Inductance and Magnetic Energy

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Outline

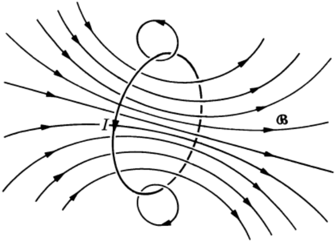


Figure: A circuit Idealized as a Loop to emphasize the self flux $\Phi_I(\mathbf{B})$

- Think of any circuit carrying a current I.
- Ampere's law guarantees that the current produces a magnetic field ($\mathbf{B}$) which, at each point, is proportional to I ($\mathbf{B} \propto I$).
- We may compute the magnetic flux through the circuit due to its own magnetic field, and call it the self flux (Φ_I).

- Φ_I is also proportional to the current so we may write

$$\Phi_I = L\,I$$

- The coefficient L depends on the geometrical shape of the conductor and is called the selfinductance of the circuit.
- L is measured in $Wb\,A^{-1}$, a unit called the henry, in honor of Joseph Henry, and abbreviated [1] H
- If the current changes in time there appears an emf in the loop given by

$$\mathcal{E} = -L\,\frac{dI}{dt} \tag{1}$$

[1] $H = Wb\,A^{-1} = m^2\,kg\,C^{-2}$

How does L work?

Let us apply the what we have just learned to figure2

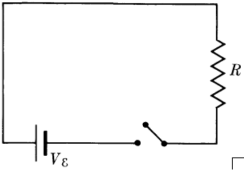


Figure: One resistor and one voltage source

After closing the switch must beginners would be tempted to write

$$RI = V_\mathcal{E} \tag{2}$$

Unfortunately, eq. 2 is forgetting the fact that there are time variations of the current. To take those into account, we must add the fem associated with the selfinductance of the circuit, namely, the equation must be modified to

$$\boxed{RI = V_{\mathcal{E}} - L\frac{dI}{dt}} \tag{3}$$

This is a differential equation describing the modified circuit of fig 3 below, showing that the current depends on time

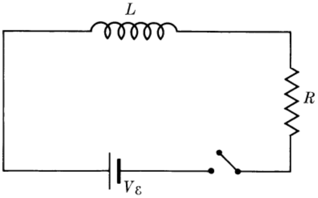


Figure: Incorporating the selfinductance

Eq. 3 is a first order separable ordinary differential equation, to solve it we reorder the terms as

$$I - \frac{V_{\mathcal{E}}}{R} = -\frac{L}{R}\frac{dI}{dt} \tag{4}$$

or

$$-\frac{L}{R}dt = \frac{dI}{I - \frac{V_{\mathcal{E}}}{R}} \tag{5}$$

at $t = 0$ there is no current $(I(t = 0) =)$ so we may integrate with limits

$$-\frac{L}{R}\int_0^t ds = \int_0^{I(t)} \frac{du}{u - \frac{V_{\mathcal{E}}}{R}} \tag{6}$$

Elementary integration yields

$$\ln(I(t) - \frac{V_{\mathcal{E}}}{R}) - \ln(-\frac{V_{\mathcal{E}}}{R}) = -\frac{R}{L}\,t \tag{7}$$

the final form of the current being:

$$\boxed{I(t) = \frac{V_{\mathcal{E}}}{R}\left(1 - e^{-\frac{R}{L}\,t}\right)} \tag{8}$$

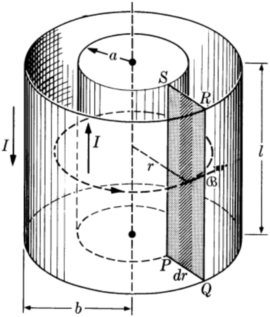
An interpretation is badly needed at this point.

$$\boxed{I(t) = \frac{V_{\mathcal{E}}}{R}\left(1 - e^{-\frac{R}{L}\,t}\right)} \tag{9}$$

At $t = 0$ the exponential term equals 1 so the current is zero as expected, from then on, the exponential term decays to zero with the parameter R/L giving a characteristic time of the decay.
As the exponential decays, the current increases slowly until for large times it approaches its asymptotic value $V_{\mathcal{E}}/R$
The selfinductance then gives the time scale for the current to reach its DC value, i.e. the inductance is responsible for the transient effects, in the limit $L \to 0$ the asymptotic state is reached in practically no time.

Calculating Inductance

It is interesting to discuss how inductance is -in principle- theoretically calculated. Let us try with the figure bellow which shows a circuit composed of two coaxial cylindrical metallic sheets of radii a and b, each carrying a current I, but in the opposite direction.



In order to calculate L we must find the flux of $\mathbf{B}$ through any section of the conductor, we will use section $PSRQ$. The field is easily calculated via Ampere's law and it is non zero in the region $a < r < b$ where it is found to be

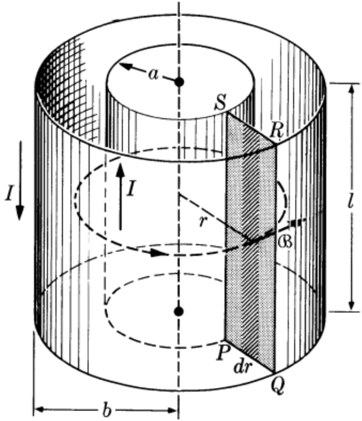
$$\mathbf{B} = \frac{\mu_0 \, I}{2\pi} \frac{\hat{\mathbf{e}}_\phi}{r}$$

We may think of the area of interest as composed of strips of length l and infinitesimal width dr and we may choose the orientation to be $\hat{\mathbf{n}} = \hat{\mathbf{e}}_\phi$, therefore

$$\Phi = \frac{\mu_0 I l}{2\pi} \int_a^b \frac{dr}{r} = \frac{\mu I l}{2\pi} \ln(b/a)$$

Therefore, the inductance per unit length of this configuration is

$$L = \frac{\mu_0}{2\pi} \ln(b/a)$$



Let us go back to our friend, the RL circuit,

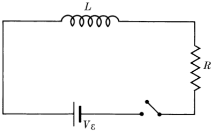


FIGURE: R-L Circuit with battery

And it's description

$$V_\mathcal{E} = RI + L\frac{dI}{dt}$$

which after multiplying by the current is

$$V_\mathcal{E} I = RI^2 + LI\frac{dI}{dt}$$

If we notice that

- $V_{\mathcal{E}} I \Rightarrow$ Power supplied by the battery
- $RI^2 \Rightarrow$ Power dissipated by the resistance. Is the energy spent in moving the electrons through the crystal lattice of the conductor and is transferred to the ions that make up the lattice (heating)

Then, it is completely natural to interpret $LI\frac{dI}{dt}$ as the energy required per unit time to build the current or to establish its associated magnetic field.

With the interpretation we have just given, the rate of increase of the magnetic energy is

$$\frac{dE_{\mathbf{B}}}{dt} = L\,I\,\frac{dI}{dt} = \frac{d}{dt}\left(\frac{LI^2}{2}\right)$$

This means that if the current builds from zero to I, the energy stored in the inductor will be

$$E_{\mathbf{B}} = \frac{1}{2}LI^2$$

This energy is -in fact- stored in the magnetic field

Just as happens with the electric field, the magnetic field stores energy and the energy density per unit volume is given by

$$u_{\mathbf{B}} = \frac{1}{2\mu_0}\mathbf{B}^2$$


LECTURE SUMMARY
- PHIESOPHENT LIKE YOU PHILSOMENS
NEVET ARE PHEDANT
- PEGROBES YOUR SACKEST FLUNT
- COOR INY ANGHANDAY THE
- COPFRY
PHILESHPERT

- Magnetic flux: $\Phi(\mathbf{B}) = L\, I$
- Voltage across an inductance $V = L\,\frac{dI}{dt}$
- Magnetic energy density $u_{\mathbf{B}} = \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}$