Gauss Law

Mario I. Caicedo

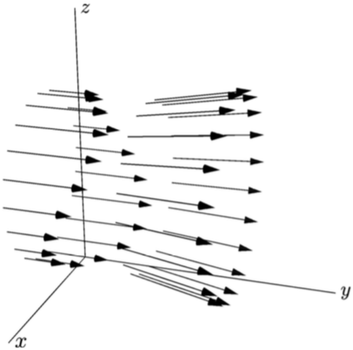
July 23, 2025

Carl Friedrich Gauss [April 30, 1777-February 23, 1855]

Extraordinary German mathematician, physicist, and astronomer, often hailed as the "Prince of Mathematicians."

Many concepts are named after him, including the Gaussian distribution (or normal distribution), Gaussian integrals, Gaussian elimination in linear algebra, and the Gaussian plane in complex analysis.

Vector Fields As Flows



We may think of a vector field as representing the flow of some liquid

What is Flux?

Imagine a river flowing. If you hold a fishing net in the water, how much water passes **through** the net?

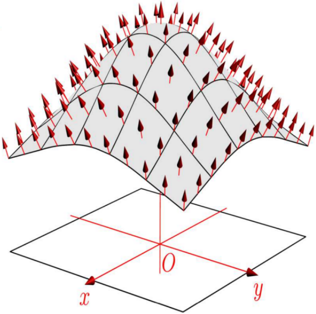
- Stronger current, lots of water.
- Bigger net more water.
- How you **hold** the net (facing flow = most, sideways = none, tilted = some).

Flux in physics is just like that! It's a measure of "how much" of a **vector field** (like an electric field) passes *through* a given surface.

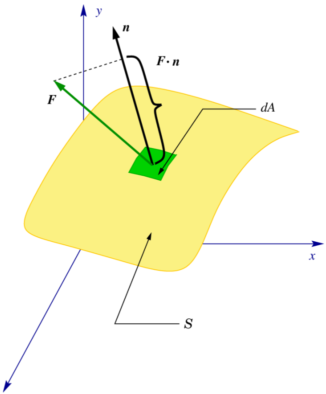
Quantifying Flow

A vector field (liquid) flowing through a surface(net). To quantify flow we might ask: how many arrows are piercing the surface?

In some -concrete- cases we might perform an experiment, but, how do we do it with a mathematical entity like a vector field?

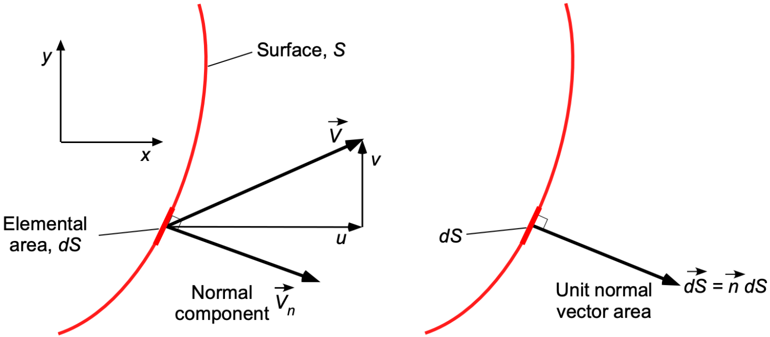


The Concept and Its Visualization



1. Divide the surface into small pieces with local normal unit vectors ($\hat{\mathbf{n}}$).
2. **Only** the component of the vector field perpendicular to the surface flows!
3. Define the flux through one piece of area dA and normal $\hat{\mathbf{n}}$ as $d\Phi_S(\mathbf{F}) = \mathbf{F}.\hat{\mathbf{n}}dA$

A vector field $\mathbf{V}$ of cartesian components $\mathbf{u}$ and $\mathbf{v}$ pierces the surface. It is only the normal component $\mathbf{V}_n$ (neither $\mathbf{u}$ nor $\mathbf{v}$) of the vector field, what actually goes through the surface. The tangential component $(\mathbf{V}_t)$ just trickles down the surface

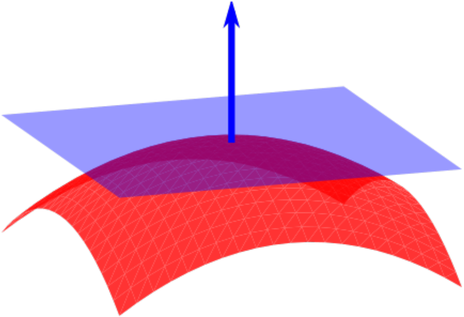


Flux: Some Math Behind the "Flow"

To define flux it is essential to know a basic mathematical result.

Theorem

At every point of any sufficiently smooth surface there is always a local unit normal vector $\hat{\mathbf{n}}$.



The Flux of The Electric Field

Take the elements we have been discussing, the electric field $\mathbf{E}$ a surface $\mathcal{S}$ and its local area vector $d\mathbf{A}$.
$d\mathbf{A}$ is the local normal to $\mathcal{S}$ with size equal to an infinitesimal element of area. We may want to write

$$d\mathbf{A} = \hat{\mathbf{n}}\, dA$$

where $\hat{\mathbf{n}}$ is the local normal of unit length. Then

Definition

The element of electric flux through $d\mathbf{A}$ is given by

$$d\Phi_A(\mathbf{E}) = \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \cdot \hat{\mathbf{n}} dA$$

Definition

The **total flux** *of Electric through a surface $\mathcal{S}$, is found adding up all these tiny contributions:*

$$\Phi_{\mathcal{S}}(\mathbf{E}) = \int_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A}$$

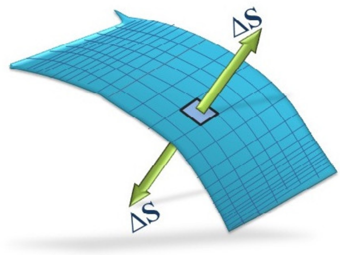
Two remarks are in order

Remark

When you do the dot product $(\mathbf{E} \cdot d\mathbf{A})$, you're finding the part of the electric field that goes **straight through** the surface.

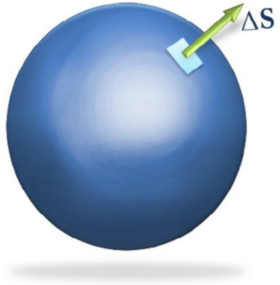
Remark

There are two cases to consider, the surface is open like a sheet, or it is closed as a balloon



Open Surface

(Any one of the *two normals* can be taken as the *outward normal*)



Closed Surface

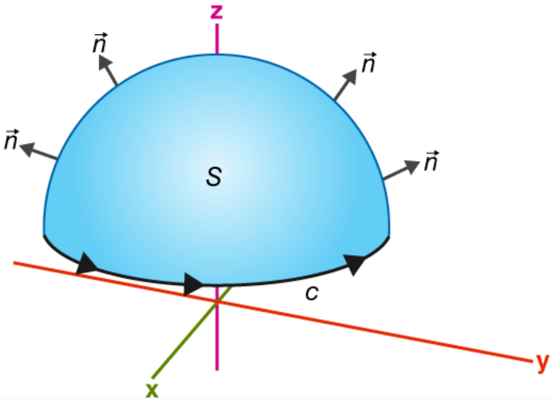
(Only *one* possible *outward normal*)

Important Note: Orientation Matters!

The concept of flux depends on being able to consistently choose **one side** of the surface (for a closed surface, this choice defines an unambiguous "in" or "out," like for a balloon). For certain special surfaces, called **non-orientable surfaces** (such as a Möbius strip, or the more complex and celebrated Klein bottle), you simply cannot make this consistent choice across the entire surface. If you try to define a normal vector and trace it around, it will flip its direction relative to your starting point! Because of this fundamental lack of consistent orientation, calculating a meaningful total flux through such surfaces is impossible.

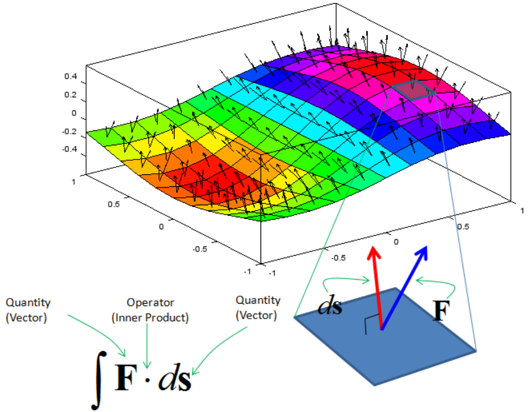
Remark

In the case of an open surface, the orientation -when it exists- is defined by the orientation of the base (boundary) curve and the right hand rule.



Actually Calculating Flux

The surface is divided in infinitesima sectors on which the element of flux is calculated and then a sum over all surface elements (integration) is found



What Does Flux Tell Us?

The sign of the flux tells us about the field's direction relative to the surface's chosen orientation:

- If field lines are coming **out** of the surface (in the direction of the chosen normal), you'll have **positive flux**.
- If field lines are going **into** the surface (opposite to the chosen normal), you'll have **negative flux**.
- If field lines are just skimming **along** the surface (parallel to it), or if there's no field at all, you'll have **zero flux**.

This applies to **both open and closed surfaces**, as long as you've established a clear "outward" direction for your surface.

Gauss's Law: A Physics Superstar!

Now for one of the absolute superstars of electromagnetism: **Gauss's Law**!

This law is a super elegant and incredibly powerful principle that connects **electric fields** directly to the **charges** that create them.

It's especially useful for finding electric fields when things have a lot of **symmetry** (like spheres, cylinders, or planes!).

Gauss's Statement

Gauss's Law

The total electric flux $\Phi_{\mathcal{S}}(\mathbf{E})$ passing *out* of any **closed surface** $\mathcal{S}$ is directly proportional to the total electric charge (Q_{enclosed}) *inside* that closed surface.

Here it is in its full mathematical glory:

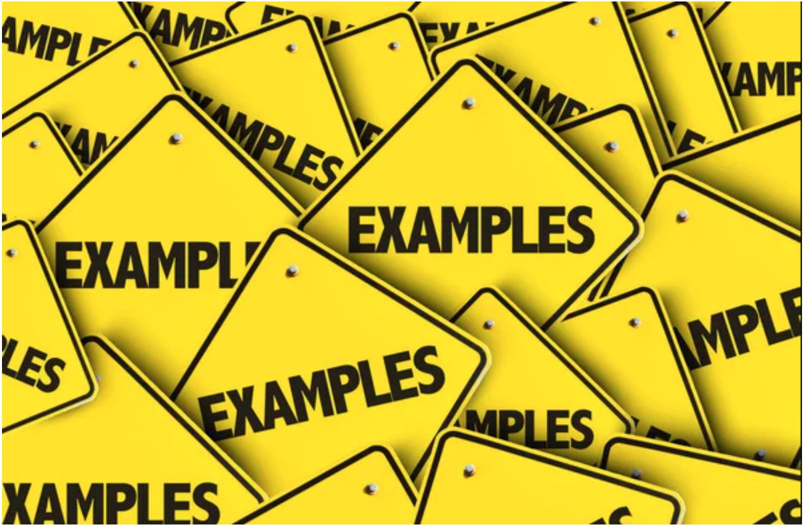
$$\Phi_{\mathcal{S}}(\mathbf{E}) = \oint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$$

- The little circle on the integral sign $(\oint)$ just reminds us that we're integrating over a **closed surface**.
- ϵ_0 (epsilon-nought) is a fundamental physicsl constant: the **permittivity of free space**.

A Quick Note on Gauss's Law

We won't be diving into a full *proof* of Gauss's Law in this lecture. Instead we refer to the following video

- If you do not want to have a look to the beautiful proof, you may take the position of accepting Gauss Law as a fundamental truth, like Newton's Laws.
- Our goal is to learn how to use its amazing power to solve problems!


EXAMPLES
EXAMPLES
EXAMPLES
EXAMPLES

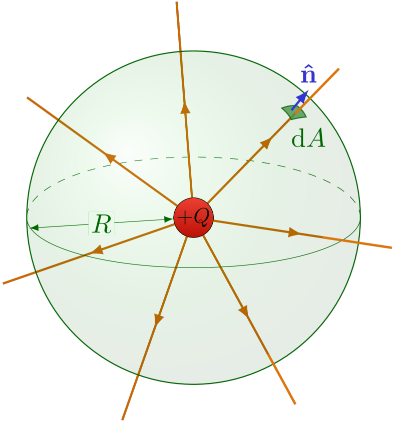
The electric field of the particle at any point is[a]

$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{e}}_r$$

where $\hat{\mathbf{e}}_r$ is a vector along the radius vector that joins the field point and the particle.
Given the geometry,

$$\hat{\mathbf{n}}dA = \hat{\mathbf{e}}_r\, dA$$

[a]The main purpose of this example is to show calculational technique. By the way, the result here is essentially what is used as tool for proving Gauss' s law



Therefore, the flux of **E** through the tiny piece of sphere is

$$d_{d\mathbf{A}}\Phi(\mathbf{E}) = \left[\frac{q}{4\pi\epsilon_0 R^2}\hat{\mathbf{e}}_r\right]\cdot\hat{\mathbf{e}}_r\, dA = \frac{q}{4\pi\epsilon_0 R^2}1\, dA = \frac{q}{4\pi\epsilon_0 R^2}\, dA$$

Summing over all tiny areas of the sphere gives

$$\begin{aligned}\Phi(\mathbf{E}) &= \frac{q}{4\pi\epsilon_0 R^2}\oint_{Sphere} dA = \\ &= \frac{q}{4\pi\epsilon_0 R^2}\times \text{Area}(Sphere) = \frac{q}{4\pi\epsilon_0 R^2}\times[4\pi R^2]\end{aligned}$$

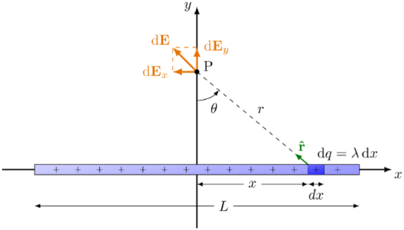
which is just

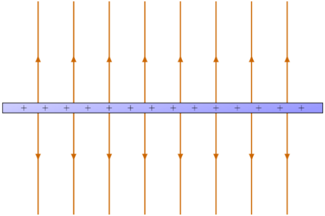
$$\boxed{\Phi(\mathbf{E}) = \frac{q}{\epsilon_0}} \tag{1}$$

Result to be expected due to Gauss's law

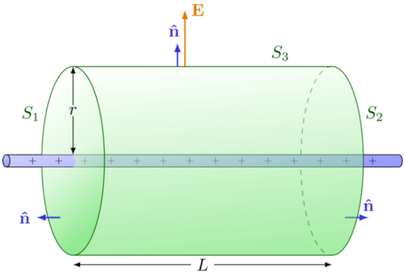
We used Coulombs law and integration to show that -using cylindrical coordinates- the electric field caused by this charge distribution is

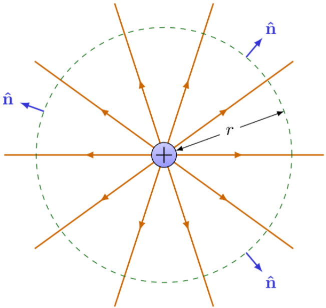
$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0\rho}\,\hat{\mathbf{e}}_\rho$$

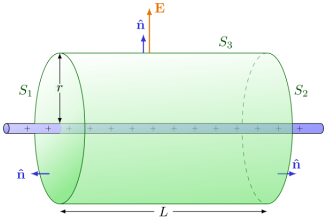




Given the symmetry, we choose a cylindrical surface coaxial with the rod







The electric field is radial ($\mathbf{E} = E(\rho)\,\hat{\mathbf{e}}_\rho$, in cylindrical coordinates). It's therefore perpendicular to the normals to the caps ($\hat{\mathbf{n}}_{caps} = \pm\hat{\mathbf{e}}_z$), so there's no flux contribution from those:

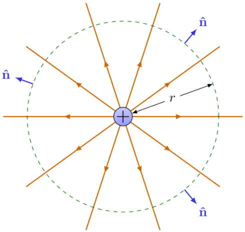
$$\hat{\mathbf{e}}_\rho \cdot [\pm\hat{\mathbf{e}}_z] = 0.$$

All that remains is the contribution from the side.

The contribution from the side is

$$\Phi_{Side}(\mathbf{E}) = \int_{Side} E(\rho)|_{\rho=r}\hat{\mathbf{e}}_\rho . [\hat{\mathbf{e}}_\rho dA] \,,$$

$$\Phi_{Side}(\mathbf{E}) = \int_{Side} E(r)\, dA = E(r) \int_{Side} dA = 2\pi r\, E(r)\, L \qquad (2)$$



We are almost done, all that is missing is the charge enclosed by the cylinder, which is given by

$$Q_{in} = \lambda L.$$

According to the right hand side of formula 2 and Gauss's law

$$2\pi r\, E(r)\, L = \frac{\text{Charge enclosed by the cylinder}}{\epsilon_o}$$

implying

$$E(r) = \frac{\lambda}{2\pi\, \epsilon_o\, r}\,,$$

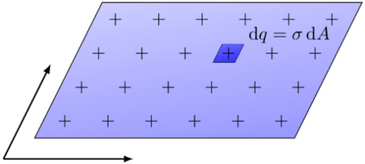
so

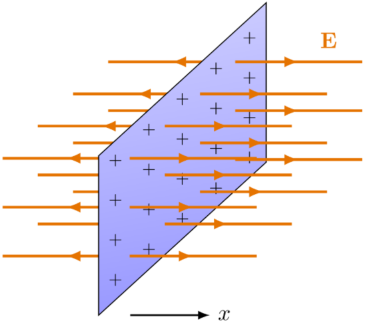
$$\mathbf{E} = \frac{\lambda}{2\pi\, \epsilon_o\, \rho}\, \hat{\mathbf{e}}_\rho$$

as we had already found with considerable more effort.

In this case, symmetry alone teaches us that the electric field must be perpendicular to the plane and dependent, at most, on the distance to the plane,

$$\mathbf{E} = E(x)\,\hat{\mathbf{e}}_x$$



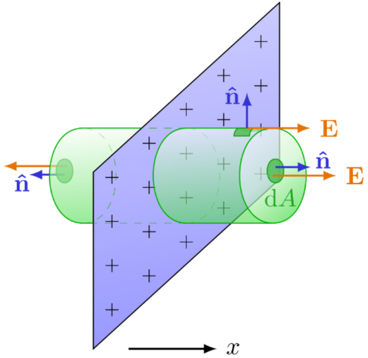


The Gaussian surface must be symmetric with respect to the plane. I chose a cylinder. In this case the side does not contribute to the flux and one gets

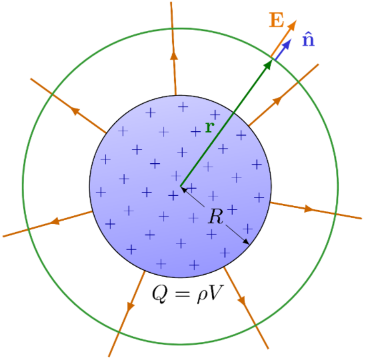
$$\Phi_E = 2E(x)\,A$$

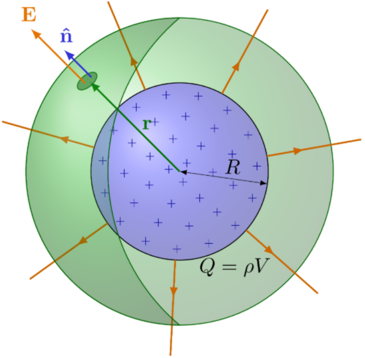
A being the transverse area of the cylinder [try a parallelogram]. Now, $q_{enclosed} = \sigma A$ Gauss's law then implies

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\mathbf{e}_x$$



This is an interesting example, there are two different regions of interest. Completely out of the sphere and inside it





Using symmetry and setting the origin of coordinates at the center of the sphere the electric field must have the form

$$\mathbf{E} = E(r)\,\hat{\mathbf{e}}_r\,, \tag{3}$$

Choosing a concentric spherical surface for gaussian should now be natural. In that case, $\hat{\mathbf{n}}\,dA = \hat{\mathbf{e}}_r\,dA$ and

$$\begin{aligned}\Phi_{Sphere}(\mathbf{E}) &= \oint_{Sphere} \mathbf{E}\cdot\hat{\mathbf{n}}\,dA = \oint_{Sphere} [E(r)\hat{\mathbf{e}}_r]\cdot[\hat{\mathbf{e}}_r]\,dA = \\ &= E(r)\hat{\mathbf{e}}_r]\cdot[\hat{\mathbf{e}}_r]\,A = \\ &= 4\pi r^2\,E(r)\end{aligned} \tag{4}$$

where r is the radius of the chosen Gaussian surface

We must ask, how much charge is enclosed by the Gaussian?. THe answer is not difficult, because the volume charge density (ρ) is uniformly distributed (i.e. constant). Therefore

$$Q_{enclosed} = \begin{cases} Q & r > R \\ \frac{4}{3}\pi\,\rho\,r^3 & r < R \end{cases} \tag{5}$$

Gauss' s Law then implies

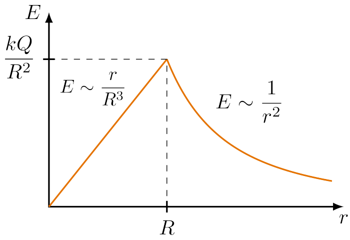
$$4\pi r^2\,E(r) = \frac{1}{\epsilon_0}\begin{cases} Q & r > R \\ \frac{4}{3}\pi\,\rho\,r^3 & r < R \end{cases} \tag{6}$$

$$4\pi r^2\, E(r) = \frac{1}{\epsilon_0}\begin{cases} Q & r > R \\ \frac{4}{3}\pi\,\rho\, r^3 & r < R \end{cases} \tag{7}$$

Finally

$$E(r) = \begin{cases} \frac{Q}{4\pi\epsilon_0 r^2} & r > R \\ \frac{\rho\, r}{3\epsilon_0} & r < R \end{cases} \tag{8}$$

with $\rho = \frac{3Q}{4\pi R^3}$



As a last example we will deduce the Coulomb's field from Gauss's law.
If we consider a pointlike charge the symmetry forces the electric field to be sphericallly symmetric, i.e. to have the form

$$\mathbf{E} = E(r)\,\hat{\mathbf{e}}_r$$

for a spherical gaussian surface of radius R centered at the charge $\hat{\mathbf{n}}\,dA = \hat{\mathbf{e}}_r\,dA$, therefore

$$\Phi_{Sphere}(\mathbf{E}) = 4\pi R^2\,E(R) = \frac{1}{\epsilon_0}$$

which implies:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{e}}_r}{r^2}$$


LESSONS
LEARNED
TODAY!

1. Gauss's Law: Enormous physical significance
2. Gausss Law$\Longleftrightarrow$Divergence (Gauss) Theorem
3. Gauss's Law tells us there are sources of electric field
4. Symmetry simplifies. Gauss's law may become a computational tool (spheres, cylinders, planes)
5. Measuring the World (Die Vermessung der Welt). Film, streaming Amazon Prime Video, Apple TV, Google Play, Plex, and MUBI, among others


ANY
QUESTIONS
?