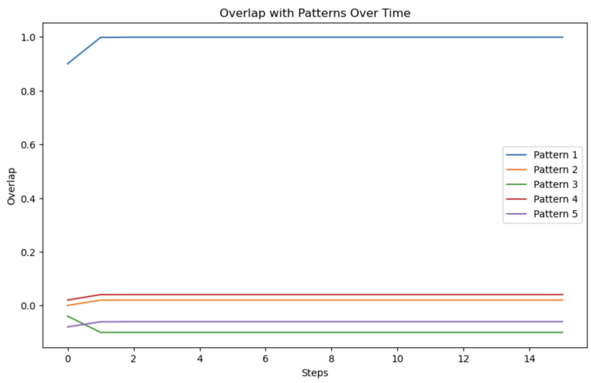
Introduction

In computational neuroscience, the Hopfield model is fundamentally used to simulate memory storage in neural networks in the form of patterns in neuronal activity. This project extends beyond the classical Hopfield model to include biologically realistic features, such as sparsity of neuronal activity and asymmetric connectivity, in compliance with Dale's law, stating that each neuron's synapse should exclusively be either excitatory or inhibitory. Here, we explore three configurations of the Hopfield model:

1. A standard symmetrical Hopfield network containing balanced random patterns with 50% of neurons active.
2. A modified network with low-activity patterns to reflect realistic, sparse activity of the neurons.
3. A network with separate excitatory and inhibitory neuronal populations with regard to Dale's law.

We evaluate the network's performance under biologically realistic constraints by quantifying memory storage capacity and pattern retrieval accuracy of the proposed networks.

Question 0



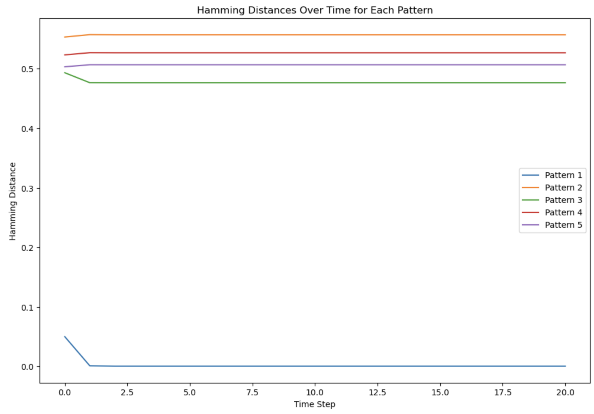
The overlaps of the final state with all the patterns. The network correctly retrieves the first pattern.

Question 1

1.1 The computational gain of a single update step varies with every run, being approximately 200, which represents how much faster the optimised function is running, compared to the original. This is expected, since the expensive weight matrix computation is avoided in the optimised state update function.

1.2 The Hamming distance corresponds to the number of neurons that differ in states between two patterns. It can be used for assessing the memory retrieval accuracy within the network, by quantifying how much the final state of the network deviates from the original pattern after the network processes and updates it.

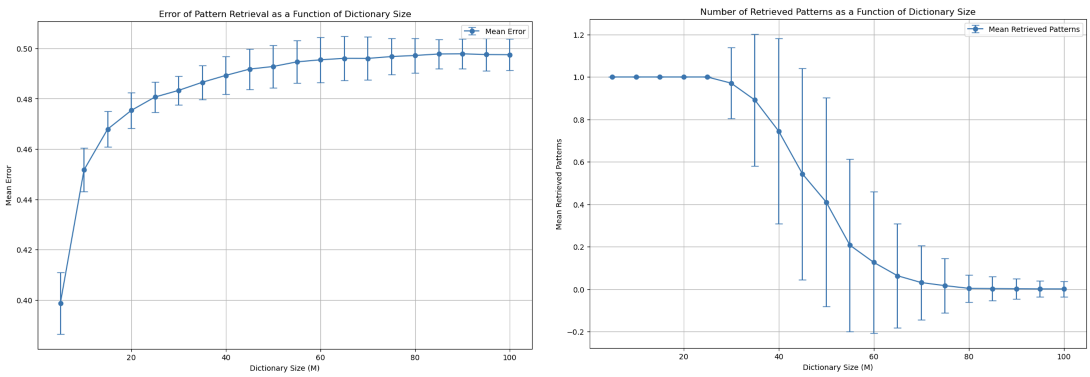
The overlap defined in lectures calculates a normalised correlation between two patters, which represents their overall similarity. Higher overlap corresponds to fewer differences, therefore the relationship between the Hamming distance and the overlap is inversely proportional.



Evolution of the Hamming distance between the network's state S(t) and each of the patterns

1.3 The first pattern is retrieved correctly by the network, as the Hamming distance between the last state and the first pattern is less than or equal to 0.05.

1.4 For a dictionary of size $M = 5$:
Mean Error of Pattern Retrieval = 0.40
Standard Deviation of Error of Pattern Retrieval = 0.01
Mean Number of Retrieved Patterns = 1.00
Standard Deviation of Number of Retrieved Patterns = 0.00



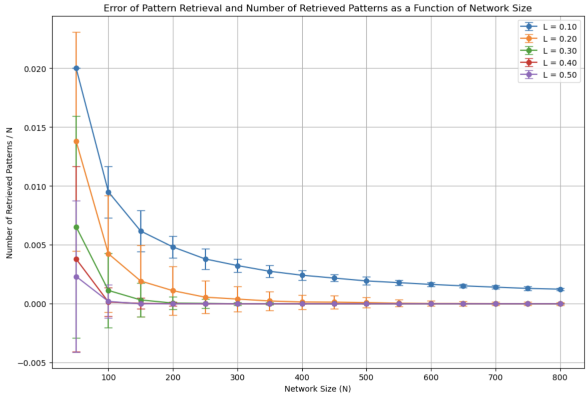
(A) Error of Pattern Retrieval as a Function of Dictionary Size

(B) Number of Retrieved Patterns as a Function of Dictionary Size

Evaluation of Varying Dictionary Sizes

1.5

1.6 The maximal number of patterns M_{max} that can be stored and retrieved in the network of 300 neurons is 41.
If the number of stored patterns increases beyond M_{max}, the retrieval error increases. The reason for this is the increase in the overlap within the synaptic weight matrix, due to the additional patterns, causing more interference during retrieval. This results in poorer network performance with its limited capacity to distinguish between similar states and the increase in interference amongst the patterns.



Network Capacity as a Function of the Loading

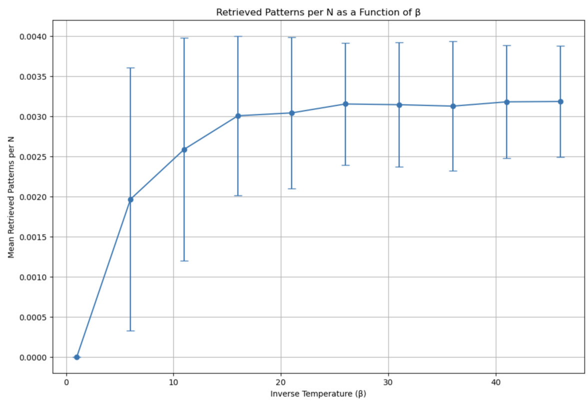
1.7 As the network size increases, the retrieval error decreases, indicating that the capacity of the Hopfield network improves. This is expected, as larger networks have more neurons to store patterns, which improves the accuracy of pattern retrieval. The decrease in retrieval error is not linear, which suggests that the efficiency of individual neurons to store patterns decreases, despite them being able to store more patterns overall. This is also expected, as theoretically, the capacity of a Hopfield network increases at a decreasing rate relative to the number of neurons.

1.8 ***(Bonus)***

$$S_i(t+1) = \phi\left(\sum_{j=1}^{N} W_{ij} S_j(t)\right) \tag{1}$$

where $\phi(h) = \tanh(\beta h)$, and we use $\beta = 4$.

The inverse temperature β in the above equation serves as a scaling factor of the input to the neuron's activation function $\phi(h)$. β has an optimal range of values, that maximise the Hopfield network's ability to retrieve patterns. Below this range, β reduces the sensitivity of $\phi(h)$, leading to neuron outputs being less distinct and less stable, as they are highly influenced by noise. If β is too high, the function becomes too sensitive with very rigid outputs, eventually leading to a plateau in retrieval performance of the model, where a further increase in β has no effect on the network capacity.



Network Capacity with Varying Inverse Temperature

Question 2

2.1 Standard Hopfield model could be described by the following equations:

$$w_{ij} = c \sum_{\mu=1}^{M} p_i^{\mu} p_j^{\mu} \tag{2}$$

$$S_i(t+1) = \phi(\sum_j w_{ij} S_j(t)) \tag{3}$$

Taking $p_i^{\mu} = 2\xi_i^{\mu} - 1$ we can substitute it into expression (1):

$$w_{ij} = c \sum_{\mu=1}^{M} (2\xi_i^{\mu} - 1)(2\xi_j^{\mu} - 1) \tag{4}$$

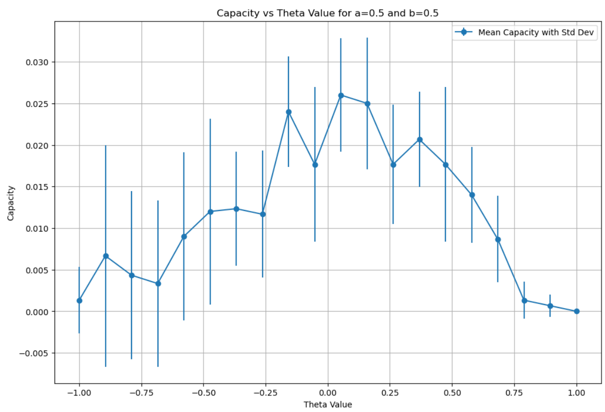
$$= 4c \sum_{\mu=1}^{M} (\xi_i^{\mu} - \frac{1}{2})(\xi_j^{\mu} - \frac{1}{2})$$

$$= 4c \sum_{\mu=1}^{M} (\xi_i^{\mu} - a)(\xi_j^{\mu} - b)$$

Hence, from this expression we can deduce that if $a = b = 0.5$ with $c^{'} = 4c$, we arrive at the Standard Hopfiel model.

The two models would be equivalent if the patterns of -1 and 1 are considered.

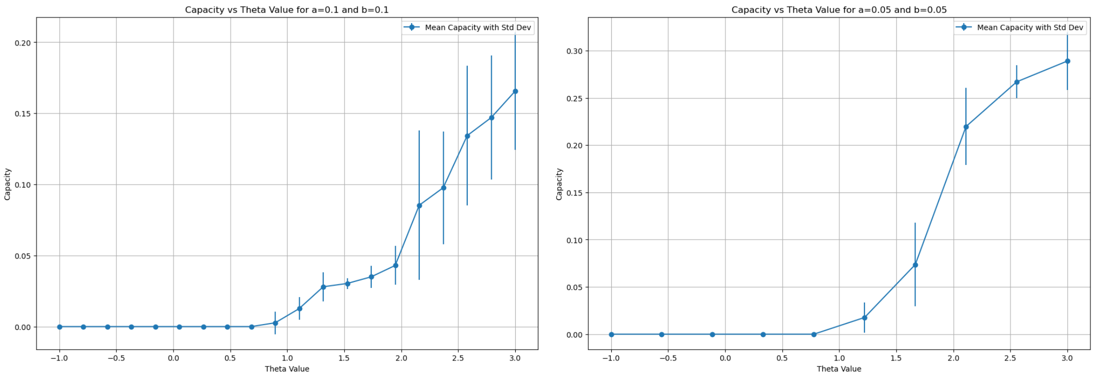
2.3 The capacity of the network for N=300 for the studied Hopfield model with added stochasticity is higher on average than the capacity of the simple Hopfield model considered in question 1. This is due to the added noise in the environment which allows the network to handle near-threshold patterns better, retrieving them more successfully in the presence of stochastic updates. The capacity in Hopfield model with stochasticity was found to be on average 0.0203 with standard deviation of 0.0086, opposed to 0.004 mean in part 1.



The capacity versus theta value of the Hopfield model with stochastic inputs and states. The best capacity observed corresponds to the theta value of about 0.1.

2.4 The best capacity of about 0.026 corresponds to the theta value of about 0.1.

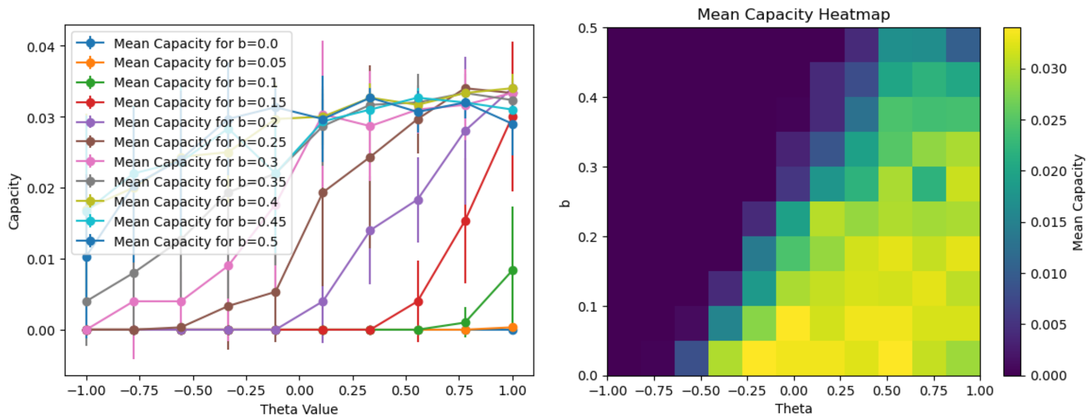
2.5 The optimal value of theta for both cases of $a = 0.1$ and $a = 0.05$ are hard to determine since as theta increases, the values of capacity rise. Nonetheless, it must be noted that for the lower values of activity, the value seems to converge sooner. For instance, for $a = 0.05$ the optimal theta seems to appear close to 3.



(A) Capacity vs. θ for $a = 0.1$ and Asymmetry value = 0.1

(B) Capacity vs. θ for $a = 0.05$ and Asymmetry value = 0.05

2.6 *(Bonus)* From the line graph we can conclude that smaller b leads to higher capacity for the reduced θ values.



(A) Capacity values for each b versus θ

(B) Capacity values for each b versus θ heatmap

Question 3

3.1 Part 1: Total Input to an Excitatory Neuron

The total input to an excitatory neuron is comprised of inputs from inhibitory and excitatory neurons:

$$h_i = h_{exc} - h_{inh}$$

Firstly, we consider h_{exc}:

$$h_{exc} = W_{o_{exc}} = \sum_{j=1}^{N} W_{ij}^{E \leftarrow E} \sigma_j(t) = \sum_{j=1}^{N} \frac{c}{N} \sum_{\mu=1}^{M} \xi_i^{\mu} \xi_j^{\mu} \sigma_j(t)$$

Secondly, we consider h_{inh}:

$$h_{inh} = W_{o_{inh}} = \sum_{k=1}^{K} W_{ik}^{E \leftarrow I} I_k(t) = \sum_{k=1}^{K} \frac{ca}{N_I} \sum_{\mu=1}^{M} \xi_i^\mu \xi_k^\mu I_k(t)$$

Total input to the excitatory neuron:

$$h_i = \sum_{j=1}^{N} W_{ij}^{E \leftarrow E} \sigma_j(t) - \sum_{k=1}^{N_I} W_{ik}^{E \leftarrow I} I_k(t)$$

Input to the inhibitory neurons:

$$h_k = \sum_{k=1}^{K} W_{ki}^{I \leftarrow E} \sigma_k(t) = \frac{1}{K} \sum_{k=1}^{K} \sigma_k(t)$$

Part 2: Model Description and Equivalence to Hopfield Model

1. Inhibiting neurons have linear gain function and fire stochastically: $\text{Prob}(\delta_k = +1 | h_k^{\text{inh}}) = h_k^{\text{inh}}$, where k is an index of an inhibitory neuron $1 \leq k \leq N_I$.
2. Each inhibitory neuron k receives input from K excitatory neurons. Connections are random and of equal weight $w_{ki}^{I \leftarrow E} = \frac{1}{K}$.
3. Thus, the input potential of the neuron k is $h_k^{\text{inh}} = \frac{1}{K} \sum_{j \in S} \sigma_j$, where S is the set of pre-synaptic neurons.
4. Connection from inhibitory neuron back to excitatory neuron $w_{ik}^{E \leftarrow I} = \frac{a}{N_I} \sum_\mu \xi_i^\mu$.
5. The total input from inhibitory neuron to excitatory is, therefore, $h_{inh} = \sum_k w_{ik}^{E \leftarrow I} \sigma_k = \frac{a}{N_I} \sum_\mu \xi_i^\mu \sigma_k$.
6. Putting everything together and considering that $k = j$ and $N = N_I$,

$$\langle h_i(t) \rangle = \frac{c}{N} \sum_{\mu}^{M} \xi_i^\mu \sum_{j}^{N} \xi_j^\mu \sigma_j(t) - \frac{ca}{N} \sum_{j}^{N} \sum_{\mu}^{M} \xi_i^\mu$$

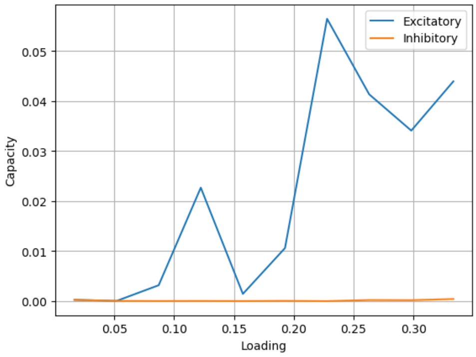
$$= \frac{c}{N} \sum_{\mu}^{M} \sum_{j}^{N} \xi_i^\mu (\xi_j^\mu - a) \sigma_j(t)$$

This is equivalent to the low-activity Hopfield model discussed in part 2:

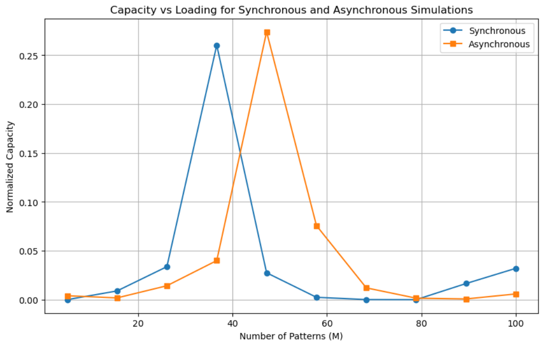
$$\langle h_i(t) \rangle = \sum_{j}^{N} w_{ij} \sigma_j(t) = \frac{c}{N} \sum_{j}^{N} \sum_{\mu}^{M} (\xi_i^\mu - b)(\xi_j^\mu - a) \sigma_j(t)$$

$$= \frac{c}{N} \sum_{j}^{N} \sum_{\mu}^{M} \xi_i^\mu (\xi_j^\mu - a) \sigma_j(t), \quad \text{QED}$$

3.3 AND 3.4 As illustrated in the plots, the synchronous updates lead to larger overall capacity. Compared to the exercise 2, the model has been able to retrieve more patterns overall.



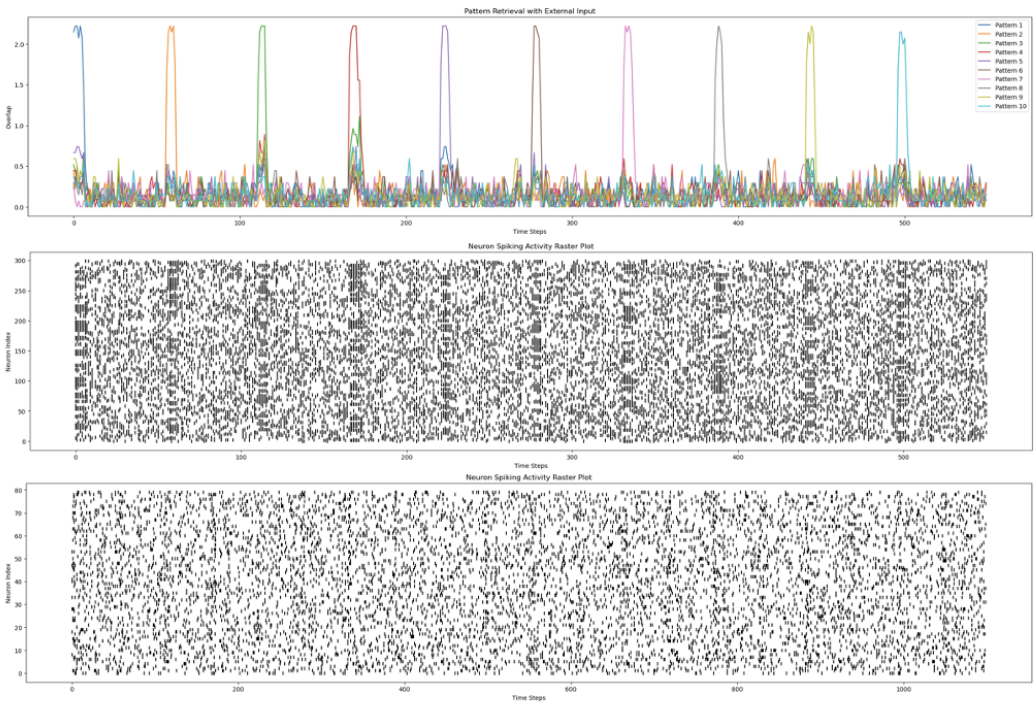
(A) Capacity values vs Loading Hopfield network with inhibitory population



(B) Capacity values vs Loading for Hopfield model with two inhibitory populations

3.5 From the retrieved patterns plot and raster plots, we can clearly see that the model has been able to sequentially retrieve one pattern at a time. This indicates that the external input only temporarily enhances the retrieval of the pattern.

Overall, the network exhibits the ability to recall patterns, but its performance is variable and seems heavily influenced by external factors and the dynamics introduced by the inputs and the network parameters.



(a) The retrieved patterns with external input and second inhibitory population; (b) Raster plot for excitatory population; (c) Raster plot for inhibitory population

3.6 An increase of loading means that more patterns are stored. This reduces the network's ability to retrieve patterns due to higher interference among the patterns and lower stability. When the number of patterns stored approaches or exceeds the network's capacity, retrieval accuracy decreases, resulting in higher errors and spurious states. Increasing sparseness improves pattern retrieval, because the overlap between patterns is decreased, and high orthogonality reduces pattern interference, so they can be distinguished better. On the other hand, low sparseness and orthogonality increase pattern overlap and interference, respectively, degrading the network's ability to retrieve patterns. To achieve optimal pattern retrieval, network's capacity should not be exceeded, loading should be well balanced and the patterns should be sparse and highly orthogonal.

3.7 In a network with a fixed number of memories and a fixed size, the capacity can be improved by increasing sparseness and orthogonality between the stored patterns. Such networks can be further optimised by using Hebbian learning rule and noise reduction. In theory, a standard Hopfield network has a storage capacity of 0.15N patterns, however more advanced models, such as Boltzmann machines, are able to improve their capacity with improved architectures and use of energy-based learning algorithms. Despite this, retrieving patterns with a very high number of patterns is still challenging due to high noise and similarity between the patterns.

Conclusion

Our findings show that the standard Hopfield network model is effective at retrieving stored patterns up to a certain capacity limit. Introducing low-activity patterns improved the network's capacity due to a more sparse neural activity. Further separating the neurons into excitatory and inhibitory populations resulted in improved capacity, yet worsened retrieval accuracy. An explanation to this could be the increased network complexity in maintaining balanced synaptic weights and interactions. These results highlight the importance of introducing biological constraints in computational neural models for better understanding the balance between memory storage capacity and retrieval accuracy.