. The truth-table for $\neg p$ is

p	$\neg p$
1	0
0	1

Conjunction "p and q". Notation $p \wedge q$.

Truth condition: $p \wedge q$ is true if both p, q true; false otherwise. The truth-table for $p \wedge q$ is

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Disjunction "p or q". Notation $p \vee q$.

Truth condition: $p \vee q$ is true if at least one of p, q is true. The truth-table is:

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Trap for young players: In English we use the word "or" in two quite different ways. If someone claimed that they would have "either fish of steak for dinner" my guess is that you would think them a liar if they had both. This is the **exclusive** use of "or".

On the other hand, the statement "running or swimming is good for you" would be true, even though doing both is probably good for you. This is the **inclusive** use of "or", that is, disjunction.

In logic and mathematics we almost always use the inclusive sense of "or".

Implication "p implies q", "if p, then q". Notation: $p \to q$ or $p \Rightarrow q$.

Truth Condition: An implication $p \to q$ is false when p is true but q is false. The truth-table is:

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

The truth-table for implication rewards serious thought. Many are surprised and confused by the fact the $p \to q$ is true in the case that p is false and q is true. Think about this. Consider the statement

If I win the lottery, then I will be happy.

It is quite consistent with the statement that you do not win the lottery but are happy for some other reason. For example you might get a good mark in a 114 assignment.

Equivalence Sometimes "biconditional". "p if and only if q" or "p is equivalent to q". Notation $p \leftrightarrow q$ or $p \Leftrightarrow q$.

Truth condition: $p \leftrightarrow q$ is true if p and q are either both true or both false. The truth-table is

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Once the truth tables for basic propositions are known we can calculate them for more complex compound propositions. Examples will be given in class.

Note that the truth table for a compound statement with three primitive statements will need 8 rows. In general, for n primitive statements 2^n rows are needed.

A **tautology** is a proposition whose truth table consists entirely of 1's. A tautology is true *independently of the truth or falsity of its primitive statements.*

You should check using truth tables that the following are tautologies.

- $p \vee \neg p$
- $p \leftrightarrow \neg\neg p$
- $((p \vee q) \wedge \neg p) \rightarrow q$
- $(p \wedge (p \rightarrow q)) \rightarrow q$
- $p \rightarrow p$
- $p \rightarrow (p \vee q)$
- $(p \wedge q) \rightarrow p$

Hot Tip Logic, and mathematics in general, should never seem like just formulae. If it seems like a bunch of meaningless symbols, *then you are not understanding it.* When you do understand it will seem concrete and straightforward. Consider the above tautologies. Using "the moon is made of green cheese" for p. We see that the first one says "either the moon is made of green cheese or it is not". It's a pretty fatuous statement, but obviously always true and we really do understand how $p \vee \neg p$ has to be a tautology.

A **contradiction** is a proposition that is always false. For example $p \wedge \neg p$ is a contradiction. In general p is a contradiction if and only if p is a tautology.

A **contingent** proposition is one that is neither a tautology or a contradiction. Obviously most propositions are contingent.

Two propositions are **logically equivalent** if they have the same truth table.

Clearly p and q are logically equivalent if and only if $p \leftrightarrow q$ is a tautology. (Why is this clear?)

Notation $p \equiv q$

Some logical equivalences

(1) Double Negation $p \equiv \neg\neg p$.
(2) De Morgan's Laws
  (a) $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  (b) $\neg(p \vee q) \equiv \neg p \wedge \neg q$
(3) $p \to q \equiv \neg p \vee q$
(4) $\neg(p \to q) \equiv p \wedge \neg q$
(5) Commutative Laws
  (a) $p \wedge q \equiv q \wedge p$
  (b) $p \vee q \equiv q \vee p$
(6) Idempotent Laws
  (a) $p \wedge p \equiv p$
  (b) $p \vee p \equiv p$
(7) Distributive Laws
  (a) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  (b) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
(8) Associative Laws
  (a) $p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$
  (b) $p \vee (q \vee r) \equiv (p \vee q) \vee r$
(9) Contrapositive $p \to q \equiv \neg q \to \neg p$
(10) Tautology. If $\mathbb{T}$ is a tautology, then

$$p \vee \mathbb{T} \equiv \mathbb{T}$$

$$p \wedge \mathbb{T} \equiv p$$

(11) Contradiction. If $\mathbb{F}$ is a contradiction, then

$$p \vee \mathbb{F} \equiv p$$

$$p \wedge \mathbb{F} \equiv \mathbb{F}$$

The **substitution rule** says that we can replace a part of a proposition by a logically equivalent one. More precisely, suppose $X \equiv Y$. Let A be a proposition and let B result by substituting Y for X in some place in A. Then $A \equiv B$.

Using repeated applications of the above laws one can derive that certain statements are logically equivalent or are tautologies. Examples will be given in class.