

Math is So Much Fun

Mario I. Caicedo S.

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Chapter 1

Introduction

1.1 A short funny story

There are many amusing stories about science in general and mathematics in particular. One of my favourites (which is graphically referred to in title page) goes somehow like this. In the late 1700's an 8 years old boy by the name of Carl Friedrich Gauss was in class at primary school. His teacher asked the class to add up all the numbers from 1 to 100, assuming that this task would occupy his students for quite a while keeping them quiet. The teacher was shocked when, after a couple of minutes, young Gauss, gave him the answer: 5050. The teacher couldn't understand how his pupil had calculated the sum so quickly in his head, Gauss pointed out that the problem was actually quite simple and proceeded to explain that he visualized the sum in his mind

$$1 + \color{blue}{2} + \color{red}{3} + 4 + 5 + 6 + \dots + 97 + \color{red}{98} + \color{blue}{99} + 100$$

and noticed that in the sum there were 50 pairs of numbers such as $1 + 100$, $\color{blue}{2} + \color{red}{99}$, $\color{red}{3} + \color{blue}{98}$, etc which added 101, and that therefore the result of the sum was $50 \times 101 = 5050$.

I find young Gauss' story terribly funny and tell it to all my friends. Most of them ask

me what happened to Gauss and when I tell them that he became one of the greatest mathematicians of all times, they usually say “ahhh no wonder he had so difficult idea” to which I customarily ask my friends whether they really find Gauss solution hard to understand.

1.2 A note about these notes

I am a Theoretical Physicist, a scientist that uses lots of mathematics for his work. I am writing these notes with the intention of showing you, the reader, that math is not only useful but amusing.

In these notes I am using two fundamental tools that I have learned on a my daily work. The first is to avoid writing all details of each line of thought¹ and the second is making references to some other places (books, articles, blogs, videos, etc) where information that may help to enrich our understanding may be found. References must be used with some care, if a reader tries to follow each reference as soon as she or he finds it, the reader will take too long to understand or even worse, to learn anything. The way scientist read a paper or a book is by leaving the references aside unless they find something they don't really grasp from the paper or the book need some help or when when they find a reason to extend their knowledge. This technique of reading or studying may be a good advice for you, don't fall into temptation of jumping to every single reference as soon as you find it.

Time and experience have taught me that we, humans, learn by doing. There is no point in reading if we do not try to do some exercises or problems. Practice is the road to perfection. The following chinese proverb explains it best:

I listen and I forget, I see and I remember, I do and I understand

¹writing all details makes every paper too bulky and hard to read

1.3 Some thoughts about Mathematics

Many, many people regard Mathematics as boring or hard to understand. The internet is full of blogs and letters of people telling stories about how math ruined their lives, I understand those people, many of them have probably been patronized by other people who claimed they did understand and hurt their feelings. Most of the time, difficulties with understanding come from one of two reasons. In first place, having being taught by someone who, besides having good intentions, do not really understand what they are trying to teach, believe me this has happened to me. The second most common reason is not having the necessary prerequisites to understand something, for example, no one can understand division without knowing addition and multiplication.

For me and for many other people **Math is great!**, full of fun and a powerful tool for approaching many, many problems.

Math is not only about numbers or calculations, **math is a way of thinking**. A branch of math called *LOGIC* deals with the ways in which thinking can go good or wrong. Nevertheless, most of us begin our contact with math with numbers and calculations and these notes are not going to be too different in that respect.

What -I hope- makes these notes somehow different is that I am trying to write them down in a conversational tone, sometimes I will signal a blog, sometimes a video. We will try to examine lots of things, some are similar to what people learn at school, some will not be the kind of things we learn at school. These notes are based on one idea, there's no limits to math but our imagination.

My goal is twofold, in one hand, I hope the notes may become helpful to learn that Math is Fun and in the other, I hope to be able to show a bit about how mathematicians think.

In these notes I will suppose (assume) that you already know a bit about products, divisions,

fractions, etc. and even learned some elementary geometry.

A last word before beginning. In a sense, Math is science and science is marvelous for several things. For me, the greatest wonder of science is that it recognizes its limits. Scientists know that all their ideas must be checked with nature, and nature can -and usually does-proves us plain wrong. Let us think of an example, a scientist has been watching -say foxes- and has found out that all foxes he has seen are white, she then goes and writes a paper reporting that all foxes are white. A couple of months later she goes through some woods and sees a red fox, then she will immediately know she was wrong in first place and will reformulate her theory about foxes.

Each time science finds it has been wrong, it rebuilds itself trying to correct its error until it find the next one. That is how science progresses. This [video](#) shows biologist [Richard Dawkins](#) talking exactly about this aspect of science.

Having finished this rather long preamble, let us begin our the journey to [Mathmagicland](#)

Chapter 2

Basic Properties of Numbers

The first thing that we should learn is that there are different kinds of numbers. *Natural* or *Counting* numbers, *Integers*, *Rational* (fractions) , *Irrationals*, *Reals*, *Complex* (or imaginary) and there are even more exotic numbers such as *Quaternions* and *Octonions*.

Some of these numbers are *contained* in others. The naturals, for example are contained in the integers. This means that any natural number is an integer number. The integers are contained in the rationals and the rationals in the real numbers.

The interesting thing is that the *converse* is not true. That means, for example, that not every rational is an integer or that not every integer is a natural number.

Let us look at an example. Rational numbers have the form a/b , thus for example,

$$\frac{1}{2}, \frac{2}{3}, \frac{5}{4} \tag{2.1}$$

are rational numbers as well as

$$\frac{14}{7}, \frac{24}{3}, \frac{3}{1}, \frac{2}{2} \tag{2.2}$$

but,

$$\boxed{\frac{14}{7} = 2, \quad \frac{24}{3} = 8, \quad \frac{3}{1} = 3, \quad \frac{2}{2} = 1} \tag{2.3}$$

and so, these numbers are integers as much as they are rationals.

There are operations, addition for example, that we can perform on numbers which yield number of the same kind, This means the following

$$\boxed{\text{some kind of number } \text{operation} \text{ same kind of number} = \text{a number of the same kind}} \quad (2.4)$$

It is important to be aware that not all operations can be applied on any kind of numbers. Addition (+) is an operation that accepts any pair of numbers, multiplication also accepts anything, but **division cannot act on integers in the sense that we cannot divide two integers and always get an integer.**

There are a pair of very special numbers that belong to all kinds, they are zero (0) and one (1). These two numbers are very special because (a) they belong to all classes of numbers and (b) they are the only numbers having the following two proprieties

$$0 + \text{any number} = \text{the same number} \quad (2.5)$$

$$1 \times \text{any number} = \text{the same number} \quad (2.6)$$

The integers have a very special property not shared by the counting numbers. If we think of an integer, any integer, there is always another integer called its *opposite* such that these couple adds zero,

$$\text{any integer} + \text{its opposite} = 0. \quad (2.7)$$

We call the opposite of 3 *minus three* -3 and say that 3 is a positive integer and -3 a negative integer.

This is a good point for some questions,

1. **Why do natural numbers have no opposite?** To be honest this question is sort of tricky, because it is easy and at the same time it is extremely hard.

2. Name and write the opposites of 2, 7, 9

3. Name the opposites of -3 , -2 and -5 .

The last question is the reason behind the expression *two negatives equal a positive*. Why?

Addition can also operate on rationals, the way it does very nicely explained in a video from [Matemáticas Portátiles](#). As happens with integers, for each rational there is an opposite.

Having talked about addition and opposites we are ready to state that addition has the following two properties

- The order of the the two numbers in one addition can always be switched without changing the result.

For example

$$1 + \frac{3}{4} = \frac{3}{4} + 1$$

This is called the *commutative* rule of addition.

- When adding several numbers we may group at our will without changing the result. For example

$$1 + \frac{3}{4} + 5 + \frac{3}{2} + 7 = (1 + \frac{3}{4}) + 5 + (\frac{3}{2} + 7) = 1 + (\frac{3}{4} + 5) + (\frac{3}{2} + 7)$$

This rule also has a name, it is called *associativity*-

2.1 Writing Math

With time Mathematicians developed a fantastic way of talking and writing about mathematics. Many people don't like this way of doing things, they say they don't understand it.

The truth about the “weird” way of expressing math is that it is very economic (it is a way to avoid too much writing), it is easy to understand and, this is even more important, it is great fun.

Mathematicians’ way of speaking and writing uses letters instead of numbers. They think that each letter represents any number you may think of as long as you respect a rule, the meaning of each letter cannot be changed at all during a “conversation” or as long as a text is not changing its subject.

Let us see how good is this way of writing or, better said, thinking.

Do you remember the rule *The order of the two numbers in one addition can always be switched without changing the result.*

Well, in mathematicians parlance this rule is:

$$\boxed{\text{For any pair of numbers, } a + b = b + a} \quad (2.8)$$

We may do better and write all addition rules.

1.

$$a + b = b + a$$

2.

$$(a + b) + c = a + (b + c)$$

3.

$$0 + a = a$$

Believe it or not, with these rules we have walked a very long road of mathematical discoveries covering more than a thousands years.

2.2 More on operations

We still need some properties of operations and numbers, to begin with, we need to learn how to multiply rational numbers.

The multiplication rule for fractions is very easy. We recall that all rationals are fractions, which means that they can always be written as

$$\frac{\text{numerator}}{\text{denominator}}$$

and it happens that the rule to multiply fractions is extremely simple, the numerator of the fraction is the product of the numerators and the denominator the product of the denominators, thus for example

$$\begin{aligned} \frac{3}{4} \times \frac{2}{5} &= \frac{3 \times 2}{4 \times 5} = \frac{6}{20} \\ &\quad (2.9) \\ \frac{17}{3} \times \frac{5}{4} &= \frac{17 \times 5}{3 \times 4} = \text{finish it} \end{aligned}$$

Just to make sure that everything is clear, a fraction is just a division, so rational numbers are decimal numbers.

There are still some tricks up the sleeve.

The first of all is that we can always write an integer as a rational, as in

$$2 = \frac{2}{1}$$

the second is this, suppose you have two integers and factor them, say,

$$\begin{aligned} 24 &= 3 \times 8 = 3 \times 2 \times 2 \times 2 \\ 14 &= 7 \times 2 \end{aligned} \tag{2.10}$$

then when we build the fraction 24/14 we may write

$$\frac{24}{14} = \frac{3 \times 2 \times 2 \times 2}{7 \times 2} = \frac{3 \times 2 \times 2}{7} \times \frac{2}{2} = \frac{3 \times 2 \times 2}{7} \times 1 = \frac{3 \times 2 \times 2}{7} = \frac{12}{7} \quad (2.11)$$

Fun, but ugly looking, Isn't it?.

Well, the above is usually written, as you have probably seen it thousands of times,

$$\frac{24}{14} = \frac{3 \times 2 \times 2 \times 2}{7 \times 2} = \frac{3 \times 2 \times 2}{7} = \frac{3 \times 2 \times 2}{7} = \frac{12}{7}. \quad (2.12)$$

This is what they call *cancelling out common factors* or simplifying a fraction.

Let's do a couple more examples, we will simplify 125/25 and 360/150.

$$\frac{125}{25} = \frac{25 \times 5}{25} = \frac{\cancel{25} \times 5}{\cancel{25}} = \frac{5}{1} = 5 \quad (2.13)$$

$$\frac{360}{150} = \frac{36 \times 10}{5 \times 3 \times 10} = \frac{\cancel{3} \times 3 \times 4 \times \cancel{10}}{5 \times \cancel{3} \times \cancel{10}} = \frac{3 \times 4}{5} = \frac{12}{5} \quad (2.14)$$

and just in case that we want a decimal, $1/5 = 0.2$ so $12/5 = 2.4$

The second trick (or definition if you will) is the *reciprocal*. This is something that the rationals (fractions have) but the integers don't

For any rational number there is a reciprocal, and the reciprocal is the number which multiplied by the original number gives 1 as result.

In mathematician's language, for a number a , the reciprocal which is written as $1/a$ is the only number that makes this equality true,

$$a \times \frac{1}{a} = 1. \quad (2.15)$$

Some examples may help. The reciprocals of 3, 5 and $1/4$ are $1/3$, $1/5$ and 4. Perhaps a little trickier is to realize that the reciprocal of $3/5$ is just $5/3$ and for any rational (we will use letters) a/b the reciprocal is b/a .

Why is this? well, let us just try multiplication,

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a}, \quad (2.16)$$

but we know that we can switch the order of multiplication without altering the result, and then

$$\frac{a}{b} \times \frac{b}{a} = \frac{a \times b}{b \times a} = \frac{b \times a}{b \times a} = \frac{b}{b} \times \frac{a}{a} = 1 \times 1 = 1, \quad (2.17)$$

the last step that we need to do is to remember that a and b are any numbers, if we think carefully we reach the realization that what we have done is always true, **with only one exception.**

2.3 A basic rule

I am sure that you are now wondering: **What is the exception?**. The answer has to do with the number zero and the fact that

$$0 \times \text{any number} = 0, \quad (2.18)$$

this fact makes impossible the existence of a reciprocal for zero, or said differently,

WE CANNOT DIVIDE BY ZERO!

Let us see the reason for this claim.

In first place, the reciprocal of any number (a) must perform like

$$a \times \frac{1}{a} = 1. \quad (2.19)$$

so if there was a reciprocal for zero we should have

$$0 \times \frac{1}{0} = 1. \quad (2.20)$$

but we already know that “zero is the easiest number to multiply for” since

$$0 \times \text{any number} = 0, \quad (2.21)$$

we can now see where is the problem. According to formula (2.19) $0/0$ should be 1 but formula (2.21) says it should be zero. If both formulas were correct, then one should be equal to zero ($1 = 0$) which we know is ridiculous.

There is an elementary rule of LOGIC (remember, logic is the science of understanding when a reasoning is or is not correct) which states that:

When one thing (A) equals something (B) and another thing (C) equals the same something (B) then the first thing (A) and the other thing (C) must be equal.

We may write this rule as

$$\text{If } A=B \text{ and } B=C \text{ then } A=C$$

This is the rule that makes the formulas $0/0 = 1$ and $0/0 = 0$ impossible to be true at the same time, because if they were, 1 should equal 0. And of course, this impossibility is what makes division by zero to be ill defined or impossible.

Time for some questions

1. What are the reciprocals of 2, 3, 4, $5/3$?
2. Write $1/5$ in decimal form. Do the same for $1/4$ and $2/8$, what about $8/2$?
3. What happens if we multiply the decimal number for $2/5$ for the decimal number for $5/2$? Is there a reason for the result?
4. Calculate
 - (a) $\frac{2}{5} + \frac{3}{2}$

(b) $\frac{2}{5} \times \frac{3}{2}$

(c) $\frac{1}{3} + \frac{2}{3}$

(d) $\frac{1}{3} + \frac{4}{6}$

2.4 Review Questions

1. Is the formula $a \times (b + c) = a \times b + a \times c$ true?. Explain your answer.
2. Calculate $3 \times (5 + 4)$ in two different ways.
3. Calculate
 - (a) $\frac{a}{b} + \frac{c}{d}$
 - (b) In the previous question set $a = 2, b = 3, c = 4, d = 6$ and verify that the “formula” you got is true.
4. Is it true that $x \times y + 5x \times z = x \times (y + 5z)$?
5. This is a harder but funnier question. An even number is an integer which after division by two gives an integer (for example 4 and 6 are even because $4/2 = 2, 6/2 = 3$, but 7 is not even, because, $7/2 = 3.5$. Do you think that **any** even number should be written as \times some integer?

Chapter 3

First order equations

First of all, we must learn that [an equation is just a question](#). For example,

What number should we add to two to get 10?

In mathematical parlance¹ that question is written as

$$2 + x = 10,$$

here x stands for the number we are asking for. And the expression $2 + x$ means (as we know) add number x to number two.

Now we understand that, since x is the number we are looking for, [2 + x = 10 just means, what is the number \(named x\) which added to two yields the result 10.](#)

It is not hard to answer the question in its original form. A little bit of thinking allows us to realize that if we add 8 to 2 we get 10 and so the answer is 8.

People disliking math would ask us a question, [why to bore ourselves by writing with math what we could have answered without math](#). This is a very good and interesting question indeed. And it has to be answered in two phrases. The first, is very simple, thinking a bit

¹we usually say **notation** instead of parlance or lingo.

and realizing the number must be eight is **doing mathematical thinking!** so we did math. The second part is that, even if we don't realize it at this very moment, mathematical notation is great because is a way to save space, is a way to write succinctly.

A question might lead to an equation like²

$$2x + 3 + 5x + 45 + 18x + 6 = 120 + x + 22x - 5, \quad (3.1)$$

in this case a little thinking might not be enough to give us the answer (x). Then how do we do find x ?

The trick is to transform this horrible looking formula into something simpler. To be successful, we shall need to remember that x is a number (the same number everywhere). Let carry on the transformation step by step to be able to see what goes on. We begin with

$$2x + 3 + \cancel{5x} + 45 + \cancel{18x} + 6 = 120 + x + 22x - \cancel{5}, \quad (3.2)$$

first we rearrange everything since changing order does not change the results

$$2x + \cancel{5x} + \cancel{18x} + 3 + 45 + 6 = 120 - \cancel{5} + x + 22x, \quad (3.3)$$

now we group

$$(2 + 5 + 18)x + 54 = 115 + x(1 + 22), \quad (3.4)$$

$$25x + 54 = 115 + 23x, \quad (3.5)$$

Our next step begins by noting that since the above is an equality, performing the same operation in both sides does not change the equality. This idea allows me to subtract 54 in both sides

$$25x + 54 - 54 = 115 + 23x - 54, \quad (3.6)$$

²try to phrase a question

which yields

$$25x = 61 + 23x, \quad (3.7)$$

finally subtract $23x$ in both sides

$$25x - 23x = 61 + 23x - 23x, \quad (3.8)$$

$$2x = 61, \quad (3.9)$$

we are almost there. Divide by two in both sides

$$2x/2 = 61/2, \quad (3.10)$$

$$x = 61/2 = 30.5, \quad (3.11)$$

we got our number!

We want to repeat the above to solve equations that have *fractional coefficients*, meaning they look like

$$\boxed{\frac{3}{2}x + 121 = \frac{5}{7}x} \quad (3.12)$$

Once again, all we need to do is to keep in mind that x means “some number”.

With that in mind let us see how might we calculate

$$\frac{3}{2}9 + \frac{2}{5}9, \quad (3.13)$$

mathematics allow many ways of performing this calculation, one of them is to distribute the products and write

$$\frac{3}{2}9 + \frac{2}{5}9 = \left(\frac{3}{2} + \frac{2}{5}\right)9 = \frac{15+4}{10}9 = \frac{20}{10}9 = 18 \quad (3.14)$$

If we use this calculational technique and remember that x is just some number, our original question can be solved with these steps

$$\begin{aligned}
 \frac{3}{2}x + 121 &= \frac{5}{7}x \\
 \frac{3}{2}x - \frac{5}{7}x &= -121 \\
 \left(\frac{3}{2} - \frac{5}{7}\right)x &= -121 \\
 \left(\frac{21 - 10}{14}\right)x &= -121 \\
 \left(\frac{11}{14}\right)x &= -121 \\
 x &= -\frac{14}{11}(11 \times 11) \\
 x &= -14 \times 11 = 154
 \end{aligned} \tag{3.15}$$

We may even do far better. What about “solving”

$$ax + b = 0 \tag{3.16}$$

this may look weird, but in reality is funny.

In the above equation, each character is just some number. To be able to solve it, we need to know which number is the one we want to find in terms of the other two.

Let's say that we want to find x . We simply forget that a and b look like letters, they are NUMBERS, so, we use math as usual ...

$$\begin{aligned}
 ax + b &= 0 \\
 ax &= -b \\
 x &= -\frac{b}{a}
 \end{aligned} \tag{3.17}$$

And now comes the fun part.

If we think carefully we realize that we have just solved any conceivable first order equation because:

a and b can be any two numbers!!

Regarding equations, in some homework, Sophie was asked the following:

Solve this equation for h

$$SA = 2\pi r^2 + 2\pi r h$$

The question means that somehow Sophie is *supposed to know* the numbers SA , π and r and she is expected to find h in terms of SA , r and π .

Well, under the assumption, the first thing to do is to subtract $2\pi r^2$ in both sides of the equation to get

$$SA - 2\pi r^2 = 2\pi r h$$

and now, multiply both sides by the reciprocal of $2\pi r$ which is

$$\frac{1}{2\pi r}$$

in that case we get;

$$\begin{aligned} \frac{1}{2\pi r} \times [SA - 2\pi r^2] &= 2\pi r h \times \frac{1}{2\pi r} \\ \frac{SA - 2\pi r^2}{2\pi r} &= \frac{2\pi r h}{2\pi r} = h, \end{aligned}$$

and so the answer is

$$h = \frac{SA - 2\pi r^2}{2\pi r}.$$

Which is indeed dependent on the known quantities π , SA and r .

Let's solve some questions

1. Solve for x : $3x + 5 = 20$
2. Find (meaning solve) t : $t - 5 = 3t + 25$

3. Find n : $3n - 8 = n$

4. This one looks hard but it is easy if you are cautious. Find x

$$\frac{3x + 5}{2x - 2} = 1$$

5. Find the solutions for the following equations (a written answer with every step explained is required)

(a) $2x + 5 = 7$

(b) $3x - 2 = 1$

(c) $2x + 1 = x + 1$

(d) $3s - 5 = s + 15$

(e) $\frac{4x-4}{x+1} = 2$

6. Is anything wrong in

$$\frac{2x + 2}{x + 3} = \frac{2x + 2}{x} + \frac{2x + 2}{3},$$

if the answer is yes, explain.

7. Same question for

$$\frac{2x + 2}{x + 3} = \frac{2x + 2}{x} + \frac{2x + 2}{3}$$

Example

Let us examine the following:

$$\frac{2x + 2}{2x + 5} = \frac{2x + 2}{2x} + \frac{2x + 2}{5},$$

This expression is wrong! Let us look at the right hand side of the formula:

$$\frac{2x+2}{2x} + \frac{2x+2}{5},$$

Using the addition rules for fractions,

$$\begin{aligned}\frac{2x+2}{2x} + \frac{2x+2}{5} &= \frac{5 \times (2x+2) + 2x \times (2x+2)}{2x \times 5} = \\ &= \frac{10x+10 + 4x \times x + 4x}{10x}\end{aligned}$$

And the last line is completely different to

$$\frac{2x+2}{2x+5}$$

Chapter 4

Integer Powers

Sometimes, when reading or watching a video we see something like 2^3 . What does that mean?.

The answer is simple,

$$2^3 = 2 \times 2 \times 2 = 8, \quad (4.1)$$

what about 3^4 ? , once again, it is simple

$$3^4 = \underbrace{3 \times 3 \times 3 \times 3}_{\text{4 times}} = 81 \quad (4.2)$$

3^2 reads as 3 **to the power of** 2 or 3 to the second power, or simply 3-squared. This means that we have just learned that, when we power one number to say 4 that means multiply the number by itself 4-times.

If we let a stand for a number,

$$a^3 = a \times a \times a \quad (4.3)$$

and so, for any positive integer (b),

$$a^b = \underbrace{a \times a \times \cdots \times a \times a}_{\text{b times}} \quad (4.4)$$

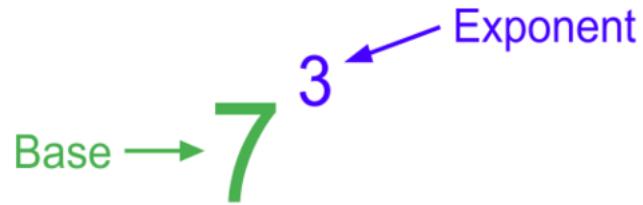


Figure 4.1: Names of numbers in powers

Thus for example,

$$a^{15} = \underbrace{a \times a \times \cdots \times a \times a}_{15 \text{ times}} \quad (4.5)$$

Some times we find a product such as $2^3 \times 2^2$, what happens in cases like this?. To find out the answer, we just multiply

$$2^3 \times 2^2 = (2 \times 2 \times 2) \times (2 \times 2) = \underbrace{2 \times 2 \times \cdots \times 2}_{5 \text{ times}} \quad (4.6)$$

[that is just multiplying 2 by itself 5 times!], which we have just learned, equals 2^5 we then conclude

$$2^3 \times 2^2 = 2^5, \quad (4.7)$$

let's try with $3^2 \times 3^2$,

$$3^2 \times 3^2 = (3 \times 3) \times (3 \times 3), \quad (4.8)$$

which is 3 times itself 4 times,

$$3^2 \times 3^2 = 3 \times 3 \times 3 \times 3 = 3^4, \quad (4.9)$$

what we are saying is this,

$$a^b \times a^c = \underbrace{a \times a \times \cdots \times a \times a}_{b+c \text{ times}} = a^{b+c}$$

(4.10)

Let us propose some exercises

$$1. \ 3^4 \times 3^5$$

$$2. \ 5^2 \times 5^3$$

$$3. \ 7^5 \times 7^2$$

$$4. \ a^4 \times a^3$$

Another thing we may find is an expression like $(3^2)^3$, and once again, we will ask ourselves: what is this? Well,

$$(3^2)^3 = (3^2) \times (3^2) \times (3^2) = (3 \times 3) \times (3 \times 3) \times (3 \times 3) = \underbrace{3 \times 3 \times 3 \times 3 \times 3 \times 3}_{6 \text{ times}} = 3^6 \quad (4.11)$$

If we think of some more examples we will conclude that for any positive integers b and c ,

$$\boxed{(a^b)^c = a^{b \times c}} \quad (4.12)$$

What happens if we divide powers?,

$$\begin{aligned} \frac{3^5}{3^2} &= \frac{3 \times 3 \times 3 \times 3 \times 3}{3 \times 3} = \\ &= \frac{3 \times 3 \times 3 \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3}} = 3 \times 3 \times 3 = 3^3 \end{aligned} \quad (4.13)$$

let us think a little bit, $3 = 5 - 2$. Hmmmm....it seems that

$$\frac{3^5}{3^2} = 3^{5-2} = 3^3 \quad (4.14)$$

And it happens that that is absolutely true.

Math is wonderful, ideas get in and they fit together as the pieces of a giant puzzle.

At some point in time, mathematicians introduced the following idea, -1 a negative integer as an exponent, they did it by stating

$$\frac{1}{a} = a^{-1} \quad (4.15)$$

To find out where this idea leads us, we think of the following expression,

$$\frac{a^b}{a^c}, \quad (4.16)$$

the rules to handle fractions tell us that

$$\frac{a^b}{a^c} = a^b \times \frac{1}{a^c}, \quad (4.17)$$

but

$$\frac{1}{a^c} = \left(\frac{1}{a}\right)^c = (a^{-1})^c, \quad (4.18)$$

and $(a^{-1})^c = a^{(-1) \times c} = a^{-c}$, therefore

$$\frac{a^b}{a^c} = a^b \times a^{-c} = a^{b-c} \quad (4.19)$$

where we have used the rule for multiplying powers of the same base.

There's still something that is many times stated as a rule,

$$a^0 = 1, \quad (4.20)$$

to see how this comes about, think of this fraction

$$\frac{a}{a} = 1 \quad (4.21)$$

we just learned that

$$\frac{a}{a} = a \times \frac{1}{a} = a \times a^{-1} = a^{1-1} = a^0, \quad (4.22)$$

we thus have

$$\frac{a}{a} = 1 \quad \text{and} \quad \frac{a}{a} = a^0, \quad (4.23)$$

so

$$a^0 = 1 \quad (4.24)$$

must be true and it is indeed true.

Let's try some exercisers

Calculate

$$1. 2^3 \times 2^4$$

$$2. 5^a \times 5^b$$

$$3. \frac{3^3}{3^2}$$

$$4. a^3 \times b^4$$

$$5. \frac{5^3}{5^4}$$

$$6. \frac{3^3}{3^4}$$

$$7. \frac{2^8 \times 3^4}{4^2 \times 9^2}$$

$$8. (a + b)^2$$

$$9. (a - b)^2$$

$$10. (a + b) \times (a - b)$$

$$11. \text{ Use what you have sone to calculate } 25 \times 15 \text{ easily}$$

This is all we need to know about integer powers, some videos may help to understand.

[Marija Kero](#)

[Franck Ives](#)

[Funny guy](#)

Chapter 5

Roots, also called: Fractional Powers

We all know (or are supposed to know) that, the square root of 4 ($\sqrt{4}$) is 2 because the square root of a number is another number which multiplied by itself gives the first number as result¹. Thus, for instance, $\sqrt{49}$ is the number which makes this equality true

$$\sqrt{49} \times \sqrt{49} = 49,$$

which obviously means that $\sqrt{49} = 7$.

Some examples and exercises will do no harm

1. $\sqrt{9} = 3$, because: $3^2 = 9$
2. $\sqrt{36} = 6$, because: $6^2 = 36$
3. $\sqrt{256} = 16$, because: $16^2 = 256$
4. Why is $\sqrt{25} = 5$?
5. Why is $\sqrt{81} = 9$?

¹Nice video

6. Find the value of $\sqrt{64}$

7. Find the value of $\sqrt{100}$

8. Of what number is 13 the square root?, what about 25?

Most square roots are pretty hard to calculate by hand, but fortunately technology (which is built on the shoulders of science) has brought in hand calculators and computers that can do the job for us.

Anyway, what I want to point out is that some mathematician (or several of them) came up with an amusing idea, think of square roots as powers. That is to say, writing $\sqrt{5} = 5^a$ for some number a , if this were possible, we should write

$$5^a \times 5^a = 5,$$

and at the same time, because the rules we have already learned,

$$5^a \times 5^a = 5^{a+a} = 5^{2a},$$

at the same time we know that

$$5^1 = 5,$$

and so because of the logical rule we have just talked about, it must happen that

$$5^{2a} = 5^1,$$

which in turn means that

$$2a = 1.$$

And now we have a question, what number multiplied by two equals 1? And the answer is the reciprocal of 2 which is 1/2. This answer means that our unknown friends that proposed the amusing idea were able to find an answer: a square root is a 1/2 power

$$\sqrt{5} = 5^{1/2}$$

and this is alway true, so if x stand for any **positive** number

$$\sqrt{x} = x^{1/2}.$$

And now something happens with is the usual thing with math (new stuff appearing everywhere if we think a little bit). This new something comes from the rule $(a^b)^c = a^{b \times c}$, indeed, since this rule is true we may apply it to power $1/2$ and see what happens

$$(x^{1/2})^a = x^{\frac{1}{2} \times a} = (x^{1/2})^a = x^{\frac{a}{2}}$$

fractional powers exist!, and indeed, they exist in many ζ , many kinds (as much as fractions).

For instance, a cubic root is

$$\sqrt[3]{17} = 17^{1/3},$$

I find this cool, and you?

With this findings all kind of cool things happen like for example, that we can do algebra with exponents (powers) as long as we take some care with the signs of the numbers we are powering (we will come to that later). Once again, our friend [Marija](#) will up review what we have just learned

Chapter 6

“Nasty” algebraic formulas



Sometimes people find the need to calculate something like

$$(3x^2 + 5)(2 + x), \quad (6.1)$$

and then begin complaining and asking why the hell do they need to even think about such awful things. Well, I give them the point that nowadays there are computer programs that can solve such and other much nastier looking formulas in no time, nevertheless I find such calculations fun and part of my everyday life so I tell those people, let's give such things a try.

Going back to the calculation proposed in formula 6.1, all that is needed is a little care, let's do it

$$\begin{aligned}
 (3x^2 + 5)(2 + x) &= \\
 = 3x^2(2 + x) + 5(2 + x) &= \\
 = 3x^2 \times 2 + 3x^2 \times x + 5 \times 2 + 5 \times x &= \\
 = 6x^2 + 3x^3 + 10 + 5x
 \end{aligned} \tag{6.2}$$

$$(3x^2 + 5)(2 + x) = 3x^3 + 6x^2 + 5x + 10 \tag{6.3}$$

Let's look at some more solved examples.

Example 1.

$$y(2y + 5) = y \times 2y + y \times 5 = 2y^2 + 5y$$

Example 2.

$$\begin{aligned}
 (2x + 3)(5x + 4) &= 2x \times 5x + 2x \times 4 + 3 \times 5x + 3 \times 4 = \\
 &= 10x^2 + (8 + 15)x + 12 = \\
 &= 10x^2 + 23x + 12
 \end{aligned}$$

Example 3.

$$\begin{aligned}
 (2x + 3)(2x - 3) &= 2x \times 2x + 2x \times (-3) + 3 \times 2x + 3 \times (-3) = \\
 &= 4x^2 + (3 - 3)x - 3^2 = \\
 &= 4x^2 - 9
 \end{aligned}$$

Let's try a harder looking exercise, this time completely symbolic, IT IS JUST MORE OF THE SAME STUFF!

Example 4.

$$\begin{aligned}
 (ax + b)(cx + d) &= ax \times cx + ax \times d + b \times cx + b \times d = \\
 &= acx^2 + (ad + bc)x + bd
 \end{aligned}$$

6.1 What are we doing?

This is a good place for some comments. We have gone through some examples, they might have been some fun, but being honest I must concede much more than a point to whoever tells me that these formulas don't look as having much to do with everyday's life but more as something that someone that likes algebra wants to force him or her to do.

To tell the truth much of the interest on these formulas have some historical reasons, many years or better said centuries (even perhaps millennia) ago, there were no smart phones or computers to handle lengthy calculations as those needed for, say banking, or government, and so life for those in charge of such calculations was probably hell. Sooner or latter, someone realized that developing formulas might be useful to make things easier.

Example 5. Consider the following product,

$$17 \times 23$$

it is certainly not the longest and awfullest calculation, neither it is the easiest calculation of all, in fact, it may be a little annoying.

Nevertheless, someone may realize (this is very easy) that

$$23 = 20 + 3 \quad \text{and}$$

$$17 = 20 - 3,$$

so the product 23×17 is just the same as calculating

$$(20 - 3)(20 + 3),$$

if this same person knows the trick (formula) $(a + b)(a - b) = a^2 - b^2$, the calculation becomes significantly easier,

$$(20 - 3)(20 + 3) = 20^2 - 3^2 = 400 - 9 = 391.$$

Product/Formula	Result	English Name	Spanish Name
$(a + b)^2$	$a^2 + 2ab + b^2$	Square of a sum	Cuadrado de una suma
$(a - b)^2$	$a^2 - 2ab + b^2$	Square of a difference	Cuadrado de una diferencia
$(a + b)(a - b)$	$a^2 - b^2$	Sum times its difference	Suma por su diferencia

Table 6.1: Most usual notable products

And how might this person know that $(a+b)(a-b) = a^2 - b^2$, the answer is, because someone else did the algebra with symbols.

Formula

$$(a + b)(a - b) = a^2 - b^2$$

is just one example of the so called **notable products**, table 6.1 shows a sample of the most common ones..

6.2 Completing the square

For some reason that will not be clear right now (so we better continue for the fun of it) we might ask if there is any way to write

$$x^2 + 2x$$

as

$$(x + 1)^2 + \text{something},$$

doing this is called **completing the square** and it turns out that it can indeed be done and that it is very useful. Let us see how to do it.

In first place, we must remember that

$$(x + a)^2 = x^2 + 2ax + a^2$$

by comparison, $x^2 + 2x$ looks like the first two terms with $a = 1$, but we are lacking a term 1^2 , if we add zero in a clever way,

$$x^2 + 2x = x^2 + 2x + (1 - 1) = (x^2 + 2x + 1) - 1 = (\textcolor{red}{x + 1})^2 - 1,$$

and we have completed the square.

6.3 Formula for the roots of a quadratic equation

A quadratic equation is an equation that looks like this:

$$ax^2 + bx + c = 0, \quad (6.4)$$

as with first order equations, this is a question where all we want to know is this:

What value (or values) of x make the right hand side of the equation true.

For instance, if the equation is

$$2x^2 + 4x + 2 = 0$$

and we try $x = 3$ we get

$$2(3)^2 + 4(3) + 2 = 18 + 12 + 2 = 32 \neq 0$$

this means, 3 is not a solution of the equation, if we try $x = -1$ we get

$$2(-1)^2 + 4(-1) + 2 = 2 + (-4) + 2 = 0,$$

meaning that $x = 1$ is indeed a solution.

Mathematicians have found that a quadratic equation can have -at most- 2 different solutions.

Thus, for example, $x = 3$ and $x = -3$ are two solutions of

$$x^2 - 9 = 0,$$

and thanks to our dearest friends the mathematicians, that there cannot be any other solution.

Now, if we think a little, we realize that there is a method to solve any first order equation and we may ask if there is any such method for quadratic equations. Fortunately our friends did the hard work and have found that the solution (solutions or roots) of the quadratic equation

$$ax^2 + bx + c = 0 \quad (6.5)$$

are given by the following formula,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (6.6)$$

where the quantity inside the square root, namely

$$D = b^2 - 4ac,$$

is referred to as the *discriminant of the equation*

Let's try this out. Imagine that we need to solve the equation

$$x^2 - x - 6 = 0,$$

to use formula 6.6 we need to identify the coefficients (numbers) a , b and c in the equation, which in the case of this discussion are $a = 1$, $b = -1$ and $c = -6$.

Therefore, the **discriminant** is

$$D = (-1)^2 - 4(1)(-6) = 1 + 24 = 25$$

which implies

$$x = \frac{-(-1) \pm \sqrt{25}}{2(1)} = \frac{1 \pm 5}{2}$$

this means that we have two roots, namely $x_1 = 6/2 = 3$ and $x_2 = -4/2 = -2$

Example 6. Let us now find the solution to

$$x^2 - 2x - 15 = 0$$

this time, $a = 1$, $b = -2$ and $c = -15$. The discriminant of the equation is

$$(-2)^2 - 4(1)(-15) = 4 + 60 = 64$$

so the formula for the solution is

$$x = \frac{-(-2) \pm 8}{2(1)}$$

yielding the solutions $x_1 = 5$ and $x_2 = -3$.

Let us now think of the “function”

$$y = x^2 - 2x - 15,$$

as any function, this is a machine that gives a value (y) to each value of x we use as input. Thus for instance,

$$y(2) = (2)^2 - 2(2) - 15 = 4 - 4 - 15,$$

$$y(2) = -15$$

Now,

$$y(-3) = (-3)^2 - 2(-3) - 15 = 9 + 6 - 15 = 0.$$

But this already knew. $y(-3)$ must be zero because we used the formula of the solution of a quadratic equation to get that -3 is one of the solutions of $x^2 - 2x - 15 = 0$.

6.4 Practicing what we have learned

Problem 1. *Solve the following quadratic equations*

- $x^2 - 13x = 30$. $x = 15, x = -2$
- $4x(2x + 3) = 36$. $x = 3/2, x = -3$
- $(x + 3)^2 - (2x - 1)^2 = 0$. $x = -2/3, x = 4$
- $x(x + 4) = 32$

Problem 2. *Find the roots of the functions*

- $y(x) = 4x^2 - 4$
- $y(x) = 2x^2 + 4x + 2$
- $y(x) = x^2 + 36$

Problem 3. This problem requires a little thinking. Consider the functions $y(x) = x^2 - 1$ and $y(x) = x + 1$. Please explain your answer.

- How do they look if drawn (just a simple sketch is enough)
- Do their graphics intersect?, if the do, where?

6.5 Factoring Quadratic Functions

Let's begin by a simple example. Look at the equality

$$x^2 - 2x + 1 = (x - 1)(x - 1) \quad (6.7)$$

Is it true?, well, let us check

$$(x - 1)(x - 1) = x \times x + x \times (-1) - 1 \times x - 1 \times (-1) = x^2 - 2x + 1$$

so the equality is indeed true. Some thinking will show us that $x = 1$ is a double root of the function $F(x) = x^2 - 2x + 1$.

Writing

$$F(x) = (x - 1)(x - 1)$$

is called **factoring** $F(x)$

Factoring a quadratic function is just re expressing it in terms of its roots.

Another example may help.

We want to factor the polynomial $P(x) = x^2 + 2x - 8$, using what we know about the roots of a quadratic polynomial, formula 6.6, the roots of the polynomial we want to factor out are given by

$$x_{1,2} = \frac{-2 \pm \sqrt{4 - 4(-8)}}{2} = \frac{-2 \pm 6}{2}$$

which means there are two different roots, $x_1 = 2$ and $x_2 = -4$.

The factoring is very simple, we just write the polynomial as a product of quantities containing its roots as

$$x^2 + 2x - 8 = (x - x_1)(x - x_2) = (x - 2)(x + 4)$$

Let us check

$$(x - 2)(x + 4) = x^2 + 4x - 2x - 8 = x^2 + 2x - 8$$

cool, isn't it

Let us look at another example, factor $Q(x) = x^2 - 4$.

This time the roots of Q are

$$x_{1,2} = \frac{0 \pm \sqrt{0 - 4(-4)}}{2} = \pm \frac{4}{2}$$

which means that

$$x^2 - 4 = (x - 2)(x + 2)$$

Let us summarize what we have done to this point.

If we have a 2nd order polynomial with first coefficient (a) equal to 1, i.e. something like

$$P(x) = x^2 + bx + c$$

$P(x)$ is factored by writing

$$P(x) = (x - x_1)(x - x_2)$$

where x_1 and x_2 are the real roots when they exist.

In the case in which $a \neq 1$ we can always re express the polynomial as

$$P(x) = a[Q(x)]$$

where $Q(x) = x^2 + \frac{b}{a}x + \frac{c}{a}$ and use what we already did for polynomials with $a = 1$.

An example is required, let us factor $P(x) = 2x^2 - 4x + 2$, we begin by factoring out $a = 2$ to write

$$P(x) = 2[x^2 - 2x + 1]$$

now we factor $Q(x) = x^2 - 2x + 1$ which is easy and yields

$$Q(x) = (x - 1)(x - 1)$$

and therefore,

$$P(x) = 2(x - 1)^2$$

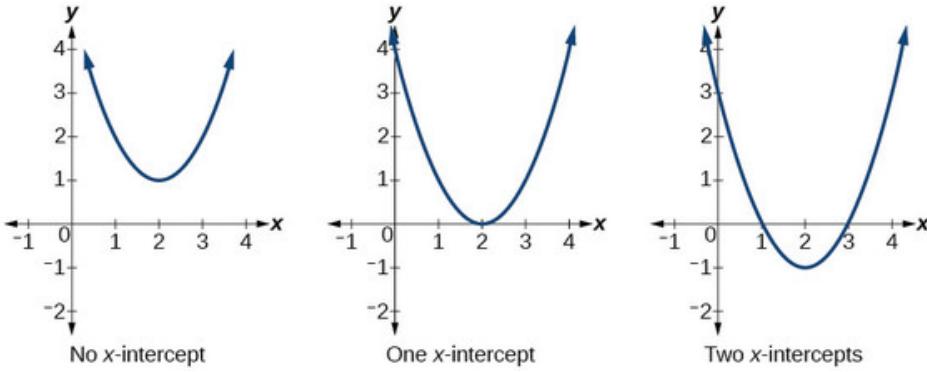


Figure 6.1: Caption

6.6 A little Geometry

Figure 6.1 shows three different curves, these particularly shaped curves are called *parabolas* and they are three of six possible graphics of quadratic functions, i.e. functions of the form

$$y(x) = ax^2 + bx + c.$$

If we carefully examine the figure, we learn two things:

- All three curves “point upwards”, mathematicians say that they are **concave upwards**
- One of the curves has tow intercepts (it crosses the x axis twice, another has just one intercept (it does not cross the x axis but just touch it, and the third has no intercept, it never crosses or touches the x axis.

Having at intercept at x_0 means $y(x_0) = 0$, i.e. x_0 is a root of $y(x)$ or, said in other words, x_0 is a solution of the quadrartic equation

$$ax^2 + bx + c = 0.$$

The problems we have solved have taught us that there are times where a quadratic equation has two solutions, sometimes it has one and only one solution and sometimes we find NO solution. Those cases correspond with the three parabolas shown in figure 6.1, we have found a very interesting connection between algebra and geometry, but more than a connection what we have found is a new tool. We may think about problems in different ways, sometimes geometrical, sometimes algebraic, sometimes there are even more ways to look at problems.

The more ways we learn to look at problems the more chances we have to understand them and solve them.

At the beginning of this section we talked about **six possible** parabolas but showed only three. The reason for this is the coefficient a in $y = ax^2 + bx + c$, when a is positive, the parabola is concave upwards, when a is negative, it is concave downwards.

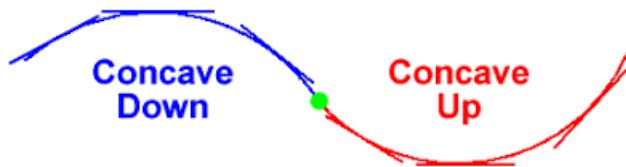


Figure 6.2: Graphical definition of concavity

We are very close to have learned a lot (but not everything) about parabolas. Two pieces of knowledge are still missing.

The first one is the question of the number of intercepts (or roots of the corresponding quadratic equation) and we will turn our attention to it. We do so by recalling the formula for the solution of the quadratic equation,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

this formula has a square root ($\sqrt{b^2 - 4ac}$) and we know that there are no square roots for negative numbers. Therefore, if $b^2 - 4ac$ turns out to be a negative number, there will be no

Sign of a	Concavity	Sign/Value of D	Number of intercepts/roots
+	Upwards	+	2
+	Upwards	0	1
+	Upwards	-	0
-	Downwards	+	2
-	Downwards	0	1
-	Downwards	-	0

Table 6.2: Classification of all parabolas of the form $y(x) = ax^2 + bx + c$. D is the discriminant, $D = b^2 - 4ac$

solution to the quadratic equation or no intercept of the parabola. There is the possibility of $b^2 - 4ac$ being equal to zero, then there is only one solution to the quadratic equation, or just one intercept of the parabola, and finally, $b^2 - 4ac$ can be positive in which case we will have two roots, or two different intercepts. Since the sign of $b^2 - 4ac$ is so important, we give a special name to this quantity (we already did) and call it the **discriminant**. With this information at hand we may now summarize our knowledge about parabolas and quadratic equations.

6.7 A nice and interesting story (real history)

Having learned the solutions to the first and second degree equations, it is natural to wonder about third, fourth, fifth or higher degree equations, of which table 6.3 shows a sample up to degree 6. The reasonable thing to do is to enquire about the existence of formulas for solving the higher degree equations.

For a long time the search for such formulas was some sort of crusade for the mathematicians, they worked incredibly hard and found formulas for the solution of the third and fourth degree

Degree	Equation
1	$a_1x + a_0$
2	$a_2x^2 + a_1x + a_0 = 0$
3	$a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
4	$a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
5	$a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$
6	$a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$

Table 6.3: Polynomial equations

equations and no matter their effort they were not able to find anything for any higher degree equations, our friends were very frustrated. Then in the nineteen century a young french mathematician and political activist by the name of [Évariste Galois](#), found what we should call an extraordinary piece of mathematics, Galois which stands as a great intellectual hero, showed that there simply doesn't exist any formula for the solution of the fifth (or higher) degree equation, meaning that he taught all of us that the search of any such formula was like the search of the mythic Chimera, i.e. a completely useless waist of effort.

The discoveries of Galois, who tragically died in a duel on May 31, 1832, at the early age of 20 constitute the foundations of two modern branches of mathematics called *Galois Theory* and *Group Theory*, the later being of fundamental importance for modern physics. [Eduardo Sáenz](#) a Mathematician and Professor of computer science gives us a very interesting and accessible account of Galois' work

6.7.1 Where does the formula for the roots of a quadratic polynomial come from?

The formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \quad (6.8)$$

is quite intriguing not to say almost magical, how does it come to be?

Well, as we said at the beginning of these notes, math is fun and the formula comes from working with math. In this case, from an extremely clever exercise of complete the square.

Indeed, let us write the equation we want to solve:

$$ax^2 + bx + c = 0$$

we begin by rewriting its first term to express the equation as

$$(\sqrt{a}x)^2 + bx + c = 0$$

now, just at the side of b we insert the number one written as

$$1 = \frac{2\sqrt{a}}{2\sqrt{a}},$$

so now the equation looks like

$$(\sqrt{a}x)^2 + bx \frac{2\sqrt{a}}{2\sqrt{a}} + c = 0$$

and now we use commutativity to move some factors a little bit,

$$(\sqrt{a}x)^2 + 2\sqrt{a}x \frac{b}{2\sqrt{a}} + c = 0,$$

now we do some grouping

$$\left[(\sqrt{a}x)^2 + 2\sqrt{a}x \frac{b}{2\sqrt{a}} \right] + c = 0,$$

and here we see that the bracketed part of the equation is where we may [complete a square](#).

To do it we need to add zero in a clever way, as

$$\left[(\sqrt{a}x)^2 + 2\sqrt{a}x \frac{b}{2\sqrt{a}} \right] + c + \left(\frac{b^2}{4a} - \frac{b^2}{4a} \right) = 0$$

we do some regrouping

$$\left[(\sqrt{a}x)^2 + 2\sqrt{a}x \frac{b}{2\sqrt{a}} + \frac{b^2}{4a} \right] + c - \frac{b^2}{4a} = 0$$

and here we see that the bracketed piece is indeed a perfect square, so now the equation looks like

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}} \right)^2 + c - \frac{b^2}{4a} = 0.$$

Our next step is taking the independent term to the right hand side of the equation,

$$\left(\sqrt{a}x + \frac{b}{2\sqrt{a}} \right)^2 = \frac{b^2}{4a} - c,$$

or,

$$\left(\frac{2ax + b}{2\sqrt{a}} \right)^2 = \frac{b^2 - 4ac}{4a},$$

we now take the square root in both sides of the equation and take care of the possibility of having two signs,

$$\left(\frac{2ax + b}{2\sqrt{a}} \right) = \pm \sqrt{\frac{b^2 - 4ac}{4a}},$$

the denominator of the right hand side can be factored out and its square root taken to yield

$$\frac{2ax + b}{2\sqrt{a}} = \pm \frac{\sqrt{b^2 - 4ac}}{2\sqrt{a}},$$

and now we can eliminate the denominators

$$2ax + b = \pm \sqrt{b^2 - 4ac},$$

and from here we immediately get

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(6.9)

Chapter 7

Equation of The Line

7.1 The Plane with Cartesian Coordinates

7.2 First Visit

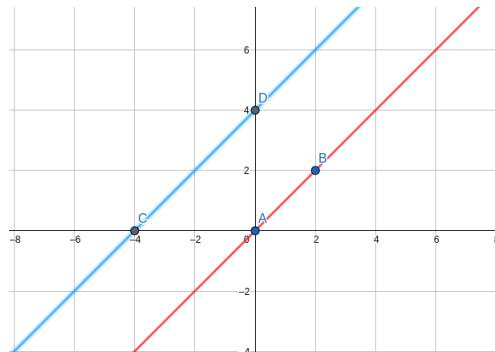


Figure 7.1: Points are given coordinates, $A(0,0)$, $B(2,2)$, $C(-4,0)$ and $D(0,4)$. The red and blue lines are parallel

Figures 7.1 and 7.2 were drawn using [GeoGebra](#), the blue and red lines in 7.1 were set using

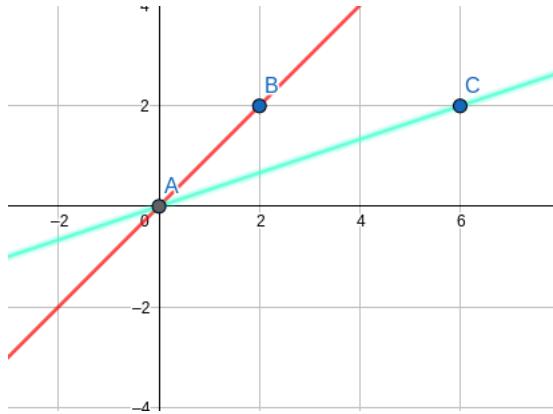


Figure 7.2: Points are given coordinates, A(0,0), B(2,2), C(6,2). The red line is steeper than the blue line

the formulas

$$\begin{aligned} & y = x \\ & y = x + 4 \end{aligned} \tag{7.1}$$

For the lines on figure 7.2 the formulas are

$$\begin{aligned} & y = x \\ & y = \frac{1}{3}x \end{aligned} \tag{7.2}$$

With a little bit of attention and if we use what we have learned about mathematical language, we will notice that all four formulas have the form

$$y = mx + n \tag{7.3}$$

in formula $y = \frac{1}{3}x$, $m = \frac{1}{3}$, while in formula $y = x + 4$, $m = 1$ and $n = 4$. These two numbers, m and n have special meanings.

- m is called **slope** (pendiente in Spanish), and it tells us how steep the line is. If m is small, say $m = 0.2$ the line will be not very steep, while if $m = 5$ we will see a quite inclined line.

- n is called the **intercept** it the value of y when $x = 0$.

Let try to perfectly understand the meaning of formula 7.3. If we think of a value for x and stick that value in the formula, we get a value for y , those two values end up being the coordinates of a point in the line. Let us see an example. The red line in figure 7.1 has the formula $y = x$, if we put $x = 4$, we get $y = 4$ and it happens that the point $(4, 4)$ is indeed in the line, the same happens with $(-2, -2)$. The blue line of the same figure has the formula $y = x + 4$, setting $x = 2$ gives $y = 6$ and $(2, 6)$ is a point belonging to the line. As we said before, for this to parallel lines, $m = 1$ which is quite steep (in fact, you may find that the angle the lines makes with the x axis is exactly 45°).

The cyan line in figure 7.2 is given by the formula $y = \frac{1}{3}x$, this time the slope has the value $m = 0.333\dots$ and that is why the line is not too inclined, and the point $(6, 2)$ belongs to the line because $y = 2 = \frac{1}{3}x$ with $x = 6$.

Ok, so we have learned how to find points belonging to a line if we know its equation. But, what else can we do?

Many, many years ago¹, geometry as we know it today, was invented. In fact, a man by the name of **Euclid** also known as Euclid of Alexandria, a Greek mathematician, that lived in the ancient and famous city of [Alexandria](#) wrote one of the most important and influential books of all times, it is called **Elements** and in it, Euclid wrote a great deal of geometry, so much in fact, that the Elements is still a fundamental work on the subject. But, what does Euclid have to do with our current discussion?

It happens that Euclid's first postulate states something that you probably already know: *given two points there is one and only one line that passes through both of them.* this result raises the following question,

[Given the coordinates of two points, is it possible to find the equation of the unique line](#)

¹Nearly 2350 years ago, to be precise

that passes through such points?

The short answer is: **yes**. But we will need to go do some amusing work before giving the complete answer. Let us consider the two triangles in figure 7.3, Thales theorem states that for

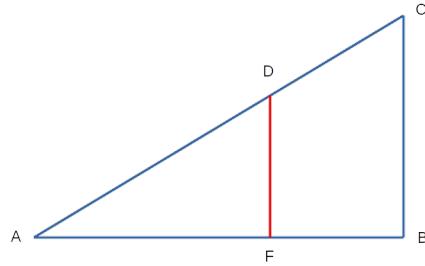


Figure 7.3: Triangles ABC and AFD are **similar** this means that all corresponding angles are equal

such similar triangles,

$$\frac{BC}{BA} = \frac{FD}{FA}. \quad (7.4)$$

and that happens with whatever couple of big and small triangles we might think of.

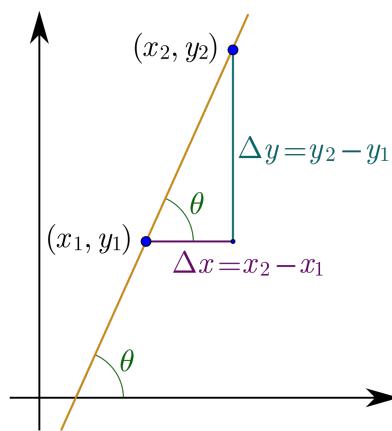


Figure 7.4: Any two pair of points along the yellow line make a triangle which is similar to another built from any other pair of points belonging to the line

Since in figure 7.4 all pair of points along the lines build similar triangles, Thales' s theorem assures us that the ratio

$$\frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (7.5)$$

will always have the same value, such ratio happens to be the slope of the line. Let us write this again with a little bit of care,

Given two points $P1(x_1, y_1)$ and $P2(x_2, y_2)$ the slope of the line to which they belong is given by

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} \quad (7.6)$$

and here we need to give a warning, the order of the coordinates in the denominator and numerator of the formula is vital.

Now imagine a generic point Q along the same line to whchich $P1$ and $P2$ belong, and let (x, y) be the coordinates of Q , then

$$\frac{y - y_1}{x - x_1} = m \quad (7.7)$$

or

$$y - y_1 = m(x - x_1) \quad (7.8)$$

i.e.

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad (7.9)$$

and this is another form for the equation of the line.

7.3 Three Ways of the Line

There are three different problems associated with finding the equation of a line.

- We are given the slope (m) and and intercepet (n) of the line.

- We are given the slope and the coordinates (x_0, y_0) of a point P_0 belonging to the line, and finally
- We are given the coordinates (x_1, y_1) (x_2, y_2) of two points belonging to the line.

The ways to build the equations for the line in each case are (respectivley)

$$y = mx + n$$

$$y - y_0 = m(x - x_0) \quad (7.10)$$

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

Chapter 8

Systems of Linear Equations

8.1 What are these?

Sometimes we find a problem such as the following, find x and y that solve the following two equations simultaneously

$$\begin{aligned}x + y &= 1 \\x - y &= 1\end{aligned}\tag{8.1}$$

in this case, the answer is $x = 0$ and $y = 1$.

Same goes to

$$\begin{aligned}2x + y &= 1 \\3x - y &= 2\end{aligned}\tag{8.2}$$

with solution, for the moment $x = 3/5$, $y = -1/5$. Things might be even be more interesting

such as in this problem,

$$\begin{aligned}x + y + z &= 0 \\x + y - z &= 0 \\x - y + z &= 1\end{aligned}\tag{8.3}$$

with solution, $x = 1/2$, $y = -1/2$ and $z = 0$

Mathematicians gave given a name to these kind of things, they call them **systems of linear equations**.

8.2 Approaching the Solution

In this sectio we will give a first look to the question: **how do we solve systems of linear equations?**

For the time being, we will concentrate in the simplest case of just two equations and two variables, i.e. things that look like

$$\begin{aligned}2x + 2y &= 0 \\x - y &= 1.\end{aligned}\tag{8.4}$$

There are several techniques to solve these type of problems.

One very simpe method consists in trying to multiply one or both equations by some, cleverly chosen numbers, and then adding the result in order to get rid of one of the variables, which leaves us wit a linear equation for the other variable, we ten solve the equation, get the value of the variable we left alone, and use such value to substitute it in any of the two original equations, to get a new simple linear equation for the remaining variable.

Let us try these method with system 8.4. A little bit of observation (and practice) lead us to note that, if we multiply the second equation by 2 we will be ready to add both equations

and cancel y , indeed, multiplication of the second equation by 2 leaves us with

$$\begin{aligned} 2x + 2y &= 0 \\ 2x - 2y &= 2, \end{aligned} \tag{8.5}$$

and if we add both equations we get,

$$2x + 2y + (2x - 2y) = 2, \tag{8.6}$$

which yields

$$4x = 2, \tag{8.7}$$

with the result, $x = 1/2$. We may now substitute this value in, say, the first original equation to get

$$2(1/2) + 2y = 0 \tag{8.8}$$

or

$$1 + 2y = 0 \tag{8.9}$$

with solution, $y = -1/2$. And as advertised, we found the solution to the system which is $x = 1/2$, $y = -1/2$.

8.2.1 Is there always a solution?

Consider the system

$$\begin{aligned} 2x + y &= 1 \\ x + \frac{1}{2}y &= 3 \end{aligned} \tag{8.10}$$

if we multiply the second equation by -2 and add we get,

$$\begin{aligned} 2x + y &= 1 \\ -2x - y &= -6 \end{aligned} \tag{8.11}$$

and

$$2x + y - 2x - y = 1 - 6 \tag{8.12}$$

which ends up as

$$0 = -5 \tag{8.13}$$

WHAT?, this is ridiculous, 0 and -5 are different, this is an impossible result. What did we do wrong?.

The answer is: WE DID NOTHING WRONG, no matter what we try, in this particular system, we will always get an impossible answer.

There is clearly a question:

Why did this happen?

We will go back to answer it in section 8.3 where we will uncover a beautiful relation between systems of linear equations and geometry.

For the time being, let us look at this other system,

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= 2 \end{aligned} \tag{8.14}$$

To attempt a solution, we multiply the first equation by -2 and add,

$$-2x - 2y + 2x + 2y = -2 + 2 \tag{8.15}$$

this time it follows that

$$0 = 0, \tag{8.16}$$

which is always true, an odd seeming result, indeed. Once again, there is nothing wrong, we just arrived to an absolute true, is there any meaning to it?, once again, we must be patient and wait for section 8.3

8.3 Systems of Linear Equations and Geometry

Let us look at the following system

$$\begin{aligned} 2x + y &= 2 \\ 16x - 2y &= 6, \end{aligned} \tag{8.17}$$

we may write it as

$$\begin{aligned} y &= -2x + 2 \\ -2y &= -16x + 6, \end{aligned} \tag{8.18}$$

or

$$\boxed{\begin{aligned} y &= -2x + 2 \\ y &= 8x - 3, \end{aligned}} \tag{8.19}$$

Let us now take a deep breath and look carefully to what we have in front of our eyes. It is a system of linear equations, yes. But we may also think that they are the equations of two straight lines, one with slope $m = -2$ and the other one with slope $m = 8$. Ohhh, they are two non parallel lines and therefore they must intersect in one and only one point.

Had we worked to solve system 8.17 we would have found the solution $x = 1/2$, $y = 1$ and if we use GeoGebra, we will be able to see that the lines intersect exactly at point $(1/2, 1)$.

In the case of system 8.10

$$\begin{aligned} 2x + y &= 1 \\ x + \frac{1}{2}y &= 3 \end{aligned} \tag{8.20}$$

or

$$\begin{aligned}y &= -2x + 1 \\y &= -2x + 6\end{aligned}\tag{8.21}$$

geometry tells us that we are dealing with two parallel lines (both have slope $m = -2$) and different intercepts, those lines will never meet each other, their meeting is impossible as the equality $0 = -5$.

Finally, in system 8.14

$$\begin{aligned}x + y &= 1 \\2x + 2y &= 2\end{aligned}\tag{8.22}$$

or

$$\begin{aligned}y &= -x + 1 \\2y &= -2x + 2\end{aligned}\tag{8.23}$$

which simplifies to

$$\begin{aligned}y &= -x + 1 \\y &= -x + 1\end{aligned}\tag{8.24}$$

what happened is that we were talking about the same line, and therefore, all its points belong to it, the system is always true.

We have found a beautiful and profound connection between algebra and geometry. Systems of 2 linear equations with two variables are equivalent to the problem of asking the point of intersection of two lines.

Geometrically we know what happens

- The lines intersect in only and only one point

- The lines are parallel and never intersect, or
- The lines are one and the same and therefore, all its points belong to the line.

From the point of view of systems of linear equations, these three possibilities translate into

- The system has one and only one solution (mathematicians say, there is a unique solution)
- There system has no solution, and
- There system has infinite solutions.

8.3.1 Systems of three equations and three variables

We have already seen systems of these kind, they look like

$$\begin{aligned} x + y + z &= 1 \\ 2x + 2y - 3z &= 2 \\ 5x + 7y - 3z &= 5 \end{aligned} \tag{8.25}$$

It happens that geometrically, an equation like

$$x + y + z = 1, \tag{8.26}$$

represents a plane sitting in the three dimensional space. Planes can also be parallel and never touch themselves, or they can intersect, in fact, two intersecting planes intersect at a line.

When we have three plane equations it may happen that two of them intersect along a line and other two at another line and these lines can meet in which case, if they meet at one point, the 3 by 3 system will have a unique solution. If the two lines are the same, there will be infinite solutions. If two of the planes are parallel and never touch we will have no solutions.

8.4 Really big systems

Think of a 4 by 4 or 5 by 5 systems

$$\begin{aligned} x + y + z + t &= 1 \\ 2x + 2y - 3z - t &= 2 \\ 5x + 7y - 3z + 2t &= 5 \\ x - y + 3z - 2t &= 5 \end{aligned} \tag{8.27}$$

and

$$\begin{aligned} x + y + z + t - 3w &= 1 \\ 2x + 7y - 3z - t + 5w &= 2 \\ 3x + 7y - 3z + t - w &= 5 \\ x - 7y - 6z + t/2 - 8w &= 5 \\ 3x + 5y - z + 2t - w &= 0 \end{aligned} \tag{8.28}$$

they look funny, and they look even more fun if we think of a geometrical interpretation.

An equation such as

$$x + y + z + t = 1 \tag{8.29}$$

can be thought of as a “plane” (they are referred to as hyperplanes) sitting in a four dimensional space, while something like

$$7x + 2y + 3z + t - w = 1 \tag{8.30}$$

is a hyperplane (a four dimensional one) sitting in a five dimensional space.

The fun does not stop here, we can imagine spaces of, say 100 dimensions, in which case, we don't have enough different characters to refer to each dimension so we use indices as in

$(x_1, x_2, x_3, \dots, x_{98}, x_{99}, x_{100})$, we may think of hyperplanes sitting in such big dimensional spaces and think of systems of 100 linear equations with 100 variables, **HAHAHAHAHA** (Mad scientists hysterical laughter).

These ideas have nothing weird in them, just imagine how many variables might influence the price of a pair of shoes (the price of leather, the price of electricity, employees salaries, price of transport, taxes, rent of the store, cost of the machinery to make the shoes, etc, etc, etc). Each one of these variables is a dimension in the space of factors influencing the price, there's no mystery about it. Scientists, engineers, economist and many other professionals keep solving huge systems of linear equations. Google in particular does it for systems consisting of billions of equations and billions of variables. Of course, no one attempts to attack big problems by hand, we use powerful computers which is just another reason to have fun with math.

Chapter 9

Functions

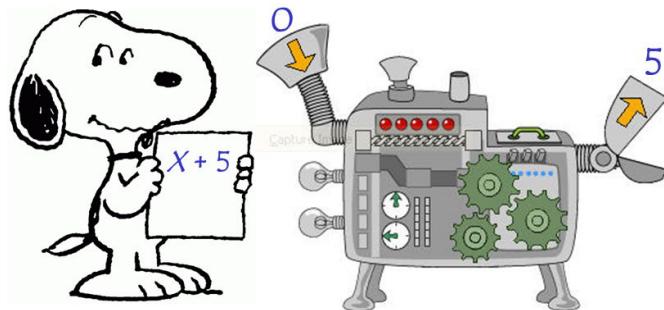


Figure 9.1: A function portrayed as a machine

9.1 First Encounter: Functions Are Machines

A function is nothing more than a machine that eats an object and turns it into something else but always into just one and the same thing.

Functions have three elements, the domain, the function itself and the range.

Domain and range are sets. The domain is the set of objects that are allowed to go into the machine and the range the set of all possible outputs of the machine.

Let us imagine a very simple example, a function whose input are the names of NFL players for the 2024 season and the output is the name of the team to which the player belongs.

In this example, the domain is the set of all NFL players and the range the set of the names of all NFL teams for the 2024 season.

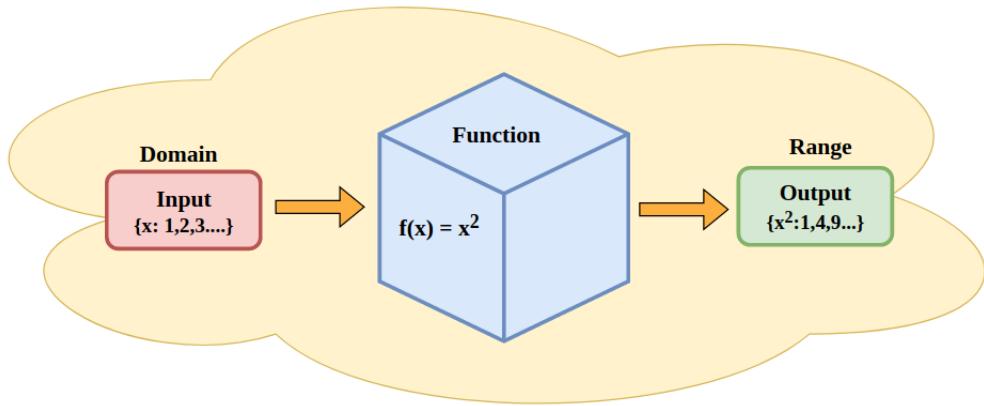


Figure 9.2: The elementary mathematical function $f(x) = x^2$

At this point of our studies (elementary mathematics), a function eats a number and produces a new number, and to talk about functions we write

$$f(x) = \text{a formula with } x \text{ inside}, \quad (9.1)$$

in this notation, x is the number eaten by the machine and $f(x)$ the product, called image. Let us try some examples

Example 7. As a first example, let us think about figure 9.2 which represents a function that squares positive integers

Example 8. Some times functions can be represented as tables,

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	1	4	9	16	25	36	49	64	81	100

It is important to realize that when talking about functions we must specify the input, i.e. we must be specific about what the function is supposed to eat, this is called the **domain**.

Example 9. Other way to specify functions is through formulas, such as

$$u(x) = \sqrt{x}, \quad x \geq 0 \quad (9.2)$$

here, the weird looking thing “ $x \geq 0$ ” is the domain specification. The formula thus means, that u is a function that takes (eats) any number greater or equal to zero and gives back the value of its square root.

Example 10. Functions are very often represented by graphics, such is the case of figure 9.3 where we are looking at two very famous functions in the domain $0 \leq x \leq 2\pi$, you might need to recall that $\pi = 3.141592654\dots$ is a special number related with the circle.

What about solving some problems?

Problem 4. Consider the domain

$$x = -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6$$

and the formula $f(x) = x^2$. Build a table of values for the function.

Problem 5. Consider the domain $-5 \leq x \leq 5$ and the formulas

$$f(x) = x + 1$$

$$g(x) = 2x + 1$$

$$h(x) = -2x + 1$$

Draw (or sketch) the corresponding graphics.

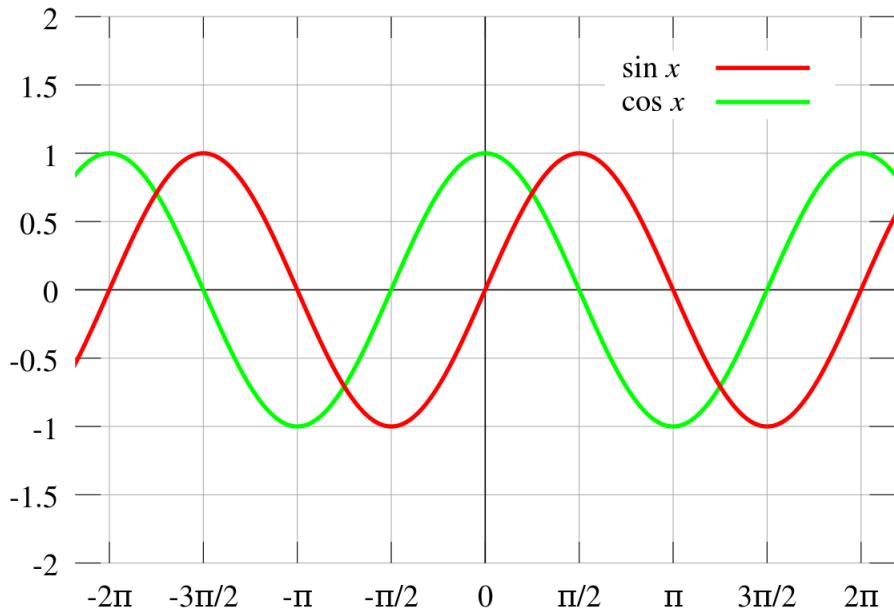


Figure 9.3: Sine and cosine functions

9.2 Improving the concept of Domain

As expressed in example 8, the domain of a function is a way to specify what is the function supposed to eat.

If we give a function by a formula and do not explicitly specify its domain, we must think about the formula and decide what things the formula can eat.

Let us consider

$$f(x) = x^2,$$

if we think a little, we realize that we can square any number, so the *a priori* domain of $f(x) = x^2$ is just all numbers.

If we now think about

$$f(x) = \sqrt{x},$$

we find a little problem. We cannot get the square root of a negative number. So the domain of \sqrt{x} is all non negative numbers, which is written as $0 \leq x$.

In a similar fashion, if we take $u(x) = 1/x$ we quickly realize that we can almost always divide one (1) by any number, zero being the only exception, then we can say that the domain of $1/x$ is $x \neq 0$.

What about this?

$$y(x) = \frac{1}{x - 1}.$$

this function is a quotient, and therefore it can only be calculated unless the denominator equals zero, therefore, the domain of y is all numbers but 1, which we write $x \neq 1$.

A much more interesting example is this,

$$f(x) = \sqrt{\frac{x+1}{x-1}}$$

now we have two conditions, in first place, the denominator of the fraction cannot be zero and in second place, the fraction must result in a non negative number, these two conditions may be written as

$$\frac{x+1}{x-1} \geq 0$$

$$x - 1 \neq 0,$$

and they must be simultaneously satisfied. This turns out to be a cute exercise in what is called *precaculus* and I will show you a technique to solve it.

The technique consists in building a table that considers signs and special values for both $x + 1$ and $x - 1$,

x	$x < -1$	-1	$-1 < x < 1$	1	$x > 1$
$x - 1$	–	–	–	0	+
$x + 1$	–	0	+	+	+
$\frac{x-1}{x+1}$	+	not defined	–	0	+

Therefore, y is defined for $x < -1$ or $1 \leq x$

9.3 The need of speaking math lingo

If we sit and take a slow and deep breath we will notice that the more we speak math, the more and more difficult it becomes to explain things in plain English, this is not an issue exclusively related to math, it happens with all fields of knowledge, we always need to introduce a particular language to express our ideas or get lost (like plenty of influencers do). Language is fundamental, think of the word **cat** and what it means or describes if you wish. If we do not say *hey look, a cat* we will be forced to say. *hey look, a four legged haired animal that meows and eats smaller animals like mice*, the same goes with almost anything you can think of, the word *cathedral* stands for a big building dedicated to certain religious activities and even that is not enough since we may have catholic, protestant or orthodox cathedrals.

Having definitively established the need of a new language, let us introduce some *math-lang*¹. We will do so by introducing some words and symbols and then trying to explain what they mean. And the more we walk the mathematical landscape, the more precise and perfect we will need to be, but that is something that will happen with time.

A list of words: set, empty set (\emptyset), union (\cup) , intersection (\cap).

¹hehehe, my way of saying mathematical language

Chapter 10

Measuring Stuff

10.1 What is measuring?

One of the most important activities of any scientist or engineer (and even cheffs) is to measure things.

We measure length, weight, electric current, area, volume and many other things.

Measuring means comparing with something. For instance, what happens if I ask: is Mario tall or short? it happens that the answer is not as simple as it looks, you might think he is tall because is taller than Sophie, but perhaps a jirafe would think he is shorter than a baby jirafe and it happens that even though the answers are different, both are right. The goal of measuring is to have a little bit more specific answer to questions like these by establishing standards of comparison.

To develop an specific answer we need to decide what are going to compare Mario's height with. We might decide we are going to compare his height with a stick. We might find that Mario is four times taller than the stick and Sophie 3 and a half times taller than the stick. In that case, we would say that the stature of Sophie and Mario are 3.5 and 4 sticks each.

With time people began to realize that we all must agree in what we use for measuring lengths (the stature of one person is a length) and the measurement standards were born.

Today there is an *intergovernmental organisation* called *The International Bureau of Weights and Measures* or in French *Bureau international des poids et mesures* which is in charge of defining the standards for measuring different quantities.

The international unit of length for example is the meter, and for mass the SI (for systeme internationale) is the kilogram.

Besides the SI system which is used in most countries there is the Imperial System in which lengths, for instance are measured in inches, feet, yards and miles.

Sophie's height, for instance is 5 ft 3' (five feet and three inches) which is 1.62 m

10.1.1 Some things we measure

We measure lots of things, some are common day life things, others are not. Some measures such as height or distance are easy to understand, some others are not as easy. Some things are measured by *basic units*, others by *derived units*. Sometimes the basic units are too small or to big to measure something and when that happens the international system of units uses multiples or sub multiples of the units and use prefixes for referring to these *rescaled units* surely you have heard about centimeters or kilometers.

Length

One of the simplest kind of quantities we measure is length. Imagine two pieces of thread. We perfectly understand what someone says that one is longer than the other, the idea of being longer or shorter is what lies in the concept of length. The SI and imperial units of length are meters and feet respectively.

To have a grasp on what things have a length close to one meter ($\ell \approx 1 \text{ m}$) place the finger

Name	kind	SI unit	Imperial Unit
Distance	length	meter	Feet
Height	length	meter	Feet
Surface	area	meter ²	Acre
Container capacity	volume	meter ³	gallons

Table 10.1: Some quantities and their units

tips of any of your hands at the center of your chest and extend (laterally) your other arm as far as you can from your body, the distance between the finger tips of your extended arm and those at your chest is quite close to a meter. Please look for a measuring tape and do the experiment, tapes usually have one side that measure in centimeters and meters and one that measures in inches and feet, you will find that your measurement is close to 1 *m* or three feet (and as you might know, three feet constitute a yard). Distances between cities are quite

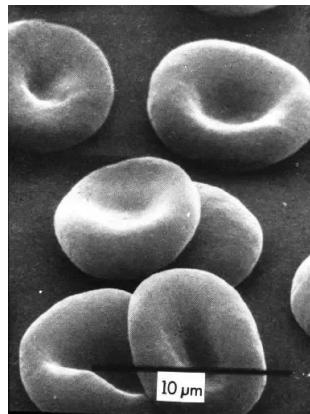


Figure 10.1: If distances are really short, using meters is inconvenient. Our blood red cells -for example- look like small disks with a diameter close to 0.000006 *m* which is six millionths of a meter, for such small distance we rather say that red blood cell diameter is around 6 micro-meter ($6 \mu\text{m}$).



Figure 10.2: The blender cup volume measuring marks

big and measuring them in feet or meters is quite inconvenient, that is why we use miles and kilometers. For example, the distance between Miami and New York is 1753 kilometers or 1089 miles. That would be 1753000 meters which is a big number which for many purposes is uncomfortable.

The earth moon distance ($384400\ km$ or $238,900\ mi$) is a much better example of the convenience of using kilometers or miles instead of meters when talking about big distances. Write that distance in meters.

Volume

Another very common quantity we measure is *volume*. The simplest notion of volume I can think of is the amount of water (or juice) that fills a blender's cup. Blender cups have marks whose meaning can be appreciated in figure 10.2 (or in your kitchen's one). The highest the level

of liquid, the more volume of liquid you will drink. Most blenders have marks for measuring volume in two different units called cups and milliliters (*ml*), those are very convenient for cooking but not for measuring the volume of water that fills a swimming pool.

The SI unit of volume (cubic meter) is rather big for everyday use. For volumes such as the amount of paint we need to decorate our houses or petrol to fill our cars we use liters or gallons. For swimming pools we begin to use cubic meters.

A natural question is, what exactly is a cubic meter. There are two precise but foolish answers for this question, one is: a cubic meter is the SI unit for volume and the other would be, a cubic meter is a volume equaling a thousand liters. None of these answers is satisfactory because none of them give us a feeling about what is a cubic meter. We need a better answer, let us attempt to develop it.

The standard blender cup can contain a volume of nearly one liter, that means that: **in order to prepare one cubic meter of juice in one shot we would need to use a thousand blenders at the same time!**, wow, a cubic meter is quite a big volume when compared to a liter. An Olympic swimming pool is 50 m long, 25 m wide and 2 m deep, using the formula for the volume of a box *length* \times *width* \times *height*, the resulting volume is

$$V = 50 \times 25 \times 2 \text{ } m^3 = 2500 \text{ } m^3 \quad (10.1)$$

that means

An olympic swimming pool can contains 2.5 million liters of water

A [video](#) may be very helpful.

Area

This is other usual quantity. Perhaps the most primitive notion of area has to do with agriculture. Imagine a farmer that wants to plant, say, corn. She needs to know how much space she has for the task. Is the space as big as a football field?, two football fields? or perhaps it is the size of a basketball field. The space she is thinking about is an area.

The SI unit for area is the square meter. A square meter is easy to picture, the only thing you need to do is to use some chalk to draw a square in the floor and make sure that each side of the square is one meter long, once you finish, the space inside the square has an area of exactly $1\ m^2$.

This idea of measuring the size of a piece of land is the origin of the word **geometry**. Indeed, the etymology of the word dates for the ancient greek $\gamma\epsilon\omega\mu\epsilon\tau\rho\alpha$; geo- “earth”, -metron “measurement”). Geometry is, with arithmetic, one of the oldest branches of mathematics.

Calculating areas for simple geometric figures like a rectangle is easy, all we have to do is multiply the length of the longest side for the length of the shortest side, in formulas

$$A_{rectangle} = \text{shortest side's length} \times \text{longest side's length} \quad (10.2)$$

10.2 Some more Geometry

Think of a square. We have just learned how to measure its area, since its longest and shortest sides are of the same length, let's call it ℓ the area of the square is

$$A_{square} = \ell^2, \quad (10.3)$$

now imagine that this square is as big as the block where you live. Certainly you can walk around this “block” and ask yourself how much distance did you walk in a round trip. The

answer has a name, the *perimeter* of the square, and it is the sum of the lengths of each side you walked by.

10.3 Measuring angles

You, the reader has probably heard someone measuring angles and using the words degrees minutes and seconds of arc length. To define a degree a circle is divided into 360 equal circular segments, the angle defined by two straight segments beginning at the center of the circle and ending at the two extremes of one of those equal circular segments defines a 1° angle. This angle is in turn divided into sixty (60) equal parts each measuring a minute of arc, finally, a minute of arc ($1'$) is divided into sixty equal tiny angles each measuring one second of arc ($1''$). This way of measuring angles is very convenient for several applications.

One can ask: [why dividing the circle into 360 equal segments?](#) and there is a simple answer: it is an arbitrary decision. There is a much more natural way of measuring angles, it is based on the following fact: given an arc of angle $\Delta\theta$ drawn on a circle of radius r , the length of the arch (Δs is proportional to the angle. This means that the length of an arc of twice the angle $\Delta\theta$ is going to be twice the length of the arc of angle θ ; if the angle is tripled, the length is tripled, if $\Delta\theta$ is divided by two then so is the arc length¹. Stated in formulas, the fact we are discussing reads:

$$\Delta s \propto \Delta\theta, \quad (10.4)$$

But there is another fact, if we have several arcs of the same angle, and different radii, it happens that their arc lengths double, triples, halves, etc according to the radii, which in formulas would be

$$\Delta s \propto r, \quad (10.5)$$

¹This eventually leads to the well known formula $circ = 2\pi r$

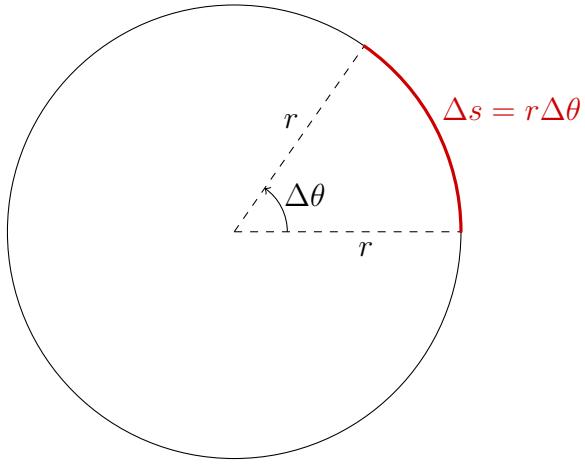


Figure 10.3: Circular arc subtended by an angle $\Delta\theta$

These formulas may be synthesized by an unique formula

$$\Delta s = r \Delta\theta, \quad (10.6)$$

from where

$$\Delta\theta = \frac{\Delta s}{r}. \quad (10.7)$$

Here we stop to suggest the reader to perform a vital experiment, take different objects of circular shape and measure the length of their circumference (ℓ) and their diameter (D) and make sure you measure them in the same units. Make a table with the pairs of number so obtained and add a column with the quotient ℓ/D , you will find that the new column gets filled with a nearly constant quantity of value 3.14, such value is an approximation to the famous irrational number

$$\pi = 3.141592654..... \quad (10.8)$$

Angles defined according to formula 10.7 are said to be measured in radians, and they are obviously dimensionless (quotients of lengths). For illustration, we note that an angle of 360°

corresponds to 2π rad, while an angle of 90° which subtends a quarter of a circle corresponds to

$$\frac{\text{Length of quarter of a circle of radius } r}{r} = \frac{1}{4} \frac{2\pi r}{r} = \frac{\pi}{2} \quad (10.9)$$

Arc	Angle in Degrees	Angle in Radians
Full circle	360°	2π
Quarter of a circle	90°	$\pi/2$
Sixth of a circle	60°	$\pi/3$
Eight of a circle	90°	$\pi/4$
Twelfth of a circle	90°	$\pi/6$

Table 10.2: Equivalence between notable angular values

Chapter 11

A Glimpse To Euclidean Geometry

We spent some time talking about geometry in the previous chapter, it is time to be a little more serious about it and we begin doing so by introducing one of the most widely reprinted and translated books in history, second only to the Bible in some accounts. This prominent book is no other than Euclid's Elements

Euclid's Elements ($\Sigma\tauοιξεῖα$) is a large set of math books about geometry and numbers, written by the ancient Greek mathematician known as Euclid (c. 325 BC–265 BC) in Alexandria (Egypt) circa 300 BC. The Elements was a core textbook taught across many cultures over a period of nearly 2000 years after its writing. Over 1000 editions and translations are estimated to have been published since the first printed edition in 1482; in fact, It was one of the very first mathematical works to be printed after the invention of the movable type printing press in the Renaissance era. Translations exist in languages ranging from Arabic, Persian, Hebrew and Turkish to Latin, Italian, English, French, German and Russian among many others.

Until the 20th century, the Elements was considered essential reading and studied by anyone wanting to learn geometry and mathematics formally. Its longevity and widespread usage are due to its remarkably clear organization, rigorous proofs, and its role as a model for deductive

reasoning. The resilience of The Elements over millennia attests to its historical importance in mathematics.



Figure 11.1: First printed edition of The Elements

The Elements is the first book in which math is discussed in the terms we understand it today. It presents everything with an approach that begins with some *primitive concepts* and **five fundamental statements called AXIOMS or POSTULATES**.

The primitive concepts are related to ideas, which are so basic that no one has ever managed to define them beyond intuition, thus for example a **point** is thought of as something really really tiny that we might draw on a sheet of paper with the sharpest pencil ever imagined.

Other primitive notions are, line, plane, and congruence¹

It is extraordinary that all Euclid's geometry is based on the following five postulates

¹Let's say that two drawings are congruent if we can put a copy of one of them on to of the other and they are identical in every respect

1. There is one and only one straight line that can be drawn from any point to any point.
2. There is one and only one line that extends a finite straight line continuously in a straight line.
3. There is one and only one circle with a given circle and radius.
4. All right angles are equal to one another.
5. [The parallel postulate]: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than two right angles.

The fifth postulate can be stated in a simpler way, given a straight line (A) and a point (p) not on the line, there is one and only one line (B) which is parallel to A and passes through p (i.e. contains p).

Axioms allows anyone to state and prove many, many other statements. Proving a statement means making sure that the statement is correct in the sense that if the postulates are correct or true if you will, the statement is also true. Proofs are achieved using the rules of logic. Mathematicians and lawyers are trained to use the rules of logic.

11.1 Logic: Rules of reasoning

People have a strong tendency to state claims, sometimes the claims are obviously false, as for example in **a red car is blue**. Other times, finding out whether a claim is true or false is not so easy. Logic is the tool that shows when a reasoning is right or wrong, and what a tool it is.

Before going back to geometry, let us do some logic.

- Imagine that you have never, ever seen a fox before. You go to Alaska and see say...40 to 50 foxes which happen to be white. You might be tempted to state: **all foxes are white**. Would that be a true statement? The answer is NO, by the rules of logic, this statement cannot be correct. The reason is as follows, if you see something happening many times and think that the next time it is going to be exactly the same, you are doing what is called an **inductive reasoning**, and it happens that experts in logic have found that inductive reasoning doesn't lead to a correct reasoning, in the case of the foxes, you might easily find red foxes which clearly shows that not all foxes are white or said differently your induction about foxes color was plain wrong.
- We all know that mammals have four limbs. Is it correct to state that any animal with four limbs is a mammal?, well, we perfectly know that such statement is completely false, indeed, alligators have four limbs and they are reptiles, not mammals.

Here we have encountered a particularly dangerous reasoning. The statement **all mammals have four limbs** or stated in other way, **any mammal has four limbs** is usually called a direct statement, while the statement **all animal with four limbs is a mammal** is called its **reciprocal**. Well, it happens that IF a direct statement is true, the reciprocal may in general, be false.

11.1.1 The notion of PROOF

Imagine you hear a statement such as **40 is an even number**

Is the statement true?, to answer this question we appeal to the definition of even numbers: **Any number that can be exactly divided by 2 is called as an even number**, given such definition, we just check whether or not the division of 40 by 2 is exact, $40/2 = 20$ which is indeed exact, and we conclude without doubt that 40 is an even number.

What we have done is to give a PROOF of the statement. A proof is a piece of reasoning that convinces ANYONE that a statement is true without any doubts regardless of whether or not a person hearing or reading the proof does not like the proof.

If a statement is not true a proof of the falseness of the statement is necessary. An example is worth here. Suppose someone claims that 3 is even, in that case you calculate the quotient $3/2$ which being 1.5 is not exact, since a number is even when divided by two the result is exact, the conclusion is that 3 is not even.

11.2 Back to Geometry

Experts in logic have found any rules for the correctness of reasoning, and geometry is a great place to learn some of those rules so let us go back to it.

In order to make the size of this notes reasonable we must remember (or perhaps learn) some concepts and some true claims.

1. A polygon is a set of line segments between given points called vertices
2. Two lines that meet at a polygon vertex are called **adjacent**
3. A closed polygon is a polygon with no free ends (a three segment closed polygon is a triangle, a four segment closed polygon is a quadrilateral. A parallelogram is a simple quadrilateral with two pairs of parallel sides, a rectangle is a parallelogram whose sides make right angles. A rectangle with all sides of equal length is a square).
4. A theorem is a statement that must be proved.

For what follows we will need a couple of definitions

Definition 1. Two polygons are congruent when the segments between their correspondent vertices are equal and the angles between successive segments are equal

Definition 2. Two polygons are similar when the angles between all corresponding segments are equal

Equilateral triangles have all their internal angles equal to 60^0 , therefore, all equilateral triangles are similar, nevertheless, not all equilateral triangles are congruent, since we can easily draw an equilateral triangle with 20 cm sides and another with 7 m sides which are evidently non congruent.

The following statement is an example of a theorem

Theorem 1. The sum of the internal angles of a triangle equals a straight angle

In these notes, we will sometimes state theorems without giving their proof.

In all problems you must explain the reasoning leading you to your answer. You can only use theorems already stated without proof or proven in these notes

Problem 6. Is it possible to have a triangle with internal angles of 30, 30 and 10 degrees?.

Problem 7. Give a proof of the following statement (Theorem) the internal angles of a rectangle add up a full rotation (360^0)

There are three fundamental theorems about congruence of triangles which we will state without proof

Theorem 2. Two triangles having corresponding sides of the same length are congruent

Theorem 3. Two triangles having one side of the same length and the corresponding angles of same measure are congruent

Theorem 4. Two triangles having one angle of equal measure and their adjacent sides of equal length are congruent

11.2.1 Two Important Theorems

Here we will comment on two of the most important theorems of Euclidean geometry

In first place we address the intercept theorem, also known as Thales's theorem, basic proportionality theorem or side splitter theorem, is an important theorem in elementary geometry about the ratios of various line segments that are created if two rays with a common starting point are intercepted by a pair of parallels. It is equivalent to the theorem about ratios in similar triangles.

Traditionally attributed to the Greek mathematician [Thales of Miletus](#).(circa 626/623 – circa 548/545 BC) It was known to the ancient Babylonians and Egyptians, although its first known proof appears in Euclid's Elements.

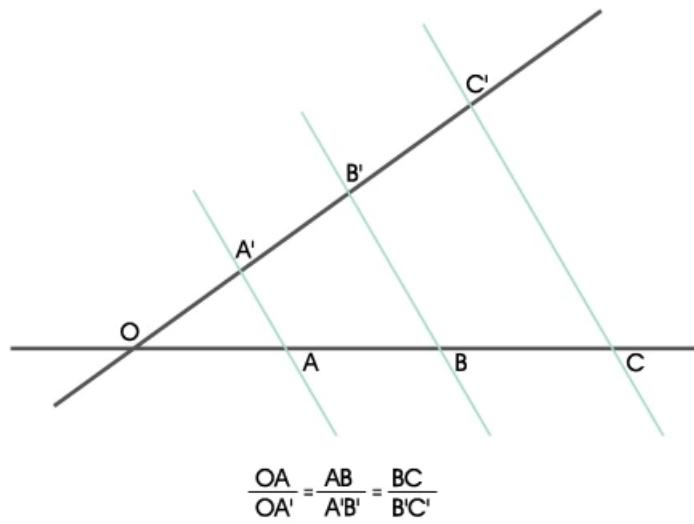


Figure 11.2: Thales Theorem

Theorem 5. *If two lines which intersect each other and are in the same plane are cut by parallel lines, segments determined in one line are proportional to the segments determined in the other line.*

The other and much more famous theorem we want to comment about is the Pythagorean theorem or Pythagoras' theorem, it establishes a fundamental relation in Euclidean geometry

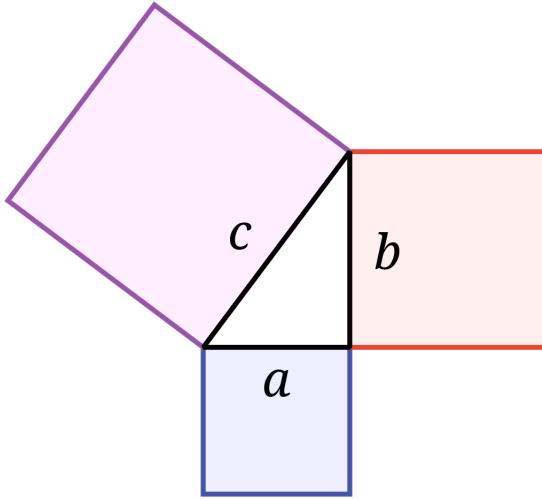


Figure 11.3: Pythagorean Theorem

between the three sides of a right triangle, namely

Theorem 6. *the area of the square whose side is the hypotenuse (the side opposite the right angle) is equal to the sum of the areas of the squares on the other two sides.*

The theorem can be written as the following identity -sometimes called the Pythagorean equation- relating the lengths of the sides a , b and the hypotenuse c ,

$$a^2 + b^2 = c^2. \quad (11.1)$$

The theorem is named for the Greek philosopher Pythagoras, born around 570 BC. The theorem has been proved numerous times by many different methods – possibly the most for any mathematical theorem. The proofs are diverse, including both geometric proofs and algebraic proofs, with some dating back thousands of years.

Chapter 12

Introduction to Trigonometry

Trigonometry is a branch of mathematics that deals with the study of triangles and the relationships between their sides and angles. It is a fundamental concept that has applications in various fields, including physics, engineering, navigation, surveying, and even computer graphics and game development.

At its core, trigonometry involves the use of ratios called trigonometric functions, such as sine, cosine, and tangent, to determine the relationships between the sides and angles of triangles. These functions allow us to calculate unknown sides or angles of a triangle, given certain information about the other parts. Trigonometry also explores the behavior of these functions over a range of angles, leading to the study of periodic functions and their properties. Additionally, trigonometry introduces concepts like the unit circle, which provides a geometric representation of these trigonometric functions and their relationships. Understanding trigonometry opens the door to more advanced mathematical concepts, such as calculus, complex analysis, and vector analysis, making it an essential tool for anyone interested in pursuing higher-level mathematics, physics, or related scientific and technical disciplines.

12.1 Three trigonometric functions: sine, cosine and tangent

Here we will define the three basic trigonometric functions sine, cosine and tangent using the method of the **unitary circle**.

We begin by considering a circle of radius $R = 1$ (that is why it is called unitary) as in figure 12.1, and draw the x and y axes. Let us additionally consider a segment beginning at the center of the circle, this segment makes an angle α^1 with the x axis and ends at the point where it meets a vertical segment of the tangent to the circle that touches it at the point $(1, 0)$, and which is therefore perpendicular to the x axis.

We now note that this construction naturally defines two **similar** right triangles. The hypotenuse of the smallest of which is the radius defining the angle α , while its legs are the orange vertical segment going from the intersection of the radius with the circle to the x axis and horizontal blue segment corresponding to the projection of the hypotenuse on the x axis. As shown in the figure, the sine and cosine of the α angle are defined by the **signed** length of those arms.

The tangent, on the other hand, is defined as the signed length of the vertical arm of the big triangle.

The first thing that happens when we use these definitions is that the fundamental formula linking the sine and cosine functions becomes almost obvious, indeed,

Corollary 1. Fundamental Trigonometric Identity *For any angle,*

$$\boxed{\sin^2 \alpha + \cos^2 \alpha = 1}$$

¹In the usual convention, angles are positive in the counterclockwise sense

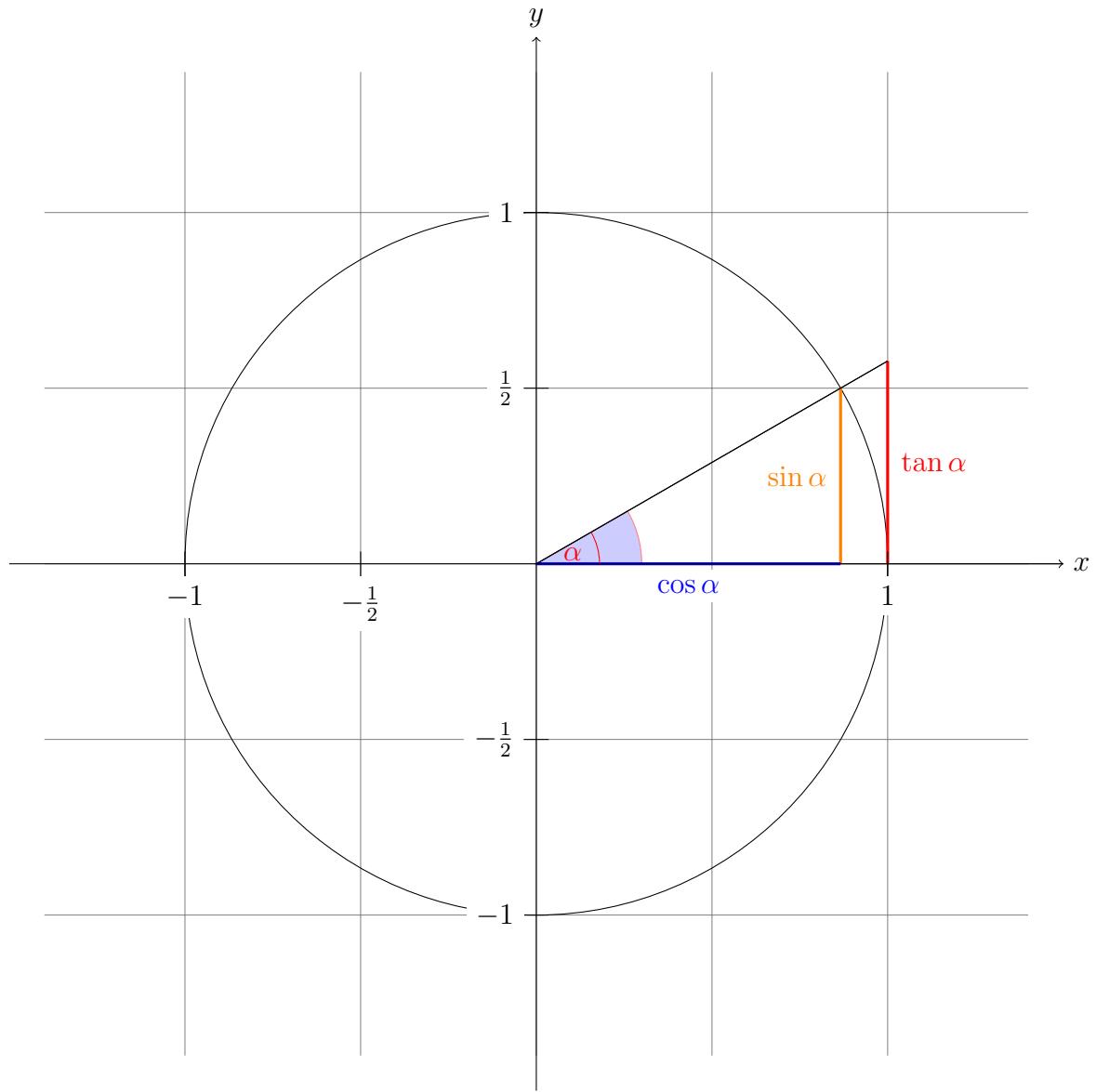


Figure 12.1: The Unitary Circle and Trigonometric Functions. **This sketch is the only thing that must be memorized about trigonometry**

The proof of this corollary is extremely simple, since the sine and cosine are defined as the signed lengths of the legs of a right triangle whose hypotenuse is of length 1, we can immediately use the Pythagorean theorem and get the wanted result.

12.2 A connection to geometry

We now take note of something apparently silly, in the big triangle, the length of the horizontal leg of triangle is 1 so we may write,

$$\tan\alpha = \frac{\tan\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{adyacent leg to } \alpha} \quad (12.1)$$

Let us now recall Thales theorem from section 11.2.1, according to Thales' theorem the ratios between corresponding sides of similar triangles are equal.

If we apply Thales theorem statement to the two triangles of figure 12.1

$$\frac{\text{opposite leg to } \alpha \text{ in big triangle}}{\text{adyacent leg to } \alpha \text{ in big triangle}} = \frac{\text{opposite leg to } \alpha \text{ in small triangle}}{\text{adyacent leg to } \alpha \text{ in small triangle}} \quad (12.2)$$

which by virtue of the figure translates into

$$\frac{\tan\alpha}{1} = \frac{\sin\alpha}{\cos\alpha}. \quad (12.3)$$

we thus end up with

$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$

(12.4)

which is the definition of the tangent of an angle.

Thales' s theorem can also be used to put the usual definitions of sine and cosine under a different light. Imagine a right triangle, any right triangle, with no other particular characteristic than being right, a little thought shows that we can always rescale the sides of such triangle to construct another triangle, similar to the one we began with, but having a hypotenuse of unit

length. Once we do that, we can put the new triangle in the unitary circle to obtain something very close to what we see in figure 12.1, in this new triangle, and since the length of hypotenuse equals one we can write the formulas

$$\begin{aligned} \sin\alpha &= \frac{\sin\alpha}{1} = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}} \\ \cos\alpha &= \frac{\sin\alpha}{1} = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}, \end{aligned} \quad (12.5)$$

but, by virtue of the similarity of the new triangle with the original one, and Thales' s theorem, we get, that no matter what triangle we are talking about, as long as it is a right triangle,

$$\boxed{\sin\alpha = \frac{\text{opposite leg to } \alpha}{\text{hypotenuse}}} \quad (12.6)$$

$$\boxed{\cos\alpha = \frac{\text{adyacent leg to } \alpha}{\text{hypotenuse}}} \quad (12.7)$$

which are the usual formulas given at the school.

It is imperative to understand and learn that the two sets of definitions we have given are different but that they define exactly the same things, i.e. there is nothing to worry when using one definition or the other. In fact, if we continue our study of math to a higher level, namely full blown calculus, we will find more definitions of the trigonometric functions. One of those is in terms of a field by the name of *differential equations*, but we will not pursue this path in these notes.

12.3 Some “not very Useful” Formulae

As time goes by and science pushes forward, some things that belonged to what we should call **fundamental knowledge** cease to be fundamental.

LOGARITHM											
	0	1	2	3	4	5	6	7	8	9	Mean Difference
	1	2	3	4	5	6	7	8	9	0	1 2 3 4 5 6 7 8 9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	1 2 3 4 5 6 7 8 9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	1 2 3 4 5 6 7 8 9
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 2 3 4 5 6 7 8 9
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	1 2 3 4 5 6 7 8 9
54	7334	7332	7340	7348	7356	7364	7372	7380	7388	7396	1 2 3 4 5 6 7 8 9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	1 2 3 4 5 6 7 8 9
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	1 2 3 4 5 6 7 8 9
57	7569	7566	7574	7582	7589	7597	7604	7612	7619	7627	1 2 3 4 5 6 7 8 9
58	7654	7642	7649	7657	7664	7672	7679	7686	7693	7701	1 2 3 4 5 6 7 8 9
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	1 2 3 4 5 6 7 8 9
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	1 2 3 4 5 6 7 8 9
61	7853	7860	7867	7875	7882	7889	7896	7903	7910	7917	1 2 3 4 5 6 7 8 9
62	7921	7928	7935	7942	7949	7956	7963	7970	7977	7984	1 2 3 4 5 6 7 8 9
63	7983	8000	8007	8014	8021	8028	8035	8041	8048	8055	1 2 3 4 5 6 7 8 9
64	8062	8059	8075	8082	8089	8096	8102	8109	8116	8122	1 2 3 4 5 6 7 8 9
65	8129	8136	8142	8149	8156	8163	8170	8176	8183	8189	1 2 3 4 5 6 7 8 9
66	8185	8202	8209	8215	8222	8228	8235	8241	8248	8254	1 2 3 4 5 6 7 8 9
67	8261	8267	8274	8281	8287	8293	8299	8305	8312	8319	1 2 3 4 5 6 7 8 9
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	1 2 3 4 5 6 7 8 9
69	8360	8365	8371	8376	8382	8387	8393	8399	8405	8411	1 2 3 4 5 6 7 8 9
70	8451	8457	8463	8470	8476	8482	8488	8494	8499	8506	1 2 3 4 5 6 7 8 9
71	8513	8519	8525	8531	8537	8543	8549	8556	8561	8567	1 2 3 4 5 6 7 8 9
72	8579	8585	8591	8597	8603	8609	8615	8621	8627	8633	1 2 3 4 5 6 7 8 9
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	1 2 3 4 5 6 7 8 9
74	8682	8698	8704	8710	8716	8722	8727	8733	8739	8745	1 2 3 4 5 6 7 8 9
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	1 2 3 4 5 6 7 8 9
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	1 2 3 4 5 6 7 8 9
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 2 3 4 5 6 7 8 9
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	1 2 3 4 5 6 7 8 9
79	8976	8982	8987	8993	8998	9003	9009	9015	9020	9025	1 2 3 4 5 6 7 8 9
80	9031	9036	9042	9047	9053	9059	9065	9070	9074	9079	1 2 3 4 5 6 7 8 9
81	9065	9079	9096	9101	9106	9112	9117	9122	9129	9135	1 2 3 4 5 6 7 8 9
82	9149	9164	9179	9194	9209	9224	9239	9254	9269	9284	1 2 3 4 5 6 7 8 9
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9236	1 2 3 4 5 6 7 8 9
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	1 2 3 4 5 6 7 8 9
85	9294	9309	9314	9319	9324	9329	9334	9339	9344	9349	1 2 3 4 5 6 7 8 9
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	1 2 3 4 5 6 7 8 9
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	1 2 3 4 5 6 7 8 9
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	1 2 3 4 5 6 7 8 9
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	1 2 3 4 5 6 7 8 9
90	9542	9547	9552	9557	9562	9567	9571	9576	9581	9586	1 2 3 4 5 6 7 8 9
91	9590	9595	9600	9605	9609	9614	9619	9624	9629	9633	1 2 3 4 5 6 7 8 9
92	9630	9643	9647	9652	9657	9661	9666	9671	9675	9680	1 2 3 4 5 6 7 8 9
93	9678	9683	9688	9693	9698	9703	9708	9713	9718	9723	1 2 3 4 5 6 7 8 9
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	1 2 3 4 5 6 7 8 9
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	1 2 3 4 5 6 7 8 9
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	1 2 3 4 5 6 7 8 9
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9906	1 2 3 4 5 6 7 8 9
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	1 2 3 4 5 6 7 8 9
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	1 2 3 4 5 6 7 8 9
	8	1	2	3	4	5	6	7	8	9	1 2 3 4 5 6 7 8 9

Figure 12.2: log tables

Log tables (fig 12.2) are a good example, in 1950 it was impossible to imagine an engineer unable to use them, they were absolutely necessary for any difficult calculation

The same happens with many other things like the following formulas

$$\cos(\theta_1 + \theta_2) = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2$$

$$\sin(\theta_1 + \theta_2) = \sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2$$

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

$$\cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2$$

$$\sin(\theta_1 - \theta_2) = \sin \theta_1 \cos \theta_2 - \cos \theta_1 \sin \theta_2$$

$$\tan(\theta_1 + \theta_2) = \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2}$$

$$\tan(\theta_1 - \theta_2) = \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2}$$

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}, \quad \cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad (12.8)$$

these formulas were extremely necessary to create tables with the values for trigonometric functions for many different angles. Nowadays, the dumbest cell phone can calculate trigonometric functions so there is no real need for these formulas in the every day work environment of any scientist or engineer.

Nevertheless, these formulas find an important application, they are of use to train the minds of young people who want to pursue STEM studies.

Intellectual training is absolutely fundamental for anyone, and if someone is interested in pursuing a STEM career, mathematics are a must. This is why we suggest in the strongest possible terms to study how the above formulas (or at least some of them) are shown to be true. We dare to suggest [Proof of angle addition formula for sine — Trigonometry — Khan Academy](#) as a good starting point

12.4 Values of basic the trigonometric functions for some angles

Mathematics is all about finding patterns and developing intuition about them. Trigonometry, being a branch of mathematics is no exception to this rule. In this section we will use some very simple reasoning to calculate the values of the trigonometric functions for some very particular angles. Let us see how far we can get with simple ideas without using any difficult stuff.

Since the sine and cosine of an angle are the signed lengths of the legs of a right triangle whose hypotenuse has unit length, the values of both functions must belong to the interval $[-1, 1]$, we may state this as a

Theorem 7. $\forall \theta,$

$$-1 \leq \sin\theta \leq 1, \quad -1 \leq \cos\theta \leq 1$$

Now, when θ is very close to 0 the leg of the triangle opposite to θ (the vertical orange segment in fig. 12.1) becomes very, very short while the horizontal blue leg becomes longer and longer almost reaching the point $(1, 0)$. In fact, when θ_0 the triangle completely collapses rendering the hypotenuse to coincide with the adjacent leg, this is just the statement

$$\sin(0) = 0, \quad \cos(0) = +1,$$

a similar reasoning leads to

$$\sin(\pi) = 0, \quad \cos(\pi) = -1,$$

Let us now think to what happens when the angle we are interested in gets very close to $\pi/2$ (a quarter of a circle). In this case the hypotenuse becomes almost vertical so its horizontal projection is close to disappear while the orange leg almost equals the hypotenuse, when θ

exactly matches $\pi/2$ we conclude

$$\sin\left(\frac{\pi}{2}\right) = +1, \quad \cos\left(\frac{\pi}{2}\right) = 0,$$

the case $\theta = 3\pi/2$ is handled similarly yielding

$$\sin\left(\frac{3\pi}{2}\right) = -1, \quad \cos\left(\frac{3\pi}{2}\right) = 0,$$

The $\theta = \pi/4$ (i.e. 45°) angle is treated a bit differently. All we must do is to realize that in this case the hypotenuse of the triangle is nothing but the diagonal of a square, expressed differently, both legs of the right triangle over the unitary circle are equal. This means that

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right),$$

the fundamental trigonometric identity for this case yields

$$\begin{aligned} 1 &= \sin^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{4}\right) = \\ &= 2 \sin^2\left(\frac{\pi}{4}\right), \end{aligned}$$

which is nothing but

$$2 \sin^2\left(\frac{\pi}{4}\right) = 1,$$

which in turn implies

$$\sin\left(\frac{\pi}{4}\right) = \pm \frac{\sqrt{2}}{2},$$

since $\pi/4$ is in the first quadrant, the sign must be $+$, and since the cosine of this angle equals its sine,

$$\sin\left(\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}.$$

There are still two notable angles $\pi/3$ (60°) and $\pi/6 = \frac{1}{2}\pi/3$ (30°), whose sines, cosines and tangents can be easily calculated by hand. To this end we draw an equilateral triangle², we call the lengths of its sides $2x$. Then we note that any of the bisector angles divides the opposite side in two segments of exactly the same length (x). Besides, the bisection builds two congruent right triangles which share a leg (the bisecting segment). The hypotenuses of these two triangles are nothing but two of the sides of original triangle.

Thinking of any of the two small right triangles it is easy to see that the leg adjacent to the 60° angle has length x . Due to the Pythagorean theorem and if we call s the length of the other leg,

$$s^2 + x^2 = 4x^2, \quad (12.9)$$

from which $s = \sqrt{3}x$, since this leg is opposite to the 60° angle, we end up with

$$\begin{aligned} \sin(\pi/3) &= \frac{\text{leg opposite to the } 60 \text{ degree angle}}{\text{hypotenuse}} = \frac{s}{2x} = \frac{\sqrt{3}x}{2x} = \frac{\sqrt{3}}{2}, \quad \text{and} \\ \cos(\pi/3) &= \frac{\text{leg adyacent to the } 60 \text{ degree angle}}{\text{hypotenuse}} = \frac{x}{2x} = \frac{1}{2}, \quad \text{finally} \\ \tan(\pi/3) &= \frac{\sin(\pi/3)}{\cos(\pi/3)} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3} \end{aligned} \quad (12.10)$$

Example 11. *Draw the necessary sketches needed to achieve the geometric reasoning given above*

Example 12. challenge *Carry out the necessary modifications to calculate $\sin(\pi/3)$, $\cos(\pi/3)$ and $\tan(\pi/3)$.*

Example 13. *Given $\sin(\pi/3) = \sqrt{3}/2$, use trigonometric identities to $\cos(\pi/3)$, $\sin(\pi/6)$ and $\cos(\pi/3)$. A way to do it is to recall that the 30 and 60 degrees are complementary,*

Example 14. *Calculate all the values shown in table 12.1*

²The internal angles of an equilateral triangle are all 60° angles

Angle ($^{\circ}$)	Angle (rad)	Quadrant	sin	cos	tan
0	0	I	0	+1	0
30	$\pi/6$	I-IV	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45	$\pi/4$	I	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60	$\pi/3$	I	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90	$\pi/2$	I-II	+1	0	$+\infty$
120	$4\pi/6$	II	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$
135	$3\pi/4$	II	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1
150	$5\pi/6$	II	$1/2$	$-\sqrt{3}/2$	-1
180	π	II-III	0	-1	0
210	$7\pi/6$	III	$-1/2$	$-\sqrt{3}/2$	$\sqrt{3}/3$
225	$5\pi/4$	III	$-\sqrt{2}/2$	$-\sqrt{2}/2$	+1
240	$4\pi/3$	III	$-\sqrt{3}/2$	$-1/2$	$+\sqrt{3}$
270	$3\pi/2$	III-IV	-1	0	$-\infty$
300	$10\pi/6$	IV	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$
315	$7\pi/4$	IV	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1
330	$7\pi/4$	IV	$-1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
360	2π	I-IV	0	+1	0

Table 12.1: Values of sine, cosine and tangent of some special angles. Angles showing two quadrants signal boundaries between quadrants

12.5 An equivalence relation

Angles are **periodic** i.e. different values of angles do in fact refer to the same point in the unitary circle, when this happens we say that the angles are **equivalent** which is expressed by

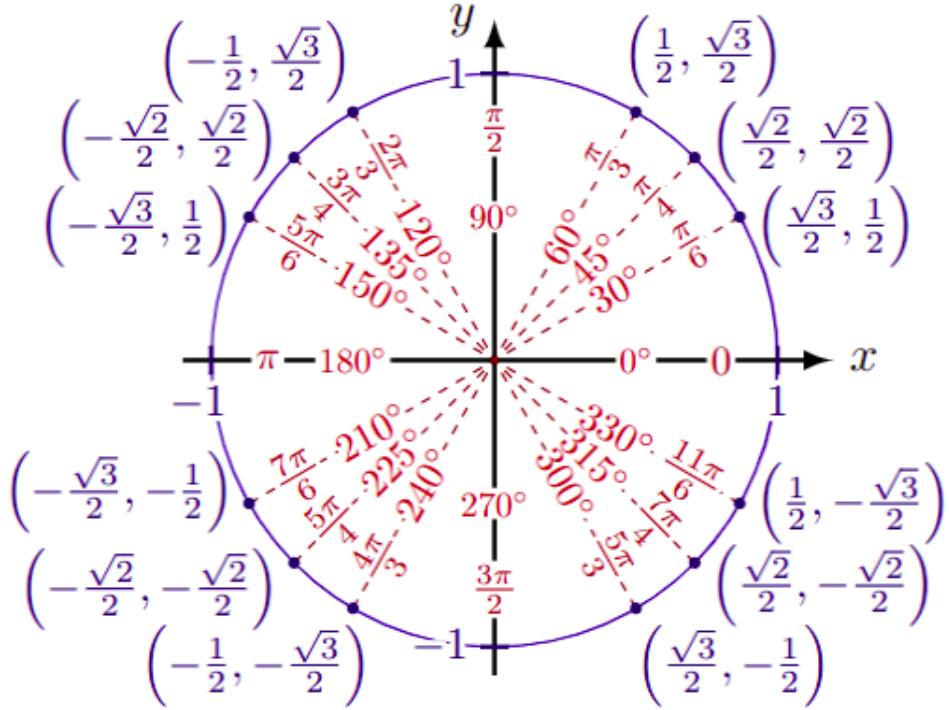


Figure 12.3: The unit circle and some special values

the symbol: \approx . Thus for instance, the angles 0 and 2π both define the angle that signals the point $(+1, 0)$ of the unit circle. In our new language we write $0 \approx 2\pi$, but, and this is extremely interesting, if we go around the unitary circle twice the corresponding angle³ is $2 \times 2\pi$ and we go back again to the point $(1, 0)$ so $0 \approx 2 \times 2\pi$.

In fact, the same happens with any number n of complete rotations which. After some thought we can generalize the idea and write

Definition 3. Two angles θ_1 and θ_2 are equivalent ($\theta_1 \approx \theta_2$) if and only if

$$\theta_2 - \theta_1 = 2k\pi, \quad k \in \mathbb{Z}$$

³The first 2 corresponding to the number of rotations around the circle

A couple of examples will help to understand this concept. Set a point in the unitary circle signaled by an angle $\theta_1 = 45^\circ$, if begin in this point and wind (counterclockwise) the circle 5 times we end up at exactly the same point, but the angle will be $\theta_2 = 45^\circ + 5 \times 360^\circ$. Measuring the angles in radians, we would say

$$\theta_1 = \pi/4, \quad \theta_2 = \pi/4 + 2(5)\pi,$$

therefore

$$\theta_2 - \theta_1 = 2(5)\pi$$

and since $5 \in \mathbb{Z}$ we can safely state

$$\theta_2 \approx \theta_1$$

which -as we already know- is just a fancy way to say that both angles mark exactly the same point in the unitary circle.

12.6 Trigonometric functions are periodic

The angular equivalence we have discussed in section 12.5 has deep geometrical roots and gives rise to a concept **periodicity of the trigonometric functions**, to introduce this concept let us carefully think about the sine function through its definition according to the unitary circle. Imagine two angles, let's say $\alpha_1 = \pi/6$ and $\alpha_2 = \pi/6 + 2\pi = 13\pi/6$. Certainly $\alpha_1 \approx \alpha_2$ (with winding $n = 1$) but the important thing here is that when we go to the unitary circle we certainly find that

$$\sin(\pi/6) = \sin(13\pi/6), \tag{12.11}$$

besides, more geometrical thinking will convince beyond any doubt that for any angle α and any $k \in \mathbb{Z}$

$$\sin(\alpha) = \sin(\alpha + n L), \tag{12.12}$$

where $L = 2\pi$ is known as the **period**. This concept is generalized to any function with real domain, and is given as⁴

Definition 4.

$$f : \mathbb{R} \rightarrow \mathbb{R},$$

is called periodic of period $L \in \mathbb{R}$ if and only if, $\forall x \in \mathbb{R}$

$$f(x) = f(x + L)$$

According to definition 4, all trigonometric functions are periodic with period $L = 2\pi$.

There is a little twist to this story but we will not push it in these notes.

12.7 Trigonometric equations

It is perfectly possible that when solving some mathematical problem we reach an equation (i.e. a question) such that

$$\sin(x) = \frac{1}{2}, \tag{12.13}$$

where we must find x .

According to the previous sections, we may be tempted to answer that x is either 30° or 150° but such answer is only partially correct because it does not really reflect all that we have learned, in particular the concept of periodicity.

For the 30° angle the correct answer should be: 30° or any other angle obtained from it by completely winding circles an arbitrary number of times. Something similar happens with the 150° angle.

In fact, the absolutely correct answer should be:

⁴The reader is urged to translate the definition to plain English

Equation eq. 12.13 has infinite number of solutions, and in radians they can be written as $x = \pi/6 = 2k\pi$ where $k \in \mathbb{Z}$, and $x = 5\pi/6 = 2m\pi$ where $m \in \mathbb{Z}$

In plainer English what we are saying is that the set \mathcal{S} of all solutions to eq. 12.13 is given by

$$\mathcal{S} = \{\dots, -23\pi/6, -11\pi/6, \pi/6, 13\pi/6, 25\pi/6, \dots\} \cup \{\dots, -7\pi/6, 5\pi/6, 17\pi/6, \dots\}. \quad (12.14)$$

There is an interesting modification to the original problem which is a **completely different** problem:

Find the solutions to

$$\sin(x) = \frac{1}{2}, \quad (12.15)$$

in the interval $[0, 2\pi)$. It happens that the apparently harmless modification of asking the solution to be in an specified interval, changes everything. Indeed, the solution of the new problem (the equation together with the domain where its solution is required) is a set with two elements only,

$$\mathcal{S} = \{\pi/6, 5\pi/6\}. \quad (12.16)$$

In fact, if we restrict the domain in which to look for the solution to be the set $[0, \pi/2]$, the solution becomes **unique** and equal to $x = \pi/6$.

At this point it is worth to comment that, **in mathematics the questions of existence and uniqueness of solutions to problems are of upmost importance.**

Let us try the following

Example 15. Determine if $x = 3\pi/8$ is a solution of the equation

$$\tan 2x = -1 \quad (12.17)$$

Example 16. Find all the solutions to the equation

$$4 \sin\theta + 1 = 2 \sin\theta \quad (12.18)$$

We begin by rewriting the equation as

$$\sin \theta = -\frac{1}{2} \quad (12.19)$$

There are two angles in the interval $[0, 2\pi)$ satisfying the equation, namely, $\theta = 7\pi/6$ (210°) and $\theta = 11\pi/6$ (330°) all the solutions to the original equation are found by adding to this angle an integer number of times 2π , i.e. the solutions are $\theta = 7\pi/6 + 2\kappa\pi, 11\pi/6 + 2\ell\pi$, $\kappa, \ell \in \mathbb{Z}$

Example 17. Solve

$$6 \cos^2 x - 3 = 0 \quad (12.20)$$

in the interval $[0, 2\pi)$

The equation can be cast as

$$\cos x = \pm \frac{\sqrt{2}}{2} \quad (12.21)$$

By mere inspection, there are clearly four solutions in the interval, 2 for each sign. indeed, $x_1 = \pi/4$ (first quadrant), and, $x_2 = 3\pi/2 + \pi/4$ (fourth quadrant) satisfy $\cos x_1 = \cos x_2 = \sqrt{2}/2$, while $x_3 = 3\pi/4$ (second quadrant) and $x_4 = 5\pi/4$ (third quadrant), satisfy $\cos x_2 = \cos x_3 = -\sqrt{2}/2$

Example 18. Solve

$$4 \sin^2 x = 9 \sin x - 5. \quad (12.22)$$

in the interval $[0, 2\pi)$

We begin by rewriting the equation as

$$4 \sin^2 x - 9 \sin x + 5 = 0, \quad (12.23)$$

the fact that $\sin x$ appears with two powers, 1 and 2 suggests⁵ that we introduce a change of variables by naming $u = \sin x$, in this way, the equation now looks as the ordinary quadratic

⁵This is a very old trick in anyone's toolbox

equation

$$4u^2 - 9u + 5 = 0, \quad (12.24)$$

with solutions

$$u = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases}, \quad (12.25)$$

which really stands for

$$\sin x = \frac{9 \pm \sqrt{81 - 4 \times 4 \times 5}}{8} = \frac{9 \pm \sqrt{81 - 80}}{8} = \frac{9 \pm 1}{8} = \begin{cases} 10/8 \\ 1 \end{cases}, \quad (12.26)$$

now, trigonometry has taught us that $-1 \leq \sin x \leq +1$, this condition rules out $\sin x = 8/10$ leaving us with $\sin x = +1$, which, along with the condition $x \in [0, 2\pi)$ means that the solution is unique, and given by: $x = \pi/2$

Example 19. *Solve*

$$2 \sin^2 x - \cos x - 1 = 0. \quad (12.27)$$

in the interval $[0, 2\pi)$

To approach this exercise we use the fundamental trigonometric identity to reach a first transformation of the left hand side

$$2 \sin^2 x - \cos x - 1 = 2[1 - \cos^2 x] - \cos x - 1 \quad (12.28)$$

after which we end up with the equation

$$\begin{aligned} 2[1 - \cos^2 x] - \cos x - 1 &= \\ -2\cos^2 x - \cos x + 1 &= 0 \end{aligned} \quad (12.29)$$

setting $z = \cos x$, we transform the equation into the standard second order equation

$$2z^2 + z - 1 = 0 \quad (12.30)$$

with solutions

$$z = \frac{-1 \pm \sqrt{1 - 4 \times 2 \times (-1)}}{4} = \frac{-1 \pm 3}{4} = \begin{cases} 1/2 \\ -1 \end{cases} . \quad (12.31)$$

From here

$$\cos x = \begin{cases} 1/2 \\ -1 \end{cases} . \quad (12.32)$$

In the interval under consideration there are three solutions to this equation. Indeed, $x = \pi$ is the one and only angle with $\cos x = -1$. On the other hand, there are two angles for which $\cos x = 1/2$, one in the first quadrant and one in the fourth, namely, $x = \pi/4 \text{ rad}$ and $x = 7\pi/4 \text{ rad}$.