Maxwell's Equations

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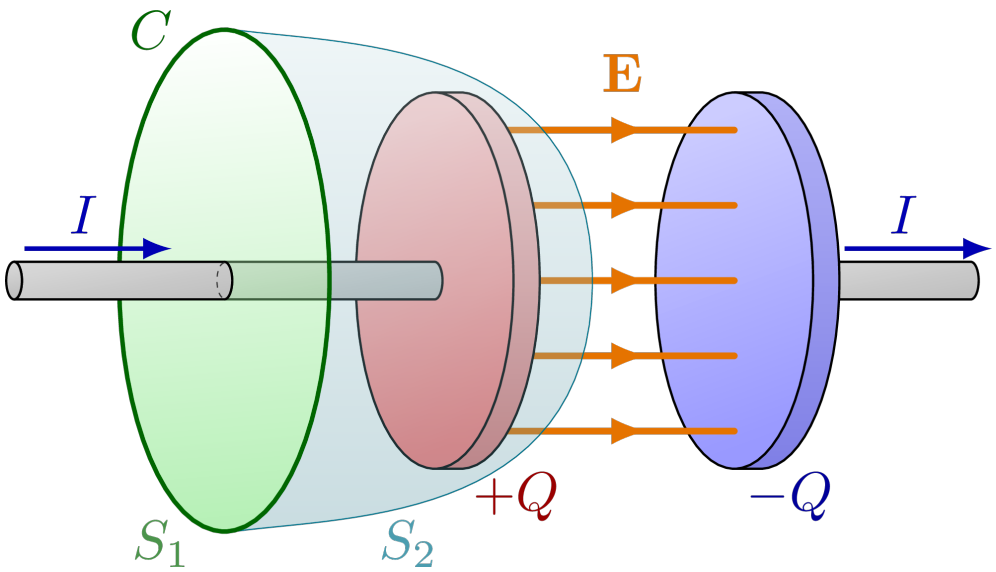
Our Current Understanding

We have arrived at a very interesting point. We are roughly in the same knowledge frame available to J. C. Maxwell in 1861. Our current knowledge adds up to: the principle of charge conservation, the Lorentz force formula and the following four field involving equations.

$$\begin{aligned}
\oint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dA &= \frac{Q_{enclosed}}{\epsilon_0}\\
\oint_{\mathcal{S}} \mathbf{B}\cdot\hat{\mathbf{n}}\,dA &= 0\\
\oint_{\partial\mathcal{S}} \mathbf{E}\cdot d\vec{\ell} &= -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot\hat{\mathbf{n}}\,dA\\
\oint_{\partial\mathcal{S}} \mathbf{B}\cdot d\vec{\ell} &= \mu_o\int_{\mathcal{S}} \mathbf{J}\cdot\hat{\mathbf{n}}\,dA = \mu_0 I
\end{aligned} \tag{1}$$

The Setup: A Charging Capacitor



Let us carefully consider the figure and note that C is the boundary of both S_1 the green flat disc, and S_2 the "bag like" light green surface.

S_1 is pierced by the cable and therefore by the current while no current traverses S_2

Before proceeding, it is crucial to recall that Ampère's Law (or the Ampère-Circuital Law) states:

$$\oint_{\partial \mathcal{S}} \mathbf{B} \cdot d\vec{\ell} = \mu_o \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA$$

$\mathcal{S}$ is any surface having $\partial \mathcal{S}$ as boundary

Applying this principle to our figure, don' t forget that C from the figure is boundary to S_1 and S_2, therefore

$$\oint_C \mathbf{B} \cdot d\vec{\ell} = \begin{cases} \mu_0 I & \text{current passing through the green disk} S_1 \\ 0 & \text{no current passing through the bag like surface} S_2 \end{cases} \tag{2}$$

For centuries, the bedrock of Western thought has been the Transitive Property:
If $A = B$ **and** $B = C$**, then** $A = C$**.**
A sacred, undeniable truth!

Yet, in our realm of electromagnetic fields, we face a horrifying paradox that threatens to make the very walls of our human intellect crumble around us!

$\oint_C \mathbf{B} \cdot d\vec{\ell} = \mu_0 I$ (from surface S_1)

$\oint_C \mathbf{B} \cdot d\vec{\ell} = 0$ (from surface S_2)

...implying $\mu_0 I = 0$, even when current I is flowing!

"The universe has gone mad! What's next, circular squares?"

SYMMETRY!, Maxwell Saves The Day

$$\oint_{\partial S} \mathbf{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int_{S} \mathbf{B} \cdot \hat{\mathbf{n}}\, dA$$

$$\oint_{\partial S} \mathbf{B} \cdot d\vec{\ell} = \mu_o I [+ \textit{missing?}]$$

Ampere-Maxwell' s Law

Maxwell's Brilliant Idea

Try

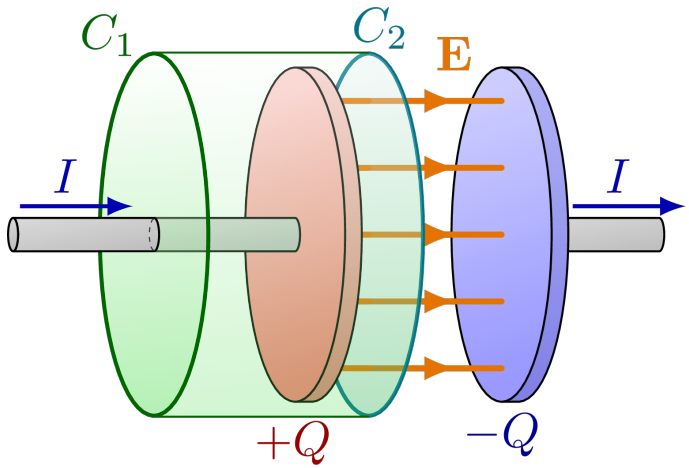
$$\oint_{\partial \mathcal{S}} \mathbf{B} \cdot d\vec{\ell} = \mu_o \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA + \alpha \frac{d}{dt} \int_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA$$

and try to find α

Determining the Constant α

Considering a cylindrical surface that begins with an open end with C_1 as boundary and ends on the disk of boundary C_2 one finds

$$\oint_{C_1} \mathbf{B} \cdot d\vec{\ell} = \mu_o \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA + \alpha \frac{d}{dt} \int_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA$$



Now:

- $\int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA = 0$ because no physical current pierces $\mathcal{S}$
- $\int_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = \int_{Disk} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA$, where *Disk* is the disk based on C_2 shown in the figure
- We know that the electric field between the plate of the capacitor has magnitude $E = \sigma/\epsilon_0$
- Since the radius of the capacitor plate is smaller than the radius of the disk (whose area is A):

$$\int_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = \int_{Disk} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = \frac{\sigma A}{\epsilon_0} = \frac{Q}{\epsilon_0}$$

We then conclude that

$$\oint_{C_1} \mathbf{B} \cdot d\vec{\ell} = \alpha \frac{d}{dt}\left(\frac{Q}{\epsilon_0}\right) = \alpha \frac{I}{\epsilon_0}$$

$$\oint_{C_1} \mathbf{B} \cdot d\vec{\ell} = \alpha \frac{I}{\epsilon_0}$$

Choosing $\alpha = \mu_0 \epsilon_0$ the inconsistency found in equation 2 is **apparently resolved**

Maxwell's Equation's

Representing the summit of the monumental unifying work of J. C. Maxwell in electromagnetism, these four equations are indeed mathematically consistent!,

$$\begin{aligned}
\oint_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dA &= \frac{Q_{enclosed}}{\epsilon_0}\\
\oint_{\mathcal{S}} \mathbf{B}\cdot\hat{\mathbf{n}}\,dA &= 0\\
\oint_{\partial\mathcal{S}} \mathbf{E}\cdot d\vec{\ell} &= -\frac{d}{dt}\int_{\mathcal{S}} \mathbf{B}\cdot\hat{\mathbf{n}}\,dA\\
\oint_{\partial\mathcal{S}} \mathbf{B}\cdot d\vec{\ell} &= \mu_o\int_{\mathcal{S}} \mathbf{J}\cdot\hat{\mathbf{n}}\,dA + \mu_0\epsilon_0\frac{d}{dt}\int_{\mathcal{S}} \mathbf{E}\cdot\hat{\mathbf{n}}\,dA
\end{aligned} \tag{3}$$

but ... What about physics?

What is a Physical Theory?

The journey to complete the consistency of Maxwell's equations might seem a mathematical *tour de force* from the powerful mind of Maxwell. But that is not the case.

Physics is not juggling with fancy math, it's about understanding the fundamental laws of nature

A physical theory serves as the mathematical representation of an experimental framework. It must accurately describe all known experimental data. But its true power, and a hallmark of its validity, is its ability to predict novel phenomena that can be experimentally verified, expanding our understanding of nature.

To powerfully illustrate predictivity, consider one of science's most stunning triumphs:

Newton's Theory of Universal Gravitation.

Anomalies in Uranus's observed orbit, a stubborn puzzle that led many to question Newton's very laws, instead fortified the conviction of John Couch Adams and Urbain Le Verrier in its correctness. Both scientists embarked on a real intellectual Odyssey. Indeed, they independently calculated the precise location of a hidden gravitational influence – a new planet – that the theory implicitly demanded.

And lo and behold, in a testament to the theory's predictive might, Neptune was triumphantly discovered on September 23, 1846, exactly where it was predicted to be.

The Experimental Foundations: Faraday's Legacy

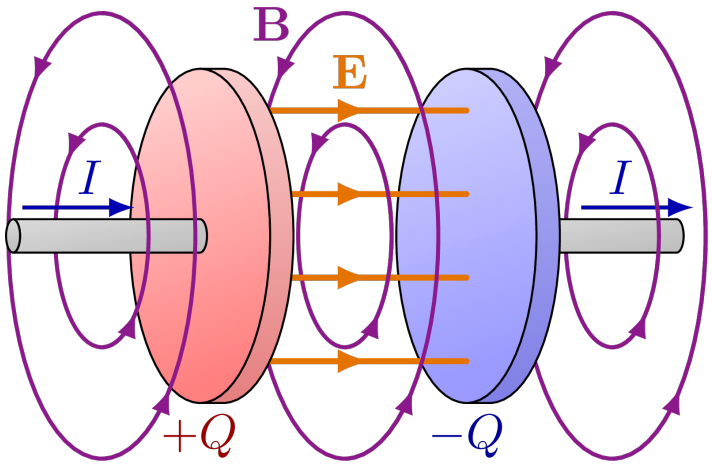
The vast experimental evidence for electric and magnetic phenomena, much of it painstakingly gathered and brilliantly demonstrated by giants like **Michael Faraday**, formed the known framework of electromagnetism.

Remarkably, Maxwell's equations, with profound accuracy, accounted for and explained every single one of these observations known at the time of their inception.

To crown Maxwell's work as a full theory in the strongest sense, bold predictions were still required. And, in a stroke of genius, Maxwell himself brought them to light.

A First Prediction: The Inseparable Dance

As soon as the displacement current term $\mu_0\epsilon_0 \frac{d}{dt}\int_S \mathbf{E}\cdot\hat{\mathbf{n}}\,dA$ was introduced to restore the consistency of the equations, Maxwell derived a crucial prediction:
Whenever time-varying fields are at stake, there must be both electric and magnetic fields. There is nothing like a standalone time-varying electric field (or magnetic field) in isolation.



A Secret Revealed: Maxwell's Electromagnetic Waves

Electromagnetic Waves

In 1864 Maxwell himself proposed that disturbances in electric and magnetic fields could propagate through space as waves, traveling at the speed of light [Physical prediction from the theory].

Heinrich Hertz

In 1887, proved the actual existence of electromagnetic waves in a series of experiments.Hertz demonstrated that transverse electromagnetic waves could propagate through free space.

This groundbreaking discovery directly paved the way for the development of radio in the closing years of the 19th century.

A Consistent Universe: Charge Conservation!

Up to this point we have considered charge conservation as an independent postulate, might Maxwell's equation tell us something about it?
To answer the question, consider a region of space where we may find charges and electromagnetic fields, and let us examine them under the light of Ampere-Maxwell law.

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\vec{\ell} = \mu_o \int_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA + \mu_0 \epsilon_0 \frac{d}{dt} \int_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA$$

Where $\mathcal{S}$ is a surface in the region whose boundary $\mathcal{C}$ is a tiny closed curve

If $\mathcal{C}$ collapses to a point,

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\vec{\ell} = 0\,,$$

and $\mathcal{S}$ will transform into a closed surface with oriented normal. Besides, the following formula will hold:

$$\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA + \epsilon_0 \frac{d}{dt} \oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = 0$$

Remember Gauss!

Gauss's Law states that the electric flux through a closed surface $\mathcal{S}$ is proportional to the enclosed charge:

$$\oint_{\mathcal{S}} \mathbf{E} \cdot \hat{\mathbf{n}}\, dA = Q_{enclosed}$$

Now, let's substitute this into our result from the Ampère-Maxwell Law (when $\mathcal{C}$ collapses and $\mathcal{S}$ becomes a closed surface):

$$\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA + \epsilon_0 \frac{d}{dt}\left(Q_{enclosed}\right) = 0$$

Rearranging terms, we get:

$$\frac{dQ_{enclosed}}{dt} = -\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA$$

Recognizing that

$$\oint_{\mathcal{S}} \mathbf{J} \cdot \hat{\mathbf{n}}\, dA$$

represents the total current (I) flowing out of the closed surface $\mathcal{S}$, we arrive at:

$$\boxed{\frac{dQ_{enclosed}}{dt} = -I}$$

This is the precise mathematical statement of the fundamental Principle of Charge Conservation

Conclusion

Charge Conservation is inherently encoded in Maxwell's Equations.

A Glance at Things to Come

The constants

$$\begin{aligned} \epsilon_0 &= 8.8542 \times 10^{-12}\, C^2\, N^{-1}\, m^{-2} \\ \mu_0 &= 4\pi \times 10^{-7}\, N\, A^{-2} \end{aligned} \tag{4}$$

Have been conspicuous all along our journey through the electromagnetic field, as we approach the end of our exploration,it is interesting to write down the following thought provoking intriguing relationship between them and the speed of light (c)

$$\boxed{1/\sqrt{\epsilon_0 \mu_0} = 2.9979 \times 10^8\, m\, s^{-1}} \tag{5}$$


LECTURE SUMMARY
- PHIESOPHENT LIKE YOU PHILSOMENS NEVET ARE PHEDANT
- PEGROBES YOUR SACKEST FLUNT
- COOR INY ANGHANDAY THE
- COPFRY
PHILESHPERT

Not Exactly A Summary

- Ampere-Maxwell Law assures the consistency of Maxwell's equations
- Electromagnetic Energy Density Formula

$$u = \frac{\epsilon_0}{2}\mathbf{E}\cdot\mathbf{E} + \frac{1}{2\mu_0}\mathbf{B}\cdot\mathbf{B}$$

- Along with Lorentz force formula, Maxwell's Equations Describe all -but quantum-Electromagnetic Phenomena