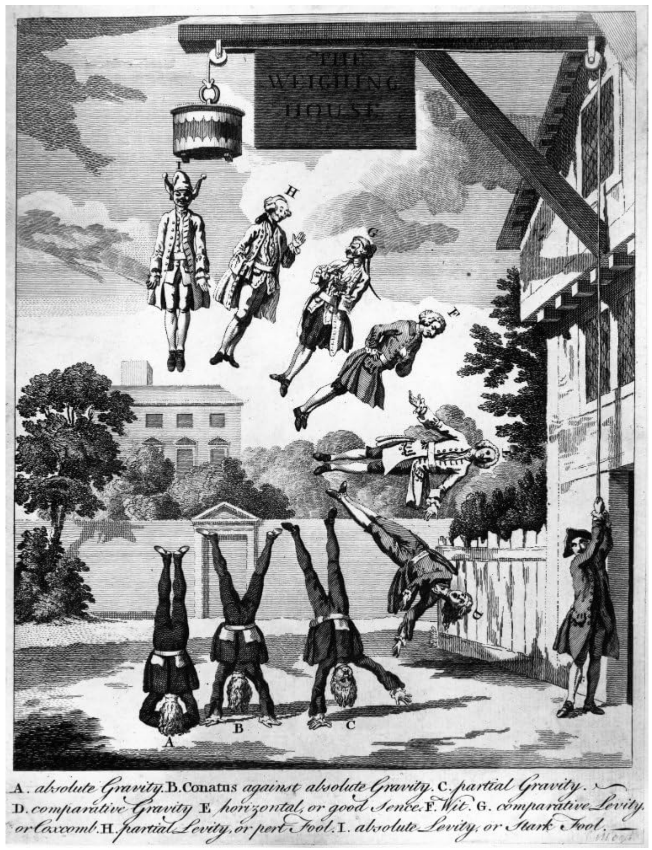

Algebra Based Physics: Mechanics

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Contents

An important note One secret to master problem solving in general and science in particular is to examine each new situation with great care looking for similarities to some situations or problems we might have already found.

Chapter 1

Introduction

1.1 What is physics all about?

When you were about four to seven years old, you probably asked questions like these:

1. How does a car move?

2. Daddy, is the moon far away?

3. Mommy, why is it light during the day and dark at night?

4. Why does my robot turn off when the *batteries run out*?

5. [If daddy and/or mommy were adventurous:] Why does my tummy feel weird when we are going fast down the roller coaster?

6. [In the kitchen after someone's mistake:] "wow... how cool, the food that was in the *pressure cooker* blew out when you opened it, can we repeat it to see if it happens again?"

Sometimes the answers were somehow easy to understand, ...“Honey, we’re forty kilometers from [write down the name of your preferred city] and we’re going a hundred kilometers per hour, if we keep going like this we’ll arrive in a little less than half an hour”. Other times things got complicated for everyone, ...“uhmmmm, son, the sky is blue ... because .... well, because you know, it is blue. What do you want me to say?”. Other times the answer was (Grandma): ...“child, you ask so many silly questions, take a deep breath and sit quietly because your mom is driving and you’re going to drive her crazy”.

The simple truth about those questions is that you were interested in physics questions, and your curiosity was a small untrained scientist that we all carry inside. Over time, that budding physicist may have dozed off and perhaps mass formal education and society managed to do their job convincing you that those questions are nonsense that only interest “nerds”, but remember something, the little worm is there inside, and you just have to release it to start asking again and maybe get super interesting answers.

How far away is the moon? Well[1] it’s far away, Honey. About 384,000 km or 238606.538 mi to be precise. Do you remember when you asked me about the fastest thing, I told you that light is the fastest thing in the universe?. Well the distance to the moon I just told you means that, if you fired a beam of light to a mirror on the moon, it would take a little over two seconds to the beam to come back.

The sun is much farther away, about 150 million km, which is nearly 391 times farther than. A ray of light that leaves the sun takes about eight minutes to get here to Earth. And the stars, whoa!, they are much, much farther away. So much so, that from the nearest star, It takes about 4 years and four months for alight ray to get here (Earth).

The color of the sky? Hmmmm ... there is no way to give a short answer.

[1] an answer for an eight-year-old girl

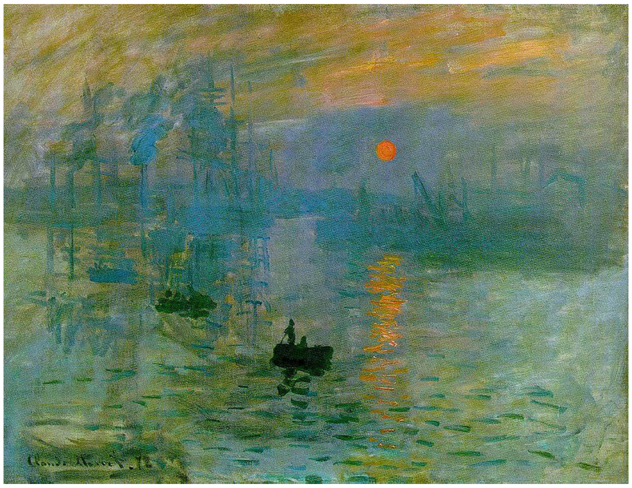
Science in general, and physics in particular, aims to describe elements of reality **quantitatively** and objectively. One step in that direction is that we all must agree on that what we measure with our instruments is objective and independent of our personal perception. If someone does not agree on such instrumental view of objectivity like this, she or he must take the discussion for what it is, a philosophical topic about the notion of reality.

Even accepting objectivity, the task of physics is enormously non-trivial and requires a high degree of abstraction. Until not long ago (late 19th century), the field of study of physics was relatively limited to certain areas, including the motion of objects such as cars or the moon, the construction of buildings, temperature and heat, and electricity and magnetism. These areas of study have the names mechanics, thermodynamics and electromagnetic theory.

More recently physics has been able to seriously address the question what is everything made of? and for attacking such difficult questions we need the theory of relativity, and quantum theory.

With the advent of inter- and transdisciplinarity, physics has invaded an enormous number of fields of great interest; biochemistry, economics, neuroscience, and many aspects of modern social sciences, all these fields are receiving very important contributions from physicists who, using the typical tools of their discipline, find and explore new aspects on countless problems of great interest.

Many of the elements of reality that interest physics also interest other areas of human activity beyond science. The observation of the daytime sky, for example, is of profound interest to both a physicist and, for instance, a painter or a writer; all three will surely notice the color changes in the sky and try to describe them in their particular terms.



Impression, Sunrise, Claude Monet. 1872 Oil on canvas, Marmottan-Monet Museum, Paris

Monet's *Impression, Sunrise* provides us with an example of a representation of reality, in this case a view of the port of Le Havre (Normandy) at dawn. Despite the enormous aesthetic beauty that suggests many things to us, the representation of the port (and the dawn) that the artist gives us is not, under any circumstances, objective; each of us will have different sensations and thoughts when contemplating the painting. In the wors of Claude Monet:

The landscape is nothing more than an impression, an instantaneous impression, hence the title, an impression that it gave me. I have reproduced an impression in Le Havre, from my window, sun in the mist and a few silhouettes of boats standing out in the background... they asked me for a title for the catalog, it couldn't really be a view of Le Havre and I said put **Impression**

Unlike the subjectivity associated with the representation of reality provided by an artist, a scientist seeks an objective (observer-independent) and quantitative description. The objectivity that science has as a partial goal has the consequence that there are not a few who perceive science as dehumanized in the most extreme cases and as impersonal in some more moderate ones. It is very generally thought that the scientific view of reality lacks aesthetics and emotions[2].

Now, what is exactly meant by that quantitative description or better yet, **model** of reality? To answer the question, let's consider a very simple example that was studied by Galileo, the pendulum.

A simple pendulum is nothing more than a mass tied by a string to a fixed suspension point. The period (T) of the pendulum is the time it takes to pass twice through the same position (let's say point A in the drawing inserted in the photo). The length of the pendulum is, by definition, the length (ℓ) of the thread that connects the mass to the suspension point.

Suppose we are interested in modeling the relationship (if any) between the period (T) and the length (ℓ) of the pendulum. Well, a quantitative model consists of establishing some mathematical relationship of the form

$$T = f(\ell), , \tag{1.1}$$

[2]nothing could be further from the truth, science is a human activity and therefore its practitioners are emotional beings with aesthetic values that often guide their research

but that's not all, like all science, physics is experimental, its results (predictions or models) must be not only quantitative but also verifiable with reality, that is, measurements must be made and the quantitative relationships that constitute our models must be compared with the numerical results obtained after conducting experiments.

In the case of the pendulum, it is necessary to measure the period when we vary the length of a real pendulum[3], keeping all other variables constant[4]. When doing the experiments, we will find that as the length increases, the period also increases.

If you try the experiment at home and take values that you write down in a table, you will surely find that if you double the length of the pendulum, the period will increase by an approximate factor of 1.4, and if you triple the length, the period will increase by something like a factor of 1.7, in fact, you will find that your table of values for oscillations with angles no greater than 30° indicates that the formula that relates the quantities we are interested in has to be[5]:

$$T = 0.1\sqrt{\ell} \tag{1.2}$$

According to Karl Popper's[6] ideas, the only thing we can do in science, and of course in physics, from the point of view of epistemology or theory of knowledge, is to propose models or theories that aim to describe phenomena and dedicate ourselves to seeking experiments -called critical- that allow us to falsify such models.

Following Popper, as long as we don't find critical experiments, the models only possess a statistical value whose content is reduced to telling us that our model fits all experimental values within a certain range and that therefore they are not demented models. Let's consider

[3] a nice clock like the one in the photo, for example

[4] one way to do this is to release the pendulum from the same initial angle in each experiment

[5] the length must be measured in *cm* and the period in seconds

[6] Sir Karl Raimund Popper (28 July 1902 – 17 September 1994) was an Austrian–British philosopher, he was in fact, one of the 20th century's most influential philosophers of science

again our example of the simple pendulum and suppose that someone proposes that the model describing the period as a function of the pendulum's length is $T = 0.1\ell$. If all the experiments with pendulums are carried out with pendulums of ℓ very close to $1.0, cm$, the model will seem true.

A Popperian experimenter will seek any way to falsify the model and will achieve it without much difficulty by measuring the period for large lengths. Let's take $\ell = 200\,cm$ to show what happens, the model predicts a period of $20; s$ but the experimental measurements will result in values close to $1.42\,, s$, which dramatically highlights that the model $T = 0.1, \ell$ has nothing to do with reality, thus determining its invalidity.

The model $T = 0.1, \sqrt{\ell}$ is more acceptable, it should be considered as one of the adequate models -for now- and of course requires the search for a critical experiment whose objective will be to invalidate the new model.

1.2 Units of measurement and quantitativity

As we mentioned in the previous section, we want to be quantitative. Imagine if we weren't quantitative at all, one would go to a design conference -aeronautical for example- and when asking about the size of the plane being planned, the answer would be something like: "huge". An answer like that would make it extremely difficult to be responsible for designing the wings of the aircraft in question.

In more elementary terms, if we say that "Joe is tall", we're giving very little information if any, perhaps in the Netherlands Joe is in fact, a rather short man. Thus, *big, small, much, little, etc.* are adjectives that, from a scientific point of view, are essentially useless. Perhaps a not so appropriate use of such adjectives in the framework of science should be in phrases like: *bigger than*, etc. but even that is not as good as we need.

The best use of any of the adjectives we are discussing would be in phrases that include the expression *so many times smaller than*, that's the idea behind being quantitative. Indeed, by saying that something weighs the same as two bricks we are establishing a standard (a unit of measure of 'weight', if when establishing a measurement standard (the weight of a brick), for scientific use, the standard becomes known to a certain group of the population, that of people who deal with bricks for example, the standard becomes conventional within that group and can be used there without much problem.

Thus, for example, in my country of birth: Venezuela, we all understand quite well that by saying: *Maritza has a height of one meter seventy* we are talking about a lady whose height is equal to 1.70 times a certain length standard called 'meter', which makes her a woman who in Venezuela is considered 'tall'. To an American, we would have to explain (to adapt to their conventional standards) that Maritza measures *five feet and eight inches*.

In physics there are certain quantities called basic from which all others are constructed or defined, which are called derived physical quantities. Associated with these quantities, there are two types of standards (or units), *basic units* and *derived units*. In the International System of Units seven basic physical quantities are used whose standards are given in table 1.2. Derived units are used to express physical quantities that are the result of combining basic quantities, it is important to note that multiples and submultiples of basic units are not derived units. Some examples of basic quantities and their corresponding units appear in table 1.2

1.3 Significant Figures

The models of reality that we construct in science must be compared with experimental results and these in turn are quantitative, that is, numerical. However, the figures reported as a result of an experiment have an associated uncertainty (error) that is intrinsic to the measurement

Physical Quantity	Dimensional Symbol	Standard Name	Symbol
Length	L	meter	m
Time	T	second	s
Mass	M	Kilogram	Kg
Temperature	Θ	Kelvin	K
Amount of substance	N	mole	mol
Electric current	I	Ampere	A
Luminous intensity	J	candela	cd

Table 1.1: SI basic units

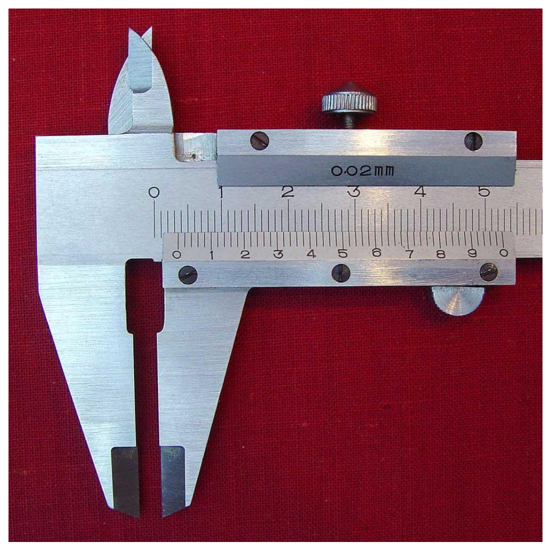
process.

Let's consider an example, the measurement of the diameter of the thumb of a person's right hand using an ordinary 30 cm ruler as a measuring instrument. Suppose the result of the measurement is 2.5; cm, it would be wrong to express this figure as $2.50\,cm$ since an ordinary ruler is only reliable to millimeters ($0.1\,cm$) and therefore it is not possible to distinguish between $2.52\,cm$, $2.55\,cm$ or $2.56\,cm$. However, if we use a vernier caliper, we can report that Mario's thumb diameter is $2.54135\,cm$.

Physical Quantity	Dimensional Symbol	Standard Name	Symbol
Speed	L/T	meters per second	m/s
Force	$M \times \frac{L}{T^2}$	Newton	N
Power	$M \times \frac{L^2}{T^3}$	watts	$watt$

Table 1.2: Some SI derived units

The vernier is a refinement of length measuring instruments that is achieved by adding a secondary scale. The name nonius refers to the Portuguese Petrus Nonius (1492-1577), the instrument was later improved by Pierre Vernier in 1631. The figure shows a Vernier capable of appreciating $0.02; mm$.



The difference between the two measurements, the one made with the ruler and the one carried out with the vernier, is the appreciation or measurement uncertainty. The measurement carried out with the vernier has a lower uncertainty than the one made with the ruler, as it is a measurement of greater appreciation. Uncertainty is commonly called measurement error because it indicates the maximum difference that is estimated to exist between the measured value and the real value of a quantity

$$error = \delta \equiv |real_{value} - measured_{value}|,, \tag{1.3}$$

the error in a measurement depends, as we have already seen, on the measurement technique used. Usually, the uncertainty in a measurement is indicated by writing the value of the measurement followed by the symbol $\pm$ followed by a second number that expresses the experimental error. If we report that Mario Sebastián's height is $1.252 \pm 0.001\,m$ we are trying to express that it is unlikely that MS's height exceeds $1.253\,m$ or is below $1.251\,m$. In a somewhat more modern notation, the expression 1.252 ± 0.001 is represented by the symbol $1.252(1)$, in this notation, the figure in parentheses shows the error in the measurement of the main figure. Uncertainty is also often expressed in terms of the maximum relative error (fractional or percentage error). Mario Sebastián's height reported as $1.252 \pm 0.001; m$ is expressed in terms of percentage error as $1.252\,m \pm 0.1\%$ since the fractional error in the measurement is $\delta = 0.001/1.251 \times 100\%$. A usual way of presenting uncertainty consists of forgetting the explicit reference to errors and indicating it through the number of *significant figures*. When we report that the diameter (D) of Sophia's thumb's first phalanx is $12.7; mm$ we are expressing D with three significant figures, this implies that there is no uncertainty in the first two figures and that the uncertainty is in the value of the third. Given that in this case the third digit is in the tenths, reporting $D = 12.7\,mm$ means that the uncertainty in the measurement is approximately $0.1\,mm$.

It's important to note that two figures with the same number of significant figures can be associated with very different uncertainties. Indeed, two lengths, the distance Maracay-Caracas $109, Km$ and the author's height, $1.72; m$ reported with the same number of significant figures (3), have associated errors of $1; Km$ and $1; cm$ respectively. The percentage errors, however, $\delta_1 = 1/109$ and $\delta_2 = 1/1.72$ are not so different $(0.5 - 1\%)$ When performing calculations involving quantities with uncertainties, the resulting values also have uncertainties, and there are certain basic rules to control them, for example:

- When adding or subtracting numbers, the location of the decimal point determines the number of significant figures. Let's consider the sum $247.35 + 2.8$, the result with the number of significant figures is $245.35 + 2.8 = 248.1$. Indeed, the uncertainty in the addend 245.32 is approximately one hundredth, while for the addend 2.8 the uncertainty is one tenth, which obviously means that the uncertainty in the sum must be in that order, that is, in tenths.

- The number of significant figures in a product or division cannot be greater than the number of them in the factor with the maximum uncertainty or the least number of significant figures. For example, $2.7183 \times 2.34 \times 0.4 = 26.7$.

The general rules for following uncertainty throughout a calculation are called error propagation norms.

A basic application of these ideas that you should practice at home is determining an experimental value for π, for this we simply use the definition of π as the ratio of the length of a circle to its diameter[7]. Given the definition, we just need to draw a circle (the bigger the better), measure its length and divide it by the diameter. At the time of writing this introduction, the author has used a circular dining table he has at home and has measured the diameter obtaining[8] $D = 3, ft; 10.46' \pm 0.05'$ while the circumference of the table turned out to be $\ell = 12, ft; 1.96' \pm 0.05'$, the ratio of these numbers is $\pi = D/\ell = 3.14 \pm 0.05$ or $\pi = 3.14(5)$ or $\pi = 3.14$ in all three ways of reporting the result it is clear that the uncertainties in the measurements leading to the final result are adequately treated. The set of rules that allows identifying significant digits in a given number are:

1. Every non-zero digit must be considered significant.

[7] We know that the true value of π to ten digits is 3.141592654

[8] One foot=12 inches, and yes, Professor Mario has and uses a tape measure that measures lengths in imperial units

2. All zeros that appear between non-zero digits are considered significant. 101.23 has five significant figures.

3. Zeros placed at the beginning of a number are not significant. 0.00012, for example, has two significant figures, 1 and 2.

4. Zeros at the end of a number that contains a decimal point are significant. The number 12.2300 has six significant figures, 1, , 2, , 2, , 3, , 0 and 0. The number 0.000122300 also has six significant figures. This convention of zeros after a decimal point allows to make clear the precision with which the given number is known. If a result accurate to four decimal places is reported as 45.22 it would seem that it is only accurate to two decimal places (four significant figures), writing the number as 45.2200 makes it clear, by virtue of this rule, that this number is known to four decimal places of precision.

5. Zeros at the end of a number without a decimal point are ambiguous. In a number like 1200 it is not clear if the precision is up to the non-zero digit and therefore it happens that the measurement is coincidentally an exact multiple of 100 or if it was just rounded to hundreds. There are several conventional methods to resolve these special cases but we won't discuss them here, we'll just say that it often becomes necessary to determine whether the zeros at the end of a number are significant or not. In this text we will try to avoid the inconvenience by appropriately handling the decimal separator.

6. A number whose digits are all zero, for example 0.0000 has no significant figures since the error represented in this notation is greater than the value of the measurement.

When performing calculations using significant figures, one must be careful. Today we use calculators and computers to perform calculations, and it's an error to present a result with all the digits that our calculation instrument's arithmetic is capable of providing, as such results do not adequately represent the uncertainty with which we know the magnitudes we use as data. The final result should be reported by rounding to the correct number of significant figures. Scientific notation is very useful for these purposes since the rules about significant figures apply to the factor that appears before the power of ten. For example, the number 6.02×10^{23} has three significant figures, while 1.609×10^{-19} has four. Scientific notation is very useful for avoiding potential ambiguity in zeros at the end of numbers. Consider, for example, an experiment where a five-significant-figure result has been reported as 15000. Here we have ambiguity in the three zeros at the end of the number fifteen thousand. By reporting the result as 1.5000×10^4 and using the rules for recognizing significant figures, the ambiguity is removed since the three zeros are placed to the right of the decimal separator and must consequently be taken into account as significant figures.

1.4 Physics and Intuition

Some argue that physics is intuitive, but in the author's opinion, this notion is dubious to say the least. Aristotelian mechanics, for example, contains intuitive elements that lead to false conclusions. According to Aristotle, the natural state of a body is rest, and for the body to move, it's necessary to exert an action on it. Do an experiment and convince yourself that this seems to be true. However, today we know this is false, and in fact, Newton's first law states the opposite. In truth, physicists, like engineers, musicians, writers, etc., develop an intuition associated with the exercise of the discipline that interests us. Physics is an experimental science, and because of this, a physicist's intuition is deeply associated with observable phenomena. The origin of the idea that physics is very intuitive lies in the undeniable fact that many of the phenomena that physics has studied have to do with anyone's day-to-day life: pushing things, turning lights on and off, looking at the sky, hearing, seeing the sea surface and observing waves, etc. However, and as we said before, this notion is not true. Many times physical phenomena contradict our intuition, and only the most complex experiments and/or the most extravagant theories alien to our intuition can give an adequate description of them.

1.5 Another Visit to Our Initial Question

We began this introduction with a tempting question: What is physics about? And right after, we asked a set of "childish" questions. In this last section, we're going to answer that question with some of the "childish" questions that physicists are interested in answering today:

1. Will we be able to use nuclear fusion as an alternative energy source?
2. Can we build a solvable model of fluid motion in turbulent regime?
3. If the answer to the previous question is affirmative, can we use these models to give long-term predictions of complex phenomena such as climate?
4. Can we understand evolution as something that naturally derives from the complexity of life itself?
5. What was the universe like about $10^{-43}\,s$ after it began its existence?
6. What is space like when examined at distances of about $10^{-35}\,m$?
7. Will we be able to travel large interstellar distances in some way?
8. What is the fate of the universe? Will it expand forever? Will it begin to contract at some point?
9. What is the origin of that little number we call mass?

Chapter 2

Classical Mechanics, A Short Visit

Classical mechanics stands as the cornerstone of modern physics, a field that unravels the mysteries of motion and "interaction" in our physical world. The evolution of classical mechanics is the captivating story of how humanity came to understand the dance of celestial bodies, the fall of an apple, and everything in between. This branch of physics is like a time capsule of scientific progress, each era adding its own discoveries and insights. Let's embark on a journey through the pivotal moments that shaped our understanding of how the universe moves:

- Ancient Greece (4th century BCE): Our journey begins with Aristotle. He believed heavier objects fall faster than lighter ones. Spoiler alert: he was wrong, but it took nearly 2000 years to prove it!

- Renaissance Italy (17th century): Enter Galileo Galilei. He challenged Aristotle's ideas by rolling balls down inclined planes and observing falling objects. Galileo showed that (ignoring air resistance) all objects fall at the same rate, regardless of their weight.

- England (late 18th century): Sir Isaac Newton takes the stage. Building on Galileo's work, Newton published his famous laws of motion and universal gravitation in 1687.

These laws explained everything from why apples fall to how planets orbit the sun.

- 18th-19th centuries: Brilliant minds like Euler, Lagrange, and Hamilton refined and expanded Newton's work, developing powerful mathematical tools to describe motion.

Classical mechanics helps us understand things like: How a car accelerates and brakes. Why a boomerang returns to the thrower. How a satellite stays in orbit. The motion of pendulums in grandfather clocks.

It's important to note that while classical mechanics works wonderfully for most everyday situations, it has limits. When dealing with very small (atomic) scales or very high speeds (near the speed of light), we need to use quantum mechanics and special relativity theory. This problem of having limits is common to all natural sciences where models always have a range of application outside which, they simply cannot be used.

Classical mechanics is like the opening chapter of an epic science saga. It set the stage for incredible discoveries and continues to be essential in engineering, space exploration, and understanding our daily lives.

Classical mechanics serves as the bedrock upon which much of modern physics is built. Its principles and methods provide a crucial foundation for understanding more advanced and specialized branches of physics:

- **Thermodynamics** The laws of motion and energy conservation in classical mechanics underpin our understanding of heat and energy transfer.

- **Electromagnetism** While dealing with different forces, electromagnetism borrows many mathematical techniques and concepts from classical mechanics.

- **Quantum Mechanics** Although it describes a very different realm, quantum mechanics often uses classical analogies and parallels to explain its more abstract concepts.

- **Relativity** Einstein's theories of special and general relativity can be seen as extensions of classical mechanics to extreme speeds and strong gravitational fields.
- **Astrophysics and Cosmology** The motion of planets, stars, and galaxies is largely described using classical mechanics, albeit with relativistic corrections for extreme cases.
- **Statistical Mechanics** This field bridges classical mechanics and thermodynamics, applying mechanical principles to large systems of particles.

Even as physics has advanced into new frontiers, classical mechanics remains invaluable for solving a wide range of practical problems in engineering, robotics, and everyday life. It provides a intuitive framework for understanding motion and forces, making it an essential starting point for anyone venturing into the world of physics. In essence, while other branches of physics may seem to overshadow it, classical mechanics continues to be the sturdy foundation upon which our understanding of the physical universe is built.

Chapter 3

Kinematics I

Kinematics is the branch of classical mechanics that describes the motion of points, objects and systems of groups of objects, without reference to the causes of motion (i.e., forces). The study of kinematics is often referred to as the "geometry of motion".

Imagine you're watching your favorite superhero zoom across the sky or a race car speeding around a track. Have you ever wondered how we can describe and understand their motion? That's where kinematics comes in! Kinematics is like the storyteller of motion in physics. It's a branch of mechanics that focuses on describing how objects move without worrying about why they move. Think of it as the "what" rather than the "why" of motion. In kinematics, we use simple concepts that you experience every day:

- **Position**: Where something is located
- **Distance:** How far or close an object is
- **Displacement:** The change in position from a starting and ending positions
- **Speed**: How fast an object is moving

- **Velocity:** Speed in a **specific direction**

- **Acceleration**: How quickly speed, direction or both is/are changing

These ideas help us answer questions like:

How long will it take you to get to school?, How high does a ball go when you throw it up?. At what point will two cars meet if they're driving towards each other?

Kinematics gives us the tools to describe the graceful arc of a dolphin leaping from the water, the erratic path of a butterfly, or the precise movements of a robot arm in a factory.

As you dive deeper into kinematics, you'll discover fascinating equations and graphs that can predict where objects will be at any given time. It's like having a crystal ball for motion!

Remember, kinematics is just the beginning. It sets the stage for understanding more complex ideas in physics, like forces and energy. But for now, let's enjoy exploring the "how" of motion in the world around us!

In this chapter we will concentrate on the motion of objects along a straight line, a picture of such might be a car racing along a straight track[1] such as the one shown in figure 3.1.

[1] While historically, the standard distance for drag racing was a quarter-mile (1,320 feet), Top Fuel dragsters and Funny Cars in the National Hot Rod Association (NHRA) and other major sanctioning bodies now race at a shorter distance of 1,000 feet (0.19 miles or 304.8 meters).

This change was implemented in 2008 for safety reasons following a fatal accident. The immense speeds reached by these nitro-methane powered vehicles made the quarter-mile distance increasingly hazardous in terms of braking and stopping within the available runoff area.

So, while some classes in drag racing still compete on the traditional quarter-mile, modern Top Fuel and Funny Car races are primarily 1,000 feet. You might still encounter some older references or discussions about quarter-mile times for these categories, but currently, the shorter distance is the standard in major professional competitions.

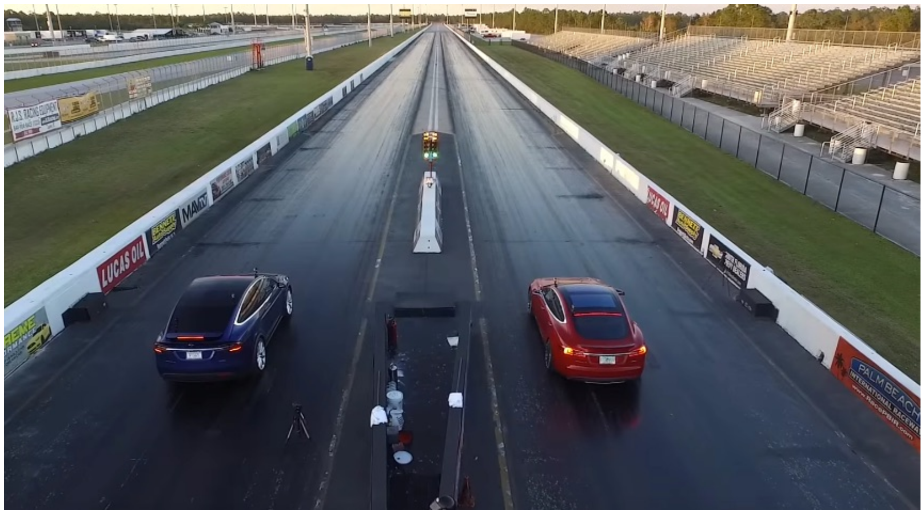


Figure 3.1: Straight racing car track. Until 2008 a coordinate system was usually set by calling $x = 0$ the starting point of the track and $x = 0.25\,mi$ the ending point of the racing

3.1 Position and displacement

Think of the track shown in figure 3.1, it has a particularly interesting point: the start, from where it opens giving space for the cars to move on, the simplest mathematical model for this is a line segment where we mark a special point and call it origin -$\mathcal{O}$-.

Let us for now imagine that we sit at the grandstands to the side of the track, in such a way that, for us, the cars race to our right. Have $\mathcal{O}$ been chosen, we let x be the signed position of a car[2](the moving body) with respect to $\mathcal{O}$, and let us for now take negative x for positions to the left of $\mathcal{O}$ and let x be positive for positions to the right of the origin, this sign assignment is called orientation. In order to give a precise description of the motion we need a clock to measure time (t), the measurement has to be done in some units, may they be, seconds, hours, microseconds, etc. As we did with the track, the description of the motion requires us to think

[2]for all practical purposes, a car is to be thought as a point

of some initial moment of time which we usually call t_0, it is quite customary to state $t_0 = 0$.

To be effective when doing physics we must learn the **dialect of physics**[3], physicists use regular words but endow them with very specific precise meanings. Imagine yourself walking along a $1\,mi$ long boulevard. let's say from $6\,pm$ to $9\,pm$. You begin your walk at some street corner, a position we call $x(6\,pm) = 0$, then you walk for a while and $10\,min$ later, you reach a library which is half a mile from the initial corner so $x(6:10\,pm) = +0.5\,mi$, where we have used our free will to choose the positive sing for x when you walk from the the initial corner to the library.

The **displacement** Δx between $t_0 = 6\,pm$ and $t_1 = 6:10\,pm$ is defined to be

$$\Delta x_{t_o t} \equiv x(t) - x(t_0) = x(6:10\,pm) - x(6:00\,pm) = 0.5\,mi\,, \tag{3.1}$$

You stay in the library for $40\,min$, go out and walk for $5\,min$ to an ice cream shop which is located $1/4\,mi$ of the initial corner in the direction to the library. In physicist dialect, your position at $t_2 = 55\,min$ is $x(55\,min) = 0.25\,mi$. At this point is where people tend to get confused because habits are deeply rooted in the mind, meaning that there is a psychological refusal to accept the new, more precise dialect. Just look at the following facts:

- **Positions**
  - $x(t_0) = x(6:00\,pm) = 0$, At six pm in the evening you are at the starting point of your walk (**motion**).
  - $x(t_1) = x(6:10\,pm) = +0.5\,mi$, 10 min later you are half a mile of the starting point in whatever you chose to be the positive direction.

[3] I have intentionally used the mathematical notation $x(t)$ to refer to the position at time t. With it, I am making a non-trivial link with math. We want to think of the particle's positions as a function of time (why a function? Well, because a particle can't be in two different places at the same time, can it?).

– $x(t_2) = x(6:55\,pm) = +0.25\,mi$. At $6:55\,pm$ you are a quarter mile from the starting point in the positive direction.

- **Displacements**

  – $\Delta_{t_0,t_1}x = x(t_1) - x(t_0) = x(6:10\,pm) - x(6:00\,pm) = +0.5\,mi$. Between $6:00\,pm$ and $6:10\,pm$ your displacement was half a mile in the positive direction.

  – $\Delta_{t_1,t_2}x = x(t_2) - x(t_1) = x(6:55\,pm) - x(6:10\,pm) = -0.25\,mi$. Between $6:10\,pm$ and $6:55\,pm$ your displacement was a quarter mile in the negative direction.

The funniest part comes in if you remain for quite a long time in the ice cream shop chatting with some friends, at $8:54\,pm$ you remember you are going to lose some tv show, say good bye, and rush back to the initial corner arriving there at exactly $9:00\,pm$, then, even though you logged a walk of $1.0\,mi$ your total displacement is

$$\Delta_{t_0,t_fin}x = x(t_fin) - x(t_0) = x(9:00\,pm) - x(6:00\,pm) = 0.00\,mi\,, \tag{3.2}$$

there is nothing wrong here, you indeed walked a mile, but the name of the total distance you walked is the **trajectory's length**, also known as the total arc length.

A simpler example would be running in an Olympic athletics track. Everyone knows the length of the track is a quarter mile, so, if you run four laps you will have logged a mile but your total displacement in the period of time it takes you to run such distance is definitively $0\,mi$

3.2 The notions of velocity and speed

Everyone understands the meaning of the words **fast** and **slow**. We are all aware that if three runners sprint in a race figure 3.2, the fastest one is that who reaches the finish line in the least

Figure 3.2: Elaine Thompson-Herah's 10.61 s to win Tokyo gold adjusts to the fastest wind-legal time ever, 10.57 s.

time, while the slowest is that who those in the longest time, the runner arriving in second place is neither the fastest nor the lowest, but is definitively slower than the fastest and faster than the slowest.

Any person who has ridden in a car has at least seen the speedometer (see figure 3.3). The numbers on the speedometer indicate the speed of the car in miles per hour (mph). For example, if the speedometer reads 30 mph, we know that it will take us about 18 minutes to travel 9 miles.

Let's examine how we know the above, i.e. that it will take 18 minutes to travel 9 miles. Most of us agree that speed (what the speedometer measures) is the rate at which the car

Figure 3.3: Instrument Cluster Meter of an SUV with the speedometer at the center

travels a distance in unit time. In our case, the car is moving at a speed of

$$30\,mph = 30\,miles/(60\,minutes) = 0.5\,miles/minute \tag{3.3}$$

In other words, at a speed of 30 mph, the car travels half a mile in one minute. So, traveling 9 miles, which is equal to 18 times half a mile, will take 18 times the time to travel half a mile, or 18 minutes.

The information we get from the speedometer is only about the rate of distance per unit time, we don't know if the motion is forward, backwards to the east or to the west. Given the above we conclude that if a car travels a distance of $50\,mi$ in $2\,h$, the speed of the car is

$$speed = \frac{50\,mi}{2\,h} = 25\,mi/h \tag{3.4}$$

3.3 Motion at constant velocity

Let us imagine an experiment, it consists on observing an object moving along a line and recording its position at certain times. The result of our *Gedankenexperiment* is presented in the form some data with two significant figures

t (s)	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0
x (m)	1.0	0.8	0.6	0.4	0.2	0.0	-0.2	-0.4

Table 3.1: Experiment measuring the position of a toy car along a straight track

Some examination of the table shows that the toy began its motion 1.0 m to the left of the origin and moved to the left along the track in such a way that its last recorder position, 7.0 seconds after the motion began to be recorded is 40 cm to the left of the origin.

It is important to note that the reported data does not say anything about what was the state of motion just before $t = 0$, the object might have being somehow moving along the track or it might have been standing still at the initial position $x(0.0) = 1.0\ m$.

During the time of the experiment, and thinking at the different instants of time at which the measurements took place, the toy went through a series of changes in position called **displacements** (Δx) defined by

$$\Delta x(t) \equiv x(t + \Delta t) - x(t)\,, \tag{3.5}$$

The quantity Δt appearing in formula 3.5 is just the time interval between two successive measurements.

We might sketch a graphic of the data [please do it as an exercise] and would find a straight line with

$$\text{slope} \equiv v = \frac{\Delta x}{\Delta t} = -0.2\ m/s\,. \tag{3.6}$$

formula 3.6 is quite similar to formula 3.4 defining the speed. There are some subtle differences though.

In formula 3.6 v stands for *velocity*, which is to be understood as a signed quantity. This is related to the fact that the quantity that enters the definition of velocity is displacement which is a signed quantity while distance which is what appears in the definition of speed is not signed.

Speed is therefore the magnitude of the velocity and as we already knew, it just talks about the motion being fast or slow.

The sign in the velocity sign tells us whether the motion is rightwards or leftwards, that is, velocity tells us about direction of motion and speed.

In our experiment, v is negative meaning -as we already knew- that the motion is to the left.

To give a better notion of speed we might quote that NASCAR cars may reach a top speed of nearly 200 mph or 321 km/h. F1 cars are quite more impressive, Honda, who took their RA106 to the Bonneville Salt Flats in the US, a site famous for top-speed runs, to try and break 400 km/h. They were unsuccessful, but set a 397.36 km/h (246.9 mph) top speed, to claim the highest speed in an F1. In actual F1 races and due to the difficulties imposed by the circuits, F1 cars have typical speeds (magnitudes of v) close to 161 mph (260 Km/h) which is 5.6 times faster than the usual city limit of 30 mph.

Our experiment corresponds to the the simplest motion of all. Known as *Uniform Rectilinear Motion* (URM) is the motion of an object that travels along a straight line always in the same direction and at constant speed. If we call x_0 the position of the particle at the initial time $t = t_0$ and v_0 the velocity at t_0, then the position of the particle $t \geq t_0$ is given by the formula

$$\boxed{x(t) = v_0\,(t - t_0) + x_0} \tag{3.7}$$

3.4 Interactive Maps and Speed: Improving Our Understanding

Imagine you're planning a quick trip across town using your favorite interactive map application. You input your destination, and the app presents you with a few route options, each displaying a slightly different length. Let's consider three possibilities:

- Route 1: 4.5 miles (approximately 7.24 kilometers)
- Route 2: 5.8 miles (approximately 9.33 kilometers)
- Route 3: 6.2 miles (approximately 9.98 kilometers)

Assuming the city has an average speed limit of 30 miles per hour (mi/hr), let's calculate the estimated travel time for each of these more realistic route lengths:

$$\text{Time} = \frac{\text{Distance}}{\text{Speed}}$$

Applying this to our routes:

- Route 1 Time: $\frac{4.5 \text{ miles}}{30 \text{ mi/hr}} = 0.15 \text{ hours} = 9 \text{ minutes}$
- Route 2 Time: $\frac{5.8 \text{ miles}}{30 \text{ mi/hr}} \approx 0.193 \text{ hours} \approx 11.6 \text{ minutes}$
- Route 3 Time: $\frac{6.2 \text{ miles}}{30 \text{ mi/hr}} \approx 0.207 \text{ hours} \approx 12.4 \text{ minutes}$

It's important to note that our calculations here are solely based on the total distance traveled and the average speed. We have made no reference whatsoever to the direction of motion along these routes. This highlights a fundamental distinction in physics: **speed** is a scalar quantity that describes how fast an object is moving, while **velocity** is a vector quantity that describes both the speed and the direction of motion. In our scenario, even if the average speed is the same, the actual velocity would be constantly changing as we navigate turns and curves along each route.

Even with these more subtle differences in distance, it becomes clear that choosing a slightly shorter route can still save you a noticeable amount of travel time. While the speed remains constant due to the imposed limit, the fundamental relationship between distance and time dictates that variations in the length of the journey will directly translate to variations in the time required to complete it. This simple exercise, mirroring our everyday use of interactive maps, reinforces the importance of considering distance when planning our travels, even within the same speed-regulated environment.

3.5 Uniformly accelerated motion

We begin this section by introducing the simplest possible generalization of formula 3.7, namely

$$\begin{aligned} v(t) &= a(t-t_0) + v_0 \\ x(t) &= \frac{a\,(t-t_0)^2}{2} + v_0\,(t-t_0) + x_0\,. \end{aligned} \tag{3.8}$$

Where x_0, v_0 and a are constants with dimensions

$$\begin{aligned} [x_0] &= \text{Length} = L \\ [v_0] &= \text{Length per unit time} = \frac{L}{T}\,, \quad \textit{and}, \\ [a] &= \text{Length per time per time} = \frac{L}{T^2} \end{aligned} \tag{3.9}$$

For formulas 3.8 to make some sense, we need to give some physical (concrete, experimental) meaning to the symbols appearing in them.

We begin with $v(t) = a(t-t_0) + v_0$, this formula expresses a motion where the velocity is not constant, in fact it changes linearly[4] with

$$\text{slope} = a \tag{3.10}$$

a the *acceleration* is a signed quantity having a quite delicate meaning, a tells us about how velocity changes in time.

3.5.1 An useful formula

Before attacking the problems we will derive an useful formula that applies for uniform accelerated motion along a line.

We begin by recalling that for this condition

$$x(t) = \frac{a(t-t_0)^2}{2} + v_0(t-t_0) + x_0 \tag{3.11}$$

$$v(t) = a(t-t_0) + v_0 \tag{3.12}$$

From 3.12

$$t - t_0 = \frac{v(t) - v_0}{a} \tag{3.13}$$

We now substitute eq. 3.13 into eq. 3.13 to get

$$\begin{aligned} 2(x(t) - x_0) &= a\left(\frac{v(t)-v_0}{a}\right)^2 + 2v_0\left(\frac{v(t)-v_0}{a}\right) = \\ &= \frac{(v(t)-v_0)^2}{a} + 2v_0\left(\frac{v(t)-v_0}{a}\right) \end{aligned} \tag{3.14}$$

[4]Please recall that one way to write the equation of a line in the $x-y$ plane is :$y - y_0 = m(x - x_0)$, where m is the slope of the line, (x_0, y_0) the coordinates of a point that we know, belongs to the line and (x, y) are the coordinates of any arbitrary point belonging to the line

But

$$
\begin{aligned}
&\left[\frac{(v(t)-v_0)^2}{a}\right]+2v_0\left(\frac{v(t)-v_0}{a}\right)= \\
&=\frac{v(t)^2-2v(t)v_0+v_0^2}{a}+\frac{2v_0v(t)-2v_0^2}{a}= \\
&=\frac{v(t)^2-2v(t)v_0+v_0^2+2v_0v(t)-2v_0^2}{a}= \\
&=\frac{v(t)^2-v_0^2}{a}
\end{aligned}
\tag{3.15}
$$

and form here we conclude that

$$2a(x(t)-x_0)=v(t)^2-v_0^2 \tag{3.16}$$

This formula is usually written as (d stands for displacement)

$$\boxed{v_f^2-v_{ini}^2=2ad} \tag{3.17}$$

3.6 Acceleration, what is it?

To develop an intuition about acceleration, imagine a drag racing car, certainly any video will show that those machines can go from zero to very high speeds in short periods of time, to be precise, a top fuel dragster changes its speed (accelerates) from a standstill to 100 mph (160.9 km/h) in as little as 0.8 seconds. In terms of a that means that the magnitude of a is 'big', with big meaning in comparison to a standard car which to go from zero to 60 mph takes nearly 5 seconds. In both examples, the magnitudes of the acceleration are

$$
\begin{aligned}
a_{dragster} &= \frac{160}{0.8}\text{ miles per second per second} = 200\text{ miles per second per second} \\
a_{standard\ car} &= \frac{60}{5}\text{ miles per second per second} = 12\,\text{miles per second per second}
\end{aligned}
\tag{3.18}
$$

said in words, a dragster acceleration is nearly 17 times bigger than a standard's car. For further comparison, the dragster acceleration is even bigger than that of a jet fighter plane during lift of from a carrier

Discussion Topic 1 *Sit with your friends or some teachers and try to go deep into the question: What exactly is acceleration?*

Discussion Topic 2 *What is the meaning of the constants v_0 and x_0 in equation 3.8.*

Discussion Topic 3 *What are the formulas for position and velocity for non accelerated motion?, How do you interpret the resulting formulas in terms of a familiar setting?*

Prob 1 *How much does it take to travel 200 miles at the 55 mph speed limit?*

Example 1 *As for today, the Olympic records for the 100 m track are 9.63 seconds, set by Usain Bolt in 2012, and 10.62 seconds, set by Florence Griffith-Joyner in 1988.*

1. *Find the speeds of said runners.*

$$v_{UB} = \frac{100\ m}{9.63\ s} = 10.39\ m/s \tag{3.19}$$

$$v_{GJ} = ? \tag{3.20}$$

2. *Transform the values to different units*

$$\begin{aligned} v_{UB} = 10.39\ m/s = 10.39 \frac{Km}{1000\,m} \frac{3600\,s}{h} = 37.38\ \ Km/h = \\ = 37.38 \frac{1\ mil}{1.609\ Km} = 23.23\ mph \end{aligned} \tag{3.21}$$

Prob 2 *Are the above the true speeds of the athletes?*

Figure 3.4: Skydiver in "free" fall

Example 2 *"Near" the surface of the Earth, an object in free fall in a vacuum will accelerate at approximately* $9.8\,,m/s^2$, **independently of its mass**. *With air resistance acting on an object that has been dropped, the object will eventually reach a terminal velocity, which is around 53 m/s (190 km/h or 118 mph) for a human skydiver.*

1. *How much time does it take to reach the terminal velocity? Since the skydiver falls with constant acceleration,*

$$v_{term} = a_{fall} t_{reach\ t.v.} \tag{3.22}$$

that means

$$t_{reach\ t.v.} = \frac{v_{term}}{a_{fall}} \tag{3.23}$$

putting the numbers together,

$$t_{reach\ t.v.} = \frac{53\ m/s}{9.8, m/s^2} = 5.4\ s \tag{3.24}$$

2. *How much distance does a skydiver fall till she reaches the terminal velocity?*

To answer this question we must assume that the initial velocity of the skydiver is 0*, setting up the observer in the plane, the initial position is also 0, so the distance is simply*

$$y = \frac{a\,t^2}{2} \tag{3.25}$$

To find the distance the skydiver falls until she reaches the terminal velocity, all that is needed is to substitute $t_{reach\ t.v.} = 5.4\ s$ *in this formula, once again, we put the numbers together to get,*

$$y = \frac{9.8\ \ m/s \times\ (5.4\ s)^2}{2} = 142.8\ m \tag{3.26}$$

Figure 3.5: The Ferrari 812 Superfast

Example 3 *A flagship Ferrari should be fast, that's obvious, but with a name like Superfast, the 812 really had to put its money where its mouth is. Fortunately for Ferrari, the 6.5-litre V12 will happily launch it from 0-60 mph in just 2.9 seconds, onto a top speed of 211 mph.*

Remark *Before going any further, we must realize that this physical situation is very similar to that of example 2 (why?)*

1. *Find the acceleration of the Ferrari 812.*

   The acceleration is just the change in speed divided by the time it takes to reach the speed,

$$a = \frac{60 \ mph}{2.9 \ s} = \frac{60 \times 1609 \ m}{3600 \ s} \times \frac{1}{2.9 \ s} = 9.24 \, m/s^2 \tag{3.27}$$

2. *In how much time does the 812 reach its maximum speed? The maximum speed of the car is*

$$211 \ \ mph = \frac{211 \times 1609 \ m}{3600 \ s} = 94 \ \ m/s \tag{3.28}$$

   To reach that speed the 812 needs

$$t = \frac{94.3 \ \ m/s}{9.24 \ s} = 10.20 \ \ s \tag{3.29}$$

Prob 3 *A car is travelling at* 36 Km/h*, the breaks are applied suddenly so the resulting* **acceleration** *is* 10 m/s^2*, how much time passes until the car stops?*

Prob 4 *On 14 October 2012 skydiver Felix Baumgartner did a freefall parachute jump from a height of* 38969.4 m*, smashing through eight world records and the sound barrier all in one go.*

Could we analyze this situation the same way we did in example 2

3.6.1 An important observation regarding acceleration

In all the examples given above, the speed increases due to the acceleration, is this always so?. The answer is **definitely NO**, indeed, we already know that acceleration signals a change in velocity. When the motion is along a straight line, a change in velocity can be either an increase or a decrease in speed.

Figure 3.6: Felix Baumgartner begining his famous jump

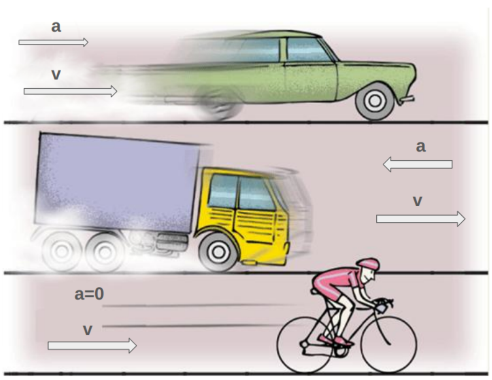


Figure 3.7: Changes in speed due to acceleration.

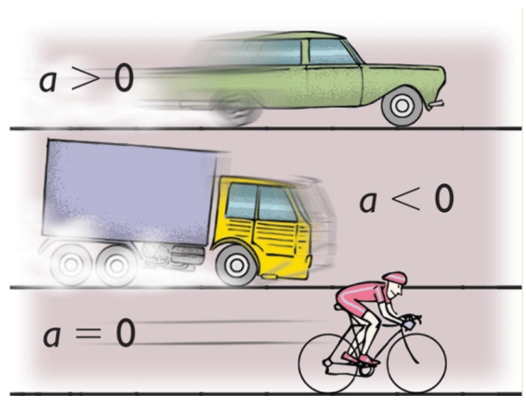


Figure 3.8: Assigning signs to velocity and acceleration

Consider figure 3.7, where three objects are moving with their velocities pointing towards the right of the figure. The acceleration of the car is parallel to its velocity and as a consequence the speed of the car increases. The acceleration of the truck is antiparallel to its velocity causing the speed to decrease. Finally, the bicycle is not accelerating and therefore its velocity is constant.

The same three motions are depicted in figure 3.8 where instead of arrows the direction of both velocity and acceleration are represented by signs. The velocities of the three bodies

Acceleration and velocity	Speed
$a = 0$	constant
parallel	increase
anti-parallel	decrease

Table 3.2: Relation betwen acceleration and speed for motion along a straight line

(right oriented) are assigned a positive sign. The acceleration of the car, being to the right is positive. For the truck, which accelerates towards the left, the acceleration is negative ($a < 0$). While the cyclist, who has cero acceleration ($a = 0$), is riding smoothly at constant velocity.

3.7 Elementary Kinematics in terms of Graphics.

Graphics are always interesting, they not only provide a different picture of phenomena we are interested in but sometimes give us extra insight on them, indeed data tables such as the position and speed of a funny car on the track are not exactly simple to interpret.

A glance at figure 3.10 clearly establishes that the funny car reached a maximum speed on the excess of 200 mi/h and this happened exactly when it was 400 m (1/4 mi) away from the starting point.

The motion of the funny car is far more complex that the kind of motion we have been studying, and one immediately wonders how the graphs of such motions may be.

Let us go back to the topic of our main interest, motion with constant acceleration by recalling the two main formulas 3.8 and setting the initial moment t_0 equal to zero for simplicity

time (s)	Position (m)	Speed (mi/h)
0.0	0.0	0.00
2.0	25.0	27.96
4.0	95.0	78.29
6.0	215.0	134.22
8.0	400.0	206.92
10.0	510.0	123.03
12.0	600.0	100.66
14.0	680.0	89.48
16.0	720.0	44.74

Table 3.3: Position and speed of a funny car during a trial in the race track

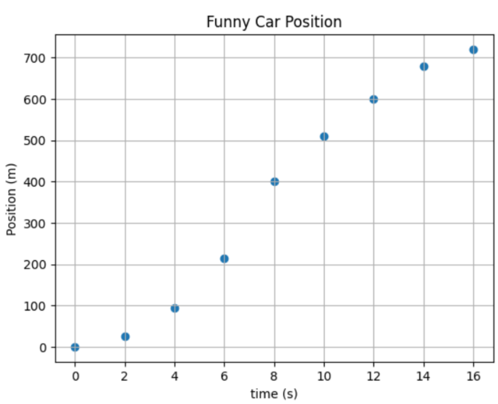


Figure 3.9: Position of the funny car

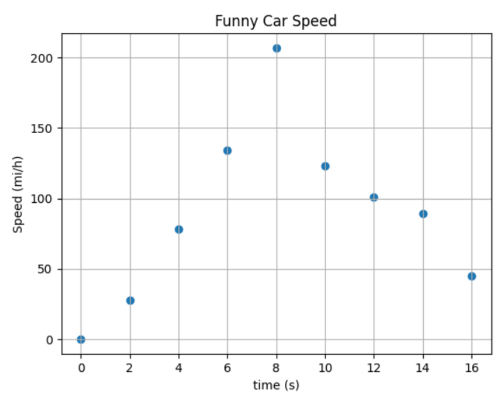


Figure 3.10: Speed of the funny car

we have:

$$x(t) = \frac{a\,t^2}{2} + v_0 t + x_0$$

$$v(t) = at + v_0$$

If we recall what we learned in high school math we will recognize that the formula for velocity is a line with slope v_0, while the formula for position corresponds to a parabola in the $t - x$ plane

In figure 3.11 we recognize exactly those graphics, including the one of the acceleration as a function of time which is represented as a horizontal line as corresponds to constant acceleration.

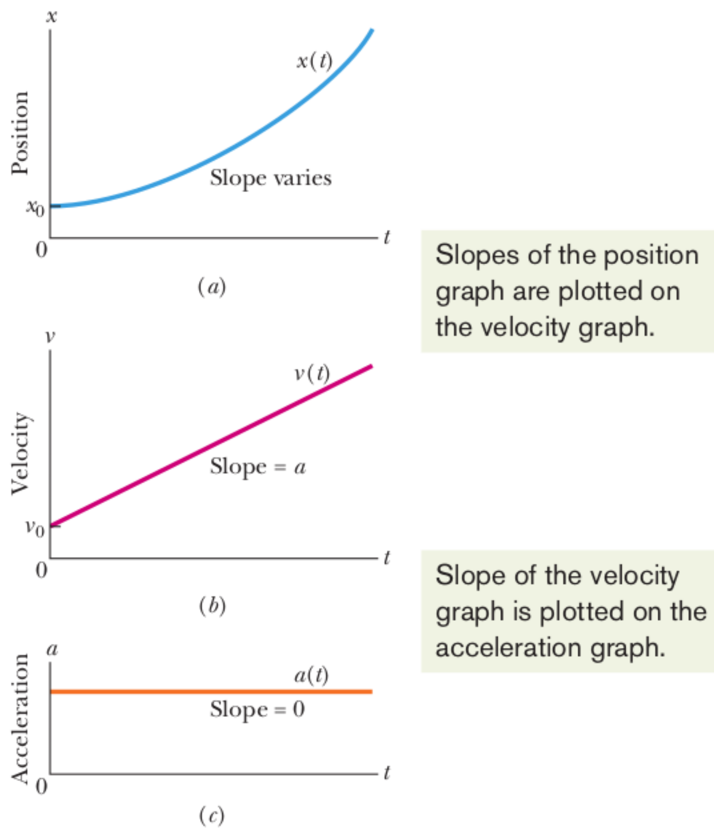


Figure 3.11: Position of the funny car

The interesting thing is that there are more relations between the physics and the geometry of the graphics that what meets the eye.

Let us think of a motion with no acceleration, then the formulas for the position and velocity are

$$x(t) = v_0 t + x_0$$

$$v(t) = v_0$$

the latter been trivial. Now, if we ask ourselves the following very simple question: what is the difference between the initial position of the moving object at time T and its position at the beginning of the motion (t=0) the answer is obviously

$$x(T) - x_0 = v_0 T$$

but, since the motion is at constant velocity, the graphic of velocity with respect to time is just a horizontal line with height v_0, and therefore $v_0 T$ is the area of a rectangle of base T (the time) and height the velocity.

Let's think of constant acceleration, when the $V - t$ graph is a line of slope a. In this case and if the initial velocity is v_0 the area under the curve for a time T that begins at $t = 0$ is the sum of the area of a rectangle of height v_0 ($A_{rectangle} = v_0 T$)and a triangle whose base is T and height is aT ($A_{triangle} = 1/2 \times T \times aT = at^2/2=$ adding the two areas we get

$$\text{Area under the v-t curve} = \frac{at^2}{2} + v_0 t\,,$$

which means

$$x(T) - x_0 = \frac{at^2}{2} + v_0 t$$

We have just found a deep physical principle.

Principle 1 *For any motion along a straight trajectory, the difference between the initial position x_0 and the position at time T ($x(T) - x_0$) equals the area under the velocity vs time graph of the motion.*

This principle is so important that it deserves some lines. In mathematics, principle 1 is called **Fundamental Theorem of Calculus**, people had been thinking of it since very old times, but they were Isaac Newton and his intellectual nemesis Gottfried Leibniz who gave the theorem the form we know today.

3.8 Dynamics, just a glance

We begin this section by stating (without any further explanations) Newton's first and second laws

1. A body will remain in constant velocity linear motion unless is acted upon by a forece.

2. When a force acts on a body of mass (inertia) M the body acquires an acceleration given by

$$\vec{F} = M\vec{a} \tag{3.30}$$

Before going any further it is important to explain that many people get confused with the concepts of **mass** and **weight**.

Mass is the inertia, i.e. the resistance to be accelerated.

Weight on the other hand is a completely different concept. The weight of an object is the force with which earth (or any astronomical object) attracts a body when it is close to the earth's surface.

The International Space Station orbits the Earth because of the gravitational pull that the earths exerts on the ISS, were not for gravity, the ISS would travel in a straight line for ever. The same happens with the moon and of course with earth which is pulled by the sun. But much more interesting (and therefore fun), to all stars in the Milky Way which are attracted to a Giant Black Hole lying at the very center of our Galaxy. According to the legend, young Newton realized that the very same pull that makes an apple fall form a branch of an apple tree is what makes the moon go around the earth, and boy, ¡he was right!

Figure 3.12: The apple and the moon are pulled towards earth. Gravity is a universal force, all objects are attracted towards each other

Example 4 *As happens with length or time, forces also have units, in fact, they have* **derived units** *which are defined by Newton' s second law. The force required to give a* $1\ Kg$ *object an acceleration of* $1\ m/s^2$ *is called a Newton (symbol: N). Therefore,*

$$1\ N = 1\ Kg \times m/s^2 \tag{3.31}$$

Figure 3.13: Balance

Figure 3.14: Dynamometer

Figure 3.15: Balance

Prob 5 *The kerb weight of the ferrari 812 (example 3) is* 1744 *kg (3,845 lb), what force is needed for pushing it with it's acceleration.*

Prob 6 *Given that a free falling body on earth falls with an acceleration very close to* $9.8\ m/s^2$. *How much does a* $100\ Kg$ *object weight?*

3.9 g forces or g as a typical measure of acceleration

If you go to the movies and watch, let's say, **Top Gun: Maverick**, you will constantly hear something like "...Mav is trying a 5g maneuver..." That means a maneuver in which the plane, and therefore the pilot, were subject to high (in terms of magnitude) accelerations. The effect of acceleration on animals, and of course humans, is very important for life.

A human can withstand up to 18g's without dying, but typically passes out at 6 to 7g's.

Any way, what is this g?. Well, the simplest answer to the question we have just posed is this: 1 g is the acceleration of an object in free fall near the earth, which is close to 32 ft per second per second or

$$1\ g = 9.78\ m/s^2 \approx 10\ m/s^2. \tag{3.32}$$

This actually means that if we let a ball fall from the top of a building it will acquire a down falling speed of 32 feet per second in the first second of its motion and will be falling a at rate of 64 ft per second after two seconds of free fall.

An interesting fact is that what we call **Weight** is nothing more that the force with which the earth attracts an object. Near the surface of the earth, the weight is simply

$$Weight \; = Mass \times 1 \; g = Mass \times 9.78 \; m/s^2 \approx Mass \times 10 \; m/s^2 \,, \tag{3.33}$$

where we must learn that the mass of an object is the resistance it offers to be accelerated.

$$Weight_{Mario} = 100 \times 10 \; m/s^2 = 1000 \; N \tag{3.34}$$

$$Weight_{rollercoaster} = 12 Weight_{Mario} = 12000 \; N \tag{3.35}$$

Let us see something more regarding g forces. Imagine a fall of 2 meters, just before reaching the floor, the speed is.

$$v = \sqrt{2 \, g \, height} = \sqrt{2 \times 10 \times 2} = 6.32 \, m/s \tag{3.36}$$

and so, if a person, say yours truly, completely stops in $0.1 \, s$, it happens that acceleration to complete stop is

$$a = v/\text{time it takes to stop} = 6.32 \, m/s/(0.1 \, s) = 63.2 \, m/s^2 \approx 6 \, g \tag{3.37}$$

Therefore, if I land on my head, the force that the floor applies on my skull equals my mass times $6g$ (6 g forces).

3.10 Interesting Videos

1. 50 min 1D Motion, lecture by Walter Lewin

2. ISS acceleration
3. Effect of Acceleration John Paul Stapp
4. Carrier Catapult, force accelerates objects
5. The Hulk, a roller coaster that uses a catapult
6. The Scuderia Ferrari Marlboro drivers visited Ferrari World Abu Dhabi, they rode another roller coaster that uses a catapult. Watch the effect of the acceleration
7. Famous Physicist Brian Cox subjet to several G' s
8. G forces felt and explained What happens when you are in a room which is in free fall?

3.11 Questions

1. A rock is thrown straight upward from the edge of a 30 m cliff, rising 10 m then falling all the way down to the base of the cliff. Find the rock's displacement.
2. a particle moves with uniform velocity. Which of the following statements about the motion of the particle is true.
   (a) its speed is zero.
   (b) its acceleration is zero.
   (c) its acceleration is opposite to the velocity.
   (d) its speed may be variable
3. In a track-and-field event, an athlete runs exactly once around an oval track, a total distance of 500 m. Find the runner's displacement for the race.

4. Assume that the runner in sample question 3 completes the race 61 in 1 minute and 20 seconds. Find her average speed and the magnitude of her average velocity.

5. A particle starts with initial velocity 10ms -1 .. it covers a distance of 20 m along a straight into two seconds. What is the acceleration of the particle.

   (a) zero

   (b) $1\,ms^{-2}$.

   (c) $10\,ms^{-2}$.

   (d) $20\,ms^{-2}$.

6. Is it possible to move with constant speed but not constant velocity? Is it possible to move with constant velocity but not constant speed?

7. A car is traveling in a straight line along a highway at a constant speed of 80 miles per hour for 10 seconds. Find its acceleration.

8. A car is traveling in a straight line along a highway at a speed of 20 m/s. The driver steps on the gas pedal, and 3 seconds later, the car's speed is 32 m/s. Find its average acceleration.

9. Spotting a police car ahead, the driver of the car in the previous question slows from 32 m/s to 20 m/s in 2 sec. Find the car's average acceleration.

10. An object with an initial velocity of 4 m/s moves along a straight axis under constant acceleration. Three seconds later, its velocity is 14 m/s. How far did it travel during this time?

11. 9. A car that's initially traveling at 10 m/s accelerates uniformly for 4 seconds at a rate of 2 m/s 2 n a straight line. How far does the car travel during this time?

12. A particle moves along a straight line path. After sometime it comes to rest. The motion is with an acceleration whose direction with respect to the direction o velocity is (a) positive throughout motion (b) negative throughout motion (c) first positive then negative

13. A car travels a distance S on a straight road in two hours and then return to the starting point in the next here hours. Its average velocity is:

    (a) S/5

    (b) 2S/5

    (c) (S/2)+(S/3)

    (d) none of the above.

Chapter 4

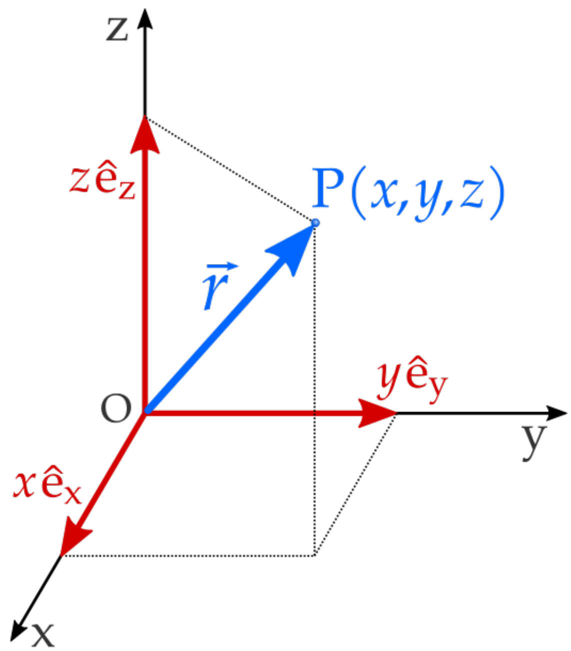
Kinematics II. Motion in 2 and 3 Dimensions: General Principles

We have already introduced some concepts to describe the motion of particles along a line in chapter 3 . We now want to be much more realistic and inquire about motion in 2 and 3 dimensions, to be success in this endeavour depends on the introduction of some new mathematical objects called *vectors*. At the most simplistic level vectors are like arrows and as such they have two defining features, direction and magnitude (size). Vectors can be added, subtracted and multiplied by a number. Being like arrows, two non parallel vectors always determine a plane containing them, besides, they make an angle.

4.1 Position

Our first modification regards the position, which now must be a vector (an arrow) with its tail based at some particular point called the origin which generalizes the one dimensional case where we also needed an origin-

For this more general motion, the role of position $x(t)$ of the 1D case is now played by a vector $\mathbf{r}(t)$ (or $\vec{r}(t)$) whose tail is located at the origin $\mathcal{O}$ and tip at the moving object which is located at a point P with coordinates (x, y, z), see figure 4.1 the position vector can be



expressed as the sum of three perpendicular vectors of unit length

$$\mathbf{r} = x\,\hat{\mathbf{e}}_x + y\,\hat{\mathbf{e}}_y + z\,\hat{\mathbf{e}}_z \tag{4.1}$$

4.2 Trajectory

As the particle moves through space its position changes, the motion follows a sequence of ordinary points that can be joined to make a curve, such curve is called the **trajectory of the particle**.

To make things tangible, picture yourself in an air show, the general atmosphere, the people, the Vroom of the Thunderbirds during a low pass (figure 4.1), etc.

Figure 4.1: Thunderbirds: USAF exhibition team

For dangerous maneuvers teams like the US NAVY Blue Angels, use colored smoke to make sure their demonstrations are clearly seen. In figure 4.2 the coloured smoke clearly shows the spiral descending motion of the parachuter, the smoke is showing the trajectory.

4.3 Velocity

As the particle moves the position vector changes in both direction and magnitude, see figure 4.3. As time changes form t to $t + \Delta t$ the particle undergoes a displacement $\Delta \mathbf{r}$, and by the

Figure 4.2: Blue Angels parachuter making spirals

addition laws of vector,

$$\mathbf{r}(t + \Delta t) = \mathbf{r}(t) + \Delta \mathbf{r}\,, \tag{4.2}$$

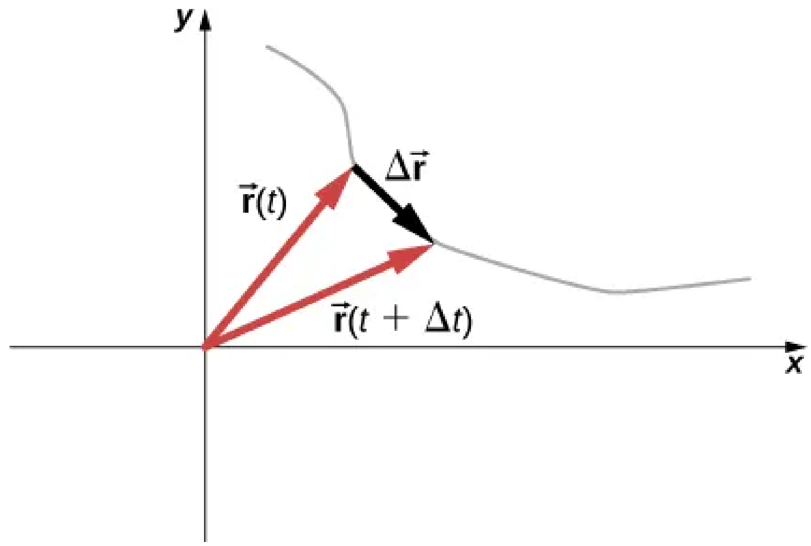
or

$$\Delta \mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)\,, \tag{4.3}$$

The average velocity $\mathbf{r}_m$ in the time interval Δt is defined as the quotient

$$\mathbf{v}_{av} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}\,, \tag{4.4}$$

at this point we have not defined how to perform the operations we have been talking about,



nevertheless, it should be quite intuitive that, whenever Δt is very, very small, the resulting displacement is a vector of small magnitude. In such cases the quotient given by the limit expression[1]

$$\lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \vec{r}(t)}{\Delta t}, \tag{4.5}$$

defines the instantaneous velocity at t.

Remark 1 *We must at all times remember that the instantaneous velocity or velocity for short is a* **vector***, it is always tangent to the trajectory of the particle and its tail is in the moving particle see figure 4.3.*

We can picture the velocity as an arrow riding the moving object in such a way that at any instant points to where the object is moving.

[1]At this point we are introducing the notion of limit, which underlies most concepts of Calculus.

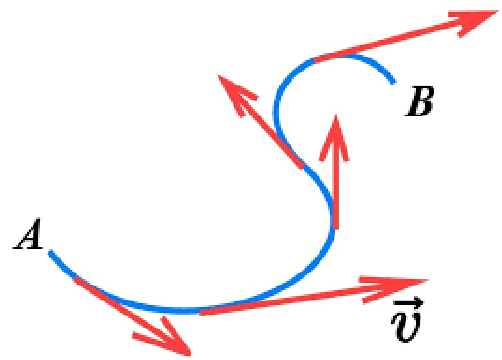


Figure 4.3: The instantaneous velocity is always tangent to the trajectory and at each instant of time, the tail of the velocity is located on the particle

4.4 Acceleration

The acceleration is a very interesting object, it measures the instantaneous change of velocity, in formulas

$$\mathbf{a}(t) \equiv \lim_{\Delta t \to 0} \frac{\mathbf{v}(t+\Delta t) - \mathbf{v}(t)}{\Delta t}\,, \tag{4.6}$$

this definition implies that the acceleration is a vector, and **just as the velocity, its tail is always on the moving object**.

There is an enormous difference between motion along a line and motion in two or three dimensions, **in two or three dimensions the vectors (position, velocity and acceleration) can change in direction and not only in magnitude**

Let us notice that -as shown in figure 4.4- at any t we can split the acceleration in two parts, a vector $\mathbf{a}_{\parallel}$ parallel to the velocity and another $\mathbf{a}_{\perp}$ perpendicular to it, i.e.

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}\,, \tag{4.7}$$

this in turn implies that for very short intervals of time,

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + [\mathbf{a}_{\parallel} + \mathbf{a}_{\perp}]\Delta t\,, \tag{4.8}$$

or

$$\mathbf{v}(t+\Delta t) = \left[\mathbf{v}(t) + \mathbf{a}_{\parallel}\Delta t\right] + \mathbf{a}_{\perp}\Delta t\,, \tag{4.9}$$

An exercise in calculus -which is beyond the scope of these notes- shows that $\mathbf{a}_{\perp}$ is responsible

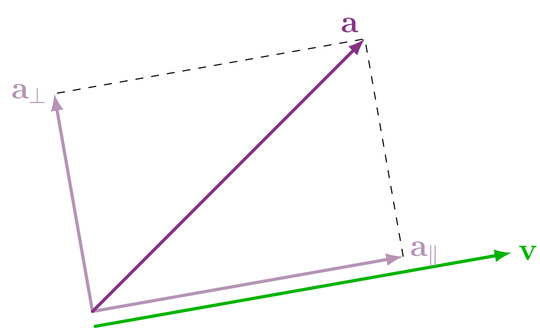


Figure 4.4: Decomposing the acceleration in its parallel and perpendicular components to the velocity

for any change in the direction of the velocity, in fact the change of direction follows the tip of the transverse acceleration.$\mathbf{a}_{\parallel}$, on the other hand, modifies the magnitude of the velocity. As happened with the case of motion along a line (see section 3.6.1), if $\mathbf{a}_{\parallel}$ is parallel to the velocity, the latter undergoes an increase in magnitude, if it is anti parallel the magnitude of the velocity decreases.

Sumarizing:

- The acceleration can always be split into perpendicular and parallel (to the velocity) components.

$$\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$$

- $\mathbf{a}_{\parallel}$ changes the speed

- $\mathbf{a}_{\perp}$ changes the direction of the velocity

Figure 4.5: Water from these fountains move under the influence of constant acceleration giving origin to what is known as ballistic motion. The parabolic trajectories are so perfect that look magical

4.5 The Simplest Possible Example: Ballistic Motion

Before embarking in our study, it is interesting to show a plethora of reasons making the study of the ballistic or parabolic motion a key topic in physics courses worldwide.

- Real-world relevance: It describes the motion of objects under gravity, which is ubiquitous in our daily lives - from throwing a ball to the arc of water from a fountain.
- Combination of concepts: It beautifully combines horizontal and vertical motion, showcasing how simple motions can create complex trajectories.
- Predictive power: It demonstrates how physics can predict the path of an object given

Figure 4.6: When the player throws the ball, it goes a motion under the influence of gravity

Figure 4.7: The ball's flight is an example of ballistic motion

initial conditions, which is crucial in many applications.

- Mathematical modeling: It provides an excellent opportunity to apply mathematical concepts like quadratic equations and vectors to a physical problem.
- Historical significance: The study of projectile motion was crucial in the development of classical mechanics.
- Practical applications: It's essential in fields like sports, military operations, and space exploration.
- Introductory complexity: It's complex enough to be challenging, yet simple enough for beginners to grasp, making it an ideal teaching tool.
- Foundation for advanced topics: Understanding ballistic motion paves the way for more complex concepts in physics.

The simplest example of two-dimensional motion is often called ballistic or parabolic motion. However, it's more accurately described as motion under constant acceleration. A familiar real-world example is a football throw, as depicted in Figure 4.6.

When a quarterback throws a football, the ball's motion after leaving their hand -ignoring, for simplicity, air resistance- is influenced only by gravity. The gravitational acceleration acts constantly and vertically downward, resulting in a parabolic trajectory that any football fan can observe during a game.

What makes this motion particularly interesting is the independence of its horizontal and vertical components. Since gravity acts only vertically, it doesn't affect the horizontal component of the ball's velocity. This separation allows us to analyze the motion using two simpler, one-dimensional equations rather than more complex vector calculations.

In the horizontal direction, the motion is uniform (constant velocity) because there's no acceleration. Vertically, the motion is uniformly accelerated due to gravity. This independence of horizontal and vertical motions is why we can describe ballistic motion using separate formulas for each direction, making it a foundational concept in understanding more complex motions.

As stated before, ballistic motion is so simple that vectors are not really needed to describe it. All we need are separated formulas for horizontal and vertical motion. If we choose the instant in which the motion begins as $t = 0$ and coordinates where upwards and right-wards are positive then, the formulas are:

$$x(t) = v_{0x}t + x_0$$

$$y(t) = -\frac{gt^2}{2} + v_{0y}t + y_0$$

Where g the acceleration of gravity has the value $g = 9.8\,m/s^2$, which for all practical purposes is $g = 32\,ft/s^2$

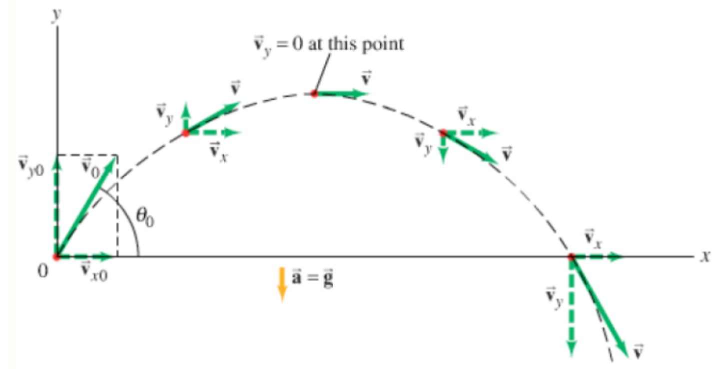


Figure 4.8: The ballistic motion is motion under the influence of gravity, this means that the acceleration is always vertical downwards

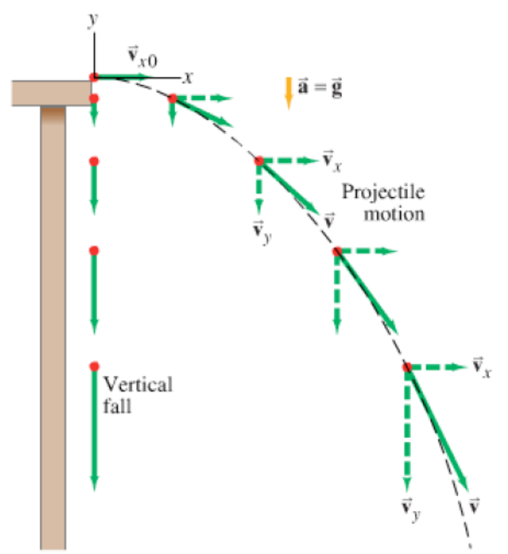


Figure 4.9: Horizontal Launch. Note that the tail of the velocity is always on the particle

Example 5 *As a simple example we consider a ball thrown horizontally form a table of height h*
The pure vertical fall is described by the formula: $y = -gt^2 + y_0$ *(what would you choose for* y_0*?)*
For the ball, choosing $x_0 = 0$*, the formulas for the ballistic motion are [why?]*

$$x(t) = v_{x0}t$$
$$y(t) = -\frac{gt^2}{2} + y_0$$

4.5.1 The Hunter and the Monkey

In this section we will explore an interesting physical situation (in this subsection we will use vector notation for this example). A hunter aims its weapon towards a monkey hanging from a branch on tree as shown in figure 4.10. As soon as the the monkey sees the hunter, he decides to open his hand to use the fall to escape the shot, at that very same moment the hunter shoots. The question is, does the monkey survive?

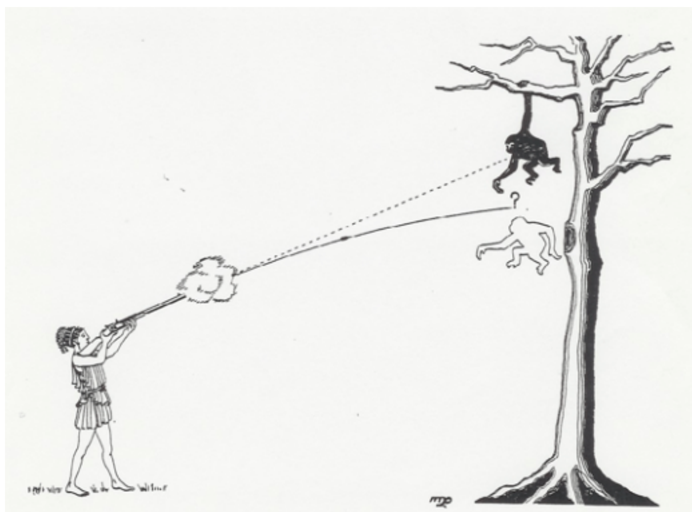


Figure 4.10: Will the monkey survive the hunter?

Let us first work out the problem with no gravity at all, in such case, the positions of the bullet and the monkey would be

$$\begin{aligned} \mathbf{r}_b &= v_{0x}t\,\hat{e}_x + v_{0y}t\,\hat{e}_y \\ \mathbf{r}_m &= L\,\hat{e}_x + H\,\hat{e}_y \end{aligned} \tag{4.10}$$

Let the hunter aim his weapon directly to the poor monkey, then the angle of sight would be,

$$\theta = arctan\left(\frac{H}{L}\right) \tag{4.11}$$

and, therefore

$$
\begin{aligned}
v_{0x} &= v_0 \frac{L}{\sqrt{L^2 + H^2}} \\
v_{0y} &= v_0 \frac{H}{\sqrt{L^2 + H^2}}
\end{aligned}
\tag{4.12}
$$

Since there's no gravity, the bullet will follow a straight trajectory and the monkey will receive the bullet no matter what.

To make sure that we fully understand the meaning of the equations we note that, in absence of gravity,

$$
\begin{aligned}
\frac{\sqrt{L^2 + H^2}}{v_0} &= \text{time to cover the horizontal distance } L\text{, but also,} \\
\frac{\sqrt{L^2 + H^2}}{v_0} &= \text{time to climb the height } H
\end{aligned}
\tag{4.13}
$$

Let us now think of the case where the monkey is hanging from a tree in some planet, then the equations describing the motions must be modified, and as we perfectly know,

$$
\begin{aligned}
\mathbf{r}_b &= v_{0x} t \, \hat{e}_x + (-\frac{gt^2}{2} + v_{0y} t) \, \hat{e}_y \\
\mathbf{r}_m &= L \, \hat{e}_x + (-\frac{gt^2}{2} + H) \, \hat{e}_y
\end{aligned}
\tag{4.14}
$$

Assuming that, once again, the hunter aims directly to the the monkey the components of the initial velocity are going to be exactly as in eq. 4.12.

For the monkey to die there must be an instant of time[2] (t_d) such that $\mathbf{r}_b = \mathbf{r}_M$, let us check for this condition, at t_d

$$
v_{0x} t_d \, \hat{e}_x + (-\frac{gt_d^2}{2} + v_{0y} t_d) \, \hat{\mathbf{e}}_y = L \, \hat{e}_x + (-\frac{gt_d^2}{2} + H) \, \hat{e}_y \tag{4.15}
$$

or

$$
v_{0x} t_d \, \hat{\mathbf{e}}_x + (-\frac{gt_d^2}{2} + v_{0y} t_d) \, \hat{\mathbf{e}}_y = L \, \hat{\mathbf{e}}_x + (-\frac{gt_d^2}{2} + H) \, \hat{\mathbf{e}}_y \tag{4.16}
$$

[2] At t_d, the bullet and the monkey must meet at the same place

i.e.

$$\boxed{v_{0x}t_d\,\hat{\mathbf{e}}_x + v_{0y}t_d\,\hat{\mathbf{e}}_y = L\,\hat{\mathbf{e}}_x + H\,\hat{\mathbf{e}}_y} \tag{4.17}$$

but these are the equations for no gravity, that means the monkey will die anyway.

What is going on? Well, the bullet and the monkey are both moving under the influence of gravity, therefore their acceleration is towards the ground and of the same magnitude. Consequently, their relative motion is independent of their falling acceleration, this in turn implies that their relative motion is exactly the same it was in the first -zero acceleration- experiment.

This is a famous situation which is used in hundreds of universities around the world to demonstrate properties of the ballistic motion. Particularly entertaining is the Large-scale demonstration of the classic experiment from Physics Force of the School of Physics and Astronomy, University of Minnesota.

4.6 Uniform Circular Motion: A First Glance

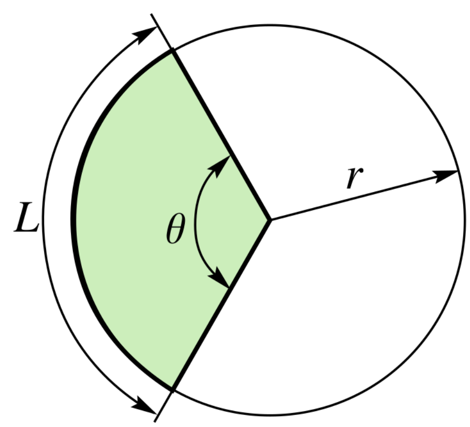


Figure 4.11: A circular sector is shaded in green. Its curved boundary of length L is a circular arc.

Motion along a circle is another example of $2D$ motion (why?), the simplest example of which is the so called **Uniform Circular Motion**, here the word uniform is used to establish that the speed is constant. At this point the reader should be able to note that constant speed does not imply constant velocity. In fact, during a uniform circular motion, the velocity is ever changing. To clearly establish what we are going to be talking about we state the following

Definition 1 *A particle is in uniform circular motion if it is going around a circle in such a way that it completes each revolution in an amount of time time T called the period of motion.*

Definition 2 *For a circular motion, the rate at which angular sectors are swept is called angular velocity and is represented by the Greek character ω*

For an uniform circular motion, the speed is of constant, implying that equal angular sectors i.e. equal arcs of circle are swept in equal amounts of time, therefore, the angular velocity is constant and can be easily calculated by noting that a full circle (arc=2π rad) is swept in one period, implying that ω has the value

$$\omega = \frac{2\pi}{T} \,. \tag{4.18}$$

It should be clear that the angular sector ($\Delta\theta$) swept in an interval of time Δt is

$$\Delta\theta = \omega \Delta t \,. \tag{4.19}$$

Just for checking, the time $\Delta t_{\pi/2}$ it takes for sweeping an angle of $\Delta\theta = \pi/2$, i.e. a quarter of a circle is

$$\pi/2 = \omega \Delta t_{\pi/2} = \frac{2\pi}{T} \Delta t_{\pi/2} \,, \tag{4.20}$$

i.e.

$$\Delta t_{\pi/2} = \frac{T}{4} \,, \tag{4.21}$$

a fourth of the time it takes for a complete revolution.

Since the speed is constant, the velocity changes in direction only and, as we already discussed in section 4.4, such changes are caused by the component of the acceleration perpendicular to the velocity ($\mathbf{a}_\perp$).

We know that the velocity is tangent to the trajectory, which is circular, elementary geometry teaches us that this implies that $\mathbf{a}_\perp$ must be in the radial direction, besides, it must be pointing towards the center of the circular motion and this is why, this acceleration is commonly known as **centripetal acceleration** (fig 4.12)

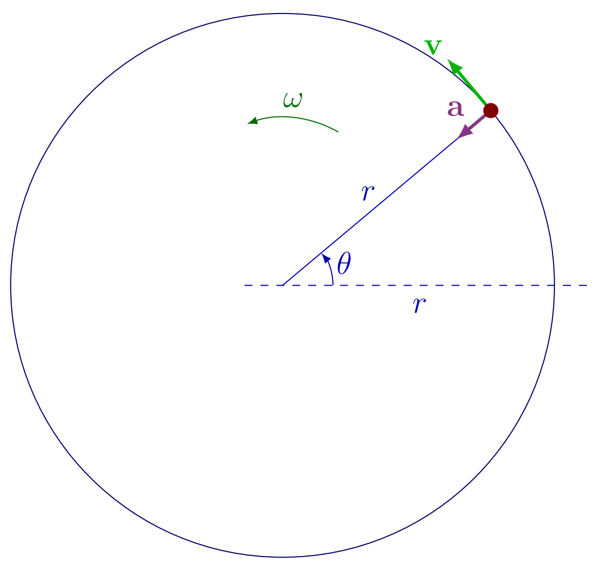


Figure 4.12: Elements of the uniform circular motion. Velocity (in green), centripetal acceleration (dark red), angular velocity (green)

As a final remark we note that, since an arc subtended by an angular sector $\Delta\theta$ has length $\Delta s = r\Delta\theta$, the speed of the motion (magnitude of the velocity) is

$$|\mathbf{v}| = r\omega\,, \tag{4.22}$$

accordingly, given a period T, the larger the radius, the larger the speed.

As a simple exercise -hated by all flat earthers- we estimate the speed of a person standing at sea level at earths equator.

The radius of the earth is 6378.1 *kilometers* so

$$speed = 6378.1\ km \times \frac{2\pi}{24\,h} \approx 1668.9\,km/h\,, \tag{4.23}$$

nearly 1670 Km/h or 1,038 miles per hour.

4.7 General 2D motion a First Visit

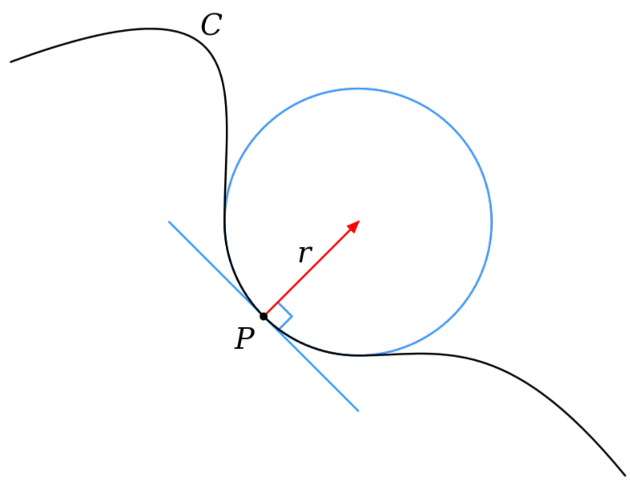


Figure 4.13: Osculating (kissing) circle

In this section, we state some facts about the general motion in 2D. To begin with, we must think that in the 2D motion trajectories are plane curves, i.e. curves that never leave a given plane where all the motion takes place. An obvious example would be circular motion.

For the discussion to be clear we need to introduce a new notion quoted from wikipedia:

... the osculating circle of a sufficiently smooth plane curve at a given point p on the curve has been traditionally defined as the circle passing through p and a pair of additional points on the curve infinitesimally close to p. Its center lies on the inner normal line, and its curvature defines the curvature of the given curve at that point. This circle, which is the one among all tangent circles at the given point that approaches the curve most tightly, was named circulus osculans (Latin for "kissing circle") by Leibniz.

The center and radius of the osculating circle at a given point are called center of curvature and radius of curvature of the curve at that point.

Figure 4.14 shows details of two different points along the trajectory of a particle. Since the trajectory is smooth it has well defined osculating circles at each point.

At each point of the trajectory, the particle may be thought of as moving in circular motion along the osculating circle at that point, whether the particle is having a change of direction is determined by the radius of curvature, the bigger the radius the lesser the change in direction, or equivalently, the smaller magnitude of centripetal acceleration similarly. The tangential acceleration, on other hand determines whether or not there is a change in velocity magnitude.

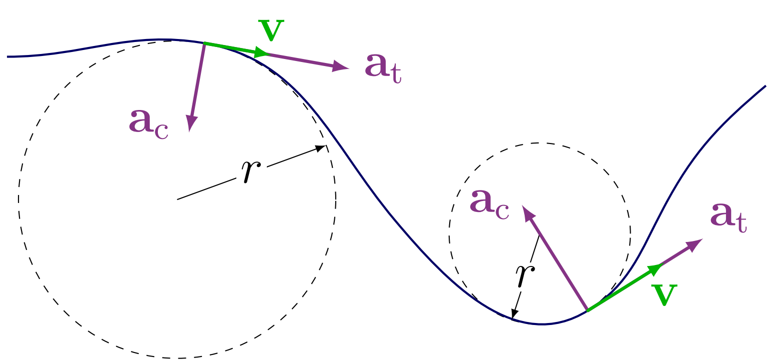


Figure 4.14: Two points along a motion

Chapter 5

Circular Kinematics

Remark 2 *In all formulas, and unless otherwise stated, angles are to be measured in radians. We must remember that given a circular segment (arc) of length ℓ and radius R, the angle in radians subtended by the arc is the quotient*

$$\theta = \frac{\ell}{R}\,, \tag{5.1}$$

those for instance, the angle corresponding to an angle of 180° *is*

$$\theta = \frac{\pi\, R}{R} = \pi\,. \tag{5.2}$$

Rotational motion with a constant nonzero acceleration is not uncommon in the world around us. For instance, many machines have spinning parts. When the machine is turned on or off, the spinning parts tend to change the rate of their rotation with virtually constant angular acceleration.

In section 4.6 we briefly discussed uniform circular motion.

Circular motion is conspicuous, we see it almost everywhere, in vehicles wheels just to mention an example.

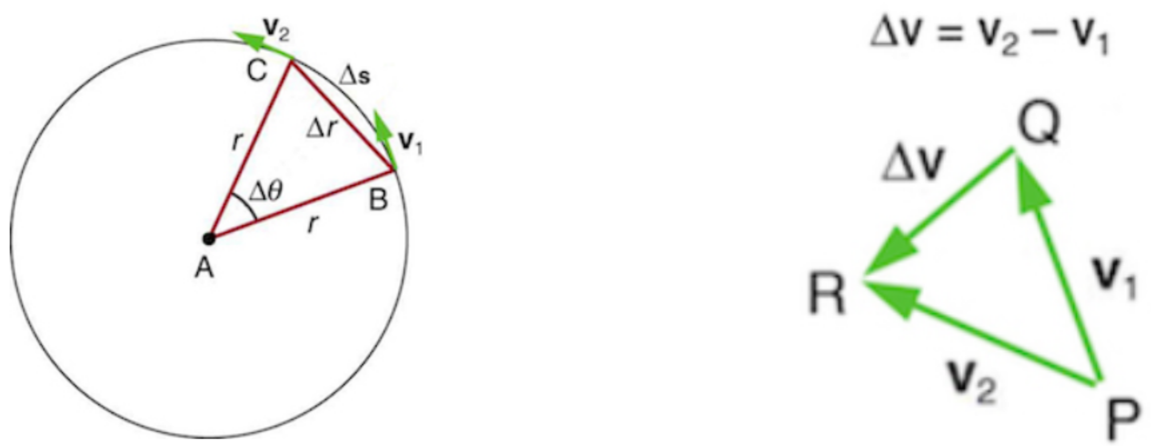


Figure 5.1: InnieMinnie

5.1 Centripetal Acceleration

Consider the triangles ΔABC and ΔPQR and note that they are similar

Since the speed is constant and the triangles are similar:

$$|\mathbf{v}_1| = |\mathbf{v}_2| = v\,, \quad \text{and} \quad \frac{|\Delta v|}{v} = \frac{\Delta s}{r}$$

from where

$$|\Delta v| = \frac{v}{r}\Delta s$$

But we know that

$$|\Delta v| = |\mathbf{a}_\perp \Delta t|$$

so

$$|\mathbf{a}_\perp| = \frac{v}{r}\frac{\Delta s}{\Delta t}$$

which means

$$\boxed{a_c = \frac{v^2}{r}}$$

5.2 Uniform Circular Motion, a more Detailed visit

Imagine a point particle moving along a circle. If the position of the particle at $t = 0$ be $(R, 0)$ the radius of the circle is clearly R.

Let now θ represent the angle that the position vector makes with the x axis, our basic knowledge of trigonometry implies that at such point

$$\begin{aligned} x &= R\,cos\theta\,, \\ y &= R\,sin\theta\,, \end{aligned} \tag{5.3}$$

Let us open our minds for a little bit new idea. Since the point particle is moving, and the radius is fixed, the only thing that changes with time is the angle, so we should write: $\theta = \theta(t)$, meaning that in vector notation, the position of the particle must be written as

$$\mathbf{r}(t) = R\,cos\theta\,\hat{\mathbf{e}}_x + R\,sin\theta\,\hat{\mathbf{e}}_y\,, \tag{5.4}$$

which may also be expressed as

$$\mathbf{r}(t) = R\,\hat{\mathbf{e}}_r\,, \tag{5.5}$$

where

$$\hat{\mathbf{e}}_r = cos\theta\hat{\mathbf{e}}_x +\ sin\theta\,\hat{\mathbf{e}}_y \tag{5.6}$$

is a unit length vector pointing in the outgoing radial direction.

Example 6 *Draw the position vector of a particle moving along a circle of radius* $R = 1\,unit$ *when* $\theta = 30°, 45°, 90°, 135°$ *and* $180°$

Example 7 *Draw the vector*

$$\hat{\mathbf{e}}_\theta = -sin\theta\,\hat{\mathbf{e}}_x + cos\theta\,\hat{\mathbf{e}}_y\,,$$

and convince your self that it is perpendicular to $\hat{\mathbf{e}}_r$ *and consequently, to the position vector and therefore tangent to the circle.*

According to example 7 and the notion of tangent velocity, it is natural to express the vector of tangent velocity as

$$\mathbf{v} = R\,\omega\,\hat{\mathbf{e}}_\theta\,,$$

while the centripetal acceleration is

$$\mathbf{a}_c = -\frac{v^2}{R}\,\hat{\mathbf{e}}_r = -R\omega^2\,\hat{\mathbf{e}}_r\,,$$

5.3 Constant Angular Aceleration

Many physical situations in rotational kinematics involve motion of a particle with constant nonzero angular acceleration.

The kinematic equations for such motion are very similar to those of linear motion with constant acceleration, namely

$$\begin{aligned}\theta(t) &= \frac{\alpha\,t^2}{2} + \omega_0 t + \theta_0 \\ \omega(t) &= \alpha\,t + \omega_0\,.\end{aligned} \tag{5.7}$$

Here, the meaning of the symbols is as follows: $\theta(t)$ is the angular position of the particle at time . θ_0 is the initial angular position of the particle. $\omega(t)$ is the angular velocity of the particle at time t. ω_0 is the initial angular velocity of the particle. α is the angular acceleration of the particle.

5.4 Questions

Prob 7 *To what radian measure does a one degree angle correspond?*

Prob 8

1. *What is the angular position in radians of the minute hand of a clock at 3:30?*[1]
2. *What is the angular position in radians of the minute hand of a clock at 1:15?*
3. *What is the angular position in radians of the minute hand of a clock at 2:55?*

Prob 9 *A child on a merry-go-round takes* $3.9\ s$ *to go around once. What is his angular displacement during a* $1.0\ s$ *time interval?*

Prob 10 *A turntable rotates counterclockwise at* $76\ rpm$ *. A speck of dust on the turntable is at* $0.47\ rad$ *at* $t = 0$*. What is the angle of the speck at* $t = 8.2\ s$ *?*

Your answer should be between 0 *and* $2\pi\ rad$ *.*

Prob 11 *A turntable is rotating at* $33\,1/3\ rpm$*. You then flip a switch, and the turntable speeds up, with constant angular acceleration, until it reaches* $78\ rpm$*.*

Would it be possible to find the amount of time, in seconds, it takes for the turntable to reach its final rotational speed?

Prob 12 *Figure 5.2 shows a merry-go-round rotating at constant angular speed. Two children are riding the merry-go-round: Ana is riding at point A and Bobby is riding at point B.*

1. *Which child moves with greater magnitude of linear velocity?*
2. *Who moves with greater magnitude of angular velocity?*
3. *Who moves with greater magnitude of tangential acceleration?*

[1] Express your answer in radians to three significant figures.

4. *Who has the greater magnitude of centripetal acceleration?*

5. *Who moves with greater magnitude of angular acceleration?*

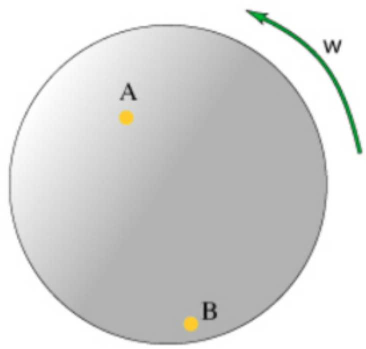
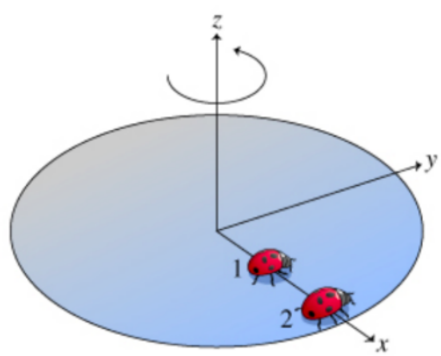


Figure 5.2:



Prob 13

Figure 5.3:

Two ladybugs sit on a rotating disk, as shown in figure 5.3. The ladybugs are at rest with respect to the surface of the disk and do not slip. Ladybug 1 is halfway between ladybug 2 and the axis of rotation.

1. *What is the angular speed of ladybug 1?*

    (a) *One-half the angular speed of ladybug 2*

    (b) *The same as the angular speed of ladybug 2*

    (c) *Twice the angular speed of ladybug 2*

    (d) *One-quarter the angular speed of ladybug 2*

2. *What is the ratio of the linear speed of ladybug 2 to that of ladybug 1? Answer numerically.*

3. *What is the ratio of the magnitude of the radial acceleration of ladybug 2 to that of ladybug 1? (Answer numerically).*

4. *What is the direction of the vector representing the angular velocity of ladybug 2? See the figure for the directions of the coordinate axes.*

5. *Now assume that at the moment pictured in the figure, the disk is rotating but slowing down. Each ladybug remains "stuck" in its position on the disk. What is the direction of the tangential component of the acceleration (i.e., acceleration tangent to the trajectory) of ladybug 2?*

Prob 14 *To throw the discus, the thrower holds it with a fully outstretched arm. Starting from rest, he begins to turn with a constant angular acceleration, releasing the discus after making one complete revolution. The diameter of the circle in which the discus moves is about* $1.9\ m$ *.*

If the thrower takes $1.2\ s$ *to complete one revolution, starting from rest, what will be the speed of the discus at release?*

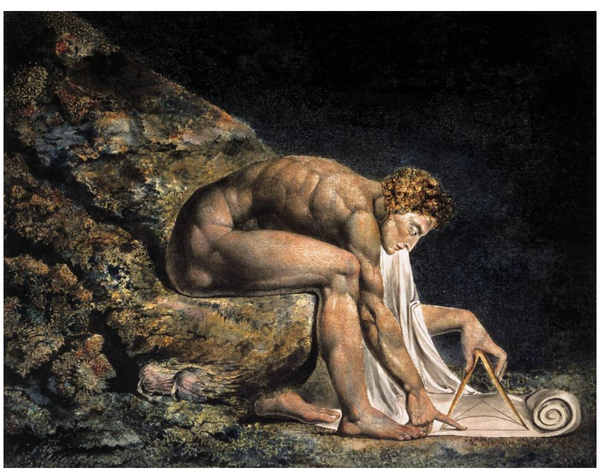
Prob 15 *A computer hard disk starts from rest, then speeds up with an angular acceleration of* $190\ rad/s^2$ *until it reaches its final angular speed of* $7200\ rpm$ *.*

How many revolutions has the disk made $10.0\ s$ *after it starts up?*

Chapter 6

Dynamics: Newton Laws

Mechanics is, probably, one of the oldest subdisciplines of physics. The primary objectives of mechanics consist of: explaining the equilibrium and movement of the bodies that we observe around us, aiming not only to describe the movements, but to understand the causes that produce them.



Newton by William Blake. Ink and watercolor on paper. Tate Gallery, London.

Mechanics is built upon two fundamental principles. The first is the principle of "causality," a philosophical principle based on experience, according to which, everything happens due to a preceding cause. The other fundamental philosophical principle is "reductionism"; in very simple terms, we can understand reductionism as an approach to the study of the nature of the complex in terms of the study of its parts and the interactions between them. Expressed in other terms, reductionism states that a complex system is nothing more than the sum of its parts and that therefore, to understand it, it is sufficient to understand its constituents and the interactions between them. According to reductionism, it is possible to study the phenomena that interest us by discarding qualities of them that could be unnecessary for the description. Neither the principle of causality nor the reductionist position are obvious or necessarily true but both are at the heart of the discipline we want to study.

Galileo [15/02/1564-8/01/1642] and others before him had studied kinematics, and had even tried to discuss some of the causes that determined the movements of particles. According to Aristotle in the fourth century BC, it is necessary to push in order to move an object, or expressed differently, it is not possible for a movement without a motive force to maintain it. In truth, our daily experience urges us to share Aristotelian ideas (if you don't believe it, try to find a box that moves eternally without requiring something to push it). However, the teachings of Aristotle and our daily experience are fallacious, and it is necessary to use a good dose of abstraction to realize this.

Galileo was the one who introduced the notion that the natural tendency of bodies was to maintain their state of motion. But it was not until Isaac Newton[1] [25/12/1642–20/3/1726/7] that this notion was established more precisely and that an adequate treatment of the problem of the causality of motion appeared. In modern terms, Newton introduced the notion of forces as causes of changes in the state of motion of bodies as well as the need to define forces and other objects of interest as what we know today as vectors.

[1] Curiously, Newton was born the year Galileo died

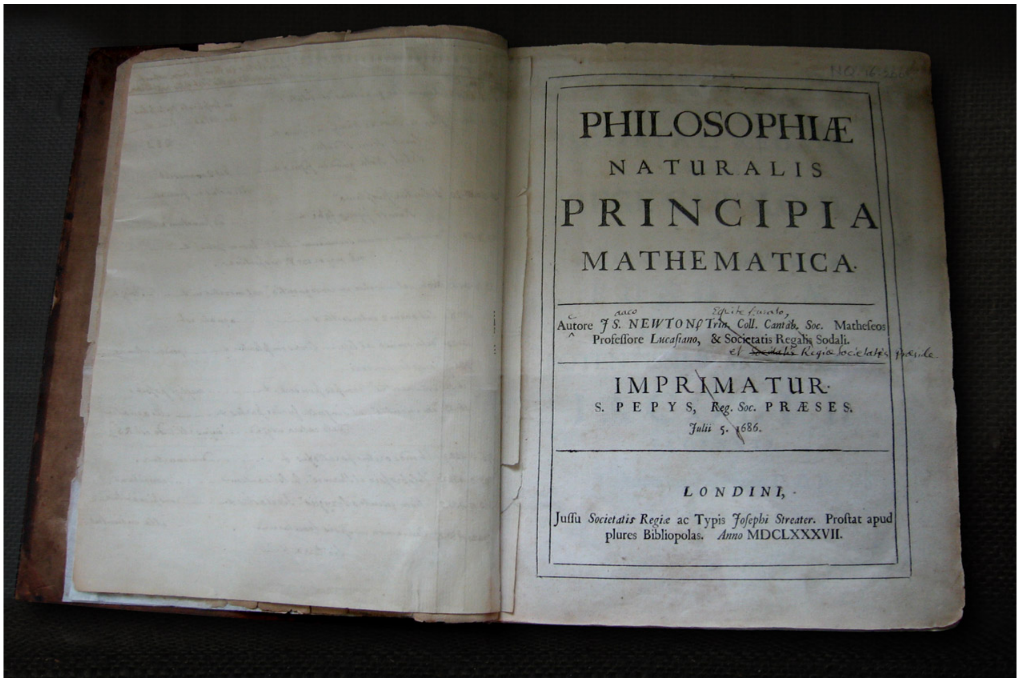


Figure 6.1: A personal copy of Newton's first edition of the *Principia* contains handwritten notes for corrections for the second edition. The work was published under the auspices of Edmond Halley on July 5, 1687.

6.1 Newton's Laws as He wrote them

Our interest in physics and our historical curiosity make it impossible for us to resist commenting a bit on the statement of Newton's laws as they appear in the first edition of Newton's work. The Principia is written in a language that is very reminiscent of mathematics textbooks, with definitions, axioms (laws), theorems, corollaries, and their rigorous proofs. In fact, the first chapter introduces the basic definitions.

In the Principia, each idea (be it a definition, a law, or anything else) is introduced with complete precision and is accompanied by some explanations that help to illustrate it. In any case, the degree of observation of nature, of abstraction from the observations, and of care with which each detail is described is enormous and is evidently influenced by Galileo. Having said this, let us take our little historical walk.

It is necessary to emphasize that the definitions associated with motion are formulated from the point of view of a terrestrial observer, which presupposes that - in modern terms - an observer fixed on the Earth is inertial (a hypothesis that, as we will see in paragraph 6.3.1, is false, although it has approximate validity). The work is written in a form that is difficult to read for users of differential and integral calculus, however, if we overlook that detail, the reasoning is quite clean and elegant.

6.1.1 Definitions

PHYLOSOPIAÆ NATURALIS PRINCIPIA MATHEMATICA

DEFINITIONES

Def. I

Quantitas materiæ est mensura ejustem orta ex illius Densitate & Magnitudine Conjunctim.

The quantity of matter is the measure of it that arises from its density and volume jointly.

Before continuing, it is very important to emphasize that, in context, the Latin expression *Densitate & Magnitudine Conjunctim* should be understood as the product of *Densitate* and *Magnitudine* meaning that the *Quantitas materiæ* is given by the formula:

$$\textit{Quantitas materiæ} = \text{density} \times \text{volume}\,,$$

and is therefore the mass as we know it today. Obviously, one can criticize that the definition is incomplete since it requires having defined density, but we will not deal with that problem here; what interests us is the fact that Newton intends to give a concrete definition of *Quantitas materiæ*

In the discussion that immediately follows the definition of *Quantitas materiæ*, Newton states "...*per experimenta pendulorum*...", that is: *by experiments with pendulums*, thus indicating an experimental basis for his work.

Def. II

Quantitas motus est mensura ejustem orta ex Velocitate et quantitate Materiæ Conjunctim.

"The quantity of motion is the measure thereof which doth arise conjointly from the celerity and quantity of matter".

According to what we have already learned about Latin, the expression *Velocitate et quantitate MateriæConjunctim* that appears in Newton's Definition II is the product of the body's mass and its velocity. *Quantitas motus* is therefore what we know today as: momentum, or (respecting Newton's choice), quantity of motion.

Def. III

Materiæ vis insita est potentia resistendi, qua corpus unumquodq;, quantum in se est, perseverat in statu suo vel quiscendi vel movendi uniformiter in directum.

"The *vis insita*, or innate force of matter, is a power of resisting, by which every body, as much as in it lies, endeavors to persevere in its present state, whether it be of rest, or of moving uniformly straight forward."

Newton continues explaining and says

"*...unde tiam vis insita nomine fignificantiffimo vis inertiæ dicci poffit*".

In English: "this innate force may more properly be called inertiæ, or force of inactivity".

6.1.2 Laws of Motion

The *Principia* continues with other definitions that we will not discuss at this time. Next, it presents the laws of motion[2].

LEGES MOTUS

[2] The latin to english translations of the laws were generated using Google's AI Gemini language model

Lex I

Corpus omne perseverare in statu suo quiescendi vel movendi uniformiter in directum, nisi quatenus illud a viribus impressis cogitur statum illuum mutare.

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.

Lex II

Mutationem motus proportionalem esse vi motrici impressæ, & fieri secundum lineam rectam qua vis illa imprimitur.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

This is an extremely important point. In almost all physics literature for science and engineering, the second law is written in the form: *force=mass×acceleration*. This is not at all what is expressed in the Lex II, which, as we can clearly read, refers to the change in *motus*, that is, in the *Quantitas motus*, that is: *force=change in the Quantitas motus*. In fact, and according to our current understanding, this "change" is truly the instantaneous rate of change, that is, the time derivative of momentum. Newton invented Calculus to work on these ideas.

Lex III

Actioni contrariam semper & æqualem esse reactionem: sive corporum duorum actiones in se mutuo semper esse æquales & in partes contrarias dirigi.

"For every action, there is always an equal and opposite reaction: or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts."

One of the most simple and direct reasonings I have the had the opportuinity to read is the proof of Corolary I of the *Principia* which states:

Corol. I

Corpus viribus conjunctis diagonallem parallelogrammi edom tempore describeri, quo latera separatis.

A body, when acted upon by two forces at once, will describe the diagonal of a parallelogram in the same time in which it would describe the sides by the forces taken separately.

In this corollary and its demonstration, we can recognize that forces are vectors.

Although the success of Newtonian mechanics is undeniable, both from a purely scientific standpoint and from a practical one (the construction of machinery, buildings, airplanes, etc. uses Newtonian mechanics as a fundamental ingredient), it is necessary to criticize it.

Firstly, we must remember that theories have ranges in which they are applicable, Newtonian mechanics is only applicable in the domain of macroscopic objects that move at speeds small compared to the speed of light ($c \approx 300.000\, Km/s$). For objects of molecular size or smaller, that is, with typical longitudinal dimensions on the order of tenths of nanometers or smaller, Newton's theory fails miserably and must be replaced by quantum mechanics. For objects that move in such a way that the ratio between their speed ($\dot{s}$) and c satisfies $\dot{s}/c \geq 0.1$, Einstein's relativistic dynamics must be introduced.

Secondly, since Poincaré's early works, we have learned that, although good as an approximate principle, reductionism is not adequate for the description of nature, whose systems are of such complexity that the whole ends up being much more than the sum of its parts.

6.2 Inertial Reference Frames

If you lived in a place with starry nights, you would notice that the positions of most of these stars relative to one another are basically fixed. This observation is nothing new and has been part of the accumulated knowledge of many and diverse civilizations before ours.

The few celestial objects that, to the naked eye, are exceptions to the rule of fixed stellar position were named wandering stars (planets). From a relatively primitive point of view, it is natural to choose the fixed stars as a basis for creating a reference system. Fortunately, this is only an illusion due to the fact that our lives are short, the stars change their relative positions and therefore cannot constitute a system of reference in absolute rest ($\mathcal{G}$) that serves as a basis for - using the techniques of the previous section - constructing other reference systems linked to $\mathcal{G}$ through Galilean transformations.

If $\mathcal{G}$ existed, any pair of reference frames in uniform motion with respect to $\mathcal{G}$ could be related to each other through Galilean transformations (prove this). Such systems are called inertial, one of the fundamental flaws of Newtonian mechanics is the need to resort to inertial reference systems.

Today we know perfectly well that the notion of an absolute inertial reference frame is something that makes no sense, however, from a practical point of view for engineering, we can assume that a reference system fixed to the Earth is inertial, this is the position that - since this is a course in Newtonian mechanics - we are forced to take in this course.

To understand the problem of the existence of inertial systems, we can take a practical point of view, the only way to know if a system is or is not inertial is to ask if it is in uniform motion with respect to some other system that is known to be inertial. Now, if the question is asked and a positive answer is obtained, it is necessary to find out if the new system is inertial, for this .... you already know what is going to happen. To solve this problem of a circular definition physicist take a practicla point of view, they admit the existence of an ideal inertial system

By the way, today we can know if a reference system is locally inertial, for this it is necessary to observe the behavior of light rays, but this friends is material for more advanced courses.

6.3 Newton Laws: A modern Perspective

To put it very precisely and in somewhat modern terms, Newtonian mechanics is solely concerned with the mechanics of a point particle. That is to say, objects are conceptualized as mathematical points in Euclidean geometry (just as you're reading this: a planet is a point). Objects with dimensions (planets, cars, people, etc.) are thought of as systems composed of many point-like particles. Integration a calculus concept underlies this notion.

It's important to note that Newton's laws contain a significant amount of implicit information. Firstly, they are laws that allow us to study the motion of point particles in terms of vectors. Secondly, they postulate the existence of a single universal time and of inertial reference frames, which are the only frames in which these laws are applicable.

To be more precise, what is described as the position of a point is nothing more than our modern position vector as a function of time, $\mathbf{r}(t)$, which we should understand as a vector in space $\mathbb{R}^3$ with its origin at the origin of the coordinate system. Having said this, and to dive into the subject matter seriously, we will assume that you are familiar with the concepts of the kinematics of a point particle: inertial reference frames, the position vector $\mathbf{r}(t)$, the velocity $\mathbf{v}(t)$, the acceleration $\mathbf{a}(t)$, and how these vectors change when described from different inertial reference frames.

Furthermore, assuming that we have a notion of the mass of a particle, we will introduce Newton's definition of momentum in modern terms.

Definition 3 *The momentum of a particle of mass m moving with velocity $\mathbf{v}$ is given by*

$$\mathbf{p} = m\mathbf{v} \tag{6.1}$$

Clearly, this definition depends on the concept of mass, which we have not yet defined, but let's sweep that "little" detail under the rug for now.

First Law of Motion *Inertial reference frames exist and describe the same mechanics (they are equivalent to one another). Moreover, the second and third laws are only valid for descriptions made from inertial reference frames.*

Second Law of Motion *The time derivative (instantaneous rate of change) of the momentum is given by:*

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} \tag{6.2}$$

where $\mathbf{F}$ is <u>*the net force*</u> *acting on the particle, and $\frac{d\mathbf{p}}{dt}$ is the calculus notation for instantaneous rate of change for all practical purposes in noncalculus physics, we may just write*

$$m\mathbf{a} = \mathbf{F} \tag{6.3}$$

.

Third Law of Motion *When a particle exerts a force on another, the latter exerts a force on the former that is equal in magnitude and opposite in direction to the force that the first particle exerts on the second.*

In other words, think of two objects pushing or pulling on each other. If object 1 pushes on object 2 with a force $\mathbf{F}_{12}$, then object 2 will push back on object 1 with exactly the same amount of force but in the opposite direction. The key point about Newton's third law is that these forces always act on different objects, $\mathbf{F}_{12}$ acts in particle 2 while $\mathbf{F}_{21}$ acts on particle 1.

Perhaps the most obvious but important observation that can be made about Newton's third law is the following: the points of application of the (paired) action-reaction forces are different. In fact, it is crucial to realize and remember that the action and reaction forces never act on the same object.

6.3.1 What do Newton's laws establish?

Law of Inertia

The first law establishes the existence and equivalence of inertial observers. The degree of abstraction of this axiom should be highlighted. An observer on Earth is not inertial, the Earth rotates on its axis at a rate of $2\pi\ rad/\text{day} \approx 7.3 \times 10^{-5}\ rad/seg$, although this rotation is slow, its effects are clearly observable, just look at a Foucault pendulum in a museum located outside the intertropical zone (just so that things happen at an appreciable rate). This implies a high degree of abstraction, indeed, to establish the first law we must imagine a motion as it would be seen by a true inertial observer.

Today we know that the most we can do is establish the local existence of inertial observers, which can be done experimentally by observing the behavior of light rays propagating in a vacuum. In any case, the fundamental thing about the first law is the following: the physics that two different inertial observers measure is the same.

Force Law

The most common way to present Newton's second law is to note that it involves an intrinsic property of matter (mass) that demonstrates that if a pattern and a way of measuring acceleration are chosen, the force can be measured through an appropriate quotient. In these notes we want to emphasize something a little bit more practical, the equality

$$\dot{\mathbf{p}} = \mathbf{F} \tag{6.4}$$

can be understood in two ways.

- First, if we know the right-hand side ($\mathbf{F}$) and the mass of the particle subjected to the force, the equality can be interpreted as a differential equation that allows us to calculate (except for the initial conditions) the law of motion $\mathbf{r}(t)$ of the particle. If this sounds complicated, let's rephrase it by saying that, knowing $\mathbf{F}$ and m, the quotient $\mathbf{F}/m$ is the acceleration, which, when known, places us in the problem posed in section 6.4.

- Second, if we know the law of motion $\mathbf{r}(t)$ we can say things about the nature of the force acting on a particle.

Law of action and reaction.

The first comment we should make regarding the third law does not have to do with physics but with psychology. When one begins to study a discipline, a strong tendency appears to memorize sentences or short phrases that supposedly summarize concepts. In the case of the third law, the summary sentence that many students memorize in a usual way says something like: "Action and reaction pairs are equal and opposite forces." In this course, we want to be extremely emphatic in asking students not to memorize or use this expression or anything similar. Pseudo-summaries are typically false (or even worse: totally erroneous), due to their brevity they are very easily incorporated into the set of preconceptions of the students and remain there in their subconscious waiting for the first opportunity to be released from their prison, pass to the conscious and make them make egregious errors.

Let's describe the correct idea once and for all, let's consider two interacting particles, and let's call the pair of action-reaction forces[3]: $\mathbf{F}_{12}$ and $\mathbf{F}_{21}$. It is important to note that both velocity and acceleration are well defined when understood as pairs of objects $(\mathbf{r}, \mathbf{v})$ and $(\mathbf{r}, \mathbf{a})$. Speaking in less technical terms, but no less correct, we must specify at what point the origin of velocity and acceleration are located. Forces are also well defined only when the point at which they are applied is specified. Thus, for example, in the case of the two interacting particles the pairs are $(\mathbf{r}_1, \mathbf{F}_{21})$ and $(\mathbf{r}_2, \mathbf{F}_{12})$, that is, the force acting on particle number 1 is $\mathbf{F}_{21}$ while the force acting on particle number 2 is $\mathbf{F}_{12}$. The equality $\mathbf{F}_{12} = -\mathbf{F}_{21}$ must be interpreted as an equality between the second member of one pair and the opposite of the second member of the other as elements of Re^3 .

Let's now go with an interpretation of the Third Law. One way to understand it is to think of it as a principle of completeness.

Consider the motion of a pair of particles isolated from the rest of the universe[4], under these circumstances, the only force that could act on one of them would have to be exerted by the other (and vice versa), The motion of the particles is described by the equations of motion:

$$\begin{aligned} \dot{\mathbf{p}}_1 &= \mathbf{F}_{21} \\ \dot{\mathbf{p}}_2 &= \mathbf{F}_{12},, \end{aligned} \tag{6.5}$$

[3]remember that in this notation, $\mathbf{F}_{12}$ is the force that particle 1 exerts on particle 2, perhaps something like $\mathbf{F}_{1\to 2}$ should be used but that would be exaggerating things

[4]if this seems too abstract, imagine a baseball cut cleanly in two equal parts that are barely held together during an experiment inside the international space station. Each of the halves exerts a force on the other

if we add both equations we obtain

$$\dot{\mathbf{p}}_1 + \dot{\mathbf{p}}_2 = \mathbf{F}_{21} + \mathbf{F}_{12} \tag{6.6}$$

The left side of this equality must be defined as the total momentum ($\mathbf{p}$) of the set formed by the two particles. As we are thinking that the set of the two particles is isolated from the rest of the universe, it can be postulated that $\mathbf{p}$ is constant, that is, it is postulated that

$$\dot{\mathbf{p}} = 0 \tag{6.7}$$

but this is equivalent to postulating the identity (third law):

$$\mathbf{F}_{21} = -\mathbf{F}_{12},, \tag{6.8}$$

6.4 Examples, dynamics along a line

Example 8 *What happens when the net force acting on a particle is zero? Recall that, according to the first law, the physics described by all inertial reference systems will be the same, this means that the net force acting on the particle is zero for all inertial systems. Taking this into account, consider some inertial reference system whose origin we will call $\mathcal{O}$. The equation of motion of the body will be (by virtue of the second law)*

$$m\mathbf{a} = 0\,, \tag{6.9}$$

so that $\mathbf{v}$ has to be a constant vector, let's call it $\mathbf{v}_0$, where $\mathbf{v}_0$ is nothing but the initial velocity of the particle which is something that has to be measured and referred to $\mathcal{O}$, this is the first law!, so there is no inconsistency between this and the second law of Newton at least for the reference system with origin $\mathcal{O}$. On the other hand, the only thing that happens when changing from one inertial reference system to another is the addition of the relative velocity $\mathbf{V}$ *between* *the two reference systems, so that, if $\mathcal{O}'$ is the origin of another inertial reference system whose velocity measured by $\mathcal{O}$ ($\mathbf{V}$) is constant, the velocity of the mobile according to $\mathcal{O}'$ will be*

$$\mathbf{v}' = \mathbf{v}_0 - \mathbf{V}\,, \tag{6.10}$$

which is also constant. Moreover, if $\mathbf{V} = \mathbf{v}_0$, that is, if $\mathcal{O}'$ is comoving with the particle, it will be described by $\mathcal{O}'$ as at rest, definitively: A particle on which no forces act remains at rest or remains in uniform rectilinear motion, which is the standard statement of Newton's first law.

Example 9 *Consider a particle of mass m subject to a constant force $\mathbf{F}$. Newton's second law gives us a constant acceleration $\mathbf{a} = \mathbf{F}/m$. Consequently, the particle's motion follows a parabolic path described by the time dependent position vector (see* **??***)*

$$\mathbf{r}(t) = \frac{\mathbf{F}}{2\,m}\,(t - t_0)^2 + \mathbf{v}_0(t - t_0) + \mathbf{r}_0\,, \tag{6.11}$$

where $\mathbf{v}_0$ and $\mathbf{r}_0$ are the initial velocity and position, respectively.

Example 10 *A "harmonic oscilator" is a particle that shows an oscillating periodic motion about a special point called the equilibrium point. The archetype of a harmonic oscillator is a particle driven by an ideal spring that excerts a force on the particle which is proportional to the the distance of the particle to the equilibrium point of the spring (the point where the spring does neither pull nor push the particle). The force of the spring is restitutive (it always points towarrds the equilibrium point), if we set coordinates in such a way that the equilibrium point is at $x = 0$, the force applied by the spring to the particle will be described as*

$$\mathbf{F} = -\kappa\,x\,\hat{\mathbf{e}}\,, \tag{6.12}$$

where $\hat{\mathbf{e}}$ is a unit vector defining the positive side of the x axis, and κ is a constant that characterizes the elasticity of the spring and has units of force/distance.

The above in turn implies the following relation between the acceleration of the particle and its position at any time

$$m\mathbf{a} = -\kappa\, x\, \hat{\mathbf{e}}\,, \tag{6.13}$$

or

$$a(t) = -\omega_0^2\, x(t)\,, \tag{6.14}$$

where clearly,

$$\omega_0^2 = \frac{\kappa}{m}\,, \tag{6.15}$$

it may be easily shown (exercise) that ω_0 *has units of 1/time and it is for this that it is called angular frequency.*

An exercise in calculus which is beyond the scope of these notes shows that the motion is given by the formula

$$x(t) = A\, cos(\omega_0\, t + \delta)\,, \tag{6.16}$$

where A *and* δ *are constants, if you, the reader, do not know trigonometry yet, don' t worry, sufficient is to say that this motion is what you appreciate in the following video*

On the tomb of Newton in Westminster Abbey in London, one can read the following inscription:

H. S. E. ISAACUS NEWTON Eques Auratus,
Qui, animi vi prope divinâ,
Planetarum Motus, Figuras,
Cometarum semitas, Oceanique Aestus. Suâ Mathesi facem praeferente
Primus demonstravit:
Radiorum Lucis dissimilitudines,
Colorumque inde nascentium proprietates,
Quas nemo antea vel suspicatus erat, pervestigavit.
Naturæ, Antiquitatis, S. Scripturæ,
Sedulus, sagax, fidus Interpres
Dei O. M. Majestatem Philosophiâ asseruit,
Evangelij Simplicitatem Moribus expressit.
Sibi gratulentur Mortales,
Tale tantumque exstitisse
HUMANI GENERIS DECUS.
NAT. XXV DEC. A.D. MDCXLII. OBIIT. XX. MAR. MDCCXXVI

According to the english translation by G.L. Smyth[5], the text of the inscription is

Here is buried Isaac Newton, Knight, who by a strength of mind almost divine, and mathematical principles peculiarly his own, explored the course and figures of the planets, the paths of comets, the tides of the sea, the dissimilarities in rays of light, and, what no other scholar has previously imagined, the properties of the colours thus produced. Diligent, sagacious and faithful, in his expositions of nature, antiquity and the holy Scriptures, he vindicated by his philosophy the majesty of God mighty and good, and expressed the simplicity of the Gospel in his manners. Mortals rejoice that there has existed such and so great an ornament of the human race! He was born on 25th December, 1642, and died on 20th March 1726/7.

[5] *The Monuments and Genii of St. Paul's Cathedral, and of Westminster Abbey* (1826), ii, 703-4

Chapter 7

Momemtum and Impulse

Chapter 8

Work and Energy

Imagine a particle moving along a straight line under the influence of only one force (**F**) parallel to the trajectory and of constant magnitude F.

Think of two points along the trajectory which we will call initial and final points, we are interested in the difference

$$v_f^2 - v_i^2 = 2ad \tag{8.1}$$

which in kinematics we learned to be true for a particle moving with constant acceleration a, v representing the velocity of the particle, and d representing the distance between the initial and final points, see formula 3.17.

If we additionally multiply the identity by the mass, and factor out 2 from the right hand side we get

$$\frac{mv_f^2}{2} - \frac{mv_i^2}{2} = Fd\,, \tag{8.2}$$

where we have used Newton' s second law

$$F = ma\,, \quad \text{or } \; a = \frac{F}{m} \tag{8.3}$$

to substitute the acceleration (a kinematical quantity) by the force, a dynamical quantity, in fact, the agent causing the acceleration.

The quantities

$$
\begin{aligned}
W &= Fd\,, \quad \text{and} \\
T &= \frac{mv^2}{2}
\end{aligned}
\tag{8.4}
$$

are respectively called *the Work performed by the force* and *Kinetic Energy*, it is worth noticing that the quantity

$$
\frac{mv_f^2}{2} - \frac{mv_i^2}{2} \tag{8.5}
$$

is nothing but the change ($\Delta T = T_f - T_i$) of kinetic energy between the initial and final points.

Equation 8.2 represents an elementary form of what it is widely known as the *work-energy theorem*. It is important to emphasise here a subtle but profound point. In physics, a 'theorem' often originates from a mathematical identity, as is the case with Equation (8.2). In this case, the mathematical result establishes a connection between a dynamical quantity (force) and a kinematical quantity the square of the magnitude of the velocity (encoded in the kinetic energy). This connection elegantly bypasses the need for a direct application of Newton's laws to determine the final velocity of a motion given its initial velocity, effectively eliminating time from the calculation. However, unlike pure mathematics, physics is an experimental science. While mathematical reasoning can lead to powerful predictions and relationships, such as the one expressed in Equation 8.2, these must ultimately be validated by experimental observation to be considered physical laws or theorems. The work-energy theorem, in its more general form, has been extensively verified through experimentation, solidifying its place as a fundamental principle of physics.

A complete and rigorous derivation of the work-energy theorem in its most general form requires advanced calculus. With these tools, the theorem can be extended to arbitrary trajectories and variable forces. At this introductory level, we state the general form of the theorem without full mathematical proof, along with two important related results that will be useful in applications.

Theorem 1 Work and Energy *Let* $\mathbf{F}$ *the net force acting on a particle that moves along a trajectory between two points* i *and* f*, then the work done by* $\mathbf{F}$ *equals the change of kinetic energy* (ΔT) *of the particle between such points.*

1. The work done by a constant force to move a particle along an arbitrary trajectory equals the product of the projection of the force along a straight segment separating the two points and the length of the segment.

2. The work done by a force acting on a particle moving along a trajectory which is always perpendicular to the force is zero.

Example 11 *Near the Earth (altitudes of no more than 30 Km, the force of gravity acting on a body of mass* M *points direclty downwards and has a magnitude* $F_G = M \times g$*, where* g *is the acceleration of gravity,* $g = 9.78\, m\, s^{-2}$*. Gravity under these conditions is therefore a constant force.*

Let us think of someone trying to climb Mount Everest (elevation: $8848\, m$*) starting from base camp North (see figure 8.1). An interesting question might be what is the work that the climber must do against gravity to reach the summit of the highest mountain on Earth?*

Clearly, the trajectory of the climber is a very complicated curve in space, fortunately, and as we have already noticed, under the circumstances of the climber, gravity is constant, a feature that allows us to use some mathematical trickery to calculate the work of gravity. We consider a trajectory made of two lines, one perfectly vertical form camp north up to 8848 m and another completely horizontal reaching the summit, according to this, the work will be the sum of two pieces $W_G = W_1 + W_2$*. Along the first trajectory, gravity does a work* $W_1 = M \times (9.78 \times$

$8848\, J/kg$). *Along the horizontal part of the trajectory, there is no work done by gravity* ($W_2 = 0$) *since it is vertical while the path is horizontal which makes the force perpendicular to the trajectory.*

If we imagine that the mass of the climber and its equipment equal 100 kg, the work of gravity will be

$$\boxed{W_G = W_1 + W_2 = 978 \times 8848\, j = 8,653,344\, j} \tag{8.6}$$

Figure 8.1: Mount Everest taken from the perspective of base camp, Tibet, China (elevation: 5,200 metres). Credit: Gunther Hagleitner

Before ending this example, we may comment on units, when dealing with fitness and exercise people are used to talk about kilocalories, 1 Kcalorie is equivalent to 4184 j, that means that the work we calculated may be cast as

$$\boxed{W_G = 2068.19\, kcal} \tag{8.7}$$

A typical plate of spaghetti Bolognese contains around 600-650 calories depending on the serving size and recipe, with a standard serving usually containing around 2 cups of pasta with sauce, this seems to mean that a good trained climber might compensate his energy consumption to conquest the Everest by eating nearly 4 plates of spaghetti.

Points of view are very diverse, to first reader this quantity, 4 plates of spaghetti might sound too little to conquer the Everest, but, on the other hand, what do the reader think would happen to a normal sedentary person including four full plates of spaghetti Bolognese a week in his/her diet?

Example 12 *The treatment of the previous example, even though illustrative, was somehow sloppy. Let us consider the vertical part of the trajectory, and instead of thinking of an able climber think of someone which is being pulled upwards by a strong line. Under these conditions there are two vertical forces acting on the "climber", gravity dangerously pulling him downwards to make him fall and the force of the rope, called tension, which is pulling him upwards. If these two forces are of the same magnitude, the climber will not accelerate and his motion (if any) would be at constant velocity. Let us suppose that the motion is upwards, then the tension will be parallel to the velocity and the weight will be antiparallel, this considerations have to do with the expression "projection along the motion" related with the calculation of the work done by a force.*

If the climber moves, say, a distance L, and since the motion is upwards, gravity will be antiparallel to the motion, or in other words, the projection of gravity along the path is negative, *this means that the work done by gravity will be*

$$W_G = -mgL\,. \tag{8.8}$$

Now, the tension as we described it for this exercise is of exactly the same magnitude as the weight (why?), but since it is parallel to the motion the work it does has a positive sing:

$$W_T = TL = mg\,, \tag{8.9}$$

here the reader can easily realize the reason of the claim that the calculation of the previous exercise was somehow sloppy, we forgot the sign of the projections and therefore all we did was find the magnitude of the work and not the work itself.

8.1 One situation three perspectives

Imagine a very simple experiment, namely, tossing a marble vertically up. We will analize the physical description of such experiment from three different points of view.

8.1.1 Kinematics

Kinematics follows from thousands of experimental observations that teach us that: the free vertical motion of a particle near the ground is an uniformly accelerated motion with acceleration of magnitude $g = 9.78\,],m\,s^{-2}$ towards the ground.

If a marble is tossed up with initial speed v_0, it will reach its highest elevation (h) above the initial point, when its speed equals its turning point value: $v_f = 0$. Therefore, we may link those values by the kinematical formula

$$v_f^2 - v_0^2 = -2gh\,, \tag{8.10}$$

where the sign in the right side of the formula stems from using the positive sign for the upwards direction combined with the fact that the acceleration is towards the ground.

We therefore conclude

$$h = \frac{v_0^2}{2g}\,, \tag{8.11}$$

8.1.2 Dynamics

Dynamical formulas can be thoroughly constructed, when we reach a point of understanding of the physics involved, to allows us to learn the interaction (force) that produces the acceleration we observe in the kinematic experiments. In the case of the marble tossing, the force is the gravitational attraction that the Earth exerts on the marble.

Gravity acts universally between all massive objects, in particular, the gravitational force that a point particle of mass M excerts on another particle of mass m, is a force attracting mass m to the point where M located with a strenght

$$F_G = G\frac{mM}{r^2}\,. \tag{8.12}$$

The methods of calculus enable us to show that, if our planet Earth is considered to be a spherical object with a spherically symmetric distribution of mass, the gravitational force with which it attracts a small mass m, obeys exactly the same formula. Therefore, if R is the radius of the Earth, M its mass and G the so called gravitational constant, the gravitational force with which the Earth acts on a small body is

$$F_G = G\frac{M}{R^2}\,m\,, \tag{8.13}$$

towards the center of the eart.

Looking for the values on the appropriate places (this is a cute exercise for the reader), we find

$$F_G = 9.78\,Nw/kg \times\ m\,, \tag{8.14}$$

towards the center of the earth.

Now, at very low altitude (compared with the 6,378 km of radius of the Earth, 10000 m is a tiny height, this is a constant vertical force and therefore, we find that an object acted upon the earths gravity accelerates vertically downwards with a magnitude

$$g = 9.78\, m\, s^{-2}\,, \tag{8.15}$$

fantastic!, we can now use the kinematical formulas and find

$$h = \frac{v_0^2}{2g}\,, \tag{8.16}$$

Let stop for a moment, someone naive would say we did nothing but repeat ourselves.

A critical thinker would immediately jump and claim: by no means!. There is a huge conceptual difference between what has been done in this section and what was done in the previous one.

The formulas of kinematics originate on many, many observations that established that the free vertical motion has a very particular character. It is a uniformly accelerated motion. Nevertheless, and this is a huge but, we do not have any idea why is this so. We just now that the observations taught that is the way things are

In this section, on the other hand, everything relied on some extreme;y interesting result about a kind of interaction from which we were able to derive an acceleration formulas.

In a sense, the universal gravitational force formula tells us how is the acceleration between two particles going to be. Said in a different way, we understand the cause of the motion.

8.1.3 Energy Methods

Let us finally explore how the work an energy theorem applies to these case.

At the beginning of the motion the kiknetic energy of the marble is

$$T_i = \frac{mv_i^2}{2}\,. \tag{8.17}$$

When the marble arrives at what we consider to be the final point of the motion, i.e. the turning point where it stops, the kinetic energy will be

$$T_f = \frac{mv_f^2}{2}\,. \tag{8.18}$$

The work energy theorem expresses the fact that any difference between those two values is due to the work done by the net force acting on m, since in the free vertical motion the only acting force is gravity, we write

$$\Delta T = T_f - T_i = \frac{mv_f^2}{2} - \frac{mv_i^2}{2} = W_G\,, \tag{8.19}$$

where, obviously, W_G is the work made by gravity ($W_G = -mgh$). We know that initially $v_i = v_0$, we also know that at the highest or turning point $v_f = 0$, so, in this case, the work energy theorem reduces to

$$-\frac{mv_0^2}{2} = -mgh\,, \tag{8.20}$$

from where we get

$$h = \frac{v_0^2}{2g}\,, \tag{8.21}$$

it would be impossible to get something different or the laws of physics would be logically inconsistent, not to say, plain wrong.

8.1.4 Energy a Way To Simplify

Energy methods are considerably simplifying, they work as some sort of ledger. You get certain valuable info at one spatial point, and get some info at the end point. The difference tells you almost effortless a oversimplified story of what happened in between. Thus for instance, if at the end you get lees kinetic energy than what you had at the beginning, you immediately know that somewhere intermediate an agent did negative on the system.

If for some reason you know a simple way to calculate the work done along a trajectory, you will be able to get valuable information at the final; point of it by just finding the work and adding it to the initial energy.

8.2 From Tycho Brahe to Adams and Leverrier, the difference between kinematics and Dynmaics

To the naked eye, the starry sky looks like a gigantic vault with the stars welded on it. Indeed, careful observation shows that at the same time on different nights, the stars that we see in the vault are different, but, their relative positions are always the same, that is what allows conceiving the constellations, groups of stars which bring in wonderful mythical images to our minds, just like the clouds do. The difference between the visuals from the clouds and those from the stars is that the former are quite ephemeral lasting at most some minutes, while those of the stars change their shape in thousands of years or more.

This simple observation brought forth the composed term *fixed stars* to the stars we see unmovable in the vault. But this is not the whole story, objects far closer to Earth can easily be seen moving with respect to the background of fixed stars, a fact that gave birth to the name *wandering stars* or planets.

In 1609 after eight years of exhausting effort analyzing the astronomical observations of Tycho Brahe, Johhanes Kepler Kepler found his published his first two laws about planetary motion, which were published in his book "Astronomia Nova". In 1619, Kepler published "Harmonices Mundi", in which he describes his "third law." The third law shows that there is a precise mathematical relationship between a planet's distance from the Sun and the amount of time it takes revolve around the Sun.

Kepler had believed in the Copernican model of the Solar System, which called for circular orbits, but he could not reconcile Brahe's highly precise observations with a circular fit to Mars' orbit – Mars coincidentally having the highest eccentricity of all planets except Mercury.

Kepler's First Law: Each planet's orbit about the Sun is an ellipse. The Sun's center is always located at one focus of the ellipse. The planet follows the ellipse in its orbit, meaning that the planet-to-Sun distance is constantly changing as the planet goes around its orbit.

Kepler's Second Law: The imaginary line joining a planet and the Sun sweeps out – or covers – equal areas of space during equal time intervals as the planet orbits. Basically, the planets do not move with constant speed along their orbits. Instead, their speed varies so that the line joining the centers of the Sun and the planet covers an equal area in equal amounts of time. The point of nearest approach of the planet to the Sun is called perihelion. The point of greatest separation is aphelion, hence by Kepler's second law, a planet is moving fastest when it is at perihelion and slowest at aphelion.

Kepler's Third Law: The orbital period of a planet, squared, is directly proportional to the semi-major axes of its orbit, cubed. This is written in equation form as $p^2 = a^3$. Kepler's third law implies that the period for a planet to orbit the Sun increases rapidly with the radius of its orbit. Mercury, the innermost planet, takes only 88 days to orbit the Sun. Earth takes 365 days, while distant Saturn requires 10,759 days to do the same.

In 1621, Kepler noted that his third law applies to the four brightest moons of Jupiter. Godefroy Wendelin also made this observation in 1643. The second law, in the "area law" form, was contested by Nicolaus Mercator in a book from 1664, but by 1670 his Philosophical Transactions were in its favour. As the century proceeded it became more widely accepted.

Kepler' s laws are no less than a master piece of kinematics, they give a detailed account of how the orbits of objects gravitating around a massive object such as a star should be. Why the emphasis on the word kinematic? Well it comes about since Kepler' s laws came to be as a fantastic synthesis of Tycho Brahe observations and not from the understanding of some interaction Law.

Astronomy had to wait until the publication of Newton's Principia to get to the point where the planetary motions could be understood as the necessary result of a dynamical principle, namely Newton' s universal law of gravity.

Dynamical principles have enormous predictive power, in the case of gravity, we should mention the discovery of Neptune byu Adams and Leverrier who in 1945 used Newton's gravitational law to explain some anomalies on Uranus orbit by proposing the existence of a new planet beyond Uranus whose gravitational effect produced the observed anomaly. Adams and Leverrier incredibly difficult calculations succeeded and Neptune was discovered, a task that could have never be achieved using Kepler's laws..

Chapter 9

Systems of Particles and Center of Mass

Consider a system composed of three particles of masses m_1, m_2 and $m_3 = 4$, let $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ their positions with respect to an inertial reference frame with origin $\mathcal{O}$.

Under these conditions define a new vector as the sum

$$\mathbf{R} \equiv \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3}{m_1 + m_2 + m_3}\,, \tag{9.1}$$

$\mathbf{R}$ has dimensions of length and we want to think of it as the position vector of a point which we will call the center of mass of the system consistent of the three particles. A first obvious observation is that the position of the center of mass is calculated as a weighted average where the masses of each particle of the system constitute the weights.

Let us now think on the forces that act on each particle of the system. The net force acting on m_1 is

$$\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_1^{(ext)}\,, \tag{9.2}$$

where $\mathbf{F}_{21}$ and $\mathbf{F}_{31}$ are the forces which m_2 and m_3 exert on m_1, while $\mathbf{F}_1^{(ext)}$ is the net force that external environment to the 3 particle system excerts on m_1. Something similar happens with the other two particles.

If we add all the forces acting on the particles of the system we get

$$\mathbf{F} = \mathbf{F}_{21} + \mathbf{F}_{12} + \mathbf{F}_{31} + \mathbf{F}_{13} + \mathbf{F}_{23} + \mathbf{F}_{32} + \mathbf{F}_1^{(ext)} + \mathbf{F}_2^{(ext)} + \mathbf{F}_3^{(ext)} \tag{9.3}$$

Newton's third law ensures that pairs such as $\mathbf{F}_{21} + \mathbf{F}_{12}$ cancell out, and so

$$\mathbf{F} = \mathbf{F}_1^{(ext)} + \mathbf{F}_2^{(ext)} + \mathbf{F}_3^{(ext)} \tag{9.4}$$

On the other hand, Newton's second law implies that

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3 = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\,, \tag{9.5}$$

the sum of the forces on each particle, but we already calculated what happens with the right hand side, and so

$$m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3 = mathbf F_1^{(ext)} + \mathbf{F}_2^{(ext)} + \mathbf{F}_3^{(ext)}\,, \tag{9.6}$$

if we multiply and divide by $M = m_1 + m_2 + m_3$, the total mass of the system in the left hand side of this identity, we get

$$M\frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3}{M} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3\,, \tag{9.7}$$

remarkably, the quantity

$$\frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + m_3\mathbf{a}_3}{M}\,, \tag{9.8}$$

is the weighted average of the acceleration of the particles of the system and cannot be anything but the "acceleration of the center of mass' ' of the system ($\mathbf{A}$), we have just found that

$$M\mathbf{A} = \mathbf{F}^{(ext)} \tag{9.9}$$

where the total external force acting on the system is

$$\mathbf{F}^{(ext)} \equiv \mathbf{F}_1^{(ext)} + \mathbf{F}_2^{(ext)} + \mathbf{F}_3^{(ext)} \tag{9.10}$$

we have made a wonderful mathematical discovery, the center of mass is a point that somehow, via averaging moves under the sole influence of the external forces acting on the system without regard to the forces of interaction between the particles of the system, callled internal formces