Sorbonne University Abu Dhabi

MAT 100, Second Midterm Examination. 120 min

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November 2024

THE EXAMINATION CONSISTS OF 3 EXERCISES AND ONE PROBLEM. IT IS PRINTED ON 2 PAGES

Note: The examination is out of 22. Whatever grade obtained out of 22 will be considered directly as the overall grade out of 20, without proportinality adjustments.

EXERCISE 1 *EXERCISE 1 (5 MARKS = 1+2+2) Let a and b be two real numbers. Define the function*

$$f : \mathbb{R} \to \mathbb{R}$$

by

$$f(x) = \begin{cases} \frac{\sin(ax)}{x} & \text{if } x < 0 \\ 1 & \text{if } x = 0 \\ e^{bx} - x & \text{if } x > 0 \end{cases}$$

1. *Show that*

$$\lim_{x \to 0} \frac{x\cos(x) - \sin(x)}{x^2} = 0$$

2. *Determine a and b so that f is continuous on $\mathbb{R}$.*

3. *Determine a and b so that f is differentiable on $\mathbb{R}$.*

The Street Fighter Solution

Before we get lost in a sea of epsilons and deltas, let's look at what is actually happening. We want to solve this using intuition and efficiency (the "Street Fighter" way).

Since we are given no rules as to how to show our results (there is no prohibition, for instance, of using L'Hopital's rule), for the first question I will use my favorite technique (it even allows finding results without writing much), namely Taylor series. For what we need:

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$
$$x\cos(x) = x - \frac{x^3}{2!} + \frac{x^5}{4!} + \dots$$

Therefore:

$$\begin{aligned} x\cos(x) - \sin(x) &= \left(x - \frac{x^3}{2!} + \dots\right) - \left(x - \frac{x^3}{3!} + \dots\right) \\ &= x^3\left(-\frac{1}{2} + \frac{1}{6}\right) + \dots \\ &= -\frac{1}{3}x^3 + \dots \end{aligned}$$

Upon division by x^2, the leading term of the full function is proportional to x. Therefore, the limit is indeed: zero.

As for part 2. We need to find the left and right limits (to zero) of $f(x)$ and compare them with $f(0) = 1$.

Using standard knowledge from calculus for the left limit ($x < 0$):

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin(ax)}{x} = a$$

For the right limit ($x > 0$), I will use again my beloved Taylor series:

$$\lim_{x\to 0^+} (e^{bx} - x) = \lim_{x\to 0^+} \left(1 + bx + \frac{(bx)^2}{2!} + \dots - x\right) = 1$$

(Note that as $x \to 0$, all terms with x vanish).

For f to be continuous, we need Left Limit = Right Limit = $f(0)$.

$$a = 1 = 1$$

Therefore, we must set $\mathbf{a = 1}$. At this stage (continuity), there is no restriction on b, so $b \in \mathbb{R}$.

Finally, for differentiability, we must ensure the "slope" matches at the junction ($x = 0$). Recall that for a function to be differentiable, it must first be continuous, so we keep $a = 1$.

Instead of calculating tedious limits of difference quotients, I will simply look at the linear term (the coefficient of x) in the Taylor expansion of each side. Why? Because locally, every differentiable function looks like a line: $f(x) \approx f(0) + f'(0)x$.

For $x > 0$: We already expanded this in the previous step:

$$f(x) = e^{bx} - x \approx 1 + bx - x = 1 + (b-1)x$$

The coefficient of x is clearly $(b - 1)$. Thus, the right derivative is $b - 1$.

For $x < 0$ (with $a = 1$):

$$f(x) = \frac{\sin(x)}{x} \approx \frac{x - \frac{x^3}{6}}{x} = 1 - \frac{x^2}{6}$$

Notice there is no term proportional to x (or rather, the coefficient is 0). Thus, the left derivative is 0.

Matching the slopes:

$$b - 1 = 0 \implies b = 1$$

Conclusion: For f to be differentiable on $\mathbb{R}$, we must have $a = 1$ and $b = 1$.

The "Standard" Textbook or Good Boy Solution

Part 2: Continuity

For the function f to be continuous at $x = 0$, we require:

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

1. The Left Limit ($x < 0$):

$$\lim_{x \to 0^-} \frac{\sin(ax)}{x}$$

We use the standard limit identity $\lim_{u \to 0} \frac{\sin(u)}{u} = 1$. Let $u = ax$. Then:

$$\lim_{x \to 0^-} a \cdot \frac{\sin(ax)}{ax} = a \cdot 1 = a$$

2. The Right Limit ($x > 0$):

$$\lim_{x \to 0^+} (e^{bx} - x)$$

By direct substitution (since exponential and polynomial functions are continuous):

$$e^{b(0)} - 0 = 1 - 0 = 1$$

3. The Value at $x = 0$:

$$f(0) = 1$$

Conclusion: For continuity, we equate the limits: $a = 1$. (Note that b can be any real number).

Part 3: Differentiability

For f to be differentiable at $x = 0$, the Left Derivative and Right Derivative must exist and be equal. We use the definition of the derivative:

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

Prerequisite: We assume $a = 1$ from the continuity requirement.

1. **The Left Derivative ($h \to 0^-$):**

$$f'_-(0) = \lim_{h \to 0^-} \frac{\frac{\sin(h)}{h} - 1}{h} = \lim_{h \to 0^-} \frac{\sin(h) - h}{h^2}$$

Substituting $h = 0$ yields form $\frac{0}{0}$, so we apply L'Hôpital's Rule:

$$\overset{L'H}{=} \lim_{h \to 0^-} \frac{\cos(h) - 1}{2h}$$

This is still $\frac{0}{0}$. We apply L'Hôpital's Rule a second time:

$$\overset{L'H}{=} \lim_{h \to 0^-} \frac{-\sin(h)}{2} = \frac{0}{2} = 0$$

2. **The Right Derivative ($h \to 0^+$):**

$$f'_+(0) = \lim_{h \to 0^+} \frac{(e^{bh} - h) - 1}{h} = \lim_{h \to 0^+} \frac{e^{bh} - h - 1}{h}$$

Substituting $h = 0$ yields form $\frac{0}{0}$, so we apply L'Hôpital's Rule:

$$\overset{L'H}{=} \lim_{h \to 0^+} \frac{be^{bh} - 1}{1}$$

By direct substitution:

$$b(1) - 1 = b - 1$$

Conclusion: For differentiability, we equate the left and right derivatives:

$$0 = b - 1 \implies b = 1$$

Thus, the required values are $\mathbf{a = 1}$ and $\mathbf{b = 1}$.

EXERCISE 2 *EXERCISE 2 (5 MARKS = 2+3)*

The aim of this exercise is to study the behaviour of the function

$$f : \mathbb{R} \to \mathbb{R}$$

defined by

$$f(x) = \ln(e^x + \sqrt{e^{2x} + 1})$$

in a neighbourhood of $+\infty$.

1. *Consider the function*

$$g(t) = \ln(1 + \sqrt{1+t})$$

   Show that, as $t \to 0$,

$$g(t) = \ln(2) + \frac{t}{4} + o(t).$$

2. *Show that*

$$f(x) = x + \ln\left(1 + \sqrt{1 + e^{-2x}}\right) .$$

Deduce, using Question 1, that f *admits a slant (oblique) asymptote as* $x \to \infty$. *Determine the relative position of the graph of* f *with respect to this asymptote.*

The Physicist's 'Back-of-the-Envelope' Solution

This exercise follows the *very French tradition* of guiding the student step by step to the solution.

I will not follow that tradition; instead, I will let the study of the asymptotic ($x \to \infty$) behavior of the function f guide us naturally to the steps listed in the exercise.

First, I note that for large x, there are clearly dominant terms in f. Namely[1]:

$$f(x) = \ln(e^x + \sqrt{e^{2x}+1}) \sim \ln(e^x + \sqrt{e^{2x}})$$
$$= \ln(e^x + e^x) = \ln(2e^x) = \ln(2) + x$$

This calculation—which is quite clear, by the way—should be sufficient to suggest the asymptote $y = x + \ln(2)$. Nevertheless, it is interesting to study f with more precision. With that goal in mind, we take the following steps:

$$\begin{aligned} f(x) &= \ln(e^x + \sqrt{e^{2x}+1}) \\ &= \ln\left[e^x + \sqrt{e^{2x}(1+e^{-2x})}\right] \\ &= \ln\left[e^x + e^x\sqrt{1+e^{-2x}}\right] \\ &= \ln\left[e^x(1+\sqrt{1+e^{-2x}})\right] \\ &= \ln(e^x) + \ln(1+\sqrt{1+e^{-2x}}) \\ &= x + \ln(1+\sqrt{1+e^{-2x}}) \end{aligned}$$

We have thus shown that

$$f(x) = x + \ln\left(1 + \sqrt{1+e^{-2x}}\right).$$

In this expression, we clearly see that x is the linear part of the slant asymptote of $f(x)$. We also become confident that the remaining behavior of f for large x is associated with the log

[1]The symbol $\sim$ used here denotes behavior for large x

function. Indeed, for very large values of x, e^{-2x} goes to zero very rapidly, allowing me to write:

$$\sqrt{1+e^{-2x}} \approx 1 + \frac{1}{2}e^{-2x}\,.$$

The above reasoning makes the introduction and study of the function g natural. For large values of x, setting $t = e^{-2x}$, the remaining term behaves as:

$$\begin{aligned}\ln(1+\sqrt{1+e^{-2x}}) &\approx \ln\left(1+1+\frac{1}{2}e^{-2x}\right)\\ &= \ln\left(2+\frac{1}{2}e^{-2x}\right)\\ &= \ln\left[2\left(1+\frac{1}{4}e^{-2x}\right)\right]\\ &= \ln(2) + \ln\left(1+\frac{1}{4}e^{-2x}\right)\end{aligned}$$

Using the standard Taylor expansion for $\ln(1+s)$ where s is small yields:

$$\ln(1+\sqrt{1+t}) = \ln(2) + \frac{1}{4}t + o(t)$$

where $t = e^{-2x}$.

Substituting this back into f yields:

$$f(x) = x + \ln(2) + \frac{1}{4}e^{-2x} + o(e^{-2x})$$

This result shows in full detail that the asymptote to f is indeed the line:

$$\boxed{y = x + \ln(2)}$$

Since the leading term of the difference $f(x) - (x + \ln(2))$ is $\frac{1}{4}e^{-2x}$, which is strictly positive, we conclude that the graph of f approaches the asymptote from **above**.

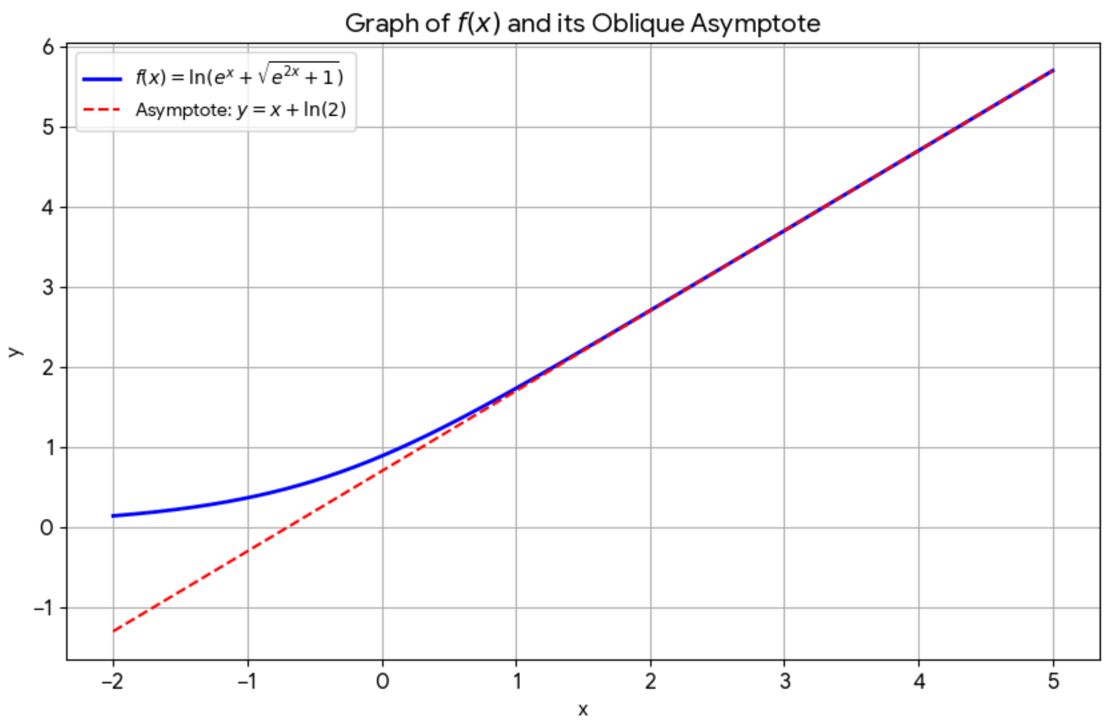

Figure 1: Graph of the function $f(x)$ (blue) approaching its oblique asymptote $y = x + \ln(2)$ (red dashed line) from above.

EXERCISE 3 (5 MARKS = 2+3) Recall that a function : $\Re \to \Re$ is uniformly continuous if and only if for any fixed $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that, for any $x, y \in \Re$,

$$|x - y| < \delta \to |f(x) - f(y)| < \epsilon$$

1. Show that $\forall x, y \in \Re$

$$||y| - |x|| \leq |y - x|$$

2. Consider the function

$$f : \Re \to \Re$$

defined by

$$f(x) = \frac{1}{1 + |x|}$$

Show that is uniformly continuous on $\Re$.

The Street Fighter Solution

Let's look at what these mathematical statements actually mean in the real world.

Part 1: The Inequality

$$||y| - |x|| \leq |y - x|$$

This looks intimidating, but it is just the **Reverse Triangle Inequality**. Geometrically, $|x|$ is the distance from the origin to x, and $|y - x|$ is the distance between x and y. Think of it as a triangle with vertices at $0, x$, and y. This inequality simply states that the difference in length between two sides of a triangle cannot be longer than the third side. If I walk to point y and you walk to point x, the difference in our travel distances cannot exceed the straight-line distance between us. It is physically obvious.

Part 2: Uniform Continuity We are asked to show that $f(x) = \frac{1}{1+|x|}$ is uniformly continuous. In the language of physics, uniform continuity essentially asks: "Does the function have a speed limit?"

If a function has a derivative that is bounded everywhere (i.e., $|f'(x)| \leq M$), then the function cannot "jump" or change too fast. It satisfies the Lipschitz condition, which guarantees uniform continuity.

Let's look at the derivative of our function for $x \neq 0$:

$$f'(x) = \frac{d}{dx}(1 + |x|)^{-1} = -(1 + |x|)^{-2} \cdot \operatorname{sgn}(x) = \frac{\mp 1}{(1 + |x|)^2}$$

The maximum absolute value of this "velocity" happens near $x = 0$, where the slope approaches ± 1. As $x \to \infty$, the slope goes to 0. Since the speed is capped at 1 everywhere (the slope never exceeds 1), the function is definitely uniformly continuous. We can simply pick $\delta = \epsilon$ and call it a day.

The "Standard" Textbook Solution

Part 1: Proving the Inequality

We rely on the standard Triangle Inequality: $|a + b| \leq |a| + |b|$.

We can write y as $(y - x) + x$. Applying the Triangle Inequality:

$$|y| = |(y - x) + x| \leq |y - x| + |x|$$

Subtracting $|x|$ from both sides gives:

$$|y| - |x| \leq |y - x| \quad \text{— (i)}$$

Similarly, we can write $x = (x - y) + y$. Applying the inequality:

$$|x| = |(x - y) + y| \leq |x - y| + |y|$$

Since $|x - y| = |y - x|$, we rearrange to get:

$$|x| - |y| \leq |y - x| \implies -(|y| - |x|) \leq |y - x| \quad \text{— (ii)}$$

Combining (i) and (ii), we have:

$$-|y - x| \leq |y| - |x| \leq |y - x|$$

Which is the definition of the absolute value inequality:

$$\boxed{||y| - |x|| \leq |y - x|}$$

Part 2: Uniform Continuity

We must show that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x-y| < \delta \implies |f(x)-f(y)| < \epsilon$.

Let us analyze the difference $|f(x) - f(y)|$:

$$|f(x) - f(y)| = \left| \frac{1}{1 + |x|} - \frac{1}{1 + |y|} \right|$$

Finding a common denominator:

$$= \left| \frac{(1 + |y|) - (1 + |x|)}{(1 + |x|)(1 + |y|)} \right|$$

$$= \frac{||y| - |x||}{(1 + |x|)(1 + |y|)}$$

We now apply two crucial observations:

1. From Part 1, we know that $||y| - |x|| \leq |y - x|$.

2. The denominator $(1 + |x|)(1 + |y|)$ is always greater than or equal to 1 (since $|x| \geq 0$ and $|y| \geq 0$). Therefore, $\frac{1}{(1+|x|)(1+|y|)} \leq 1$.

Using these estimates:

$$|f(x) - f(y)| \leq \frac{|y - x|}{1} = |x - y|$$

To satisfy the definition of uniform continuity, we simply choose $\delta = \epsilon$.

$$\text{If } |x - y| < \delta, \text{ then } |f(x) - f(y)| \leq |x - y| < \delta = \epsilon$$

Conclusion: Since δ depends only on ϵ (and not on x or y), f is uniformly continuous on $\mathbb{R}$.

PROBLEM (7 MARKS = 4(1+0.5+1+1.5)+3(1+0.5+1.5))

Let $f : (0, +\infty) \to \Re$ be a twice continuously differentiable function such that

$$\lim_{x \to +\infty} f(x)$$

exists and is finite.

The aim of this exercise is to show that if f'' is bounded, then

$$\lim_{x \to +\infty} f'(x) = 0$$

1. Let $M > 0$ be such that for all $x > 0$, $|f''(x)| \leq M$, and define the sequence $(x_n)_{n \in N}$

$$x_n := \frac{n\epsilon}{M},$$

where $\epsilon > 0$ is fixed.

(a) Show that $\exists m \in N$ such that

$$n > m \to |f(x_{n+1}) - f(x_n)| \leq \frac{\epsilon^2}{M}$$

(b) Deduce that for $n > m$, $\exists c_n \in (x_n, x_{n+1})$ such that

$$|f'(c_n)| \leq \epsilon$$

(c) Deduce in turn that

$$u \in [x_n, x_{n+1}] \to |f'(u) - f'(c_n)| \leq M|u - c_n| \leq \epsilon$$

(d) Combine all the findings above to show that

$$u \in [x_n, x_{n+1}] \rightarrow |f'(u)| \leq 2\epsilon$$

and conclude

2. Consider the function

$$h : (0, +\infty) \rightarrow \Re : x \mapsto \frac{cos(x^2)}{x+1}$$

(a) Show that $f \in \mathcal{C}^2((0, \infty))$

(b) Show that

$$\lim_{x \to +\infty} f(x) \text{ is finite}$$

(c) Study

$$\lim_{x \to +\infty} h'(x)$$

and explain the apparent contradiction with Question 1.

Intuition Behind Question 1 of the the Problem

The first things to note are that being $f \in \mathcal{C}^2[(0, +\infty)]$ a twice continuously differentiable function f is pretty smooth, besides, the condition

$$\lim_{x \to +\infty} f(x)$$

exists and is finite. Is telling us that f has a horizontal asymptote (this is quite exactly what we must prove).

The Big Picture

We want to prove that if f approaches a finite limit as $x \to +\infty$ and f'' is bounded, then f' must approach zero.

Why should this be true intuitively?

- If $f(x)$ converges to some value L as $x \to +\infty$, the function is "settling down" to a horizontal asymptote
- If f' didn't go to zero, the function would keep rising or falling at a persistent rate, contradicting the fact that it has a finit limit for large values of x
- The bounded second derivative f'' prevents f' from "wiggling" too much — it can't suddenly spike up or down

The Strategy

We construct a sequence of points $x_n = \frac{n\epsilon}{M}$ that march off to infinity. The spacing between consecutive points is $\frac{\epsilon}{M}$. And build an associated succession of points $f(x_n)$

Since f has a limit the succession converges, and this means that eventually $|f(x_{n+1}) - f(x_n)|$ becomes tiny. By the Mean Value Theorem, this means f' is small at *some point* in each interval.

The bounded f'' then ensures f' can't vary too much within each interval, so f' is small *everywhere* on these intervals.

Since ϵ is arbitrary, we conclude $f' \to 0$.

Rigurous Discussion

Part (a)

Since $\lim_{x\to+\infty} f(x)$ exists and is finite, say $\lim_{x\to+\infty} f(x) = L$, the function f satisfies the *Cauchy property* at infinity. This means:

For any $\delta > 0$, there exists $X > 0$ such that for all $x, y > X$:

$$|f(x) - f(y)| < \delta$$

Choose $\delta = \frac{\epsilon^2}{M}$. Then there exists $X > 0$ such that for all $x, y > X$:

$$|f(x) - f(y)| < \frac{\epsilon^2}{M}$$

Since $x_n = \frac{n\epsilon}{M} \to +\infty$ as $n \to \infty$, there exists $m \in \mathbb{N}$ such that $x_m > X$.

Therefore, for all $n > m$, we have $x_n > X$ and $x_{n+1} > X$, which gives:

$$|f(x_{n+1}) - f(x_n)| \leq \frac{\epsilon^2}{M}$$

Part (b)

By the Mean Value Theorem applied to f on the interval $[x_n, x_{n+1}]$, there exists $c_n \in (x_n, x_{n+1})$ such that:

$$f'(c_n) = \frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}$$

Now, $x_{n+1} - x_n = \frac{(n+1)\epsilon}{M} - \frac{n\epsilon}{M} = \frac{\epsilon}{M}$.

Therefore:

$$|f'(c_n)| = \left|\frac{f(x_{n+1}) - f(x_n)}{x_{n+1} - x_n}\right| = \frac{|f(x_{n+1}) - f(x_n)|}{\epsilon/M} \leq \frac{\epsilon^2/M}{\epsilon/M} = \epsilon$$

Part (c)

Apply the Mean Value Theorem to f' on the interval between c_n and u (where $u \in [x_n, x_{n+1}]$).

There exists a point ξ between c_n and u where:

$$f''(\xi) = \frac{f'(u) - f'(c_n)}{u - c_n}$$

Since $|f''(x)| \leq M$ for all $x > 0$, we have:

$$|f'(u) - f'(c_n)| = |f''(\xi)| \cdot |u - c_n| \leq M|u - c_n|$$

Now, since both u and c_n are in $[x_n, x_{n+1}]$, we have:

$$|u - c_n| \leq x_{n+1} - x_n = \frac{\epsilon}{M}$$

Therefore:

$$|f'(u) - f'(c_n)| \leq M \cdot \frac{\epsilon}{M} = \epsilon$$

Part (d)

For any $u \in [x_n, x_{n+1}]$ with $n > m$, by the triangle inequality:

$$|f'(u)| = |f'(u) - f'(c_n) + f'(c_n)| \leq |f'(u) - f'(c_n)| + |f'(c_n)| \leq \epsilon + \epsilon = 2\epsilon$$

Conclusion: Let $\eta > 0$ be arbitrary. Choose $\epsilon = \frac{\eta}{2}$. Then with this choice, there exists $m \in \mathbb{N}$ such that for all $n > m$:

$$u \in [x_n, x_{n+1}] \Rightarrow |f'(u)| \leq 2\epsilon = \eta$$

Since $x_n \to +\infty$ as $n \to \infty$, the intervals $[x_n, x_{n+1}]$ eventually cover all sufficiently large values of x. Specifically, for $x \geq x_{m+1}$, there exists some $n > m$ with $x \in [x_n, x_{n+1}]$, and thus $|f'(x)| \leq \eta$.

Since $\eta > 0$ was arbitrary, we conclude:

$$\boxed{\lim_{x \to +\infty} f'(x) = 0}$$

Intuition Behind Question 2 2

What's happening here

This is a **counterexample construction** designed to show that the hypotheses in Question 1 are *necessary*. The function $h(x) = \frac{\cos(x^2)}{x+1}$ is crafted to satisfy *most* but not *all* the conditions from Question 1, and we'll see that the conclusion fails.

The apparent contradiction

- Question 1 says: if $f \in C^2$, $\lim_{x\to\infty} f(x)$ exists and is finite, and f'' is bounded, then $f' \to 0$
- Here, $h(x) = \frac{\cos(x^2)}{x+1}$ will satisfy the first two conditions but h' will **not** converge to 0
- The resolution: h'' is **not bounded**!

Why this function?

The numerator $\cos(x^2)$ oscillates rapidly (faster and faster as x increases), while the denominator $x+1$ grows linearly. The oscillations get "compressed" and eventually damped out, so $h(x) \to 0$.

But the derivative will have terms involving $-2x\sin(x^2)$ from the chain rule, which oscillates with amplitude growing linearly. Even after dividing by $(x+1)^2$, this creates persistent oscillations in $h'(x)$ that don't die out.

The second derivative will be even worse, showing h'' is unbounded.

Solution

Part (a): Show that $h \in C^2((0, +\infty))$

The function $h(x) = \frac{\cos(x^2)}{x+1}$ is composed of:

- The numerator: $\cos(x^2)$ is infinitely differentiable (composition of smooth functions)
- The denominator: $x+1 > 1$ for all $x > 0$, so it never vanishes and is infinitely differentiable

Since h is the quotient of two C^∞ functions with non-vanishing denominator on $(0, +\infty)$, we have $h \in C^\infty((0, +\infty))$, and in particular $h \in C^2((0, +\infty))$.

Part (b): Show that $\lim_{x\to+\infty} h(x)$ is finite

For all $x > 0$:

$$|h(x)| = \left|\frac{\cos(x^2)}{x+1}\right| \leq \frac{|\cos(x^2)|}{x+1} \leq \frac{1}{x+1}$$

Since $\lim_{x\to+\infty} \frac{1}{x+1} = 0$, by the squeeze theorem:

$$\lim_{x\to+\infty} h(x) = 0$$

which is finite.

Part (c): Study $\lim_{x\to+\infty} h'(x)$ and explain the apparent contradiction

Computing the derivative

Using the quotient rule:

$$h'(x) = \frac{d}{dx}\left[\frac{\cos(x^2)}{x+1}\right] = \frac{-\sin(x^2)\cdot 2x\cdot(x+1) - \cos(x^2)\cdot 1}{(x+1)^2}$$

$$= \frac{-2x(x+1)\sin(x^2) - \cos(x^2)}{(x+1)^2}$$

$$= \frac{-2x\sin(x^2)}{x+1} - \frac{\cos(x^2)}{(x+1)^2}$$

Analyzing the behavior as $x \to +\infty$

The first term dominates: $\frac{-2x\sin(x^2)}{x+1} \approx \frac{-2x\sin(x^2)}{x} = -2\sin(x^2)$ for large x.

Since $\sin(x^2)$ oscillates between -1 and 1 indefinitely (and increasingly rapidly as $x \to \infty$), the term $-2\sin(x^2)$ oscillates between -2 and 2.

The second term $\frac{\cos(x^2)}{(x+1)^2} \to 0$ as $x \to +\infty$.

Therefore, $h'(x)$ oscillates approximately between -2 and 2 for large x, and:

$$\boxed{\lim_{x\to+\infty} h'(x) \text{ does not exist}}$$

Explaining the apparent contradiction

There is no contradiction! Question 1 requires **three conditions**:

1. ✓ $h \in C^2((0, +\infty))$ — satisfied

2. ✓ $\lim_{x\to+\infty} h(x)$ exists and is finite — satisfied

3. × h'' is bounded — **NOT satisfied**

Let's verify that h'' is unbounded:

$$h''(x) = \frac{d}{dx}\left[\frac{-2x\sin(x^2)}{x+1} - \frac{\cos(x^2)}{(x+1)^2}\right]$$

The dominant term from differentiating $-2x\sin(x^2)$ gives:

$$-2\sin(x^2) - 4x^2\cos(x^2) + \text{lower order terms}$$

The term $-4x^2\cos(x^2)$ oscillates with amplitude $4x^2 \to \infty$, so h'' is **unbounded**.

Conclusion

This example shows that the boundedness of f'' is **essential** in Question 1. Without it, even though $f(x)$ converges to a finite limit, the derivative $f'(x)$ can fail to converge to zero. "'

Python Code for Visualization

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the range for x
x = np.linspace(-2, 5, 400)

# Define the function f(x) and the asymptote
y_f = np.log(np.exp(x) + np.sqrt(np.exp(2*x) + 1))
y_asymp = x + np.log(2)

# Plotting
plt.figure(figsize=(10, 6))
plt.plot(x, y_f, label='f(x)', color='blue')
plt.plot(x, y_asymp, label='Asymptote', color='red', linestyle='--')

plt.title('Graph of f(x) and its Oblique Asymptote')
plt.xlabel('x')
plt.ylabel('y')
plt.legend()
plt.grid(True)
plt.show()
```

Listing 1: Plotting the Asymptote