<u>Exercitii</u> - Trudeau: „Introduction to Graph Theory" - pg <u>67-75</u>

1. List all subsets of the set $\{1;2;3\}$

<u>Solution</u>: $\{\phi; \{1;2;3\}; \{1\}; \{2\}; \{3\}; \{1;2\}; \{1;3\}; \{2;3\}\}$.
1 2 3 4 5 6 7 8

2. It follows from the Law of Excluded Middle that to prove
a mathematical statement true, it suffices to show that it cannot
be false. Letting A be a set and $\emptyset$ an empty set, show that the
statement „$\emptyset$ is a subset of A" cannot be false and so it is true.

<u>Solution</u>: $\emptyset$ is an empty set. A is a set. By definition every set
has in it contained the empty set. Suppose $\emptyset \not\subset A$ ($\not\subset$:= is not
included in), then the empty set is not included in A. Contradiction.
Therefore, $\emptyset \subset A$ ($\subset$:= is included in)

3. The village barber shaves those and only those men who live
in the village and who not shave themselves. The village barber
is a man and he lives in the village. Consider the question:
"Who shaves the barber?"

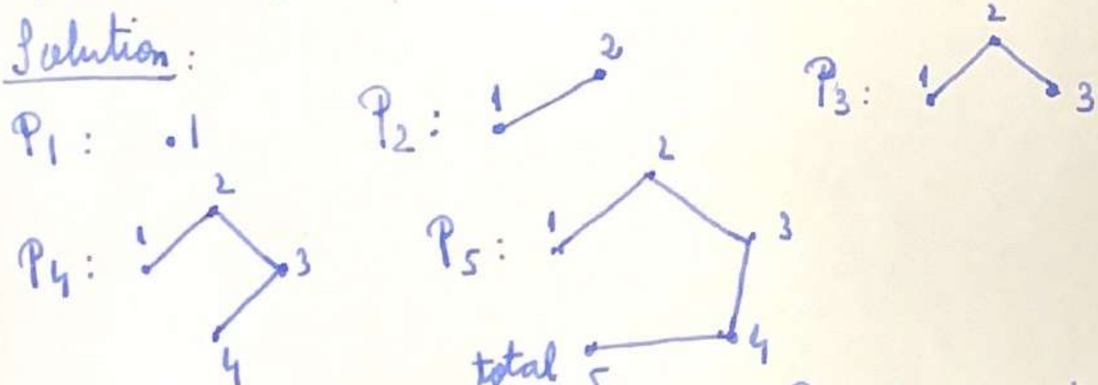
<u>Possibly erronated solution</u>:
Throughout this reasoning I will assume that there exists only one
barber in the village, namely: the village barber.
If the barber doesn't shave himself, he will be shaved by
himself according to the exercise's construction.
In my view for the barber to shave himself, he will need to
be a female (so that he will be excluded from the category of
men)

4. Let „S" be the collection of all sets that can be described in an english sentence of 25 words or less. Is S a set? Why or why not?

<u>Solution</u>: No, the set „S" would not be a set because the description of „S" is less than 25- words, thus „S" is in the set „S". Contradiction. By the definition of a set, any set cannot be in the same set

5. If v is an integer greater than or equal to 2, the path graph on v vertices, denoted „P_V", is the graph having vertex set $\{1;2;\dots;V\}$ and the edge set: $\{\{1;2\};\{2;3\};\dots;\{v-1;V\}\}$. Draw the first five path graphs. Then find and prove a formula analogous to Theorem: „The number of edges in a complete graph K_V is given by: $\underline{e = \frac{1}{2}V(V-1)}$."

<u>Solution</u>:



<u>Conjecture</u>: The number of total edges in P_V is given by the formula: $e = V-1$.

<u>Corollary</u>: The number of total vertices in P_V is given by the formula: $V = e+1$

7. Use Theorem: „The total number of edges in a complete graph K_v is given by: $\underline{e = \frac{1}{2} v(v-1)}$" to prove that:

$1 + 2 + \ldots + (v-1) = \frac{1}{2} v(v-1)$.

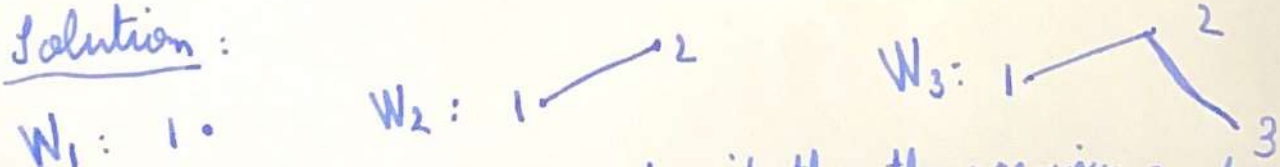
Solution:

$1 + 2 + \ldots + (v-1) = \frac{1}{2} v(v-1) = e$

The statement $1+2+\ldots+(v-1) = e$ is true for a complete graph of v- vertices: K_v. (because the graph having v vertices as its total number of vertices, it will be legally allowed to have a total number of $v-1$ edges for each vertex)

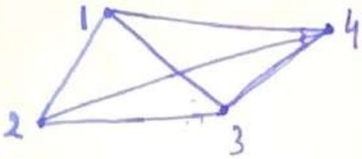
6. If v is an integer greater than or equal to 4, the wheel graph on v vertices, denoted "W_v" is the graph having the vertex set: $\{1; 2; 3; \ldots; v\}$ and the edge set $\{\{1;2\};\{1;3\};\ldots;\{1;v\}; \{2;3\};\{2;4\}; \ldots; \{2;v\};\ldots\ldots;\{1-v;v\};\{v;2\}\}$. Draw the first five wheel graphs. Then find and prove a formula analogous to Theorem: „the total number of edges in a complete graph K_v is given by: $e = \frac{1}{2} v(v-1)$".

Solution:



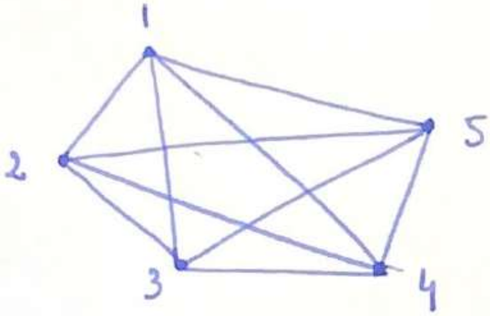
Attention W_i for $i = \overline{1,3}$ do not exist, thus the previous graphs $W_1; W_2; W_3$ are undefined.

W_4:



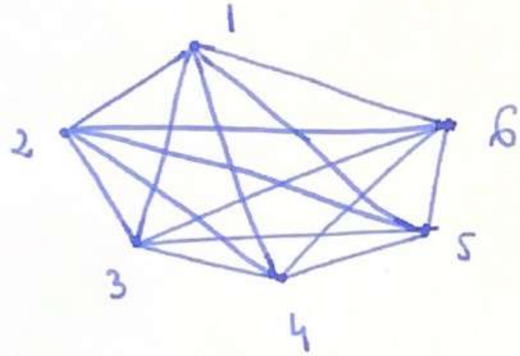
W_4:
Vertex set $\{1;2;3;4\}$
Edge set $\{\{1;2\};\{1;3\};\{1;4\};\{2;3\};\{2;4\};\{3;4\}\}$.

W_5:



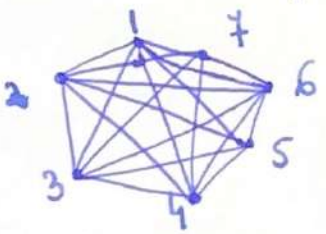
W_5:
Vertex set $\{1;2;3;4;5\}$
Edge set $\{\{1;2\};\{1;3\};\{1;4\};\{1;5\};\{2;3\};\{2;4\};\{2;5\};\{3;4\};\{3;5\};\{4;5\}\}$

W_6:



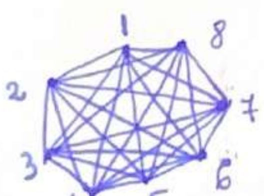
W_6:
Vertex set $\{1;2;3;4;5;6\}$.
Edge set $\{\{1;2\};\{1;3\};\{1;4\};\{1;5\};\{1;6\};\{2;3\};\{2;4\};\{2;5\};\{2;6\};\{3;4\};\{3;5\};\{4;5\};\{3;6\};\{4;6\};\{5;6\}\}$

W_7:



W_7.
Vertex set. $\{1;2;3;4;5;6;7\}$
Edge set: $\{\{1;2\};\{1;3\};\{1;4\};\{1;5\};\{1;6\};\{1;7\};\{2;3\};\{2;4\};\{2;5\};\{2;6\};\{2;7\};\{3;4\};\{3;5\};\{3;6\};\{3;7\};\{4;5\};\{4;6\};\{4;7\};\{5;6\};\{5;7\};\{6;7\}\}$

W_8:



W_8: Vertex set: $\{1;2;3;4;5;6;7;8\}$
Edge set: $\{\{1;2\};\{1;3\};\{1;4\};\{1;5\};\{1;6\};\{1;7\};\{1;8\};\{2;3\};\{2;4\};\{2;5\};\{2;6\};\{2;7\};\{2;8\};\{3;4\};\{3;5\};\{3;6\};\{3;7\};\{3;8\};\{4;5\};\{4;6\};\{4;7\};\{4;8\};\{5;6\};\{5;7\};\{5;8\};\{6;7\};\{6;8\}\{7;7\};\{7;8\}\}$.

From W_4, we observe that $e_4 = 5 = \frac{1}{2}(1+2+3+4)$

W_5, we observe that $e_5 = 10 = 1+2+3+4 = \frac{1}{2}\cdot 4\cdot 5$

W_6, we observe that $e_6 = 15 = \frac{1}{2}\cdot 36 = \frac{1}{2}5\cdot 6$

W_7, we observe that $e_7 = 21 = \frac{1}{2}\cdot 42 = \frac{1}{2}6\cdot 7$

W_8, we observe that $e_8 = 28 = \frac{1}{2}\cdot 56 = \frac{1}{2}\cdot 7\cdot 8$

Conjecture: The total number of edges for wheel graphs is:

$e = \frac{1}{2}v(v-1)$.

8. Let G be a graph with v vertices and e edges. In terms of v and e, how many vertices and edges does $\overline{G}$ have?

Solution: $\overline{G} = K_v \setminus G$ (in terms of edges)

G has v vertices and e edges

K_v has v (the same as G) vertices and a total of

$e = \frac{1}{2}(v-1)v$ edges, thus $\overline{G}$ has: $\begin{cases} v\text{- vertices} \\ \frac{1}{2}(v-1)v - e \text{ edges} \end{cases}$

9. Let G be a graph with $v=6$ vertices.

Prove that G or $\overline{G}$ (or both) has a subgraph isomorphic to K_3.

Solution: If G has $v=6$ then K_6 has $e = \frac{1}{2}5\cdot 6 = 15$ total edges (legal ones).

Remark discussed in the book: Any graph (legal one) with v vertices is a subgraph of K_v, thus any legal graph has at most $\frac{1}{2}v(v-1)$ edges.

So $\cdot\, e_G :=$ number of total edges for G

$\cdot\, e_{\overline{G}} :=$ number of total edges for $\overline{G}$ (complement of G)

with the condition that: $e_G + e_{\overline{G}} = 15$.

10. Use exercise 9 to prove that in any gathering of six people there are either three people who are mutually acquainted or three people who are mutually unacquainted.

Solution: Problem 9 proves that either G or $\overline{G}$ (the complement of G) contains a subgraph isomorphic to K_3. Consider the 6 people as vertices and the relationship between each person as an edge. Problem 9 seen in the context of people (vertices) and their relationships (edges) claims that there are 3 people who are mutual friends (or mutually acquainted) [G contains a subgraph of K_3] or there are 3 people who are not mutual friends (or mutually unacquainted) [$\overline{G}$ contains a subgraph of K_3]. Although abstractly explained, this is a sufficient explanation.

11. Prove: the sum of the degrees of the vertices of a graph is $2 \cdot e$

Solution: Whenever an edge is introduced in a graph, it will connect two vertices. The degree of both of these vertices is equal to "one". Summing up the degrees of the two vertices we will have that the sum is equal to "two". So for each edge the degree of the total number of vertices will increase with 2. If there are e edges in total, then $\deg(v) = \underbrace{2+2+\ldots+2}_{e \text{ times}} = 2e$.

12. Use Exercise 11 to answer these questions:

a) If a graph has $V = 9$; 4 vertices of degree 3, 2 vertices of degree 5, 2 vertices of degree 6 and 1 vertex of degree 8, how many edges has the graph?

b) The utility graph has $V = 6$ and every vertex has degree 3. Prove that there are no graphs with $V = 7$ in which every vertex has degree 3.

Solution:

a) There are: 4 vertices of degree 3, 2 vertices of degree 5, 2 vertices of degree 6, 1 vertex of degree 8 $\xrightarrow[\text{degrees up we get}]{\text{Summing all the}}$

$$\sum_{v \in \text{Graph}} \deg(v) = 2e = 3+5+6+8 \Leftrightarrow e = \frac{1}{2}(22) = 11$$

There are 11 edges.

b) We need to prove that there doesn't exist a graph with $V = 7$, with each vertex having a degree of 3.

Suppose that there exists such graph, then:

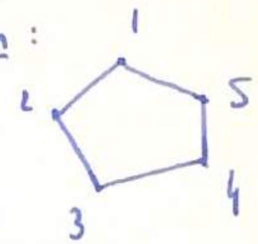
$$\sum_{v \in \text{graph}} \deg(v) = \underbrace{3+3+\dots+3}_{7 \text{ times}} = 7 \cdot 3 = 2 \cdot e \text{, but } 2 \nmid 7 \text{ and } 2 \nmid 3,$$

therefore we have a contradiction.

13. Prove that $C_5 \cong C_5$ (where „$\cong$" means „is isomorphic to").
Then prove that no other cyclic graph is isomorphic to its complement.

<u>Solution</u>:

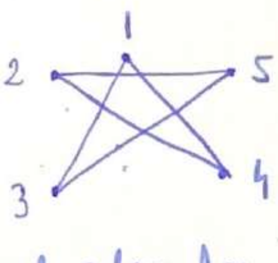
C_5:



Because $C_5 = C_5$ we get that $C_5 \cong C_5$

Attention: There is a mistake regardin the exercise. It was meant to be $C_5 \cong \overline{C_5}$.

$\overline{C_5}$:



- Number of edges for $\overline{C_5}$ is 5 which is equal to the number of edges for C_5.
- Number of vertices for $\overline{C_5}$ is 5 which is equal to the number of vertices for C_5.
- Each vertex for $\overline{C_5}$ has degree of 2 which is equal to the degree of each vertex of C_5
- $\overline{C_5}$ and C_5 are made in one piece (that means that there doesn't exist a disconnected piece of the graph)

All the 4 properties have been satisfied in order for $C_5 \cong \overline{C_5}$.

It is needed to be proved that $C_i \ncong \overline{C_i}$ for $i \in \mathbb{N}^* \setminus \{5\}$.
We know that $e_{\overline{C}} = e_K - e_c$ where: $e_{\overline{C}}$:= total number of edges for $\overline{C}$; e_K := total number of edges for K; e_c := total number of edges for C.

$e_K = \frac{1}{2}(v-1)v$.

$e_{\overline{C}} = e_K - e_c \Leftrightarrow e_{\overline{C}} = \frac{1}{2}(v-1)v - e_c$

for $C_i \ncong \overline{C_i}$ for $i \in \mathbb{N}^* \setminus \{5\}$ we will need $e_{\overline{C}} \neq e_c$

Suppose $C_i \cong \overline{C_i}$ for $i \in \mathbb{N}^* \setminus \{5\}$, then:

$e_{\overline{C}} = e_c \overset{\text{not}}{=\!=} e$, so: $e_{\overline{C}} = \frac{1}{2}(v-1)v - e_c \Leftrightarrow e = \frac{1}{2}(v-1)v - e$

$\Leftrightarrow 4e = (v-1)(v)$ so either $v|4$ or $v-1|4$.
The necessary condition for this to be satisfied is that $e = v$
therefore $v = 5 = e$ but that graph is C_5 or $\overline{C_5}$ in this context,
thus we have a contradiction. $C_i \ncong \overline{C_i}$ for $i \in \mathbb{N}^* \setminus \{5\}$.

14. Prove that if $G \cong \overline{G}$ then v or $v-1$ is a multiple of 4.
Solution: if $G \cong \overline{G}$ then $v_G = v_{\overline{G}}$ and $e_G = e_{\overline{G}}$.
We know that $e_{\overline{G}} = e_{K_G} - e_G$ where e_{K_G} is the complete graph of G with v_G vertices.

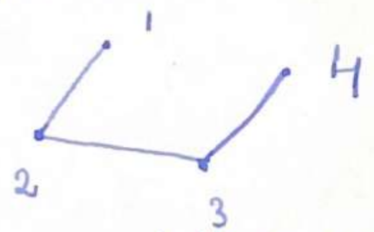
$$\begin{cases} e_{\overline{G}} = e_{K_G} - e_G \\ e_G = e_{\overline{G}} \\ e_{K_G} = \frac{1}{2}(v_G)(v_G - 1) \end{cases} \Rightarrow 2e_G = \frac{1}{2} v_G(v_G - 1) \text{ so } 4e_G = v_G(v_G - 1)$$

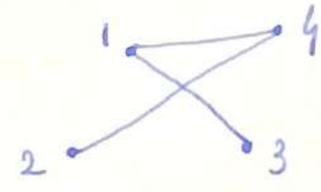
so $4|v_G$ or $4|v_G - 1$
v_G in our context is the v the exercise requested.

15. If $G \cong \overline{G}$; G is called a „self-complementary graph".
Exercise 13 tells us that C_5 is a self-complementary graph.
Find other two self complementary graphs (using the help of exercise 14).

Solution: Exercise 14 tells us that: if $G \cong \overline{G}$ then v or $v-1$ is a multiple of 4.

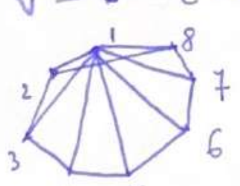
I. For $v = 4$ we have $e_{\overline{G}} = \frac{1}{4} \cdot 4 \cdot 3 = 3 = e_G$

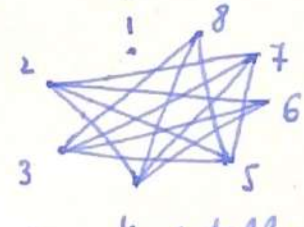
G_4: 

$\overline{G_4}$: 

We can verify the 4 properties, but the visual graph tells us that $G_3 \cong \overline{G_3}$.

II. For $v = 8$ $(4|v)$ we have that $e_{\overline{G}} = \frac{1}{4} \cdot 8 \cdot 7 = 14 = e_G$

G_8: 

$\overline{G_8}$: 

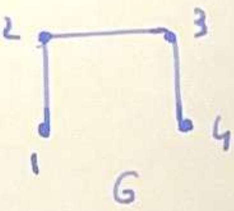
We can verify the 4 properties, but the visual graph tells us that $G_8 \cong \overline{G_8}$.

16. Find a self-complementary graph with $V=8$. Of the 12,346 graphs with $V=8$ only four are self-complementary.
(the underlined text is just a remark).

Solution: The self-complementary graph was found in the second example in exercise 15. (for $V=8$).

17. Since there are an odd number of graphs having $V=4$ (drawn in Figure 45 (page 67)), one of them must be self-complementary. Which one is it?

Solution: The self-complementary graph is:



where $\overline{G}$:

The pictures of G and $\overline{G}$ indicate that. $G \cong \overline{G}$.

18. How many different one-to-one correspondence between:
$\{a;b;c;d;e;f;g;h;i;j\}$ and $\{1;2;3;4;5;6;7;8;9;10\}$ exist?
Attention: Two one-to-one correspondences are „different" if there is at least one element that they associate with distinct elements.

Solution: To generalize:

There are n choices for the first mapping $\cdot$ $n-1$ choices for the second mapping $\cdot$ .... $\cdot$ 2 choices for the penultimate mapping

$\cdot$ 1 choice for the last mapping $= n\cdot(n-1)\cdot\ldots\cdot 2\cdot 1 = n!$

mapping = correspondence

penultimate = the position before the last one.

19. We said in the text that the existence of a one-to-one correspondence between two finite sets implies that they contain the same number of elements. The situation for infinite sets are somewhat different. The set of positive integers $\{1;2;3;\dots\}$ in some sense contains "twice as many" elements as the set of even numbers $\{2;4;6;\dots\}$ yet it is possible to put the set into an one-to-one correspondence. Do so.

Solution: Let $A = \{1;2;3;\dots\}$; $B = \{2;4;6;\dots\}$ and

$$\left\{\begin{matrix} x_1 \in A \ ; x_2 \in A \\ y \in B \end{matrix}\right\}$$ Then let: $x_1 = 2x_2 = y$

By checking injectivity and surjectivity, we get that A and B are in an one-to-one correspondence.

20. Besides the four properties mentioned in the text, another property preserved by isomorphism is the distribution of subgraphs. That is, if two graphs are isomorphic and you select a subgraph at random from either one, then the other will necessarily have an isomorphic subgraph. Hence you can prove that two graphs are not isomorphic by finding a subgraph of the first graph that is not isomorphic to a subgraph of the second graph. Use this fact to devise a proof, shorter than the one in the text, that the graphs of Figure 35 are not isomorphic:

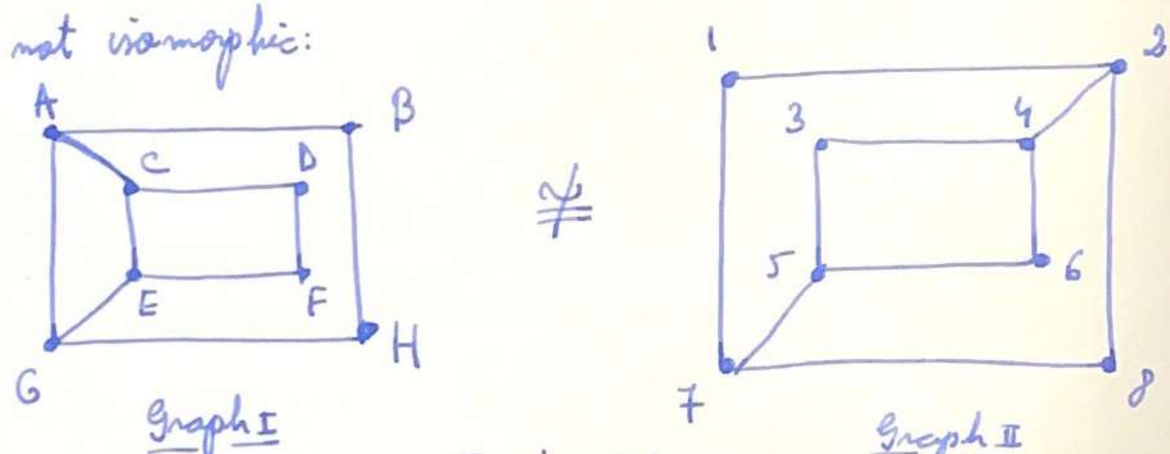
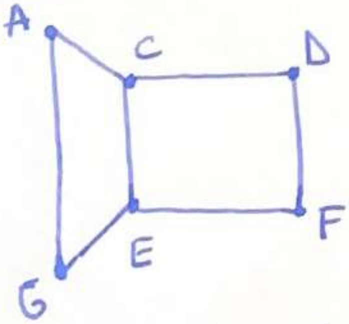


Figure 35

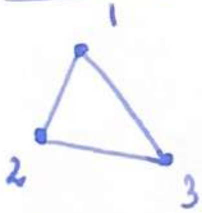
Solution: We will pick from Graph I the subgraph:



By inspection (of Graph II) we conclude that there doesn't exist a subgraph of Graph II isomorphic to our picked subgraph of Graph I (because there doesn't exist any combination of a subgraph in Graph II that has two nearby vertices with degree of 3)

21. K_3 (with its vertices labeled) has 17 unequal subgraphs. Draw them.

Solution: Firstly I will draw K_3:



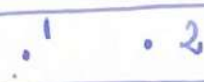
The unequal subgraphs:

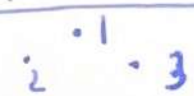
1) Order Zero Graph = $\{\phi\}$	(14) 1 —— 2
2) •1	(15) 2 —— 3
3) •2	(16) 1 —— 3
4) •3	(17) •1 2 —— 3
5) •1 •2	
6) •1 •3	
7) •2 •3	
8) 1• 2• •3	
9) 2 —— 1 •3	
10) 2• 1 —— 3	
11) 2 —— 1, 2 —— 3	
12) 2 —— 1 —— 3	
13) 1 —— 3, 2 —— 3	

22. The number of nonisomorphic subgraphs of K_3 is only 7. Draw them.

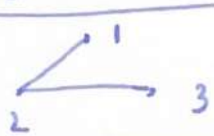
Solution :

1) 

2) 

3) 

4) 

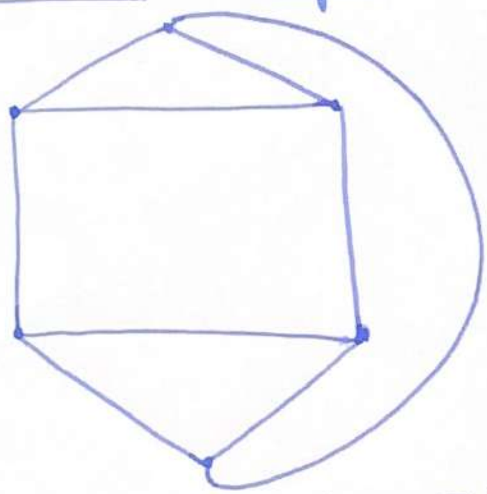
5) 

6) 

7) 

23. The graphs of Figure 46 are not isomorphic. Prove this by finding a subgraph of one that is not a subgraph of the other.

Solution : Graph I



Graph II

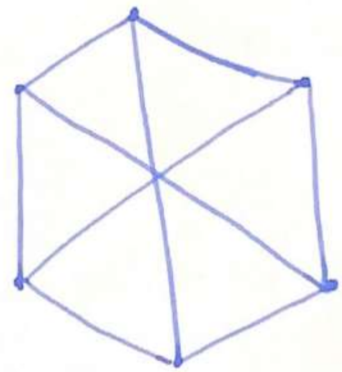
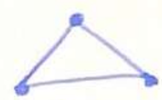


Figure 46

We will pick from graph I the subgraph :  .

We remark that there isn't any legal triangle that can be formed as a subgraph from Graph II, thus Graph I $\ncong$ Graph II.

24. Satisfy yourself that isomorphic graphs have isomorphic complements and nonisomorphic graphs have nonisomorphic complements. Then use this fact to create a proof, different from the one in the text, that the graphs of Figure 31 are not isomorphic.

<u>Solution</u>: Firstly, I will prove „Isomorphic graphs have isomorphic complemets" and „nonisomorphic graphs have nonisomorphic complements".

I will use this definition:

„Two graphs $G_1 = (V_1; E_1)$ and $G_2 = (V_2; E_2)$ are isomorphic if there is a bijectiation (a one-to-one, onto map) f from V_1 to V_2 such that $\{v; w\} \in E_1 \Leftrightarrow \{f(v); f(w)\} \in E_2$. In this case f will be called an isomorphism from G_1 to G_2."

1. If G_1 is isomorphic to G_2 then $(\exists)\ f : V(G_1) — V(G_2)$, a bijection [where $V(G_i)$ represents the vertex set of the graph G_i with $i = \overline{1,2}$] such that $(\forall)\{u_1; v_1\} \in E(G_1)$ we have $\{f(u_1); f(v_1)\} \in E(G_2)$ where $E(G_i)$ represents the edge set of the graph G_i with $i = \overline{1,2}$.

We have that $(\exists)\ f : V(G_1) \to V(G_2)$, a bijection such that $(\forall)\ \{u_1; v_1\} \in E(G_1)$ we have $\{f(u_1); f(v_1)\} \in E(G_2)$ which is equivalent to saying: $(\exists)\ f : V(G_1) \to V(G_2)$, a bijection such that $(\forall)\ \{u_1; v_1\} \notin E(G_1)$ we have $\{f(u_1); f(v_1)\} \notin E(G_2)$.

By saying: $\{u_1; v_1\} \notin E(G_1)$ we mean $\{u_1; v_1\} \in \overline{E(G_1)}$

$\{f(u_1); f(v_1)\} \notin E(G_2)$ we mean $\{f(u_1); f(v_1)\} \in \overline{E(G_2)}$

where: $\overline{A}$ means the conjugate of A.

Because the vertices of a graph G equal to the vertices of the complement of the graph G (namely: $\overline{G}$) we have that:

$$\textcircled{c}\begin{cases} V(G_1) = V(\overline{G_1}) \\ V(G_2) = V(\overline{G_2}) \end{cases}$$

Therefore: $(\exists)$ $f: V(\overline{G_1}) \longrightarrow V(\overline{G_2})$, a bijection such that $(\forall)$ $\{u_1; v_1\} \in E(\overline{G_1})$ we have $\{f(u_1); f(v_1)\}$ which is equivalent to saying: „If $G_1 \cong G_2$ then $\overline{G_1} \cong \overline{G_2}$".

2. If G_1 is not isomorphic to G_2 then $(\nexists)$ $f: V(G_1) \longrightarrow V(G_2)$, a bijection such that $(\forall)$ $\{u_1; v_1\} \in E(G_1)$ we have $\{f(u_1); f(v_1)\} \in E(G_2)$ which is equivalent to saying that $(\nexists)$ $f: V(G_1) \longrightarrow V(G_2)$, a bijection such that $(\forall)$ $\{u_1; v_1\} \notin E(G_1)$ we have $\{f(u_1); f(v_1)\} \notin E(G_2)$

By saying: $\{u_1; v_1\} \notin E(G_1)$ we mean $\{u_1; v_1\} \in E(\overline{G_1})$

$\{f(u_1); f(v_1)\} \notin E(G_2)$ we mean $\{f(u_1); f(v_1)\} \in E(\overline{G_2})$

Using the fact that $V(G_1) = V(\overline{G_1})$ and $V(G_2) = V(\overline{G_2})$ we have that: If G_1 is not isomorphic to G_2 then $(\nexists)$ $f: V(\overline{G_1}) \to V(\overline{G_2})$, a bijection such that $(\forall)$ $\{u_1; v_1\} \in E(\overline{G_1})$ we have $\{f(u_1); f(v_1)\} \in E(\overline{G_2})$ which is equivalent to: If $G_1 \ncong G_2$ then $\overline{G_1} \ncong \overline{G_2}$

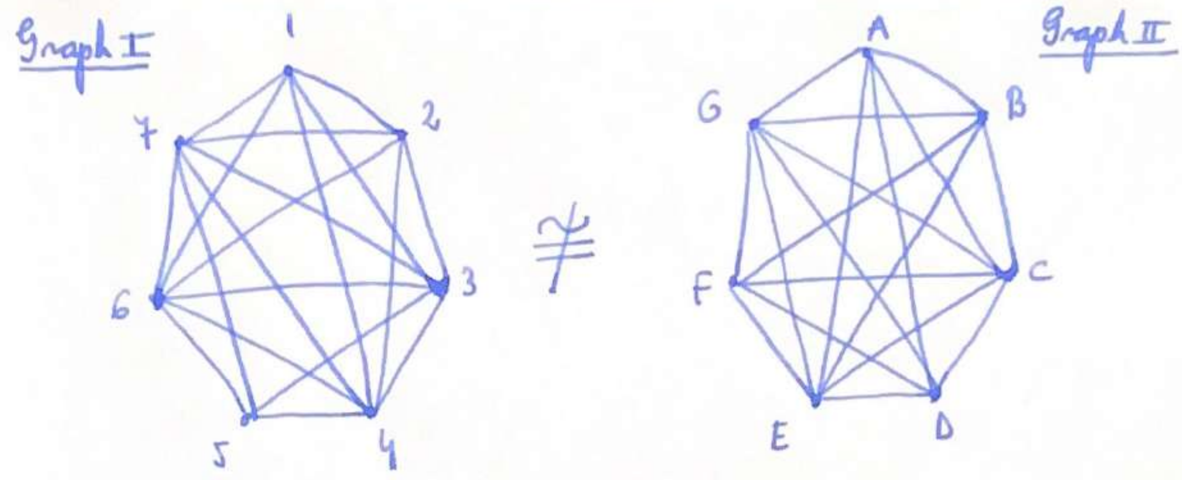


Figure 39

Using the fact that "$G_1 \cong G_2 \Longrightarrow \overline{G_1} \cong \overline{G_2}$" we will suppose that Graph I $\cong$ Graph II and check if $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$

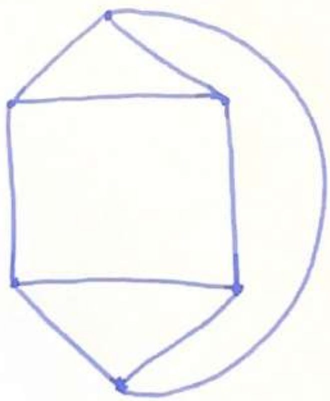
$\overline{\text{Graph I}}$:

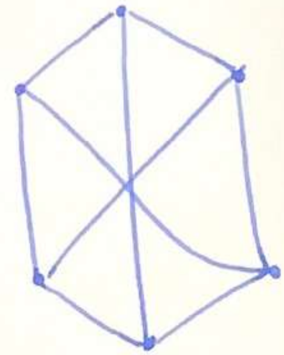
$\overline{\text{Graph II}}$:

We observe that $\overline{\text{Graph I}} \ncong \overline{\text{Graph II}}$, because $\overline{\text{Graph I}}$ contains a vertex of degree 2 (namely 5) which (even after a different labelling) won't be present in $\overline{\text{Graph II}}$.

25. Use the technique of Exercise 24 to prove that the graphs of Figure 46 are not isomorphic.

Solution:





Graph II

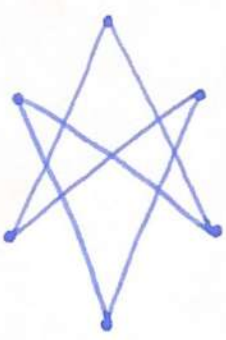
Graph I

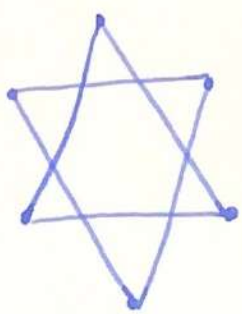
Figure 46

Using the fact that " $G_1 \cong G_2 \Longrightarrow \overline{G_1} \cong \overline{G_2}$ " we will assume . Graph I $\cong$ Graph II and check if $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$.

$\overline{\text{Graph II}}$

$\overline{\text{Graph I}}$





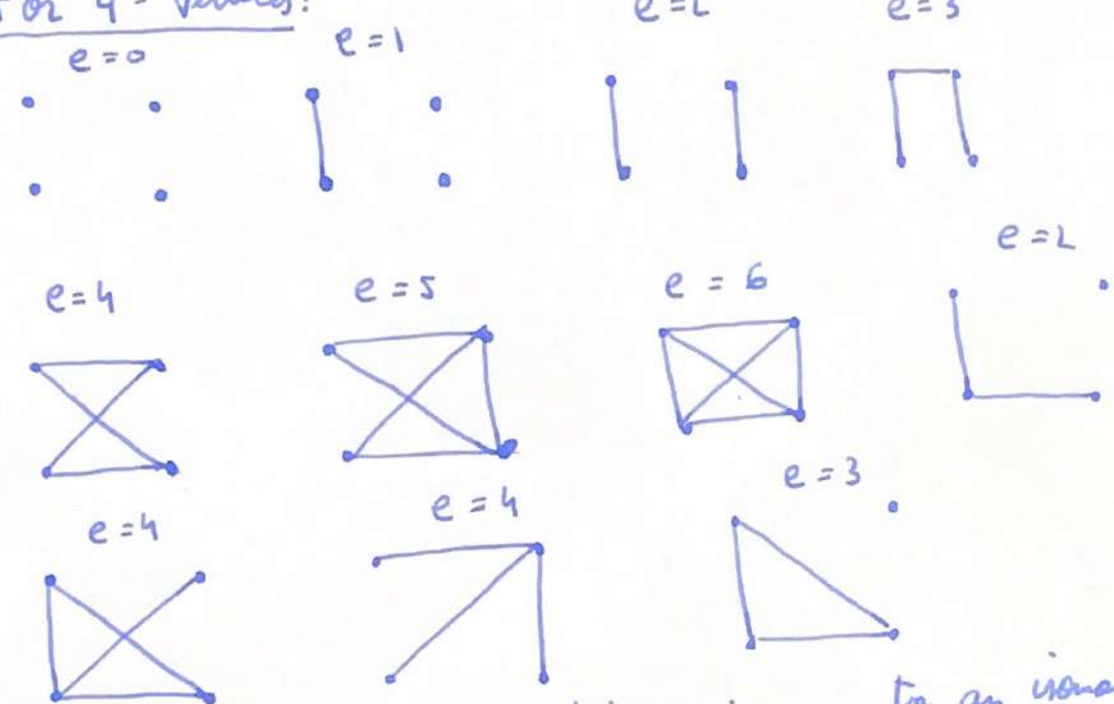
It is a difference between the $\overline{\text{Graph I}}$ and $\overline{\text{Graph II}}$.

$\overline{\text{Graph II}}$ can be decomposed as :  which makes it a graph of two pieces while $\overline{\text{Graph I}}$ is made out of one single piece, thus $\overline{\text{Graph I}} \ncong \overline{\text{Graph II}}$ which contradicts our assumption that Graph I $\cong$ Graph II $\Longrightarrow$ $\overline{\text{Graph I}} \ncong \overline{\text{Graph II}}$. Therefore : Graph I $\cong$ Graph II

26. Draw all graphs having $V=5$. There are 34 of them. Imitate the procedure I used in the text to find all graphs with $V=4$.

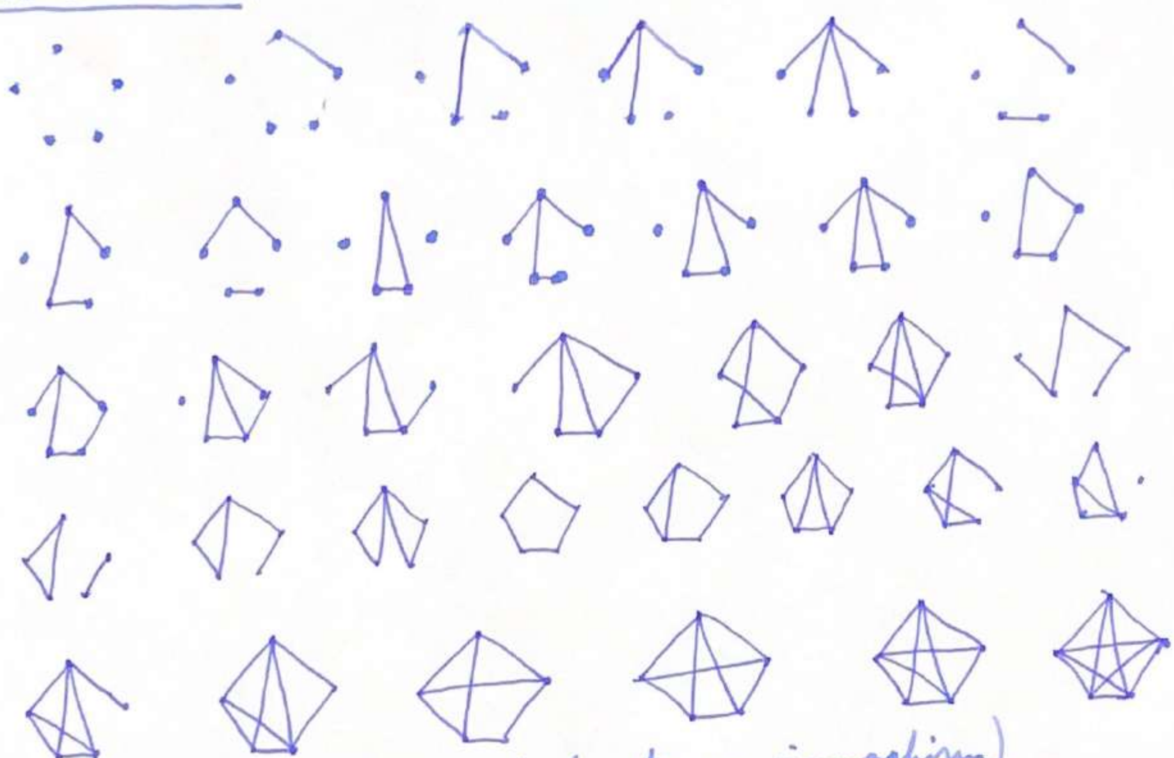
Solution: To see the procedure head to page 67, and look at Figure 45:

For 4 - vertices:



These are all graphs of 4-vertices unique up to an isomorphism.

For 5-vertices:



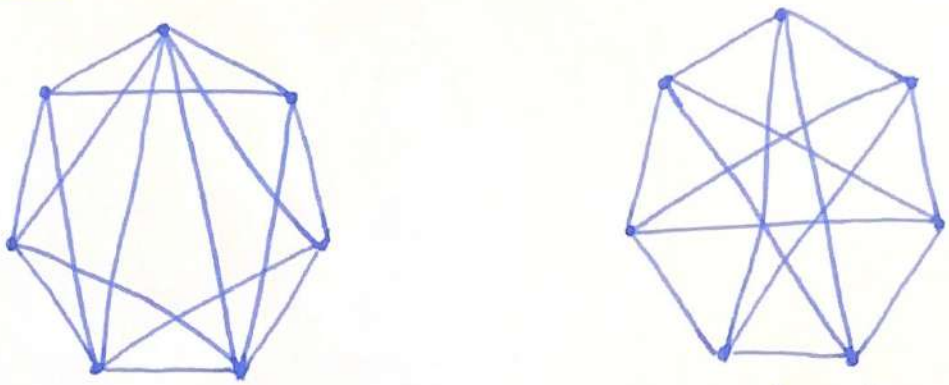
These are all 34 unique graphs (up to an isomorphism)

27. In the table on page 66, the numbers in the second column are mostly even. If we ignore the first row with $v=1$ because of the triviality of the answer, we will be left with $v=4$ as the only number of vertices listed for which there are an odd number of graphs. Do you think this is due to chance, or can you think of a reason why $v=4$ should be unique? If the table were continued do you think more odd numbers would turn up in the second column?

Solution: For an explicit answer, it will be required to know „Polya's Enumeration Theorem"

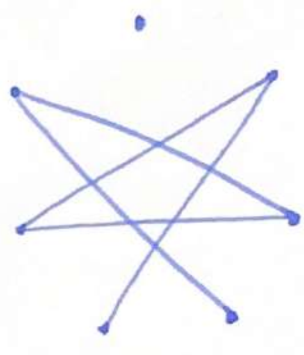
28. Prove that the graphs of Figure 47 are isomorphic

Solution:

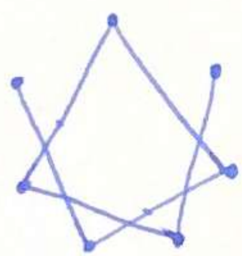


Graph1 Figure 47 Graph2

Following the fact " Two isomorphic graphs have isomorphic complements" wich is equivalent to saying " $G_1 \cong G_2 \Longleftrightarrow \overline{G_1} \cong \overline{G_2}$", we will have to prove that $\overline{G_1} \cong \overline{G_2}$ which will demonstrate that $G_1 \cong G_2$. We will shorten Graph1 with G_1 and Graph2 with G_2 (while $\overline{\text{Graph}_1}$ will be $\overline{G_1}$ and $\overline{\text{Graph}_2}$ will be $\overline{G_2}$).



$\overline{\text{Graph}_1}$



$\overline{\text{Graph}_2}$

Visually, $\overline{\text{Graph}_1} \cong \overline{\text{Graph}_2}$

29. Prove that the graphs of Figure 48 are all isomorphic to the utility graph.

Solution:

a) b) c)

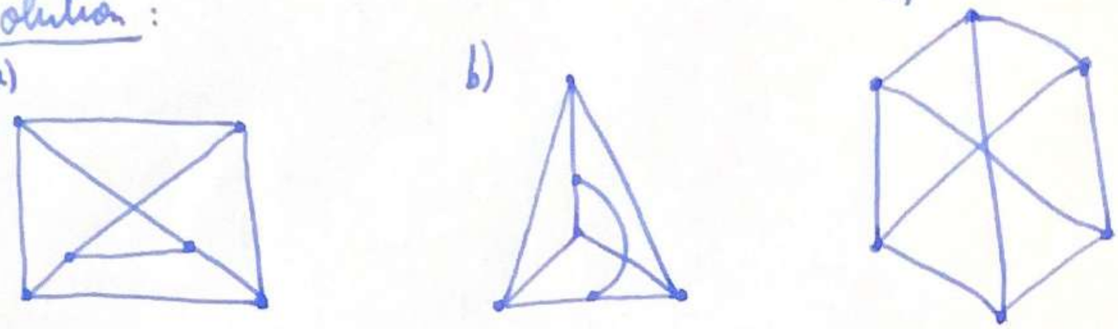
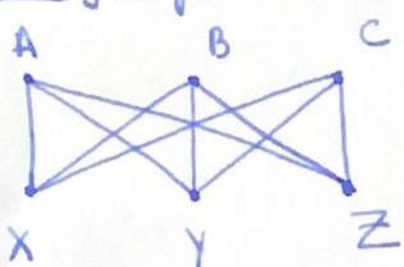
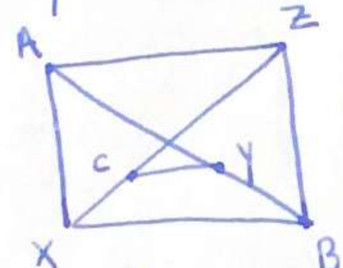


Figure 48.

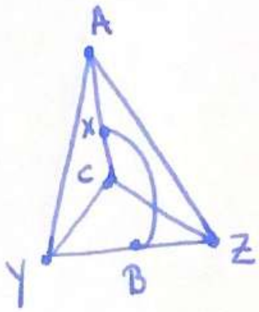
Utility Graph:



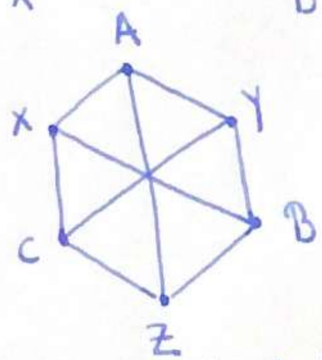
For a)



b)



c)



Visually, all properties of isomorphism hold for: a) $\cong$ U. G.

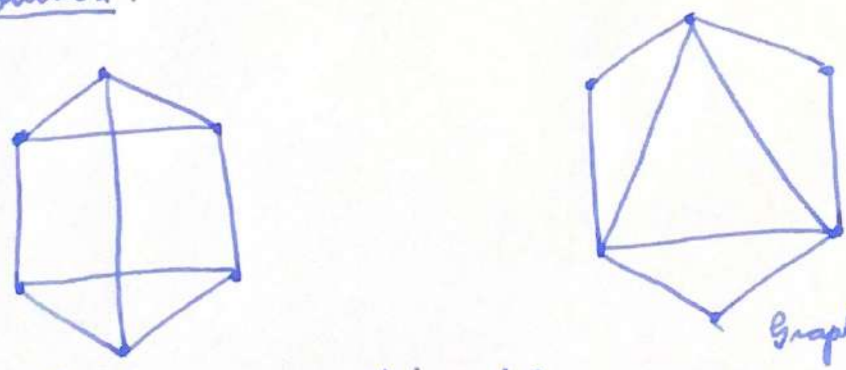
b) $\cong$ U. G.

c) $\cong$ U. G.

where „U. G." stands for „Utility Graph".

30-37. In each of the Figures 49-56, decide whether or not the two graphs are isomorphic. If you decide they are isomorphic, prove it by finding a label under which they would be equal. If you decide they are not isomorphic, prove it by finding a property which is preserved by isomorphism but in terms of which the two graphs differ (We now have six such properties: the four mentioned in the text, and two more introduced in "Exercise 20" and "Exercise 24")

Solution:

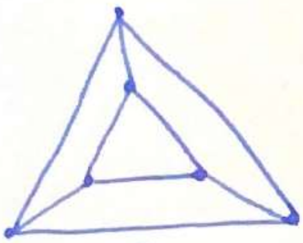


Graph I Figure 49 Graph II

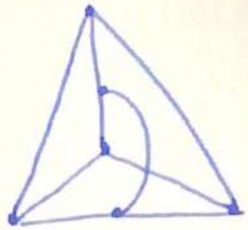
Visually, Graph I $\ncong$ Graph 2 because Graph I only has vertices with degrees equal to 3, while Graph II has 3 vertices with degrees equal to 2.

Graph I Figure 50 Graph II

Visually, Graph I $\cong$ Graph II. As a matter of fact the Graphs (Graph I and Graph II) are equal.



Graph I

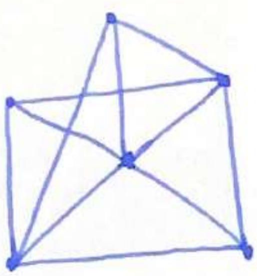


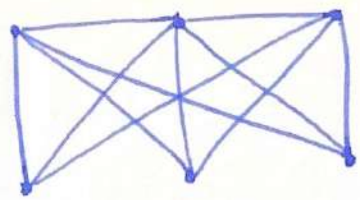
Graph II

Figure 51

We know from Figure 48 that the Graph II (from Figure 51) is isomorphic to the Utility Graph, thus Graph I (from Figure 51) must be isomorphic to the Utility Graph. Inspecting Graph I we come to the conclusion that is isomorphic to the Utility Graph.

Therefore Graph I $\cong$ Graph II.



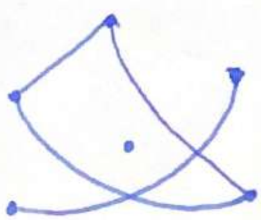


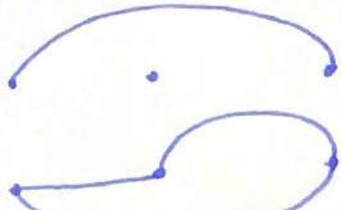
Graph II

Graph I

Figure 52

Visually, Graph I $\cong$ Graph II. To convince ourselves we will check that $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$

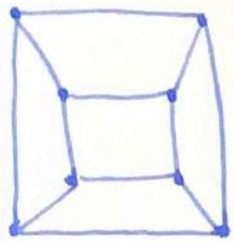




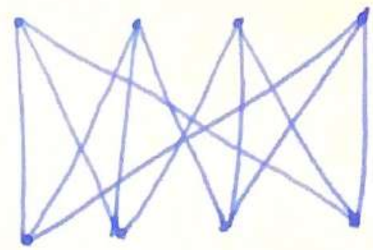
$\overline{\text{Graph II}}$

$\overline{\text{Graph I}}$

Visually $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$, thus Graph I $\cong$ Graph II.



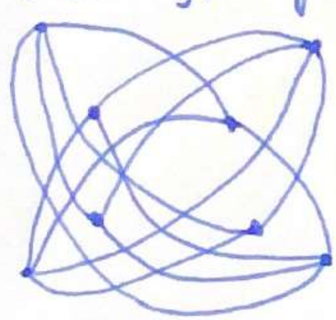
Graph I



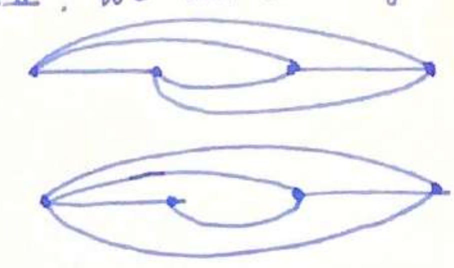
Graph II

Figure 53

Visually, Graph I $\cong$ Graph II. We can check if $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$

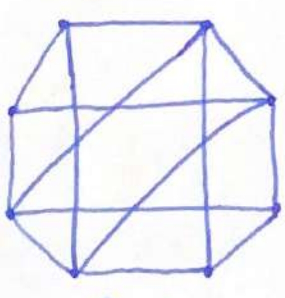


$\overline{\text{Graph I}}$

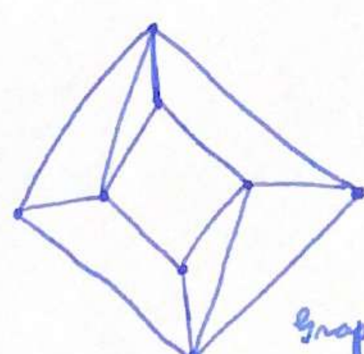


$\overline{\text{Graph II}}$

Visually, $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$, thus Graph I $\cong$ Graph II.



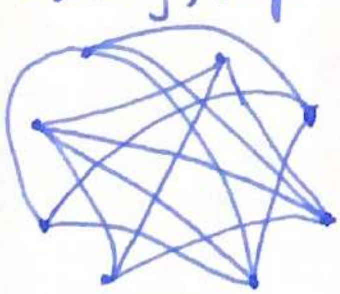
Graph I



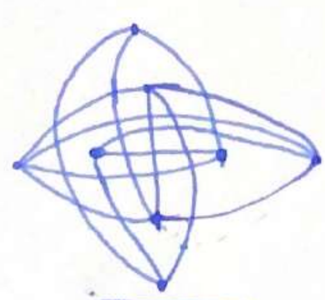
Graph II

Figure 54

Visually, Graph I $\cong$ Graph II. We can check if $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$.



$\overline{\text{Graph I}}$

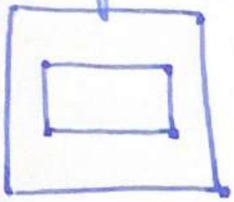


$\overline{\text{Graph II}}$

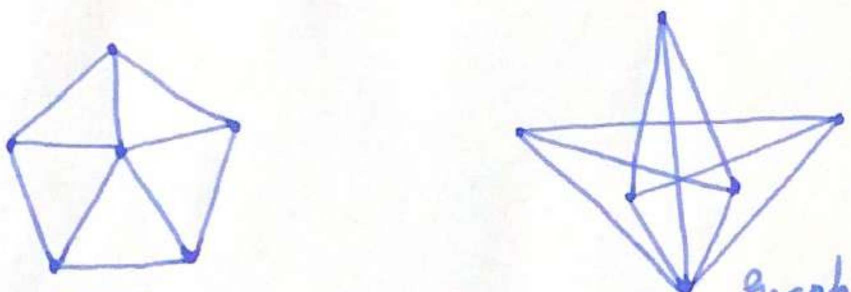
Visually $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$, thus Graph I $\cong$ Graph II.

Graph I Figure 55 Graph II

Graph I $\ncong$ Graph II. As we take the subgraph (from Graph I):



; we observe that there is no possible (and legal) subgraph from Graph II, that will be isomorphic to the subgraph from Graph I we selected.



Graph II

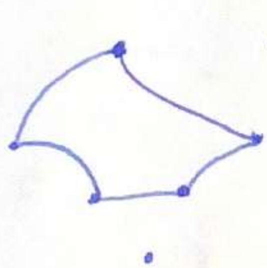
Graph I

Figure 56

Visually, Graph I $\cong$ Graph II. We can check if $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$.

$\overline{\text{Graph I}}$

Visually, $\overline{\text{Graph I}} \cong \overline{\text{Graph II}}$, thus Graph I $\cong$ Graph II



$\overline{\text{Graph II}}$

38. A chess tournament consists of 25 players, each of whom plays one game with every other player. How many games are played during the tournament?

Solution: In this case, the edges between the vertices (who represent the players) represent the games played. The graph is K_{25} (the complete graph of 25 vertices) which has a total of : $e = \frac{1}{2}(V-1)V \Leftrightarrow e = \frac{1}{2} 24 \cdot 25 = 12 \cdot 25 = 300$ games.

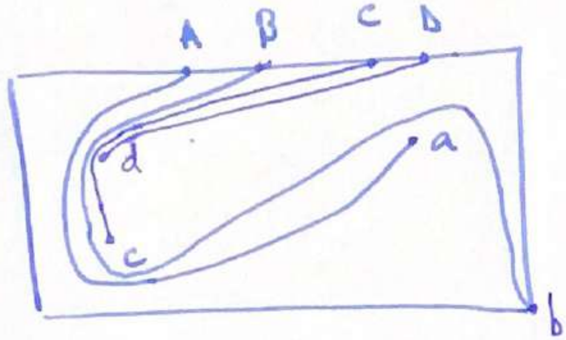
39. When I said on page 67„ There is no simple way of determining the number of graphs having a given v" I meant of course that there is no simple way of determining the number of nonisomorphic graphs having a given v. Prove that there are exactly $2^{\frac{1}{2}v(v-1)}$ unequal graphs whose v vertices have been labeled with the integers from 1 to v.

Solution: We know that a graph with v vertices has $\binom{v}{2}$ possible edges (as each edge binds two vertices), thus there are exactly $2^{\binom{v}{2}}$ graphs with v vertices (where „$2^{\square}$" signify the two option for the total amount of legal edges: first option (we can have the total amount of possible edges), second option (we cannot have the total amount of possible edges). Final result: There are exactly $2^{\binom{v}{2}}$ unequal graphs.

Remark: $\binom{v}{2} = \frac{1}{2}(v-1)v$.

40. The utility puzzle mentioned on page 35 was invented by Henry Ernest Dudeney (1857-1930), a self-taught matematician who was the foremost puzzlemaker of his time. Here is another of Dudeney's puzzles, one which, unlike the utility puzzle, has a solution. In Figure 56A connect a to A; b to B; c to C and d to D by edges which are inside the rectangle and don't cross one another.

Solution:



Page 109-113:

1. Prove the following statements:

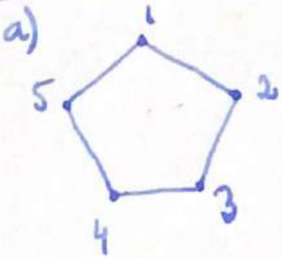
a) If three edges are added to the graph of Figure 63 a, then at least two of the new edges will be adjacent.

b) Every graph with $v=5$ and $e=3$ has at least two adjacent edges.

c) If v is an odd number then every graph with v vertices and $\frac{1}{2}v(v-1)$ edges has at least two adjacent edges.

Solution:

a)



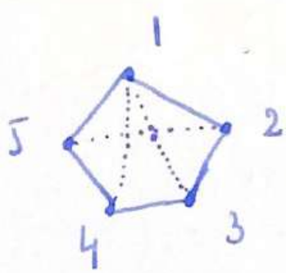
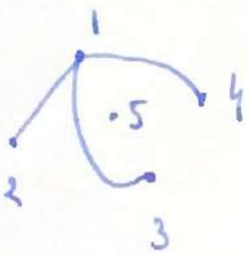


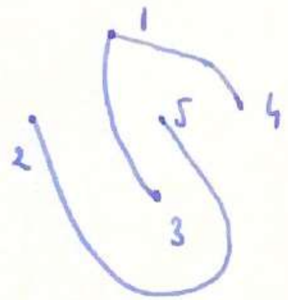
Figure 63 a)

We can speak in terms of a visual proof. We observe that the vertex we choose to draw new edges from doesn't matter, because any vertex can be labelled to be any remaining vertex, as each vertex of the graph of Figure 63 a) is identical to each other remaining vertex. We can observe that any combination of new edges we make, there will be at least one common vertex that will be shared with at least two of the total three edges that we have.

b) We will draw all graphs with vertices $V = 5$ and edges $e = 3$ (that are unique, up to an isomorphism)



Graph I



Graph II

We have drawn all unique graphs with $V=5$ and $e=3$ that are unique (up to an isomorphism). As we can see every graph (unique, up to isomorphism) has at least two adjacent vertices.

c) We know that a graph with v vertices and $e = \frac{1}{2} v(v-1)$ edges is a complete graph (abbreviated with K_v). Graphically, a complete graph, despite the value of v (and dismissing the case $v=1$ and $V=0$) will have at least two adjacent edges. Therefore, the statement: „If v is an odd number then the graph with v vertices and $e = \frac{1}{2} v(v-1)$ edges has at least two adjacent edges" holds true only if $v \neq 1$ for $v =$ odd.

2. Prove that in Figure 58, graph a) is planar, and that graphs b) and c) are nonplanar.

Solution:

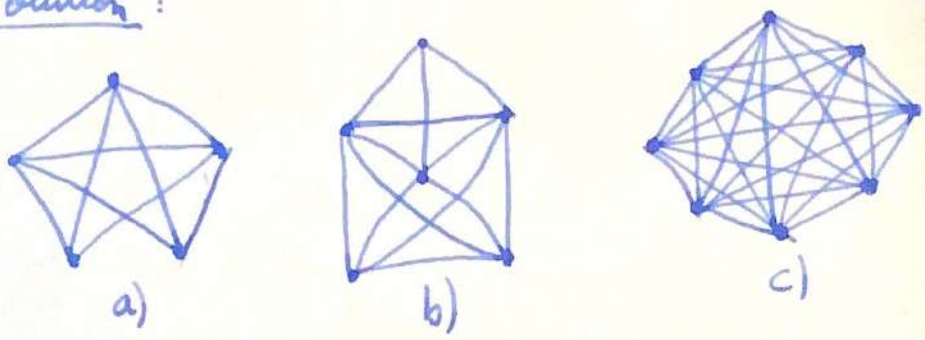
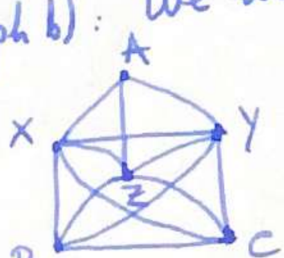


Figure 58

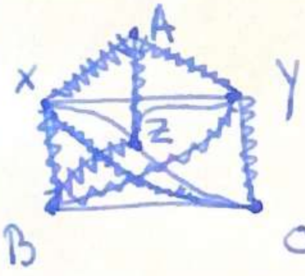
For graph a):

We managed to uncross the overlapping edghes and our graph has become planar.

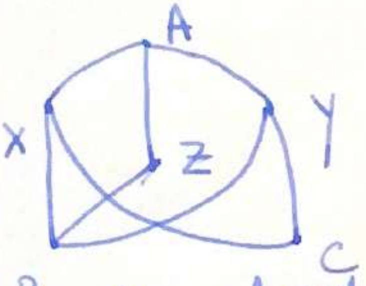
For graph b): We will label the graph b) in the following manner:



We will mark all uncrossed (which is the same as saying they are disjoint) paths from the three houses A; B; C (A being the first house, B being the second house, C baing the third house) to the three utility companies X; Y; Z (where the ascendant ordering of the utility companies match the alphabetical ordering of X; Y; Z) in the following manner:

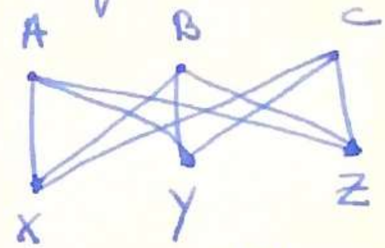


All non-marked things will disappear; leaving us with:



Our subgraph

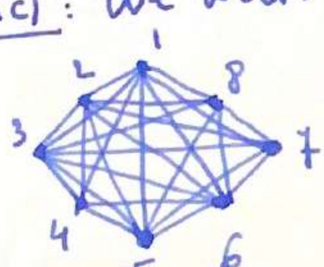
=



the utility graph.

We know that the utility graph is a nonplanar graph.
Using Kuratowski's Theorem: A subgraph of the expansion of either the utility graph or the complete graph K_5 is nonplanar. Because our graph is a supergraph of our subgraph which is nonplanar, we will get to the conclusion: Graph b) is nonplanar.

For graph c): We will label de graph c) in the following manner:



We acknowledge the fact that graph c) represents K_8.
Using Kuratowski's Supergraph Theorem: „Every nonplanar graph is a supergraph of an expansion of U.G (the utility graph) or K_5". K_8 is a supergraph of K_5 (which is an expansion of itself)

Therefore K_8 is nonplanar.

3. The expansions of K_3 are the cyclic graphs: C_3; C_4; C_5;... (in other words: the expansions of K_3 are the cyclic graphs C_n, where $n \in \mathbb{N} \setminus \{0;1;2\}$). Satisfy yourself that except for C_3 (which is isomorphic to K_3), no other subgraph is a supergraph of K_3. Thus K_3 has the property that none of its expansions [except for K_3 and <u>C_3</u> (which is isomorphic to K_3)] is also a supergraph. Find a simple graph having the „reverse" property, that is, find a graph G such that every expansion of G is also a supergraph of G.

<u>Solution</u>: We will prove every statement.

1. „The expansions of K_3 are the cylic graphs: C_3; C_4; C_5;... , where „..." = „and so on"."

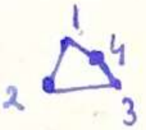
We will draw the graph of K_3:  .

K_3 C_3

We observe that we can easily insert a vertex in one of the edges. After inserting a vertex between the two vertices 3 and 1, we get C_4. After inserting another vertex between the two vertices 4 and 1, we get C_5. We can continue this process forever, therefore proving the statement.

2. "Satisfy yourself that except for C_3 (which is isomorphic to K_3), no other cyclic graph is a supergraph of K_3."

We will look at C_4:



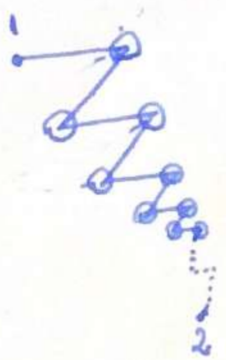
We observe that the vertex 4 is an illegal vertex for C_4 to be considered a supergraph of K_3, because <u>spliced vertices</u> [which are vertices that one position it on an edge between two <u>original vertices</u> (vertices from the original graph)].

Because C_4 is the building block from which we derive $C_5; C_6; \dots$; we cannot have a supergraph as any cyclic graph as the rest of the cyclic graphs ($C_5; C_6; \dots$) derive from the cyclic Graph C_4 which we already proved that it isn't a supergraph.

3. "Thus K_3 has the property that none of its expansions [except itself and C_3 (which is isomorphic to K_3)] is also a supergraph".

We will need to check if all of K_3 expansions are just cyclic graphs C_n, with $n \in \mathbb{N} \setminus \{1;2\}$. Because of the fact that expansions are just original graphs with vertices added to preexisting edges, we come to the conclusion that indeed cyclic graphs of the form C_n with $n \in \mathbb{N} \setminus \{1;2\}$ are the only expansions of K_3 (which we have proved at the second statement that they don't are supergraphs of K_3, excepting C_3)

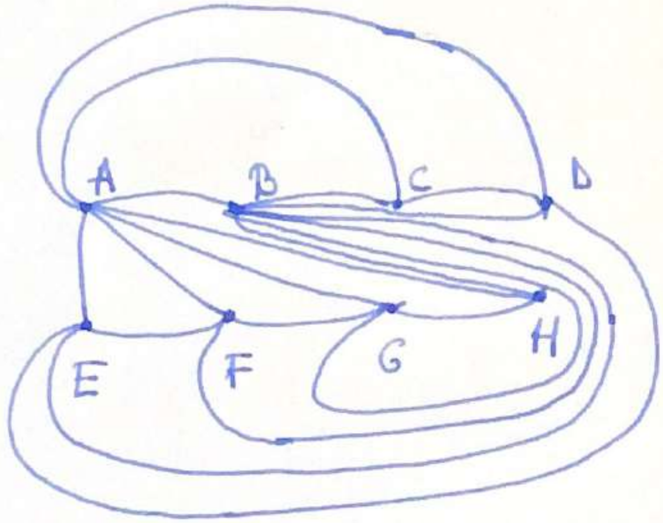
Instead of statements to prove, we will have to come up with some ideas for the remaining exercise. For finding a graph for which every expansion of the graph (we will name it G) is also a supergraph of the graph that we named G, we will consider a line: . Notice that on the line we can add infinitely many vertices, thus obtaining infinitely many expansions which visually proves the fact that every expansion of the graph G is also a supergraph of G. Another visually pleasing graph that satisfies the property that every expansion of the graph is also a supergraph of that graph is a thunder shaped graph:



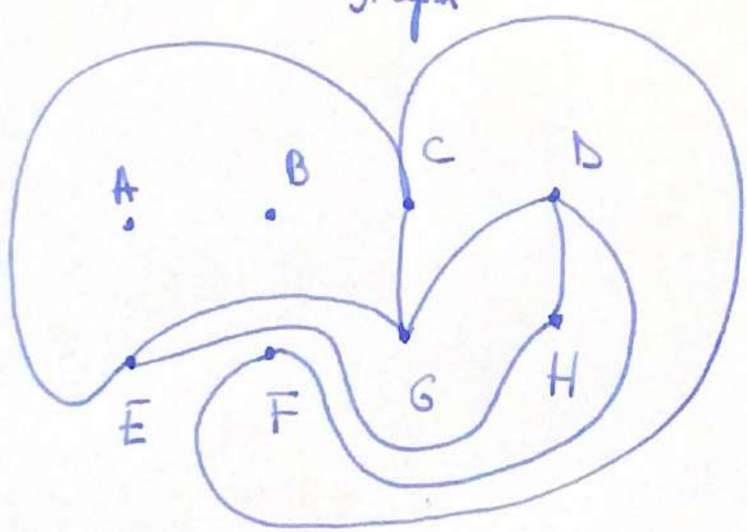
in which we add a vertex to any unlabelled corner (which we encircled). The number of corners matches the number of expansions that we want to add.

4. Find a planar graph with $V=8$ whose complement is also planar.

Solution :



Graph

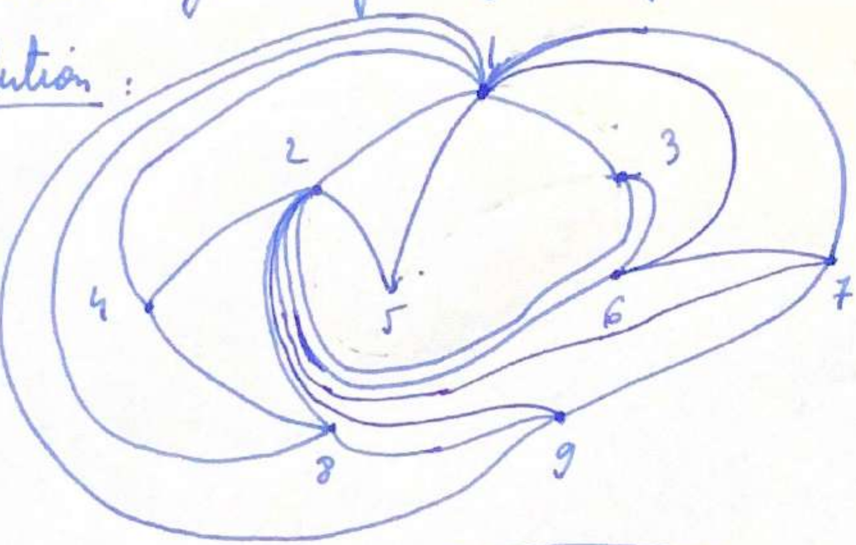


$\overline{\text{Graph}}$

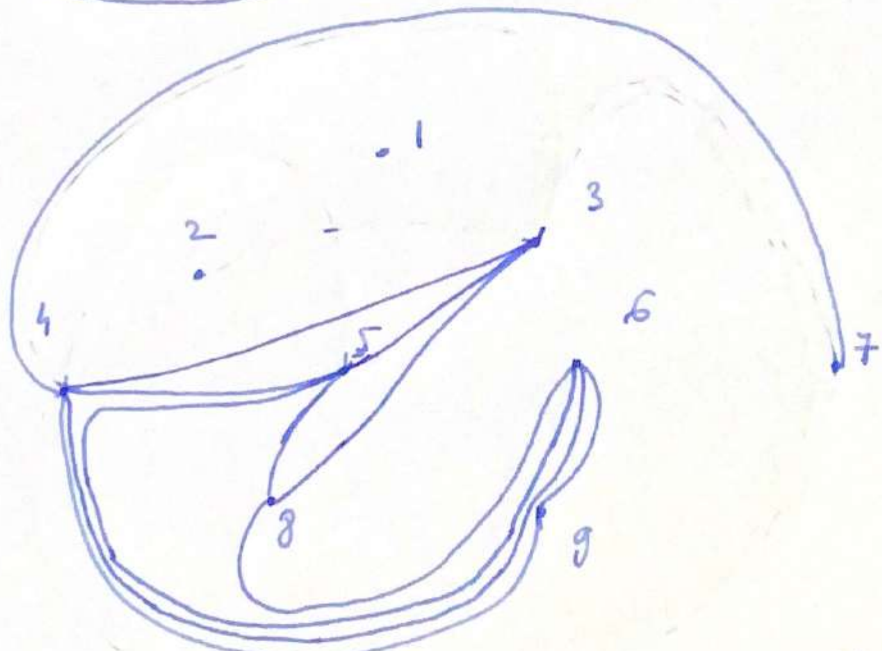
As we can see, all the edges are noncrossing (for both Graph and $\overline{\text{Graph}}$), so both of them are non planar

5. It is a fact that any planar graph with $V=9$ has a nonplanar complement. Verify this in one case by drawing a planar graph with $V=9$ (make it complicated or the exercise will be boring) and proving its complement is nonplanar.

Solution:



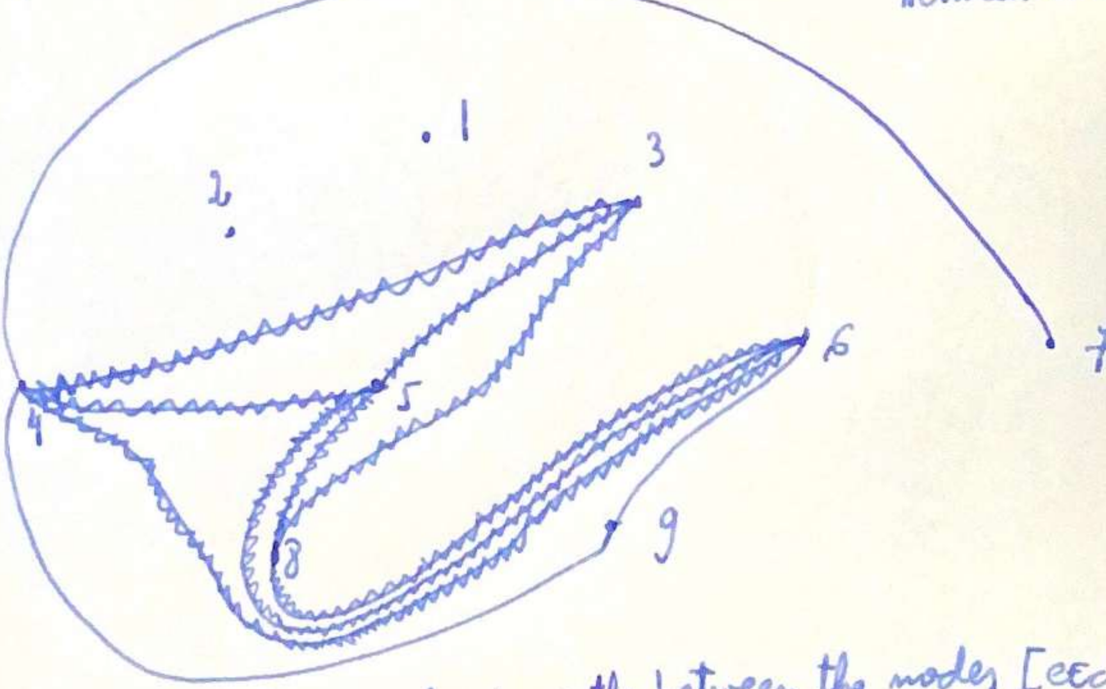
Graph



$\overline{\text{Graph}}$

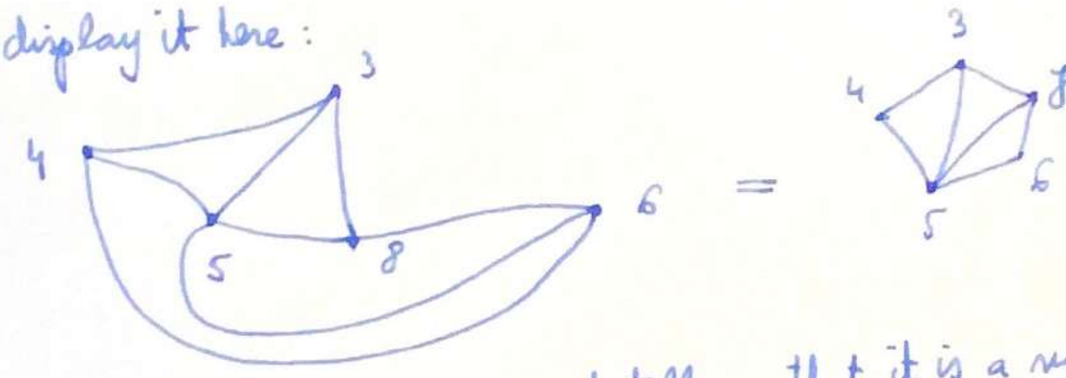
We can observe that for $\overline{\text{Graph}}$ there is no way of tracing a path from the vertex 7 to the vertex 5 without crossing either the edge set {4;5} or the edge set {5;8}. Therefore the complement of our graph is probably nonplanar.

For it to be certainly nonplanar we will pick 5 vertices with degrees at least equal to 3 and check if we can subtract a subgraph from the original Graph in order to prove [Using Kuratowski's Subgrap Theorem : A graph that has a subgraph of an expansion of either K_5 (the complete graph with 5- vertices), or U.G. (the utility graph) is nonplanar] the nonplanarity of Graph. We will pick the vertices 4;5;6;8;9. (because each one of them has degree at least equal to 3). We will mark all nonoverlapping paths between the nodes with jagged lines [excluding the common nodes each path shares between each other]



After marking the nonoverlapping paths between the nodes [excluding the common nodes each path shares between each other], we will get rid of everything that isn't marked with jagged lines, getting as a result a subgraph of K_5. Knowing that K_5 is an expansion of itself, we have gotten ourselves a subgraph of an expansion of K_5.

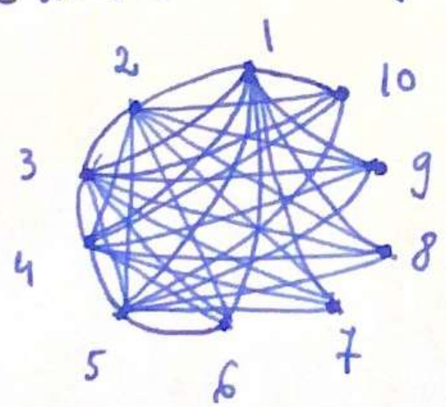
Using Kuratowski's Subgraph Theorem: A graph that has a subgraph of an expansion of either K_5 or U.G. is nonplanar and the fact that we extracted a subgraph of an expansion of K_5 proves the exercise. For the visual of our selected subgraph, we will display it here:



Visually our marked indeed tells us that it is a subgraph of K_5 (which is an expansion of K_5 itself)

6. Draw a nonplanar graph whose complement is nonplanar.

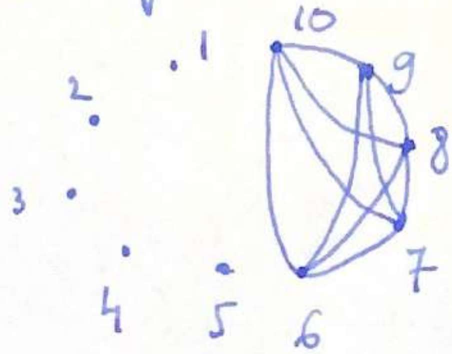
Solution: We will consider the following graph:



Graph.

We will like to know if the graph is nonplanar. Visually, by picking the vertices 1;2;3;4;5 we remark that, taking into account all the edges between the picked vertices (1;2;3;4;5), together with the vertices themselves create K_5.

Using the fact that our graph has a subgraph of K_5 (which is an expansion of itself) we conclude (using Kuratowski's Subgraph Theorem: A graph that has a subgraph of an expansion of either K_5 or U.G., is nonplanar) that our graph is nonplanar. Taking the complement of our graph we will get K_5 formed out of all edges that bind the vertices 10, 9, 8, 7, 6. To confirm this view, we will represent the complement graphically:

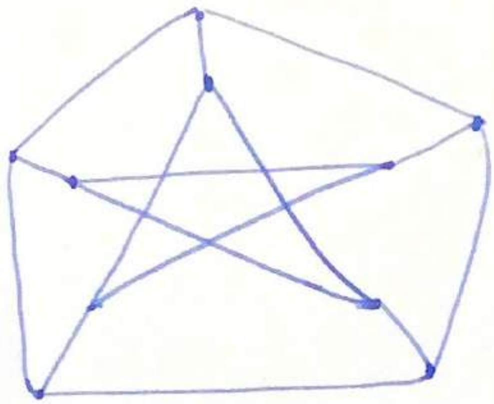


$\overline{\text{Graph}}$

We see that our complemented graph has a subgraph of K_5 (which is an expansion of itself). We conclude (using Kuratowski's Subgraph Theorem: A grap that has a subgraph of an expansion of either K_5 or U.G., is nonplanar) that our complemented graph is nonplanar. Therefore, we have found a graph that is nonplanar, with a nonplanar complement.

7. Prove that the „Petersen graph" of Figure 88 is nonplanar.

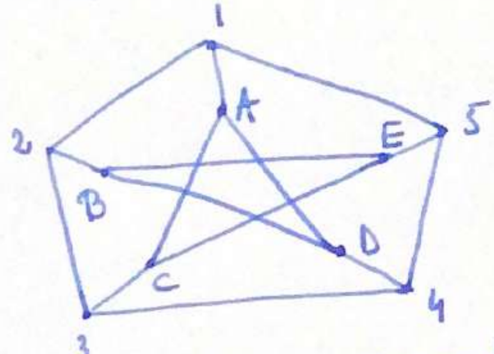
Solution:



Petersen Graph

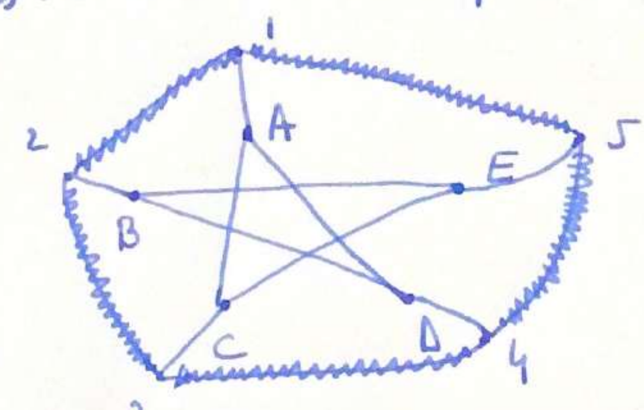
Figure 88

We will label each vertex in order to bring more clarity.



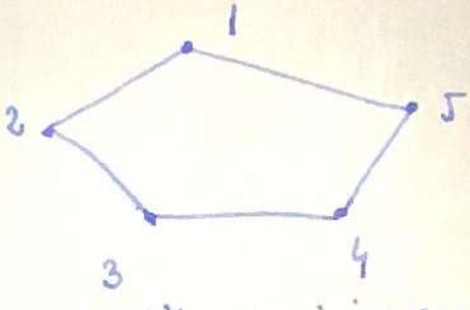
Labelled Petersen Graph

We will choose vertices 1,2,3,4,5 and mark with jagged lines the nonoverlapping paths between them (excluding the common vertices between each one of the paths).



Marked Petersen Graph

Everything outside the marked paths (excluding the common vertices between each one of the paths) will vanish, resulting in the following graph:

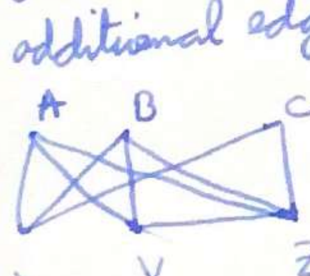


The remaining graph.

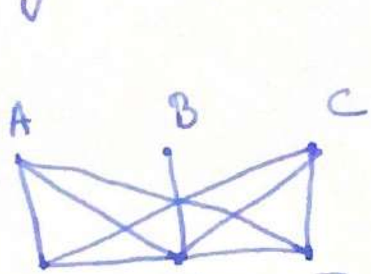
The remaining graph represents a subgraph of the original graph. Using Kuratowski's Supergraph Theorem (A graph that has a subgraph of an expansion of either: K_5 or U.G., is nonplanar) and the fact that our graph has a subgraph of K_5 (which is an expansion of itself) we conclude that: The Petersen Graph is nonplanar.

8. Of the 34 graphs with $V=5$; K_5 is of course the only one that is nonplanar. But of the 156 graphs with $V=6$, 13 are nonplanar besides U.G.. Find them:

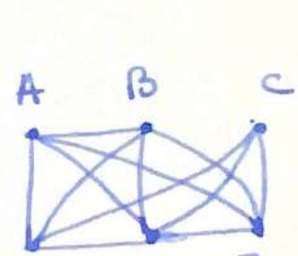
<u>Solution</u>: Knowing that U.G. is nonplanar. We will use U.G. as our building block from which we will add additional edges:



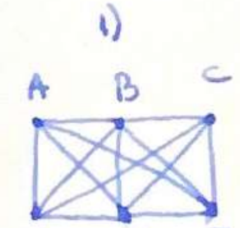
1)



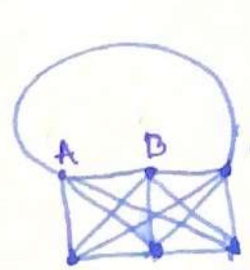
2)



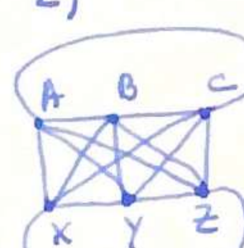
3)



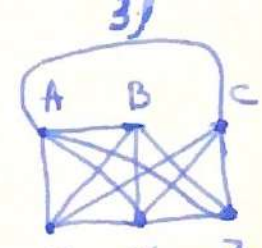
4)



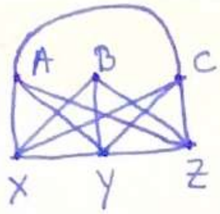
5)



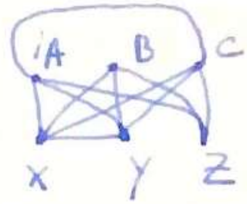
6)



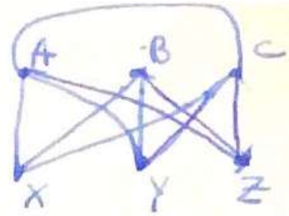
7)



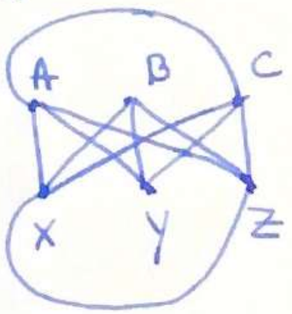
8)



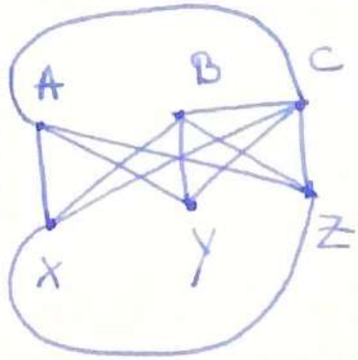
9)



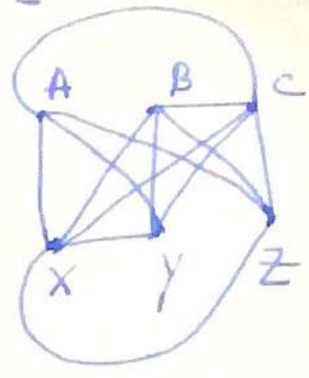
10)



11)



12)



13)

9. Prove: if H is an expansion of G then $V_G + e_H = V_H + e_G$
(where V_G = the number of vertices of G ~~…~~ displayed, e_H = the number of edges of H displayed, V_H = the number of vertices of H displayed, e_G = the number of edges of G displayed)

<u>Solution</u>: Let our graph be a line: 1 —— 2. As we can observe $G = \{V; E\}$, where $V = \{1; 2\}$ and $E = \{\{1; 2\}\}$.
To create an expansion, let „3" be the vertex added in between the vertices „1" and „2". Therefore we have:

G H

$$\left.\begin{array}{l} V_H = 3 \\ e_H = 2 \\ V_G = 2 \\ e_G = 1 \end{array}\right\} \Rightarrow V_H + e_G = V_G + e_H$$

We observe that e_G and v_G will be of a fixed value where $e_G=1; v_G=2$.

We have the equality: $v_G+e_H=v_H+e_G$ that can be written as: $v_H-e_H=v_G-e_G \Leftrightarrow v_H-e_H=2-1 \Leftrightarrow v_H-e_H=1$

We observe that this equality will hold because no matter how many new vertices we introduce in G, the "number of edges with the new vertices" will always equal to "the number of vertices (new and old) minus one". In mathematical terms $e_H=v_H-1$. Remark: As we add new vertices between old vertices we create an expansion (which we denoted by "H"). Our equality: $v_G+e_H=v_H+e_G$ for the selected graph (denoted by "G") is therefore proved.

10. Prove: If H and J are expansions of G then $v_H+e_j=v_j+e_G$ (where v_H = the number of vertices of H displayed; e_j = the number of edges of J displayed; v_j = the number of vertices of J displayed; e_G = the number of edges of G displayed)

Solution: If we consider the graph G a line: 

G

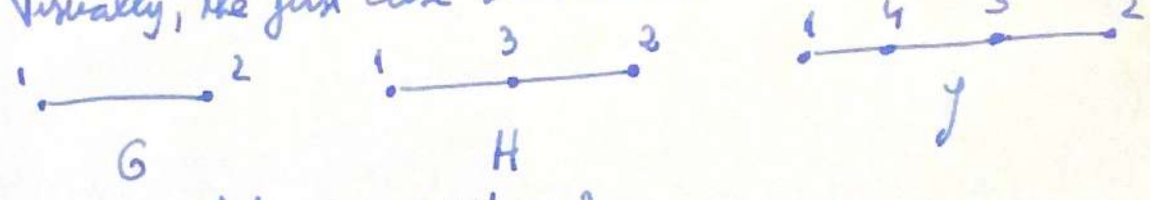
If we have J = H then we situate ourselves at the equality: $v_H+e_H=v_H+e_G \Leftrightarrow e_H=e_G$. This equality holds only if H = G, otherwise $e_H \neq e_G$ (because at each addition of a new vertex between two original vertices we will get a new edge, therefore $e_H \neq e_G$).

If we have J ≠ H, we will consider $e_H<e_j$ and $v_H<v_j$.

Because of our imposed conditions on H and J (namely, that $e_H<e_j$ and $v_H<v_j$), for each adition of one vertex to our original

graph G (in order to create the expansion H), we will have to add a second new vertex to our original graph G (in order to create the expansion of J).

Visually, the first case will look like this:



We have that: $e_G = 1;\ V_G = 2$

$e_H = 2;\ V_H = 3$

$e_J = 3;\ V_J = 4$

We will verify the equation: $V_H + e_J = V_J + e_G \Leftrightarrow 2+3 = 4+1$ $\Leftrightarrow 5=5$. (True)

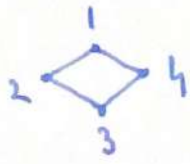
Therefore we have proved that the equality: $V_H + e_J = V_J + e_G$ holds if $H \neq J$ and $e_J = e_H + 1;\ V_J = V_H + 1$.

11. Find all integers v for which $\overline{C_v}$ (which is the complement of C_v) is nonplanar. Remark: C_v = the cyclic graph of v vertices. Prove that your answer is correct.

<u>Solution</u>: Using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either K_5 or U.G., is nonplanar", we will see if it brings us any help.

We will start testing every cyclic graph until we reach the first cyclic graph that is nonplanar and has a nonplanar complement. We will omit the first 4 cyclic graphs: $C_0;\ C_1;\ C_2;\ C_3$ as they are easy to verify that they are indeed planar, with planar complements.

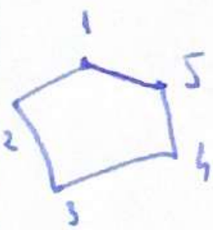
Starting at C_4:



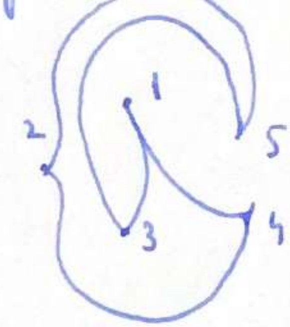
C_4

$\overline{C_4}$

We observe that C_4 is planar and $\overline{C_4}$ is planar.

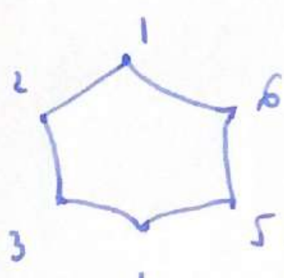


C_5



$\overline{C_5}$

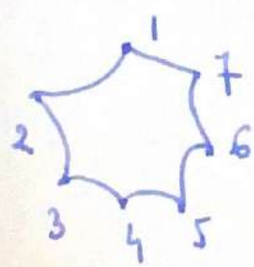
We observe that both C_5 and $\overline{C_5}$ are indeed planar.



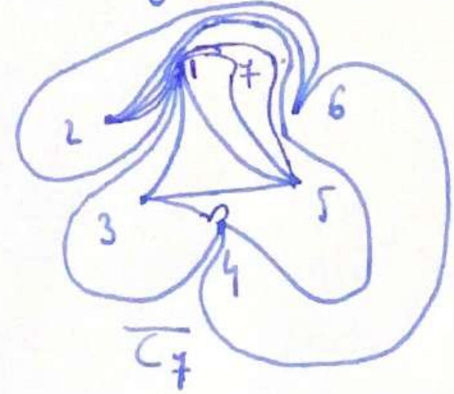
C_6

$\overline{C_6}$

We observe that both C_6 and $\overline{C_6}$ are indeed planar.



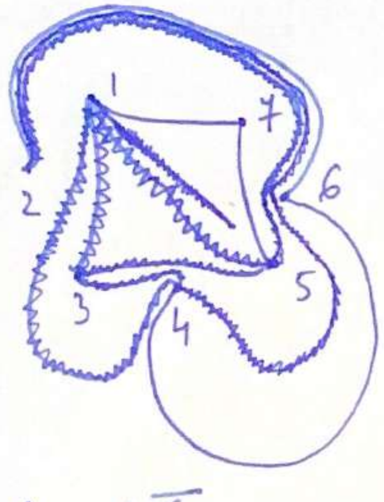
C_7



$\overline{C_7}$

For $\overline{C_7}$ we observe that there is no path from the vertex 3 to the vertex 7 without crossing other paths, thus our $\overline{C_7}$ is "probably nonplanar".

For it to be "certainly planar", we will pick the vertices 1, 2, 3, 4, 5 and mark each path that doesn't overlap with one another (belonging to the selected vertices, excluding the common vertices each path shares between each other) with jagged lines.



Jogged $\overline{C_7}$

The jagged $\overline{C_7}$ will represent a subgraph of $\overline{C_7}$.
Using Kuratowski's Subgraph Theorem: "A graph that has a subgraph of an expansion of K_5 or U.G., is nonplanar", and the fact that K_5 is an expansion of itself (and the fact that our jagged $\overline{C_7}$ is a subgraph of K_5), we conclude that $\overline{C_7}$ is nonplanar.

Using $\overline{C_7}$ as the building block for the following complemented cyclic graphs $\overline{C_n}$ (where $n \in \mathbb{N} \setminus \{0;1;2;3;4;5;6;7\}$), we arrive at the conclusion that $\overline{C_7}$ and $\overline{C_n}$ (where $n \in \mathbb{N} \setminus \{0;1;2;3;4;5;6;7\}$) are nonplanar graphs. Therefore all $V = n$ for which $\overline{C_n}$ is non planar are $V \in \mathbb{N} \setminus \{0;1;2;3;4;5;6\}$.
which we denoted by

12. Here is an explicit statement of the „pigeonhole principle" mentioned on the page 83: If m objects are distributed into n boxes and m is larger than n, then at least one box contains at least $\frac{m}{n}$ objects. Use this principle to prove that there are at least two red maples in the United States having the same number of leaves.

Solution: Let m = number of leaves (that are equal) of two trees $= 240$
n = pair of two threes $= 2$
Using the pigeonhole principle here, we will also consider that one three has 120 leaves. Applying the pigeonhole principle where $m > n$, we will conclude that at least a pair of trees has a number of $240 \cdot \frac{1}{2} = 120$ leaves, which is true, because one tree has 120 leaves. May the species of trees be red maples. With this proof, we arrived at our conclusion: „There are at least two red maples in the United States having the same number of leaves".

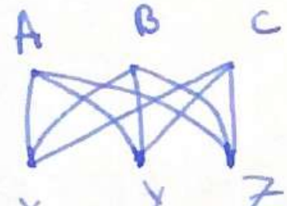
13. Prove the following statements:

a) Except for U.G. itself, no expansion of U.G. is also a supergraph of U.G..

b) Except for K_5 itself, no expansion of K_5 is also a supergraph of K_5.

Solution:

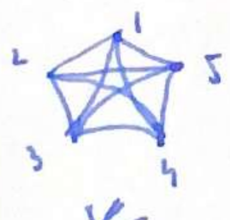
a) We will firstly draw U.G.:



The U.G. graph.

Even if we inserted a vertex between two original vertices (that belong to the original U.G. graph) binded by a path (thus obtaining an expansion), we know for a fact that supergraphs allows someone to add vertices only outside of paths that bind the original vertices of the U.G. graph. Therefore, we arrived at the desired statement (that we now have proved), namely: Except for U.G. itself, no expansion of U.G. is also a supergraph of U.G.

b) We will firstly draw K_5:

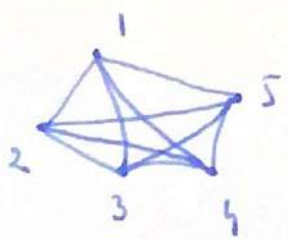


K_5

Even if we inserted a vertex between two original vertices (that belong to the original graph U. G.) binded by a path (thus obtaining an expansion), we know for a fact that supergraphs allows someone to add vertices only outside of paths that bind the original vertices of the U. G. graph. Therefore, we proved the statement: Except for K_5 itself, no expansion of K_5 is also a supergraph of K_5.

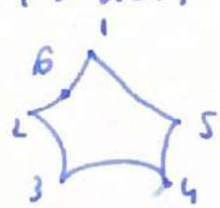
14. Let S be the set of all expansions of supergraphs of U. G. or K_5, and let T_5 be the set of all supergraphs of expansions of U. G. or K_5. We mentioned on page 97 that every expansion of a supergraph of U. G. or K_5 is also a supergraph of an expansion of U. G. or K_5. So „$S \subseteq T$" (where „$\subseteq$" means a left-to-right inclusion with the possibility of an equality). Prove (however) that T is not a subset of S, therefore obtaining „$S \neq T$" by finding a supergraph of an expansion of K_5 that is not an expansion of a supergraph of K_5.

Solution: We consider the original K_5 graph as a starting graph:

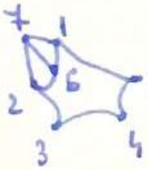


K_5

We add a vertex between the vertices 1 and 2 (on the path between the vertices 1 and 2), thus obtaining an expansion of K_5:



From the obtained expansion of K_5 we will consider a supergraph
of the expansion of K_5: A new vertex labelled with 7 joining with
distinct paths the vertices 2; 1; 6. Therefore we obtain:



A supergraph of an extension of K_5.

We have created a supergraph of an extension of K_5 that
is not an extension of a supergraph of K_5 (because supergraphs
cannot add new vertices on an original path between two original
vertices. Therefore the set of all supergraphs of expansions of K_5
(that we named „T") does not equal the set of all expansions of
supergraphs of K_5.

15. Isomorphism is „transitive", that is, if $G \cong H$ and $H \cong J$,
then $G \cong J$. This enables us to prove the following theorem:

<u>Theorem: If H is planar and $G \cong H$, then G is also planar.</u>

<u>Proof</u>: H is planar, so H is isomorphic to graph J which drawn
in a plane without edge-crossings. [Definition 18 (from page 77):
A graph is planar iff it is isomorphic to a graph that has been drawn
in a plane without edge-crossings. Otherwise a graph is nonplanar]
Since $G \cong H$ and $H \cong J$; we get that $G \cong J$ (from the transitivity
of isomorphisms: if $G \cong H$ and $H \cong J$, then $G \cong J$.), that is G is isomorphic
to a graph that has been drawn without edge-crossing, thus
(from Definition 18: A graph is planar iff it is isomorphic to a graph
that has been drawn in a plane without edge-crossing. Otherwise
a graph is planar.) G is planar.

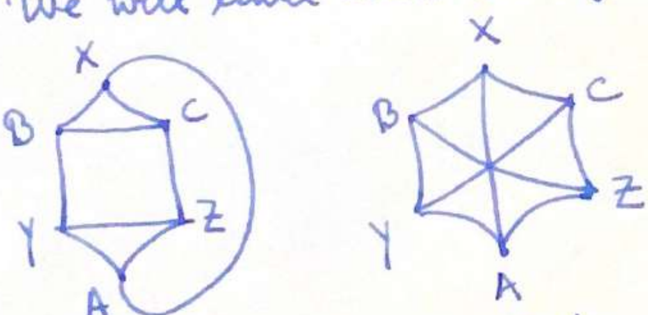
Thus planarity is yet another property preserved by isomorphism. Use this fact to devise new proofs that the pairs of graphs in Figure 46; Figure 51; Figure 54; Figure 55 are not isomorphic.

Solution:

Graph II

Graph I   Figure 46

We will label each vertex for a cleaner view:

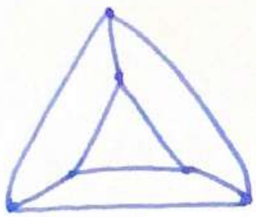


Graph I   Graph II

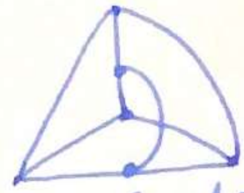
We will suppose that Graph I $\cong$ Graph II. We remark that Graph II $\cong$ U.G.. We have that:

$$\left.\begin{array}{l}\text{Graph I} \cong \text{Graph II} \\ \text{Graph II} \cong \text{U. G.}\end{array}\right\} \xrightarrow[\text{of graph isomorphisms}]{\text{Out of the transitivity}} \text{Graph I} \cong \text{U. G.}$$

We remark the fact that Graph I is planar and know [from Theorem 3: U.G. is nonplanar (page 82)] that U. G. is non planar, therefore Graph I $\not\cong$ Graph II (because planarity is a property preserved by isomorphisms of graphs).



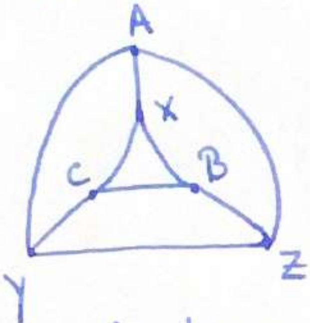
Graph I



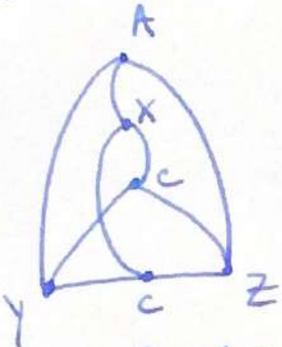
Graph II

Figure 51

We will label each vertex for clarity:

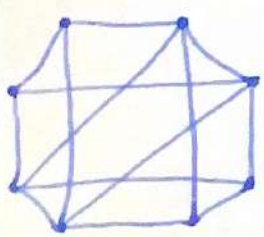


Graph I

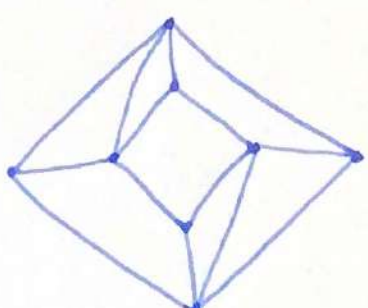


Graph II

We will suppose that Graph I $\cong$ Graph II. We remark that Graph II $\cong$ U. G. From the transitivity of isomorphisms we get the fact that Graph I $\cong$ Graph II and Graph II $\cong$ U. G. results: Graph I $\cong$ U. G.. As we can see Graph I is planar and as we know U. G. is nonplanar, which contradicts the fact that planarity under isomorphisms of graph is preserved. Therefore: Graph I $\ncong$ Graph II.



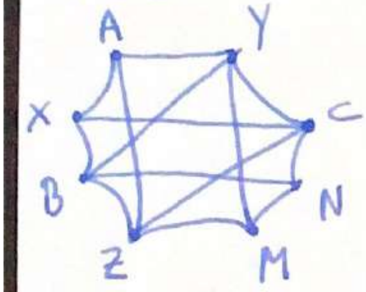
Graph I



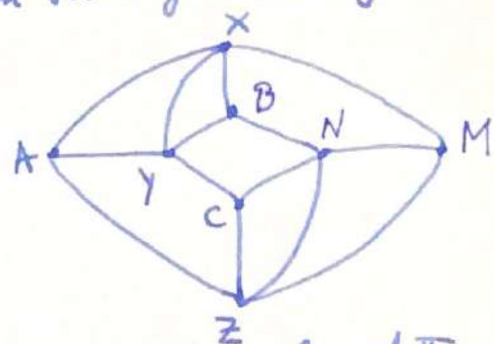
Graph II

Figure 54

We will label each vertex for clarity:

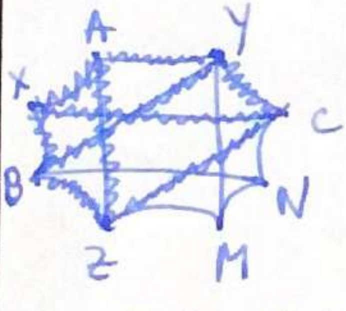


Graph I

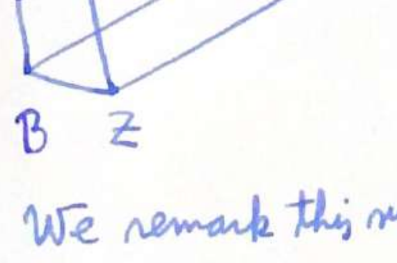


Graph II

We will suppose Graph II ≅ Graph I. We remark the nonplanarity of Graph I. In order to confirm with certainty the nonplanarity of Graph I, we will pick the vertices A;B;C; X;Y;Z and mark each path between these vertices with jagged lines:

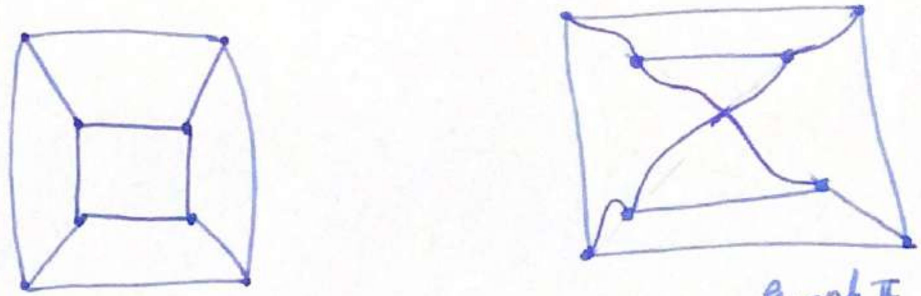


All that isn't jagged will vanish, resulting in an extracted subgraph:



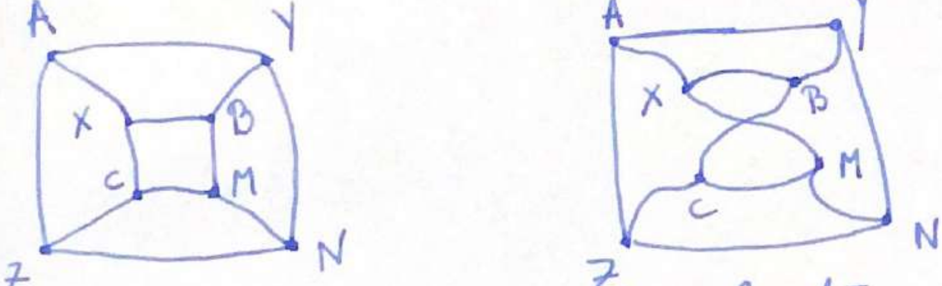
We remark this subgraph to be exactly the graph of U.G.. Therefore using Kuratowski's Subgraph Theorem; „A graph that has a subgraph of an expansion of either K_5 or U.G., is nonplanar.";

we proved that Graph I is nonplanar. Because planarity is preserved under isomorphisms of graphs, we get a contradiction with the hypothesis that Graph I (which is nonplanar) $\cong$ Graph II (which is planar).



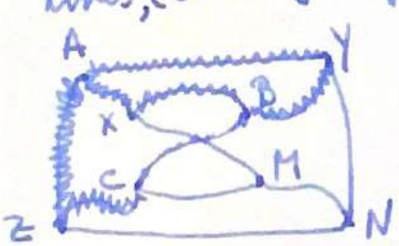
Graph I   Figure 55   Graph II

We will label each vertex for clarity:



Graph I   Graph II

We suppose Graph I $\cong$ Graph II. We remark Graph II may be nonplanar. In order to confirm with certainty the nonplanarity of Graph II, we will pick the vertices A; B; C; X; Y; Z and mark each path between the chosen vertices (A;B;C;X;Y;Z) with jagged lines, (omitting the paths between „A and B"; „B and C"; „A and C"):



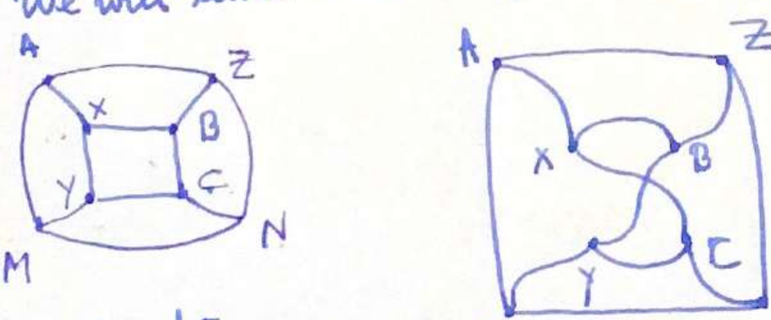
we proved that Graph I is nonplanar. Because planarity is preserved under isomorphisms of graphs, we get a contradiction with the hypothesis that Graph I (which is nonplanar) ≅ Graph II (which is planar).

Graph I

Graph II

Figure 55

We will label each vertex for clarity

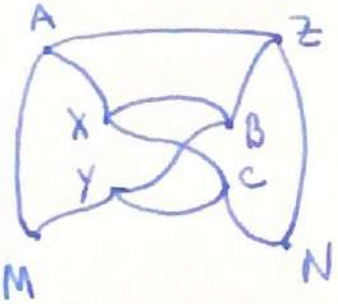


Graph I

Graph II

We suppose Graph I ≅ Graph II. We remark that Graph II may be nonplanar. In order to confirm with certainty the nonplanarity of Graph II, we will pick the vertices A; B; C; X; Y; Z and mark each path between the chosen vertices (A; B; C; X; Y; Z) [excluding any possible path for the following edge combinations: {{A; B}; {A; C}; {B; C}; {X; Y}; {X; Z}; {Y; Z}}] with jagged lines:

Everything outside the jagged paths will vanish, resulting in an extracted subgraph from our original graph:



We visually recognise that our selected subgraph is an expansion of U.G.. Therefore, using Kuratowski's Subgraph Theorem: "A graph that has a subgraph of either an expansion of K_5 or U.G., is nonplanar", we conclude that Graph II is nonplanar.
The Nonplanarity of Graph II implies Graph I to be nonplanar (under the preservation of planarity under graph isomorphisms), but Graph I is planar, thus Graph I $\not\cong$ Graph II

16. Prove that the graphs in Figure 89 are planar.

<u>Solution</u>:

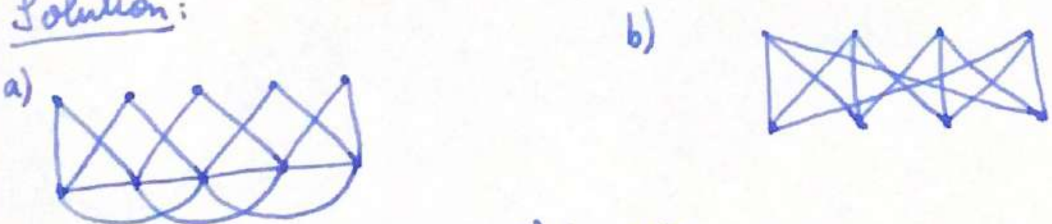
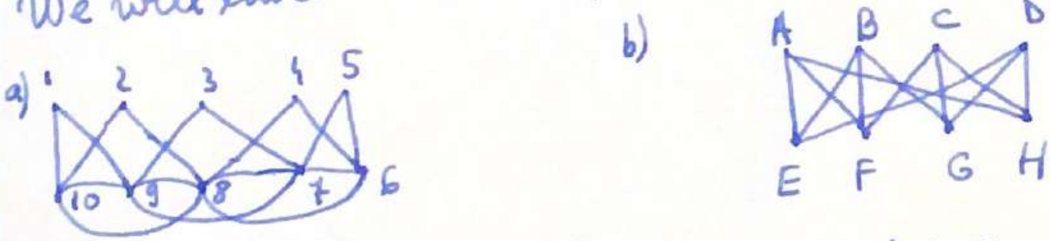


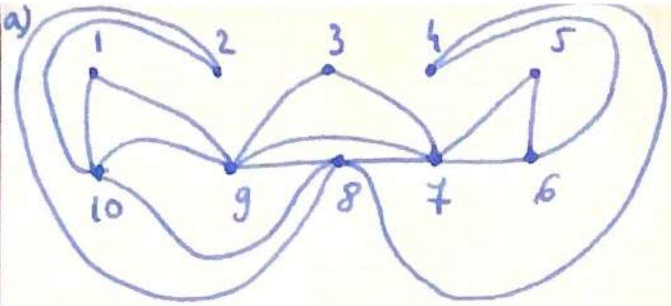
Figure 89

We will label each vertex for a better visualization

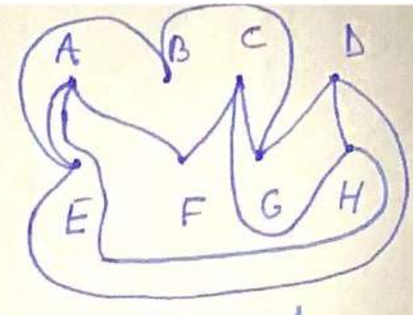


Next, we will try to untangle both graphs so that they will be planar.

a)



b)



We have untangled the seemingly nonplanar graphs and have visually proved that the two graphs from Figure 89 are planar.

17. Prove that the graphs in Figure 90 are nonplanar

a) b)

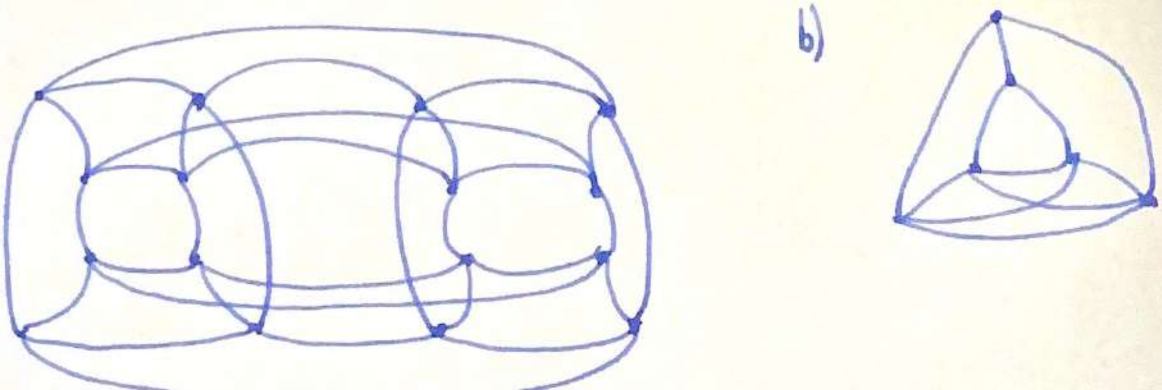
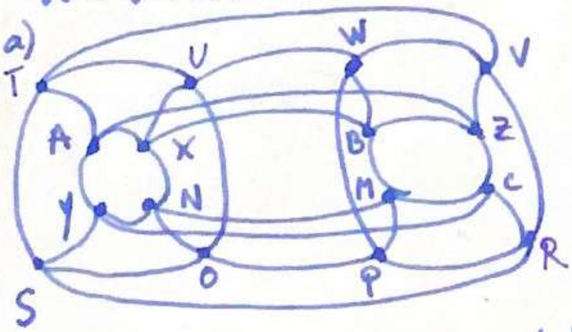


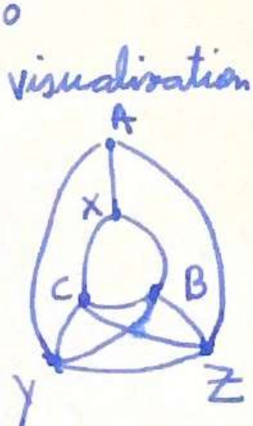
Figure 90

We will label each vertex for a better visualisation.

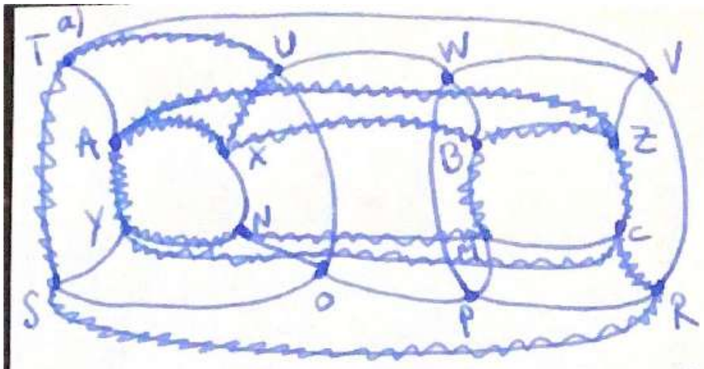
a)

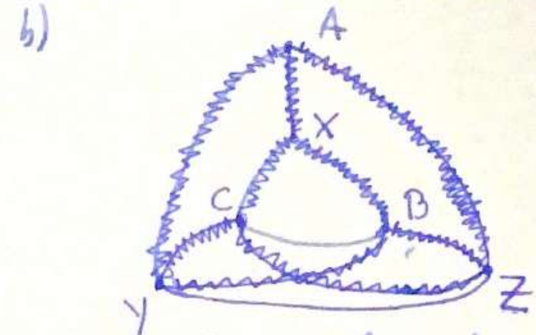


b)

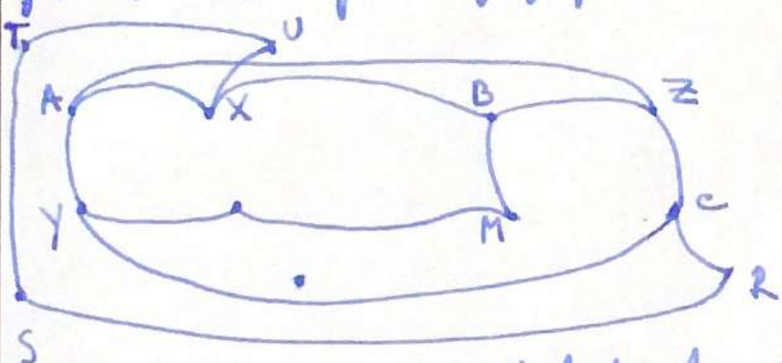


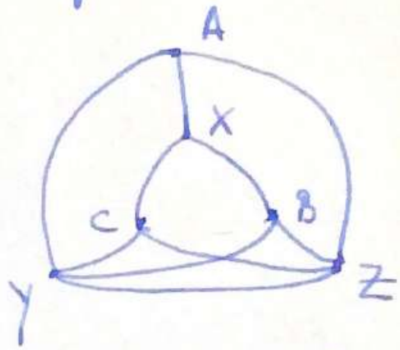
For Graph a) and Graph b) we will pick the vertices A;B;C;X;Y;Z and mark each path between the picked vertices (A;B;C;X;Y;Z) [excluding any possible path for the following edge combinations: {{A;B};{A;C};{B;C};{X;Y};{X;Z};{Y;Z}}] with jogged lines:





All that isn't jagged will vanish, resulting in two subgraphs from our corresponding graphs (Graph a) and Graph b)):





We observe that our selected subgraphs are an expansion of U.G.. Therefore, using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either K_5 or U.G., is nonplanar" we conclude that Graph a) and Graph b) are nonplanar.

18.-20. In Figure 91, Figure 92, Figure 93, decide wether each graph is planar or nonplanar and then prove that your choice is correct.

Solution:

a) b)

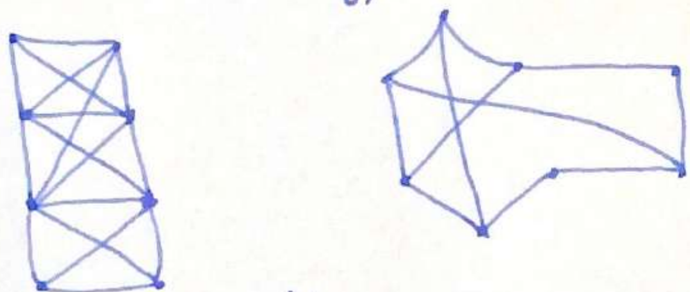
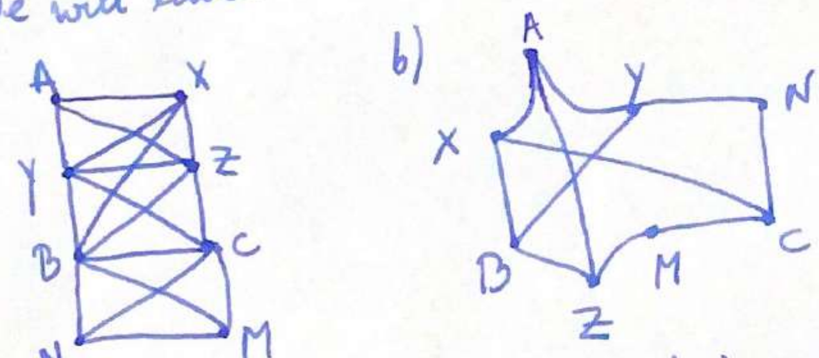


Figure 91

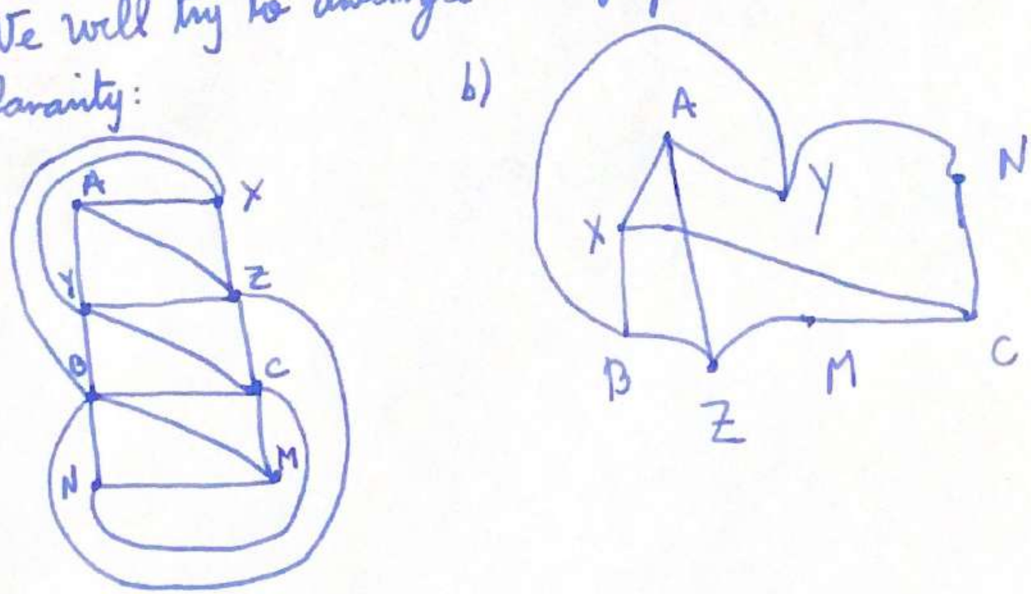
We will label each vertex for a clearer visualisation.

a) b)

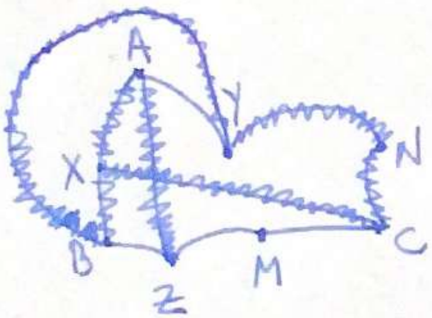


We will try to untangle both graphs in order to check their planarity:

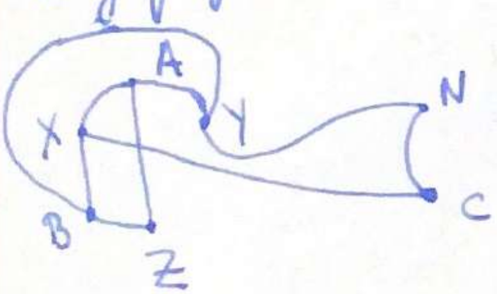
a) b)



We managed to untangle Graph a), thus Graph a) is planar. Graph b) is probably nonplanar. For Graph b), we will pick the vertices A; B; C; X; Y; Z and mark each path between the picked vertices (A; B; C; X; Y; Z) [excluding any possible path for the following edge combinations: {{A;B}; {A;C}; {B;C}; {X;Y}; {X;Z}; {Y;Z}}] with jagged lines:



Anything unjagged vanishes next, so that we obtain a subgraph from our original graph (Graph b)):



We observe that our chosen subgraph is an expansion of U.G. Using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either K_5 or U.G., is nonplanar", we conclude: Graph b) is nonplanar.

a)

b)

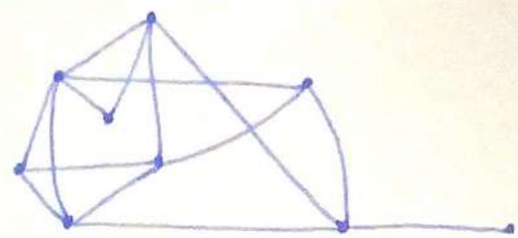
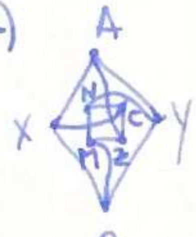


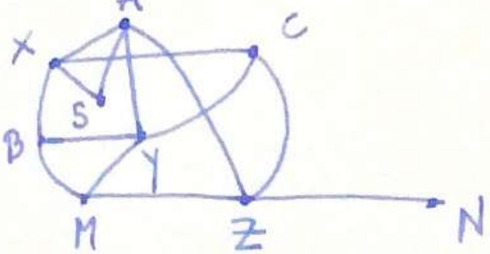
Figure 92

We will label each vertex for a clearer visualisation:

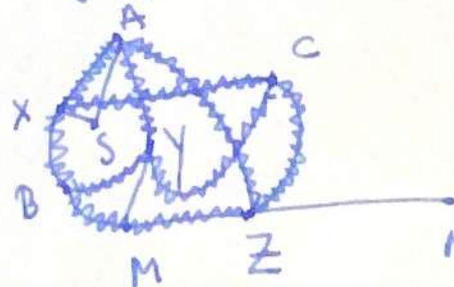
a)



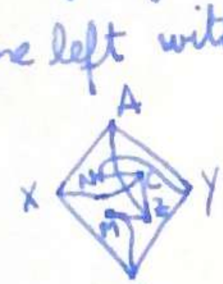
b)

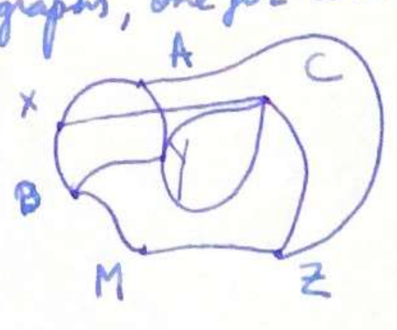


Visually, both Graph a) and Graph b) seem to have a subgraph of an expansion of U.G. For both graphs (Graph a) and Graph b)) we will pick the vertices: A;B;C;X;Y;Z and mark each path between the picked vertices (A;B;C;X;Y;Z) [excluding any possible path for the following edge combinations: {{A;B};{A;C}; {B;C};{X;Y};{X;Z};{Y;Z}}] with jagged lines:



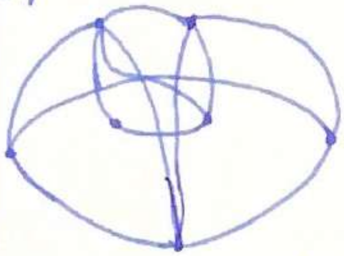
Everything outside the jagged paths shall vanish, so that we are left with two subgraphs, one for each of Graph a) and Graph b).



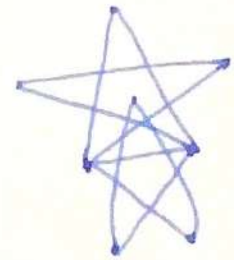


We remark that the selected subgraphs (from Graph a) and Graph b)) are expansions of U. G.. Therefore, using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either K_5 or U. G., is nonplanar", we conclude: Graph a) and Graph b) are nonplanar.

a)

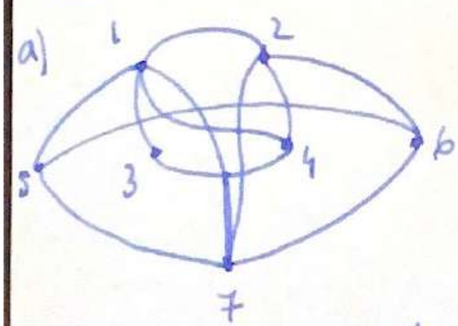


b)

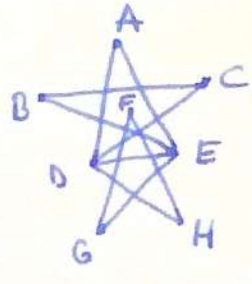


We will label each vertex for a clearer visualisation.

a)

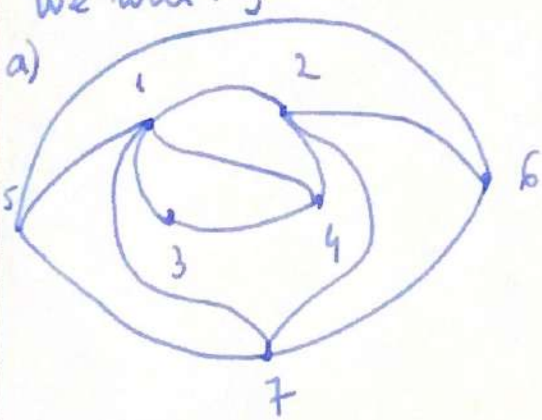


b)

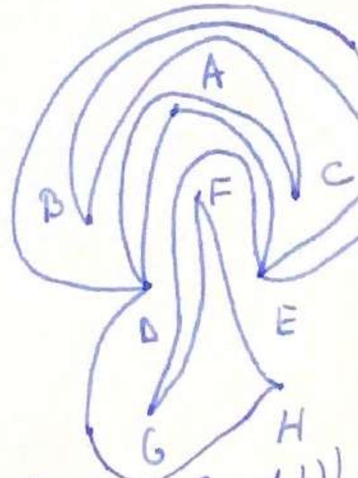


We will try to untagle both of the graphs (Graph a) and Graph b)):

a)



b)



We remark that both of the graphs (Graph a) and Graph b)) are completely untangled, thus we conclude: Graph a) and Graph b) are planar.

Page 130 – 135

1. N_1 is certainly planar, and we proved on page 115 that it is connected. Prove now that is polygonal by proving that the statement „every edge of N_1 borders on two different faces" is true.

Solution: Because N_1 doesn't contain any edges the statement „every edge of N_1 borders on two different faces" is not false. Using the Law of Excluded Middle which states that: „a logical statement (where a mathematical statement is a logical statement) is either true or false, no in-between" the statement: „every edge of N_1 borders of two different faces" is therefore true.

2. Find the error in the following „proof":

Theorem: „If A is a set of horses, then all horses in A are of the same color".

Proof: Let S be a statement, „If A is a set of n horses, then all the horses in A are of the same color". S is a statement about positive integers n, and we shall prove that S is true for every positive integer by the principle of mathematical induction, this will establish the theorem. The first step is to prove:

1) S is true for $n = 1$

Rephrased, this is „If A is a set containing one horse, then all the horses in A are of the same color". Obviously this is true. Next, we have to prove:

2) If S is true for $n = k$, then S is true for $n = k+1$.

That is, we are given the truth of „ if A is a set of k horses, then all the horses in A are of the same color", and we have to deduce from this truth of „ if A is the set of k+1 horses, then all the horses in A are of the same color". So let A be a set of k+1 horses. For the sake of reference we shall number the horses from 1 to k+1. Remove horse number 1. There remains a set of k horses. We are given that all the horses in any set of k horses are of the same color, so horse number 2, horse number three, ..., and horse number (k+1) must be all of the same color. Now replace horse number 1 and remove horse number (k+1). Again, we are left with a set of k horses, so horse number 1, horse number 2, ..., horse number k, must be all of the same color. Obviously then all the horses in A, from horse number 1 to horse number k+1, must be of the same color. Invoking the principle of mathematical induction it follows that S is true for every positive integer and the theorem is proved.

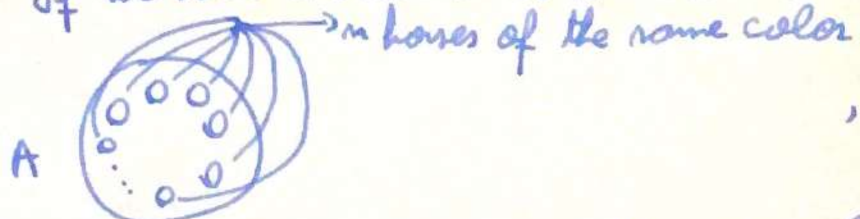
Corollary: All the horses of the world are of the same color.

Proof: Let A be the set of all the horses in the world, and then apply the theorem to A. This corollary can be used to prove a wide range of things, for example:

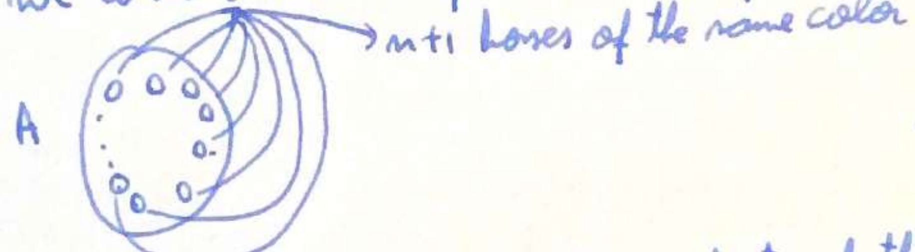
Theorem: $1+1=3$

Proof: If $1+1=3$, fine. If not, that would be a horse of a different color, which doesn't exist by the corollary.

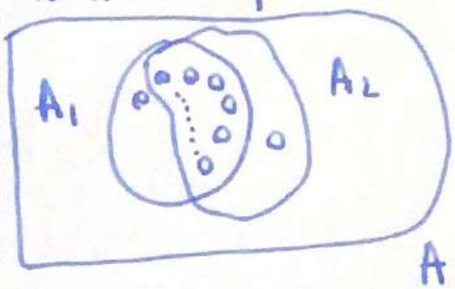
Solution: If we have a visual set A with n horses of the same color:



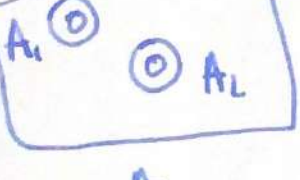
, and we consider the case of $n+1$ horses of the same color:



, we notice that we can form at least two subsets of the set of A with horses of the same color:



We notice an overlap between the set of A_1 and A_2 (A_1 and A_2 having a total of n horses of the same color), but for the case $n=2$, we observe no overlap:



, thus the induction fails (and so do the rest of the corollaries and theorems that are based on it)

3. Believe it or not, the graph of Figure 104 a is planar. Find its number of faces. (If you use Euler's Formula you won't need to draw it without edge crossings).

Solution:

a)

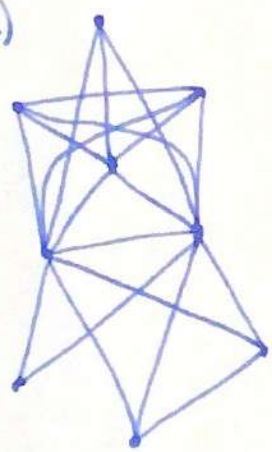


Figure 104.

The exercise tells us that the graph is planar. We observe the fact that the graph is also connected. Using Theorem 9 (from page 121), we get that (If G is planar and connected, but not polygonal, then $v+f-e=2$): $v+f-e=2$

We remark that for our graph (Graph a)): $v=9$
$e=20$

Thus: $f=2+20-9 \Leftrightarrow f=22-9 \Leftrightarrow f=13$.

Remark:
We concluded that $v=9; e=20$ by simply counting on the graph (the number of edges and the number of vertices)

4. Imitate a proof of Corollary 12 to construct a proof, independent of the results of Chapter 3 (Planar Graphs), that the graph of Figure 104 b is nonplanar.

Solution:

We will state Theorem 12: If G is planar and connected with $V \geq 3$ and G is not a supergraph of K_3, then $2f \leq e \leq 2V-4$.

We will also restate Corollary 12 (with its proof): U. G. is nonplanar.

Proof: Suppose that U. G. were planar. U. G. is connected and is not a supergraph of K_3 (because no triangle shape can be formed out of de eciting edges of U. G.), so by Theorem 12 it would have to be true that $e \leq 2V-4$. But $e = 9$ and $2V-4 = 2\cdot 6-4 = 8$, a contradiction, hence U. G. is nonplanar.

Back to our ecercise:

b)

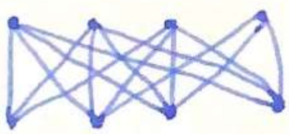


Figure 104

We suppose Graph b) to be planar. Graph b) is also connected and is not a supergraph of K_3 (because no triangle can be formed out of the eciting edges of Graph b)), thus, by applying Theorem 12: "If G (in our case Graph b)) is planar and connected with $V \geq 3$ and G is not a supergraph of K_3, then $2f \leq e \leq 2V-4$."

Using „Euler's Formula For Planar And Connected Graphs: $V+f-e=2$", we have that:

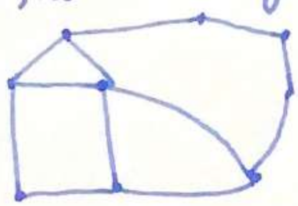
$V = 8$; $e = 4\cdot 8 = 32 \Longrightarrow f = 2+e-V \Leftrightarrow f = 34-8 \Leftrightarrow f = 26$.

Our inequality then is: $2\cdot 26 \leq 32 \leq 2\cdot 8-4$

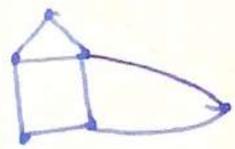
but $2\cdot 26 \leq 32$ and $32 \leq 2\cdot 8-4$ are both false, thus Graph b) is not planar.

5. Crucial to the proof of Euler's Formula, the following step involves more than is immediately apparent:

"Now let G be an arbitrary polygonal graph having $k+1$ faces. Remove some of the edges and vertices bordering the infinite face of G to produce a new polygonal graph H having one less face than G, so H has k faces." (Page 120)



G

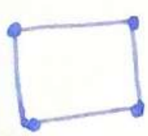


H

Find a polyogonal graph G having a face bordering the infinite face which, if removed, results in a subgraph H which is not polygonal. This shows that we must be choosy about the exterior face of G that is removed.

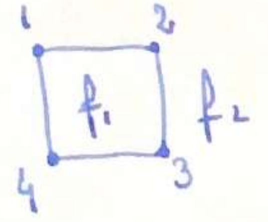
Solution: We will remind ourselves of the definition of a polygonal graph: Definition 24: A graph is polygonal if it is planar, connected, and has the property that every edge borders on exactly two faces (which are different).

An example of a polygonal graph G having a face bordering the infinite face, which, if removed, results in a subgraph H which is not polyonal is:

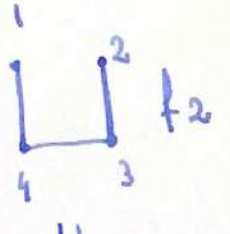


G

We will label each vertex for clarity, and note with f_1 the face inside of the graph G, and with f_2, the infinite face:



By eliminating any one of de edges $\{1;2\}$; $\{2;3\}$; $\{3;4\}$; $\{4;1\}$ we will manage to eliminate f_1. We will eliminate the edge $\{1;2\}$, thus forming a subgraph H (which is not polygonal, because any of the available edges borders not exactly two faces, as we only have the infinite face left).



6. Prove this partial converse of Euler's Formula: If a graph is planar and $v+f-e=2$, then the graph is connected.

<u>Solution</u>: We will assume that we have a planar disconnected Graph. We will further consider the disconnected planar Graph to be separated by two planar connected Subgraphs (of the planar disconnected Graph). Because the two Subgraphs are planar (naming one G_1 and the other one G_2), we will have that:

$$\begin{cases} v_{G_1}+f_{G_1}-e_{G_1}=2 & (1) \\ v_{G_2}+f_{G_2}-e_{G_2}=2 & (2) \end{cases}$$

We also know that the disconnected original planar graph is entirely made out of G_1 and G_2, therefore:

$$\begin{cases} v = v_{G_1} + v_{G_2} \\ e = e_{G_1} + e_{G_2} \\ f = f_{G_1} + f_{G_2} \end{cases}$$

By adding up (1) with (2) we get: $v+f-e=4$ do (with the fact that $v+f-e=2$).

7. Definition: A component of a graph is a connected subgraph that is not contained in a larger connected subgraph.

Corollary: A connected graph is its own single component, and the components of the disconnected graph are what we have been calling the "pieces" that comprise it.

Examples: Figure 105 a) has three components which are displayed separately in 105 b); 105 c); 105 d). Figure 106 has only one component, itself.

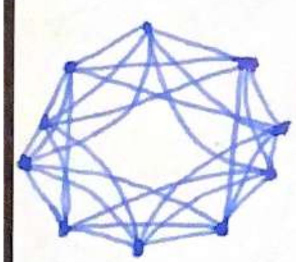


Figure 106

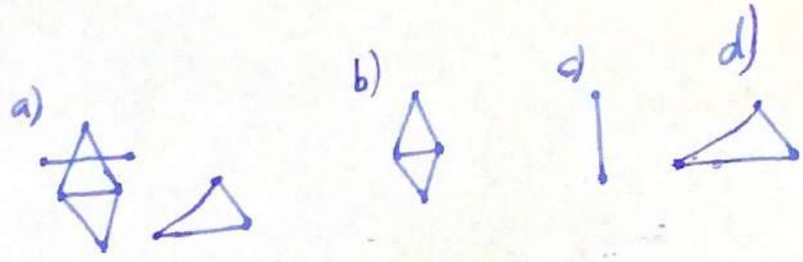


Figure 105

Let "p" denote the number of components of a graph, and prove this generalisation of Euler's Formula: if a graph is planar, then:

$v+f-e=1+p$.

Solution:

If a graph is planar, it is either:

1) Connected

2) Disconnected

For 1), we will have only one component of the graph, itself, thus $p=1$: $v+f-e=1+p \iff v+f-e=2$. Therefore, the case for the planar connected graph is proved.

For 2), we will consider Euler's Formula for each individual component $i=\overline{1,p}$:

$v_i+f_i-e_i=2$ (because individually, each component is planar)

Summing all components up, we get:

$$\sum_{i=1}^{p} v_i + \sum_{i=1}^{p} f_i - \sum_{i=1}^{p} e_i = v+f-e = 2\cdot p \quad (*)$$

We have come to this expression (*), because:

- $\sum_{i=1}^{p} v_i$ coincides with the number of total vertices of the original planar graph (which we denote by v)
- $\sum_{i=1}^{p} f_i$ coincides with the number of total faces of the original planar graph (which we denote by f).
- $\sum_{i=1}^{p} e_i$ coincides with the number of total edges of the original planar graph (which we denote by e).

We remark the fact that there is a face we count for each component of the graph, repeatedly, for a number of p times. That is: the infinite face. Therefore, we will divide "$\sum_{i=1}^{p} f_i$" in a sum:
$\sum_{i=1}^{p} f_i = f + (p-1)$; where "$f$ = the number of total faces counted one time only" and "$p-1$ = the remaining $p-1$ times we have counted the infinite face, given that the p-th time belongs to f)."

Remark: In the total number of distinct faces (which we denoted by f) we have included the infinite face strictly once leaving the rest of $p-1$ times of its counting separate. We conclude the exercise with:

$$\sum_{i=1}^{p} v_i + f - \sum_{i=1}^{p} e_i + p - 1 = 2p$$

$$\sum_{i=1}^{p} v_i + f - \sum_{i=1}^{p} e_i = p+1 \quad \odot$$

We know that:

$$\begin{cases} \sum_{i=1}^{p} v_i = v & \odot\odot \\ \sum_{i=1}^{p} e_i = e & \odot\odot\odot \end{cases}$$

Thus, combining: $\odot$, $\odot\odot$, $\odot\odot\odot$ we get that:

$$v + f - e = p+1$$

8. Corollary 13 can sometimes be used to prove a graph nonplanar. Use it on the graph on Figure 106.

<u>Solution</u>:

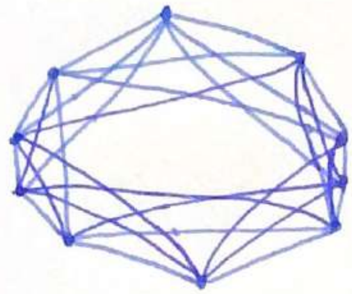


Figure 106

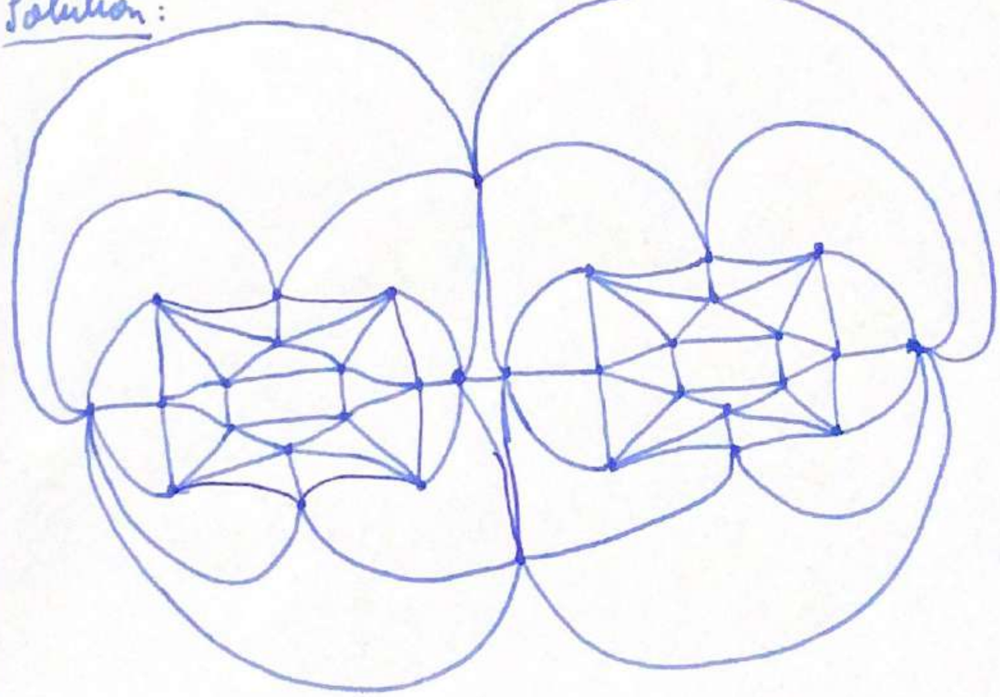
We will remind ourselves of Corollary 13.

<u>Corollary 13</u>: If G is planar, then G has a vertex of degree at most 5.

We observe that all vertices of our graph have degree of 6, thus none of the vertices available to our graph have the degree at most 5, which (from using Corollary 13: If G is planar, then G has a vertex of degree at most 5) concludes the nonplanarity of our graph. Therefore: The graph of Figure 106 is nonplanar.

9. Show by example that the statement: „ every planar graph has a vertex with degree less than or equal to 4 " is false. That is, find a planar graph in which every vertex has degree greater than or equal to 5. This shows that, in this fashion at least, Corollary 13 can't be improved upon, that it is already the strongest statement that can be possibly made.

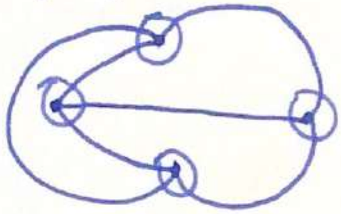
Solution:



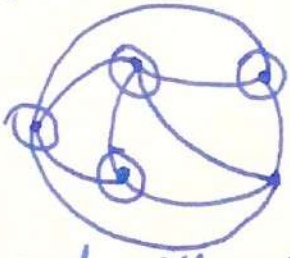
10. There is another manner, however, in which Corollary 13 can be improved upon. Prove: Every planar graph with $V \geq 4$ has at least four vertices of degree at most 5.

Solution:

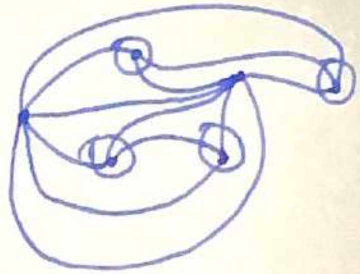
$V = 4$



$V = 5$



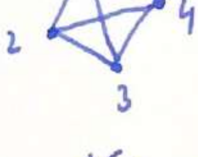
$V = 6$



By observing that planar graphs with $V \geq 4$ have at least four vertices of degree at most 5, we obtain a conjecture: „Every planar graph with $V \geq 4$ has at least four vertices of degree at most 5".

11. Definition: The connectivity of the graph is the smallest number of vertices whose removal (together with their incident edges) results in either K_1 or a disconnected graph. We shall denote the connectivity of a graph by „c".

Examples: K_4 has $c = 3$, because removing vertices with their incident edges never disconnects K_4, but after removing three vertices (with the incident edges), we are left with K_1.



K_4

K_4 (after removing 2;3;4 with the incident edges, thus obtaining K_1)

The connectivity of a graph indicates the extent to which the graph is connected, in some sense. Note that c is a minimum. If you were to remove successively vertices 3; 1; 6 and 4 from Figure 79 b), you might erroneously conclude that the graph has $c = 4$, as the graph was disconnected only after the fourth removal. Starting over and removing vertices 2 and 9 shows that c can be at most 2. c is exactly two as the removal of any one vertex leaves a connected subgraph. For visual orientation, we will display Figure 79 b):

b)

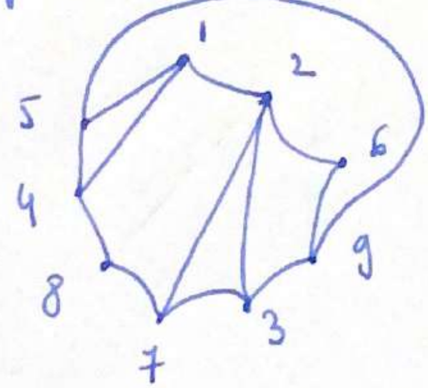


Figure 79

Find the connectivity c, for each graphs in Figure 91; Figure 92, Figure 93

Solution:

a)

b)

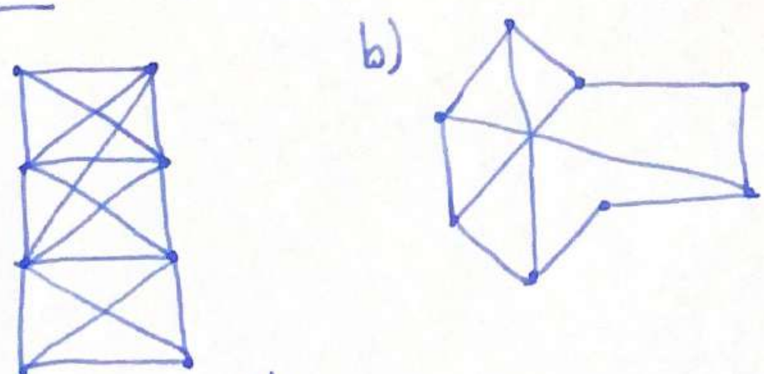


Figure 91

We will label each vertex for a greater clarity:

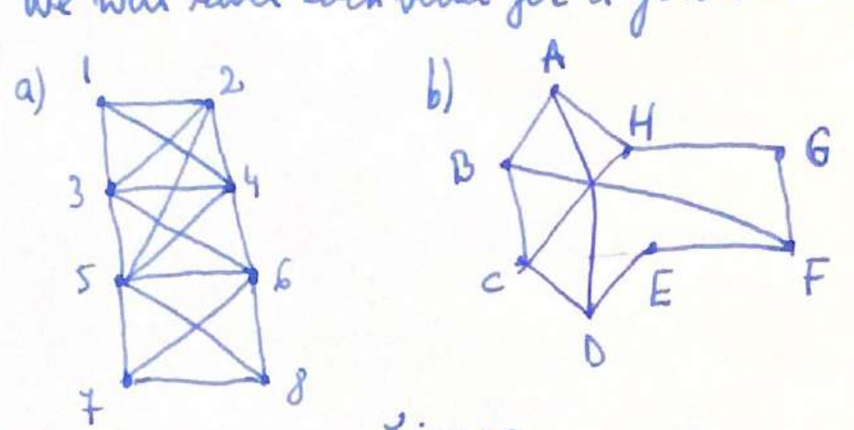


Figure 91

For Graph 91 a) we will consider eliminating 5, 6, 8 (and their incident edges), leaving the vertex 7 as an isolated vertex, thus for $c = 3$ on Graph 91(a), we have a disconnected subgraph of Graph 91(a).

For Graph 91 b), we will consider eliminating F and H (and their incident edges), leaving the vertex G as an isolated vertex, thus for $c = 2$ on Graph 91 b), we have a disconnected subgraph of Graph 91 b).

We remark that the general procedure of finding the connectivity of a graph often lies in finding a vertex with the least number of incident edges, and eliminating all vertices that connect to that particular found vertex.

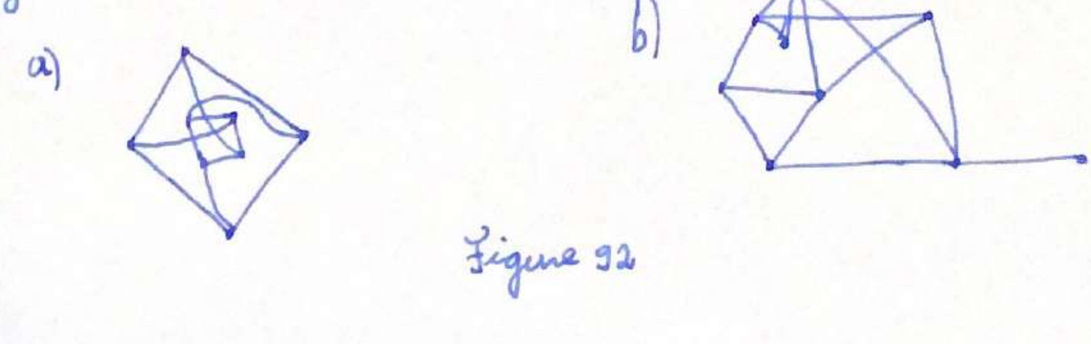


Figure 92

We will label each vertex for clarity:

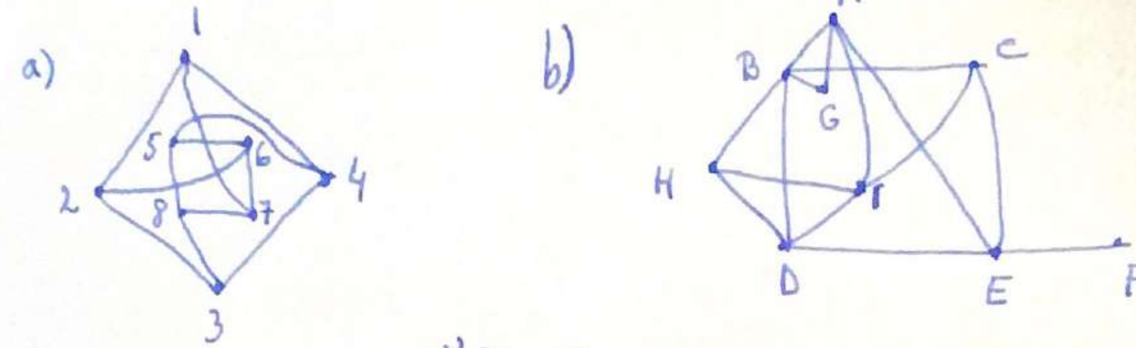


Figure 92

For Graph 92 a), we will consider eliminating vertices: 8;1;6 (and all their incident edges), leaving the vertex 7 isolated, thus for $c=3$ on Graph 92 a), we have a disconnected subgraph of Graph 92 a).

For Graph 92 b), we will consider eliminating the vertex E (and all its incident edges), leaving the vertex F isolated, thus for $c=1$ on Graph 92 b), we have a disconnected subgraph of Graph 92 b).

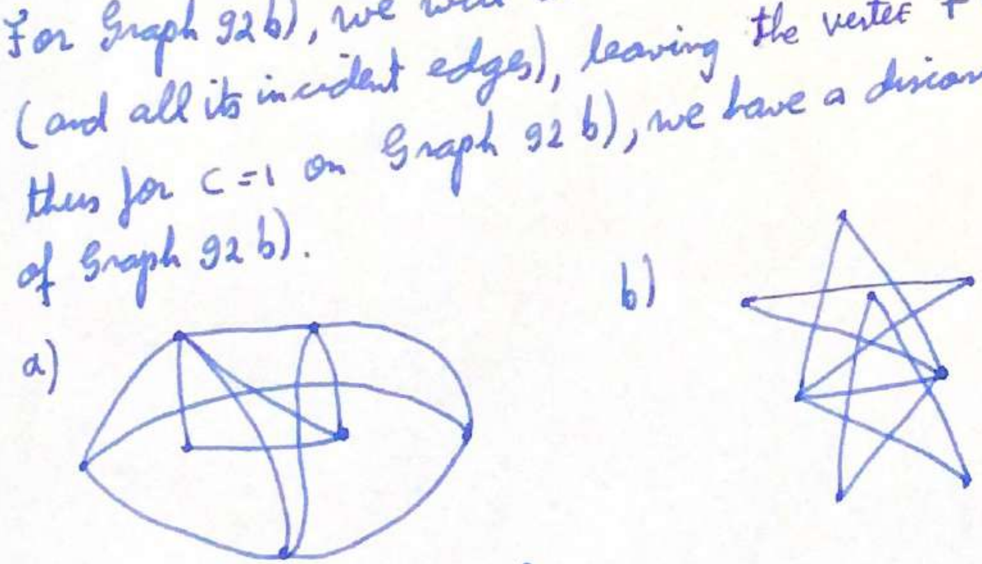
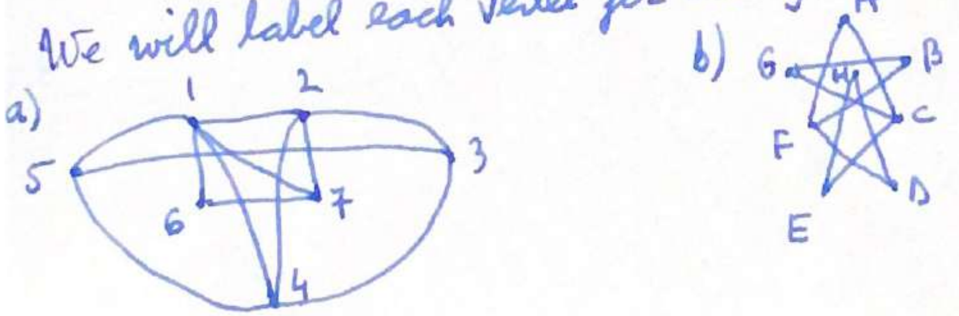


Figure 93

We will label each vertex for clarity:



For Graph 93 a), we will consider eliminating the vertices 7 ; 1 (and their incident edges), leaving the vertex 6 isolated, thus for $c = 2$ on Graph 93 a), we have a disconnected subgraph of Graph 93 a).

For Graph 93 b), we will consider eliminating F ; C (and their incident edges), leaving the vertex A isolated, thus for $c = 2$ on Graph 93 b), we have a disconnected subgraph of Graph 93 b).

12. By Exercise 11 of Chapter 2 (Graphs), $\frac{2e}{v}$ is the the average of the degrees of a graph. Prove that if a graph has connectivity c, then c is less than or equal to $\frac{2e}{v}$.

Solution: We observe that the maximum value of connectivity (for a graph) is for K_v, because there we will be obliged to remove $v-1$ vertices in order to have a disconnected graph. We know that $e = \frac{1}{2}(v-1)v$, thus $v-1 = \frac{2e}{v}$. Therefore the maximum value for c is $\frac{2e}{v}$, and for graphs that differ from K_v, the value of c will only decrease. We arrive at the conclusion: $c \leq \frac{2e}{v}$.

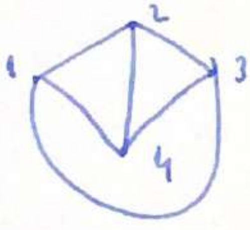
13. Use the previous exercise that there is no graph with : $e=7$ and $c=3$, and none with $e=11$ and $c=4$

<u>Solution</u>:

Using the inequality $c \le \frac{2e}{v}$ we will have:

1) for $e=7$ and $c=3$; $3 \le \frac{14}{v} \Leftrightarrow v \le \frac{14}{3} \Leftrightarrow v \le 4$.

We will consider $v=4$ with $e=7$ and $c=3$.



We observe that the total number of edges is $e=6$, thus $e \neq 7$.

Considering the fact that <u>the number</u> of vertices (besides <u>it</u> being equal to 4) could only be less than 4, the number of edges won't increase from $e=6$ (for $v=4$), thus:

There is no graph with $e=7$ and $c=3$.

2) for $e=11$ and $c=4$: $4 \le \frac{22}{v} \Leftrightarrow v \le \frac{22}{4} \Leftrightarrow v \le 5$.

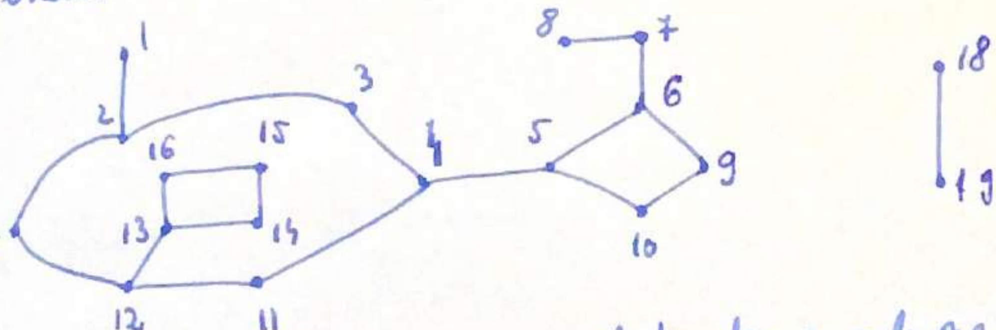
We will consider $v=5$ with $e=11$ and $c=4$.

The largest possible graph with $v=5$ and the largest number of edges is K_v (where $v=5$). We know that $e=\frac{1}{2}(v-1)v$, thus (for $v=5$): $e=\frac{1}{2}\cdot 4\cdot 5=\frac{1}{2}\cdot 20=10$, but $e=11$ (from the hypothesis of our exercise) so (with the fact that $e=10$), thus:

There is no graph with $e=11$ and $c=3$.

14. Definition : A bridge in a graph is an edge whose removal would increase the number of <u>components</u> (where a component of a graph is a directed subgraph that is not contained in any larger contained subgraph).

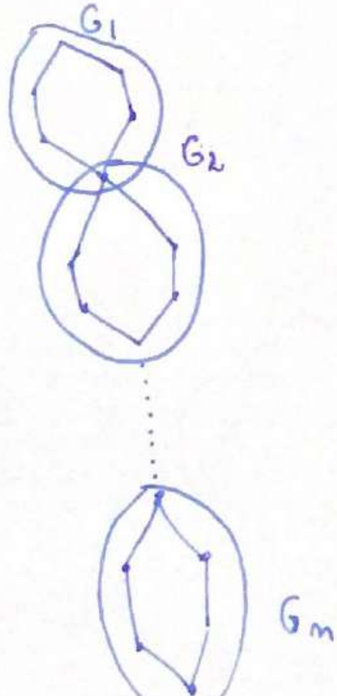
<u>Example</u> : In Figure 107, there are six bridges : $\{1;2\}$; $\{12;13\}$; $\{4;5\}$; $\{6;7\}$; $\{7;8\}$; $\{18;19\}$. The graph has two components; if any bridge were removed the resulting subgraph would have three components:



In a planar graph a bridge necessarily borders on only one face, and an edge bordering on only one edge is neccesarily a bridge. Thus bridges are the things that prevent planar connected graphs from being polygonal (because polygonal grahs are: connected, planar and every edge borders two faces). Use this fact to prove that if a planar and connected graph G has the property that the boundary (or in other terms, the contour) of every face is a cyclic graph, then G is polygonal. Then show that the converse statement is false (if <u>G</u> is polygonal, then every one of <u>its</u> faces is a cyclic graph).

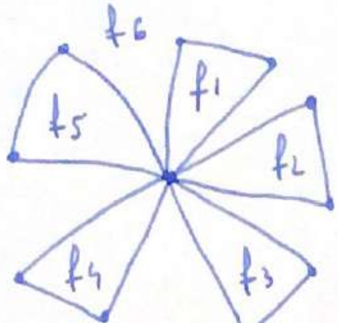
Solution:

For the proof of the statement: „If a planar and a connected graph G has the property that the boundary (or in other terms, the contour) of every face is a cyclic graph, then G is polygonal" we will use a visual demonstration [by letting G be constructed out of n cyclic graphs (named: $G_1; \dots; G_n$):



We observe that by taking any edge of any of $G_1; \dots; G_n$; it will border exactly two faces (distinct ones). Thus G being already planar, connected (and now having every edge border exactly two faces) we conclude that G is polygonal. Therefore we proved: „If a planar and connected graph G has the property that the boundary (or in other terms, the contour) of every face is a cyclic graph, G is then polygonal".

To disprove the statement: „if G is polygonal, then every one of its faces' boundaries is a cyclic graph", we will use the following polygonal graph:



We observe that the boundary of f6 is not a cyclic graph.
We know that for any cyclic graph the degree of each vertex
is exactly equal to 2. We observe that one vertex has degree 10 to
(with the fact that every vertex of a cyclic graph has degree exactly
equal to 2). Therefore we have shown that: „if G is polygonal,
then every boundary of every face (of G) is a cyclic graph" is false.

15. By Theorem 11 (If G is planar and connected with $V \geq 3$, then $\frac{3}{2}f \leq e \leq 3V-6$) we know that every planar, connected graph with $V \geq 3$ has $e \leq 3V-6$. Prove that if such a graph G has the additional property that every supergraph of G with with one more edge is nonplanar, then the boundary of every face of G is C_3; G is polygonal and G has $e = 3V-6$.

Solution: If one edge bordered only one face, we remark the fact that we could add another edge and still preserve the planarity of graph G. Therefore, all edges border two ore more faces. We know for a fact that edges can border a maximum of 2 faces, thus our graph G is polygonal. If the boundary of each face was not C_3, then adding a diagonal to any C_n (with $n > 3$) would lead to a contradiction, thus G must be made out of C_3 graphs. For $f \cdot \frac{3}{2} = e$ and the fact that $v - f + e = 2$ (because G is planar and connected) will lead to:

$$\begin{cases} V + f - e = 2 \\ f = \frac{2}{3}e \end{cases} \quad (*)$$

From (*) we will get that:

$$V + \frac{2}{3}e - e = 2$$

$$V - \frac{1}{3}e = 2$$

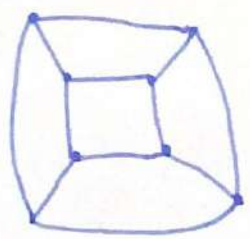
$$e = 3V - 6$$

I must admit, I didn't know how to solve this exercise, so I tried piecing up preexisting solutions.

16. Prove: If G is planar and connected with $c=3$ and the boundary of every face is C_4, then G is polygonal and $e = 2V-4$.

Solution:

One example of a planar graph (named G) that is connected with $c=3$ and the boundary of each face being C_4 is:



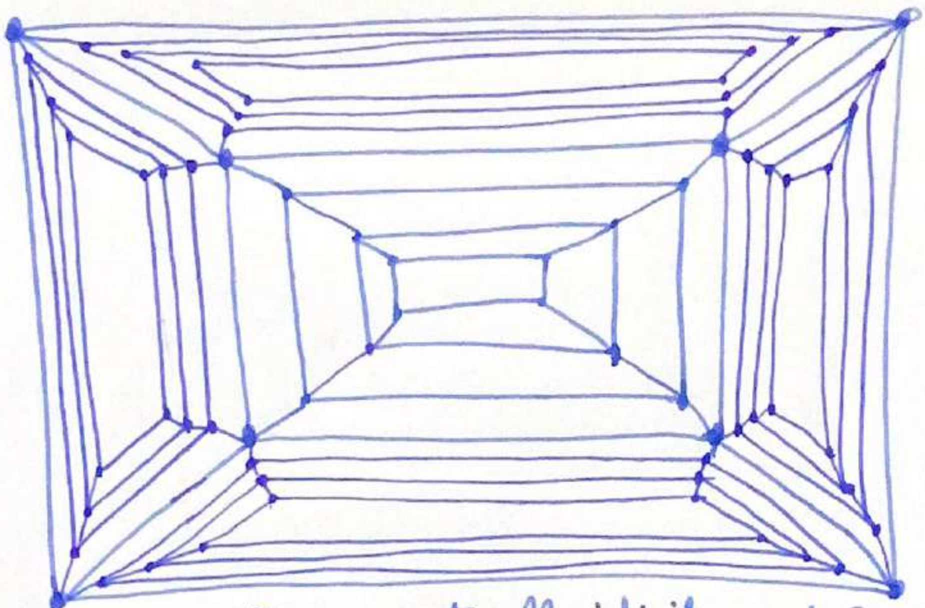
G

We observe the fact that $f = \frac{1}{2}e$ and that G is planar, connected, every edge bordering exactly 2 distinct faces. Because graph G is planar and connected we can use Euler's Formula for planar and connected graphs: $V+f-e=2$. We know that $f = \frac{1}{2}e$, thus

$$V+f-e=2 \Leftrightarrow V+\frac{1}{2}e-e=2 \Leftrightarrow -\frac{1}{2}e = 2-V \Leftrightarrow e = 2V-4.$$

Therefore, we can conclude that: „If G is planar and connected with $c=3$ and the boundary of every face is C_4, then G is polygonal and $e=2V-4$.

We also remark the fact that by inserting infinitely many rectangle graphs to the inside faces of graph G, we have managed to create an infinite planar and connected graph G with $c=3$ and the boundary of each face being C_4 that is polygonal with $e=2V-4$:



The conceptually infinite graph G
that satisfies the statement:
„An infinite, planar and connected graph G
with connectedness $c = 3$ and the boundary
of each face being C_4, that is also polygonal
with $e = 2V - 4$"

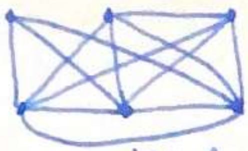
17. Prove: If the connectivity of a graph is at least 6; the graph is nonplanar.

Solution: We have proved that K_n is nonplanar for $n \in \mathbb{N} \setminus \{0;1;2;3;4;5;6\}$. We also know that each degree of each vertex of K_7 (which is the smallest complete graph that is nonplanar) is equal to 6, thus to make the connected graph disconnected (within the smallest possible combinations of eliminating vertices and their incident edges) we will have to eliminate at least 6 vertices. Therefore, the smallest value for c, in order for the graph to be nonplanar is 6. We managed to prove the statement: „If the connectivity of a graph is at least 6, the graph is nonplanar".

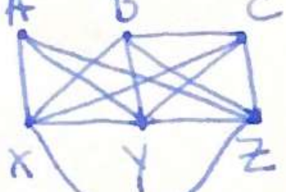
18. Prove: If a nonplanar graph has $v \geq 6$, $c \geq 3$ and it also has a subgraph which is an expansion of K_5 (which has been proved to be nonplanar), then it is also has a subgraph which is an expansion of U.G. This is what I had in mind on page 86 when I advised you that: „the vast majority of nonplanar graphs contain an expansion of U.G., so start by looking for that".

Solution:

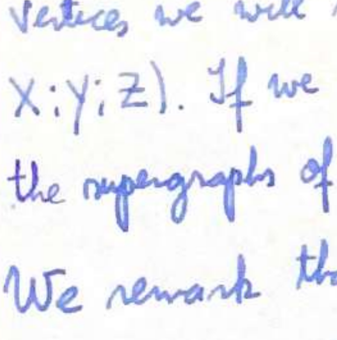
We will consider the graph with the minimum requirements (that is, $V=6$; $C=3$ and our graph must contain a subgraph which is an expansion of K_5 and also a subgraph that is an expansion of U.G.:



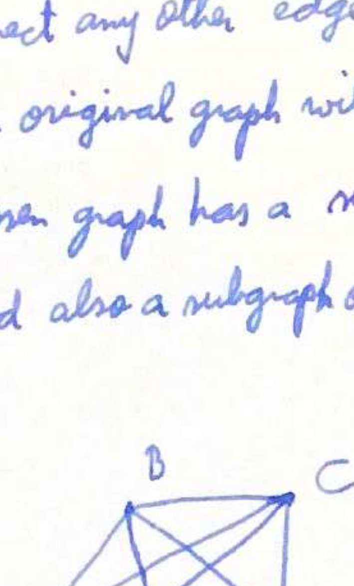
We will label each vertex for clarity:



We remark the fact that $c = 3$, because the minimum number of vertices we will need to eliminate from our graph is 3 (namely: $X; Y; Z$). If we won't connect any other edge to our vertex A, the supergraphs of our chosen original graph will always have $c = 3$.
We remark that our chosen graph has a subgraph of K_5 (which is an expansion of itself) and also a subgraph of U.G. (which is an expansion of itself):



The subgraph of U.G.



The subgraph of K_5

If we set out a rule for the supergraphs of our chosen graph:
Whichever new vertices we will add to our original graph,
"we will not bind anything to vertex A whatsoever."

We will need another rule to preserve the fact that $c \geq 3$, namely: „Whichever new vertices we will add to the original graph, we will need all the new vertices to have a degree of at least 3".

After these 2 imposed rules, we can consider our chosen original graph as the chosen building-block for other cases of graphs (which are supergraphs of our original chosen graph). Therefore, we have proved the statement: „If a nonplanar graph has $v \geq 6$; $c \geq 3$ and has a subgraph of an expansion of K_5 then it also has a subgraph of an expansion of O.G.".

Page: 144-146:

1. Draw all connected graphs that are regular of degree 1

Solution:

The line binding two points represents the only connected graph that is regular with degree 1:

2. Satisfy yourself that every connected graph which is regular of degree 2 is a cylic graph. Show by example that deleting the word „connected" results in a false statement.

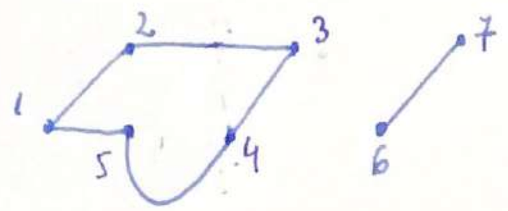
Solution:

In order to satisfy ourselves that: „Every connected graph which is regular of degree 2 is a cyclic graph", we will start with the smallest example of a graph that is regular with degree equal to 2:

We remark that the graph corresponds to C_3. By adding a new vertex (which we will name: 4), we observe that in order to keep every vertex of degree 2, we will need to erase one of the existing edges: $\{\{1;2\};\{1;3\};\{3;2\}\}$ and add one of these combinations of edges $\underbrace{\{\{1;4\};\{2;4\}\}}_{\text{for replacing } \{1;2\}.}$; $\underbrace{\{\{1;4\};\{3;4\}\}}_{\text{for replacing } \{1;3\}.}$; $\underbrace{\{\{3;4\};\{2;4\}\}}_{\text{for replacing } \{3;2\}.}$

Continuing this process inductively, we arrive at the conclusion that: „Every connected graph which is regular of degree 2 is a cyclic graph".

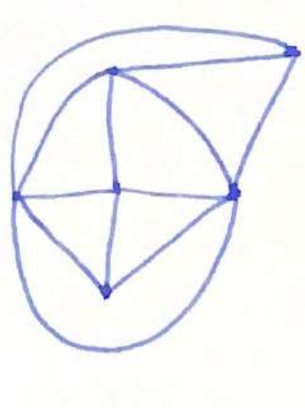
We will know show by example that deleting the word „connected" results in a false statement.
An example would be:

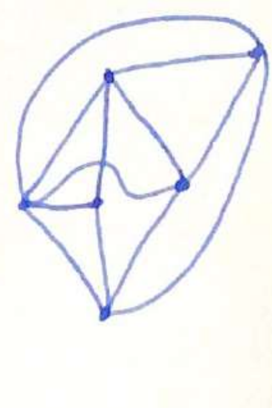


While the example describes a cyclic graph of 5 vertices and a cyclic graph of 2 vertices, the whole graph (where we include the cyclic graph of 5 vertices and the cyclic graph of 2 vertices) which is disconnected isn't a cyclic graph in its entirety.

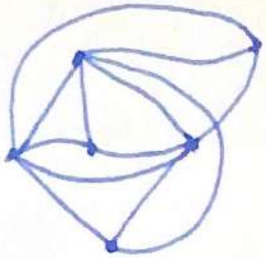
3. It's easy to see that K_4 is the only platonic graph corresponding to the first row of the table on the page 141, as K_4 is the only graph having $v=4$ and $e=6$. Now verify that there is only one platonic graph corresponding to the fourth row by drawing all graphs with $v=6$ and $e=12$ (there are five of them) and checking that only the octahedron is platonic.

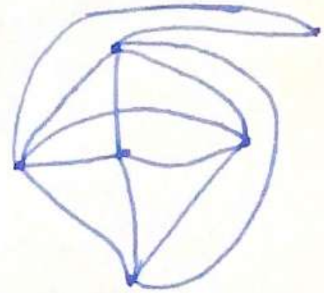
<u>Solution</u>: We will draw all five graphs with $v=6$ and $e=12$.





this is an octahedron





We remark that for all other graphs except the first graph (which is the octahedron), the degrees of the vertices aren't all equal, thus the rest of the graphs (excluding the octahedron) aren't regular. Because every platonic graph must be regular, we get that the octahedron is the only graph with $v=6$ and $e=12$ that is platonic.

4. Prove: There is no regular graph with $v=6$ and $e=10$ (This can be done algebraically, without drawing a simple graph).

Solution: Using Lemma 14 (from page 138): If the graph G is regular of degree d, then $e=\frac{dv}{2}$, we get (for $e=10$; $v=6$): $d=\frac{2\cdot 10}{6} \Leftrightarrow d=\frac{2\cdot 5}{3}$; but $d\in\mathbb{N}$ and in our case $d=\frac{10}{3}\notin\mathbb{N}$, thus we have proved that: „There is no regular graph with $v=6$ and $e=10$".

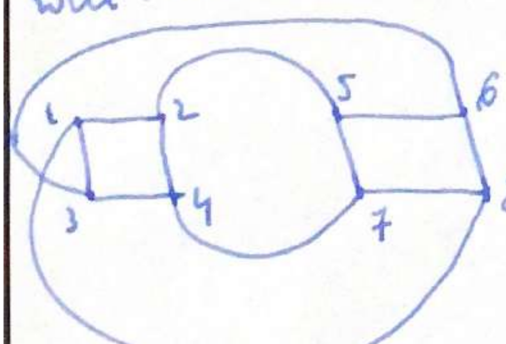
5. Prove: If a graph has an odd number of vertices and is regular of degree d, then d must be even.

Solution:

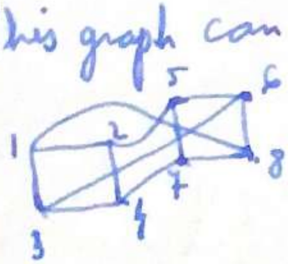
Using Lemma 14: If a graph G is regular of degree d, then $e = \frac{dV}{2}$, and the condition that $V = 2k+1$; where $k \in \mathbb{N}$, the only way of $e \in \mathbb{N}$ is for $2 | d$, thus d must be even.
e must belong to the natural numbers, as a graph can't be build on any set of numbers other than the natural ones.

6. Find a graph other than the cube that has $V = 8$ and is regular of degree 3

Solution: We consider the following graph (that we will label with: 1; 2; 3; 4; 5; 6; 7; 8)



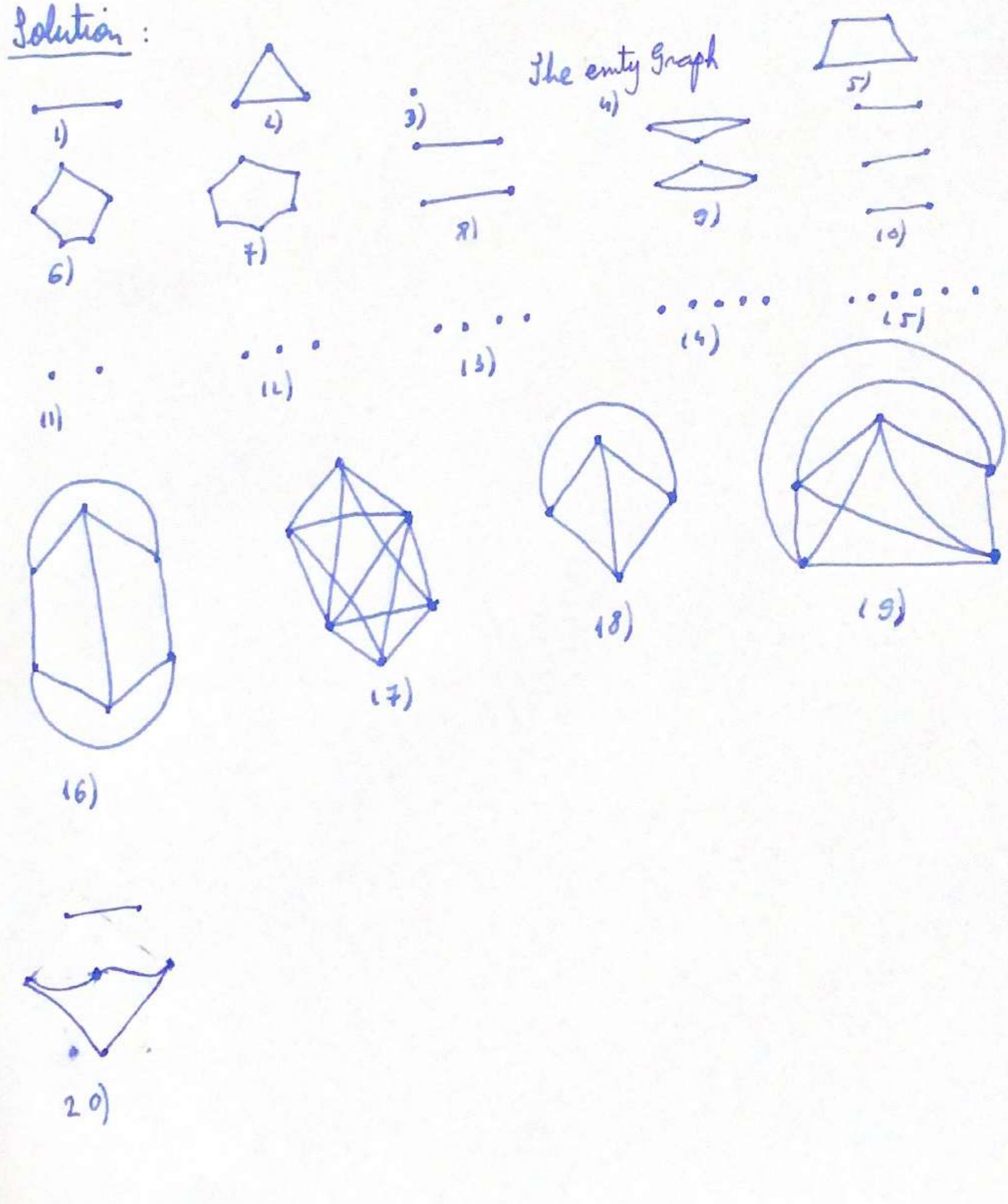
This graph can be reshaped as the following figure:



It is easy to see from the refigured graph, that our selected graph doesn't equal to the graph of a cube.

7. Draw all regular graphs with six vertices or less. There are twenty of them.

Solution :



8. Definition: A dual graph of a planar graph is formed by taking a crossing-free plane drawing of the planar graph, placing a dot inside each face, and joining two dots whenever the borders of the corresponding faces have one or more edges in common.

Example: The planar graph of Figure 112a has been redrawn in 112b along with a dual; the chosen dual is drawn by itself in 112c.

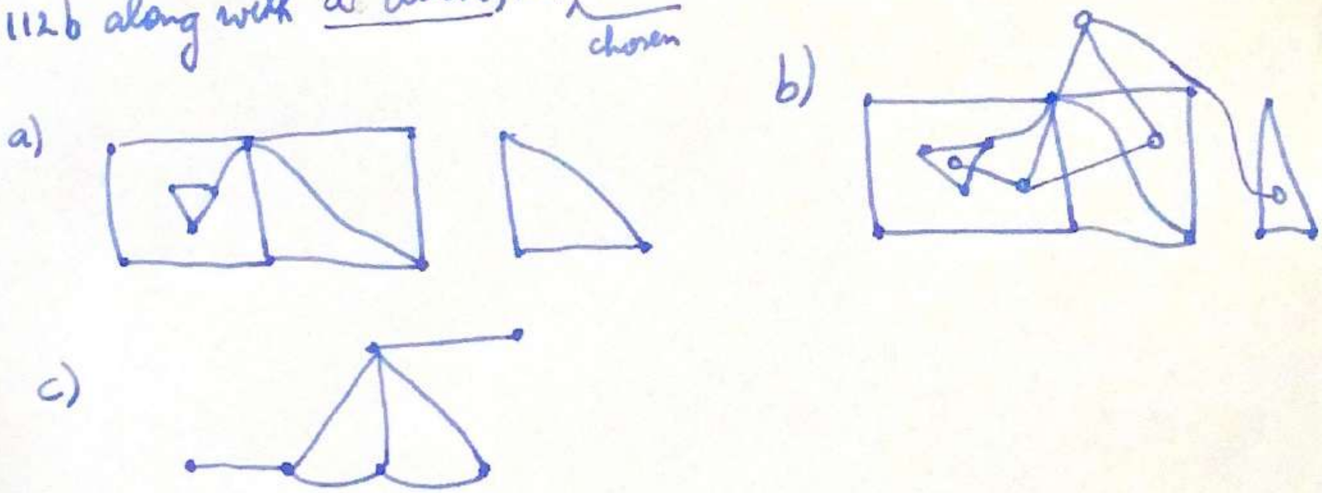
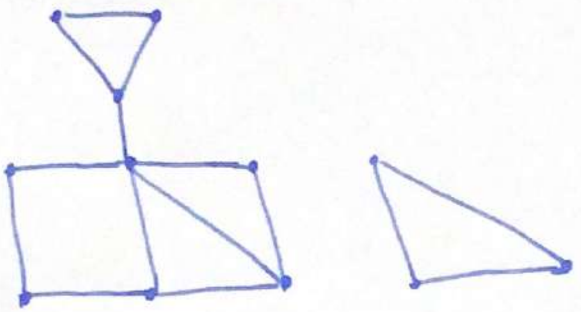


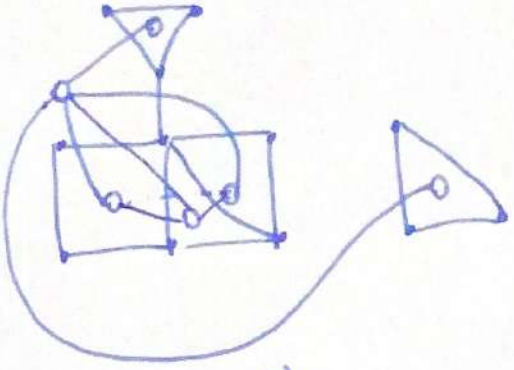
Figure 112

Note that only a planar graph can have a dual, and that dual is always planar and connected. A dual is not always unique, some planar graphs have several different duals, each arising from a different crossing-free plane drawing. This is why in the definition I said „a dual" instead of „the dual".

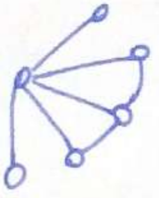
Example: Figure 113a is isomorphic to Figure 112a, yet the dual formed in 113b and shown by itself in 113c is not isomorphic to the dual of Figure 112c.



a)



b)

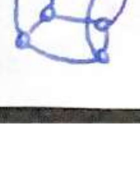


c)

Figure 113

Each of the platonic graphs does, however, have a unique dual. For example, K_1 has only one dual, itself, and each cyclic graph has only one dual, K_2. The octahedron and its unique dual are shown in the three dimensional drawing of Figure 114:

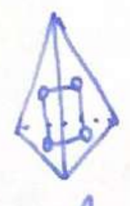
Figure 114



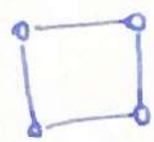
The dual of the octahedron

Draw the duals of the other four platonic graphs (regular tetrahedron, cube, regular dodecahedron, regular icosahedron). Your results will help explain the curious symmetry of the table on page 141.

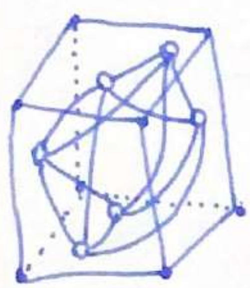
<u>Solution</u>:



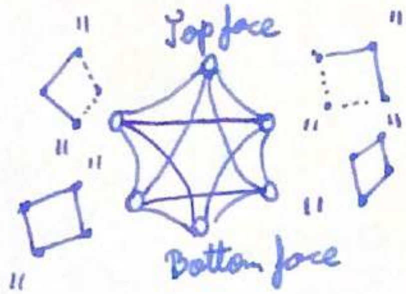
regular tetrahedron



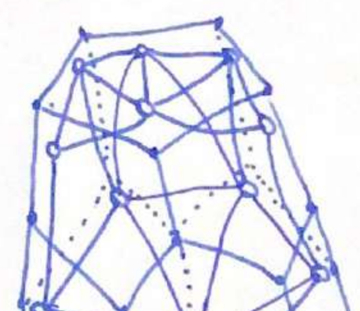
the dual of the regular tetrahedron



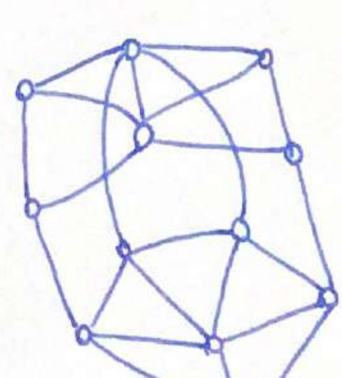
regular cube



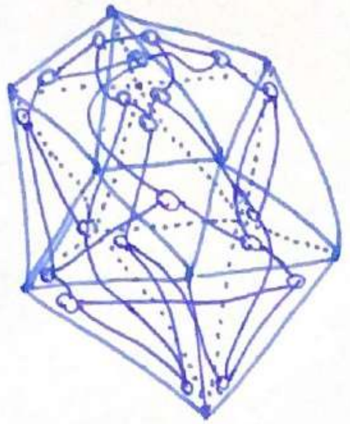
the dual of the regular cube



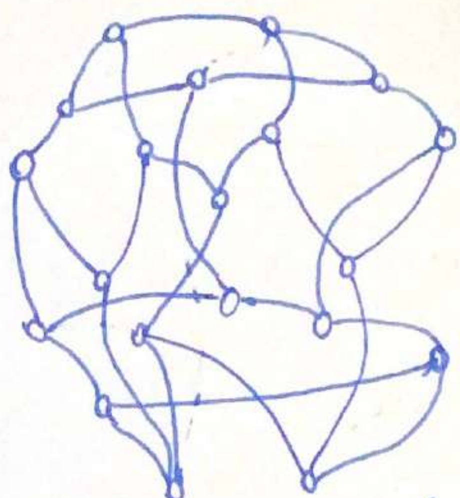
regular dodecahedron



the dual of the regular dodecahedron



regular icosahedron



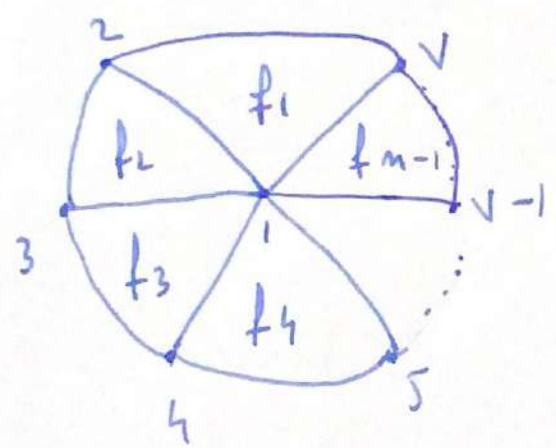
dual of a regular icosahedron.

9. Prove that every wheel W_v (see Chapter 2: Graphs, Exercise 6) is isomorphic to its (unique) dual.

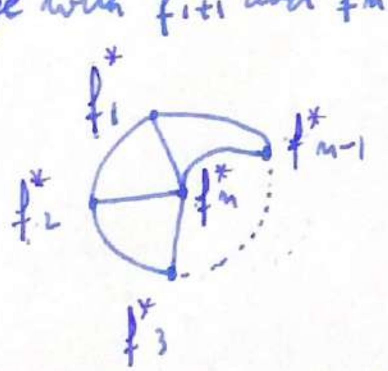
Solution: We will remind ourselves what a wheel graph looks like: If v is an integer greater than or equal to 4, the wheel graph with v vertices (denoted „W_v") is the graph having the vertex set: $\{1;2;3;\ldots;v\}$ and the edge set: $\{\{1;2\};\{1;3\};\{1;4\};\ldots;\{1;v\};\{2;3\};\{3;4\};\{4;5\};\{5;6\};\{6;7\};\ldots;\{v-1;v\}\}$.

Visually, we have:

We will make W_v larger and denote by „f_i" (with $i=\overline{1,n}$) each face:



We remark that f_n corresponds to the infinite face.
We also remark that every f_i (for $i=\overline{1,n-1}$) shares a common edge with the infinite face „f_n"
We additionally observe that every face f_i shares a common edge with f_{i+1} and f_{n-i}, thus the unique dual will look like:



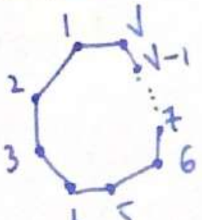
<u>The notation</u> of each vertex has been intentionally been with f_i^* with $i=\overline{1,n}$, because it tells to what face was the „dual" vertex placed and what „dual" edges does the dual graph have.
We can easily remark that by changing the dual vertices f_i^* with $i=\overline{1,n}$ $\{1;2;3;\dots;v\}$ respectively (that is: $f_1^* \to 1$; $f_2^* \to 2; \dots; f_n^* \to v$)

we will end up with W_v, thus the dual graph is isomorphic to W_v. We also remark that the dual graph is unique. Therefore, we have proved the statement: „Every wheel W_v is isomorphic to its dual".

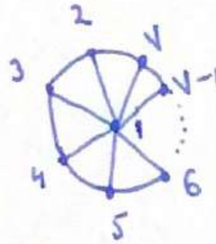
Page 161-163

1. Find X for each of the following graph: C_V; W_V; U.G., octahedron, dodecahedron, icosahedron

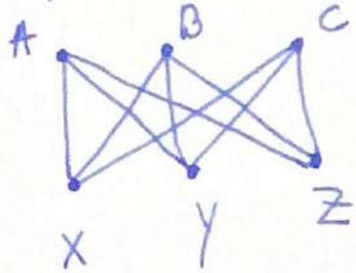
Solution: We will draw each of the graph first.



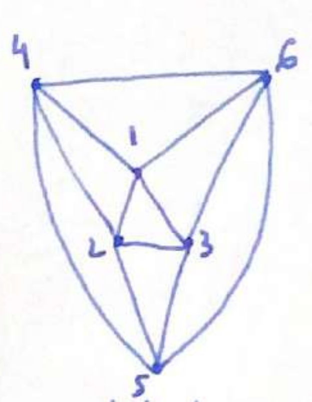
C_V



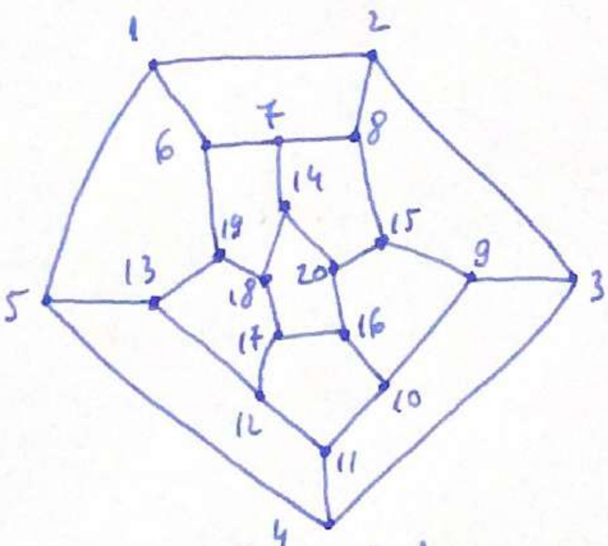
W_V



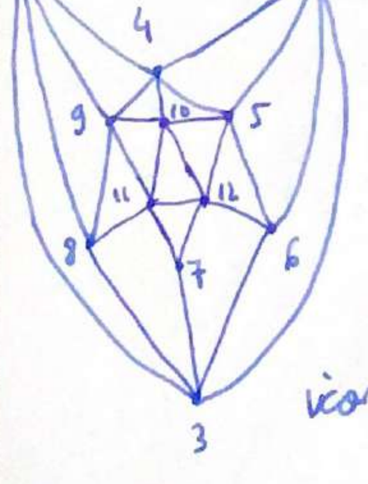
U.G.



octahedron
(in a planar manner)



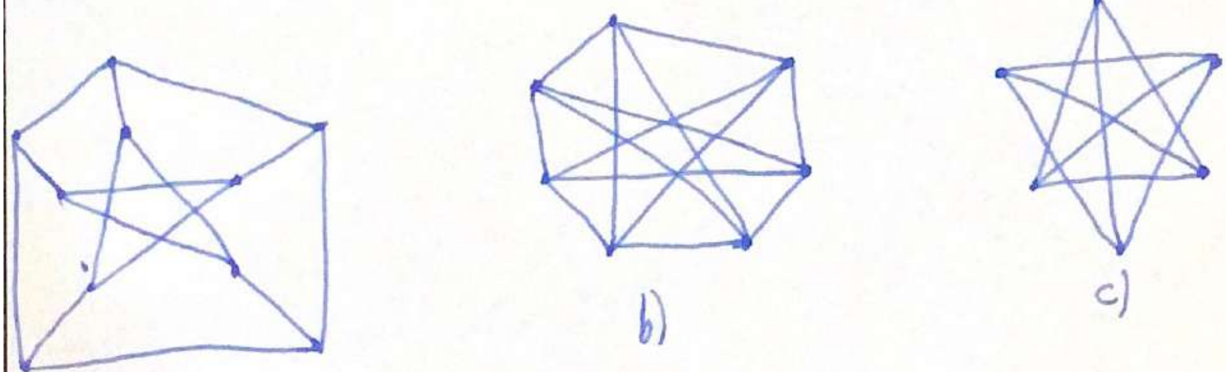
dodecahedron
(in a planar manner)



icosahedron (in a planar manner)

We remark that for C_V the chromatic number is equal to 2.
We remark that for W_V the chromatic number is equal to 3.
We remark that for U. G. the chromatic number is equal to 2.
We remark that for the octahedron the chromatic number is equal to 3.
We remark that for the dodecahedron the chromatic number is equal to 3.
We remark that for the icosahedron the chromatic number is equal to 4

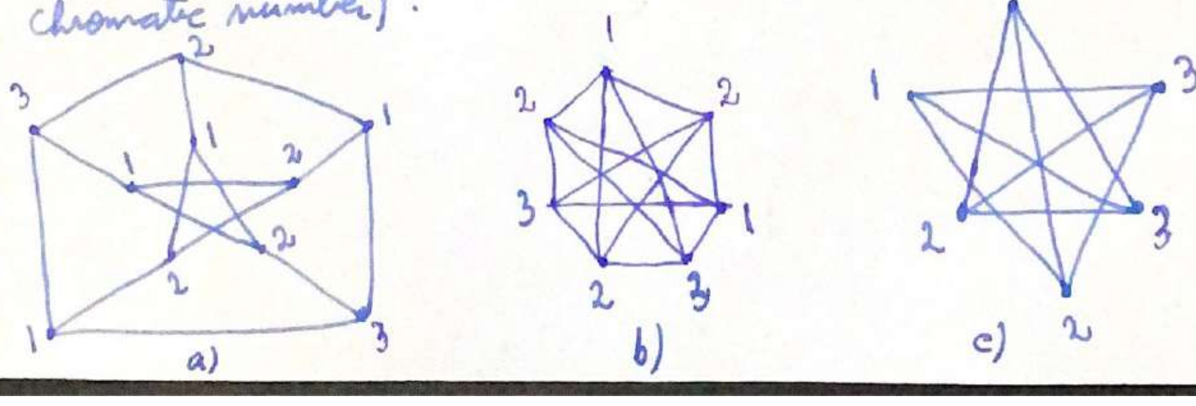
2. Find the chromatic number of the graphs in Figure 126:



a) b) c)

Figure 126

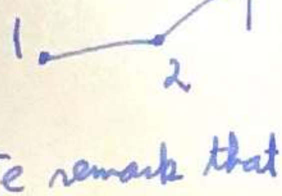
Solution: We will "color" the graph in order to find χ (= the chromatic number):



a) b) c)

We remark the fact that for all of the graphs in exercise 2, $X = 3$.
3. Obviously the set of all graphs with $X = 1$ is equal to the set of all null graphs N_V. We might call this one the „One Color Theorem". Prove the „Two Color Theorem" : the set of all graphs with $X = 2$ is equal to the set of all graphs having at least one edge and no „odd-numbered vertices" cyclic subgraphs. If only there were a „Three Color Theorem" the Four Color Conjecture would be practically resolved. For this reason much of today's four-color research is directed towards finding a characterisation of graphs with $X = 3$ analogous to the Two Color Theorem.

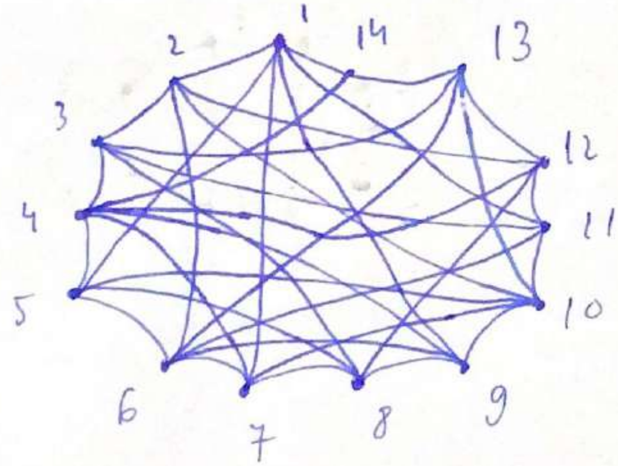
<u>Solution</u> : The first legal example for which the „Two Color Theorem" holds is a line graph: 1 —— 2. Although our matter of choice won't affect the final conclusion, we will choose vertex „2" from the line graph and add a new vertex which we will bind to the chosen vertex (that being the vertex „2"). Because the new vertex isn't bound to the vertex „1" of the original line graph, we can label our new vertex „1", thus obtaining:



We remark that by joining the new vertex „1" to the original vertex „1", we will obtain a cyclic graph with an odd valued vertex. By joining the new vertex „1" to the original vertex „1", we will have $X = 3$, as two adjacent vertices cannot have the same labelling. By continuing this process (in an inductive manner) we can conclude that: „The set of all graphs with $X = 2$ is equal to the set of all graphs having at least one edge and no „odd-numbered vertices" cyclic subgraphs".

4. K_3 has $X = 3$, so every supergraph of K_3 has $X \geq 3$. On the other hand C_5 is a graph with $X = 3$ that is not a supergraph of K_3. Find a graph with $X = 4$ that is not a supergraph of K_3.

Solution: We will consider the following graph:



After thorough analysis we will conclude that the considered graph has $X = 4$ and doesn't contain a subgraph of K_3.

5. If a graph G has chromatic number χ and its complement $\overline{G}$ has chromatic number $\overline{\chi}$, prove that $\chi \cdot \overline{\chi} \geq v$. Then use the fact that $\frac{1}{2}(m+n) \geq \sqrt{mn}$ ("this is the „arithmetic mean – geometric mean inequality") whenever m and n are positive integers to prove that: $\chi + \overline{\chi} \geq 2\sqrt{v}$.

Solution: For proving $\chi \cdot \overline{\chi} \geq v$ we will firstly consider the easiest graph [excluding the „Zero Graph" (the graph with 0 vertices and 0 edges) which has $\chi = 0$ and $\overline{\chi} = 0$, thus it satisfies the inequality $\chi \cdot \overline{\chi} \geq v$, because we will have: $0 \cdot 0 \geq 0$], namely, N_v. $\chi_{N_v} = 1$; $\overline{\chi_{N_v}} = \chi_{K_v} = v$. Any other alteration of graphs will increase in terms of the values of χ and $\overline{\chi}$.

Because N_v represents the minimal case for which the inequality $\chi \cdot \overline{\chi} \geq v$ is satisfied (as $\chi_{N_v} = 1$; $\overline{\chi}_{N_v} = \chi_{K_v} = v$, we get $\chi_{N_v} \cdot \overline{\chi}_{N_v} = v \geq v$), as we remarked any other alteration of graphs from N_v will only increase $\chi \cdot \overline{\chi}$ from the minimal case $\chi_{N_v} \cdot \overline{\chi}_{N_v} = v \geq v$, thus concluding that: „If a graph G has a chromatic number χ and its complement $\overline{G}$ has chromatic number $\overline{\chi}$, $\chi \cdot \overline{\chi} \geq v$" will be satisfied).

We will now prove the following statement: „$X + \overline{X} \geq 2\sqrt{v}$."

We have proved that $X \cdot \overline{X} \geq v$ (·)

We know that $\frac{1}{2}(m+n) \geq \sqrt{mn}$ for $\begin{cases} m \in \mathbb{N} \\ n \in \mathbb{N} \end{cases}$ (··)

We know that : $\begin{matrix} 1 \leq X \leq v \\ 1 \leq \overline{X} \leq v \end{matrix}$, thus $\begin{cases} X \in \mathbb{N} \\ \overline{X} \in \mathbb{N} \end{cases}$; (···)

(that is because $v \in \mathbb{N}$)

Combining (·); (··); (···) and letting $m = X$; $n = \overline{X}$; we get that: $\frac{1}{2}(X+\overline{X}) \geq \sqrt{X\overline{X}} \Leftrightarrow$ „$2\sqrt{X \cdot \overline{X}} \leq X + \overline{X}$" (····)

Using: $X \cdot \overline{X} \geq v \quad |\cdot\sqrt{}$

$$\sqrt{X \cdot \overline{X}} \geq \sqrt{v} \quad | \cdot 2$$

$$2\sqrt{X \cdot \overline{X}} \geq 2\sqrt{v} \quad (·····)$$

From (····) and (·····) we get that:

$$2\sqrt{v} \leq 2\sqrt{X \cdot \overline{X}} \leq X + \overline{X} \Longrightarrow 2\sqrt{v} \leq X + \overline{X}$$

(We have taken both of the extremes of : $2\sqrt{v} \leq 2\sqrt{X \cdot \overline{X}} \leq X + \overline{X}$)

Therefore, we proved the statement: „$X + \overline{X} \geq 2\sqrt{v}$" and more explicity we proved the statement: „$X + \overline{X} \geq 2\sqrt{X + \overline{X}} \geq 2\sqrt{v}$"

6. Find a graph for which $\chi \cdot \overline{\chi} = v$ and a graph for which $\chi + \overline{\chi} = 2\sqrt{v}$. Could a single graph satisfy both equations? (If a graph has chromatic number „χ", then „$\overline{\chi}$" denotes the number of its complement).

<u>Solution</u>:

A graph for which $\chi \cdot \overline{\chi} = v$ is N_v.
That is because $\chi_{N_v} = 1$ and $\overline{\chi}_{N_v} = \chi_{K_v} = v$, thus we will get that: $\chi_{N_v} \cdot \overline{\chi}_{N_v} = 1 \cdot v = v$.

A graph for which $\chi + \overline{\chi} = 2\sqrt{v}$ is a square:
$\chi = 2$; while $\overline{\chi} = 2$ (because the complement of a square is the graph of two „edge crossing" paths).
A square has $V = 4$. Therefore $\chi + \overline{\chi} = 2 + 2 = 4 = 2\sqrt{4} = 2 \cdot 2 = 4$
$\Longleftrightarrow \chi + \overline{\chi} = 2\sqrt{v}$.

A single (particular, not general) graph can satisfy both:
"$\chi \cdot \overline{\chi} = v$" and "$\chi + \overline{\chi} = 2\sqrt{v}$". That graph is N_1.
"That is because $\chi_{N_1} = 1$ and $\overline{\chi_{N_1}} = \chi_{K_1} = 1$, thus we will get that $\chi_{N_1} \cdot \overline{\chi}_{N_1} = 1 \cdot 1 = \underbrace{1}_{v \text{ of } N_1}$ and $\chi_{N_v} + \overline{\chi_{N_v}} = 1 + 1 = 2 \cdot \sqrt{\underbrace{1}_{v \text{ of } N_1}}$

7. Obviously the „Four Color Conjecture" is true for planar graphs with four vertices or less. Of the graphs with five vertices only, the complete graph K_5 has $X=5$; K_5 is nonplanar so the „Four Color Conjecture" is true for planar graphs with five vertices. To prove that the „Four Color Conjecture" is true for planar graphs with six vertices we argue as follows: K_6 is regular of degree 5, but is nonplanar; every other graph with 6 vertices has at least one vertex of degree ≤ 4; in particular, every planar graph with six vertices has at least one vertex of degree ≤ 4, and (by using Theorem 17: „Every planar graph having a vertex of degree ≤ 4, has $X \leq 4$", on page 158) $X \leq 4$. Prove that the „Four Color Conjecture" is true for planar graphs with seven vertices.

Solution: We know that every planar graph has a vertex of degree at most 5. We remark that K_7 is regular with $d=6$ where d represents the constant value of all degrees of each one of K_7 vertices, every other planar graph with 7 vertices has at least one vertex of degree ≤ 4, and (by using Theorem 17: „Every planar graph having a vertex of degree ≤ 4, has $X \leq 4$", on page 158) $X \leq 4$.
We have proved that: „The Four Color Conjecture" is true for planar graphs with seven vertices".

8. Color the faces of Figure 123b with four colors. Don't forget the infinite face:

Solution: We will denote each of the four colours with an abbreviation:

B - blue, R - red, Br - brown, O - orange:

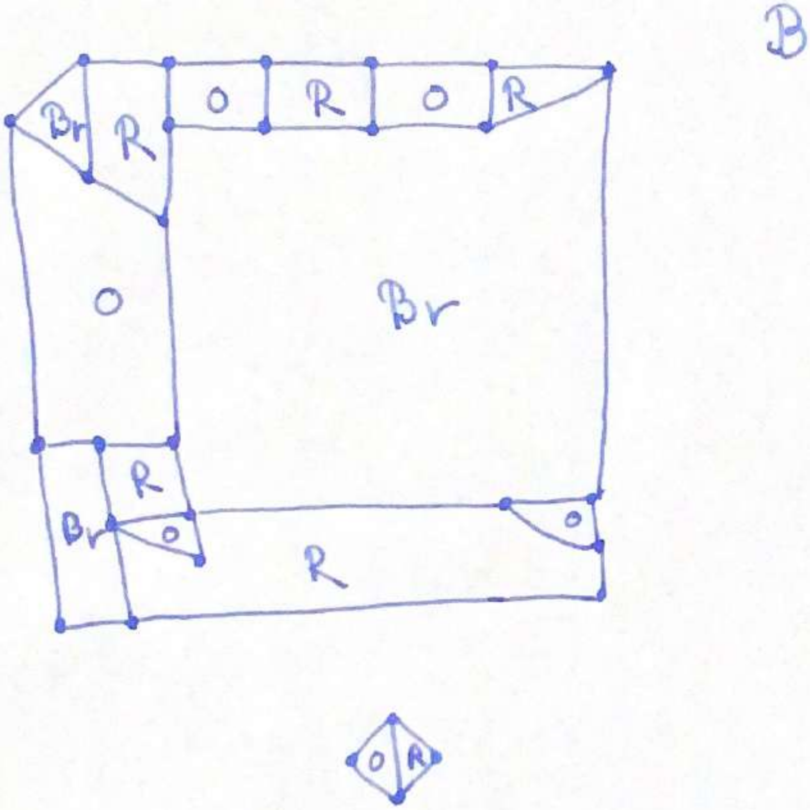
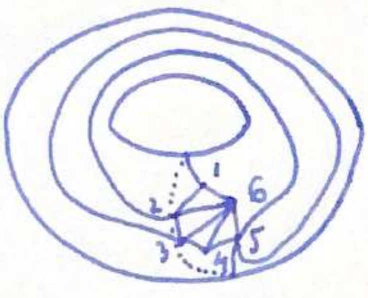


Figure 123b

Page 193 - 196

1. Make a crossing-free drawing of K_6 on S_1 and count its faces. Use Euler's Second Formula to verify your count.

Solution: We know that S_1 represents a Torus:



Our K_6 on S_1 is now crossing-free.

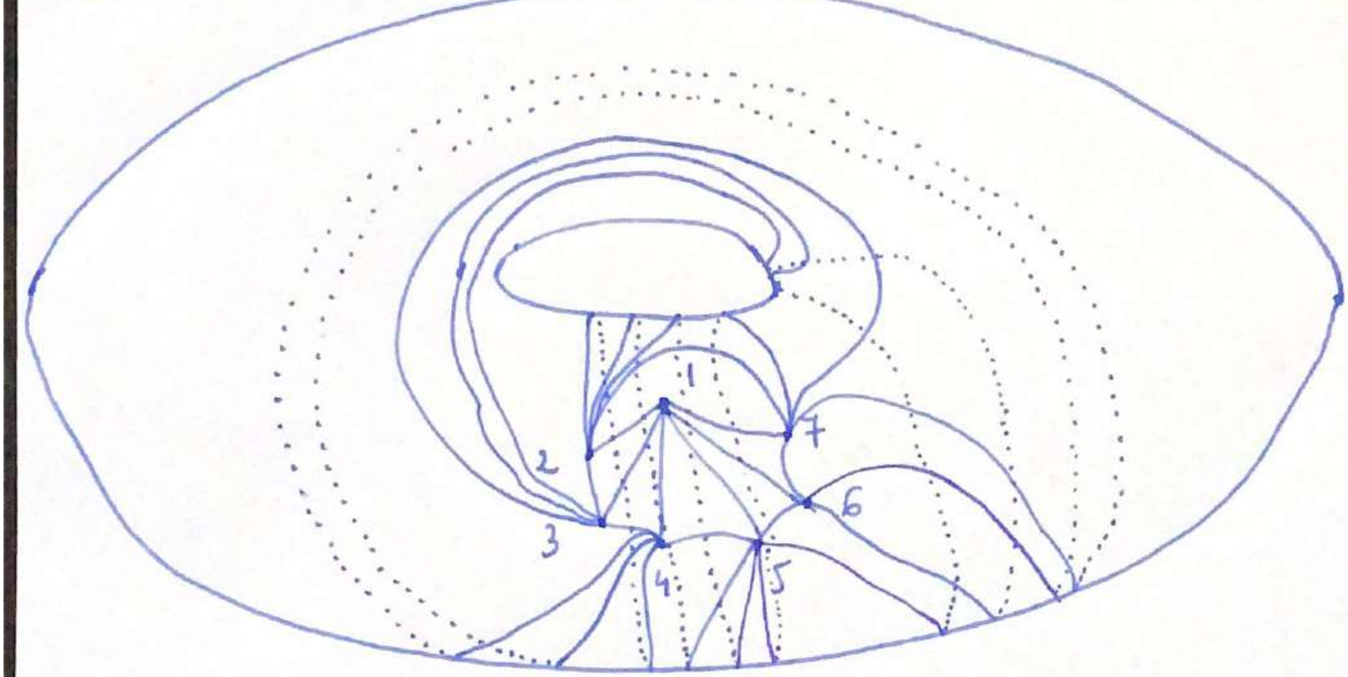
Using Euler's Second Formula, the number of faces of K_6 will be easily obtained:

$$\left.\begin{array}{l} v+f-e=2-2g \\ g=1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} v+f-e=0 \\ e=\frac{1}{2}v(v-1) \end{array}\right\} \Leftrightarrow \begin{array}{l} f=\frac{1}{2}v(v-1)-v;\ v=6 \\ f=\frac{1}{2}\cdot 6\cdot 5-6 \\ f=15-6 \Leftrightarrow \underline{f=9.} \end{array}$$

Making the K_6 embedding on a Torus, we can easily see that the number of faces is indeed equal to nine.

2. Make a crossing-free drawing of K_7 on S_1 and count its faces. Use Euler's Second Formula to verify your count.

Solution :



Our K_7 on S_1 is now crossing-free.
Using Euler's Second Formula, the number of faces of K_7 will be easily obtained:

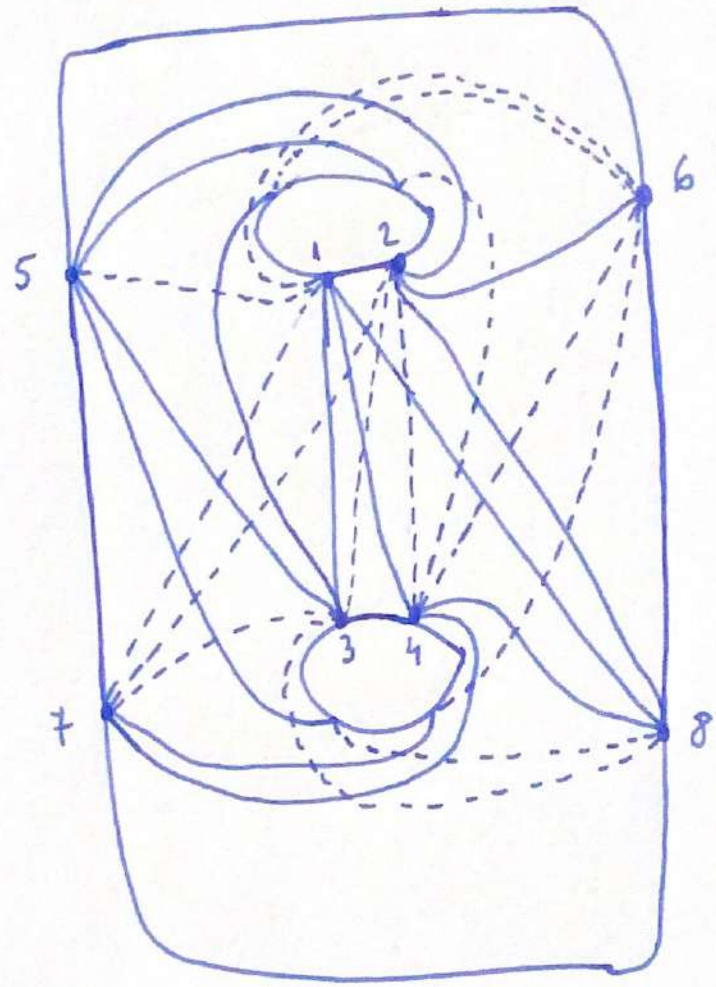
$$\left.\begin{array}{l} v+f-e=2-2g \\ g=1 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} f=e-v \\ e=\frac{1}{2}v(v-1) \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} f=\frac{1}{2}v(v-1)-v \\ v=7 \end{array}\right\} \Leftrightarrow$$

$$f=\frac{1}{2}\cdot 7\cdot 6-7 \Leftrightarrow f=3\cdot 7-7 \Leftrightarrow \underline{f=14}$$

Making the K_7 embedding on a Torus, we can easily see the number of faces is indeed equal to fourteen.

3. Make a crossing-free drawing of K_8 on S_2.

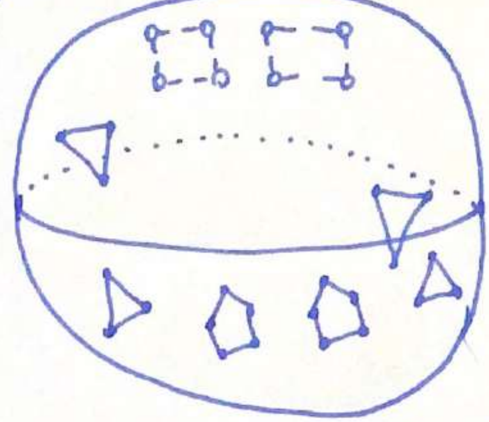
Solution:



We managed to draw K_8 crossing-free on S_2.

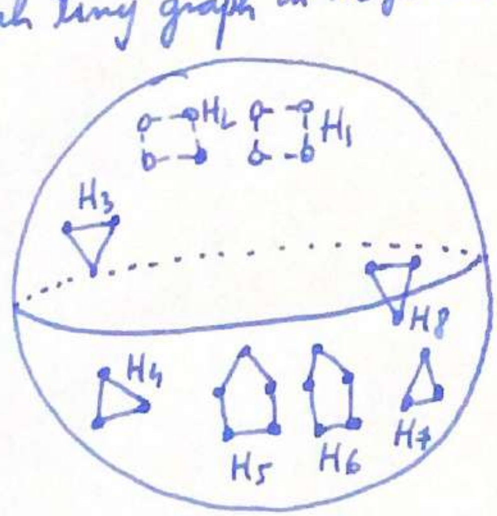
4. Prove that the graph H mentioned in the proof of Euler's Second Formula is connected.

Solution: We will look at H as a graph on S_0 (the sphere) on page 177 (illustrated in Figure 139, on the left side):



" The graph on the right side illustrated in Figure 139 "

In this graph, the graph H represents all tiny graphs belonging to S_0. Counting the number of vertices of H, we will get a total of $V_H = 30$; $e_H = 30$; $f_H = 16$. We will all each tiny graph on S_0 and write down their Euler's Second Formula. Firstly, we will label each tiny graph in the following manner:



Writing the Euler's Second Formula for each H_i with $i = \overline{1,8}$, we get that:

for H_1: $v_{H_1} + f_{H_1} - e_{H_1} = 2-2g$ (1)

for H_2: $v_{H_2} + f_{H_2} - e_{H_2} = 2-2g$ (2)

for H_3: $v_{H_3} + f_{H_3} - e_{H_3} = 2-2g$ (3)

for H_4: $v_{H_4} + f_{H_4} - e_{H_4} = 2-2g$ (4)

for H_5: $v_{H_5} + f_{H_5} - e_{H_5} = 2-2g$ (5)

for H_6: $v_{H_6} + f_{H_6} - e_{H_6} = 2-2g$ (6)

for H_7: $v_{H_7} + f_{H_7} - e_{H_7} = 2-2g$ (7)

for H_8: $v_{H_8} + f_{H_8} - e_{H_8} = 2-2g$ (8)

---

for H_1: $v_{H_1} = 4$; $e_{H_1} = 4$; $f_{H_1} = 2$; $g = 0$, thus (1) is satisfied.

for H_2: $v_{H_2} = 4$; $e_{H_2} = 4$; $f_{H_2} = 2$; $g = 0$; thus (2) is satisfied.

for H_3: $v_{H_3} = 3$; $e_{H_3} = 3$; $f_{H_3} = 2$; $g = 0$; thus (3) is satisfied.

for H_4: $v_{H_4} = 3$; $e_{H_4} = 3$; $f_{H_4} = 2$; $g = 0$; thus (4) is satisfied.

for H_5: $v_{H_5} = 5$; $e_{H_5} = 5$; $f_{H_5} = 2$; $g = 0$; thus (5) is satisfied.

for H_6: $v_{H_6} = 5$; $e_{H_6} = 5$; $f_{H_6} = 2$; $g = 0$; thus (6) is satisfied.

for H_7: $v_{H_7} = 3$; $e_{H_7} = 3$; $f_{H_7} = 2$; $g = 0$; thus (7) is satisfied

for H_8: $v_{H_8} = 3$; $e_{H_8} = 3$; $f_{H_8} = 2$; $g = 0$; thus (8) is satisfied.

Because all H_i's with $i = \overline{1,8}$ satisfy Euler's Second Formula: $v + f - e = 2-2g$, for connected graphs and $H = \sum_{i=1}^{8} H_i$, we conclude that H is connected.

5. Show that Theorem 22 is false for $V=1$ and $V=2$.

Solution: We remind ourselves what Theorem 22 says:

Theorem 22: If $V \geq 3$, the complete graph K_V has genus:

$$g = \left\{ \frac{(V-3)(V-4)}{12} \right\}; \text{ where } \underset{x \in \mathbb{R}}{\{x\}} = \min \{ n \in \mathbb{Z} \mid n \geq x \}.$$

For $V=1$, we have: $g = \left\{ \frac{(-2)(-3)}{12} \right\} = \left\{ \frac{6}{12} \right\} = \left\{ \frac{1}{2} \right\} = 1$, so K_1 has $g=1$, but K_1 is planar, thus K_1 has $g=0$ (and not $g=1$) Therefore, Theorem 22 fails for $V=1$. ⊙

For $V=2$, we have: $g = \left\{ \frac{(-1)(-2)}{12} \right\} = \left\{ \frac{2}{12} \right\} = \left\{ \frac{1}{6} \right\} = 1$; but K_2 is planar, thus K_2 has $g=0$ (and not $g=1$). Therefore, Theorem 22 fails for $V=2$. ⊙⊙

From ⊙ and ⊙⊙, we conclude the following statement:

"Theorem 22 fails for $V=1$ and $V=2$."

6. Prove that the graph of Figure 135 has genus $g=2$.

Solution:

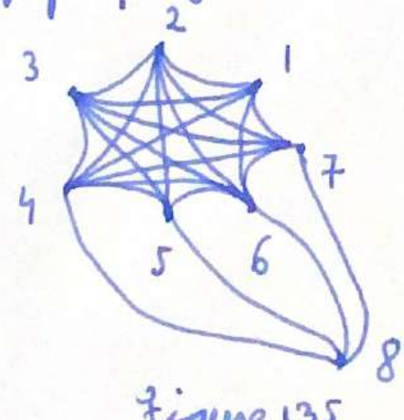


Figure 135

We remark the fact that the graph of Figure 135 has 8 vertices.

Applying Theorem 22 (If $V \geq 3$, the complete graph K_V has genus

$$g = \left\{ \frac{(V-3)(V-4)}{12} \right\}; \text{ where } \underset{x \in \mathbb{R}}{\{x\}} = \min \{ n \in \mathbb{Z} \mid n \geq x \}.$$

We remark that our graph of Figure 135 has $v = 8 \geq 3$, thus

$$g = \left\{ \frac{(v-3)(v-4)}{12} \right\} \Longleftrightarrow g = \left\{ \frac{5 \cdot 4}{12} \right\} \Longleftrightarrow g = \left\{ \frac{5}{3} \right\} \Longleftrightarrow$$

$$\Longleftrightarrow g = \{1,(6)\} = 2.$$

Therefore, we proved that the graph of Figure 135 has genus $g = 2$.

7\. Make a table showing the values of v; e; f and g for K_v; for $v = \overline{1,20}$. For what values of v does v exceed g?

<u>Solution</u>: We will remind ourselves Euler's Second Formula:

$v + f - e = 2 - 2g$ (for any graph of genus g, that is connected).

For K_v, we have $e_v = \frac{1}{2} v(v-1)$ and according to Theorem 22: if $v \geq 3$, the complete graph K_v has genus $g = \left\{ \frac{(v-3)(v-4)}{12} \right\}$.

We will use the following formulas for the list:

$$f = 2 - 2g - v + \frac{1}{2} v(v-1)$$

$$e = \frac{1}{2} v(v-1)$$

$$g = \left\{ \frac{(v-3)(v-4)}{12} \right\}; v \geq 3 \quad \text{and} \quad g = 0 \text{ for } v \in \{1;2\}$$

for $v = 1$: $e = 0$; $g = 0$; $f = 1$

for $v = 2$: $e = 1$; $g = 0$; $f = 1$

for $v = 3$: $e = 3$; $g = 0$; $f = 2$

for $v = 4$: $e = 6$; $g = 0$; $f = 4$

for $v = 5$: $e = 10$; $g = 1$; $f = 5$

for $v = 6$: $e = 15$; $g = 1$; $f = 9$

for $v = 7$: $e = 21$; $g = 1$; $f = 14$

for $v=8$: $e=28$; $g=2$; $f=18$

for $v=9$: $e=36$; $g=3$; $f=23$

for $v=10$: $e=45$; $g=4$; $f=31$

for $v=11$: $e=55$; $g=5$; $f=36$

for $v=12$: $e=66$; $g=6$; $f=44$

for $v=13$: $e=78$; $g=8$; $f=51$

for $v=14$: $e=91$; $g=10$; $f=59$

for $v=15$; $e=105$; $g=11$; $f=70$

for $v=16$: $e=120$; $g=13$; $f=80$

for $v=17$: $e=136$; $g=16$; $f=89$

for $v=18$; $e=153$; $g=18$; $f=101$

for $v=19$: $e=171$; $g=20$; $f=114$

for $v=20$: $e=190$; $g=23$; $f=126$

We observe that $v>g$ for $v\in\{1;2;3;4;5;6;7;8;9;10;11;12;13;14;15;16\}$

8. If G is a graph with 994 vertices and 492,753 edges, estimate its genus.

Solution: We are not certainly sure if graph G is connected or not.

If G is a connected graph (and has $v \geq 3$, which it does) then by Corollary 22a: „If G is connected with $v \geq 3$ and genus g, then:

$$\left\{\frac{1}{6}e - \frac{1}{2}(v-2)\right\} \leq g \leq \left\{(v-3)(v-4)\frac{1}{12}\right\}"$$

$e = 492{,}753$ and $v = 994$, therefore:

$$\left\{\frac{1}{6}\cdot 492{,}753 - \frac{1}{2}\cdot 992\right\} \leq g \leq \left\{991 \cdot 990 \cdot \frac{1}{12}\right\}$$

$$81630 \leq g \leq 490545$$

If G is not connected (and has $v \geq 3$, which it does) then by Corollary 22: „If G has $v \geq 3$ and genus g, then: $g \leq \left\{\frac{(v-3)(v-4)}{12}\right\}$.

$e = 492{,}753$ and $v = 994$, therefore:

$$g \leq \left\{991 \cdot 990 \cdot \frac{1}{12}\right\} \iff g \leq 490545 \quad \odot$$

Knowing that $0 \leq g$, we will get (with $\odot$) the double inequality:

$$0 \leq g \leq 490545.$$

9. Prove that a graph for which $L = U$ is a complete graph.
"L" and "U" are defined on page 181.

Solution: On page 181 we defined "L" and "U" as:

$$L = \frac{1}{6}e - \frac{1}{2}(v-2)$$

$$U = \frac{(v-3)(v-4)}{12}$$

$$L = U \iff \frac{1}{6}e - \frac{1}{2}(v-2) = \frac{1}{12}(v-3)(v-4)$$

$$2e - 6(v-2) = (v-3)(v-4)$$

$$e = \frac{1}{2}\left[v^2 - 4v - 3v + 12 + 6v - 12\right]$$

$$e = \frac{1}{2}\left[v^2 - v\right]$$

$$e = \frac{1}{2}v(v-1)$$

We know that a complete graph K_v has $e = \frac{1}{2}v(v-1)$, thus our graph for which $L = U$ is now a complete graph.

L represents the lower bound of a graph of genus g that satisfies Corollary 22 a

U represents the upper bound of a graph of genus g that satisfies Corollary 22 a.

We will remind ourselves what Corollary 22 a says:

Corollary 22 a: If G is connected with $v \geq 3$ and genus g, then:

$$\underbrace{\left\{\frac{1}{6}e - \frac{1}{2}(v-2)\right\}}_{=L} \leq g \leq \underbrace{\left\{(v-3)(v-4)\frac{1}{12}\right\}}_{=U}$$

10. Prove that for any integer $g>1$, $P(3;7) > P(7;3)$; $P(3;7) > P(5;4)$; $P(3;7) > P(4;5)$; $P(3;7) > P(4;6)$, where $P(d;n) = \dfrac{(4g-4)n}{nd-2d-2n}$.

Solution:

$$P(3;7) = \frac{(4g-4)\cdot 7}{21-6-14} = 28(g-1)$$

$$P(7;3) = \frac{(4g-4)\cdot 3}{21-14-6} = 12(g-1)$$

$$P(5;4) = \frac{(4g-4)\cdot 4}{20-10-8} = \frac{16(g-1)}{2} = 8(g-1)$$

$$P(4;5) = \frac{(4g-4)\cdot 5}{20-8-10} = \frac{20(g-1)}{2} = 10(g-1)$$

$$P(4;6) = \frac{(4g-4)\cdot 6}{24-8-12} = \frac{4\cdot 6(g-1)}{4} = 6(g-1).$$

We remark that all of the calculated $P(d;n)$ are multiplies of „$g-1$", thus the biggest coefficient of $(g-1)$ will be the largest out of all the calculated $P(d;n)$. We remark than that: $P(3;7) > P(7;3)$; $P(3;7) > P(5;4)$; $P(3;7) > P(4;5)$; $P(3;7) > P(4;6)$ for any $g>1$. With the last remark we solve our problem.

11. The graph of Figure 126a is nonplanar, so its genus is at least $g > 1$. Prove that its genus is exactly S_1 by drawing it on S_1 without edge-crossings.

<u>Solution</u>: We will draw Figure 126a:

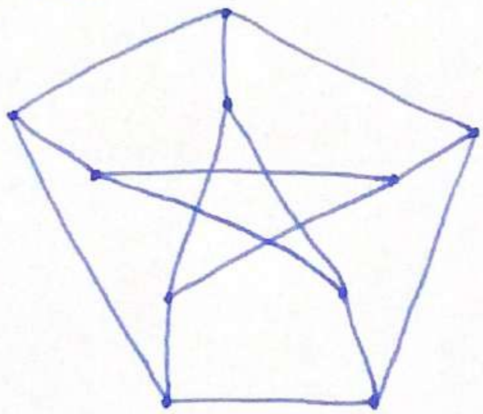
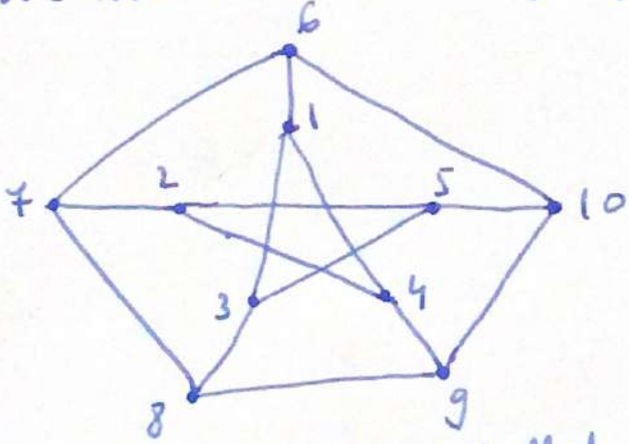
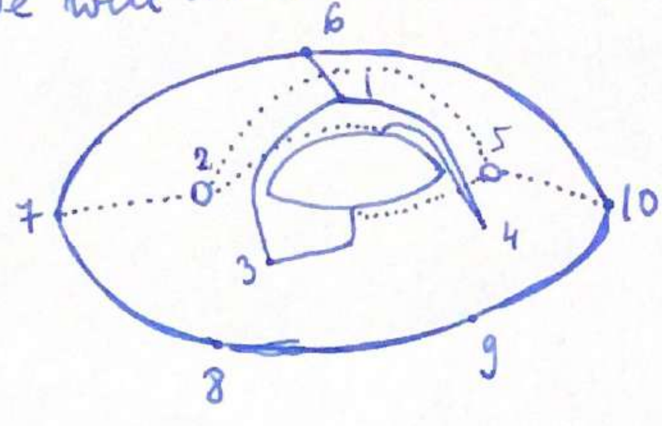


Figure 126a

We will label each vertex for greater clarity:



We will embed the labelled graph on a Torus:

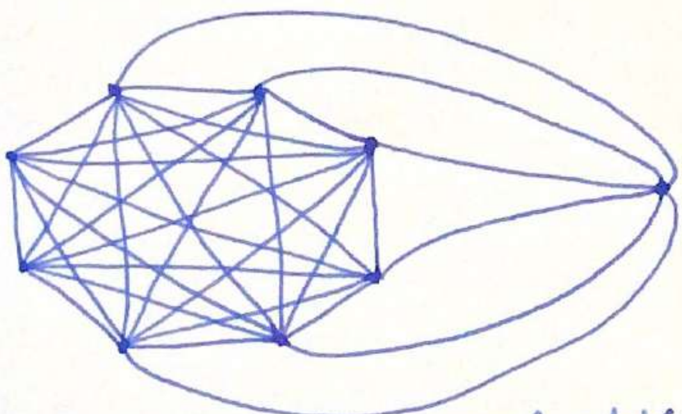


We remark that vertex 2 and vertex 5 are on the lower half of the Torus and the dotted lines are lines situated on the lower half of the Torus. We managed to draw the graph of Figure 126 a without edge-crossing, thus we arrived at the following conclusion: „The graph of Figure 126a has genus g equal to 1".

12\. Prove that the graph of Figure 143 has genus $g=3$.

Solution: We will draw the graph of Figure 143:

Figure143:



We remark the fact that our graph is almost K_9, missing two more edges in order to actually be K_9. Therefore:

$$\begin{cases} v=9 \\ e=\frac{1}{2}\,v\,(v-1)-2=\frac{1}{2}\cdot 8\cdot 9-2=4\cdot 9-2=36-2=34 \end{cases}$$

Because the graph of Figure 143 is connected and $v\geq 3$ and has genus g, we will use Corollary 22a that says: „If G is a connected graph with $v\geq 3$ and genus g then:

$$\left\{\frac{1}{6}e-\frac{1}{2}(v-2)\right\}\leq g\leq\left\{(v-3)(v-4)\cdot\frac{1}{12}\right\}".$$

$e = 34$
$v = 9$
Graph of Figure 143 is connected and $v \geq 3$ $\Big\}$ $\xrightarrow{\text{Corollary 22a}}$

$$\left\{ \frac{1}{6} \cdot 34 - \frac{1}{2} \cdot 8 \cdot 9 \right\} \leq g \leq \left\{ 6 \cdot 5 \cdot \frac{1}{12} \right\}$$

$$3 \leq g \leq 3$$

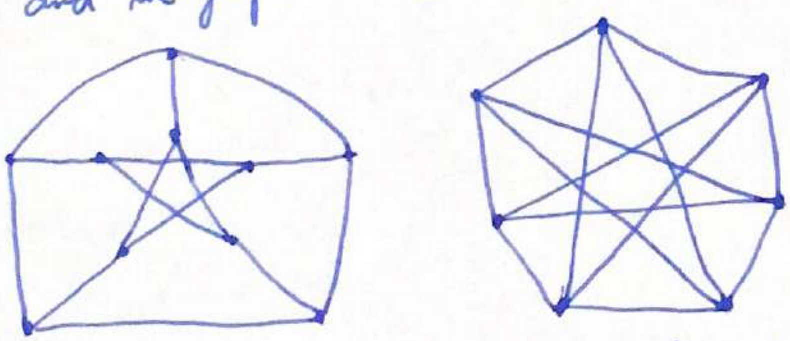
$$3$$

Therefore $g = 3$.

We arrived at the conclusion: "Graph of Figure 143 has genus $g = 3$".

13. Find f for the first two graphs in Figure 126, and for graph in Figure 143

Solution: We will draw the first two graphs in Figure 126 and the graph in Figure 143.



first graph

second graph

Figure 126

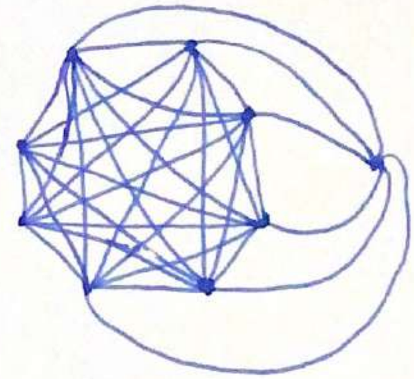


Figure 143

For all three graphs we will use the following steps:
1. Identify how many vertices does the graph have.
2. Identify how many edges does the graph have.
3. Identify the genus of the graph using Corollary 22a
4. Use Euler's Second Formula to find out the number of faces of the graph.

Solution:
For the first graph (of Figure 126) we have: $v = 10$ (by counting)
$e = 14$ (by counting)
$g = 1$ (We proved it at exercise 11).

Therefore, using Euler's Second Formula (because the graph meets the condition of it being connected) we have that:
„For every graph that is connected of genus g : $v+f-e=2-2g$".

$$v+f-e=2-2g \Leftrightarrow f=2-2g-v+e$$
$$f=2-2-10+14$$
$$f=4.$$

We have concluded: „The first graph of Figure 126 has $f=4$".

For the second graph (of Figure 126) we have: $v=7$ (by counting)
$e=14$ (by counting)

We will use Corollary 22a (because our graph G is connected with $v \geq 3$ and genus g) that states the following affirmation:
„If G is a connected graph with $v \geq 3$ and genus g then:

$$\left\{\frac{1}{6}e-\frac{1}{2}(v-2)\right\} \leq g \leq \left\{(v-3)(v-4)\cdot\frac{1}{12}\right\}.$$

Because the second graph is nonplanar (as it visually contains a subgraph of U.G.) it has genus $g > 0$.

Using "Corollary 22a"'s double inequality, we get:

$$\left\{\frac{1}{6}e - \frac{1}{2}(V-2)\right\} \leq g \leq \left\{(V-3)(V-4)\cdot\frac{1}{12}\right\}$$

$$\left.\begin{matrix} e = 14 \\ V = 7 \end{matrix}\right\} \Rightarrow \left\{\frac{1}{6}\cdot 14 - \frac{1}{2}\cdot 5\right\} \leq g \leq \left\{\frac{4\cdot 3}{12}\right\}$$

$$0 \leq g \leq 1$$

So with $0 \leq g \leq 1$ and the fact that $g > 0$, we get:

$g = 1$.

Using Euler's Second Formula (because the graph meets the condition of it being connected) we have that:

„For every graph that is connected of genus g: $V + f - e = 2 - 2g$"

$$V + f - e = 2 - 2g \Leftrightarrow f = 2 - 2g - V + e$$

$$f = 2 - 2 - 7 + 14$$

$$f = 7.$$

We have concluded: „The second graph of Figure 126 has $f = 7$".

For graph of Figure 143 we have: $V = 9$ (by counting)

$e = 34$ (by counting)

$g = 3$ (We proved it at exercise 12)

Using Euler's Second Formula (because the graph meets the condition of it being connected) we have that: „For every graph that is connected of genus g: $V + f - e = 2 - 2g$".

$v+f-e=2-2g \Longleftrightarrow f=2-2g-v+e$

$$f=2-6-9+34$$
$$f=36-6-9$$
$$f=21$$

We have concluded: „The graph of Figure 143 has $f=21$".

Final answers: •) „The first graph of Figure 116 has $f=4$"

•) „The second graph of Figure 116 has $f=7$"

•) „The graph of Figure 143 has $f=21$"

14. If a graph has 18 edges and 7 vertices, what is its genus?

If a graph has 52 edges and 11 vertices, what is its genus?

<u>Solution</u>: Because of the <u>number</u> of edges (which is big) in each of the two graphs (the graph with 18 edges and 7 vertices and the graph with 52 edges and 11 vertices) we can deduce that both graphs are connected. While finding the exact value of g is possible, we will arrive at an estimation of g using

<u>Corollary 22a</u>: „If G is connected with $v \geq 3$ and genus g, then: $\left\{\frac{1}{6}e-\frac{1}{2}(v-2)\right\} \leq g \leq \left\{(v-3)(v-4)\cdot\frac{1}{12}\right\}$.

for: $v=7$ and $e=18$, we get:

$$\left\{\frac{1}{6}\cdot 18-\frac{1}{2}\cdot 5\right\} \leq g \leq \left\{4\cdot 3\cdot\frac{1}{12}\right\}$$

$$1 \leq g \leq 1 \Longrightarrow g=1$$

Therefore, we conclude: „For a graph with 18 edges and 7 vertices $g=1$".

for : $V=11$ and $e=52$, we get:

$$\left\{\frac{1}{6}\cdot 52-\frac{1}{2}\cdot 9\right\}\leq g\leq\left\{\frac{8\cdot 7}{12}\right\}$$

$$5\leq g\leq 5\implies g=5$$

Therefore, we conclude: „For a graph with 52 edges and 11 vertices $g=5$"

Final answers: •) „For a graph with 18 edges and 7 vertices $g=1$"

•) „For a graph with 52 edges and 11 vertices $g=5$"

15. Prove: „If a graph is connected of genus g and the boundary of every face is K_3, then $e=3(V-2+2g)$"

Solution: Because the boundary of each face is K_3 our graph will have each edge bordering two distinct faces. We will use Lemma 21: „If a connected graph G has $V\geq 3$ and genus g then $3f\leq 2e$" with the remark that if each edge borders two distinct faces then $3f=2e \iff f=\frac{2}{3}e$.

Because graph G is connected (having genus g) we can use Euler's Second Formula: „If G is connected then $V+f-e=2-2g$".

$$V+f-e=2-2g \iff V+\frac{2}{3}e-e=2-2g$$

$$-\frac{1}{3}e=2-2g-V$$

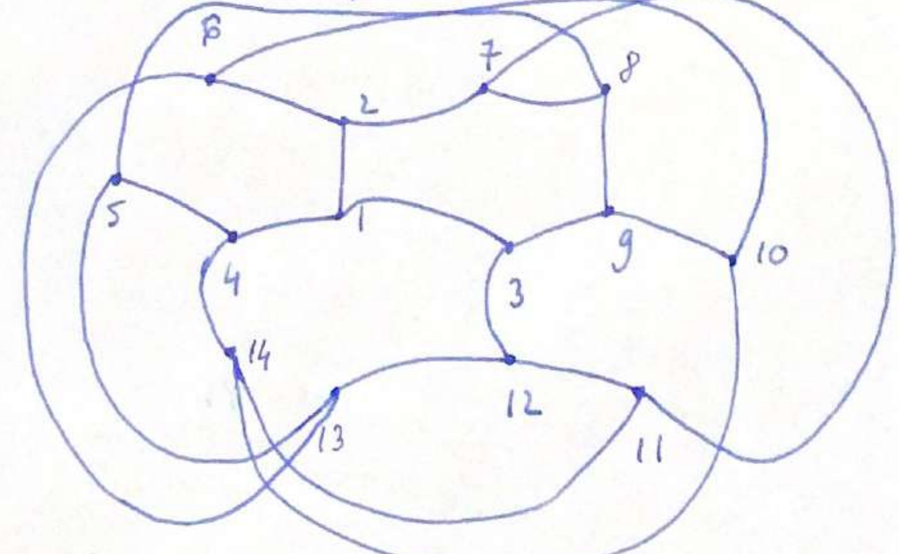
$$e=3(V-2+2g)$$

We have concluded: „If a graph is connected of genus g and the boundary of every face is K_3, then $e=3(V-2+2g)$".

Attention: The remark that $3f=2e$ only if each edge borders two distinct faces derives from the fact that the maximum number of edges occurs when all edges border two distinct faces from which we can deduce that the maximum number of edges will create the maximum number of faces which will thus create the equality: $3f = 2e$".

"16. Prove that the graph L of Figure 140 is 1-platonic with $d=3$ and $n=6$.

Solution: We will draw graph L of Figure 140:



We will remind ourselves what a g-platonic graph is. According to Definition 32: "A graph is g-platonic if it is connected, regular graph of genus g such that every edge borders two distinct faces and every face is bounded by the same number of edges".

Counting the number of vertices and edges we arrive at the following values (for v and e): $v = 14$
$e = 21$

Because our graph G is g-platonic we will use Lemma 24 that states the following: „If G is g-platonic, then $e = \frac{d \cdot V}{2}$ and $f = \frac{d \cdot V}{n}$" . n = the number of edges (that is constant in case of g-platonic graphs, because g-platonic graphs impose the condition that every face is bounded by the same number of edges) that borders each distinct face.

We have that $e = 21$; $V = 14$; thus $d = \frac{2e}{V} \Leftrightarrow d = \frac{2 \cdot 7 \cdot 3}{2 \cdot 7} \Leftrightarrow$

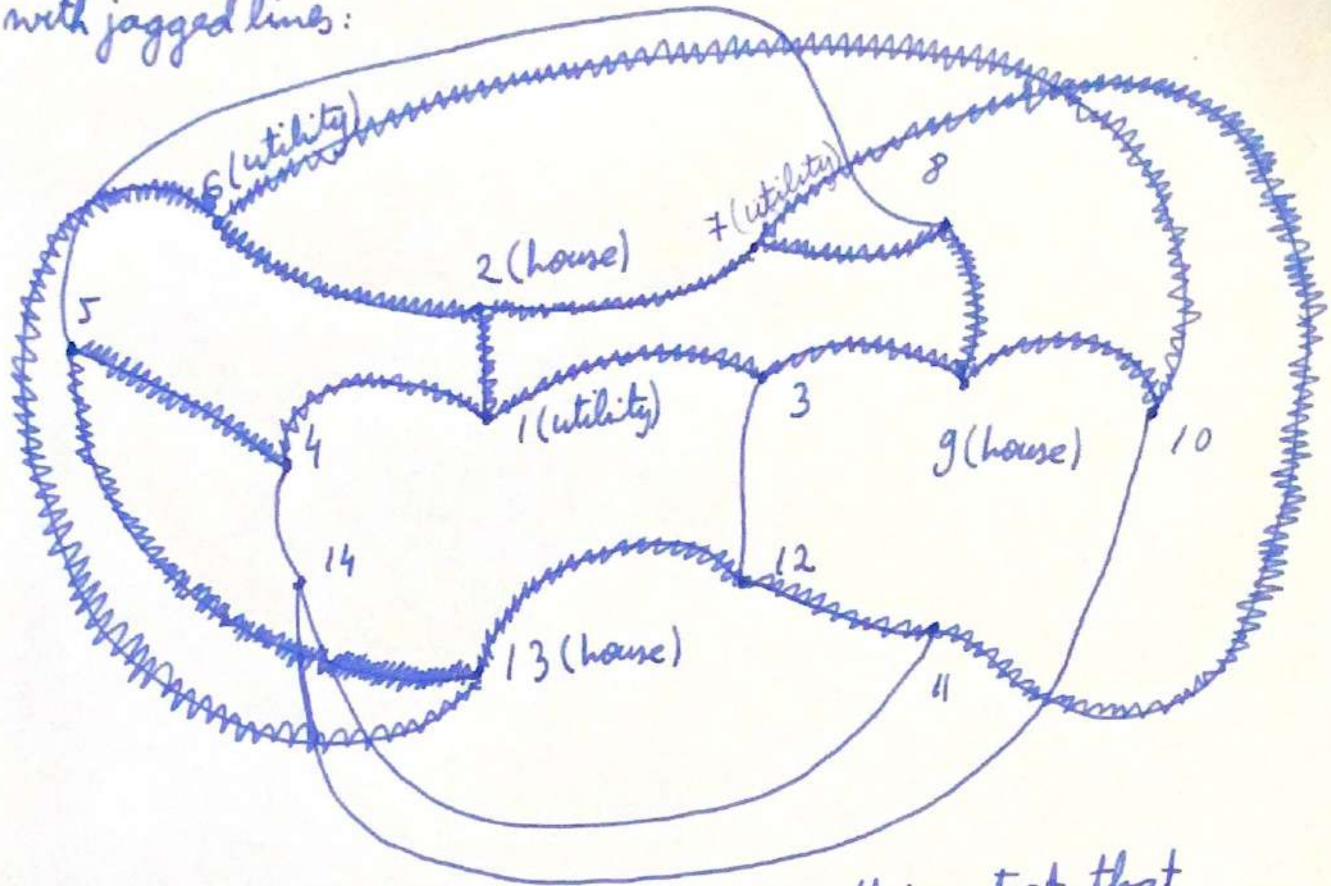
$d = 3$.

We have concluded: „The graph L of Figure 140 has $d = 3$".

Because graph L is connected, and of genus g we will also be able to use Euler's Second Formula (which we will use after finding the genus g).

We will prove that the graph L of Figure 140 is nonplanar (which will tell us that $g > 0$). We remark that by fixing the vertices 2; 9; 13 (which represent the three houses in U. G., the utility graph) and by fixing 6; 7; 1 (which represent the three utility companies in U. G., the utility graph), we can create a distinct path from each of the three houses to each of the three utility companies in a manner that each path won't overlap with each other, thus obtaining a subgraph of graph L in Figure 140 that is an expansion of U. G..

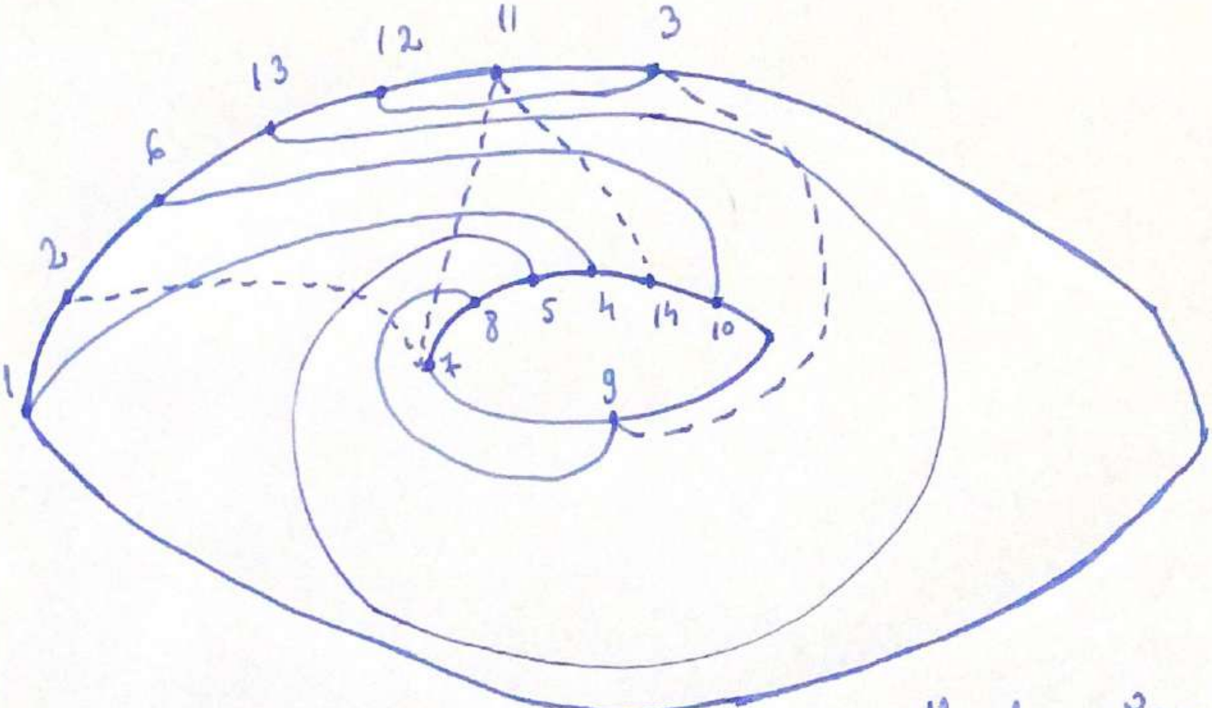
We will mark each path from each house to each utility company with jagged lines:



Remark: „Non-overlapping paths" means in this context that between each possible path (between the houses and the utility companies) there doesn't exist any common vertex and edge, each path having unique combinations of vertices and edges. The jagged lines represent a subgraph of graph L of Figure 140 that is also an expansion of U. G.. Using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either U. G. or K_5, is nonplanar", we conclude that:

„Graph L of Figure 140 has genus $g>0$".

We will try to embed a graph L of Figure 140 into a Torus to verify "graph L of Figure 140 has genus $g=1$" (because a Torus is a genus $g=1$ surface), by trying to draw L without edge-crossing.



We remark that graph L (embedded in the drawn Torus) is now drawn without edge-crossing, making it have genus $g=1$.

We conclude that: "Graph L of Figure 104 has $g=1$"

We are now ready to use Euler's Second Formula in order to find out the number of faces our graph L (embedded in the drawn Torus).

$$v+f-e=2-2g \Leftrightarrow f=e-v \Leftrightarrow f=21-14 \Leftrightarrow f=7.$$

Because our graph L of Figure 140 satisfies Lemma 24 ("If G is a graph that is g-platonic, then $e = \frac{dv}{2}$; $f = \frac{dv}{m}$), we will have

$f = \frac{dv}{m} \Leftrightarrow m = \frac{dv}{f} \Leftrightarrow m = \frac{3 \cdot 2 \cdot 7}{7} \Leftrightarrow m = 6$.

We conclude: „Graph L of Figure 140 is 1-platonic with $d=3$ and $m=6$".

17. Prove that K_7 is 1-platonic with $d=6$ and $m=3$.

Solution: We know that K_7 is nonplanar (because it contains a subgraph of an expansion of K_5 and by using Kuratowski's Subgraph Theorem: „A graph that has a subgraph of an expansion of either U.G. or K_5, is nonplanar", we conclude: „K_7 is nonplanar)

Using Exercise 2: „Make a crossing-free drawing of K_7 on S_1 and count its faces. Use Euler's Second Formula to verify your count", we proved K_7 has genus $g=1$ and $f=7$.

It is clear (from the fact that the degree of every vertex of K_7 is one numbers less than the number of vertices) that $d=6$.

From the drawing in Exercise 2, we have that:
K_7 has genus $g=1$; is connected; has every edge bordering exactly two (distinct) faces; has every face bounded by the same number of edges, thus we conclude: „K_7 is 1-platonic".

Using Lemma 24 („If G is g-platonic then $e = \frac{dv}{2}$ and $f = \frac{dv}{m}$"), and knowing $e = \frac{1}{2} \cdot 6 \cdot 7 \Leftrightarrow e = 21$; $v=7$; $d=6$; $f = 14$

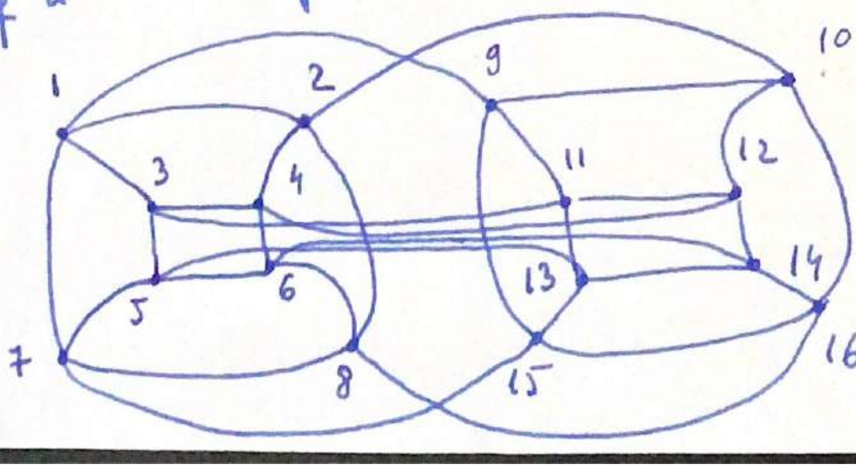
we will find „m" (the constant number of edges that borders each face):

$$n = \frac{dv}{f} \Leftrightarrow n = \frac{6 \cdot 7}{14} \Leftrightarrow n = 3.$$

We conclude: „K_7 is 1-platonic with $d=6$ and $n=3$".

18. The 1-platonic graph Q of Figure 141a is the skeleton of a „toroidal polyhedron", i.e. a polyhedron which, if inflated, would look like S_1. The „toroidal polyhedron" has been drawn in Figure 144; it look like a cube through which a square hole has been made, causing the top and bottom of the cube to collapse a bit around the hole. K_7 is also the skeleton of a toroidal polyhedron, called „Császár polyhedron"; read Martin Gardner's article about Császár polyhedron in the „May, 1975, Scientific American" and use his pattern to make a cardboard model of it.

<u>Solution</u>: For visualization purposes we will draw graph Q of Figure 141a, we will also draw the graph of Figure 144 and provide a thorough manual of instructions for the construction of a Császár polyhedron.



Graph Q of Figure 141

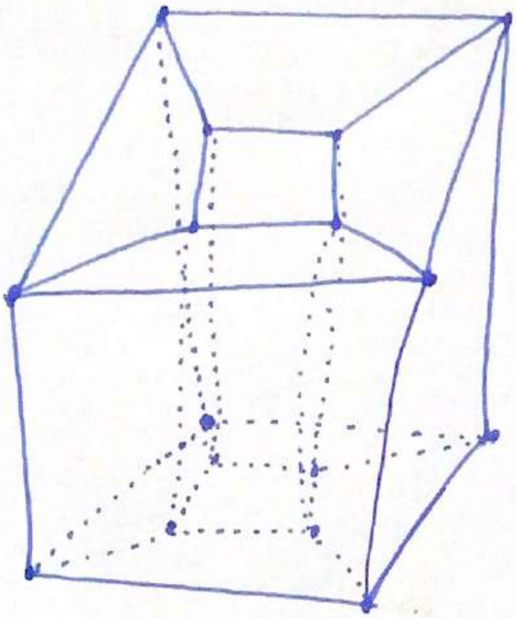
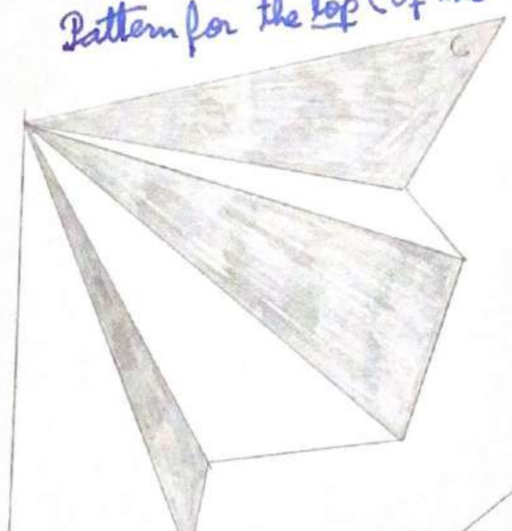


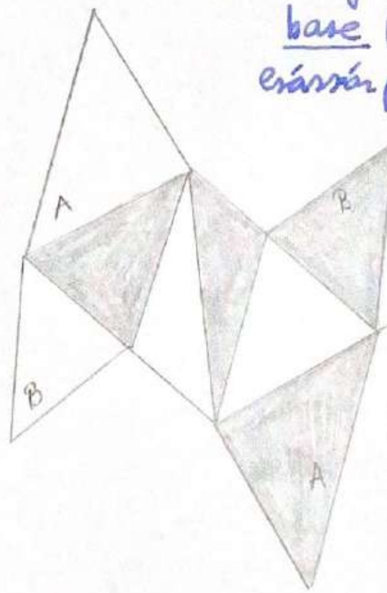
Figure 144

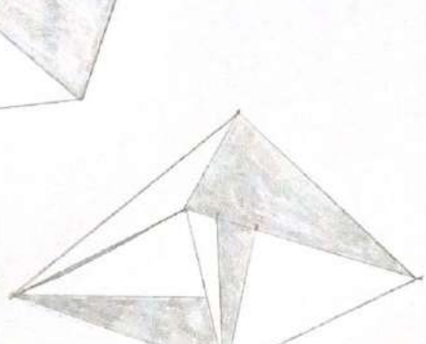
We will cite from „Mathematical Games" by Martin Gardner (from „May 1975, Scientific American":

Pattern for the top (of the Császár polyhedron)

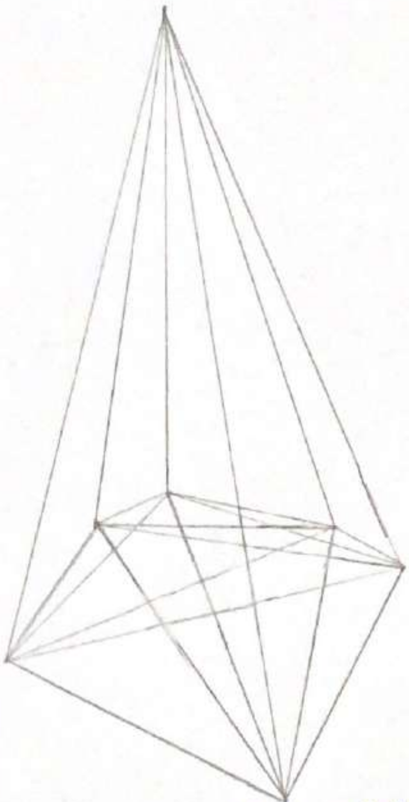


Pattern for the base (of the Császár polyhedron)





Completed base (of the Császár polyhedron)



The completed polyhedron

<u>Steps</u>:

<u>Preliminary Steps</u>: ·) Copy the two following patterns preserving the chromatic displayed originally (white-black):

„Pattern for the <u>top</u> (of the Császár polyhedron)",

„Pattern for the <u>bottom</u> (of the Császár polyhedron)".

·) After drawing them (on a piece of paper), cut the two patterns („Pattern for the top"; „Pattern for the bottom") then ondulate the two patterns along each dotted line to create „<u>mountain folds</u>" [which look like this: (from the original paper, which looks like this:)] and for non-dotted lines create „valley folds" [which look like this: (from the original paper, which looks like this:)].

Actual Steps:

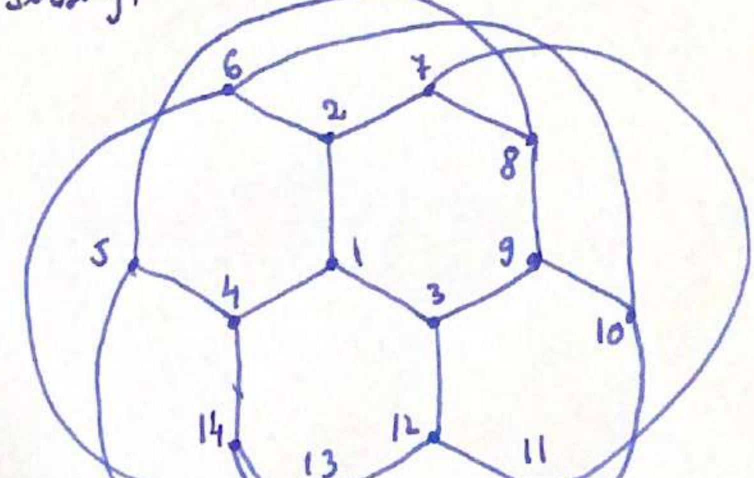
1. With the pattern for the base, fold the two largest triangles to the center and tape the A edges together. Turn the paper over. Fold the two smaller triangles to the center and tape the B edges together to obtain the „Completed base (of the Császár polyhedron)"

2. The six-faced conical top is formed by taping the C edges together. Place it on the base as shown in the drawing of „The completed polyhedron". It will fit in two ways. Choose the fit that joins triangles of opposite colors, then tape each of its six edges of the base.

We are done constructing the „Császár polyhedron".

19. Is the 1-platonic graph L of Figure 140 also the skeleton of a toroidal polyhedron? If so, draw it, otherwise explain why not.

Solution: Firstly, we will draw graph L of Figure 140:



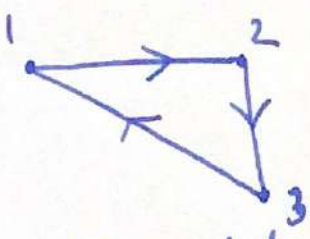
The toroidal nature of a polyhedron can be verified by its orientability and by its Euler characteristic (which is denoted by „χ" and which is equal to $v-e+f$) being <u>non-positive</u> (which means less than (or equal to) 0).

We will define what orientable graphs mean.

Firstly, we will define what a digraph is.

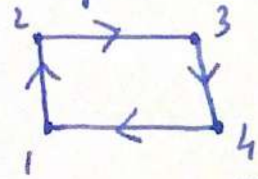
<u>Digraph</u>: A graph that has a direction (and where the edges $\{A;B\} \neq \{B;A\}$)

An example of such graph is the following:



Now we can define what an orientable graph is.

<u>Orientable Graph</u>: A digraph (which we will name with „D") is said to be orientable if D is strongly connected [„strongly connected" means that exists a <u>directed path</u> (<u>a path with a direction</u>) between each pair of vertices]. An example of an orientable graph is:



. We can remark that from each vertex (if we follow a directed path, a corresponding one) we can arrive at all other vertices.

In terms of the orientability of graph L of Figure 140, we notice that no matter how many attempts [of making graph L of Figure 140 strongly connected (which means having a directed path between each pair of vertices)] we will fail, thus graph L of Figure 140 is not a skeleton of a toroidal polyhedron (that is because the toroidal polyhedra needs to satisfy the orientability condition: our graph to be strongly connected, which it fails to be). We conclude: „ The 1-platonic graph L of Figure 140 is not a skeleton of a toroidal polyhedron due to its (referring to graph L of Figure 140) nonorientable nature".

20. There are infinitely many 1-platonic graphs. Prove this by describing and drawing the first few members of an infinite family of 1-platonic graphs having $d = 4$ and $n = 4$.

Solution: For an explicit solution, we will define a polygon, a polyhedron and a cyclic polygon.

Polygon = a flat, two-dimensional closed shape bounded by straight lines

Cyclic polygon: a polygon whose vertices (all of them) sit on one single circle.

Polyhedron : a 3-dimensional closed shape with polygonal faces.

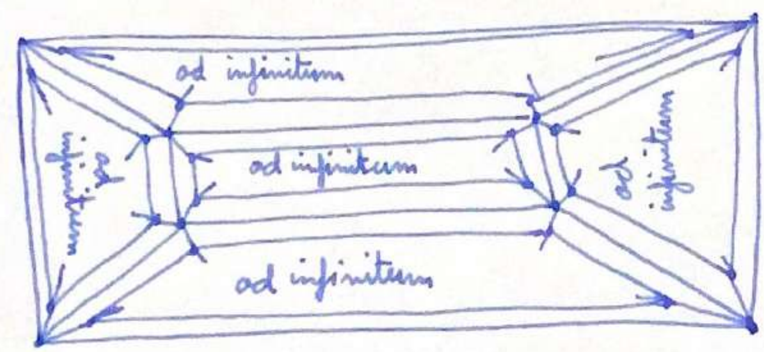
We know that g-platonic graphs (for $g \geq 0$) represent a polyhedron.

We will suggest the following approach:

We will start by stating Theorem 25.

Theorem 25: All 1-platonic graphs have either $d=3$ and $n=6$, or $d=4$ and $n=4$; or $d=6$ and $n=3$.

A simpler way to illustrate an infinite family of 1-platonic graphs having $d=4$ and $n=4$ is with the following graph:



This graph (which continues forever) contains an infinite family of 1-platonic graphs having $d=4$ and $n=4$.

We conclude: „There are infinitely many 1-platonic graphs".

21. A graph has $V = 12$ and $X = 10$ [where X represents the chromatic number, which is the number equal to the least amount of (legally) colours a graph can have]. Prove that its genus is no less than 4 and no more than 6.

<u>Solution</u>: According to the „Heawood Coloring Theorem":

„If $n>0$, then $X(S_n) = \left[\frac{7+\sqrt{1+48n}}{2}\right]$."

According to „Theorem 28": „$X(S_0)$ is either 4 or 5."

We then conclude: „$n>0$", thus we can apply the chromatic number formula of a surface S_n (with n-holes).

$$X(S_g) = 10$$
$$X(S_g) = \left[\frac{7+\sqrt{1+48g}}{2}\right]$$

Using <u>Definition 33</u>: If $n \geq 0$ is an integer then the chromatic number of surface „S_n" (<u>which means: „surface with n-holes"</u>) denoted „$X(S_n)$" is the largest among the chromatic numbers of graphs having $g \leq n$.

If $g \leq 3$ we will check $g=3$ and if $X(S_g) < 10$, we will deduce that for $g \leq 3$ we will have $X(S_g) < 10$, which contradicts the fact that $X(S_g) = 10$.

$$X(S_g) = \left[\frac{7+\sqrt{1+48n}}{2}\right] \xleftrightarrow{\text{by picking: } g=3\text{"}} X(S_{g_1}) = \left[\frac{7+\sqrt{145}}{2}\right]$$

$\Rightarrow X(S_g) = \left[\frac{7+11,04}{2}\right] = \left[\frac{19,04}{2}\right] = [9,52] = 9.$

Therefore $X(S_g) < 10$ for $g \leq 3$. We conclude:

„$g \geq 4$" ⊙

„Using Corollary 22: If G has $v \geq 3$ and genus g, then:

$g \leq \left\{\frac{(v-3)(v-4)}{12}\right\}$"

$v = 12 \Rightarrow g \leq \left\{\frac{9 \cdot 8}{12}\right\} \Leftrightarrow g \leq \left\{\frac{3 \cdot 3 \cdot 4 \cdot 2}{12}\right\} \Leftrightarrow$ „$g \leq 6$." ⊙⊙

From ⊙ and ⊙⊙ we get: $4 \leq g \leq 6$.

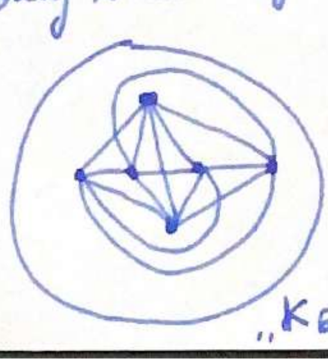
$g \geq 4$ $\quad$ $g \leq 6$

We conclude: „A graph with $v = 12$ and $X = 10$ has its genus g:

$4 \leq g \leq 6$".

22. Definition: The minimum number of edge-crossings with which a graph can be drawn in a plane is called the crossing number of the graph, denoted „k".

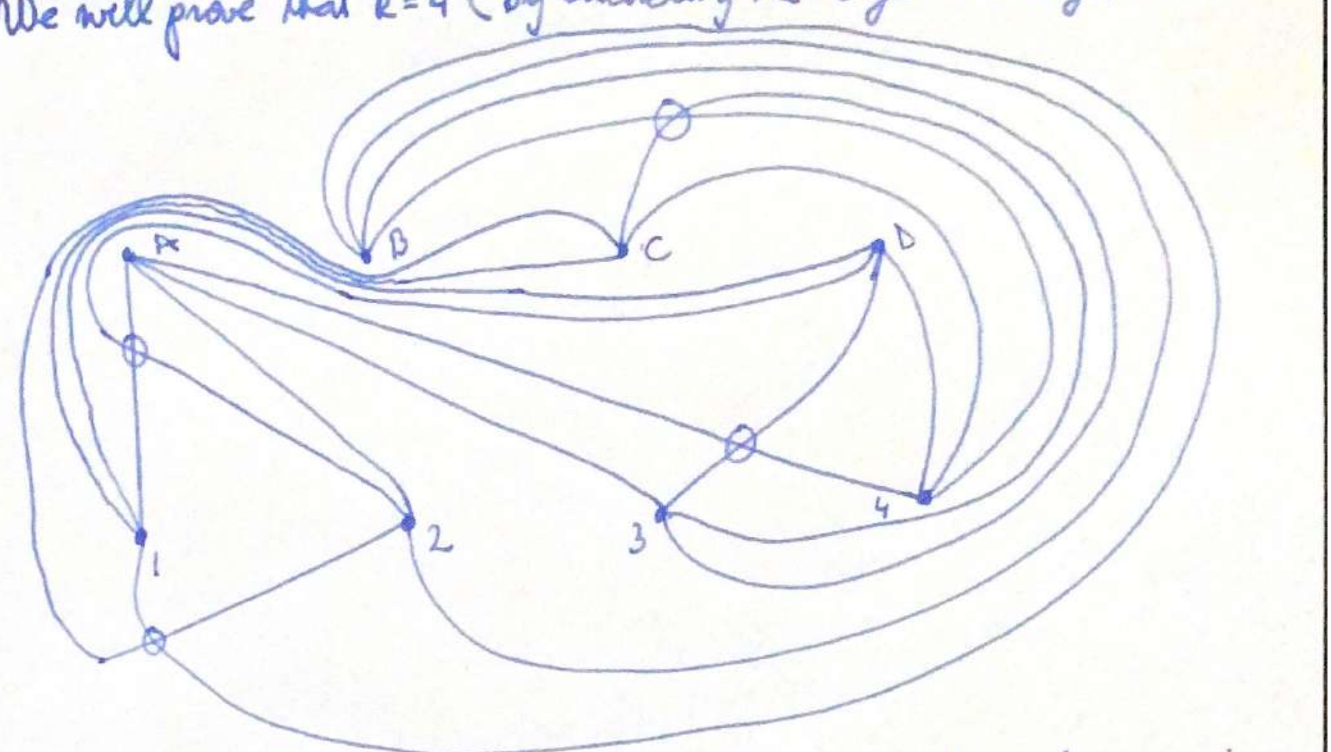
Examples: Every planar graph has $k = 0$. K_5 and U. G. (the utility graph) have $k = 1$. K_6 has $k = 3$; it has been drawn with only three edge-crossings in Figure 130a.



„K_6 on S_0"

Figure 130 a.

We will prove that $k=4$ (by encircling the edge-crossings):



As we observe $k=4$ as there are a minimum of 4 edge-crossings

The graph is nonplanar [as it contains a subgraph, namely:

$G_1 \subseteq \underbrace{G}_{\text{the original graph}}$ where $G_1 = \{ \underbrace{A;B;C}_{\text{vertexset}}; \underbrace{\{A;1\};\{A;2\};\{A;3\};\{B;1\};\{B;2\};\{B;3\};\{C;1\};\{C;2\};\{C;3\}}_{\text{edge set}}$

and G_1 represents actually U. G. (which is an expansion of itself)

Using „Kuratowski's Subgraph Theorem: A graph that has a subgraph of an expansion of either U. G. or K_5, is nonplanar", we conclude: „Graph of Figure 104b is nonplanar". Therefore $g>0$. We will check if $g=1$ by trying to draw the original graph of Figure 104b without edge-crossings on a genus $g=1$ surface, namely, a torus.

Prove that for every graph $g \le k$.

Solution: When we consider a graph G on a Surface „S_g" (a surface with genus g) and we have k crossings (k being the number of minimum edge-crossings a graph can have) for graph G, we can easily construct a handle for each edge crossing (thus obtaining the equality $g = k$). We remark that a handle can contain more than one edge-crossing (thus obtaining the inequality $g \le k$).

We conclude: „Every graph has $g \le k$".

23. The graph of Figure 104b has $k = 4$. Draw it in a plane with only four edge-crossings. Then prove it has genus $g = 1$

Solution: Firstly, we will draw the graph of Figure 104b.

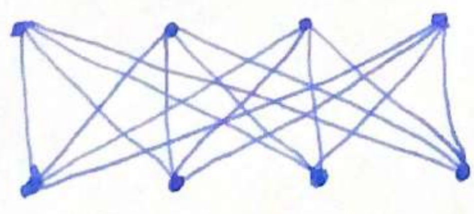
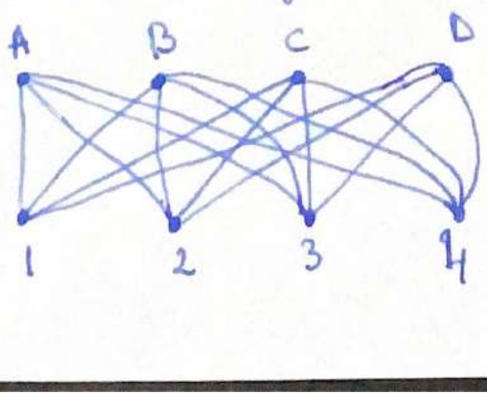
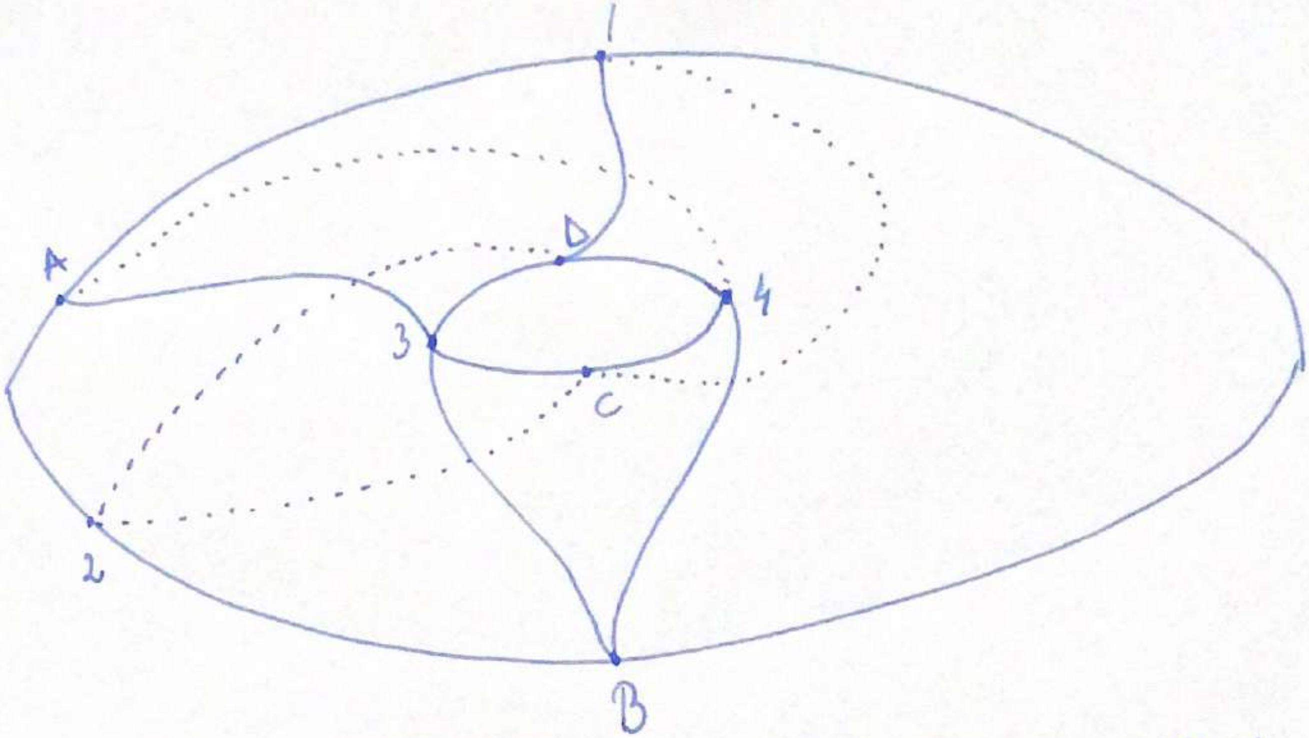


Figure 104 b

We will label each vertex for clarity:





We managed to draw the original graph of Figure 104b without edge-crossings on a surface with genus $g=1$, thus our graph of Figure 104b has genus $g=1$.

We conclude: „The graph of Figure 104b has $k=4$ and genus $g=1$".

24. Find the genus and crossing number of the Peterson graph (Figure 88)

<u>Solution</u>: We will draw the Peterson graph first:

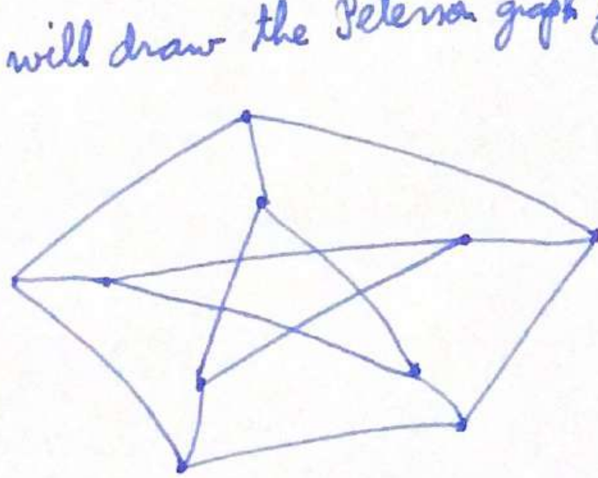
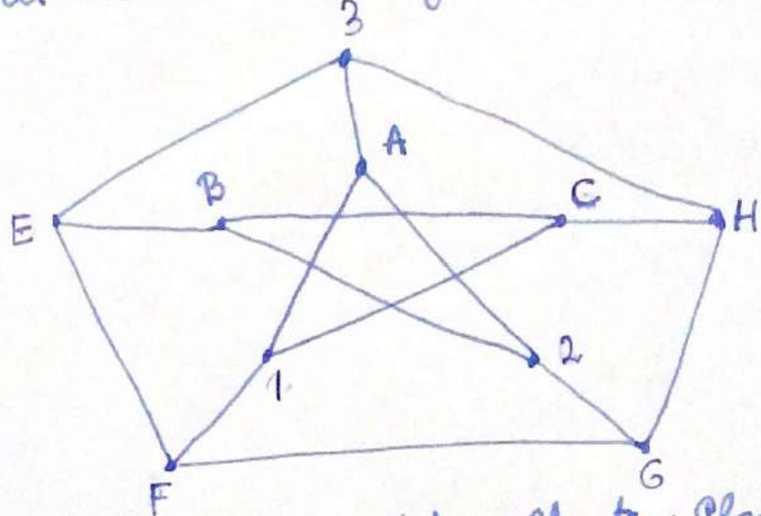
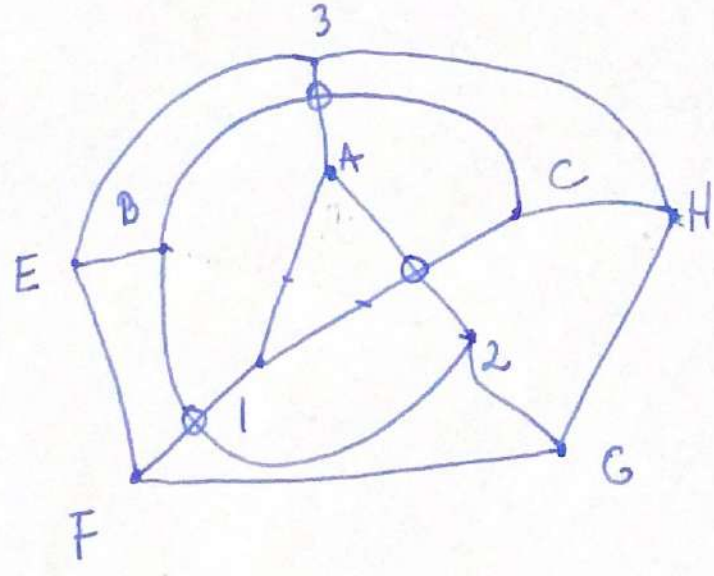


Figure 88.

We will label each vertex for clarity:

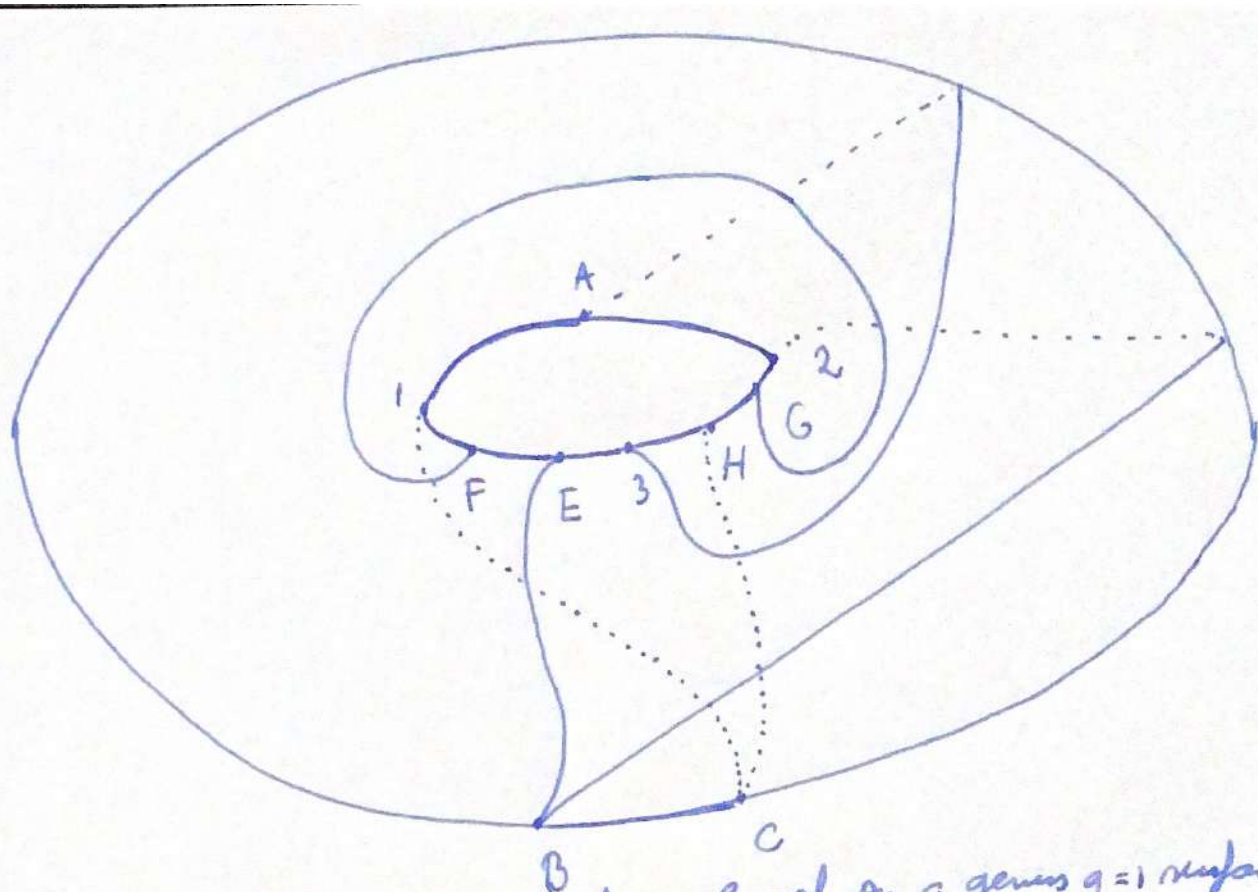


We proved at Exercise 7 (from Chapter: "Planar Graphs") that the Petersen Graph is nonplanar, thus $g > 0$. In order to find out k, we will slightly change the graphs edges position, namely, $\{B; C\}$ and $\{1; C\}$:



We remark that $k = 3$.

We will check if the Petersen Graph has genus $g = 1$, by trying to embed it (without edge-crossing) into a surface of genus $g = 1$, namely, a torus.



We managed to draw the Petersen Graph on a genus $g=1$ surface, namely a Torus, thus the Petersen Graph has genus $g=1$. We conclude: „The Petersen Graph has $k=3$ and genus $g=1$".

25. In the course of drawing K_V, where $V \geq 3$, what is the maximum number of edges that can be drawn without an edge-crossing? (For K_5, the answer is 9; for K_6, the answer is 12). Though this question can be answered with a simple formula (once you find it, by the way, don't forget to prove that it is correct), no one has found yet a formula for the crossing number k of K_V. This is because there doesn't seem to be any way of predicting how many crossings will be caused when the remaining edges are drawn; for example K_{10} has 45 edges, 24 of which can be drawn without edge-crossings, but the remaining 21 edges will cause k to be equal to 60 crossings.

Solution: Using Theorem 11 (which has been proven in the book): "If G is a planar, connected graph with $V \geq 3$ then $\frac{3}{2} f \leq e \leq 3V-6$."

Proof: Let G be planar and connected with three or more vertices.

Case 1: G has a face bounded by fewer than three edges. Then a little reflection will reveal that G must be the path graph P_3 shown in Figure 101a. For P_3: $V=3$, $f=1$, $e=2$. Hence: $\frac{3}{2} f = \frac{3}{2}$, $e=2$, $3V-6=3$ and the theorem ($\frac{3}{2} f \leq e \leq 3V-6$ for planar, connected graphs with $V \geq 3$).

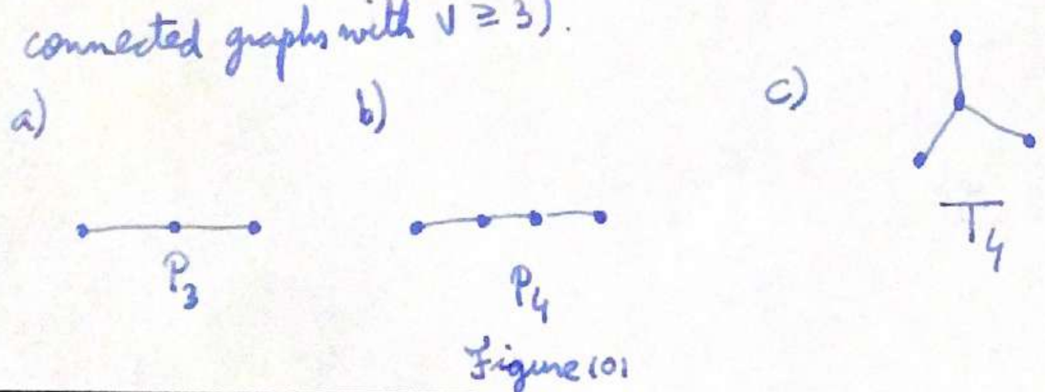


Figure 101

Case 2: Every face of G is bounded by three or more edges. Then numbering the faces of G from 1 to f we can make a series of statements:

$$\odot \begin{cases} 3 \leq \text{the number of edges bounding } 1 \\ 3 \leq \text{the number of edges bounding } 2 \\ \vdots \\ 3 \leq \text{the number of edges bounding } f \end{cases}$$

Hence $3f$, the sum of the first column, is less than or equal to the sum of the second column, which we can denote by „D". In shorthand, $3f \leq D$. If G were polygonal D would be equal to $2e$ because every edge of a polygonal graph borders on exactly two (distinct) faces and so each edge would have been counted exactly twice in the second column, once for each of the two faces it borders. If G were not polygonal D would be smaller than $2e$, because planar-connected-nonpolygonal graphs contain one or more edges that border on only one face, and such edges would have been counted only once in the second column. If G were polygonal, we would have $D = 2e$, whereas (which means „in contrast") if G were not polygonal, we would have $D < 2e$. In either case we can safely say that $D \leq 2e$. Combining „$D \leq 2e$" with the fact that „$3f \leq D$" (from $\odot$) we conclude: $3f \leq 2e \Leftrightarrow$ $\Leftrightarrow$ „$\frac{3}{2} f \leq e$".

From $\frac{3}{2}f \le e$, we multiply both sides by "$\frac{2}{3}$" to get: "$f \le \frac{2}{3}e$". We will add "$v-e$" to both sides and "the result is":

"$v+f-e \le v+\frac{2}{3}e-e$". G satisfies the hypothesis of Euler's Formula, so: "$v+f-e=2$" (if G is planar and connected). Therefore we get: $2 \le v-\frac{1}{3}e \Leftrightarrow$ "$e \le 3v-6$" which combined with: "$\frac{3}{2}f \le e$" will get us: "$\frac{3}{2}f \le e \le 3v-6$ for a planar graph G with $v \ge 3$, that is connected". Having proved Theorem: "If G is planar and connected with $v \ge 3$ then $\frac{3}{2}f \le e \le 3v-6$", we conclude:

"For K_v where $v \ge 3$ the maximum number of edges that can be drawn without edge-crossing is $e = 3v-6$".

26. Prove the following analog of Corollary 13 (which states: "If G is planar then G has a vertex of degree less than (or equal to) 5")
which is: If a graph is nonplanar (by having $g=1$), then at least one vertex has degree ≤ 6.

Solution: If we are talking about disconnected graphs, then certainly the disconnected vertex has degree less than or equal to 6.
If we are talking about connected graphs (with $g=1$), we will look at connected graphs with $V \geq 3$ since there doesn't exist any graph that is nonplanar (with $g=1$) wit $V<3$.

Using Corollary 22: "If G has $V \geq 3$ and genus g then:
$g \leq \left\{\frac{(V-3)(V-4)}{12}\right\}$; where $\{x\}_{x \in \mathbb{R}} = \min\{n \in \mathbb{Z} \mid n \geq x\}$ and
$\{x\} = x$ if and only if $x \in \mathbb{Z}$."

$g=1 \Rightarrow 1 \leq \left\{\frac{(V-3)(V-4)}{12}\right\} \Rightarrow V \geq 5$ (because for $V=5$, we get: $1 \leq \left\{\frac{2 \cdot 1}{12}\right\} \Leftrightarrow 1 \leq 1$, which is correct.)

Using "Theorem 21: If G is connected with $V \geq 3$ and genus g then: $g \geq \frac{1}{6}e - \frac{1}{2}(V-2)$"

$g=1 \Rightarrow 1 \geq \frac{1}{6}e - \frac{1}{2}(V-2) \Leftrightarrow \frac{1}{6}e \leq 1 + \frac{V}{2} - 1$
$\Leftrightarrow$ "$e \leq 3V$"

We will multiply "$e \leq 3V$" with "2" on both sides, resulting in: $2e \leq 6V$. We know that the sum of the degrees of all vertices in any graph is: $\sum_{i=1}^{m} \deg(V_i) = 2e$; where m = the last subscript of the last counted vertex. Therefore, we get: $\sum_{i=1}^{n} \deg(v_i) \leq 6V$.

We also know that $V \geq 6$, so for $V=6$, we get:

$\sum_{i=1}^{n} \deg(v_i) \leq 36$.

Because $V=6$ there can be a maximum of $e = \frac{1}{2}V(V-1)$ edges which is equivalent to saying: $e = 15$. We know that $\sum_{i=1}^{n} \deg(v_i) = 2e$

$\Leftrightarrow \sum_{i=1}^{n} \deg(v_i) = 30 \leq 36$, thus we can conclude:

"If a graph is nonplanar (by having $g=1$), then at least one vertex has degree ≤ 6".

Page 215-221

1. Definition: A vertex of a graph is odd if its degree is an odd number

Use Exercise 11 of Chapter 2 to prove that every graph has an even number of odd vertices.

Solution: At Exercise 11 of Chapter 2 („Graphs") we proved that the sum of all degrees (of all vertices) in a graph is equal to $2e$. Formally: $\sum_{i=1}^{m} deg(V_i) = 2e$. We will divide: $\sum_{i=1}^{m} deg(V_i)$ into $\sum_{i=1}^{n} deg(V_i)$ and $\sum_{i=n+1}^{m} deg(V_i)$ where $n < m$.

$\sum_{i=1}^{n} deg(V_i)$ will represent the sum of all degrees (of all vertices) that are even.

$\sum_{i=n+1}^{m} deg(V_i)$ will represent the sum of all degrees (of all vertices) that are odd.

$\sum_{i=1}^{m} deg(V_i) = 2a$ (since we are counting even degrees) where

$2a < 2e \Leftrightarrow a < e$

We know that: $\sum_{i=1}^{m} deg(V_i) = \sum_{i=1}^{n} deg(V_i) + \sum_{i=n+1}^{m} deg(V_i) = 2e$

$\Leftrightarrow \sum_{i=n+1}^{m} deg(V_i) = 2e - 2a \Leftrightarrow \sum_{i=n+1}^{m} deg(V_i) = 2(e-a)$. Since

$2 \mid \sum_{i=n+1}^{m} deg(V_i)$ and $\sum_{i=n+1}^{m} deg(V_i)$ represents the sum of all degrees (of all vertices that are odd), we get that: „Every graph has an even number of odd vertices, where odd vertices represent the odd-valued degrees of the corresponding vertices ".

2. Find all integers $v \geq 2$ for which:

a) K_v has an open euler walk

b) K_v has a closed euler walk

c) K_v has an open hamiltonian walk

d) K_v has a closed hamiltonian walk.

<u>Solution</u>: For „a) K_v has an open euler walk" we will
use Theorem 31 („If a connected graph has an euler walk then it
has exactly two odd vertices. Conversely, if a connected graph
has exactly two odd vertices then that particular graph has
an open euler walk"). <u>K_v</u> is connected and every available
vertex of <u>it</u> has an equal-valued degree, thus for K_v to have
exactly two odd-valued degrees (of two vertices) we can only
look at K_2. Graphically it would look like this:

K_2

. We observe that K_2 has exactly two odd-valued
degrees (of two vertices), namely „1" and „2". Therefore we
conclude: „K_2 is the only connected, complete graph that has an
open euler walk". In other words: „For $v=2$; K_v has an
open euler walk".

For „b) K_V has a closed euler walk" we will use Theorem 30 („If a connected graph has a closed euler walk, then every vertex has an even-valued degree. Conversely, if a graph is connected and every vertex has an even-valued degree, then it has a closed euler walk"). K_V is a connected graph and we know that every available vertex of it has an equal-valued degree. We also know that the value of each degree (of each vertex) of K_V is equal to one value less than the number of total vertices. Formally: $deg(V_1) = deg(V_2) = ... = deg(V_m)$ where $deg(V_i) = V-1$ $(\forall)$ $i = \overline{1,m}$. In order for every vertex to have an even-valued degree V must be odd (because: $(\forall)$ $i = \overline{1,m}$ $deg(V_i) = odd - 1 \Leftrightarrow deg(V_i) = even$). Therefore $V = 2k+1$ where $k \in \mathbb{N} = \{0;1;2;....\}$. We conclude: „For $V = 2k+1$; K_V has a closed euler walk, where $k = \{0;1;2;.....\}$".

For „c) K_V has an open hamiltonian walk" we will use Theorem 32 („If the sum of the degrees of every pair [meaning one vertex tied up with another distinct vertex] of a graph G is at least $V-1$, then G has an open walk"). We know that for any graph K_V, we have $deg(V_i) = V-1$; $(\forall)$ $i = \overline{1,m}$. Therefore having a pair of vertices, we will get (for every pair of vertices the fact that:

$$deg(V_i) + deg(V_j) = (V-1)+(V-1) \Leftrightarrow deg(V_i) + deg(V_j) = 2(V-1)$$

$(\forall)$ $i \neq j$ and $i = \overline{1,m}$; $j = \overline{1,m}$. Because every pair of vertices of the

graph K_v is equal to „$2(v-1)$" (and because $2(v-1) > v-1$
(∀) $v>0$) we can conclude: „for $v \in \mathbb{N}$; K_v has an open hamiltonian
walk".

For „d) K_v has a closed hamiltonian walk" we will use
Theorem 33 („If the sum of the degrees of every pair of vertices of
a graph G is at least v, then G has a closed hamiltonian walk").
We know that for any graph K_v, we have $deg(v_i) = v-1$ (∀) $i=\overline{1,n}$.
Therefore having a pair of vertices, we will get (for every pair of
vertices the fact that: $deg(v_i) + deg(v_j) = (v-1) + (v-1) \iff$
$deg(v_i) + deg(v_j) = 2(v-1)$. For $v=0$, we won't be able to
form a pair of distinct vertices, thus $v \neq 0$. For $v=1$, we
won't be able to form a pair of distinct vertices, thus $v \neq 1$.
For $v=2$: $deg(v_1) + deg(v_2) = 2$. We notice that for $v=2$
the sum of the degrees all pairs [which in this case is only one pair (v_1 and
v_2)] is exactly equal to the number of vertices. For $v>2$,
the sum of the degrees of all pairs of vertices will be bigger
than the number of vertices (because of the fact that the
number „2" from „$2(v-1)$" will increase it more than the
number of vertices). We conclude: „for $v \geq 2$, K_v has a closed
hamiltonian walk".

3. Trace each drawing in Figure 155 without lifting your pencil from the paper or going any lines more than once, a) is a bad puzzle, because it can't be done no matter where you start (explain why); b) is a bad puzzle, because it can be done no matter where you start (explain why); c) is a good puzzle because it can be done, but only if you start at one of two places (explain why).

<u>Solution</u>: We will firstly illustrate each of the drawings in

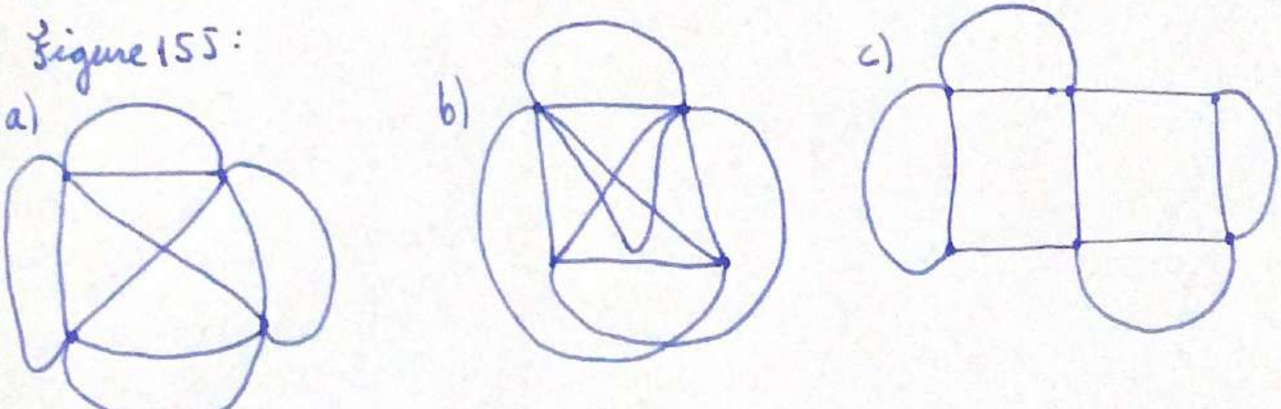


Figure 155

We will label each vertex of each graph for clarity:

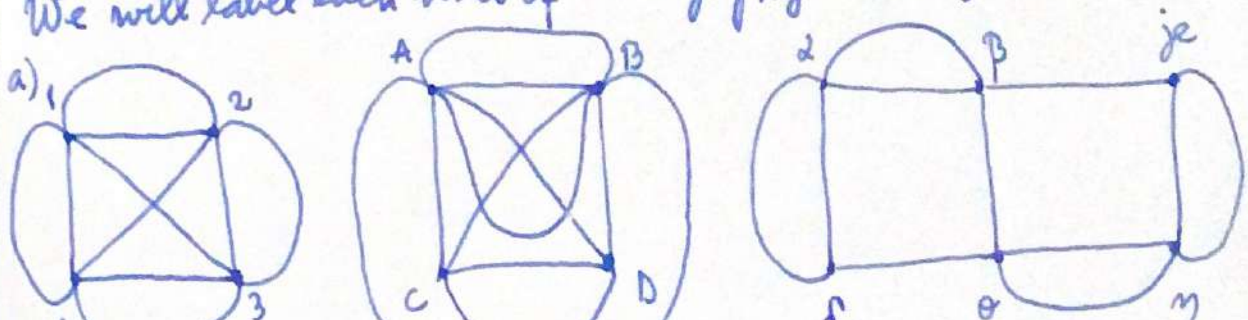


Figure 155

For graph a), we observe that: ·) it is connected
·) it has all degrees of all

vertices unevenly-valued (that means every degree of every vertex has an uneven value). Using Theorem 34 („If a connected multigraph has a closed euler walk, then every vertex is even. Conversely, if a multigraph is connected and every vertex is even, then it has a closed euler walk") and Theorem 35 („ If a connected multigraph has an open euler walk, then it has exactly two odd-valued degrees of vertices. Conversely, if a connected multigraph has exactly two odd valued degrees of vertices, then it has an open euler walk), we can remark the fact that our graph a) of Figure 155 doesn't meet the conditions of neither Theorem 34, nor of Theorem 35. We can conclude: „ Graph a) is a) a bad puzzle, because it can't be done no matter where you start" (as it doesn't contain neither an open euler walk, nor a closed euler walk).

For graph b), we observe that: ·) graph b) is connected
·) graph b) has all degrees of all

vertices evenly-valued (that means every degree of every vertex has an even value). Using Theorem 34 („If a connected multigraph has a closed euler walk, then every vertex is even. Conversely, if a multigraph is connected and every vertex is even, then it has a closed euler walk") we can remark the fact that graph b) meets all of the conditions of Theorem 3.4. Using Corollary 34 („If a particular vertex is selected from a connected multi-graph, having every vertex even, then it is possible to find a closed euler walk beginning and ending at that particular vertex"). We can conclude: „Graph b) is „b) a bad puzzle, because it can be done no matter where you start" (as it satisfies Theorem 3.4 and Corollary 3.4)."

For graph c), we observe that: ·) graph c) is connected
·) graph c) has exactly two odd-valued degrees (of vertices), namely the vertex „δ" and the vertex „je". (both having degree equal to three).

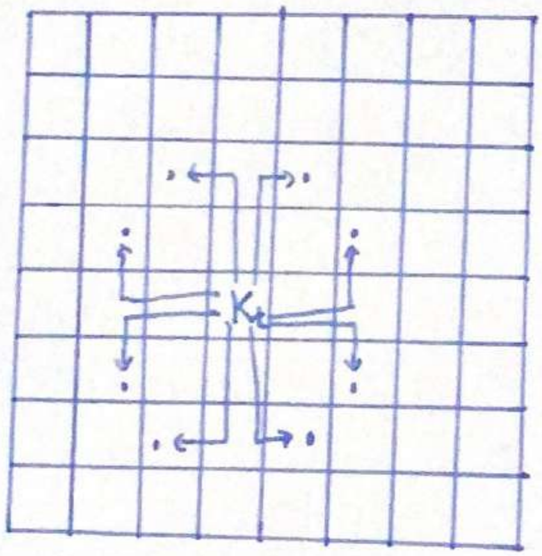
Using Theorem 35 („If a connected multigraph has an open euler walk then it has exactly two vertices with odd-valued degrees. Conversely, if a multigraph is connected and has exactly two vertices, both with odd-degree, then the multigraph has an open euler walk"), we can remark that graph c) meets all of the conditions of Theorem 35). Using Corollary 35 („If a connected multigraph has an open euler walk, the open euler walk begins at one of the two odd-valued degree vertex and ends at the other odd-valued degree vertex").

In our case, the open euler walk begins at either „δ" or „η" and ends at „η" or „δ".

We can conclude that: „Graph c) is „c) a good puzzle because it can be done, but only if you start at one of two places (namely, starting at $\underbrace{\text{„}\delta\text{" or „}\eta\text{"}}_{\text{one of}}$ from $\underbrace{\text{„}\delta\text{" and „}\eta\text{"}}_{\text{two places}}$), as it satisfies both Theorem 35 and Corollary 35".

4. In chess, a „knight's move" consists of two squares either vertically or horizontally and then one square in a perpendicular direction. Depending on where the knight is situated, he has a minimum mobility of two moves (when the knight is situated in one of the four corners of our chessboard) and a maximum mobility of eight moves, as shown in Figure 156. Let C be joined by an edge whenever a knight can go from one of the corresponding square (the in which the knight is situated currently) to another distinct square in one move. Does C have an euler walk? (you don't have to draw C to answer)

Solution: We will firstly draw Figure 156:



Kt := Knight

„→ ." := the places the „Knight" can move.

Attention: C is a graph with $v = 64$, its vertices corresponding to the square of a chess board.

We will label our chessboard in the following manner:

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64

By situating the „Knight" on 36 we can jump into the following squares (from 36): 19;21;30;26;46;42;50;53.

On the other hand if we situate the „Knight" one of these vertices: 2; 9; 7; 16; 49; 58; 56; 63, we will remark that we can only perform 3 legal moves, thus vertices 2; 9; 7; 16; 49; 58; 56; 63 share the same degree, that is equal to „3".

So far we have the two following facts: „Graph C has one vertex (namely 36) of degree equal to and 8 vertices (namely 2; 9; 7; 16; 49; 58; 56; 63) all of degrees equal to 3".

Using Theorem 30: „If a connected graph has a closed euler walk, then every vertex is even. Conversely, if that connected graph has every vertex even, then it has a closed euler walk" and

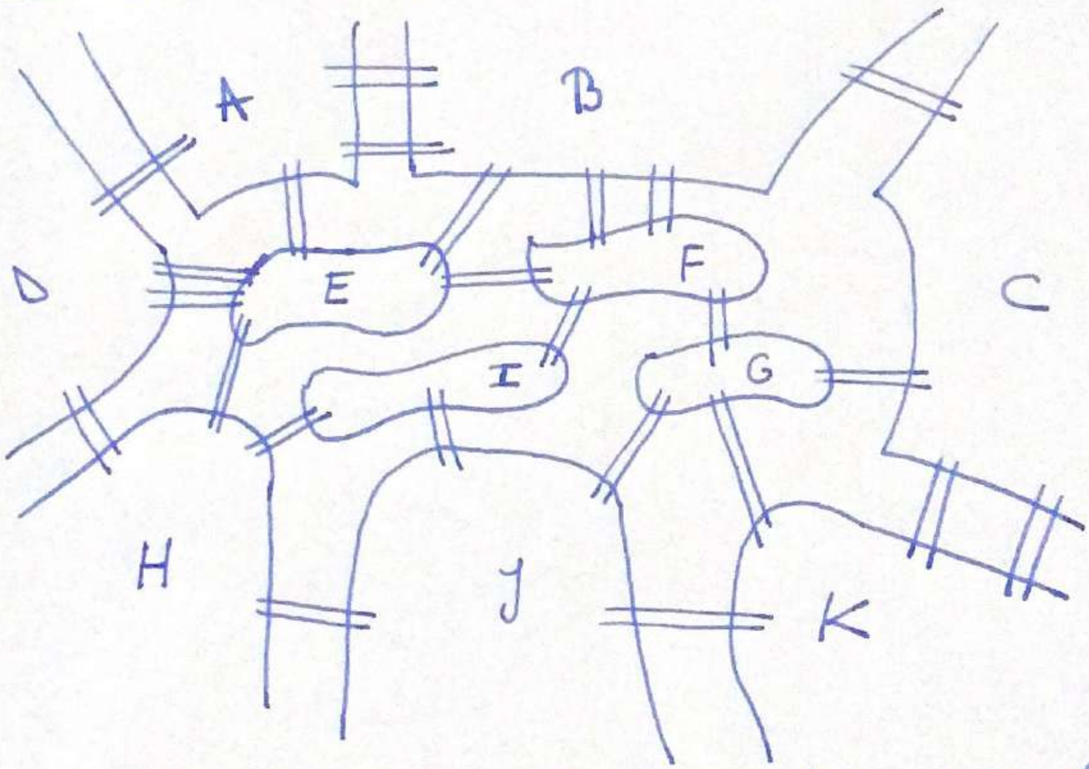
Theorem 31: „If a <u>connected graph</u> has an open euler walk, then <u>it</u> has exactly two odd vertices. Conversely, if <u>a connected graph</u> has exactly two odd vertices, then it has an open euler walk", we remark that graph C doesn't satisfy neither Theorem 30, nor Theorem 31, thus we conclude: „Graph C doesn't have any euler walks".

5. Prove that the graph C of „Exercise 4" has a closed hamilton walk. Such a walk is called a „knight's tour" by puzzle enthusiasts.

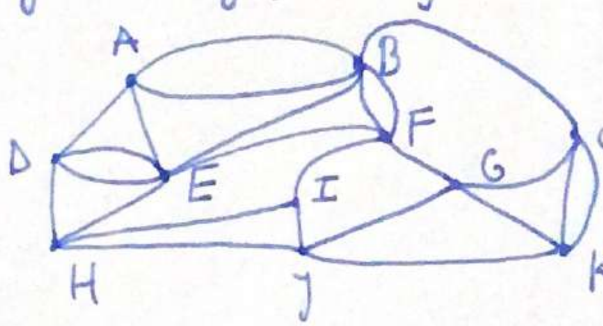
<u>Solution</u>: So far we have 8 vertices of degrees equal to 3 (namely 2; 9; 7; 16; 49; 58; 56; 63) and 1 vertex of degree equal to 8 (namely 36). The sum of the 8 vertices of degree 3 together with the vertex of degree 8 will be equal to „3·8+8" which is equivalent to „32". We will find 4 other vertices with degrees equal to 8 in order to get to a sum of 8 vertices of degree 3 and 5 vertices of degree 8 (that will be equal to „64"). Upon further inspection we will pick 37; 38; 35; 28 as 4 vertices of degrees equal to 8. Summing up the 8 vertices of degree 3 together with the 5 vertices of degree 8, we will get the sum being equal to „64". Using Theorem 33 („If the sum of the degrees of every pair of vertices of a graph G is at least V, then G has a closed hamiltonian walk") and the fact our graph C has at least the degrees of every pair of vertices equal to 64 (which is also the number of vertices), we conclude „Graph C has a closed hamiltonian walk".

6. Figure 157 depicts a system of bridges and land areas. Can you take a walk and cross each bridge exactly once? If so, where do you start and finish? Blow up the bridge from H to I and answer the same two questions.

<u>Solution</u>: We will firstly display Figure 157:

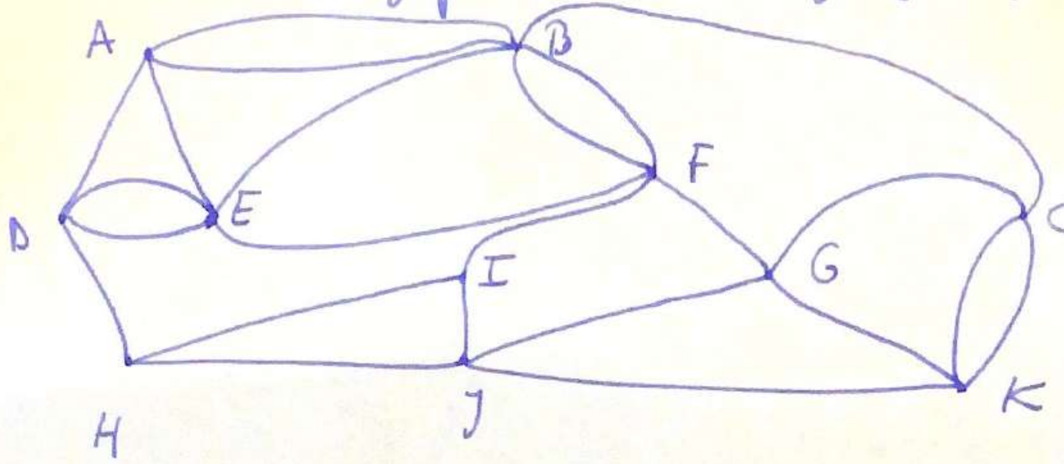


We will consider each land a vertex and each bridge an edge, coming with the following graph:



In order to check if our graph crosses the bridge exactly once, we will need to check if the graph has an euler walk.
We observe that our graph has all vertices of even degrees and is connected. Using Theorem 30 („If a connected graph has a closed euler walk, then every vertex has an even-valued degree. Conversely, if a graph is connected and every vertex is even, then it has a closed euler walk") we conclude: „The system of bridges of Figure 157 that can be transformed into a graph, can take a walk and cross each bridge exactly once".
Applying Corollary 30 („If a particular vertex is selected from a connected graph having every vertex of even-valued degree, then it is possible to find a closed euler walk beginning and ending at that particular vertex), we conclude: „The walk that crosses each bridge exactly once ends at the same land (this being one of the following: A; B; C; D; E; F; G; H; I; J) it started the walk."
Next, we will have to get rid of the edge {H; E} and answer the questions: Does the graph without the edge {H; E} still have an euler walk? If so, where do you start and finish?

We will draw the graph without the edge {H;E} :



We note that by erasing one edge from the vertices H and I, we will make their degrees be odd (as the original graph consisted only of even-valued degree vertices). Our new graph has precisely two odd-valued degrees vertices (H and I). Using Theorem 31 („If a connected graph has an open euler walk, then it has exactly two odd vertices. Conversely, if a connected graph has exactly two odd vertices, then it has an open euler walk") and the fact that our new graph is connected and has exactly two odd-valued degrees of vertices, we conclude: „The new graph (which represents the original graph without the edge {E;H}) can take a walk and cross each bridge exactly once".

Applying Corollary 31 („If a connected graph has an open euler walk, then the open euler walk begins at one of the two odd-valued degrees vertices and ends at the other odd-valued degree vertex), we conclude:

" The walk that crosses each bridge exactly once begins at either H or E and ends at E (in case it began at H) and H (in case it began at E)".

7. For each integer $v \geq 2$, find a graph with v vertices in which the sum of the degrees of every pair of vertices is at least $v-1$, but which has no closed hamiltonian walk. Of course the graph will have an open hamilton walk by Theorem 32: („If the sum of the degrees of every pair of vertices of a graph G is at least $v-1$, then G has an open hamilton walk).

Solution:

For $v=2$, we will consider the graph: 

We remark that $deg(1) + deg(2) = 1+1 = 2 \geq \underbrace{v-1}_{=1}$ and it has an open hamiltonian walk: 12, but no closed hamiltonian walk since 121 is not allowed (since a hamiltonian walk requires to travel one edge exactly once).

For $v=3$, we will consider the graph: 

We remark that $deg(1) + deg(2) + deg(3) = 1+2+1 = 4 \geq \underbrace{v-1}_{2}$ and it has an open hamiltonian walk: 123, but no closed hamiltonian walk, since 1231 is not allowed (neither 232 nor 121) [as a hamiltonian walk requires to travel an edge exactly once].

We continue this construction (which we will call the Path Graph, having the vertex set: $\{1;2;3;...;v\}$; and the edge set: $\{\{1;2\};\{2;3\};....;\{v-1;v\}\}$) and conclude: „The Path Graph is a graph for which each integer $v \geq 2$, the sum of degrees of every pair of vertices is at least $v-1$, and doesn't have any closed hamiltonian walk".

8. Show that every path P_v (see Chapter 2, Exercise 5) has an open hamiltonian walk, though Theorem 32 is applicable only to P_2 and P_3. Then show that every wheel graph W_v (see Chapter 2, Exercise 6) has a closed hamiltonian walk, though Theorem 33 is applicable only to W_4; W_5 and W_6.

Solution: We will restate Theorem 32 and Theorem 33.

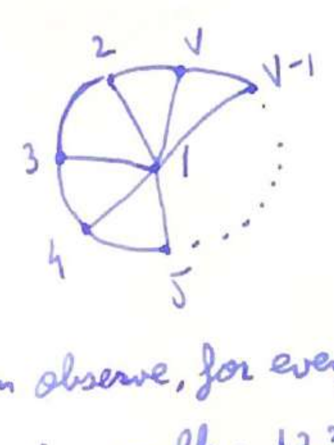
Theorem 32: If the sum of the degrees of every pair of a graph G is at least $v-1$, then G has an open hamilton walk.

Theorem 33: If the sum of the degrees of every pair of vertices of a graph G is at least v; then G has a closed hamilton walk.

We will draw P_v:

As we can observe, for every v for P_v we can have an open hamiltonian walk: 1234.... v-1v

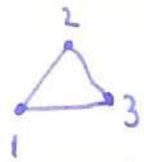
We will draw W_v:



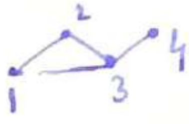
As we can observe, for every v for W_v we can have a closed hamiltonian walk: 12345.... v-1 v 1

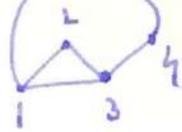
9. Satisfy yourself that every graph with v vertices and a closed hamilton walk is a supergraph of the cyclic graph C_v. Then use this fact to prove that if a graph has a closed hamilton walk then its connectivity c (meaning the least amount edges to be removed from a connected graph in order for it to become disconnected) is at least equal to two.

Solution: To satisfy ourselves that every graph with v vertices and a closed hamilton walk is a supergraph of the cyclic graph C_v we will consider firstly the minimum graph that is a closed hamiltonian walk:



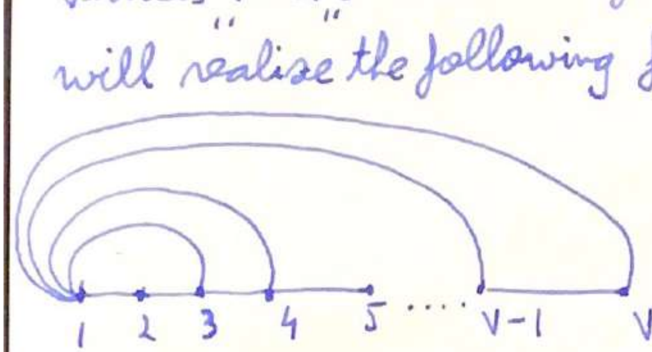
We remark that the minimum graph of a closed hamiltonian walk is the cyclic graph C_3. We will add a new vertex which we will name „4" and connect it with „3" arriving at:



We remark the fact that we cannot form any closed hamilton walk as by fixing any of the available vertices we cannot visit all the vertices without using an edge two times. By creating a new edge {1;4} we will have the following graph: 

. We remark that this graph

is actually a supergraph of C_4 (by connecting the vertices "1" and "3" with an edge). Continuing this procedure, we will realise the following final graph:



We remark that this final graph represents a supergraph of C_v (by connecting: "1" with "3"; ... ; "V-1"). We conclude: "Every graph with v vertices and a closed hamiltonian walk is a supergraph of the cyclic graph C_v".

For proving "if a graph has a closed hamiltonian walk, then its connectivity is at least equal to 2." we will look at the last graph we drew and realise that the minimum amount of edges that are needed to be removed in order for the last drawn graph to be disconnected is equal to 2 (namely: {V-1; V} and {1; V}). Because we can add "V-2" edges more to the vertex V, we conclude: "A graph with a closed hamilton walk has connectivity c at least equal to 2".

10. There are v guests to be seated at a single round table. Each guest is acquainted with at least $\left\{\frac{v}{2}\right\}$ of the others. Prove that they can be seated in such a way that each guest is between two acquaintances. We remark that $\{\cdot\}: \mathbb{R} \to \mathbb{Z}$ has the following properties:

- $\underset{x\in\mathbb{R}}{\{x\}} = \min\{n \in \mathbb{Z} \mid n \geq x\}$
- $\{x\} = x$ if and only if $x \in \mathbb{Z}$

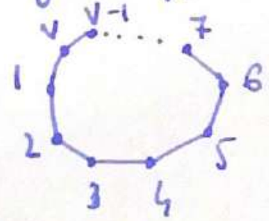
Solution: In order for a guest to be between two acquaintaces (and representing all guests as vertices and the acquaintaces relation with each guest as edges) we will need at least 3 vertices (each vertex having degree equal to 2).

We will define $\left\{\frac{v}{2}\right\}$ as follows (keeping in mind that $v \geq 3$):

$$\underset{\substack{v\geq 3\\ v\in\mathbb{N}}}{\left\{\frac{v}{2}\right\}} = \begin{cases} v - \left\{\frac{v-2}{2}\right\}; 2 \nmid v \\ \\ \frac{v}{2}; \text{ for } 2 \mid v \end{cases}$$

We also remark that for $v \geq 3$ $\left\{\frac{v}{2}\right\} \geq 2$, thus by increasing the value of v the number of acquaintaces a guest has will only be larger.

A simple graph for which every guest will sit between two acquaintaces is the cyclic graph C_v (as every vertex has a degree equal to „2").



We conclude: „The cyclic graph can be used in order to sit every guest between two acquaintances, where each vertex represents a guest and the value of each degree of a vertex illustrates the number of acquaintaces each guest has".

11. Prove: „If $n \geq 2$ and G is connected with $2n$ odd vertices then G has n open walks which, together, use every edge of G exactly once.

Solution: We will firstly consider a multigraph of our original graph G (which we will denote by $M(G)$) that consists of n new edges, each edge joining two odd-valued degree vertices. By adding a new edge to each of the odd-valued degree vertices, we will obtain a multigraph with all vertices having even-valued degrees. Using Theorem 34 („If a connected multigraph has a closed euler walk, then every vertex is even. Conversely, if a multigraph is connected and every vertex is even, then that multigraph has a closed euler walk") and the fact that our multigraph is connected and has all vertices of even-valued degree, then we can remark that our multigraph has a closed euler walk. Removing the n added edges from the closed euler walk of $M(G)$, we will be left with n open walks in G, each of which begins at an odd vertex and ends at another odd vertex (distinct from the the odd vertex we began at), that will use every edge exactly once.

12. Draw four graphs with $v=8$: the first graph with an euler walk but no hamilton walk, the second graph with a hamilton walk but no euler walk, the third graph with both of an euler walk and a hamilton walk, the fourth graph with no hamilton walk and no euler walk.

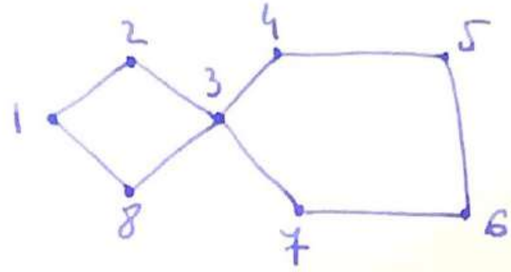
Solution: We will firstly state the theorems which we will use in this exercise:

Theorem 30: „If a connected graph has a closed euler walk, then every vertex is even. Conversely, if a graph is connected and every vertex is even, then it has a closed euler walk".

Theorem 31: „If a connected graph has an open euler walk, then it has exactly two odd vertices. Conversely, if a connected graph has exactly two odd vertices then it has an open euler walk"

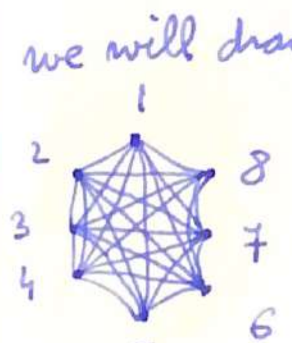
Theorem 32: „If the sum of the degrees of every pair of vertices of a graph G is at least $v-1$, then G has an open hamilton walk".

Theorem 33: „If the sum of the degrees of every pair of vertices of a graph is at least V, then G has a closed hamilton walk", We will label each of the four graphs for greater clarity. We will begin with the first graph (with $V=8$, an open euler walk, but no hamilton walk):



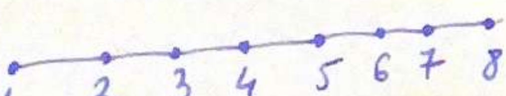
We remark the fact that the first graph has a closed euler walk : 123456781, but there is no way to establish a hamilton walk (as whatever walk you will choose to take to obtain all vertices, the vertex „3" will inevitably get repeated at least two times (but not at the end and at the beginning of the chosen walk), thus failing to have any hamilton walk.

For the second graph (with $V=8$, a hamilton walk, but no euler walk) we will draw the following graph:

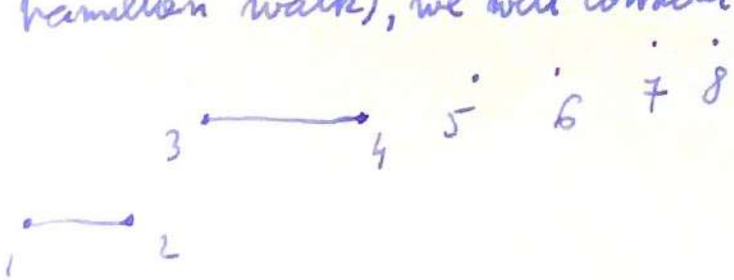


We remark the fact that our chosen graph represents the complete graph with $v=8$ (namely: K_8).

We know that K_8 is connected and every degree of every vertex of it is equal to „V-1", which is equal to „7".
We observe that the (second) chosen graph has a hamilton walk: 12345678. By inspecting Theorem 30 and Theorem 31, we remark that our graph doesn't satisfy neither Theorem 30, nor Theorem 31, thus concluding that our graph doesn't contain any euler walk.
For the third graph (with V=8, an euler walk and a hamilton walk) we will consider the following graph:



We remark that this graph has a hamiltonian walk (namely: 12345678) that is also an euler walk.
For the fourth graph (with V=8, no euler walk and no hamilton walk), we will consider the following graph:



Even if we would weaken Theorem 30, Theorem 31 from the original statements to:

·) Theorem 30 (weakened): „If a graph has a closed euler walk and every vertex is even, then it has a closed euler walk. Conversely, if a graph has every vertex even, then it has a closed euler walk."

·) Theorem 31 (weakened) : „ If a graph has an open euler walk, then it has exactly two odd vertices. Conversely, if a graph has exactly two odd vertices, then it has a closed euler walk" ; we will still remark the fact that our chosen graph doesn't satisfy Theorem 30 (weakened) and Theorem 31 (weakened), thus concluding that our chosen graph has no euler walks. Because the vertex set {1;2;3;4;5;6;7;8} of our graph contains isolated vertices (namely: 5;6;7;8) we cannot travel through them making it impossible to have a walk that contains all vertices. Therefore our graph contains no hamilton walk.

13. Definition: If G is a graph with $e \neq 0$ then the line graph of G, denoted "$L(G)$", is the graph having one vertex corresponding to each edge of G and such that two vertices of $L(G)$ are joined by an edge whenever the corresponding edges of G share a vertex.

Examples: 1) Figure 158a) is a graph whose line graph has been drawn in Figure 158b). Each vertex of $L(G)$ corresponds to an edge of G. In the figure this correspondence has been made explicit by numbering the edges of G and giving the same number to the corresponding vertices of $L(G)$. Two vertices are joined whenever the corresponding edges of G share a vertex.

2) Figure 158 c) is the line graph of K_4. K_4 has six edges, each adjacent to four of the others, thus its line graph has six vertices, each adjacent to four of the others.

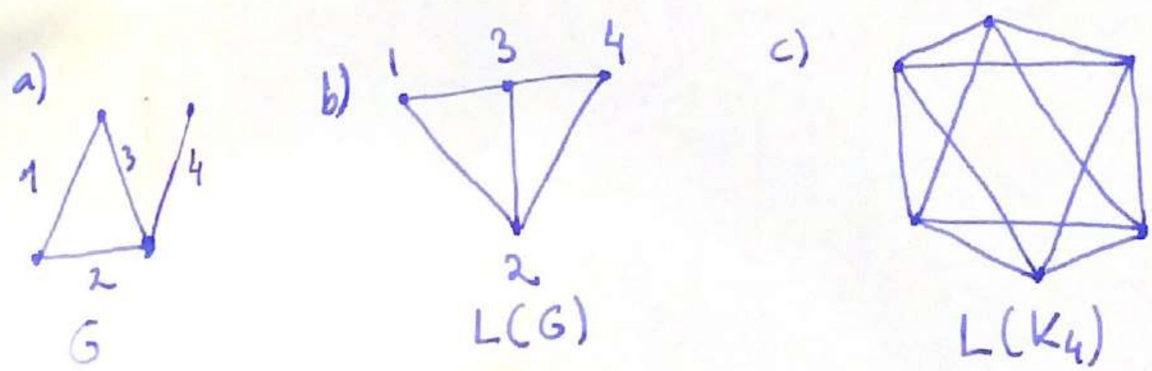


Figure 158.

Of the eleven graphs with $v=4$ (see Figure 45), N_4 has no line graph because it has $e=0$ and we have just drawn the line graph of K_4. Draw the line graphs of the remaining nine.

<u>Solution</u>:

We will firstly draw all eleven graphs from Figure 45 (including N_4 and K_4):

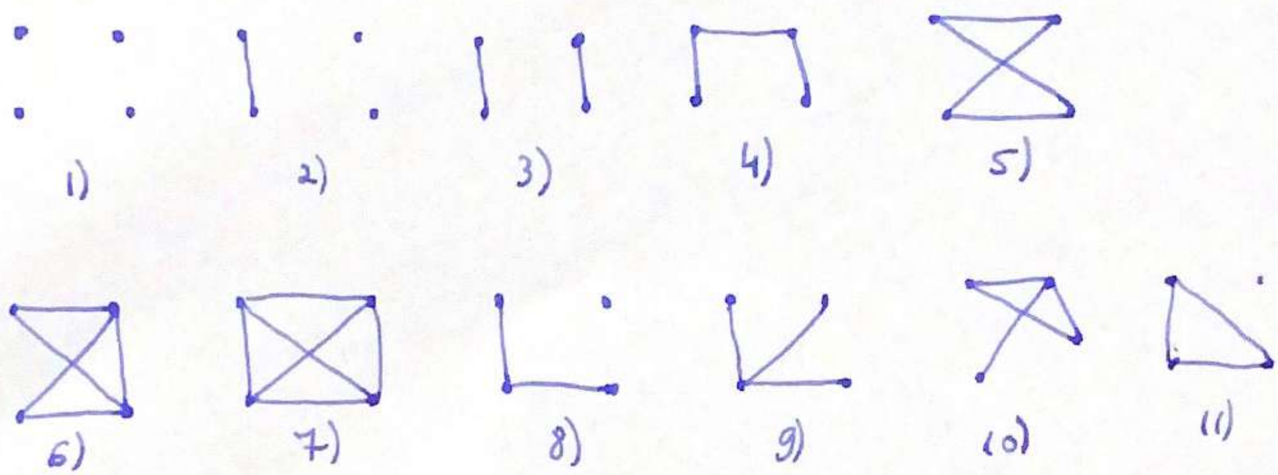
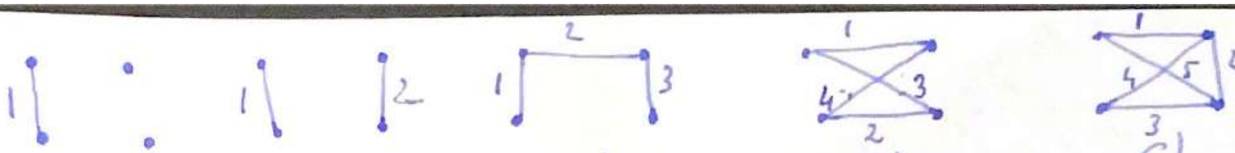


Figure 45

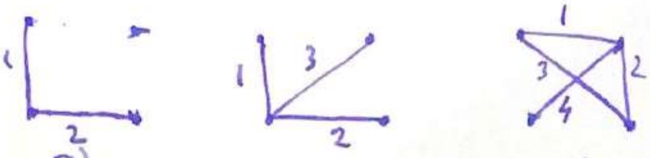
N_4 coincides to graph 1)

K_4 coincides to graph 7)

Eliminating graph 1) and graph 7), we will label the edges of the remaining graphs:

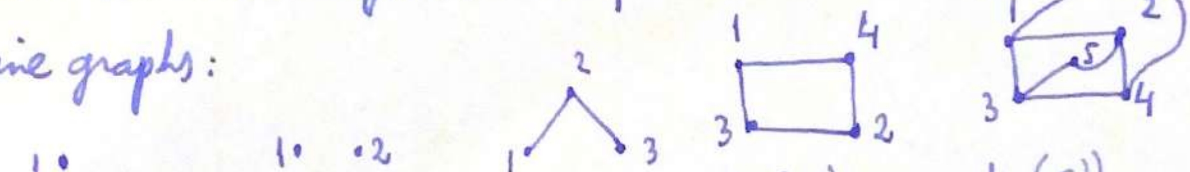


2) 3) 4) 5) 6)

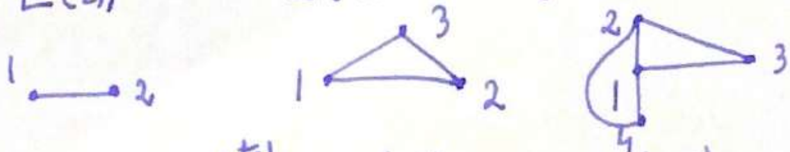


8) 9) 10)

Now we will transform all of the labeled graphs into line graphs:



L(2)) L(3)) L(4)) L(5)) L(6))



L(8)) L(9)) L(10))

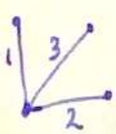
14. Prove: If (connected the graph) G has a closed euler walk then L(G) has both a closed euler walk and a closed hamilton walk.

Solution: If the graph G has a closed euler walk then according to Theorem 30: "If a connected graph has a closed euler walk, then every vertex is even. Conversely, if a graph is connected and has every vertex even, then that graph will have a closed euler walk". we will have that each vertex is even which will lead to the fact that the line graph of G (L(G)) will also have even vertices, which will mean that L(G) has a closed euler walk.

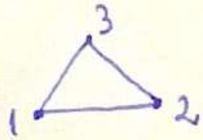
Because graph G has a closed euler walk we can denote: $E_1 E_2 \dots E_e$ the sequence of all the edges travelled by the graph G (while „e" = the total number of edges the graph G has). Transforming the edge sequence into a vertex sequence (once we go from graph G to graph L(G)), we will have: $V_1 V_2 \dots V_e$. G having a closed euler walk will mean that „$E_1 = E_e$" which in L(G) will be: „$V_1 = V_e$". We conclude: „If the connected graph G has a closed euler walk, then L(G) has both a closed euler walk and a closed hamilton walk".

15. Find a graph G such that L(G) has both a closed euler walk and a closed hamilton walk, but G doesn't have a closed euler walk. This show that the converse of Exercise 14 is false [namely: „If L(G) has both a closed euler walk and a closed hamilton walk, then G will have a closed euler walk" (this statement being false)].

Solution : Upon inspection we remark that graph of Figure 45 g) will satisfy the exercises conditions.



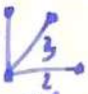
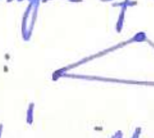
g)



L(g))

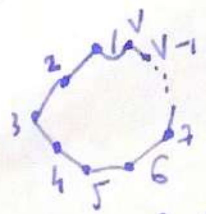
Using Theorem 30 („If a connected graph has a closed euler walk, then every vertex is even. Conversely, if a graph has every vertex even, then that graph has a closed euler walk") and Theorem 31 („If a connected graph has an open euler walk, then it has exactly two odd vertices. Conversely, if a graph is connected and has exactly two odd vertices, then that graph has an open euler walk"), we remark that our graph of Figure 45 g) doesn't meet any of the two theorems, thus it does not have any euler walks.

For the line graph L(G) we remark the fact that we can easily construct a closed euler walk (namely: 1231) and also a closed hamilton walk (namely: 1231).

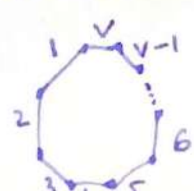
We conclude: „The graph G:  is an example of a graph that has no closed euler walk, while $L(G)$:  has both a closed euler walk and a closed hamilton walk".

16. Prove: „If G has <u>a closed hamilton walk</u>, then $L(G)$ has <u>one</u> too".

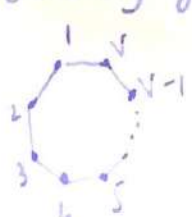
<u>Solution</u>: We will use the fact (proved at Exercise 9.) that: „Every graph with v vertices that has a closed hamilton walk is a supergraph of the cyclic graph C_v". The minimum graph that is a hamiltonian graph with v vertices is C_v:



We will label each edge now of C_v:



By transforming C_v into a line graph $L(G)$ we will actually get the same graph C_v.



$L(G) = C_v$

Because no matter how many extra edges we will add between each two vertices in graph G (and transform these edges into vertices in graph L(G)) we will still be able to preserve the closed hamiltonian walk (present both in L(G) and G), namely, 1234... V-1 V 1. We conclude: „If a graph G has a closed hamiltonian then L(G) will have also a closed hamiltonian".

18. Definition: If G is a graph with $e \neq 0$, then the trisection graph of G, denoted „T(G)", is the expansion of G formed by splicing two vertices of degree 2 into every edge of G.

Example: „The graph of Figure 159a) is the trisection graph G of figure 158a). Figure 159b) shows the line graph L(T(G)) of T(G).

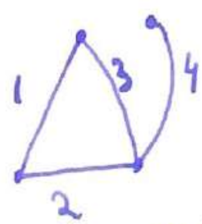


Figure (158a)
G

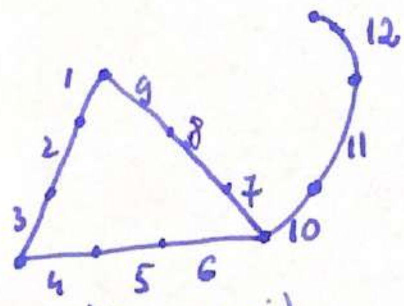


Figure 159a)
T(G)

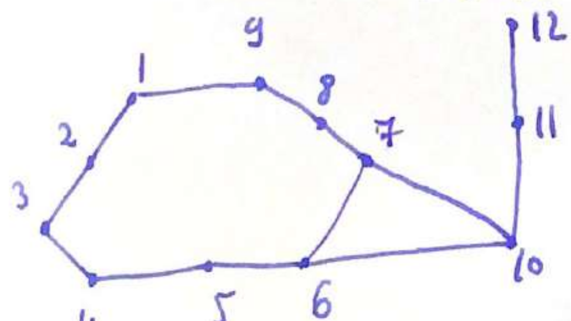


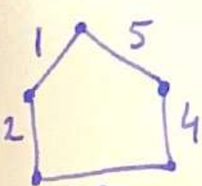
Figure 159b)
L(T(G))

Exercises 14 and Exercises 16 establish a partial relationship between euler walks and hamilton walks. The following theorem gives a more profound relationship:

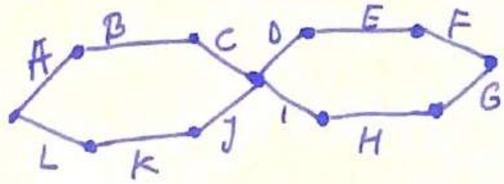
Theorem: A graph G with $e \neq 0$ does or does not have a closed euler walk according as $L(T(G))$ does or does not have, respectively, a closed hamilton walk.

Draw some graphs G with closed euler walks and in each case check that $L(T(G))$ has a closed hamilton walk. Then draw a few graphs G without closed euler walks and check that in each case $L(T(G))$ does not have a closed hamilton walk. With these examples try to prove the Theorem previously mentioned.

Solution: We will pick 2 graphs with closed euler walks and 2 graph without closed euler graphs. We will also label their edges for a greater clarity:



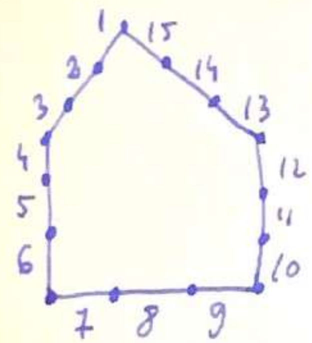
Graph I



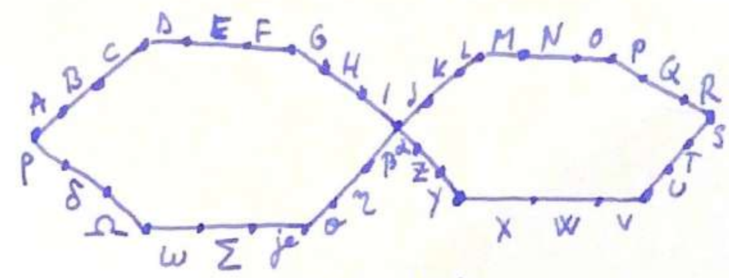
Graph II

The two last graphs drawn represent the two graphs with a closed euler walk.

We will draw the Trisection Graphs of the two graphs:

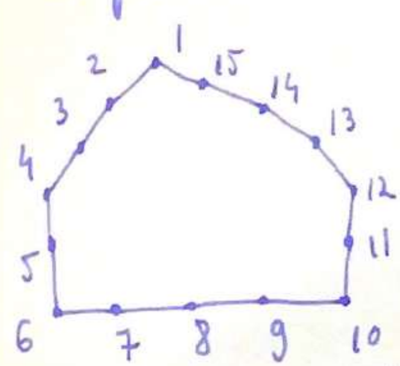


T(Graph I)

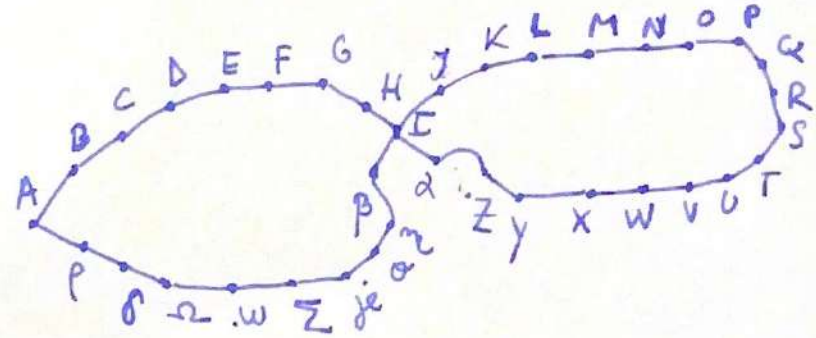


T(Graph II)

We will now draw the Line Graphs of the two trisection Graphs:



L(T(Graph I))



L(T(Graph II))

We observe that the Line Graph of the Trisection Graph doesn't change much the graph of the trisection graph.
We started with graphs that have euler walks and ended with the. Line Graph of the Trisection Graph of our initial graphs that have a closed hamilton walk:

L(T(Graph I)) has a following closed hamilton walk:

"1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 1"

L(T(Graph II)) has a following closed hamilton walk:

"ABCDEFGHIJKLMNOPQRSTUVWXYZ$\alpha\beta\varkappa\Sigma\omega\Omega\delta\rho$"

Thus the following Theorem is satisfied:

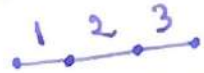
Theorem: "A graph G with $e \neq 0$ does or does not have a closed euler walk according as L(T(G)) does or does not have, respectively, a closed hamilton walk.

For the next two graphs we will pick them in a way such that they don't contain any closed euler graphs:"

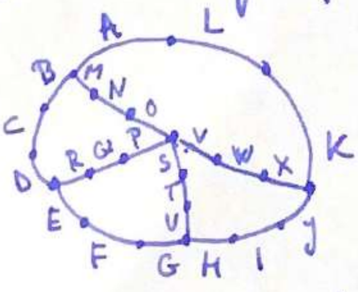
Graph III

Graph IV

We will now draw the Trisection Graphs of Graph III and Graph IV:



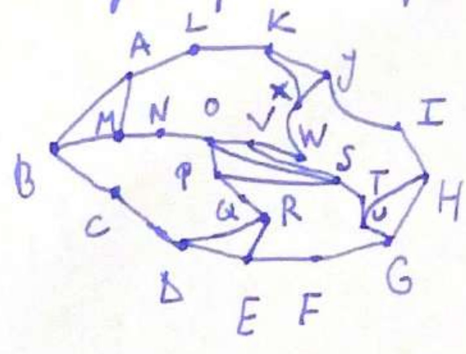
T(Graph III)



T(Graph IV)

We will now draw the Line Graphs of T (Graph III) and T (Graph IV) :

L(T(Graph III))



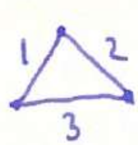
L(T(Graph IV))

We remark that the Line Graphs of the Trisection Graphs of Graph III and Graph IV do not contain any closed hamilton walk. Therefore the following Theorem is satisfied:

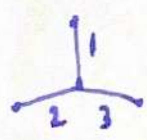
Theorem: „A graph G with $e \neq 0$ does or does not have a closed euler walk according as L(T(G)) does or does not have, respectively, a closed hamilton walk.

19. It is usually true that different (i.e. nonisomorphic) connected graphs have different line graphs. In fact there is only one exception: „There are two different connected graphs having line graph K_3. Find them".

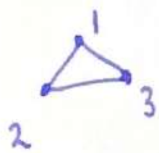
Solution: The two different connected graphs (that are nonisomorphic) having as line graph K_3 are the following:



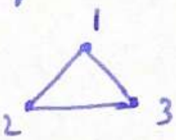
Graph I



Graph II



L(Graph I)



L(Graph II)

We labelled with 1;2;3 the edges of Graph I and Graph II.

20. Let G be a graph with v vertices and e edges, and let the degrees of the vertices of graph G be „d_1"; „d_2"; ...; „d_v". Of course $L(G)$ has e vertices. Prove now that the number of edges of $L(G)$ is equal to $\frac{1}{2}(d_1^2 + d_2^2 + \dots + d_v^2) - e$.

Solution: Consider one vertex of the graph G having degree d_1. Choose one particular edge belonging to the considered vertex of degree d_1. That particular chosen edge sits next to „$d_1 - 1$" different edges belonging to the considered vertex of degree d_1. The „particular edge" together with the other „different edges" will be converted to vertices. Upon inspection we will arrive that the degrees of the edges that belong to one particular vertex of the graph G, will all be equal to $d_1(d_1-1)$. Considering now all the vertices of graph G, we will sum up all the degrees of the vertices of the line Graph $L(G)$, obtaining the following equality:

$$\sum_{i=1}^{v} deg(v_{i,L(G)}) = d_1(d_1-1) + d_2(d_2-1) + \dots + d_v(d_v-1)$$

$$\Leftrightarrow \sum_{i=1}^{v} deg(v_i; L(G)) = d_1^2 + d_2^2 + \dots + d_v^2 - \underbrace{(d_1 + d_2 + \dots + d_v)}_{:= \sum_{i=1}^{v} deg(v_i)}$$

(for graph G)

We will use the proven fact that: $\sum_{i=1}^{v} deg(v_i) = 2e$

for any graph, thus getting:

$$\sum_{i=1}^{e} deg(v_i; L(G)) = d_1^2 + d_2^2 + \ldots + d_v^2 - 2e$$

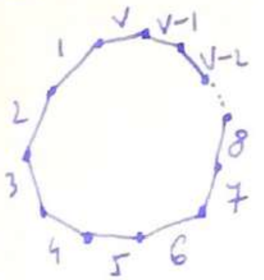
$$\Leftrightarrow 2e_{L(G)} = d_1^2 + d_2^2 + \ldots + d_v^2 - 2e$$

$$\Leftrightarrow e_{L(G)} = \frac{1}{2}(d_1^2 + d_2^2 + \ldots + d_v^2) - e$$

We conclude: „For G, a graph, with v vertices and e edges and for which we denote each degree of a vertex in the following manner: $d_1; d_2; \ldots; d_v$; we will have that its Line Graph $L(G)$, which has $e_{L(G)}$ edges and $v_{L(G)}$ vertices, has the total number of edges $e_{L(G)} = \frac{1}{2}(d_1^2 + d_2^2 + \ldots + d_v^2) - e$".

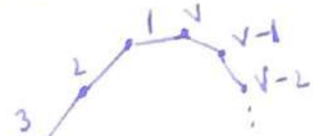
21. Prove: Cyclic graphs are isomorphic to their line graphs, and are the only graphs satisfying that property.

Solution: We will draw the cyclic graph C_V (labelling its edges rather than its vertices, for ease of transforming it into its Line Graph: $L(C_V)$):



C_V

We will draw the Line Graph of C_V, "$L(C_V)$":



We remark the fact that :

- the number of edges is preserved from C_V to $L(C_V)$.
- the number of vertices is preserved from C_V to $L(C_V)$
- both graphs are made out of exactly one piece
- the degree of each vertex is preserved from C_V to $L(C_V)$.

Therefore we conclude: "Cyclic graphs are isomorphic to their Line Graphs and are the only graphs satisfying this property, as other graphs will not satisfy the necessary four properties together with their Line Graphs".