A Brief Account of Trigonometry

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The purpose of these notes is to give an introduction of trigonometry. If someone is reading these is because she/he have a need of trigonometry for some purpose. Since I want to make these notes as short and concrete as possible, I present as few applications of subject as possible. I will leave to the reader to decide whether or not this material feels helpful in any way.

1 Measuring angles

Angles are usually measured in degrees minutes and seconds of arc length. To define a degree a circle is divided into 360 equal circular segments, the angle defined by two straight segments beginning at the center of the circle and ending at the two extremes of a one of those equal circular segments defines a 1° angle. This angle is in turn divided into sixty (60) equal parts each measuring a minute of arc, finally, a minute of arc ($1'$) is divided into sixty equal tiny angles each measuring one second of arc ($1''$). This way of measuring angles is very convenient for several applications.

One can ask why dividing the circle into 360 equal segments and there is a simple answer:

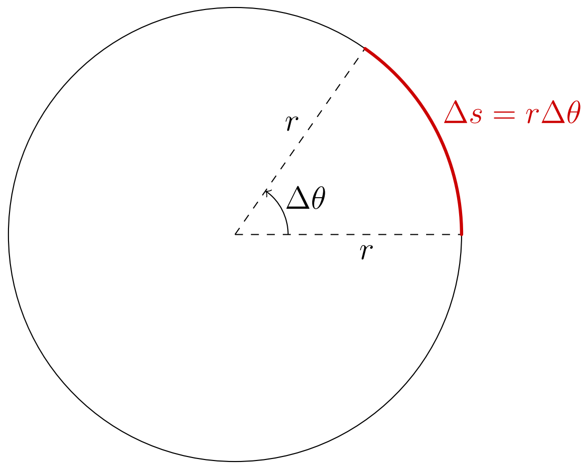

Figure 1: Circular arc subtended by an angle $\Delta\theta$

an arbitrary decision. There is a much more natural way of measuring angles, it is based on the fact that the circumference of a circle of radius r is given by the well known formula $circ = 2\pi r$. This new definition goes as follows, a circular arc of length Δs, subtends an angle $\Delta\theta$ such that:

$$\Delta s = r\,\Delta\theta\,, \tag{1}$$

from where

$$\Delta\theta = \frac{\Delta s}{r}\,. \tag{2}$$

Angles defined in this way are said to be measured in radians, and obviously they are dimensionless (quotients of lengths). For illustration, we note that an angle of 360° corresponds to 2π rad, while an angle of 90° which subtends a quarter of a circle corresponds to

$$\frac{\text{Lenght of quarter of a circle of radius } r}{r} = \frac{1}{4}\frac{2\pi\,r}{r} = \frac{\pi}{2} \tag{3}$$

Arc	Angle in Degrees	Angle in Radians
Full circle	$360°$	2π
Quarter of a circle	$90°$	$\pi/2$
Sixth of a circle	$60°$	$\pi/3$
Eight of a circle	$90°$	$\pi/4$
Twelfth of a circle	$90°$	$\pi/6$

Table 1: Equivalence between notable angular values

2 Three trigonometric functions: sine, cosine and tangent

Here we will define the three basic trigonometric functions sine, cosine and tangent using the method of the **unitary circle**.

We begin by considering a circle of radius $R = 1$ (that is why it is called unitary) as in figure 2, and draw the x and y axes. Let us additionally consider a segment beginning at the center of the circle, this segment makes an angle angle[1] α with the x axis and ends at the point where it meets a vertical segment of the tangent to the circle that touches it at the point $(1, 0)$, and which is therefore perpendicular to the x axis.

We now note that this construction naturally defines two **similar** right triangles. The hypotenuse of the smallest of which is the radius defining the angle α, while its legs are the orange vertical segment going from the intersection of the radius with the circle to the x axis and horizontal blue segment corresponding to the projection of the hypotenuse on the x axis. As shown in the figure, the sine and cosine of the α angle are defined by the **signed** length of

[1] In the usual convention, angles are positive in the counterclockwise sense

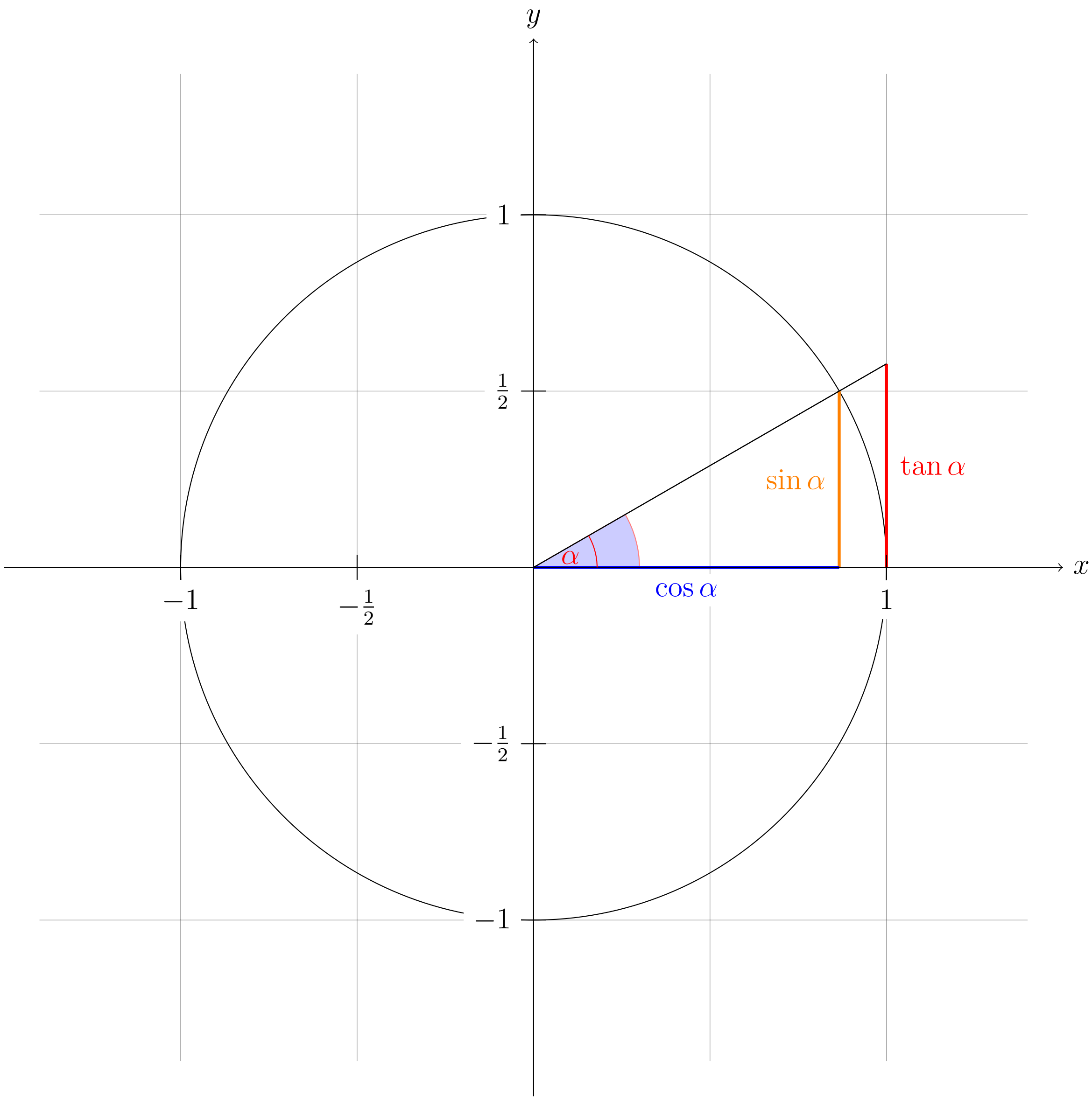

Figure 2: The Unitary Circle and Trigonometric Functions. **This sketch is the only thing that must be memorized about trigonometry**

those arms.

The tangent, on the other hand, is defined as the signed length of the vertical arm of the big triangle.

The first thing that happens when we use these definitions is that the fundamental formula linking the sine and cosine functions becomes almost obvious, indeed,

Corollary 1 Fundamental Trigonometric Identity *For any angle,*

$$\boxed{sin^2\alpha + cos^2\alpha = 1}$$

The proof of this corollary is extremely simple, since the sine and cosine are defined as the signed lengths of the legs of a right triangle whose hypotenuse is of length 1, we can immediately use the Pythagorean theorem and get the wanted result.

3 A connection to geometry

We now take note of something apparently silly, in the big triangle, the length of the horizontal leg of triangle is 1 so we may write,

$$tan\alpha = \frac{tan\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{adyacent leg to } \alpha} \tag{4}$$

Let us now turn our attention to the a famous theorem known by the name of the presocratic greek philosopher Thales of Miletus, according to Thales' theorem the ratios between corresponding sides of similar triangles are equal. If we apply this statement for our two triangles we get

$$\frac{\text{opposite leg to } \alpha \text{ in big triangle}}{\text{adyacent leg to } \alpha \text{ in big triangle}} = \frac{\text{opposite leg to } \alpha \text{ in small triangle}}{\text{adyacent leg to } \alpha \text{ in small triangle}} \tag{5}$$

which by virtue of the figure translates into

$$\frac{tan\alpha}{1} = \frac{sin\alpha}{cos\alpha}. \tag{6}$$

we thus end up with

$$\boxed{tan\alpha = \frac{sin\alpha}{cos\alpha}} \tag{7}$$

which is the definition of the tangent of an angle.

Thales' s theorem can also be used to put the usual definitions of sine and cosine under a different light. Imagine a right triangle, any right triangle, with no other particular characteristic than being right, a little thought shows that we can always rescale the sides of such triangle to construct another triangle, similar to the one we began with, but having a hypotenuse of unit length. Once we do that, we can put the new triangle in the unitary circle to obtain something very close to what we see in figure 2, in this new triangle, and since the length of hypotenuse equals one we can write the formulas

$$\begin{aligned} sin\alpha &= \frac{sin\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}} \\ cos\alpha &= \frac{sin\alpha}{1} = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}, \end{aligned} \tag{8}$$

but, by virtue of the similarity of the new triangle with the original one, and Thales' s theorem, we get, that no matter what triangle we are talking about, as long as it is a right triangle,

$$\boxed{sin\alpha = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}}} \tag{9}$$

$$\boxed{cos\alpha = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}} \tag{10}$$

which are the usual formulas given at the school.

It is imperative to understand and learn that the two sets of definitions we have given are different but that they define exactly the same things, i.e. there is nothing to worry when using one definition or the other. In fact, if we continue our study of math to a higher level, namely full blown calculus, we will find more definitions of the trigonometric functions. One of those is in terms of a field by the name of *differential equations*, but we will not pursue this path in these notes.

4 Some "not very Useful" Formulae

As time goes by and science pushes forward, some things that belonged to what we should call **fundamental knowledge** cease to be fundamental.

Log tables (fig 3) are a good example, in 1950 it was impossible to imagine an engineer unable to use them, they were absolutely necessary for any difficult calculation

LOGARITHM

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Figure 3: log tables

The same happens with many other things like the following formulas

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$
$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$$
$$\sin(\theta_1 - \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$$
$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$
$$\tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}$$

$$sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - cos\theta}{2}}, \qquad cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + cos\theta}{2}} \tag{11}$$

these formulas were extremely necessary to create tables with the values for trigonometric functions for many different angles. Nowadays, the dumbest cell phone can calculate trigonometric functions so there is no real need for these formulas in the every day work environment of any scientist or engineer.

Nevertheless, these formulas find an important application, they are of use to train the minds of young people who want to pursue STEM studies.

Intellectual training is absolutely fundamental for anyone, and if someone is interested in pursuing a STEM career, mathematics are a must. This is why we suggest in the strongest possible terms to study how the above formulas (or at least some of them) are shown to be true. We dare to suggest Proof of angle addition formula for sine — Trigonometry — Khan Academy as a good starting point

5 Values of basic the trigonometric functions for some angles

Mathematics is all about finding patterns and developing intuition about them. Trigonometry, being a branch of mathematics is no exception to this rule. In this section we will use some very simple reasoning to calculate the values of the trigonometric functions for some very particular angles. Let us see how far we can get with simple ideas without using any difficult stuff.

Since the sine and cosine of an angle are the signed lengths of the legs of a right triangle whose hypotenuse has unit length, the values of both functions must belong to the interval $[-1,1]$, we may state this as a

Theorem 1 *$\forall\theta$,*

$$-1 \leq sin\theta \leq 1\,, \qquad -1 \leq cos\theta \leq 1$$

Now, when θ is very close to 0 the leg of the triangle opposite to θ (the vertical orange segment in fig. 2) becomes very, very short while the horizontal blue leg becomes longer and longer almost reaching the point $(1,0)$. In fact, when θ_0 the triangle completely collapses rendering the hypotenuse to coincide with the adjacent leg, this is just the statement

$$sin(0) = 0\,, \qquad cos(0) = +1\,,$$

a similar reasoning leads to

$$sin(\pi) = 0\,, \qquad cos(\pi) = -1\,,$$

Let us now think to what happens when the angle we are interested in gets very close to $\pi/2$ (a quarter of a circle). In this case the hypotenuse becomes almost vertical so its horizontal projection is close to disappear while the orange leg almost equals the hypotenuse, when θ exactly matches $\pi/2$ we conclude

$$sin\left(\frac{\pi}{2}\right) = +1\,, \qquad cos\left(\frac{\pi}{2}\right) = 0\,,$$

the case $\theta = 3\pi/2$ is handled similarly yielding

$$sin\left(\frac{3\pi}{2}\right) = -1\,, \qquad cos\left(\frac{3\pi}{2}\right) = 0\,,$$

The $\theta = \pi/4$ (i.e. $45°$) angle is treated a bit differently. All we must do is to realize that in this case the hypotenuse of the triangle is nothing but the diagonal of a square, expressed differently, both legs of the right triangle over the unitary circle are equal. This means that

$$sin\left(\frac{\pi}{4}\right) = cos\left(\frac{\pi}{4}\right)\,,$$

the fundamental trigonometric identity for this case yields

$$\begin{aligned} 1 &= sin^2\left(\frac{\pi}{4}\right) + cos^2\left(\frac{\pi}{4}\right) = \\ &= 2\,sin^2\left(\frac{\pi}{4}\right)\,, \end{aligned}$$

which is nothing but

$$2sin^2\left(\frac{\pi}{4}\right) = 1\,,$$

which in turn implies

$$sin\left(\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2}\,,$$

since $\pi/4$ is in the first quadrant, the sign must be $+$, and since the cosine of this angle equals its sine,

$$sin\left(\frac{\pi}{4}\right) = cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\,.$$

There are still two notable angles $\pi/3$ ($60°$) and $\pi/6 = \frac{1}{2}\pi/3$ (30^0), whose sines, cosines and tangents can be easily calculated by hand. To this end we draw an equilateral triangle[2], we call the lengths of its sides $2x$. Then we note that any of the bisector angles divides the opposite

[2] The internal angles of an equilateral triangle are all $60°$ angles

side in two segments of exactly the same length (x). Besides, the bisection builds two congruent right triangles which share a leg (the bisecting segment). The hypotenuses of these two triangles are nothing but two of the sides of original triangle.

Thinking of any of the two small right triangles it is easy to see that the leg adjacent to the 60° angle has length x. Due to the Pythagorean theorem and if we call s the length of the other leg,

$$s^2 + x^2 = 4x^2\,, \tag{12}$$

from which $s = \sqrt{3}x$, since this leg is opposite to the 60° angle, we end up with

$$\begin{aligned} sin(\pi/3) &= \frac{\text{leg opposite to the 60 degree angle}}{hypotenuse} = \frac{s}{2x} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}\,, \quad \text{and} \\ cos(\pi/3) &= \frac{\text{leg adyacent to the 60 degree angle}}{hypotenuse} = \frac{x}{2x} = \frac{1}{2}\,, \quad \text{finally} \\ tan(\pi/3) &= \frac{sin(\pi/3)}{cos(\pi/3) = \frac{\frac{\sqrt{3}}{2}}{1}2} = \sqrt{3} \end{aligned} \tag{13}$$

Exercise 1 *Draw the necessary sketches needed to achieve the geometric reasoning given above*

Exercise 2 challenge *Carry out the necessary modifications to calculate* $sin(\pi/3)$*,* $cos(\pi/3)$ *and* $tan(\pi/3)$*.*

Exercise 3 *Given* $sin(\pi/3) = \sqrt{3}/2$*, use trigonometric identities to* $cos(\pi/3)$*,* $sin(\pi/6)$ *and* $cos(\pi/3)$*. A way to do it is to recall that the* 30 *and* 60 *degrees are complementary,*

Exercise 4 *Calculate all the values shown in table 2*

Angle (°)	Angle (rad)	Quadrant	sin	cos	tan
0	0	I	0	$+1$	0
30	$\pi/6$	I-IV	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45	$\pi/4$	I	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	I	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90	$\pi/2$	I-II	$+1$	0	$+\infty$
120	$4\pi/6$	II	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$
135	$3\pi/4$	II	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
150	$5\pi/6$	II	$1/2$	$-\sqrt{3}/2$	-1
180	π	II-III	0	-1	0
210	$7\pi/6$	III	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$
225	$5\pi/4$	III	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$+1$
240	$4\pi/3$	III	$-\sqrt{3}/2$	$-1/2$	$+\sqrt{3}$
270	$3\pi/2$	III-IV	-1	0	$-\infty$
300	$10\pi/6$	IV	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$
315	$7\pi/4$	IV	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
330	$7\pi/4$	IV	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
360	2π	I-IV	0	$+1$	0

Table 2: Values of sine, cosine and tangent of some special angles. Angles showing two quadrants signal boundaries between quadrants

6 An equivalence relation

Angles are **periodic** i.e. different values of angles do in fact refer to the same point in the unitary circle, when this happens we say that the angles are **equivalent** which is expressed by

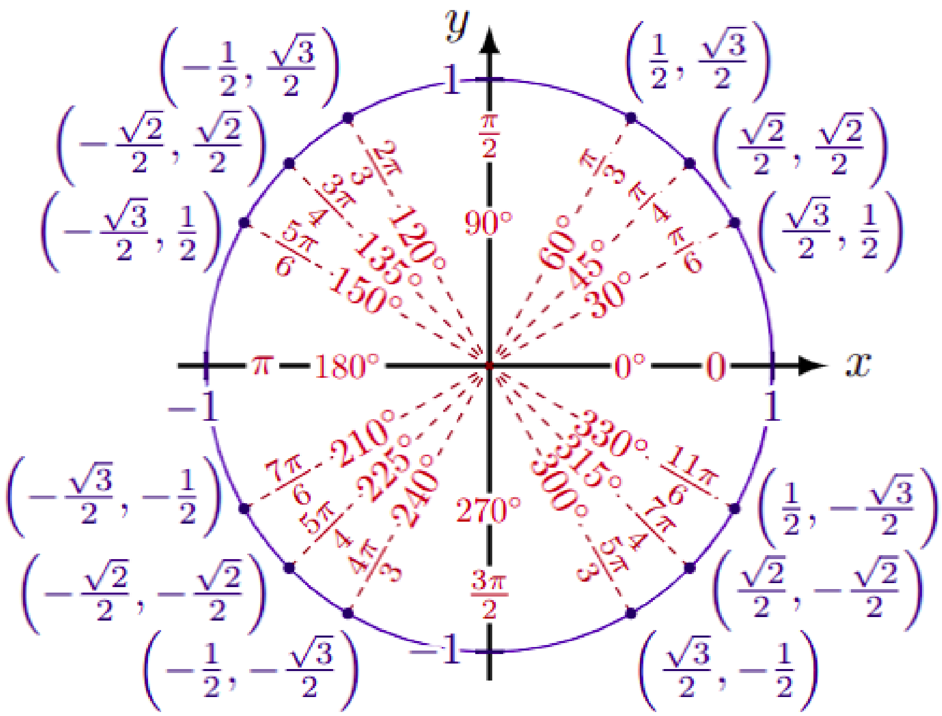

Figure 4: The unit circle and some special values

the symbol: $\approx$. Thus for instance, the angles 0 and 2π both define the angle that signals the point $(+1, 0)$ of the unit circle. In our new language we write $0 \approx 2\pi$, but, and this is extremely interesting, if we go around the unitary circle twice the corresponding angle[3] is $2 \times 2\pi$ and we go back again to the point $(1, 0)$ so $0 \approx 2 \times 2\pi$.

In fact, the same happens with any number n of complete rotations which. After some thought we can generalize the idea and write

Definition 1 *Two angles θ_1 and theta$_2$ are equivalent ($\theta_1 \approx \theta_2$) if and only if*

$$\theta_2 - \theta_1 = 2\,k\,\pi\,, \qquad k \in \mathbb{Z}$$

[3]The first 2 corresponding to the number of rotations around the circle

A couple of examples will help to understand this concept. Set a point in the unitary circle signaled by an angle $\theta_1 = 45°$, if begin in this point and wind (counterclockwise) the circle 5 times we end up at exactly the same point, but the angle will be $\theta_2 = 45° + 5 \times 360°$. Measuring the angles in radians, we would say

$$\theta_1 = \pi/4\,, \quad \theta_2 = \pi/4 + 2(5)\pi\,,$$

therefore

$$\theta_2 - \theta_1 = 2(5)\pi$$

and since $5 \in \mathbb{Z}$ we can safely state

$$\theta_2 \approx \theta_1$$

which -as we already know- is just a fancy way to say that both angles mark exactly the same point in the unitary circle.

7 Trigonometric functions are periodic

The angular equivalence we have discussed in section 6 has deep geometrical roots and gives rise to a concept **periodicity of the trigonometric functions**, to introduce this concept let us carefully think about the sine function through its definition according to the unitary circle. Imagine two angles, let's say $\alpha_1 = \pi/6$ and $\alpha_2 = \pi/6 + 2\pi = 13\pi/6$. Certainly $\alpha_1 \approx \alpha_2$ (with winding $n = 1$) but the important thing here is that when we go to the unitary circle we certainly find that

$$sin(\pi/6) = sin(13\pi/6)\,, \tag{14}$$

besides, more geometrical thinking will convince beyond any doubt that for any angle α and any $k \in \mathbb{Z}$

$$sin(\alpha) = sin(\alpha + n\,L)\,, \tag{15}$$

where $L = 2\pi$ is known as the **period**. This concept is generalized to any function with real domain, and is given as[4]

Definition 2

$$f : \Re \rightarrow \Re\,,$$

is called periodic of period $L \in \Re$ *if and only if,* $\forall x \in \Re$

$$f(x) = f(x + L)$$

According to definition 2, all trigonometric functions are periodic with period $L = 2\pi$. There is a little twist to this story but we will not push it in these notes.

8 Trigonometric equations

It is perfectly possible that when solving some mathematical problem we reach an equation (i.e. a question) such that

$$sin(x) = \frac{1}{2}\,, \tag{16}$$

where we must find x.

According to the previous sections, we may be tempted to answer that x is either $30°$ or $150°$ but such answer is only partially correct because it does not really reflect all that we have learned, in particular the concept of periodicity.

For the $30°$ angle the correct answer should be: $30°$ *or any other angle obtained from it by completely winding circles an arbitrary number of times.* Something similar happens with the $150°$ angle.

In fact, the absolutely correct answer should be:

[4]The reader is urged to translate the definition to plain English

Equation eq. 16 has infinite number of solutions, and in radians they can be written as $x = \pi/6 = 2\,k\pi$ *where* $k \in \mathbb{Z}$, and $x = 5\pi/6 = 2\,m\pi$ *where* $m \in \mathbb{Z}$

In plainer English what we are saying is that the **set** $\mathcal{S}$ of all solutions to eq. 16 is given by

$$\mathcal{S} = \{..., -23\pi/6\,, -11\pi/6\,, \pi/6\,, 13\pi/6\,, 25\pi/6\,, ...\} \cup \{..., -7\pi/6\,, 5\pi/6\,, 17\pi/6\,, ...\}\,. \tag{17}$$

There is an interesting modification to the original problem which is a **completely different** problem:

Find the solutions to

$$sin(x) = \frac{1}{2}\,, \tag{18}$$

in the interval $[0, 2\pi)$. It happens that the apparently harmless modification of asking the solution to be in an specified interval, changes everything. Indeed, the solution of the new problem (the equation together with the domain where its solution is required) is a set with two elements only,

$$\mathcal{S} = \{\pi/6\,, 5\pi/6\}\,. \tag{19}$$

In fact, if we restrict the domain in which to look for the solution to be the set $[0, \pi/2]$, the solution becomes **unique** and equal to $x = \pi/6$.

At this point it is worth to comment that, **in mathematics the questions of existence and uniqueness of solutions to problems are of upmost importance**.

Let us try the following

Exercise 5 *Determine if* $x = 3\pi/8$ *is a solution of the equation*

$$tan\,2x = -1 \tag{20}$$

Exercise 6 *Find* **all the solutions** *to the equation*

$$4\,sin\theta + 1 = 2\,sin\theta \tag{21}$$

We begin by rewriting the equation as

$$sin\theta = -\frac{1}{2} \tag{22}$$

There are two angles in the interval $[0, 2\pi)$ *satisfying the equation, namely,* $\theta = 7\pi/6$ $(210°)$ *and* $\theta = 11\pi/6$ $(330°)$ *all the solutions to the original equation are found by adding to this angle an integer number of times* 2π, *i.e. the solutions are* $\theta = 7\pi/6 + 2\kappa\pi, 11\pi/6 + 2\ell\pi$, $\kappa, \ell \in Z$

Exercise 7 *Solve*

$$6\,cos^2 x - 3 = 0 \tag{23}$$

in the interval $[0, 2\pi)$

The equation can be cast as

$$cos\,x = \pm\frac{\sqrt{2}}{2} \tag{24}$$

By mere inspection, there are clearly four solutions in the interval, 2 for each sign. indeed, $x_1 = \pi/4$ *(first quadrant), and,* $x_2 = 3\pi/2 + \pi/4$ *(fourth quadrant) satisfy* $cos\,x_1 = cos\,x_2 = \sqrt{2}/2$*, while* $x_3 = 3\pi/4$ *(second quadrant) and* $x_4 = 5\pi/4$ *(third quadrant), satisfy* $cos\,x_2 = cos\,x_3 = -\sqrt{2}/2$

Exercise 8 *Solve*

$$4\,sin^2\,x = 9\,sin\,x - 5. \tag{25}$$

in the interval $[0, 2\pi)$

We begin by rewriting the equation as

$$4\,sin^2\,x - 9\,sin\,x + 5 = 0\,, \tag{26}$$

the fact that $sin\,x$ appears with two powers, 1 and 2 suggests[5] that we introduce a change of variables by naming $u = sin\,x$, in this way, the equation now looks as the ordinary quadratic

[5] This is a very old trick in anyone's toolbox

equation

$$4\,u^2 - 9\,u + 5 = 0\,, \tag{27}$$

with solutions

$$u = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases}\,, \tag{28}$$

which really stands for

$$sin\,x = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases}\,, \tag{29}$$

now, trigonometry has taught us that $-1 \leq sin\,x \leq +1$, this condition rules out $sin\,x = 8/10$ leaving us with $sin\,x = +1$, which, along with the condition $x \in \{0, 2\pi)$ means that the solution is unique, and given by: $x = \pi/2$

Exercise 9 *Solve*

$$2\,sin^2\,x - cos\,x - 1 = 0. \tag{30}$$

in the interval $[0, 2\pi)$

To approach this exercise we use the fundamental trigonometric identity to reach a first transformation of the left hand side

$$2\,sin^2\,x - cos\,x - 1 = 2\,[1 - cos^2\,x] - cos\,x - 1 \tag{31}$$

after which we end up with the equation

$$\begin{gathered} 2\,[1 - cos^2\,x] - cos\,x - 1 = \\ -2\,cos^2\,x - cos\,x + 1 = 0 \end{gathered} \tag{32}$$

setting $z = cos\,x$*, we transform the equation into the standard second order equation*

$$2\,z^2 + z - 1 = 0 \tag{33}$$

with solutions

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-1)}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} 1/2 \\ \\ -1 \end{cases} . \tag{34}$$

From here

$$cos\, x = \begin{cases} 1/2 \\ \\ -1 \end{cases} . \tag{35}$$

In the interval under consideration there are thre solutions to this equation. Indeed, $x = \pi$ is the one and one angle only with $cos\, x = -1$. On the other hand, there are two angles for which $cos\, x = 1/2$, one in the first quadrant and one in the fourth, namely, $x = \pi/4\ rad$ and $x = 7\pi/4\ rad$.