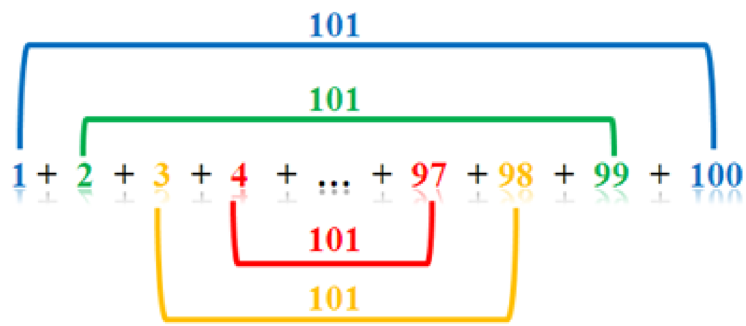

Math is So Much Fun

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Chapter 1

Introduction

1.1 A short funny story

There are many amusing stories about science in general and mathematics in particular. One of my favourites (which is graphically referred to in title page) goes somehow like this. In the late 1700's an 8 years old boy by the name of Carl Friedrich Gauss was in class at primary school. His teacher asked the class to add up all the numbers from 1 to 100, assuming that this task would occupy his students for quite a while keeping them quiet. The teacher was shocked when, after a couple of minutes, young Gauss, gave him the answer: 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, Gauss pointed out that the problem was actually quite simple and proceeded to explain that the he visualized the sum in his mind

$$1 + 2 + 3 + 4 + 5 + 6 + ... + 97 + 98 + 99 + 100$$

and noticed that in the sum there were 50 pairs of numbers such as $1 + 100$, $2 + 99$, $3 + 98$, etc which added 101, and that therefore the result of the sum was $50 \times 101 = 5050$.

I find young Gauss' story terribly funny and tell it to all my friends. Most of them ask me what happened to Gauss and when I tell them that he became one of the greatest mathematicians of all times, they usually say “ahhh no wonder he had so difficult idea” to which I customarily ask my friends whether they really find Gauss solution hard to understand.

In relation to the the young Gauss story that we discussed on this section, there is a 2012 German-Austrian 3D film directed by Detlev Buck with the english title *Measuring the World* (German: Die Vermessung der Welt), based on the eponymous novel by Daniel Kehlmann. The movie is based on a novel by Daniel Kehlmannthat which follows and contrasts the life of two eminent scientists. Gauss, the mathematician, founder of number theory, and the adventurer and explorer of South America (and the rest of the world) Alexander von Humboldt. They both came from Brunswick, Germany, and were contemporaries, and of about the same age, but Gauss rises from poverty whereas Humboldt is privileged. And whereas Gauss didn’t like to leave his office and rarely traveled, Humboldt did practically nothing but to travel.

1.2 A note about these notes

I am a Theoretical Physicist, a scientists that uses lots of mathematics for his work. I am writing these notes with the intention of showing you, the reader, that math is not only useful but amusing.

In these notes I am using two fundamental tools that I have learned on a my daily work. The first is to avoid writing all details of each line of thought[1] and the second is making references to some other places (books, articles, blogs, videos, etc) where information that may help to enrich our understanding may be found. References must be used with some care, if a reader tries to follow each reference as soon as she or he finds it, the reader will take too long to understand or even worse, to learn anything. The way scientist read a paper or a book is by

[1]writing all details makes every paper too bulky and hard to read

leaving the references aside unless they find something they don't really grasp from the paper or the book and they need some help to get the idea or when when they find a reason to extend their knowledge. This technique of reading or studying may be a good advice for you, don't fall into temptation of jumping to every single reference as soon as you find it.

Time and experience have taught me that we, humans, learn by doing. There is no point in reading if we do not try to do some exercises or problems. Practice is the road to perfection. The following proverb, usually attributed to the chinese philosopher Confucius (Kong Qiu), expresses the idea in the best possible way:

I listen and I forget, I see and I remember, I do and I understand

1.3 Some thoughts about Mathematics

Many, many people regard Mathematics as boring or hard to understand. The internet is full of blogs and letters of people telling stories about how math ruined their lives, I understand those people. Most of them were probably patronized by other people who claimed they did understand in a way that might have hurt their feelings.

Most of the time, difficulties with understanding come from one of two reasons. In first place, having being taught by someone who, besides having good intentions, do not really understand what they are trying to teach, believe me, this has happened to me. The second most common reason is not having the necessary prerequisites to understand something, for example, no one can understand division without knowing addition and multiplication.

For me and for many other people **Math is great!**, full of fun and a powerful tool for approaching many, many problems.

Math is not only about numbers or calculations, **math is a way of thinking**. A branch of math called *LOGIC* deals with the ways in which thinking can go good or wrong. Nevertheless, most of us begin our contact with math with numbers and calculations and these notes are not going to be too different in that respect.

What -I hope- makes these notes somehow different is that I am trying to write them down in a conversational tone, sometimes I will signal a blog, sometimes a video. We will try to examine lots of things, some are similar to what people learn at school, some will not be the kind of things we learn at school. These notes are based on one idea, there' s no limits to math but our imagination.

My goal is twofold, in one hand, I hope the notes may become helpful to learn that Math is Fun and in the other, I hope to be able to show a bit about how mathematicians think.

In these notes I will suppose (assume) that you already know a bit about products, divisions, fractions, etc. and even learned some elementary geometry.

A last word before beginning. In a sense, Math is science and science is marvelous for several things. For me, the greatest wonder of science is that it recognizes its limits. Scientists know that all their ideas must be checked with nature, and nature can -and usually does-proves us plain wrong. Let us think of an example, a scientist has been watching -say foxes- and has found out that all foxes he has seen are white, she then goes and writes a paper reporting that all foxes are white. A couple of months later she goes through some woods and sees a red fox, then she will immediately know she was wrong in first place and will reformulate her theory about foxes.

Each time science finds it has been wrong, it rebuilds itself trying to correct its error until it find the next one. That is how science progresses. This video shows biologist Richard Dawkins talking exactly about this aspect of science.

Having finished this rather long preamble, let us begin our the journey to Mathmagicland

Chapter 2

Basic Properties of Numbers

The first thing that we should learn is that there are different kinds of numbers. *Natural* or *Counting* numbers, *Integers*, *Rational* (fractions) , *Irrationals*, *Reals*, *Complex* (or imaginary) and there are even more exotic numbers such as *Quaternions* and *Octonions*.

Some of these numbers are *contained* in others. The naturals, for example are contained in the integers. This means that any natural number is an integer number. The integers are contained in the rationals and the rationals in the real numbers.

The interesting thing is that the *converse* is not true. That means, for example, that not every rational is an integer or that not every integer is a natural number.

Let us look at an example. Rational numbers have the form[1] a/b, thus for example,

$$\frac{1}{2}, \frac{2}{3}, \frac{5}{4} \tag{2.1}$$

are rational numbers as well as

$$\frac{14}{7}, \frac{24}{3}, \frac{3}{1}, \frac{2}{2} \tag{2.2}$$

[1] The symbol / meaning division, that is $a/b = a \div b$, so $1/2$ is nothing but the number 0.5, which is called the decimal for of the fraction.

but,

$$\boxed{\frac{14}{7} = 2, \quad \frac{24}{3} = 8, \quad \frac{3}{1} = 3, \quad \frac{2}{2} = 1} \tag{2.3}$$

and so, these numbers are integers as much as they are rationals.

There are operations, addition for example, that we can perform on numbers which yield number of the same kind, This means the following

$$\boxed{\text{some kind of number operation same kind of number} = \text{a number of the same kind}} \tag{2.4}$$

It is important to be aware that not all operations can be applied on any kind of numbers. Addition (+) is an operation that accepts any pair of numbers, multiplication also accepts anything, but division cannot act on integers in the sense that we cannot divide two integers and always get an integer.

There are a pair of very special numbers that belong to all kinds, they are zero (0) and one (1). These two numbers are very special because (a) they belong to all classes of numbers and (b) they are the only numbers having the following two proprieties

$$0 + \text{any number} = \text{ the same number} \tag{2.5}$$

$$1 \times \text{any number} = \text{ the same number} \tag{2.6}$$

The integers have a very special property not shared by the counting numbers. If we think of an integer, any integer, there is always another integer called its *opposite* such that these couple adds zero,

$$\text{any integer} + \text{its opposite} = 0\,. \tag{2.7}$$

We call the opposite of 3 *minus three* -3 and say that 3 is a positive integer and -3 a negative integer.

2.1 Questions

1. **Why do natural numbers have no opposite?**. To be honest this question is sort of tricky, because it is easy and at the same time it is extremely hard.

2. Name and write the opposites of 2, 7, 9

3. Name the opposites of -3, -2 and -5. This last question is the reason behind the expression *two negatives equal a positive*. Why?

Addition can also operate on rationals, the way it does very nicely explained in a video from Matemáticas Portátiles. As happens with integers, for each rational there is an opposite.

Having talked about addition and opposites we are ready to state that addition has the following two properties

- The order of the the two numbers in one addition can always be switched without changing the result.

  For example
  $$1 + \frac{3}{4} = \frac{3}{4} + 1$$
  This is called the *commutative* rule of addition.

- When adding several numbers we may group them at our will without changing the result. For example

$$1 + \frac{3}{4} + 5 + \frac{3}{2} + 7 = (1 + \frac{3}{4}) + 5 + (\frac{3}{2} + 7) = 1 + (\frac{3}{4} + 5) + (\frac{3}{2} + 7)$$

This rule also has a name, it is called *associativity*-

2.2 Writing Math

With time Mathematicians developed a fantastic way of talking and writing about mathematics. Many people don't like this way of doing things, they say they don't understand it.

The truth about the "weird" way of expressing math is that it is very economic (it is a way to avoid too much writing), it is easy to understand and, this is even more important, it is great fun.

Mathematicians' way of speaking and writing uses letters instead of numbers. They think that each letter represents any number you may think of as long as you respect a rule, the meaning of each letter cannot be changed at all during a "conversation" or as long as a text is not changing its subject.

Let us see how good is this way of writing or, better said, thinking.

Do you remember the rule *The order of the the two numbers in one addition can always be switched without changing the result.*

Well, in mathematicians parlance this rule is:

$$\boxed{\text{For any pair of numbers,} \qquad a+b=b+a} \tag{2.8}$$

We may do better and write all addition rules.

1.
$$a+b=b+a$$

2.
$$(a+b)+c=a+(b+c)$$

3.

$$0 + a = a$$

Believe it or not, with these rules we have walked a very long road of mathematical discoveries covering more than a thousands years.

2.3 More on operations

We still need some properties of operations and numbers. To begin with, we need to learn how to multiply rational numbers.

The multiplication rule for fractions is very easy. We recall that all rationals are fractions, which means that they can always be written as

$$\frac{numerator}{denominator}$$

and it happens that the rule to multiply fractions is extremely simple, the numerator of the fraction is the product of the numerators and the denominator the product of the denominators, in formulas:

$$\frac{numerator}{denominator} \times \frac{numerator}{denominator} = \frac{\text{product of the numerators}}{\text{product of the denominator}}$$

thus for example

$$\begin{gathered} \frac{3}{4} \times \frac{2}{5} = \frac{3 \times 2}{4 \times 5} = \frac{6}{20} \\ \frac{17}{3} \times \frac{5}{4} = \frac{17 \times 5}{3 \times 4} = \text{finish it} \end{gathered} \tag{2.9}$$

Just to make sure that everything is clear, we repeat the statement that a fraction is just a division, so rational numbers are decimal numbers.

There are still some tricks up the sleeve.

The first of all is that we can always write an integer as a rational, as in

$$2 = \frac{2}{1}$$

the second is this, suppose you have two integers and factor them, say,

$$\begin{aligned} 24 &= 3 \times 8 = 3 \times 2 \times 2 \times 2 \\ 14 &= 7 \times 2 \end{aligned} \tag{2.10}$$

then when we build the fraction 24/14 we may writte

$$\frac{24}{14} = \frac{3 \times 2 \times 2 \times 2}{7 \times 2} = \frac{3 \times 2 \times 2}{7} \times \frac{2}{2} = \frac{3 \times 2 \times 2}{7} \times 1 = \frac{3 \times 2 \times 2}{7} = \frac{12}{7} \tag{2.11}$$

Fun, but ugly looking, Isn't it?.

Well, the above is usually written, as you have probably seen it thousands of times,

$$\frac{24}{14} = \frac{3 \times 2 \times 2 \times \cancel{2}}{7 \times \cancel{2}} = \frac{3 \times 2 \times 2}{7} = \frac{3 \times 2 \times 2}{7} = \frac{12}{7}\,. \tag{2.12}$$

This is what they call *cancelling out common factors* or simplifying a fraction.

Let's do a couple more examples, we will simplify 125/25 and 360/150.

$$\frac{125}{25} = \frac{25 \times 5}{25} = \frac{\cancel{25} \times 5}{\cancel{25}} = \frac{5}{1} = 5 \tag{2.13}$$

$$\frac{360}{150} = \frac{36 \times 10}{5 \times 3 \times 10} = \frac{\cancel{3} \times 3 \times 4 \times \cancel{10}}{5 \times \cancel{3} \times \cancel{10}} = \frac{3 \times 4}{5} = \frac{12}{5} \tag{2.14}$$

and just in case that we want a decimal, $1/5 = 0.2$ so $12/5 = 2.4$

The second trick (or definition if you will) is the *reciprocal*. This is something that the rationals (fractions have) but the integers don't

For any rational number there is a reciprocal, and the reciprocal is the number which multiplied by the original number gives 1 as result.

In mathematician's language, for a number a, the reciprocal which is written as $1/a$ is the only number that males this equality true,

$$a \times \frac{1}{a} = 1\,. \tag{2.15}$$

Some examples may help. The reciprocals of 3, 5 and 1/4 are 1/3, 1/5 and 4. Perhaps a little trickier is to realize that the reciprocal of 3/5 is just 5/3 and for any rational (we will use letters) a/b the reciprocal is b/a.

Why is this? well, let us just try multiplication,

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a}\,, \tag{2.16}$$

but we know that we can switch the order of multiplication without altering the result, and then

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{b \times a}{b \times a} = \frac{b}{b} \times \frac{a}{a} = 1 \times 1 = 1\,, \tag{2.17}$$

the last step that we need to do is to remember that a and b are any numbers, if we think carefully we reach the realization that what we have done is always true, **with only one exception**.

2.4 A basic rule

I am sure that you are now wondering: **What is the exception?**. The answer has to do with the number zero and the fact that

$$0 \times \text{ any number} = 0\,, \tag{2.18}$$

this fact makes impossible the existence of a reciprocal for zero, or said differently,

WE CANNOT DIVIDE BY ZERO!

Let us see the reason for this claim.

In first place, the reciprocal of any number (a) must perform like

$$a \times \frac{1}{a} = 1\,. \tag{2.19}$$

so if there was a reciprocal for zero we should have

$$0 \times \frac{1}{0} = 1\,. \tag{2.20}$$

but we already know that "zero is is the easiest number to multiply for" since

$$0 \times \text{any number} = 0\,, \tag{2.21}$$

we can now see where is the problem. According to formula (2.19) 0/0 should be 1 but formula (2.21) says it should be zero. If both formulas were correct, then one should be equal to zero ($1 = 0$) which we know is ridiculous.

There is an elementary rule of LOGIC (remember, logic is the science of understanding when a reasoning is or is not correct) which states that:

When one thing (A) equals something (B) and another thing (C) equals the same something (B) then the first thing (A) and the other thing (C) must be equal.

We may write this rule as

If A=B and B=C then A=C

This is the rule that makes the formulas $0/0 = 1$ and $0/0 = 0$ impossible to be true at the same time, because if they were, 1 should equal 0. And of course, this impossibility is what makes division by zero to be ill defined or impossible. Mathematicians call impossibilities such as defining 0/0 "indeterminate form".

2.5 Time for some questions

1. What are the reciprocals of 2, 3, 4, 5/3?

2. Write 1/5 in decimal form. Do the same for 1/4 and 2/8, what about 8/2?.

3. What happens if we multiply the decimal number for 2/5 for the decimal number for 5/2?, is there a reason for the result?

4. Calculate

    (a) $\frac{2}{5} + \frac{3}{2}$

    (b) $\frac{2}{5} \times \frac{3}{2}$

    (c) $\frac{1}{3} + \frac{2}{3}$

    (d) $\frac{1}{3} + \frac{4}{6}$

2.5.1 Review Questions

1. Is the formula $a \times (b + c) = a \times b + a \times c$ true?. Explain your answer.

2. Calculate $3 \times (5 + 4)$ in two different ways.

3. Calculate

    (a) $\frac{a}{b} + \frac{c}{d}$

    (b) In the previous question set $a = 2$, $b = 3$, $c = 4$, $d = 6$ and verify that the "formula" you got is true.

4. Is it true that $x \times y + 5x \times z = x \times (y + 5z)$?

5. This is a harder but funnier question. An even number is an integer which after division by two gives an integer (for example 4 and 6 are even because $4/2 = 2$, $6/2 = 3$, but 7 is not even, because, $7/2 = 3.5$. Do you think that **any** even number should be written as $\times$some integer?

Chapter 3

First order equations

First of all, we must learn that an equation is just a question. For example,

What number should we add to two to get 10?

In mathematical parlance[1] that question is written as

$$2 + x = 10\,,$$

here x stands for the number we are asking for. And the expression $2 + x$ means (as we know) add number x to number two.

Now we understand that, since x is the number we are looking for, $2 + x = 10$ just means, what is the number (named x) which added to two yields the result 10.

It is not hard to answer the question in its original form. A little bit of thinking allows us to realize that if we add 8 to 2 we get 10 and so the answer is 8.

People disliking math would ask us a question, why to bore ourselves by writing with math what we could have answered without math. This is a very good and interesting question indeed. And it has to be answered in two phrases. The first, is very simple, thinking a bit

[1] we usually say **notation** instead of parlance or lingo.

and realizing the number must be eight is doing mathematical thinking! so we did math. The second part is that, even if we don' t realize it at this very moment, mathematical notation is great because is a way to save space, is a way to write succinctly.

A question might lead to an equation like[2]

$$2x + 3 + 5x + 45 + 18x + 6 = 120 + x + 22x - 5\,, \tag{3.1}$$

in this case a little thinking might not be enough to give us the answer (x). Then how do we do find x?

The trick is to transform this horrible looking formula into something simpler. To be successful, we shall need to remember that x is a number (the same number everywhere). Let carry on the transformation step by step to be able to see what goes on. We begin with

$$2x + 3 + 5x + 45 + 18x + 6 = 120 + x + 22x - 5\,, \tag{3.2}$$

first we rearrange everything since changing order does not change the results

$$2x + 5x + 18x + 3 + 45 + 6 = 120 - 5 + x + 22x\,, \tag{3.3}$$

now we group

$$(2 + 5 + 18)x + 54 = 115 + x(1 + 22)\,, \tag{3.4}$$

$$25x + 54 = 115 + 23x\,, \tag{3.5}$$

Our next step begins by noting that since the above is an equality, performing the same operation in both sides does not change the equality. This idea allows me to substract 54 in both sides

$$25x + 54 - 54 = 115 + 23x - 54\,, \tag{3.6}$$

[2]try to phrase a question

which yields

$$25x = 61 + 23x\,, \tag{3.7}$$

finally substract $23x$ in both sides

$$25x - 23x = 61 + 23x - 23x\,, \tag{3.8}$$

$$2x = 61\,, \tag{3.9}$$

we are almost there. Divide by two in both sides

$$2x/2 = 61/2\,, \tag{3.10}$$

$$x = 61/2 = 30.5\,, \tag{3.11}$$

we got our number!

We want to repeat the above to solve equations that have *fractional coefficients*, meaning they look like

$$\boxed{\frac{3}{2}x + 121 = \frac{5}{7}x} \tag{3.12}$$

Once again, all we need to do is to to keep in mind that x means "some number".

With that in mind let us see how might we calculate

$$\frac{3}{2}9 + \frac{2}{5}9\,, \tag{3.13}$$

mathematics allow many ways of performing this calculation, one of them is to distribute the products and write

$$\frac{3}{2}9 + \frac{2}{5}9 = \left(\frac{3}{2} + \frac{2}{5}\right)9 = \frac{15+4}{10}9 = \frac{20}{10}9 = 18 \tag{3.14}$$

If we use this calculational technique and remember that x is just some number, our original question can be solved with these steps

$$\begin{aligned}
\frac{3}{2}x + 121 &= \frac{5}{7}x \\
\frac{3}{2}x - \frac{5}{7}x &= -121 \\
\left(\frac{3}{2} - \frac{5}{7}\right) x &= -121 \\
\left(\frac{21-10}{14}\right) x &= -121 \\
\left(\frac{11}{14}\right) x &= -121 \\
x &= -\frac{14}{11}\left(11 \times 11\right) \\
x &= -14 \times 11 = 154
\end{aligned} \tag{3.15}$$

We may even do far better. What about "solving"

$$ax + b = 0 \tag{3.16}$$

this may look weird, but in reality is funny.

In the above equation, each character is just some number. To be able to solve it, we need to know which number is the one we want to find in terms of the other two.

Let's say that we want to find x. We simply forget that a and b look like letters, they are NUMBERS, so, we use math as usual ...

$$\begin{aligned}
ax + b &= 0 \\
ax &= -b \\
x &= -\frac{b}{a}
\end{aligned} \tag{3.17}$$

And now comes the fun part.

If we think carefully we realize that we have just solved any conceivable first order equation because:

a and b can be any two numbers!!

Example 1. *In some homework, Sophie was asked the following:*

Solve the folloing equation for h

$$SA = 2\pi r^2 + 2\pi r h \tag{3.18}$$

The question means that somehow Sophie is supposed to know the numbers SA, π *and* r *and she is expected to find* h *in terms of* SA, r *and* π.

Well, under the assumption, the first thing to do is to subtract $2\pi r^2$ *in both sides of the equation to get*

$$SA - 2\pi r^2 = 2\pi r h$$

and now, multiply both sides by the reciprocal of $2\pi r$ *which is*

$$\frac{1}{2\pi r}$$

in that case we get;

$$\frac{1}{2\pi r} \times \left[SA - 2\pi r^2\right] = 2\pi r h \times \frac{1}{2\pi r}$$

$$\frac{SA - 2\pi r^2}{2\pi r} = \frac{\cancel{2}\cancel{\pi}\cancel{r}h}{\cancel{2}\cancel{\pi}\cancel{r}} = h\,,$$

and so the solution to 3.18 is

$$\boxed{h = \frac{SA - 2\pi r^2}{2\pi r}\,.}$$

Which is indeed dependent on the known quantities π, SA *and* r.

3.1 A Little observation

Let us examine the following:

$$\frac{2x+2}{2x+5} = \frac{2x+2}{2x} + \frac{2x+2}{5},$$

This expression is wrong! Let us look at the right hand side of the formula:

$$\frac{2x+2}{2x} + \frac{2x+2}{5},$$

Using the addition rules for fractions,

$$\begin{aligned}\frac{2x+2}{2x} + \frac{2x+2}{5} &= \frac{5\times(2x+2) + 2x\times(2x+2)}{2x\times 5} = \\ &= \frac{10x+10+4x\times x+4x}{10x}\end{aligned}$$

And the last line is completely different to

$$\frac{2x+2}{2x+5}$$

If you did not get the point of this example, jump to section 14.2.1

3.2 Some Questions

1. Solve for x: $3x+5=20$
2. Find (meaning solve) t: $t-5=3t+25$
3. Find n: $3n-8=n$

4. This one looks hard but it is easy if you are cautious. Find x

$$\frac{3x+5}{2x-2} = 1$$

5. Find the solutions for the following equations (a written answer with every step explained is required)

   (a) $2x + 5 = 7$

   (b) $3x - 2 = 1$

   (c) $2x + 1 = x + 1$

   (d) $3s - 5 = s + 15$

   (e) $\frac{4x-4}{x+1} = 2$

6. Is anything wrong in

$$\frac{2x+2}{x+3} = \frac{2x+2}{x} + \frac{2x+2}{3},$$

   if the answer is yes, explain.

7. Same question for

$$\frac{2x+2}{x+3} = \frac{2x+2}{x} + \frac{2x+2}{3}$$

Chapter 4

Integer Powers

Sometimes, when reading or watching a video we see something like 2^3. What does that mean?. The answer is simple,

$$2^3 = 2 \times 2 \times 2 = 8\,, \tag{4.1}$$

what about 3^4?, once again, it is simple

$$3^4 = \underbrace{3 \times 3 \times 3 \times 3}_{\text{4 times}} = 81 \tag{4.2}$$

3^2 reads as 3 **to the power of** 2 or 3 to the second power, or simply 3-squared. This means that we have just learned that, when we power one number to say 4 that means multiply the number by itself 4-times.

If we let a stand for a number,

$$a^3 = a \times a \times a \tag{4.3}$$

and so, for any positive integer (b),

$$\boxed{a^b = \underbrace{a \times a \times \ \cdots \times a \times a}_{\text{b times}}} \tag{4.4}$$

Figure 4.1: Names of numbers in powers

Thus for example,

$$a^{15} = \underbrace{a \times a \times \cdots \times a \times a}_{\text{15 times}} \tag{4.5}$$

Some times we find a product such as $2^3 \times 2^2$, what happens in cases like this?. To find out the answer, we just multiply

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = \underbrace{2 \times 2 \times \cdots \times 2}_{\text{5 times}} \tag{4.6}$$

¡that is just multiplying 2 by itself 5 times!, which we have just learned, equals 2^5 we then conclude

$$2^3 \times 2^2 = 2^5 \,, \tag{4.7}$$

let's try with $3^2 \times 3^2$,

$$3^2 \times 3^2 = (3 \times 3) \times (3 \times 3) \,, \tag{4.8}$$

which is 3 times itself 4 times,

$$3^2 \times 3^2 = 3 \times 3 \times 3 \times 3 = 3^4 \,, \tag{4.9}$$

what we are saying is this,

$$\boxed{a^b \times a^c = \underbrace{a \times a \times \cdots \times a \times a}_{\text{b+c times}} = a^{b+c}} \tag{4.10}$$

Let us propose some exercises

1. $3^4 \times 3^5$

2. $5^2 \times 5^3$

3. $7^5 \times 7^2$

4. $a^4 \times a^3$

Another thing we may find is an expression like $(3^2)^3$, and once again, we will ask ourselves: what is this? Well,

$$(3^2)^3 = (3^2) \times (3^2) \times (3^2) = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = \underbrace{3 \times 3 \times 3 \times 3 \times 3 \times 3}_{\text{6 times}} = 3^6 \quad (4.11)$$

If we think of some more examples we will conclude that for any positive integers b and c,

$$\boxed{(a^b)^c = a^{b \times c}} \quad (4.12)$$

What happens if we divide powers?,

$$\begin{aligned} \frac{3^5}{3^2} &= \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = \\ &= \frac{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3}} = 3 \times 3 \times 3 = 3^3 \end{aligned} \quad (4.13)$$

let us think a little bit, $3 = 5 - 2$. Hmmmm....it seems that

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3 \quad (4.14)$$

And it happens that that is absolutely true.

Math is wonderful, ideas get in and they fit together as the pieces of a giant puzzle.

At some point in time, mathematicians introduced the following idea, -1 a negative integer as an exponent, they did it by stating

$$\frac{1}{a} = a^{-1} \quad (4.15)$$

To find out where this idea leads us, we think of the following expression,

$$\frac{a^b}{a^c}\,, \tag{4.16}$$

the rules to handle fractions tell us that

$$\frac{a^b}{a^c} = a^b \times \frac{1}{a^c}\,, \tag{4.17}$$

but

$$\frac{1}{a^c} = \left(\frac{1}{a}\right)^c = (a^{-1})^c\,, \tag{4.18}$$

and $(a^{-1})^c = a^{(-1)\times c} = a^{-c}$, therefore

$$\frac{a^b}{a^c} = a^b \times a^{-c} = a^{b-c} \tag{4.19}$$

wher we have used the rule for multyplying powers of the same base.

There's still something that is many times stated as a rule,

$$a^0 = 1\,, \tag{4.20}$$

to see how this comes about, think of this fraction

$$\frac{a}{a} = 1 \tag{4.21}$$

we just learned that

$$\frac{a}{a} = a \times \frac{1}{a} = a \times a^{-1} = a^{1-1} = a^0\,, \tag{4.22}$$

we thus have

$$\frac{a}{a} = 1 \quad \text{and} \quad \frac{a}{a} = a^0\,, \tag{4.23}$$

so

$$a^0 = 1 \tag{4.24}$$

must be true and it is indeed true.

4.1 Exercises

Calculate

1. $2^3 \times 2^4$
2. $5^a \times 5^b$
3. $\frac{3^3}{3^2}$
4. $a^3 \times b^4$
5. $\frac{5^3}{5^4}$
6. $\frac{3^3}{3^4}$
7. $\frac{2^8 \times 3^4}{4^2 \times 9^2}$
8. $(a+b)^2$
9. $(a-b)^2$
10. $(a+b) \times (a-b)$
11. Use what you have sone to calculate 25×15 easily

   This is all we need to know about integer powers, some videos may help to understand.

   Marija Kero

   Franck Ives

   Funny guy

Chapter 5

Roots, also called: Fractional Powers

We all know (or are supposed to know) that, the square root of 4 ($\sqrt{4}$) is 2 because the square root of a number is another number which multiplied by itself gives the first number as result[1]. Thus, for instance, $\sqrt{49}$ is the number which makes this equality true

$$\sqrt{49} \times \sqrt{49} = 49\,,$$

which obviously means that $\sqrt{49} = 7$.

Some examples and exercises will do no harm

1. $\sqrt{9} = 3$, because: $3^2 = 9$
2. $\sqrt{36}$=6, beacuse: $6^2 = 36$
3. $\sqrt{256} = 16$, because: $16^2 = 256$
4. Why is $\sqrt{25} = 5$?
5. Why is $\sqrt{81} = 9$?

[1] Nice video

6. Find the value of $\sqrt{64}$

7. Find the value of $\sqrt{100}$

8. Of what number is 13 the square root?, what about 25?

Most square roots are pretty hard to calculate by hand, but fortunately technology (which is built on the shoulders of science) has brought in hand calculators and computers that can do the job for us.

Anyway, what I want to point out is that some mathematician (or several of them) came up with an amusing idea, think of square roots as powers. That is to say, writing $\sqrt{5} = 5^a$ for some number a, if this were possible, we should write

$$5^a \times 5^a = 5\,,$$

and at the same time, because the rules we have already learned,

$$5^a \times 5^a = 5^{a+a} = 5^{2a}\,,$$

at the same time we know that

$$5^1 = 5\,,$$

and so because of the logical rule we have just talked about, it must happen that

$$5^{2a} = 5^1\,,$$

which in turn means that

$$2a = 1\,.$$

And now we have a question, what number multiplied by two equals 1? And the answer is the reciprocal of 2 which is $1/2$. This answer means that our unknown friends that proposed the amusing idea of writing the square root of five as a power were able to find an answer: a square root is a 1/2 power

$$\sqrt{5} = 5^{1/2}$$

and this is always true, so if x stand for any **positive** number

$$\sqrt{x} = x^{1/2}\,.$$

And now something happens with is the usual thing with math (new stuff appearing everywhere if we think a little bit). This new something comes from the rule $(a^b)^c = a^{b\times c}$, indeed, since this rule is true we may apply it to power 1/2 and see what happens

$$(x^{1/2})^a = x^{\frac{1}{2}\times a} = (x^{1/2})^a = x^{\frac{a}{2}}$$

fractional powers exist!, and indeed, they exist in many ¿, many kinds (as much as fractions). For instance, a cubic root is

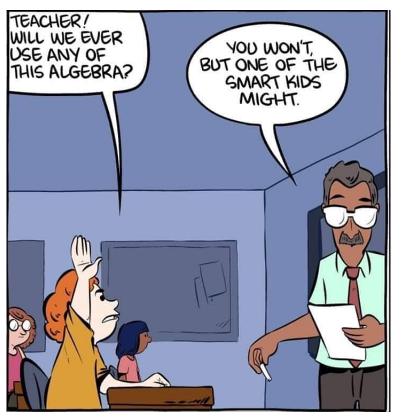
$$\sqrt[3]{17} = 17^{1/3}\,,$$

I find this cool, and you?

With this findings all kind of cool things happen like for example, that we can do algebra with exponents (powers) as long as we take some care with the signs of the numbers we are powering (we will come to that later). Once again, our friend Marija will up review what we have just learned

Chapter 6

"Nasty" algebraic formulas



Sometimes people find the need to calculate something like

$$(3x^2 + 5)(2 + x)\,, \tag{6.1}$$

and then begin complaining and asking why the hell do they need to even think about such awful things. Well, I give them the point that nowadays there are computer programs that can solve such and other much nastier looking formulas in no time, nevertheless I find such calculations fun and part of my everyday life so I tell those people, let's give such things a try.

Going back to the calculation proposed in formula 6.1, all that is needed is a little care, let's do it

$$
\begin{aligned}
&(3x^2+5)(2+x) = \\
&= 3x^2(2+x) + 5(2+x) = \\
&= 3x^2 \times 2 + \mathbf{3x^2} \times \mathbf{x} + 5 \times 2 + 5 \times x = \\
&= 6x^2 + 3x^3 + 10 + 5x
\end{aligned} \tag{6.2}
$$

$$\boxed{(3x^2+5)(2+x) = 3x^3 + 6x^2 + 5x + 10} \tag{6.3}$$

Let's look at some more solved examples.

Example 2.

$$y(2y+5) = y \times 2y + y \times 5 = 2y^2 + 5y$$

Example 3.

$$
\begin{aligned}
(2x+3)(5x+4) &= 2x \times 5x + 2x \times 4 + 3 \times 5x + 3 \times 4 = \\
&= 10\,x^2 + (8+15)x + 12 = \\
&= 10\,x^2 + 23x + 12
\end{aligned}
$$

Example 4.

$$
\begin{aligned}
(2x+3)(2x-3) &= 2x \times 2x + 2x \times (-3) + 3 \times 2x + 3 \times (-3) = \\
&= 4\,x^2 + (3-3)x - 3^2 = \\
&= 4\,x^2 - 9
\end{aligned}
$$

Let's try a harder looking exercise, this time completely symbolic, ¡IT IS JUST MORE OF THE SAME STUFF!

Example 5.

$$
\begin{aligned}
(ax+b)(cx+d) &= ax \times cx + ax \times d + b \times cx + b \times d = \\
&= ac\,x^2 + (ad+bc)x + bd
\end{aligned}
$$

6.1 What are we doing?

This is a good place for some comments. We have gone through some examples, they might have been some fun, but being honest I must concede much more than a point to whoever tells me that these formulas don't look as having much to do with everyday's life but more as something that someone that likes algebra wants to force him or her to do.

To tell the truth much of the interest on these formulas have some historical reasons, many years or better said centuries (even perhaps millennia) ago, there were no smart phones or computers to handle lengthy calculations as those needed for, say banking, or government, and so life for those in charge of such calculations was probably hell. Sooner or latter, someone realized that developing formulas might be useful to make things easier.

Example 6. *Consider the following product,*

$$17 \times 23$$

it is certainly not the longest and awfullest calculation, neither it is the easiest calculation of all, in fact, it may be a little annoying.

Nevertheless, someone may realize (this is very easy) that

$$23 = 20 + 3 \quad \textit{and}$$

$$17 = 20 - 3\,,$$

so the product 23×17 *is just the same as calculating*

$$(20 - 3)(20 + 3)\,,$$

if this same person knows the trick (formula) $(a + b)(a - b) = a^2 - b^2$, *the calculation becomes significantly easier,*

$$(20 - 3)(20 + 3) = 20^2 - 3^2 = 400 - 9 = 391\,.$$

Product/Formula	**Result**	**English Name**	**Spanish Name**
$(a+b)^2$	$a^2+2ab+b^2$	Square of a sum	Cuadrado de una suma
$(a-b)^2$	$a^2-2ab+b^2$	Square of a difference	Cuadrado de una diferencia
$(a+b)(a-b)$	a^2-b^2	Sum times its difference	Suma por su diferencia

Table 6.1: Most usual notable products

And how might this person know that $(a+b)(a-b) = a^2-b^2$, the answer is, because someone else did the algebra with symbols.

Formula

$$\boxed{(a+b)(a-b) = a^2-b^2}$$

is just one example of the so called **notable products**, table 6.1 shows a sample of the most common ones..

6.2 Completing the square

For some reason that will not be clear right now (so we better continue for the dun of it) we might ask if there is any way to write

$$x^2+2x$$

as

$$(x+1)^2 + \text{something}\,,$$

doing this is called **completing the square** and it turns out that it can indeed be done and that it is very useful. Let us see how to do it.

In first place, we must remember that

$$(x+a)^2 = x^2+2ax+a^2$$

by comparison, $x^2 + 2x$ looks like the first two terms with $a = 1$, but we are lacking a term 1^2, if we add zero in a clever way,

$$x^2 + 2x = x^2 + 2x + (1 - 1) = (x^2 + 2x + 1) - 1 = (x + 1)^2 - 1\,,$$

and we have completed the square.

6.3 Formula for the roots of a quadratic equation

A quadratic equation is an equation that looks like this:

$$ax^2 + bx + c = 0\,, \tag{6.4}$$

as with first order equations, this is a question where all we want to know is this:
What value (or values) ox x make the right hand side of the equation true.

For instance, if the equation is

$$2x^2 + 4x + 2 = 0$$

and we try $x = 3$ we get

$$2(3)^2 + 4(3) + 2 = 18 + 12 + 2 = 32 \neq 0$$

this means, 3 is not a solution of the equation, if we try $x = -1$ we get

$$2(-1)^2 + 4(-1) + 2 = 2 + (-4) + 2 = 0\,,$$

meaning that $x = 1$ is indeed a solution.

Mathematicians have found that a quadratic equation can have -at most- 2 different solutions.

Thus, for example, $x = 3$ and $x = -3$ are two solutions of

$$x^2 - 9 = 0\,,$$

and thanks to our dearest friends the mathematicians, that there cannot be any other solution.

Now, if we think a little, we realize that there is a method to solve any first order equation,we might ask whether there is any such method for quadratic equationsand would find a positive answer.

Our dear friends the mathematicians did the hard work and have found that the solution (solutions or roots) of the quadratic equation

$$\boxed{ax^2 + bx + c = 0} \tag{6.5}$$

are given by the following formula,

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}, \tag{6.6}$$

where the quantity inside the square root, namely

$$D = b^2 - 4ac\,,$$

is referred to as the *discriminant of the equation*

Let's try this out. Imagine that we need to solve the equation

$$x^2 - x - 6 = 0\,,$$

to uae formula 6.6 we need to identify the coefficients (numbers) a, b and c in the equation, which in the case of this discussion are $a = 1$, $b = -1$ and $c = -6$.

Therefore, the **discriminat** is

$$D = (-1)^2 - 4(1)(-6) = 1 + 24 = 25$$

which implies

$$x = \frac{-(-1) \pm \sqrt{25}}{2(1)} = \frac{1 \pm 5}{2}$$

this means that we have two roots, namely $x_1 = 6/2 = 3$ and $x_2 = -4/2 = -2$

Example 7. *Let us now find the solution to*

$$x^2 - 2x - 15 = 0$$

this time, $a = 1$*,* $b = -2$ *and* $c = -15$*. The discriminant of the equation is*

$$(-2)^2 - 4(1)(-15) = 4 + 60 = 64$$

so the formula for the solution is

$$x = \frac{-(-2) \pm 8}{2(1)}$$

yielding the solutions $x_1 = 5$ *and* $x_2 = -3$*.*

Let us now think of the "function"

$$y = x^2 - 2x - 15\,,$$

as any function, this is a machine that gives a value (y) to each value of x we use as input. Thus for instance,

$$y(2) = (2)^2 - 2(2) - 15 = 4 - 4 - 15\,,$$

$$y(2) = -15$$

Now,

$$y(-3) = (-3)^2 - 2(-3) - 15 = 9 + 6 - 15 = 0\,.$$

But this already knew. $y(-3)$ must be zero because we used the formula of the solution of a quadratic equation to get that -3 is one of the solutions of $x^2 - 2x - 15 = 0$.

6.4 Practicing what we have learned

Problem 1. *Solve the following quadratic equations*

- $x^2 - 13x = 30$. $x = 15$, $x = -2$
- $4x(2x + 3) = 36$. $x = 3/2$, $x = -3$
- $(x + 3)^2 - (2x - 1)^2 = 0$. $x = -2/3$, $x = 4$
- $x(x + 4) = 32$

Problem 2. *Find the roots of the functions*

- $y(x) = 4x^2 - 4$
- $y(x) = 2x^2 + 4x + 2$
- $y(x) = x^2 + 36$

Problem 3. This problem requires a little thinking. *Consider the functions $y(x) = x^2 - 1$ and $y(x) = x + 1$. Please explain your answer.*

- *How do they look if drawn (just a simple sketch is enough)*
- *Do their graphics intersect?, if the do, where?*

6.5 Factoring Quadratic Functions

Let's begin by a simple example. Look at the equality

$$x^2 - 2x + 1 = (x - 1)(x - 1) \tag{6.7}$$

Is it true?, well, let us check

$$(x-1)(x-1) = x \times x + x \times (-1) - 1 \times x - 1 \times (-1) = x^2 - 2x + 1$$

so the equality is indeed true. Some thinking will show us that $x = 1$ is a double root of the function $F(x) = x^2 - 2x + 1$.

Writing

$$F(x) = (x-1)(x-1)$$

is called **factoring** $F(x)$

Factoring a quadratic function is just re expressing it in terms of its roots.

Another example may help.

We want to factor the polynomial $P(x) = x^2 + 2x - 8$, using what we know about the roots of a quadratic polynomial, formula 6.6, the roots of the polynomial we want to factor out are given by

$$x_{1,2} = \frac{-2 \pm \sqrt{4 - 4(-8)}}{2} = \frac{-2 \pm 6}{2}$$

which means there are two different roots, $x_1 = 2$ and $x_2 = -4$.

The factoring is very simple, we jut write the polynomial as a product of quantities containing its roots as

$$x^2 + 2x - 8 = (x - x_1)(x - x_2) = (x-2)(x+4)$$

Let us check

$$(x-2)(x+4) = x^2 + 4x - 2x - 8 = x^2 + 2x - 8$$

cool, isn't it?

Let us look at another example, factor $Q(x) = x^2 - 4$.

This time the roots of Q are

$$x_{1,2} = \frac{0 \pm \sqrt{0 - 4(-4)}}{2} = \pm\frac{4}{2}$$

which means that

$$x^2 - 4 = (x - 2)(x + 2)$$

Let us summarize what we have done to this point.

If we have a 2nd order polynomial with first coefficient (a) equal to 1, i.e. something like

$$P(x) = x^2 + bx + c$$

$P(x)$ is factored by writing

$$P(x) = (x - x_1)(x - x_2)$$

where x_1 and x_2 are the real roots when they exist.

In the case in which $a \neq 1$ we can always re express the polynomial as

$$P(x) = a[Q(x)]$$

where $Q(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ and use what we already did for polynomials with $a = 1$.

Since examples are always good for understanding, let us factor $P(x) = 2x^2 - 4x + 2$.

We begin by factoring out $a = 2$ to write

$$P(x) = 2\left[x^2 - 2x + 1\right]$$

now we factor $Q(x) = x^2 - 2x + 1$ which is easy and yields

$$Q(x) = (x - 1)(x - 1)$$

and therefore,

$$P(x) = 2(x - 1)^2$$

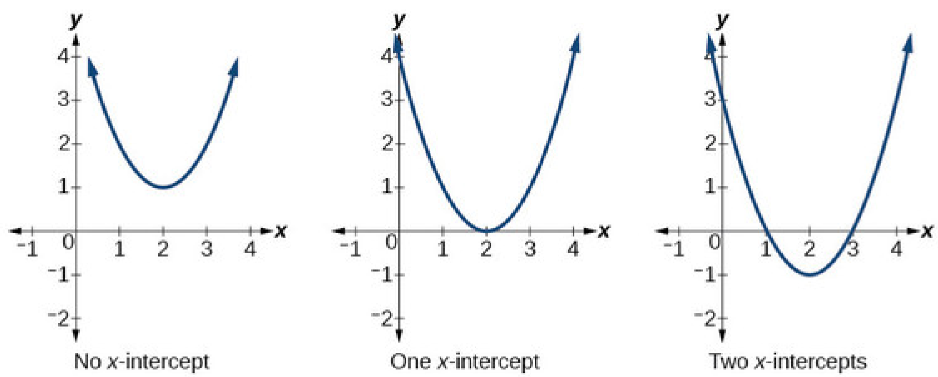


Figure 6.1: Caption

6.6 A little Geometry

Figure 6.1 shows three different curves, these particularly shaped curves are called *parabolas* and they are three of six possible graphics of quadratic functions, i.e. functions of the form

$$y(x) = ax^2 + bx + c\,.$$

If we carefully examine the figure, we learn two things:

- All three curves "point upwards", mathematicians say that they are **concave upwards**
- One of the curves has tow intercepts (it crosses the x axis twice, another has just one intercept (it does not cross the x axis but just touch it, and the third has no intercept, it never crosses or touches the x axis.

Having at intercept at x_0 means $y(x_0) = 0$, i.e. x_0 is a root of $y(x)$ or, said in other words, x_0 is a solution of the quadrartic equation

$$ax^2 + bx + c = 0\,.$$

The problems we have solved have taught us that there are times where a quadratic equation has two solutions, sometimes it has one and only one solution and sometimes we find NO solution. Those cases correspond with the three parabolas shown in figure 6.1, we have found a very interesting connection between algebra and geometry, but more than a connection what we have found is a new tool. We may think about problems in different ways, sometimes geometrical, sometimes algebraic, sometimes there are even more ways to look at problems. **The more ways we learn to look at problems the more chances we have to understand them and solve them.**

At the beginning of this section we talked about six possible parabolas but showed only three. The reason for this lies in the coefficient a in $y = ax^2 + bx + c$, when a is positive, the parabola is concave upwards, when a is negative, it is concave downwards.

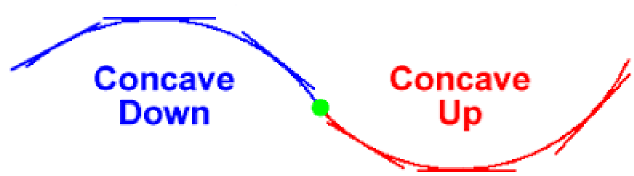


Figure 6.2: Graphical definition of concavity

We are very close to have learned a lot (but not everything) about parabolas. Two pieces of knowledge are still missing.

The first one is the question of the number of intercepts (or roots of the corresponding quadratic equation) and we will turn our attention to it. We do so by recalling the formula for the solution of the quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

this formula has a square root ($\sqrt{b^2 - 4ac}$) and we know that there are no square roots for negative numbers. Therefore, if $b^2 - 4ac$ turns out to be a negative number, there will be no

Sign of a	Concavity	Sign/Value of D	Number of intercepts/roots
+	Upwards	+	2
+	Upwards	0	1
+	Upwards	-	0
-	Downwards	+	2
-	Downwards	0	1
-	Downwards	-	0

Table 6.2: Clasification of all parabolas of the form $y(x) = ax^2 + bx + c$. D is the discriminant, $D = b^2 - 4ac$

solution to the quadratic equation or no intercept of the parabola. There is the possibility of $b^2 - 4ac$ being equal to zero, then there is only one solution to the quadratic equation, or just one intercept of the parabola, and finally, $b^2 - 4ac$ can be positive in which case we will have to roots, or two different intercepts. Since the sign of $b^2 - 4ac$ is so important, we give a special name to this quantity (we already did) and call it the **discriminant**. With his information at hand we may now summarize out knowledge about parabolas and quadratic equations.

6.7 A nice and interesting real story

Having learned the solutions to the first and second degree equations, it is natural to wonder about third, fourth, fifth or higher degree equations, of which table 6.3 shows a sample up to degree 6. The reasonable thing to do is to enquire about the existence of formulas for solving the higher degree equations.

For a long time the search for such formulas was some sort of crusade for the mathematicians, they worked incredibly hard and found formulas for the solution of the third and fourth degree

Degree	Equation
1	$a_1x + a_0$
2	$a_2x^2 + a_1x + a_0 = 0$
3	$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
4	$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
5	$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
6	$a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

Table 6.3: Polynomial equations

equations and no matter their effort they were not able to find anything for any higher degree equations, our friends were very frustrated. Then in the nineteen century a young french mathematician and political activist by the name of Évariste Galois , found what we should call an extraordinary piece of mathematics, Galois which stands as a great intellectual hero, showed that there simply doesn't exist any formula for the solution of the fifth (or higher) degree equation, meaning that he taught all of us that the search of any such formula was like the search of the mythic Chimera, i.e. a completely useless waist of effort.

The discoveries of Galois, who tragically died in a duel on May 31, 1832, at the early age of 20 constitute the foundations of two modern branches of mathematics called *Galois Theory* and *Group Theory*, the later being of fundamental importance for modern physics. Eduardo Sáenz a Mathematician and Professor of computer science gives us a very interesting and accessible account of Galois' work in spanish. For an english review I recommend the following video.

6.7.1 Where does the formula for the roots of a quadratic polynomial come from?

The formula

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}\,, \tag{6.8}$$

is quite intriguing not to say almost magical, how does it come to be?

Well, as we said at the beginning of these notes, math is fun and the formula comes from working with math. In this case, from an extremely clever exercise of completing a square.

Indeed, let us write the most general form of the second degree equation with real coefficients:

$$ax^2 + bx + c = 0\,, \quad a, b, c \in \Re \tag{6.9}$$

we begin by rewriting eq. 6.9 as

$$(\sqrt{a}\,x)^2 + bx + c = 0 \tag{6.10}$$

now, just at the side of b we insert the number one written as

$$1 = \frac{2\sqrt{a}}{2\sqrt{a}}\,,$$

so now the equation looks like

$$(\sqrt{a}\,x)^2 + b\,\frac{2\sqrt{a}}{2\sqrt{a}}x + c = 0$$

making use ofcommutativity we move some factors a little bit,

$$(\sqrt{a}\,x)^2 + 2\sqrt{a}\,x\,\frac{b}{2\sqrt{a}} + c = 0\,,$$

now we do some grouping

$$\left[(\sqrt{a}\,x)^2 + 2\sqrt{a}\,x\,\frac{b}{2\sqrt{a}}\right] + c = 0\,,$$

and here we see that the bracketed part of the equation is where we may complete a square. To do it we need to ad zero in a clever way, as

$$\left[(\sqrt{a}\,x)^2 + 2\sqrt{a}\,x\,\frac{b}{2\sqrt{a}}\right] + c + \left(\frac{b^2}{4a} - \frac{b^2}{4a}\right) = 0$$

and do a little bit of more reordering and regrouping

$$\left[(\sqrt{a}\,x)^2 + 2\sqrt{a}\,x\,\frac{b}{2\sqrt{a}} + \frac{b^2}{4a}\right] + c - \frac{b^2}{4a} = 0$$

and here we see that the bracketed piece is indeed a perfect square, so now the equation looks like

$$\left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2 + c - \frac{b^2}{4a} = 0\,.$$

Our next step is taking the independent term to the right hand side of the equation,

$$\left(\sqrt{a}\,x + \frac{b}{2\sqrt{a}}\right)^2 = \frac{b^2}{4a} - c\,,$$

or,

$$\left(\frac{2a\,x + b}{2\sqrt{a}}\right)^2 = \frac{b^2 - 4ac}{4a}\,,$$

we now take the square root in both sides of the equation and take care of the possibility of hafing two signs,

$$\left(\frac{2a\,x + b}{2\sqrt{a}}\right) = \pm\sqrt{\frac{b^2 - 4ac}{4a}}\,,$$

the denominator of the right hand side can be factored out and its square root taken to yield

$$\frac{2a\,x + b}{2\sqrt{a}} = \pm\frac{\sqrt{b^2 - 4ac}}{2\sqrt{a}}\,,$$

and now we can eliminate the denominators

$$2a\,x + b = \pm\sqrt{b^2 - 4ac}\,,$$

and from here we immediately get

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}} \tag{6.11}$$

Chapter 7

Equation of The Line

7.1 The Plane withCartesian Coordinates

7.2 First Visit

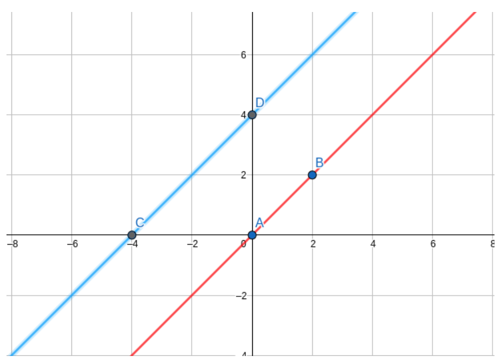


Figure 7.1: Points are given coordinates, A(0,0), B(2,2), C(-4,0) and D(0,4). The red and blue lines are parallel

Figures 7.1 and 7.2 were drawn using GeoGebra, the blue an red lines in 7.1 were set using

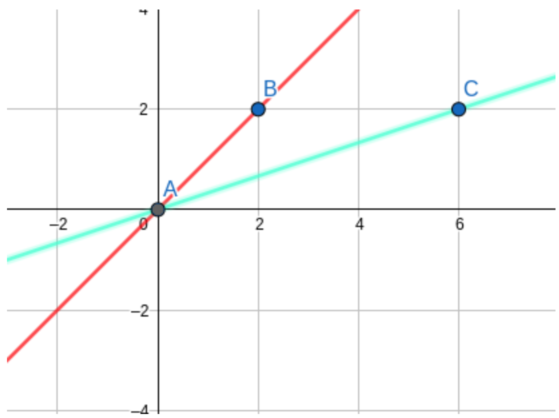


Figure 7.2: Points are given coordinates, A(0,0), B(2,2), C(6,2). The red line is steeper than the blue line

the formulas

$$\begin{aligned} y &= x \\ y &= x+4 \end{aligned} \tag{7.1}$$

For the lines on figure 7.2 the formulas are

$$\begin{aligned} y &= x \\ y &= \frac{1}{3}x \end{aligned} \tag{7.2}$$

Wit a little bit of attention and if we use what we have learned about mathematical language, we will notice that all four formulas have the form

$$\boxed{y = mx + n} \tag{7.3}$$

in formula $y = \frac{1}{3}x$, $m = \frac{1}{3}$, while in formula $y = x + 4$, $m = 1$ and $n = 4$. These two numbers, m and n have special meanings.

- m is called **slope** (pendiente in spnish), and it tells us how steep the line is. If m is small, say $m = 0.2$ the line will be not very steep, while if $m = 5$ we will see aquite inclined line.

- n is called the **intercept** it the value of y when $x = 0$.

Let try to perfeclty understand the meaning of formula 7.3. If we think of a value for $x$and stick that value in the formula, we get a value for y, those two values end up being the coordinates of a point in the line. Let us see an example. The red line in figure7.1 has the for,ula $y = x$, if we put $x = 4$, we get $y = 4$ and it happens that the point $(4, 4)$ is indeed in the line, te same happens with $(-2, -2)$. THe blue line of the same figure has the formula $y = x + 4$, setting $x = 2$ gives $y = 6$ and $((2, 6)$ is a point belonging to the line. As we said before, for this to parallel lines, $m = 1$ which is quite steeep (in fact, you may find that the angle the lines makes with the x axis is exactly 45^o.

The cyan line in figure 7.2 is given by the formula $y = \frac{1}{3}x$, this time the slope has the value $m = 0.333...$ and that is why the line is not too inclined, and the pont $(6, 2)$ belongs to the line because $y = 2 = \frac{1}{3}x$ with $x = 6$.

Ok, so we have learned how to find points belonging to a line if we know its equation. But, what else can we do?

Many, many years ago[1], geometry as we know it to day, was invented. In fact, a man by the name of **Euclid** also known as Euclid of Alexandria, a Greek mathematician, that lived in the ancient and famous city of Alexandria wrote one of the most important and influential books of all times, it is called **Elements** and in it, Euclid wrote a great deal of geometry, so much in fact, that the Elements is still a fundamental work on the subject. But, what does Euclid have to do with our current discussion?

It happens that Euclide's first postulate states something that you probably already know: *given two points there is one and only one line that passes through both of them.* this result raises the following question,

Given the coordinates of two points, is it possible to find the equation of the unique line

[1]Nearly 2350 years ago, to be precise

that passes through such points?

The short answer is: **yes**. But we will need to go do some amusing work before giving the complete answer. Letus consider the two triangles in figure 7.3, Thales theorem states that for

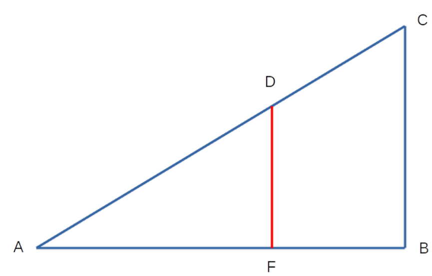


Figure 7.3: Triangles ABC and AFD are **similar** this means that all corresponding anlges are equal

such similar triangles,

$$\frac{BC}{BA} = \frac{FD}{FA}\,. \tag{7.4}$$

and that happens with whatever couple of big and small triangles we might think of.

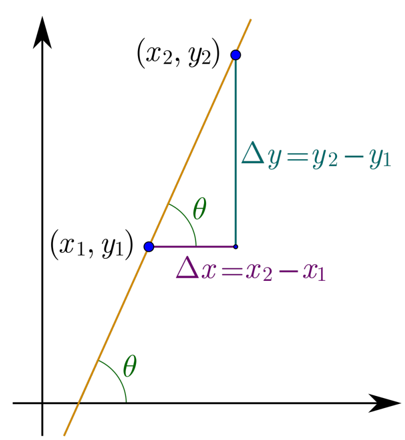


Figure 7.4: Any two pair of points along the yellow line make a triangle which is similar to another built from any other pair of points belonging to the line

Since in figure 7.4 all pair of points along the lines build similar triangles, Thales' s theorem assures us that the ratio

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \tag{7.5}$$

will always have the same value, such ratio happens to be the slope of the line. Let us write this again with a little bit of care,

Given two points $P1(x_1, y_1)$ and $P2(x_2, y_2)$ the slope of the line to which they belong is given by

$$\boxed{m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}} \tag{7.6}$$

and here we need to give a warning, the order of the coordinates in the denominator and numerator of the formula is vital.

Now imagine a generic point Q along the same line to whchich $P1$ and $P2$ belong, and let (x, y) be the coordinates of Q, then

$$\frac{y - y_1}{x - x_1} = m \tag{7.7}$$

or

$$y - y_1 = m(x - x_1) \tag{7.8}$$

i.e.

$$\boxed{y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)} \tag{7.9}$$

and this is another form for the equation of the line.

7.3 Three Ways of the Line

There are three different problems associated with finding the equation of a line.

- We are given the slope (m) and and intercepet (n) of the line.

- We are given the slope and the coordinates (x_0, y_0) of a point $P0$ belonging to the line, and finally

- We are given the coordinates (x_1, y_1) (x_2, y_2) of two points belonging to the line.

The ways to build the equations for the line in each case are (respectivley)

$$
\begin{aligned}
y &= mx + n \\
y - y_0 &= m(x - x_0) \\
y - y_1 &= \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)
\end{aligned}
\tag{7.10}
$$

Chapter 8

Systems of Linear Equations

8.1 What are these?

Sometimes we find a problem such as the following, find x and y that solve the following two equations simultaneously

$$\begin{aligned} x + y &= 1 \\ x - y &= 1 \end{aligned} \tag{8.1}$$

in this case, the answer is $x = 0$ and $y = 1$.

Same goes to

$$\begin{aligned} 2x + y &= 1 \\ 3x - y &= 2 \end{aligned} \tag{8.2}$$

with solution, for the moment $x = 3/5$, $y = -1/5$. Things might be even be more interesting such as in this problem,

$$\begin{aligned} x + y + z &= 0 \\ x + y - z &= 0 \\ x - y + z &= 1 \end{aligned} \tag{8.3}$$

with solution, $x = 1/2$, $y = -1/2$ and $z = 0$

Mathematicians gave given a name to these kind of things, they call them **systems of linear equations**.

8.2 Approaching the Solution

In this sectio we will give a first look to the question: **how do we solve systems of linear equations?**

For the time being, we will concentrate in the simplest case of just two equations and two variables, i.e. things that look like

$$\begin{aligned} 2x + 2y &= 0 \\ x - y &= 1 \,. \end{aligned} \tag{8.4}$$

There are several techniques to solve these type of problems.

One very simpe method consists in trying to multiply one or both equations by some, cleverly chosen numbers, and then adding the result in order to get rid of one of the variables, which leaves us wit a linear equation for the other variable, we ten solve the equation, get the value of the variable we left alone, and use such value to substitute it in any of the two original equations, to get a new simple linear equation for the remaining variable.

Let us try these method with system 8.4. A little bit of observation (and practice) lead us to note that, if we multiply the second equation by 2 we will be ready to add both equations and cancel y, indeed, multiplication of the second equation by 2 leaves us with

$$\begin{aligned} 2x + 2y &= 0 \\ 2x - 2y &= 2\,, \end{aligned} \tag{8.5}$$

and if we add both equations we get,

$$2x + 2y + (2x - 2y) = 2\,, \tag{8.6}$$

which yields

$$4x = 2\ , \tag{8.7}$$

with the result, $x = 1/2$. We may now substitute this value in, say, the first original equation to get

$$2(1/2) + 2y = 0 \tag{8.8}$$

or

$$1 + 2y = 0 \tag{8.9}$$

with solution, $y = -1/2$. And as advertised, we found the solution to the system which is $x = 1/2$, $y = -1/2$.

8.2.1 Is there always a solution?

Consider the system

$$\begin{aligned} 2x + y &= 1 \\ x + \frac{1}{2}y &= 3 \end{aligned} \tag{8.10}$$

if we multiply the second equation by -2 and add we get,

$$\begin{aligned} 2x + y &= 1 \\ -2x - y &= -6 \end{aligned} \tag{8.11}$$

and

$$2x + y - 2x - y = 1 - 6 \tag{8.12}$$

which ends up as

$$0 = -5 \tag{8.13}$$

WHAT?, this is ridiculous, 0 and -5 are different, this is an impossible result. What did we do wrong?.

The answer is: WE DID NOTHING WRONG, no matter what we try, in this particular system, we will always get an impossible answer.

There is clearly a question:

Why did this happen?

We will go back to answer it in section 8.3 where we will uncover a beautiful relation between systems of linear equations and geometry.

For the time being, let us look at this other system,

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned} \tag{8.14}$$

To attempt a solution, we multiply the first equation by -2 and add,

$$-2x - 2y + 2x + 2y = -2 + 2 \tag{8.15}$$

this time it follows that

$$0 = 0\,, \tag{8.16}$$

which is always true, an odd seeming result, indeed. Once again, there is nothing wrong, we just arrived to an absolute true, is there any meaning to it?, once again, we must be patient and wait for section 8.3

8.3 Systems of Linear Equations and Geometry

Let us look at the following system

$$\begin{aligned} 2x + y &= 2 \\ 16x - 2y &= 6\,, \end{aligned} \tag{8.17}$$

we may write it as

$$\begin{aligned} y &= -2x + 2 \\ -2y &= -16x + 6\,, \end{aligned} \tag{8.18}$$

or

$$\boxed{\begin{aligned} y &= -2x + 2 \\ y &= 8x - 3\,, \end{aligned}} \tag{8.19}$$

Let us now take a deep breath and look carefully to what we have in front of our eyes. It is a system of linear equations, yes. But we may also think that they are the equations of two straight lines, one with slope $m = -2$ and the other one with slope $m = 8$. Ohhh, they are two non parallel lines and therefore they must intersect in one and only one point.

Had we worked to solve system 8.17 we would have found the solution $x = 1/2$, $y = 1$ and if we use GeoGebra, we will be able to see that the lines intersect exactlay at point $(1/2, 1)$.

In the cse of system 8.10

$$\begin{aligned} 2x + y &= 1 \\ x + \frac{1}{2}y &= 3 \end{aligned} \tag{8.20}$$

or

$$
\begin{aligned}
y &= -2x + 1 \\
y &= -2x + 6
\end{aligned}
\tag{8.21}
$$

geometry tells us that we are dealing with two parallel lines (both have slope $m = -2$) and differnt intercepts, those lines will never meet each other, their meeting is impossible as the equality $0 = -5$.

Finally, in system 8.14

$$
\begin{aligned}
x + y &= 1 \\
2x + 2y &= 2
\end{aligned}
\tag{8.22}
$$

or

$$
\begin{aligned}
y &= -x + 1 \\
2y &= -2x + 2
\end{aligned}
\tag{8.23}
$$

which simplifies to

$$
\begin{aligned}
y &= -x + 1 \\
y &= -x + 1
\end{aligned}
\tag{8.24}
$$

what happened is that we were talking about the same line, and therefore, all its points belong to it, the system is always true.

We have found a beautiful and profound connection between algebra and geometry. Systems of 2 linear equations with two variables are equivalent to the problem of asking the point of intersection of two lines.

Geometrically we know what happens

- The lines intersect in only and only one point

- The lines are parallel and never intersect, or

- The lines are one and the same and therefore, all its points belong to the line.

Form the point of view of systems of linear equations, these three possibilities translate into

- The system has one and only one solution (mathematicians say, there is a unique solution)

- There system has no solution, and

- There system has infinite solutions.

8.3.1 Systems of three equations and three variables

We have already seen systems of these kind, they look like

$$\begin{aligned} x+y+z&=1 \\ 2x+2y-3z&=2 \\ 5x+7y-3z&=5 \end{aligned} \tag{8.25}$$

It happens that geometrically, an equation like

$$x+y+z=1\,, \tag{8.26}$$

represents a plane sitting in the three dimensional space. Planes can also be parallel and never touch themselves, or they can intersect, in fact, two intersecting planes intersect at a line.

When we have three plane equations it may happen that two of them intersect along a line and other two at another line and these lines can meet in which case, if they meet at one point, the 3 by 3 system will have a unique solution. If the two lines are the same, there will be infinite solutions. If two of the planes are parallel and never touch we will have no solutions.

8.4 Really big systems

Think of a 4 by 4 or 5 by 5 systems

$$\begin{aligned} x + y + z + t &= 1 \\ 2x + 2y - 3z - t &= 2 \\ 5x + 7y - 3z + 2t &= 5 \\ x - y + 3z - 2t &= 5 \end{aligned} \tag{8.27}$$

and

$$\begin{aligned} x + y + z + t - 3w &= 1 \\ 2x + 7y - 3z - t + 5w &= 2 \\ 3x + 7y - 3z + t - w &= 5 \\ x - 7y - 6z + t/2 - 8w &= 5 \\ 3x + 5y - z + 2t - w &= 0 \end{aligned} \tag{8.28}$$

they look funny,and they look even more fun if we think of a geometrical interpretation.

An equation such as

$$x + y + z + t = 1 \tag{8.29}$$

can be thought of as a "plane" (they are referred to as hyperplanes) sitting in a four dimensional space, while something like

$$7x + 2y + 3z + t - w = 1 \tag{8.30}$$

is a hyperplane (a four dimensional one) siting in a five dimensional space.

The fun does not stop here, we can imagine spaces of, say 100 dimesnions, in which case, we don have enough different characters to refer to each dimension so we use indices as in

$(x_1, x_2, x_3, ..., x_{98}, x_{99}, x_{100})$,we may think of hyperplanes sitting in such big dimensional spaces and think of systems of 100 linear equations with 100 varibales, **HAHAHAHAHA** (Mad scientsts hysterical laughter).

These ideas have nothing weird in them, just imagine how many variables might influence the price of a pair of shoes (the price of leather, the price of electricity, employees salaries, price of transport, taxes, rent of the store, cost of the machinery to make the shoes, etc, etc, etc). Each one of these variables is a dimension in the space of factors influencing the prize, there's no mystery about it. scientists, engineers, economist and many other professionals keep solving huge systems of linear equations. Google in particular does it for systems consisting on billions of equations and billions of variables. Of course, no one attempts to attack big problems by hand, we use powerful computers which is just another reason to have fun with math.

8.5 Actually Solving Linear Systems

Big systems are allways solved by computers, neverhteless, for small systems there are some tricks that may be helpful.

Example 8. *Let us think of the system*

$$2x - y + 3z = 9 \tag{8.31}$$

$$x - 3y - 2z = 0 \tag{8.32}$$

$$3x + 2y - z = -1 \tag{8.33}$$

There is a first trick to know that this system is going to have a unique solution, but it need some geometry that we have not yet discussed.

We will forget that we know there is a solution and will attempt to solve the system using the 3 methods which are usually taught at high school.

- **First Method:** *We may equation 8.32 to find x as function of y and z which yields*

$$x = 3y + 2z\,, \tag{8.34}$$

we now substitute this value of x in equations 8.31 and 8.33 to get

$$\begin{aligned} 2(3y + 2z) - y + 3z &= 9 \\ 3(3y + 2z) + 2y - z &= -1 \end{aligned} \tag{8.35}$$

after some bit of algebra we end up with a smaller system, in fact, a system of two liner equations and two unknowns

$$5y + 7z = 9 \tag{8.36}$$

$$11y + 5z = -1\,, \tag{8.37}$$

equation 8.37 implies that

$$z = \frac{1}{5}(-11y - 1)\,, \tag{8.38}$$

which in turn implies, after substitution in equation 8.36

$$5y - \frac{1}{5}(11y + 1) = 9\,, \tag{8.39}$$

a standard first order equation of one variable which results in

$$\boxed{y = -1} \tag{8.40}$$

We now substitute this value of y back in equation 8.37

$$11(-1) + 5z = -1\,, \tag{8.41}$$

and trivially get

$$\boxed{z = 2} \tag{8.42}$$

Finally, we go equation 8.31 and substitute the values of y and z we have found

$$2x - (-1) + 3(2) = 9\,, \tag{8.43}$$

an elementary equation giving us the value of x. At the end of the day, we have found that the solution of the original system is

$$x = 1\,, \quad y = -1\,, \quad z = 2 \tag{8.44}$$

- **Second Method:** *We must remember that each time that we operate the same way on two sides of an equation, the resulting equation is completely equivalent as the original. With these in mind we display the system of interest*

$$2x - y + 3z = 9 \tag{8.45}$$

$$x - 3y - 2z = 0 \tag{8.46}$$

$$3x + 2y - z = -1 \tag{8.47}$$

We first note that if we multiply equation 8.46 by $-1/3$ on both sides we get an equation when y appears as $+y$. Similarly, if we multiply equation 8.45 by 2 we will see $-2y$ instead of y. This suggests the following:

  - *Add eqn. 8.46 multiplied by $-1/3$ to eqn. and 8.45*
  - *Add twice eqn 8.45 to eqn. 8.47, i.e.*

$$\begin{aligned} 2x - y + 3z - \frac{1}{3}(x - 3y - 2z) &= 9 \\ 2(2x - y + 3z) + 3x + 2y - z &= -1 + 2(9)\,, \end{aligned} \tag{8.48}$$

after a little bit of algebra we get

$$\begin{aligned} \frac{5}{3}x + \frac{11}{3}z &= 9 \\ 7x + 5z &= 17 \end{aligned} \tag{8.49}$$

As happened with the first method, we have found a 2 by 2 system. At this point we might find one variable, say z in terms of x and repeat what we did before, but instead we will continue with the process of combining the equations in order to get rid of one of the two variables.

Take the first equation of the new system and multiply it by $-7 \times 3/5$ to get

$$\begin{aligned} -7 \times 3/5 \times (\frac{5}{3}x + \frac{11}{3}z) &= (-7 \times 3/5) \times 9 \\ 7x + 5z &= 17 \end{aligned} \tag{8.50}$$

or

$$\begin{aligned} -7x - \frac{77}{5}z &= -189/5 \\ 7x + 5z &= 17 \end{aligned} \tag{8.51}$$

now add both equations and obtain

$$\begin{aligned} -\frac{52}{5}z &= -104/5 \quad \textit{so} \\ z &= 2 \end{aligned} \tag{8.52}$$

From here we go backwards to eq. $7x + 5z = 17$ to get

$$7x + 5(2) = 17 \tag{8.53}$$

implying $x = 1$.

Finally, substitution in equation 8.46 of the original system

$$(1) - 3y - 2(2) = 0 \tag{8.54}$$

gives $y = -1$.

- **Third Method**

In this method we try to find express one variable in terms of the other two for all three equations.

Beginning with the system with which we have been playing

$$\begin{aligned} 2x - y + 3z &= 9 \\ x - 3y - 2z &= 0 \\ 3x + 2y - z &= -1 \end{aligned} \tag{8.55}$$

we find x in each equation

$$\begin{aligned} x &= \frac{9 + y - 3z}{2} \\ x &= 3y + 2z \quad \textit{and} \\ x &= \frac{-1 - 2y + z}{3} \end{aligned} \tag{8.56}$$

The next step on this method is to use these inequalities to build a new system of two equations and two unknowns

$$\begin{aligned} 3y + 2z &= \frac{9 + y - 3z}{2} \quad \textit{and} \\ 3y + 2z &= \frac{-1 - 2y + z}{3} \end{aligned} \tag{8.57}$$

or

$$\begin{aligned} 5y + 7z &= 9 \quad \textit{and} \\ 11y + 5z &= -1 \end{aligned} \tag{8.58}$$

the trick is repeated again to get

$$\begin{aligned} z &= \frac{9 - 5y}{7} \quad \textit{and} \\ z &= -\frac{1 + 11y}{5} \end{aligned} \tag{8.59}$$

which at the end yields the linear equation

$$\frac{9-5y}{7} = -\frac{1+11y}{5} \tag{8.60}$$

i.e.

$$45 - 25y = -7 - 77y \tag{8.61}$$

$$52y = -52\,, \tag{8.62}$$

implying $y = -1$. *I am sure that the reader knows what follows. Exactly: go backwards.*

$$\begin{aligned} z &= \frac{9-5y}{7} \quad \textit{which, for } y = 1 \\ z &= \frac{9-5(-1)}{7} = 2 \end{aligned} \tag{8.63}$$

and finally, we stick $y = -1$ *and* $z = 2$

in

$$x = \frac{9+y-3z}{2} \tag{8.64}$$

to end up with

$$x = \frac{9+(-1)-3(2)}{2} = \frac{9-7}{2} = 1 \tag{8.65}$$

Each one of these three methods is presented in the school where students are usually taught to follow them step by step.

A procedure that is to be rigorously followed step by step is called an *Algorithm*, up to this point in this notes we have learned several algorithms but never stopped to think to much about that. Algorithms are very good for some kind of computer programming called imperative. but that is for some other notes.

The point that we want to make to finish this section is that there is no need to follow these algorithms blindly, we may combine them as long as all our mathematical operations are legal.

Example 9. *Solve the system*

$$x + 2y + 3z = 1 \tag{8.66}$$

$$3x + 2y + z = 7 \tag{8.67}$$

$$2x + y + 2z = 1 \tag{8.68}$$

We begin by adding -three times eq8.66 from eq8.67 and substracting twice eq8.66 from eq8.68 to get the new system

$$x + 2y + 3z = 1 \tag{8.69}$$

$$-4y - 8z = 4 \tag{8.70}$$

$$-3y - 4z = -1 \tag{8.71}$$

Now add $-3/4$ *of eq8.70 to eq8.71 to end up with the system*

$$x + 2y + 3z = 1 \tag{8.72}$$

$$-4y - 8z = 4 \tag{8.73}$$

$$2z = -4 \tag{8.74}$$

this system is -for obvious reason- in what is known as upper triangular form.

As always we must recall that, since all the operations we did on the system were legal, the new upper triangular system is equivalent to the original one in the sense that the solutions to the latter are the solutions of the former.

In its upper triangular form the system is trivially solved, we get $z = -2$ which is now substituted on eq8.74 which yields

$$-4y - 8(-2) = 4$$

so $y = -5$, finally:

$$x + 2(-5) + 3(-2) = 1$$

implies

Chapter 9

Functions

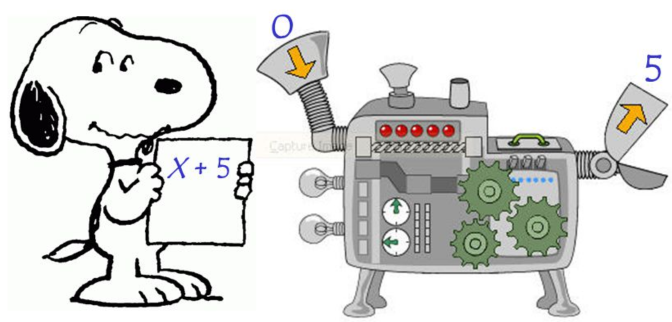


Figure 9.1: A function portrayed as a machine

9.1 First Encounter: Functions Are Machines

A function is nothing more than a machine that eats an object and turns it into something else but always into just one and the same thing.

Functions have three elements, the domain, the function itself and the range.

Domain and range are sets. The domain is the set of objects that are allowed to go into the machine and the range the set of all possible outputs of the machine.

Let us imagine a very simple example, a function whose input are the names of NFL players for the 2024 season and the output is the name of the team to which the player belongs.

In this example, the domain is the set of all NFL players and the range the set of the names of all NFL teams for the 2024 season.

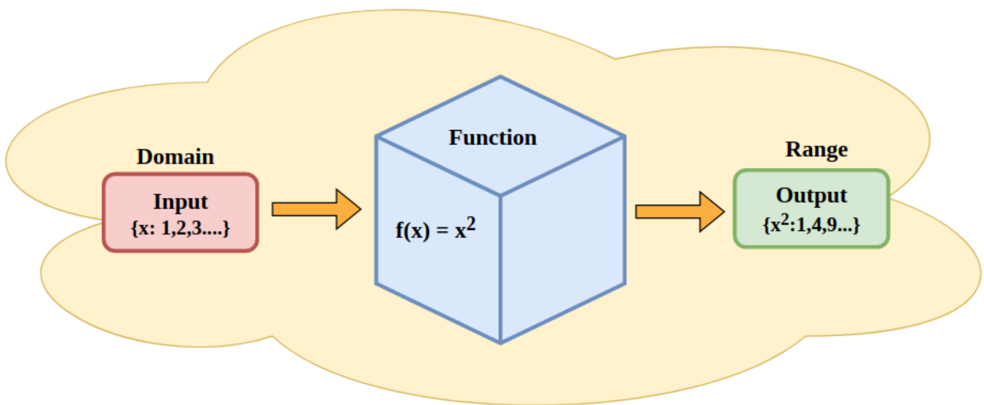


Figure 9.2: The elementary mathematical function $f(x) = x^2$

At this point of our studies (elementary mathematics), a function eats a number and produces a new number, and to talk about functions we write

$$\boxed{f(x) = \text{a formula with } x \text{ inside}}, \tag{9.1}$$

in this notation, x is the number eaten by the machine and $f(x)$ the product, called image. Let us try some examples

Example 10. *As a first example, let us think about figure 9.2 which represents a function that squares positive integers*

Example 11. *Some times functions can be represented as tables,*

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	4	9	16	25	36	49	64	81	100

It is important to realize that when talking about functions we must specify the input, i.e. we must be specific about what the function is supposed to eat, this is called the **domain***.*

Example 12. *Other way to specify functions is through formulas, such as*

$$u(x) = \sqrt{x}\,, \quad x \geq 0 \tag{9.2}$$

here, the weird looking thing "$x \geq 0$" is the domain specification. The formula thus means, that u is a function that takes (eats) any number greater or equal to zero and gives back the value of its square root.

Example 13. *Functions are very often represented by graphics, such is the case of figure 9.3 where we are looking at two very famous functions in the domain $0 \leq x \leq 2\pi$, you might need to recall that pi $= 3.141592654...$ is a special number related with the circle.*

What about solving some problems?

Problem 4. *Consider the domain*

$$x = -6, -5, -4, -3, -3, -1, 0, 1, 2, 3, 4, 5, 6$$

and the formula $f(x) = x^2$. Build a table of values for the function.

Problem 5. *Consider the domain $-5 \leq x \leq 5$ and the formulas*

$$f(x) = x + 1$$

$$g(x) = 2x + 1$$

$$h(x) = -2x + 1$$

Draw (or sketch) the corresponding graphics.

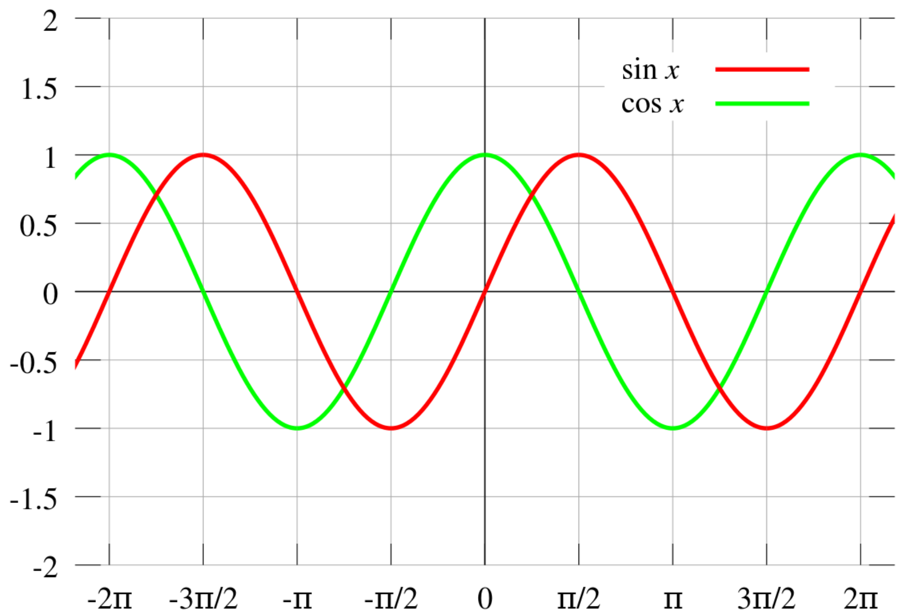


Figure 9.3: Sine and cosine functions

9.2 Improving the concept of Domain

As expressed in example 11, the domain of a function is a way to specify what is the function supposed to eat.

If we give a function by a formula and do not explicitly specify its domain, we must think about the formula and decide what things the formula can eat.

Let us consider

$$f(x) = x^2 \,,$$

if we think a little, we realize that we can square any number, so the *a priori* domain of $f(x) = x^2$ is just all numbers.

If we now think about

$$f(x) = \sqrt{x}\,,$$

we finf a little problem. We cannot get the square root of a negative number. So the domain of $\sqrt{x}$ is all non negative numbers, which is written as $0 \leq x$.

In a similar fashion, if we take $u(x) = 1/x$ we quickly realize that we can almost always divide one (1) by any number, zero being the only exception, then we can say that the domain of $1/x$ is $x \neq 0$.

What about this?

$$y(x) = \frac{1}{x-1}\,.$$

this function is a quotient, and therefore it can only be calculated unless the denominator equals zero, therefore, the domain of y is all numbers but 1, which we write $x \neq 1$.

A much more interesting example is this,

$$f(x) = \sqrt{\frac{x+1}{x-1}}$$

now we have two conditions, in first place, the denominator of the fraction cannot be zero and in second place, the fraction must result in a non negative number, these two conditions may be written as

$$\frac{x+1}{x-1} \geq 0$$
$$x - 1 \neq 0\,,$$

and they must be simultaneously satisfied. This turns out to be a cute exercise in what is called *precalculus* and I will show you a technique to solve it.

The technique consists in building a table that considers signs and special values for both $x+1$ and $x-1$,

x	< -1	-1	$-1 < x < 1$	1	> 1
$x-1$	$-$	$-$	$-$	0	$+$
$x+1$	$-$	0	$+$	$+$	$+$
$\frac{x-1}{x+1}$	$+$	not defined	$-$	0	$+$

Therefore, y is defined for $x < -1$ or $1 \leq x$, something we can say by writing

$$Dom(y) = x \in (-\infty, -1) \cup (1, \infty)$$

9.3 The need of speaking math lingo

If we sit and take a slow and deep breath we will notice that the more we speak math, the more and more difficult it becomes to explain things in plain English, this is not an issue exclusively related to math, it happens with all fields of knowledge, we always need to to introduce a particular language to express our ideas or get lost (like plenty of influencers do). Language is fundamental, think of the word **cat** and what it means or describes if you wish. If we do not say *hey look, a cat* we will be forced to say. *hey look, a four legged haired animal that meows and eats smaller animals like mice*, the same goes with almost anything you can think of, the word *cathedral* stands for a big building dedicated to certain religious activities and even that is not enough since we may have catholic, protestant or orthodox cathedrals.

Having definetively established the need of a new language, let us introduce some *math-lan* [1]. We will do so by introducing some words and symbols and then trying to explain what they mean. And the more we walk the mathematical landscape, the more precise and perfect we will need to be, but that is something that will happen with time.

A list of words: set, empty set ($\emptyset$), union ($\cup$) , intersection ($\cap$).

[1] hehehe, my way of saying mathematical language

Chapter 10

Measuring Stuff

10.1 What is measuring?

One of the most important activities of any scientist or engineer (and even cheffs) is to measure things.

We measure length, weight, electric current, area, volume and many other things.

Measuring means comparing with something. For instance, what happens if I ask: is Mario tall or short? it happens that the answer is not as simple as it looks, you might think he is tall because is taller than Sophie, but perhaps a jirafe would think he is shorter than a baby jirafe and it happens that even though the answers are different, both are right. The goal of measuring is to have a little bit more specific answer to questions like these by establishing standards of comparison.

To develop an specific answer we need to decide what are going to compare Mario's height with. We might decide we are going to compare his height with a stick. We might find that Mario is four times taller than the stick and Sophie 3 and a half times taller than the stick. In that case, we would say that the stature of Sophie and Mario are 3.5 and 4 sticks each.

Name	kind	SI unit	Imperial Unit
Distance	length	meter	Feet
Height	length	meter	Feet
Surface	area	meter^2	Acre
Container capacity	volume	meter^3	gallons

Table 10.1: Some quantities and their units

With time people began to realize that we all must agree in what we use for measuring lengths (the stature of one person is a length) and the measurement standards were born.

Today there is an *intergovernmental organisation* called *The International Bureau of Weights and Measures* or in French *Bureau international des poids et mesures* which is in charge of defining the standards for measuring different quantities.

The international unit of length, for example, is the meter, and for mass the SI (for systeme internationale) is the kilogram.

Besides the SI system which is used in most countries there is the Imperial System in which lengths, for instance are measured in inches, feet, yards and miles.

Sophie's height, for instance is 5 ft 3' (five feet and three inches) which is 1.62 m

10.1.1 Some things we measure

We measure lots of things, some are common day life things, others are not. Some measures such as height or distance are easy to understand, some others are not as easy. Some things are measured by *basic units*, others by *derived units*. Sometimes the basic units are too small or to big to measure something and when that happens the international system of units uses multiples or sub multiples of the units and use prefixes for referring to these *rescaled units* surely you have heard about centimeters or kilometers.

Length

On of the simplest kind of quantities we measure is length. Imagine two pieces of thread. We perfectly understand what someone says that one is longer than the other, the idea of being longer or shorter is what lies in the concept of length. The SI and imperial units of length are meters and feet respectively.

There is an old method to get a grasp on what things have a length close to one meter ($\ell \approx 1\ m$) called the “arm span” or “outstretched arm” technique. Place the finger tips of any

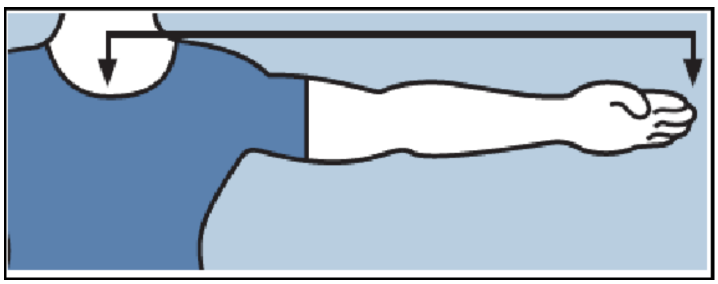


Figure 10.1: arm span Estimation of one meter, the distance between the two arrow tips is very close to 1 m for most adults

of your hands at the center of your chest and extend (laterally) your other arm as far as you can from your body, the distance between the finger tips of your extended arm and those at your chest is quite close to a meter. Please look for a measuring tape and do the experiment, tapes usually have one side that measure in centimeters and meters and one that measures in inches and feet, you will find that your measurement is close to 1 m or three feet (and as you might know, three feet constitute a yard).

Mountains have heights that reach at most $8848\,m$ which is not too much larger than the size of a human being, that is why we use meters or feet to measure them. Now, distances between cities are quite large as compared to our height, measuring them in feet or meters is quite inconvenient, that is why we use miles and kilometers to measure them. For example, the

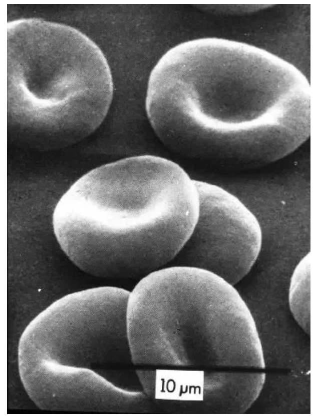


Figure 10.2: If distances are really short, using meters is inconvenient. Our blood red cells -for example- look like small disks with a diameter close to 0.000006 m which is six millionths of a meter, for such small distance we rather say that red blood cell diameter is around 6 micro-meter (6 μm).

driving distance between Miami and New York is 2054 kilometers or 1276 miles. That would be 2054000 meters, a big number which for many purposes is uncomfortable.

The earth moon distance (384400 km or $238,900$ mi) is a much better example of the convenience of using kilometers or miles instead of meters when talking about big distances(as an exercise on convenience, write that distance in meters).

Volume

Another very common quantity we measure is *volume*. The simplest notion of volume I can think of is the amount of water (or juice) that fills a blender's cup. Blender cups have marks whose meaning can be appreciated in figure 10.3 (or in your kitchen's one). The highest the level of liquid, the more volume of liquid you will drink. Most blenders have marks for measuring volume in two different units called cups and milliliters (ml), those are very convenient for

Figure 10.3: The blender cup volume measuring marks

cooking but not for measuring the volume of water that fills a swimming pool.

The SI unit of volume (cubic meter) is rather big for everyday use. For volumes such as the amount of paint we need to decorate our houses or petrol to fill our cars we use liters or gallons. For swimming pools we begin to use cubic meters.

A natural question is, what exactly is a cubic meter. There are two precise but foolish answers for this question, one is: a cubic meter is the SI unit for volume and the other would be, a cubic meter is a volume equaling a thousand liters. None of these answers is satisfactory because none of them give us a feeling about what is a cubic meter. We need a better answer, let us attempt to develop it.

The standard blender cup can contain a volume of nearly one liter, that means that: **¡in order to prepare one cubic meter of juice in one shot we would need to use a thousand blenders at the same time!**, wow, a cubic meter is quite a big volume when compared to a liter. An Olympic swimming pool is 50 m long, 25 m wide and 2 m deep, using the formula for the volume of a box $length \times width \times height$, the resulting volume is

$$V = 50 \times 25 \times 2 \ m^3 = 2500 \ m^3 \tag{10.1}$$

that means

An olympic swimming pool can contains 2.5 million liters of water

A video may be very helpful.

Area

This is other usual quantity. Perhaps the most primitive notion of area has to do with agriculture. Imagine a farmer that wants to plant, say, corn. She needs to know how much space she has for the task. Is the space as big as a football field?, two football fields? or perhaps it is the size of a basketball field. The space she is thinking about is an area.

The SI unit for area is the square meter. A square meter is easy to picture, the only thing you need to do is to use some chalk to draw a square in the floor and make sure that each side of the square is one meter long, once you finish, the space inside the square has an area of exactly 1 m^2.

This idea of measuring the size of a piece of land is the origin of the word **geometry**. Indeed, the etymology of the word dates for the ancient greek $\gamma\epsilon\omega\mu\epsilon\tau\rho\acute{\iota}\alpha$; geo- "earth", -metron "measurement"). Geometry is, with arithmetic, one of the oldest branches of mathematics.

Calculating areas for simple geometric figures like a rectangle is easy, all we have to do is multiply the length of the longest side for the length of the shortest side, in formulas

$$A_{rectangle} = \text{shortest side's length} \times \text{longest side's length} \tag{10.2}$$

10.2 Some more Geometry

Think of a square. We have just learned how to measure its area, since its longest and shortest sides are of the same length, let's call it ℓ the area of the square is

$$A_{square} = \ell^2 \,, \tag{10.3}$$

now imagine that this square is as big as the block where you live. Certainly you can walk around this "block" and ask yourself how much distance did you walk in a round trip. The answer has a name, the *perimeter* of the square, and it is the sum of the lengths of each side you walked by.

10.3 Measuring angles

You, the reader has probably heard someone measuring angles and using the words degrees minutes and seconds of arc length. To define a degree a circle is divided into 360 equal circular segments, the angle defined by two straight segments beginning at the center of the circle and ending at the two extremes of a one of those equal circular segments defines a 1° angle. This angle is in turn divided into sixty (60) equal parts each measuring a minute of arc, finally, a minute of arc ($1'$) is divided into sixty equal tiny angles each measuring one second of arc ($1''$). This way of measuring angles is very convenient for several applications.

One can ask: why dividing the circle into 360 equal segments? and there is a simple answer: it is an arbitrary decision. There is a much more natural way of measuring angles, it is based on the following fact: given an arc of angle $\Delta\theta$ drawn on a circle of radius r, the length of the arch (Δs is proportional to the angle. This means that the length of an arc of twice the angle $\Delta\theta$ is going to be twice the length of the arc of angle θ; if the angle is tripled, the length is tripled, if $\Delta\theta$ is divided by two then so is the arc length[1]. Stated in formulas, the fact we are discussing reads:

$$\Delta s \propto \Delta\theta\,, \tag{10.4}$$

But there is another fact, if we have several arcs of the same angle, and different radii, it happens that their arc lengths double, triples, halves, etc according to the radii, which in formulas would be

$$\Delta s \propto r\,, \tag{10.5}$$

These formulas may be synthesized by an unique formula

$$\Delta s = r\,\Delta\theta, \tag{10.6}$$

[1]This eventually leads to the the well known formula $circ = 2\pi r$

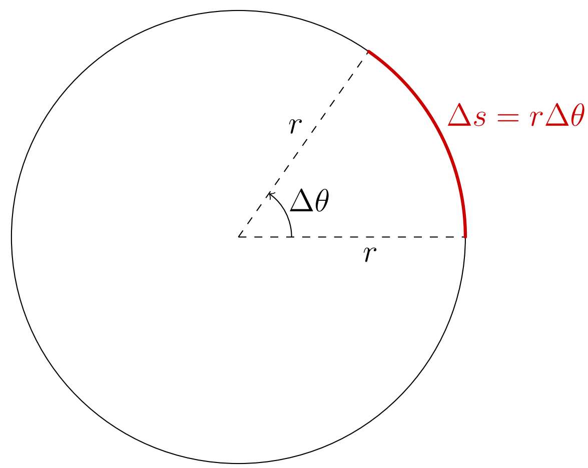


Figure 10.4: Circular arc subtended by an angle $\Delta\theta$

from where

$$\Delta\theta = \frac{\Delta s}{r}\,. \tag{10.7}$$

Here we stop to suggest the reader to perform a vital experiment, take different objects of circular shape and measure the length of their circumference (ℓ) and their diameter (D) and make sure you measure them in the same units. Make a table with the pairs of number so obtained and add a column with the quotient ℓ/D, you will find that the new column gets filled with a nearly constant quantity of value 3.14, such value is an approximation to the famous irrational number

$$\pi = 3.141592654..... \tag{10.8}$$

Angles defined according to formula 10.7 are said to be measured in radians, and they are obviously dimensionless (quotients of lengths). For illustration, we note that an angle of 360° corresponds to 2π rad, while an angle of 90° which subtends a quarter of a circle corresponds to

$$\frac{\text{Lenght of quarter of a circle of radius } r}{r} = \frac{1}{4}\frac{2\pi\, r}{r} = \frac{\pi}{2} \tag{10.9}$$

Arc	Angle in Degrees	Angle in Radians
Full circle	360°	2π
Quarter of a circle	90°	$\pi/2$
Sixth of a circle	60°	$\pi/3$
Eight of a circle	90°	$\pi/4$
Twelfth of a circle	90°	$\pi/6$

Table 10.2: Equivalence between notable angular values

Chapter 11

A Glimpse To Eucliden Geometry

We spent some time talking about geometry in the previous chapter, it is time to be a little more serious about it and we begin doing so by introducing one of the most widely reprinted and translated books in history, second only to the Bible in some accounts.This prominent book is no other than Euclid's Elements

Euclid's Elements ($\Sigma\tau o\iota\xi\epsilon\tilde{\iota}\alpha$) is a large set of math books about geometry and numbers, written by the ancient Greek mathematician known as Euclid (c. 325 BC–265 BC) in Alexandria (Egypt) circa 300 BC. The Elements was a core textbook taught across many cultures over a period of nearly 2000 years after its writing. Over 1000 editions and translations are estimated to have been published since the first printed edition in 1482; in fact, It was one of the very first mathematical works to be printed after the invention of the movable type printing press in the Renaissance era. Translations exist in languages ranging from Arabic, Persian, Hebrew and Turkish to Latin, Italian, English, French, German and Russian among many others.

Until the 20th century, the Elements was considered essential reading and studied by anyone wanting to learn geometry and mathematics formally. Its longevity and widespread usage are due to its remarkably clear organization, rigorous proofs, and its role as a model for deductive reasoning. The resilience of The Elements over millennia attests to its historical importance in mathematics.

Figure 11.1: First printed edition of The Elements

The Elements is the first book in which math is discussed in the terms we understand it today. It presents everything with an approach that begins with some *primitive concepts* and **five** fundamental statements called **AXIOMS** or **POSTULATES**.

The primitive concepts are related to ideas, which are so basic that no one has ever managed to define them beyond intuition, thus for example a **point** is thought of as something really really tiny that we might draw on a sheet of paper with the sharpest pencil ever imagined.

Other primitive notions are, line, plane, and congruence[1]

It is extraordinary that all Euclid's geometry is based on the following five postulates

[1]Let's say that two drawings are congruent if we can put a copy of one of them on to of the other and they are identical in every respect

1. There is one and only one straight line that can be drawn from any point to any point.
2. There is one and only one line that extends a finite straight line continuously in a straight line.
3. There is one and only one circle with a given circle and radius.
4. All right angles are equal to one another.
5. [The parallel postulate]: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The fifth postulate can be stated in a simpler way, given a straight line (A) and a point (p) not on the line, there is one and only one line (B) which is parallel to A and passes through p (i.e. contains p).

Axioms allows anyone to state and prove many, many other statements. Proving a statement means making sure that the statement is correct in the sense that if the postulates are correct or true if you will, the statement is also true. Proofs are achieved using the rules of logic. Mathematicians and lawyers are trained to use the rules of logic.

11.1 Logic: Rules of reasoning

People have a strong tendency to state claims, sometimes the claims are obviously false, as for example in **a red car is blue**. Other times, finding out whether a claim is true or false is not so easy. Logic is the tool that shows when a reasoning is right or wrong, and what a tool it is.

Before going back to geometry, let us do some logic.

- Imagine that you have never, ever seen a fox before. You go to Alaska and see say...40 to 50 foxes which happen to be white. You might be tempted to sate: **all foxes are white**. Would that be a true statement? Te answer is NO, by the rules of logic, this statement cannot be correct. The reason is as follows, if you see something happening many times and think that the next time it is going to be exactly the same, you are doing what is called an **inductive reasoning**, and it happens that experts in logic have found that inductive reasoning doesn't lead to a correct reasoning, in the case of the foxes, you might easily find red foxes which clearly shows that not all foxes are white or said differently your induction about foxes color was plain wrong.

- We all know that mammals have four limbs. Is it correct to state that any animal with four limbs is a mammal?, well, we perfectly know that such statement is completely false, indeed, alligators have four limbs and they are reptiles, not mammals.

  Here we have encountered a particularly dangerous reasoning. The satement **all mammals have four limbs** or stated in other way, **any mammal has four limbs** is usually called a direct statement, while the statement **all animal with four limbs is a mammal** is called its **reciprocal**. Well, it happens that IF a direct statement is true, the reciprocal may in general, be false.

11.1.1 The notion of PROOF

Imagine you hear a statement such as 40 is an even number

Is the statement true?, to answer this question we appeal to the definition of even numbers: Any number that can be exactly divided by 2 is called as an even number, given such definition, we just check whether or not the division of 40 by 2 is exact, $40/2 = 20$ which is indeed exact, and we conclude without doubt that 40 is an even number.

What we have done is to give a PROOF of the statement. A proof is a piece of reasoning that convinces ANYONE that a statement is true without any doubts regardless of whether or not a person hearing or reading the proof does not like the proof.

If a statement is not true a proof of the falseness of the statement is necessary. An example is worth here. Suppose someone claims that 3 is even, in that case you calculate the quotient 3/2 which being 1.5 is not exact, since a number is even when divided by two the result is exact, the conclusion is that 3 is not even.

11.2 Back to Geometry

Experts in logic have found any rules for the correctness of reasoning, and geometry is a great place to learn some of those rules so let us go back to it.

In order to make the size of this notes reasonable we must remember (or perhaps learn) some concepts and some true claims.

1. A polygon is a set of line segments between given points called vertices
2. Two lines that meet at a polygon vertex are called **adyacent**
3. A closed polygon is a polygon with no free ends (a three segment closed polygon is a triangle, a four segment closed polygon is a quadrilateral. A parallelogram is a simple quadrilateral with two pairs of parallel sides, a rectangle is a parallelogram whose sides make right angles. A rectangle with all sides of equal length is a square).
4. A theorem is a statement that must be proved.

For what follows we will need a couple of definitions

Definition 1. *Two polygons are congruent when the segments between their correspondent vertices are equal and the angles between successive segments are equal*

Definition 2. *Two polygons are similar when the angles between all corresponding segments are equal*

Equilateral triangles have all their internal angles equal to 60^0, therefore, all equilateral triangles are similar, neverthelss, not all equilateral triangles are congruent, since we can easily draw an equilateral triangle with 20 cm sides and another with 7 m sides which are evidently non congruent.

The following statement is an example of a theorem

Theorem 1. *The sum of the internal angles of a triangle equals a straight angle*

In these notes, we will sometimes state theorems without giving their proof.

In all problems you must explain the reasoning leading you to your answer. You can only use theorems already stated without proof or proven in these notes

Problem 6. *Is it possible to have a triangle with internal angles of* 30, 30 *and* 10 *degrees?.*

Problem 7. *Give a proof of the follwing statement (Theorem)* **the internal angles of a rectangle add up a full rotation** (360^o)

There are three fundamental theorems about congruence of triangles which we will state without proof

Theorem 2. *Two triangles having corresponding sides of the same length are congruent*

Theorem 3. *Two triangles having one side of the same length and the corresponding angles of same measure are congruent*

Theorem 4. *Two triangles having one angle of equal measure and their adjacent sides of equal length are congruent*

11.2.1 Two Important Theorems

Here we will comment on two of the most important theorems of Euclidean geometry

In first place we address the intercept theorem, also known as Thales's theorem, basic proportionality theorem or side splitter theorem, is an important theorem in elementary geometry about the ratios of various line segments that are created if two rays with a common starting point are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles.

Traditionally attributed to the Greek mathematician Thales of Miletus.(circa 626/623 – circa 548/545 BC) It was known to the ancient Babylonians and Egyptians, although its first known proof appears in Euclid's Elements.

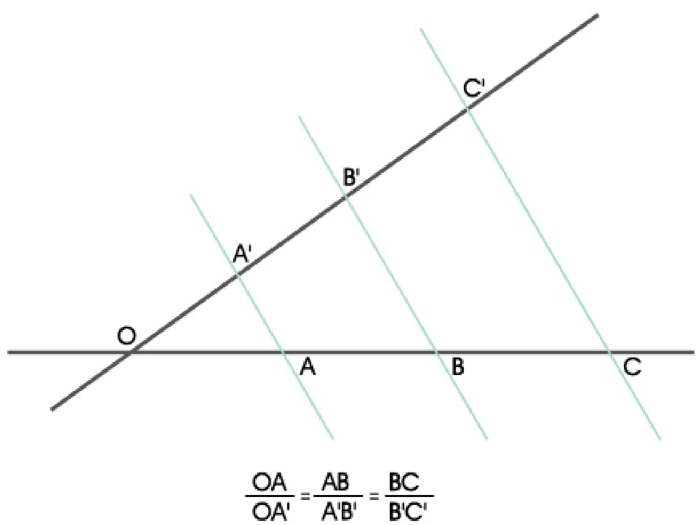


Figure 11.2: Thales Theorem

Theorem 5. *If two lines which intersect each other and are in the same plane are cut by parallel lines, segments determined in one line are proportional to the segments determined in the other line.*

The other and much more famous theorem we want to comment about is the Pythagorean theorem or Pythagoras' theorem, it establishes a fundamental relation in Euclidean geometry

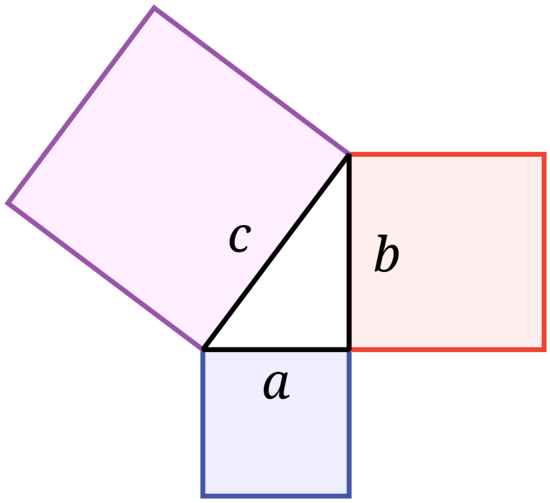


Figure 11.3: Pythagorean Theorem

between the three sides of a right triangle, namely

Theorem 6. *the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares on the other two sides.*

The theorem can be written as the following identity -sometimes called the Pythagorean equation- relating the lengths of the sides a, b and the hypotenuse c,

$$a^2 + b^2 = c^2. \tag{11.1}$$

The theorem is named for the Greek philosopher Pythagoras, born around 570 BC. The theorem has been proved numerous times by many different methods – possibly the most for any mathematical theorem. The proofs are diverse, including both geometric proofs and algebraic proofs, with some dating back thousands of years.

(3, 4, 5)	(5, 12, 13)	(8, 15, 17)	(7, 24, 25)
(20, 21, 29)	(12, 35, 37)	(9, 40, 41)	(28, 45, 53)
(11, 60, 61)	(16, 63, 65)	(33, 56, 65)	(48, 55, 73)
(13, 84, 85)	(36, 77, 85)	(39, 80, 89)	(65, 72, 97)

Table 11.1: There are 16 primitive Pythagorean triples of numbers up to 100

Even though the theorem is evidently of geometrical contents, it can be thought of as a special kind of equation asking for the existence of a triple of integer numbers (a, b, c) such that they satisfy the formula

$$a^2 + b^2 = c^2. \tag{11.2}$$

When thought of as an equation for triplets, an obvious question comes to everyone's minds. Are there triplets of integers that satisfy the identity

$$a^n + b^n = c^n. \tag{11.3}$$

for some integer n?

This is a question of a subfield of mathematics under the name number theory. The cases n = 1 and n = 2 have been known since antiquity to have infinitely many solutions.

Nevertheless, around 1637, the great mathematician Pierre de Fermat stated his second last theorem according to which, there are no solutions for such equations for $n > 2$.

The proof of this seemingly innocent theorem remained a great mathematical mystery for 385 years until 1994 when, the British mathematician, Andrew Wiles succeeded in proving Fermat's theorem.

Chapter 12

Introduction to Trigonometry

Trigonometry is a branch of mathematics that deals with the study of triangles and the relationships between their sides and angles. It is a fundamental concept that has applications in various fields, including physics, engineering, navigation, surveying, and even computer graphics and game development.

At its core, trigonometry involves the use of ratios called trigonometric functions, such as sine, cosine, and tangent, to determine the relationships between the sides and angles of triangles. These functions allow us to calculate unknown sides or angles of a triangle, given certain information about the other parts. Trigonometry also explores the behavior of these functions over a range of angles, leading to the study of periodic functions and their properties. Additionally, trigonometry introduces concepts like the unit circle, which provides a geometric representation of these trigonometric functions and their relationships. Understanding trigonometry opens the door to more advanced mathematical concepts, such as calculus, complex analysis, and vector analysis, making it an essential tool for anyone interested in pursuing higher-level mathematics, physics, or related scientific and technical disciplines.

12.1 Three trigonometric functions: sine, cosine and tangent

Here we will define the three basic trigonometric functions sine, cosine and tangent using the method of the **unitary circle**.

We begin by considering a circle of radius $R = 1$ (that is why it is called unitary) as in figure 12.1, and draw the x and y axes. Let us additionally consider a segment beginning at the center of the circle, this segment makes an angle angle[1] α with the x axis and ends at the point where it meets a vertical segment of the tangent to the circle that touches it at the point $(1, 0)$, and which is therefore perpendicular to the x axis.

We now note that this construction naturally defines two **similar** right triangles. The hypotenuse of the smallest of which is the radius defining the angle α, while its legs are the orange vertical segment going from the intersection of the radius with the circle to the x axis and horizontal blue segment corresponding to the projection of the hypotenuse on the x axis. As shown in the figure, the sine and cosine of the α angle are defined by the **signed** length of those arms.

The tangent, on the other hand, is defined as the signed length of the vertical arm of the big triangle.

The first thing that happens when we use these definitions is that the fundamental formula linking the sine and cosine functions becomes almost obvious, indeed,

Corollary 1. Fundamental Trigonometric Identity *For any angle,*

$$\boxed{sin^2\alpha + cos^2\alpha = 1}$$

[1] In the usual convention, angles are positive in the counterclockwise sense

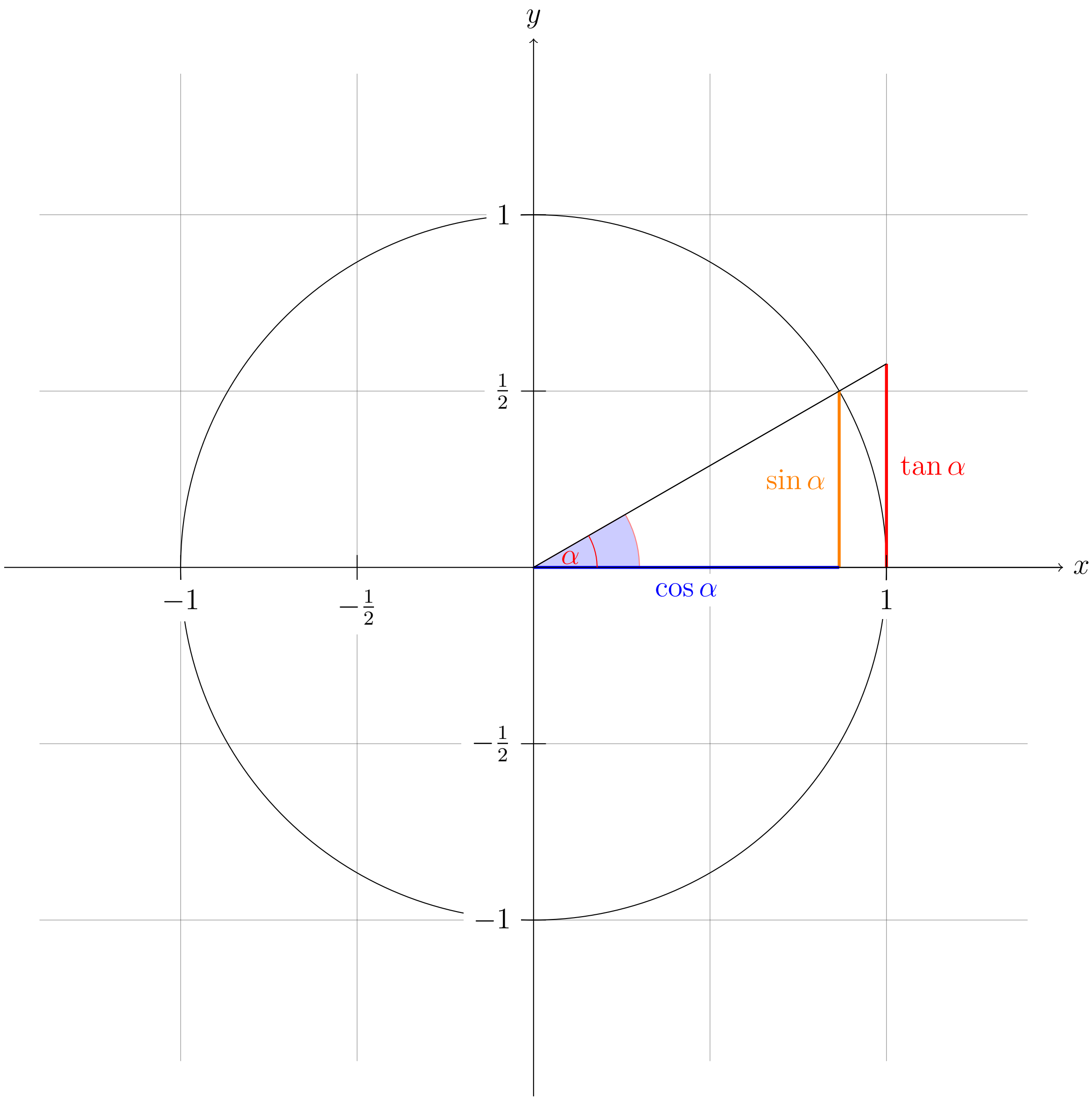


Figure 12.1: The Unitary Circle and Trigonometric Functions. **This sketch is the only thing that must be memorized about trigonometry**

The proof of this corollary is extremely simple, since the sine and cosine are defined as the signed lengths of the legs of a right triangle whose hypotenuse is of length 1, we can immediately use the Pythagorean theorem and get the wanted result.

12.2 A connection to geometry

We now take note of something apparently silly, in the big triangle, the length of the horizontal leg of triangle is 1 so we may write,

$$tan\alpha = \frac{tan\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{adyacent leg to } \alpha} \tag{12.1}$$

Let us now recall Thales theorem from section 11.2.1, according to Thales' theorem the ratios between corresponding sides of similar triangles are equal.

If we apply Thales theorem statement to the two triangles of figure 12.1

$$\frac{\text{opposite leg to } \alpha \text{ in big triangle}}{\text{adyacent leg to } \alpha \text{ in big triangle}} = \frac{\text{opposite leg to } \alpha \text{ in small triangle}}{\text{adyacent leg to } \alpha \text{ in small triangle}} \tag{12.2}$$

which by virtue of the figure translates into

$$\frac{tan\alpha}{1} = \frac{sin\alpha}{cos\alpha}\,. \tag{12.3}$$

we thus end up with

$$\boxed{tan\alpha = \frac{sin\alpha}{cos\alpha}} \tag{12.4}$$

which is the definition of the tangent of an angle.

Thales' s theorem can also be used to put the usual definitions of sine and cosine under a different light. Imagine a right triangle, any right triangle, with no other particular characteristic than being right, a little thought shows that we can always rescale the sides of such triangle to construct another triangle, similar to the one we began with, but having a hypotenuse of unit length. Once we do that, we can put the new triangle in the unitary circle to obtain something very close to what we see in figure 12.1, in this new triangle, and since the length of hypotenuse equals one we can write the formulas

$$\begin{aligned} sin\alpha &= \frac{sin\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}} \\ cos\alpha &= \frac{cos\alpha}{1} = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}, \end{aligned} \tag{12.5}$$

but, by virtue of the similarity of the new triangle with the original one, and Thales' s theorem, we get, that no matter what triangle we are talking about, as long as it is a right triangle,

$$\boxed{sin\alpha = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}}} \tag{12.6}$$

$$\boxed{cos\alpha = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}} \tag{12.7}$$

which are the usual formulas given at the school.

It is imperative to understand and learn that the two sets of definitions we have given are different but that they define exactly the same things, i.e. there is nothing to worry when using one definition or the other. In fact, if we continue our study of math to a higher level, namely full blown calculus, we will find more definitions of the trigonometric functions. One of those is in terms of a field by the name of *differential equations*, but we will not pursue this path in these notes.

12.3 Some "not very Useful" Formulae

As time goes by and science pushes forward, some things that belonged to what we should call **fundamental knowledge** cease to be fundamental.

LOGARITHM

	0	1	2	3	4	5	6	7	8	9	Mean Difference								
											1	2	3	4	5	6	7	8	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1	2	3	3	4	5	6	7	8
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1	2	3	3	4	5	6	7	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1	2	2	3	4	5	6	7	7
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1	2	2	3	4	5	6	6	7
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	1	2	2	3	4	5	6	6	7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1	2	2	3	4	5	5	6	7
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1	2	2	3	4	5	5	6	7
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	1	2	2	3	4	5	5	6	7
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	1	1	2	3	4	4	5	6	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1	1	2	3	4	4	5	6	7
60	7782	7789	7769	7803	7810	7818	7825	7832	7839	7846	1	1	2	3	4	4	5	6	6
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	1	1	2	3	4	4	5	6	6
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	1	1	2	3	3	4	5	6	6
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	1	1	2	3	3	4	5	5	6
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1	1	2	3	3	4	5	5	6
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	1	1	2	3	3	4	5	5	6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	1	1	2	3	3	4	5	5	6
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	1	1	2	3	3	4	5	5	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1	1	2	3	3	4	4	5	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	1	1	2	2	3	4	4	5	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	1	1	2	2	3	4	4	5	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	1	1	2	2	3	4	4	5	5
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	1	1	2	2	3	4	4	5	5
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1	1	2	2	3	4	4	5	5
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	1	1	2	2	3	4	4	5	5
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1	1	2	2	3	3	4	5	5
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1	1	2	2	3	3	4	5	5
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1	1	2	2	3	3	4	4	5
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1	1	2	2	3	3	4	4	5
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	1	1	2	2	3	3	4	4	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	1	1	2	2	3	3	4	4	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	1	1	2	2	3	3	4	4	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	1	1	2	2	3	3	4	4	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	1	1	2	2	3	3	4	4	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1	1	2	2	3	3	4	4	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	1	1	2	2	3	3	4	4	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1	1	2	2	3	3	4	4	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	0	1	1	2	2	3	3	4	4
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	0	1	1	2	2	3	3	4	4
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	0	1	1	2	2	3	3	4	4
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	0	1	1	2	2	3	3	4	4
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	0	1	1	2	2	3	3	4	4
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	0	1	1	2	2	3	3	4	4
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	0	1	1	2	2	3	3	4	4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	0	1	1	2	2	3	3	4	4
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	0	1	1	2	2	3	3	4	4
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	0	1	1	2	2	3	3	4	4
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	0	1	1	2	2	3	3	4	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	0	1	1	2	2	3	3	4	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	0	1	1	2	2	3	3	3	4
	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9

Figure 12.2: log tables

Log tables (fig 12.2) are a good example, in 1950 it was impossible to imagine an engineer unable to use them, they were absolutely necessary for any difficult calculation

The same happens with many other things like the following formulas

$$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$$

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

$$
\begin{aligned}
\cos(\theta_1 - \theta_2) &= \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \\
\sin(\theta_1 - \theta_2) &= \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \\
\tan(\theta_1 + \theta_2) &= \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2} \\
\tan(\theta_1 - \theta_2) &= \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2}
\end{aligned}
$$

$$
sin\frac{\theta}{2} = \pm\sqrt{\frac{1 - cos\theta}{2}}, \qquad cos\frac{\theta}{2} = \pm\sqrt{\frac{1 + cos\theta}{2}} \tag{12.8}
$$

these formulas were extremely necessary to create tables with the values for trigonometric functions for many different angles. Nowadays, the dumbest cell phone can calculate trigonometric functions so there is no real need for these formulas in the every day work environment of any scientist or engineer.

Nevertheless, these formulas find an important application, they are of use to train the minds of young people who want to pursue STEM studies.

Intellectual training is absolutely fundamental for anyone, and if someone is interested in pursuing a STEM career, mathematics are a must. This is why we suggest in the strongest possible terms to study how the above formulas (or at least some of them) are shown to be true. We dare to suggest Proof of angle addition formula for sine — Trigonometry — Khan Academy as a good starting point

12.4 Values of basic the trigonometric functions for some angles

Mathematics is all about finding patterns and developing intuition about them. Trigonometry, being a branch of mathematics is no exception to this rule. In this section we will use some very simple reasoning to calculate the values of the trigonometric functions for some very particular angles. Let us see how far we can get with simple ideas without using any difficult stuff.

Since the sine and cosine of an angle are the signed lengths of the legs of a right triangle whose hypotenuse has unit length, the values of both functions must belong to the interval $[-1, 1]$, we may state this as a

Theorem 7. *$\forall\theta$,*

$$-1 \leq sin\theta \leq 1\,, \qquad -1 \leq cos\theta \leq 1$$

Now, when θ is very close to 0 the leg of the triangle opposite to θ (the vertical orange segment in fig. 12.1) becomes very, very short while the horizontal blue leg becomes longer and longer almost reaching the point $(1, 0)$. In fact, when θ_0 the triangle completely collapses rendering the hypotenuse to coincide with the adjacent leg, this is just the statement

$$sin(0) = 0\,, \qquad cos(0) = +1\,,$$

a similar reasoning leads to

$$sin(\pi) = 0\,, \qquad cos(\pi) = -1\,,$$

Let us now think to what happens when the angle we are interested in gets very close to $\pi/2$ (a quarter of a circle). In this case the hypotenuse becomes almost vertical so its horizontal projection is close to disappear while the orange leg almost equals the hypotenuse, when θ

exactly matches $\pi/2$ we conclude

$$sin\left(\frac{\pi}{2}\right) = +1\,, \qquad cos\left(\frac{\pi}{2}\right) = 0\,,$$

the case $\theta = 3\pi/2$ is handled similarly yielding

$$sin\left(\frac{3\pi}{2}\right) = -1\,, \qquad cos\left(\frac{3\pi}{2}\right) = 0\,,$$

The $\theta = \pi/4$ (i.e. $45°$) angle is treated a bit differently. All we must do is to realize that in this case the hypotenuse of the triangle is nothing but the diagonal of a square, expressed differently, both legs of the right triangle over the unitary circle are equal. This means that

$$sin\left(\frac{\pi}{4}\right) = cos\left(\frac{\pi}{4}\right)\,,$$

the fundamental trigonometric identity for this case yields

$$\begin{aligned} 1 &= sin^2\left(\frac{\pi}{4}\right) + cos^2\left(\frac{\pi}{4}\right) = \\ &= 2\,sin^2\left(\frac{\pi}{4}\right)\,, \end{aligned}$$

which is nothing but

$$2sin^2\left(\frac{\pi}{4}\right) = 1\,,$$

which in turn implies

$$sin\left(\frac{\pi}{4}\right) = \pm\frac{\sqrt{2}}{2}\,,$$

since $\pi/4$ is in the first quadrant, the sign must be $+$, and since the cosine of this angle equals its sine,

$$sin\left(\frac{\pi}{4}\right) = cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\,.$$

There are still two notable angles $\pi/3$ ($60°$) and $\pi/6 = \frac{1}{2}\pi/3$ (30^0), whose sines, cosines and tangents can be easily calculated by hand. To this end we draw an equilateral triangle[2], we call the lengths of its sides $2x$. Then we note that any of the bisector angles divides the opposite side in two segments of exactly the same length (x). Besides, the bisection builds two congruent right triangles which share a leg (the bisecting segment). The hypotenuses of these two triangles are nothing but two of the sides of original triangle.

Thinking of any of the two small right triangles it is easy to see that the leg adjacent to the $60°$ angle has length x. Due to the Pythagorean theorem and if we call s the length of the other leg,

$$s^2 + x^2 = 4x^2\,, \tag{12.9}$$

from which $s = \sqrt{3}x$, since this leg is opposite to the $60°$ angle, we end up with

$$\begin{aligned} sin(\pi/3) &= \frac{\text{leg opposite to the 60 degree angle}}{hypotenuse} = \frac{s}{2x} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}\,, \quad \text{and} \\ cos(\pi/3) &= \frac{\text{leg adyacent to the 60 degree angle}}{hypotenuse} = \frac{x}{2x} = \frac{1}{2}\,, \quad \text{finally} \\ tan(\pi/3) &= \frac{sin(\pi/3)}{cos(\pi/3) = \frac{\frac{\sqrt{3}}{2}}{1}2} = \sqrt{3} \end{aligned} \tag{12.10}$$

Example 14. *Draw the necessary sketches needed to achieve the geometric reasoning given above*

Example 15. challenge *Carry out the necessary modifications to calculate $sin(\pi/3)$, $cos(\pi/3)$ and $tan(\pi/3)$.*

Example 16. *Given $sin(\pi/3) = \sqrt{3}/2$, use trigonometric identities to $cos(\pi/3)$, $sin(\pi/6)$ and $cos(\pi/3)$. A way to do it is to recall that the 30 and 60 degrees are complementary,*

Example 17. *Calculate all the values shown in table 12.1*

[2]The internal angles of an equilateral triangle are all $60°$ angles

Angle ($^\circ$)	Angle (rad)	Quadrant	sin	cos	tan
0	0	I	0	$+1$	0
30	$\pi/6$	I-IV	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45	$\pi/4$	I	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	I	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90	$\pi/2$	I-II	$+1$	0	$+\infty$
120	$4\pi/6$	II	$\sqrt{3}/2$	-1/2	$-\sqrt{3}$
135	$3\pi/4$	II	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
150	$5\pi/6$	II	$1/2$	$-\sqrt{3}/2$	-1
180	π	II-III	0	-1	0
210	$7\pi/6$	III	-1/2	$-\sqrt{3}/2$	$\sqrt{3}/3$
225	$5\pi/4$	III	$-\sqrt{2}/2$	$-\sqrt{2}/2$	$+1$
240	$4\pi/3$	III	$-\sqrt{3}/2$	$-1/2$	$+\sqrt{3}$
270	$3\pi/2$	III-IV	-1	0	$-\infty$
300	$10\pi/6$	IV	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$
315	$7\pi/4$	IV	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
330	$7\pi/4$	IV	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
360	2π	I-IV	0	$+1$	0

Table 12.1: Values of sine, cosine and tangent of some special angles. Angles showing two quadrants signal boundaries between quadrants

12.5 An equivalence relation

Angles are **periodic** i.e. different values of angles do in fact refer to the same point in the unitary circle, when this happens we say that the angles are **equivalent** which is expressed by

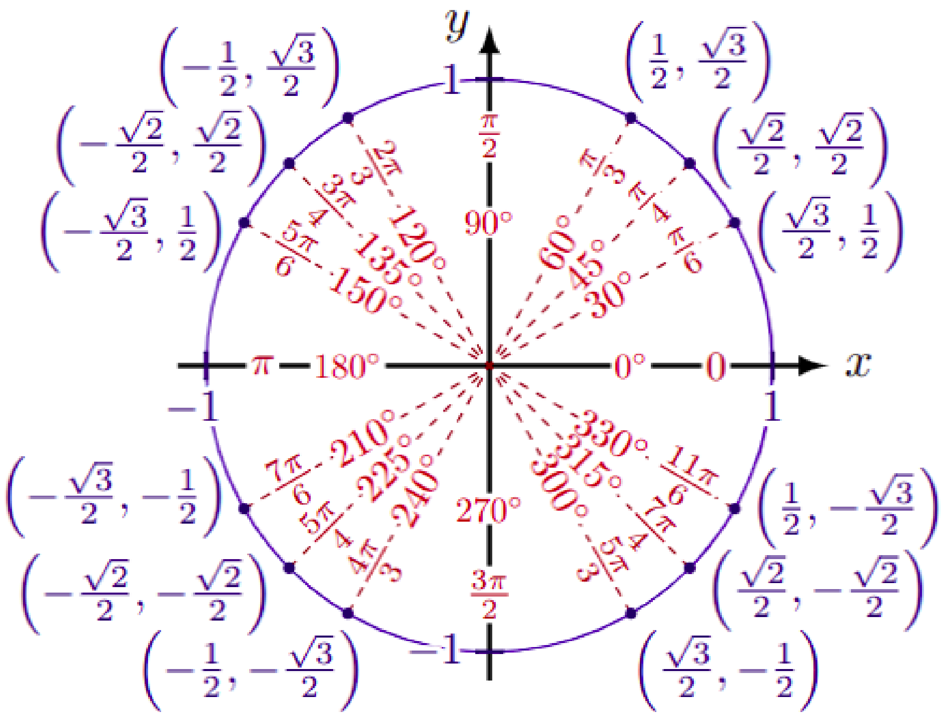


Figure 12.3: The unit circle and some special values

the symbol: $\approx$. Thus for instance, the angles 0 and 2π both define the angle that signals the point $(+1, 0)$ of the unit circle. In our new language we write $0 \approx 2\pi$, but, and this is extremely interesting, if we go around the unitary circle twice the corresponding angle[3] is $2 \times 2\pi$ and we go back again to the point $(1, 0)$ so $0 \approx 2 \times 2\pi$.

In fact, the same happens with any number n of complete rotations which. After some thought we can generalize the idea and write

Definition 3. *Two angles θ_1 and theta$_2$ are equivalent ($\theta_1 \approx \theta_2$) if and only if*

$$\theta_2 - \theta_1 = 2\,k\,\pi\,, \qquad k \in \mathbb{Z}$$

[3] The first 2 corresponding to the number of rotations around the circle

A couple of examples will help to understand this concept. Set a point in the unitary circle signaled by an angle $\theta_1 = 45°$, if begin in this point and wind (counterclockwise) the circle 5 times we end up at exactly the same point, but the angle will be $\theta_2 = 45° + 5 \times 360°$. Measuring the angles in radians, we would say

$$\theta_1 = \pi/4 \,, \quad \theta_2 = \pi/4 + 2(5)\pi \,,$$

therefore

$$\theta_2 - \theta_1 = 2(5)\pi$$

and since $5 \in \mathbb{Z}$ we can safely state

$$\theta_2 \approx \theta_1$$

which -as we already know- is just a fancy way to say that both angles mark exactly the same point in the unitary circle.

12.6 Trigonometric functions are periodic

The angular equivalence we have discussed in section 12.5 has deep geometrical roots and gives rise to a concept **periodicity of the trigonometric functions**, to introduce this concept let us carefully think about the sine function through its definition according to the unitary circle. Imagine two angles, let's say $\alpha_1 = \pi/6$ and $\alpha_2 = \pi/6 + 2\pi = 13\pi/6$. Certainly $\alpha_1 \approx \alpha_2$ (with winding $n = 1$) but the important thing here is that when we go to the unitary circle we certainly find that

$$sin(\pi/6) = sin(13\pi/6) \,, \tag{12.11}$$

besides, more geometrical thinking will convince beyond any doubt that for any angle α and any $k \in \mathbb{Z}$

$$sin(\alpha) = sin(\alpha + n\,L) \,, \tag{12.12}$$

where $L = 2\pi$ is known as the **period**. This concept is generalized to any function with real domain, and is given as[4]

Definition 4.

$$f : \Re \to \Re\,,$$

is called periodic of period $L \in \Re$ if and only if, $\forall x \in \Re$

$$f(x) = f(x + L)$$

According to definition 4, all trigonometric functions are periodic with period $L = 2\pi$.

There is a little twist to this story but we will not push it in these notes.

12.7 Trigonometric equations

It is perfectly possible that when solving some mathematical problem we reach an equation (i.e. a question) such that

$$sin(x) = \frac{1}{2}\,, \tag{12.13}$$

where we must find x.

According to the previous sections, we may be tempted to answer that x is either $30°$ or $150°$ but such answer is only partially correct because it does not really reflect all that we have learned, in particular the concept of periodicity.

For the $30°$ angle the correct answer should be: $30°$ *or any other angle obtained from it by completely winding circles an arbitrary number of times.* Something similar happens with the $150°$ angle.

In fact, the absolutely correct answer should be:

[4]The reader is urged to translate the definition to plain English

Equation eq. 12.13 has infinite number of solutions, and in radians they can be written as $x = \pi/6 = 2\,k\pi$ *where* $k \in \mathbb{Z}$, and $x = 5\pi/6 = 2\,m\pi$ *where* $m \in \mathbb{Z}$

In plainer English what we are saying is that the **set** $\mathcal{S}$ of all solutions to eq. 12.13 is given by

$$\mathcal{S} = \{..., -23\pi/6\,, -11\pi/6\,, \pi/6\,, 13\pi/6\,, 25\pi/6\,, ...\} \cup \{..., -7\pi/6\,, 5\pi/6\,, 17\pi/6\,, ...\}\,. \quad (12.14)$$

There is an interesting modification to the original problem which is a **completely different** problem:

Find the solutions to

$$sin(x) = \frac{1}{2}\,, \quad (12.15)$$

in the interval $[0, 2\pi)$. It happens that the apparently harmless modification of asking the solution to be in an specified interval, changes everything. Indeed, the solution of the new problem (the equation together with the domain where its solution is required) is a set with two elements only,

$$\mathcal{S} = \{\pi/6\,, 5\pi/6\}\,. \quad (12.16)$$

In fact, if we restrict the domain in which to look for the solution to be the set $[0, \pi/2]$, the solution becomes **unique** and equal to $x = \pi/6$.

At this point it is worth to comment that, **in mathematics the questions of existence and uniqueness of solutions to problems are of upmost importance**.

Let us try the following

Example 18. *Determine if* $x = 3\pi/8$ *is a solution of the equation*

$$tan\,2x = -1 \quad (12.17)$$

Example 19. *Find* **all the solutions** *to the equation*

$$4\,sin\theta + 1 = 2\,sin\theta \quad (12.18)$$

We begin by rewriting the equation as

$$sin\theta = -\frac{1}{2} \tag{12.19}$$

There are two angles in the interval $[0, 2\pi)$ *satisfying the equation, namely,* $\theta = 7\pi/6$ $(210°)$ *and* $\theta = 11\pi/6$ $(330°)$ *all the solutions to the original equation are found by adding to this angle an integer number of times* 2π*, i.e. the solutions are* $\theta = 7\pi/6 + 2\kappa\pi, 11\pi/6 + 2\ell\pi$, $\kappa, \ell \in Z$

Example 20. *Solve*

$$6\,cos^2 x - 3 = 0 \tag{12.20}$$

in the interval $[0, 2\pi)$

The equation can be cast as

$$cos\,x = \pm\frac{\sqrt{2}}{2} \tag{12.21}$$

By mere inspection, there are clearly four solutions in the interval, 2 for each sign. indeed, $x_1 = \pi/4$ *(first quadrant), and,* $x_2 = 3\pi/2 + \pi/4$ *(fourth quadrant) satisfy* $cos\,x_1 = cos\,x_2 = \sqrt{2}/2$*, while* $x_3 = 3\pi/4$ *(second quadrant) and* $x_4 = 5\pi/4$ *(third quadrant), satisfy* $cos\,x_2 = cos\,x_3 = -\sqrt{2}/2$

Example 21. *Solve*

$$4\,sin^2\,x = 9\,sin\,x - 5. \tag{12.22}$$

in the interval $[0, 2\pi)$

We begin by rewriting the equation as

$$4\,sin^2\,x - 9\,sin\,x + 5 = 0\,, \tag{12.23}$$

the fact that $sin\,x$ appears with two powers, 1 and 2 suggests[5] that we introduce a change of variables by naming $u = sin\,x$, in this way, the equation now looks as the ordinary quadratic

[5] This is a very old trick in anyone's toolbox

equation

$$4\,u^2 - 9\,u + 5 = 0\,, \tag{12.24}$$

with solutions

$$u = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases} \quad , \tag{12.25}$$

which really stands for

$$sin\,x = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases} \quad , \tag{12.26}$$

now, trigonometry has taught us that $-1 \leq sin\,x \leq +1$, this condition rules out $sin\,x = 8/10$ leaving us with $sin\,x = +1$, which, along with the condition $x \in \{0, 2\pi)$ means that the solution is unique, and given by: $x = \pi/2$

Example 22. *Solve*

$$2\,sin^2\,x - cos\,x - 1 = 0. \tag{12.27}$$

in the interval $[0, 2\pi)$

To approach this exercise we use the fundamental trigonometric identity to reach a first transformation of the left hand side

$$2\,sin^2\,x - cos\,x - 1 = 2\,[1 - cos^2\,x] - cos\,x - 1 \tag{12.28}$$

after which we end up with the equation

$$\begin{gathered} 2\,[1 - cos^2\,x] - cos\,x - 1 = \\ -2\,cos^2\,x - cos\,x + 1 = 0 \end{gathered} \tag{12.29}$$

setting $z = cos\,x$*, we transform the equation into the standard second order equation*

$$2\,z^2 + z - 1 = 0 \tag{12.30}$$

with solutions

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-1)}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} 1/2 \\ \\ -1 \end{cases} \quad . \tag{12.31}$$

From here

$$cos\, x = \begin{cases} 1/2 \\ \\ -1 \end{cases} \quad . \tag{12.32}$$

In the interval under consideration there are thre solutions to this equation. Indeed, $x = \pi$ is the one and one angle only with $cos\, x = -1$. On the other hand, there are two angles for which $cos\, x = 1/2$, one in the first quadrant and one in the fourth, namely, $x = \pi/4\ rad$ and $x = 7\pi/4\ rad$.

Chapter 13

One of Many Wonders of Math Prime Numbers: The Math Pieces of LEGO

We have been discussing topics in which math seems rather close to every day applicaions. But mathis not allways that way, there are topics that may seem far from something we wold need, such is the case of a branch of mathematics called *number theory*

Number theory is the branch of mathematics that deals with the properties of integers. It's often seen as a pure branch of mathematics, meaning it's focused on understanding the relationships and patterns between numbers for their own sake, rather than for immediate practical applications.

The nowadays famous Indian mathematician Srinivasa Ramanujan was a pioneer in the field of number theory. His work was groundbreaking and made significant contributions to the understanding of integers and their properties.

Ramanujan was particularly known for his intuition and ability to discover deep mathematical truths through his own unique methods. His work had a profound impact on the field of number theory.

While it might seem esoteric, number theory has deep connections to other areas of mathematics and has even found surprising applications in fields like cryptography and computer science.

13.1 Factors

Imagine two integer numbers, n and m, we say that m is a factor of n if the quotient $n \div m$ leaves no reminder.

Thus for example,

- 1,2 and 4 are factors of 4.
- 1,2,3 and 6 are factors of 6, while
- 1, 2, 3, 4, 6, and 12 are factors of 12.

13.2 Prime Numbers

Definition 5. *A prime number is a natural number greater than 1 that has no positive divisors (factors) other than 1 and itself. In simpler terms, a prime number is a number that can only be divided evenly by 1 and itself.*

Some prime numbers are for, example:

2, 3, 5, 7, 11 and 13

Non-prime numbers are called composite numbers. They can be factored into two smaller natural numbers. For example, 4 is composite because it can be factored into 2×2.

Prime numbers have the following key properties

- There are infinitely many prime numbers.
- 2 is the only even prime number. All other prime numbers are odd
- **Fundamental Theorem of Arithmetic:** Every natural number greater than 1 is either a prime number or can be uniquely factored into a product of prime numbers.

Let us briefly explore the fundamental theorem of arithmetic by studying the numbers 21, 24, and 37.

- If we make a list of divisions of 21 by integers in ascending order we find that 21 is divisible by 1 $21 \div 1 = 21$, the value of the quotient (21) is divisible by 3: $21 \div 3 = 7$ the three factors 1, 3 and 7 are prime so the decomposition of 21 in prime factors is

  $$21 = 3 \times 7$$

  showing that 21 is not a prime number
- When we try with 24, we get
  $$24 = 2 \times 2 \times 3$$
- and, when the same procedure is attempted with 37 one finds that 37 is divisible without reminders just by 1 and 37 itself, meaning that 37 is a prime number

13.2.1 Decomposing an integer into its prime factors

The decomposition is the process of breaking down a number into its smallest prime number components. It is done by following these steps

1. Start with the integer.

2. Divide the integer by the smallest prime number that divides it evenly.

3. Repeat step 2 with the quotient until you reach 1.

4. The prime factors are the divisors you used in steps 2 and 3.

Let us try some examples:

- We begin by decomposing 8 into its prime factors. First step $8 \div 2 = 4$, the second step implies dividing the quotient by 2, $4 \div 2 = 2$, we try dividing this quotient by two and since the result is $2 \div 2 = 1$, the process has ended with the decomposition

$$8 = 2 \times 2 \times 2 = 2^3$$

- THe decompostion of the number 24 into its prime factors goes as follows: $24 \div 2 = 12$, $12 \div 2 = 6$, $6 \div 2 = 3$. Since 3 is a prime number, the process stops.

  Therefore, the prime factorization of 24 is

$$2 \times 2 \times 2 \times 3 = 2^3 \times 3$$

13.3 Simple applications

In school, during our learning of math it is usual to get our teachers asking to simplify the expression

$$\frac{81 \times 14}{45 \times 7}$$

something like this is easily addressed by decomposing each factor in its prime factors and then cancelling the common factors that might appear in the numerator and denominator of the fraction.

Lets work out a couple of examples beginning with something very simple.

- Simplify 30/15. We begin by noting that the prime factor decompostition of both the numerator and denominator are $30 = 2 \times 3 \times 5$ and $15 = 3 \times 5$, therefore

$$\frac{30}{15} = \frac{2 \times 3 \times 5}{3 \times 5}$$

so

$$\frac{30}{15} = \frac{2 \times \cancel{3} \times \cancel{5}}{\cancel{3} \times \cancel{5}} = 2\,,$$

- Simplify the expression

$$\frac{8^3\, 24^2}{4^4}\,. \tag{13.1}$$

Here we will use the power of all that we have been studying up to this point.

Beginning with the following s in prime factors $4 = 2^2$, $8 = 2^3$ ans $24 = 3 \times 2^3$, which upon substitution, yield

$$\frac{8^3\, 24^2}{4^4} = \frac{(2^3)^3 \times (3 \times 2^3)^2}{((2)^2)^4}\,, \tag{13.2}$$

using negative exponents this turns into

$$\frac{8^3\, 24^2}{4^4} = (2^3)^3 \times (3 \times 2^3)^2 \times ((2)^2)^{-4}\,, \tag{13.3}$$

reordering and operating on all the powers one finally gets

$$\begin{aligned} \frac{8^3\, 24^2}{4^4} &= 2^9 \times 2^6 \times 2^{-8} \times 3^2 = 2^{9+6-8} \times 3^2 = \\ &= 2^7 \times 3^2\,, \end{aligned} \tag{13.4}$$

which is a formula expressed in terms of primes only

Chapter 14

Blunders That People Make

In this chapter, we will show some crazy (deadly) mistakes we can make if using math without the care and respect it deserves.

Reading this article might be of interest.

14.1 Flawed Factoring

Factoring a term out form an expression is an operation where people make many mistakes. Below we have a simple example

$$
\begin{aligned}
a + a^2 + b &= a(1 + a + b)\,, \quad \text{wrong} \\
a + a^2 + b &= a(1 + a + \frac{b}{a}) \quad \text{correct.}
\end{aligned}
\tag{14.1}
$$

The mistake in the first line is that a was never a factor in the third term, in fact, if we take the right hand side of the first equality and distribute the a as is must be done, we get

$$a(1 + a + b) = a + a^2 + ab\,,$$

which is obviously different to the left hand side of the original equality.

The lesson is simple, when detecting a candidate for a common factor, take must be taken in finding the candidate everywhere in the expression form where we want to factor it out.

14.2 Wrecked Fractions

14.2.1 Splitting the Denominator

This is one of the most common mistakes when operating with fractions. And it is worth to explaining by an specific example

What I typically, find is people doing something like

$$\frac{1}{3+4} = \frac{1}{3} + \frac{1}{4}\,, \tag{14.2}$$

an equality that must horrify anyone, to say the least.

Why is it so wrong? Fractions represent parts of a whole. The denominator tells us how many parts the whole is divided into, and the numerator tells us how many of those parts we have. In the left hand side of equation 14.2 the whole has been split in $3+4=7$ parts, and as the numerator indicates, we are talking just one of those parts. In the right hand side we are talking about two wholes, one divided into 4 parts and one divided into 3 parts, besides we are talking about one of the pieces of the division by 4 to one of the pieces of the splitting in 3.

In order to make the error in the pretended equality 14.2 more vivid, we may think of its left hand side as representing one slice of a pizza divided into seven equal slices while the right hand side represents **two** slices of pizza, one taken from a pizza divided in three equal parts, the other one taken from a pizza divided into four equal parts.

Another way of noticing the error is to consider the decimal expressions of the fractions on both sides,

$$\begin{aligned} \frac{1}{3+4} &= \frac{1}{7} = 0.142857142857.... \\ \frac{1}{3} &= 0.333333.... \\ \frac{1}{4} &= 0.25 \end{aligned} \tag{14.3}$$

$$\frac{1}{3}+\frac{1}{4} = 0.58333333... > 0.142857142857.... \tag{14.4}$$

It is easy to imagine how the mistake is made, fractions showing a numerator expressed as an addition can be split into fractions of the same denominator as in

$$\frac{1+9}{3} = \frac{1}{3}+\frac{9}{3}, \tag{14.5}$$

so an easy split up of the mind would be doing the same with the denominators, which is terrible as we have been discussing.

The point is that the splitting of the numerator as in equation is nothing but the way in which fractions of common denominator are added.

We perfectly know that when denominators are different we must find a common denominator of for the fractions to be added, by the way, some people (like many school teachers) tend to use the least common multiple (LCM) of the denominators when adding or subtracting fractions because it simplifies the resulting fraction.

How to avoid this mistake?

Simple, just work systematically and think of every step, do not rush to show off that you are super smart.

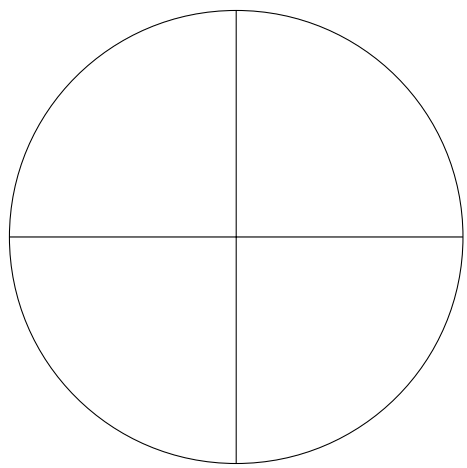


Figure 14.1: A circle (Pizza) divided into four equal parts

To finish this discussion with a *Coup de grâce* think of the wrongful equality

$$\frac{1}{2+2} = \frac{1}{2} + \frac{1}{2} \tag{14.6}$$

in terms of figs 14.1 and 14.2 and the way fractions are usually taught to children in elementary school.

$1/(2+2)$ is $1/4$ meaning 1 slice of a pizza cut into 4 equal parts (fig 14.1), $1/2+1/2$ means taking two slices of a pizza cut into 2 equal parts, i.e. a whole pizza.

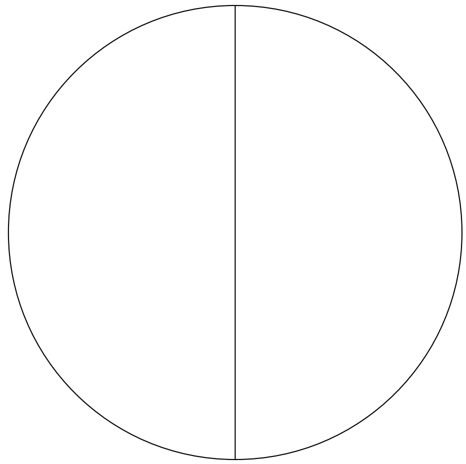


Figure 14.2: A circle (Pizza) divided into halfs.

He who writes these notes is sure that, unless the reader is on a dietary regime, she/he will rather go for the two halves (i.e. 1 full pizza) than for the first option of one slice.

So, please don' t forget

$$\boxed{\frac{1}{2+2} \neq \frac{1}{2} + \frac{1}{2}}$$

14.3 The Cancellation Trap

Let us begin by an example without error. When a fraction contains a common factor in both numerator and denominator, the common factor can be cancelled out from both the numerator and denominator as follows.

$$\frac{125}{25} = \frac{25 \times 5}{25} = \frac{\cancel{25} \times 5}{\cancel{25}} = \frac{5}{1} = 5\,, \tag{14.7}$$

another way of looking at this is by factoring out the common factor from the fraction to build a fraction with a value of 1 as

$$\frac{125}{25} = \frac{25 \times 5}{25} = \frac{25}{25} \times 5 = 1 \times 5\,, \tag{14.8}$$

which is the real content of the cancellation.

Let us know show the wrong cancellation mistake in the form it is most usually found, which is when people are operating in symbolic form, or as many young students say, "with letters instead of numbers".

In is most common form, the trouble begins with a fraction of the form

$$\frac{a+b}{a+c}\,, \tag{14.9}$$

which is awfully wrong treated as

$$\frac{a+b}{a+c} = \frac{\cancel{a}+b}{\cancel{a}+c} = \frac{b}{c}\,. \tag{14.10}$$

Some times the mistake is a bit more provocative such as in

$$\frac{a+ba^2}{a+c} = \frac{a(1+ba)}{a+c}\frac{\cancel{a}(1+ba)}{\cancel{a}+c} = \frac{1+ba}{c}\,. \tag{14.11}$$

In both cases, the error of faulty cancelation is similar to that shown in sec.14.2.1, i.e. denominator splitting.

The way to avoid this mistake is the same, being careful and thinking of each step along a calculation before taking it.

14.4 How to "Prove" 1=2 (and Why You Shouldn't)

Let two real numbers be a and b, and suppose that: $a = b$.

Multiply both sides of the equality by a to get:

$$a^2 = ab\,,$$

now subtract b^2 from both sides to get:

$$a^2 - b^2 = ab - b^2\,.$$

Clearly, the left side can be factored out as

$$a^2 - b^2 = (a+b)(a-b)\,.$$

On the right hand side of the identity we may factor b out

$$ab - b^2 = b(a-b)\,,$$

thus

$$(a+b)(a-b) = b(a-b)\,,$$

Since $(a-b)$ appears on both sides, we can cancel it to get:

$$a+b = b\,,$$

and since $a = b$, we can substitute b in for a to get: $b + b = 2b$, consequently :

$$2b = b$$

This clearly implies

$$2 = 1\,.$$

What an absurdity!, where did it come from?

Detecting the error in this deduction is quite subtle. If we check line by line, there are no algebraic mistakes anywhere along the calculation. It seems that we are facing a mystery.

Mystery?, no such a thing. As always, we must be very careful at each step of any mathematical reasoning (well, we better be careful in any kind of reasoning). Notice the line

$$(a - ba)(a - b) = b(a - b)\,,$$

up to that point all operations have been perfectly legal, so to speak. But the next apparently logical step, canceling out $a - b$ is plain wrong. The reason should be apparent to anyone paying attention to each step of a line of reasoning: From the very beginning, there was the assumption that $a = b$ and therefore $a - b = 0$ which renders division by $a - b$ ill defined, i.e. the last step of reasoning is inconsistent with the rules of good mathematical reasoning and therefore, if used, we get an inconsistent result such as $1 = 2$