Electromagnetic Waves

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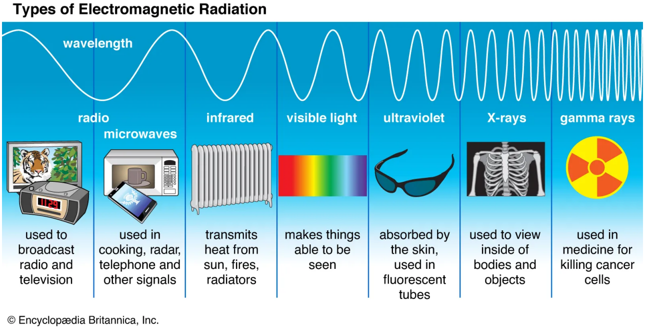
When we introduced Maxwell's equations we enthusiastically mentioned that one of the great triumphs of the Scottish scientist was his prediction of electromagnetic waves, proved to be a physical reality by the brilliant experiments of the German physicist Hertz.



- The introduction of the displacement current, taught us that the field configuration within the volume of the capacitor was such that the electric and magnetic fields were perpendicular.
- We have found the same situation ($\mathbf{E} \perp \mathbf{B}$) in other highly symmetrical situations.
- Our next topic of interest is the existence and properties of electromagnetic waves. For this endeavor, based on our previous experience, we will assume that the time-varying electric and magnetic fields are orthogonal to each other and to the direction of propagation.

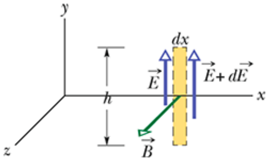


Figure: An infinitesimally wide rectangle through which a magnetic field is perpendicularly flowing. The boundary of the rectangle is $\mathcal{C}$

Let us carefully calculate the circulation of the electric field along $\mathcal{C}$, to that goal

- We will orient $\mathcal{C}$ as to make sure the normal to the yellow rectangle points along the positive z axis just as the magnetic induction does.
- Accordingly $\mathcal{C}$ is made up of four segments, two parallel to the x axis, the left vertical segment pointing in the negative y direction and the right one pointing in the positive y direction.

The electric field points along the positive y axis, $\mathbf{E} = \mathcal{E}\,\hat{\mathbf{e}}_y$. This electric field configuration implies that the horizontal segments of $\mathcal{C}$ do not contribute to the circulation, the calculation goes as follows

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{E}\cdot d\vec{\ell} &= \int [\mathcal{E} + d\mathcal{E}]\,\hat{\mathbf{e}}_y \cdot dy\,\hat{\mathbf{e}}_y + \int \mathcal{E}\,\hat{\mathbf{e}}_y \cdot (-dy\,\hat{\mathbf{e}}_y) = \\ &= [\mathcal{E} + d\mathcal{E}]\,h - \mathcal{E}\,h = hd\mathcal{E} = \frac{\partial \mathcal{E}}{\partial x}\,dx\end{aligned} \tag{1}$$

The flux of the magnetic induction through the yellow strip is quite easy to calculate, indeed

$$\Phi(\mathbf{B}) = \int_{strip} \mathcal{B}\,\hat{\mathbf{e}}_z \cdot \hat{\mathbf{e}}_z\, dA = \mathcal{B}\,A = \mathcal{B}\,h\,dx$$

Therefore

$$\frac{\partial \Phi(\mathbf{B})}{dt} = h\,dx\,\frac{\partial B}{\partial t} \tag{2}$$

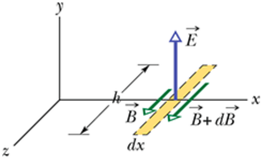


Figure: A rectangle, this time in the $z-x$ plane, showing the flux of electric field. Note the orientation of the boundary $\mathcal{C}'$

Taking care of the orientation of $\mathcal{C}'$ to make the normal to the strip point along the positive y action, allows to get

$$\oint_{\mathcal{C}'} \mathbf{B}\cdot d\vec{\ell} = -\left[\mathcal{B}+d\mathcal{B}\right]h + \mathcal{B}\,h = -h\,d\mathcal{B} = -h\,\frac{\partial \mathcal{B}}{\partial x}\,dx \qquad (3)$$

$$\Phi(\mathbf{E}) = \mathcal{E}(h\,dx) \quad \text{so:}\quad \frac{d\Phi(\mathbf{E})}{dt} = h\,dx\,\frac{\partial E}{\partial t} \qquad (4)$$

Equations 1, 2 and Faraday's Law imply

$$\frac{\partial \mathcal{E}}{\partial x} + \frac{\partial \mathcal{B}}{\partial t} = 0 \tag{5}$$

Similarly equations 3 and 4 under the light of Ampere-Maxwell's Law yield

$$\frac{\partial \mathcal{B}}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial \mathcal{E}}{\partial t} \tag{6}$$

Let us collect and display equations 5 and 6 for close examination.

$$\boxed{\begin{aligned} \frac{\partial \mathcal{E}}{\partial x} &= -\frac{\partial \mathcal{B}}{\partial t} \\ \frac{\partial \mathcal{B}}{\partial x} &= -\mu_0 \epsilon_0 \frac{\partial \mathcal{B}}{\partial t} \end{aligned}} \tag{7}$$

The above constitute a system of first order partial differential equations with constant coefficients, where, by the way, the speed of light (c) appears disguised as

$$\epsilon_0 \mu_0 = \frac{1}{c^2}$$

.

Our goal is to find (by any means) a field configuration that may solve the system

Experience shows that when confronted with a system such as eq. 7, attempting monochromatic harmonic wave solutions i.e. proposing

$$\mathcal{E} = \mathcal{E}_0 cos(kx - \omega t) \qquad \mathcal{B} = \mathcal{B}_0 cos(kx - \omega t) \tag{8}$$

where: $\mathcal{E}_0$, $\mathcal{E}_0$, k and ω are all constants may prove fruitful. Upon substitution, the system of differential equations changes to an algebraic system

$$k\mathcal{E}_0 = -\omega\mathcal{B}_0$$
$$k\mathcal{B}_0 = -\frac{\omega}{c^2}\mathcal{E}_0$$

Multiplication of the first equation by k and the second by ω to get

$$k^2\mathcal{E}_0 = -\omega k\mathcal{B}_0$$
$$k\omega\mathcal{B}_0 = -\frac{\omega^2}{c^2}\mathcal{E}_0$$

And from here we arrive to a highly relevant result

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathcal{E} = 0$$

In passing note that given $\mathcal{E}_0$, the magnetic intensity must be $\mathcal{B}_) = -\frac{k}{\omega}\mathcal{E}$, of course, if $\mathcal{B}_0$ is given, then $\mathcal{E}_0 = -\frac{\omega}{k}\mathcal{B}_0$

We have just discovered that independently of the amplitudes of the electric (or magnetic) field, a harmonic right traveling wave is a solution provided that the wave number (k) and the angular frequency (ω) satisfy the so called dispersion relation:

$$\boxed{k^2 - \frac{\omega^2}{c^2} = 0}$$

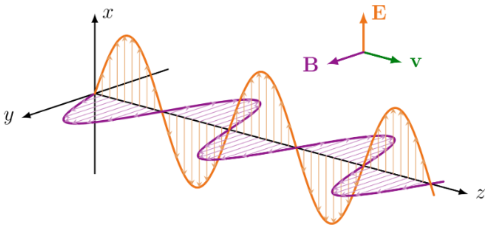
The Character of Electromagnetic Waves

- The solution we have found is such that, both fields are **perpendicular**
- They are varying in phase (their space time dependence is identical).
- The wave number (k) and the angular frequency (ω) of the solution are related to the speed of light through the dispersion relation

$$k = \frac{\omega}{c}$$

- Finally, the direction of propagation (x) axis, is perpendicular to the plane in which the waves vibrate (y-z), kinds that vibrate this way are called transverse waves.

Character of the Electromagnetic Waves



- Electromagnetic waves in vacuum are transverse waves with $\mathbf{E} \perp \mathbf{B}$ and $\mathcal{B} = -\frac{k}{\omega}\mathcal{E} = -\frac{1}{c}\mathcal{E}$
- The direction of propagation is given by the wave vector $\mathbf{k}$
- For harmonic waves the frequency and wave vector satisfy $|\mathbf{k}| = \frac{\omega}{c}$

Harmonic waves and energy

We have learned that the electromagnetic field stores energy in a way described by the electromagnetic energy density

$$u = u_E + u_B$$

Let us study the distribution of energy for the harmonic wave of equation 8

$$u_E = \frac{\epsilon_0}{2}\mathcal{E}^2 cos^2(kx - \omega t)\,, \quad u_B = \frac{1}{2\mu_0}\mathcal{B}^2 cos^2(kx - \omega t)$$

Let us make the substitutions and jiggle a little with the constants,

$$
\begin{aligned}
u &= \frac{1}{2}\left[\epsilon_0\,\mathcal{E}_0^2 + \frac{\mathcal{B}_0^2}{\mu_0}\right] cos^2(kx-\omega t) = \\
&= \frac{1}{2}\left[\epsilon_0\,\mathcal{E}_0^2 + \frac{\mathcal{E}_0^2}{\mu_0\,c^2}\right] cos^2(kx-\omega t) = \\
&= \frac{1}{2}\left[\epsilon_0\,\mathcal{E}_0^2 + \frac{\epsilon_0\mu_0\mathcal{E}_0^2}{\mu_0}\right] cos^2(kx-\omega t)
\end{aligned}
$$

The energy clearly "travels" with the wave

$$\boxed{u = \epsilon_0\,\mathcal{E}_0^2\,cos^2(kx-\omega t)} \tag{9}$$

but ... how?

An interesting quantity is the time average of u,

$$\langle u \rangle = \frac{1}{T} \int_0^T dt \, u$$

where $T(= 2\pi/\omega)$ is the period of oscillation of the wave, the calculation is elementary,

$$\langle u \rangle = \frac{\epsilon_0 \mathcal{E}}{T} \int_0^T dt \, cos^2(kx - \omega t) = \frac{\epsilon_0 \mathcal{E}}{2}$$

Energy Transport

The answer to the question of how electromagnetic energy was transported was given by John Henry Poynting (1852-1914) who developed the concept of a vector that describes the direction and magnitude of electromagnetic energy flow. This work was first published in 1884.

The Poynting vector

$$\mathbf{S} \equiv \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$$

Describes the rate of energy transport per unit area carried by an electromagnetic wave.

In the case of an harmonic wave propagating along the wave vector **k**

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \frac{\mathbf{E}}{c} = \frac{1}{\mu_0 c}\mathcal{E}^2\, cos^2(kx - \omega t)\,\hat{\mathbf{k}}$$
$$= \frac{\epsilon_0}{\mu_0 \epsilon_0 c}\mathcal{E}^2\, cos^2(kx - \omega t)\,\hat{\mathbf{k}} = \frac{\epsilon_0\, c^2}{c}\mathcal{E}^2\, cos^2(kx - \omega t)\,\hat{\mathbf{k}} =$$
$$= c\,\epsilon_0 \mathcal{E}^2\, cos^2(kx - \omega t)\,\hat{\mathbf{k}} = c\, u\; \hat{\mathbf{k}}$$

$$\boxed{\mathbf{S} = c\, u\; \hat{\mathbf{k}}} \tag{10}$$

Note that **S** is just the energy density multiplied by the velocity of the wave $c\hat{\mathbf{k}}$ as it should be. Consequently, the time average of thee Poynting vector is $\langle \mathbf{S} \rangle = c\frac{\epsilon_0 \mathcal{E}}{2}\hat{\mathbf{k}}$

The Energetics of a Discharging Capacitor: Where Does the Energy Go?

When we studied electric fields and energy storage, we learned about capacitors. Now, let's consider a familiar setup: a parallel-plate capacitor, initially charged, connected to a resistor. As the capacitor discharges, current flows, and the stored electric energy within the capacitor plates decreases.

The central question for our final challenge is:

- How does this energy leave the capacitor volume?
- Can we describe this energy flow using the powerful tools Maxwell gave us?

This situation will require us to connect many concepts we've learned throughout the course!

We learned that the discharging law of the capacitor in an RC circuit is

$$Q(t) = Q_0\, e^{-t/RC}$$

We need, in first place, the electric field ($\mathbf{E}$) between the plates, to calculate it, we take $\hat{\mathbf{n}}$ to be the normal between the plates pointing from the positive to the negative plate, then

$$\mathbf{E} = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}} = \frac{Q}{\epsilon_0\, A}\,\hat{\mathbf{n}}$$

$$\boxed{\mathbf{E} = \frac{Q_0}{\epsilon_0\, A}\, e^{-t/RC}\,\hat{\mathbf{n}}} \tag{11}$$

Given **E** we may now turn our attention to the magnetic induction field **B** between the plates. Ampere-Maxwell law is perfectly suited to this end.
We begin by considering an amperian curve in the form of a circle of radius r smaller than the radius of the plates and use the obvious cylindrical symmetry ($\mathbf{B} = B_\phi \hat{\mathbf{e}}_\phi$) to get

$$\oint_{\mathcal{C}} \mathbf{B} \cdot d\vec{l} = 2\pi r B_\phi$$

since here is no current in the interplate volume this must equal the time derivative of the flux of the electric field (times $\epsilon_0\mu_0$),

$$\phi(\mathbf{E}) = \pi r^2 \, \frac{Q_0}{\epsilon_0 \, A} \, e^{-t/RC}$$

and so

$$2\pi r B_\phi = \epsilon_0\mu_0 \frac{d}{dt}\left[\pi r^2 \, \frac{Q_0}{\epsilon_0 \, A} \, e^{-t/RC}\right] = \frac{\mu_0 \, r}{2A} \, \frac{d}{dt} \left(Q_0 \, e^{-t/RC} \right)$$

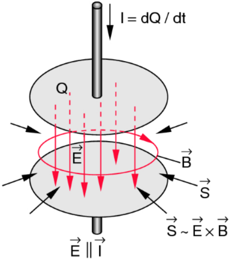
$$B_\phi = -\frac{\mu_0 \, r}{2A} \, I(t) \, , \quad I(t) = \frac{Q_0}{RC} \, e^{-t/RC} = I_0 \, e^{-t/RC}$$

$$
\begin{aligned}
\mathbf{S} &= -\frac{1}{\mu_0}\left[\frac{Q_0}{\epsilon_0\, A}\, e^{-t/RC}\, \hat{\mathbf{n}}\right] \times \left[\frac{\mu_0\, r}{2A}\, I_0\, e^{-t/RC} \hat{\mathbf{e}}_\phi\right] = \\
&= -\left[\frac{Q_0 RC}{\epsilon_0 ARC}\right]\left[\frac{r}{2A}\, I_0\, e^{-2t/RC}\right]\hat{\mathbf{n}} \times \hat{\mathbf{e}}_\phi = \\
&= -\left[\frac{I_0 RC}{\epsilon_0 A}\right]\left[\frac{r}{2A}\, I_0\, e^{-2t/RC}\right]\hat{\mathbf{n}} \times \hat{\mathbf{e}}_\phi = \\
&= -\left[\frac{RC}{2\epsilon_0 A^2}\right]\left[r\, I_0^2\, e^{-2t/RC}\right](-\hat{\mathbf{e}}_r)
\end{aligned} \tag{12}
$$

The conclusion is amazing,

$$\mathbf{S} = \frac{RCI_0^2}{2\epsilon_0 A^2}\, r\; e^{-2t/RC}\, \hat{\mathbf{e}}_r \tag{13}$$

S is oriented outwards from the capacitor, energy is leaving the capacitor because the capacitor is discharging so the energy it stores is decreasing.



An advanced project for the students. Find the Poynting vector in the interplate region of a charging parallel plate capacitor.
Remeber: The voltage across a capacitor initially discharges and connected in series with a resistance and a source of voltage is

$$v_c(t) = V_S(1 - e^{-\frac{t}{RC}})$$


LECTURE SUMMARY
- PHIESOPHENT LIKE YOU PHILSOMENS NEVET ARE PHEDANT
- PEGROBES YOUR SACKEST FLUNT
- COOR INY ANGHANDAY THE
- COPFRY
PHILESHPERT

Most of this lecture can be summarized in the figure