Electrostatic Potential

Mario I. Caicedo

July 25, 2025

Table of Contents (Part 1)

Table of Contents (Part 2)

Table of Contents (Part 3)


GARAGE SALE!

- As revealed by the primal principles of mechanics: work is done when a force causes a displacement.
- When the work done by a force $\mathbf{F}$ is independent of the trajectory, the force is said to be conservative and it has an associated potential energy which at a point P is given by the line integral

$$U(P) = -\int_{arb}^{P} \mathbf{F} \cdot d\mathbf{r}\,,$$

where *arb* is an arbitrary point, that has to be left fixed once it is chosen.

As a direct consequence of the definition, in the presence of conservative forces the work done by a force to move a particle from point A to point B obeys the relation

$$W_{AB} = U_A - U_B = -\Delta U_{AB} \,.$$

Indeed,

$$\begin{aligned} W_{AB} &= \int_A^B \mathbf{F} \cdot d\mathbf{r} = \int_A^{arb} \mathbf{F} \cdot d\mathbf{r} + \int_{arb}^B \mathbf{F} \cdot d\mathbf{r} = \\ &= -\int_{arb}^A \mathbf{F} \cdot d\mathbf{r} + \int_{arb}^B \mathbf{F} \cdot d\mathbf{r} = U_A - U_B = -\Delta U_{AB} \,. \end{aligned}$$

Climbing El Capitan changes the gravitational potental energy of Lynn Hill. Arbitrarily choosing U=0 at the base, the gravitational potental energy at height h above the base will be

$$U = mgh\,,$$

where m is the mass of Lynn and g the acceleration of gravity

So far, we've focused on the electric force (F) and the electric field (E), which are vector quantities.

- While powerful, calculating forces or fields directly can be complex, especially with multiple charges and non-simple geometries.
- Just as in mechanics, where understanding potential energy can simplify problems involving forces, a similar concept exists in electrostatics.
- This new concept is the Electric Potential (V), a scalar quantity that simplifies many calculations and offers a different perspective on electrostatic interactions.

It can be rigorously shown that electrostatic forces are conservative and therefore have an associated electrostatic potential energy U.

- Since,

$$W_{AB} = \int_A^B q\mathbf{E} \cdot d\mathbf{r}$$

is the work done *by the electric field* as the charge q moves from A to B.

- And we want the change in electric potential energy to satisfy

$$W_{AB} = U_A - U_B = -\Delta U_{AB}$$

Mechanics teaches us what to do

Definition

The electrostatic potential energy at point P is given by

$$U(P) = -\int_{arb}^{P} q\mathbf{E} \cdot d\mathbf{r}$$

where arb is an arbitrary point to be left fixed once that it is chosen

Accordingly, if the electric force does positive work, the potential energy of the system decreases. In contrast, if an external agent has to do positive work against the electric force, the potential energy increases.

Definition

The Electric Potential V is the electric potential energy per unit charge, letting U be the electrostatic potential energy

$$V \equiv \frac{U}{q}$$

- Clearly this definition is closely related to the definition os electric field as the electric force per unit charge.
- The electric potential is a property of the electric field itself, independent of any test charge placed within it.
- The electric potential difference (ΔV) between two points A and B is the change in potential energy per unit positive test charge q_0:

$$\Delta V = V_B - V_A = \frac{U_B - U_A}{q_0} = \frac{\Delta U}{q_0}$$

- Combining with the work definition:

$$\Delta V = -\frac{W_{AB}}{q_0}$$

- This means that electric potential is essentially the **potential energy "landscape"** created by source charges.

The SI unit for electric potential is the **Volt (V)**.

- From the definition $\Delta V = \Delta U / q_0$, we can see that:

$$1 \text{ Volt (V)} = 1 \text{ Joule (J)} \;/\; \text{Coulomb (C)}$$

- This unit honors Alessandro Volta, who invented the first electric battery.
- **Analogy:** Think of electric potential like gravitational height.
  - Just as higher ground has more gravitational potential energy for a given mass, a point with higher electric potential has more electric potential energy for a given positive charge.
  - Charges "fall" from high potential to low potential, similar to how objects fall from high height to low height.

While we can speak of an absolute electric potential at a point (often by defining $V = 0$ at infinity), what truly matters in most practical applications is the **potential difference (ΔV)** between two points.

- This is also commonly called **voltage**.
- A potential difference between two points means that work must be done to move a charge from one point to the other against the electric field.
- Conversely, if a charge moves between points with a potential difference, the electric field does work on it.
- Circuits, batteries, and power sources all rely on creating and maintaining potential differences to drive charge flow.

Remark

The electric potential $V(P)$ at a point P is defined relative to a reference point where the potential is set to zero.
Conventionally, we choose this reference point to be **infinity**, *so* $V(\infty) = 0$.

Remark

With this convention, the electric potential $V(P)$ is the **negative of the work done by the electric force** *in bringing a positive test charge (q_0) from infinity to point P:*

$$V(P) - V(\infty) = -\frac{W_{\infty,P}}{q_0}$$

Since $V(\infty) = 0$, we have:

$$V(P) = -\frac{W_{\infty,P}}{q_0}$$

We also know that the work done by the electric force **F** is

$$W_{\infty,P} = \int_{\infty}^{P} \mathbf{F} \cdot d\mathbf{r}\,, \text{ where } \mathbf{F} = q_0\mathbf{E}$$

Substituting these into the definition yields

$$\boxed{V(P) = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{r}} \tag{1}$$

Let us be a little more explicit.
Consider a field point P and its cartesian coordinates (x,y,z), then we may write

$$\boxed{V(P) = V(x,y,z) = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{r}} \tag{2}$$

Just as in one variable calculus, the ordinary derivative and the integral are somehow inverse operations, there is a vector derivation that plays the role of the inverse to the line integrals of conservative vector fields. Such operation is called *grad* and it is usually represented by the operator ∇ in terms of which we write

$$\boxed{\mathbf{E}(P) = -\nabla V(P)} \tag{3}$$

The gradient has different expressions for each coordinate system, consequently the formula

$$\mathbf{E}(P) = -\nabla V(P) \tag{4}$$

may be expressed in any of the forms shown below

$$\mathbf{E} = - \begin{cases} \frac{\partial V(x,y,z)}{\partial x}\, \hat{\mathbf{e}}_x + \frac{\partial V(x,y,z)}{\partial y}\, \hat{\mathbf{e}}_y + \frac{\partial V(x,y,z)}{\partial z}\, \hat{\mathbf{e}}_z \\ \\ \frac{\partial V(\rho,\theta,z)}{\partial \rho}\, \hat{\mathbf{e}}_\rho + \frac{1}{\rho}\frac{\partial V(\rho,\theta,z)}{\partial \theta}\, \hat{\mathbf{e}}_\theta + \frac{\partial V(\rho,\theta,z)}{\partial z}\, \hat{\mathbf{e}}_z \\ \\ \frac{\partial V(r,\theta,\phi)}{\partial r}\, \hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial V(r,\theta,\phi)}{\partial \theta}\, \hat{\mathbf{e}}_\theta + \frac{1}{r\sin\theta}\frac{\partial V(r,\theta,\phi)}{\partial \phi}\, \hat{\mathbf{e}}_\phi \end{cases} \tag{5}$$

Let's apply formula 1 to find the potential created by a single point charge q located at the origin.
The electric field is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{e}}_r$$

. Therefore,

$$V(r_P) = -\int_{\infty}^{r_P} \mathbf{E}\cdot d\mathbf{r} = -\int_{\infty}^{r_P}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{e}}_r\right)\cdot(dr\,\hat{\mathbf{e}}_r + d\mathbf{r}_\perp)$$

$$V(r_P) = -\frac{q}{4\pi\epsilon_0}\int_{\infty}^{r_P}\frac{1}{r^2}dr = -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{\infty}^{r_P}$$

Evaluating the limits:

$$V(r_P) = -\frac{q}{4\pi\epsilon_0}\left[-\frac{1}{r_P} - \left(-\frac{1}{\infty}\right)\right]$$

Since

$$\frac{1}{\infty} = 0\,,$$

$$\mathbf{V}(r_P) = -\frac{q}{4\pi\epsilon_0}\left(-\frac{1}{r_P} - 0\right)$$

This gives us the final formula for the electric potential due to a point charge:

$$\boxed{\mathbf{V}(r_P) = \frac{q}{4\pi\epsilon_0 r_P}}$$

Summary

- The electric potential produced by a point charge is found by integrating the electric field from infinity (where potential is defined as zero) to a point at distance r from the charge. In SI units

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

- **Key features:**
  - The Potential is a **scalar quantity**, so it has no direction.
  - The Potential depends inversely on distance $(1/r)$, unlike the electric field $(1/r^2)$.
  - The Potential is **positive** for a positive source charge $(q > 0)$.
  - The Potential is **negative** for a negative source charge $(q < 0)$.

For the sake of completeness, let us calculate the electric field of the point charge form the potential, we have learned that

$$\mathbf{E} = -\nabla V$$

the expression

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

is obviously written in spherical coordinates so it is convenient to use the spherical form of the gradient which, in this case, where all the dependence is on the radius r reduces the calculation to

$$\mathbf{E} = -\left[\frac{\partial V(r,\theta,\phi)}{\partial r}\,\hat{\mathbf{e}}_r\right] = -\frac{\partial}{\partial r}\left[\frac{q}{4\pi\epsilon_0 r}\right]\,\hat{\mathbf{e}}_r = \frac{q}{4\pi\epsilon_0 r^2}\,\hat{\mathbf{e}}_r$$

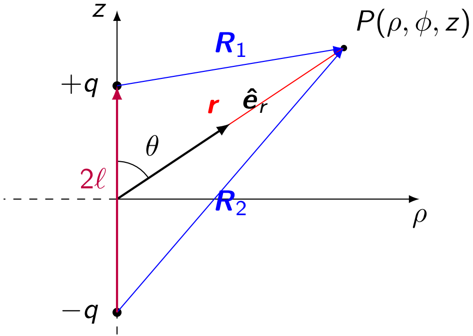
Just like electric fields, electric potentials also obey the superposition principle.

- For a system of multiple point charges, the total electric potential at any point P is simply the **algebraic sum** of the potentials created by each individual charge.
- This is much simpler than vector addition of electric fields!
- For n point charges $(q_1, q_2, \ldots, q_n)$:

$$V_{total} = V_1 + V_2 + \cdots + V_n = \sum_{i=1}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i}{r_i}$$

where r_i is the distance from charge q_i to point P.

Consider a system consisting of two charges of equal magnitude, but of opposite sign, each situated a distance ℓ from the origin, which is taken to lie on the line connecting the charges. Such a system is the simplest example of an electric dipole.



$$V(\rho, \phi, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{\rho^2 + (z-\ell)}} - \frac{1}{\sqrt{\rho^2 + (z+\ell)}}\right]$$

We are interested in the limit $\ell \to 0$ with $p = 2q\ell$ being finite. In such limit, we may approximate

$$V(\rho, \phi, z) = -\frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial z}\left[\frac{1}{\sqrt{\rho^2 + z^2}}\right]2\ell = \frac{1}{4\pi\epsilon_0}\frac{z(2q\ell)}{(\rho^2 + z^2)^{3/2}}$$

Defining the dipole moment vector as $\mathbf{p} = 2q\ell\,\hat{\mathbf{e}}_z$ we may write

$$\boxed{V(\rho, \phi, z) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}}$$

Remark

We already know Faraday's Maxwell's lines of force or field lines. They constitute a very good way to visualize electric fields; nevertheless, the vector character of the field might become an obstacle due to the technical difficulties associated to represent vectors in 3 space

Remark

The existence of the potential has the consequence of providing another tool "equipotential surface" qhich is particularly useful to visualize the electric field.

Definition

An **equipotential surface** *is a surface on which all points have the same electric potential.*

- Equipotential surfaces are always **perpendicular** to electric field lines.
- The electric field always points in the direction of **decreasing potential**.
- No work is done by the electric field when a charge moves along an equipotential surface. Indeed, for anu path $W = -q_0 \Delta V$, and $\Delta V = 0$ for any two points on the same equipotential surface.

One Positive Charge

The simplest possible case is the point charge, we know that in such case and using the right coordinates

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

If we ask for all points such that $V = V_0$ a fixed value, we get

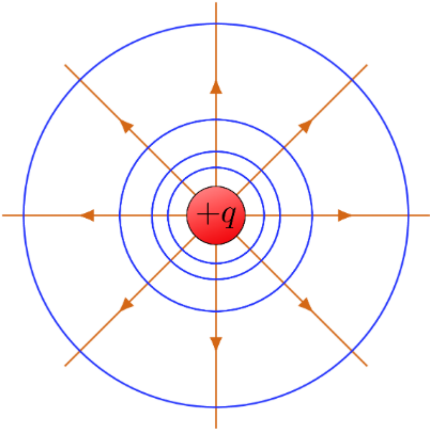
$$r = \frac{1}{4\pi\epsilon_0}\frac{q}{V_0},$$

which describes a sphere. Consequantly the equipotential surfaces constitute a family of concentric spheres.

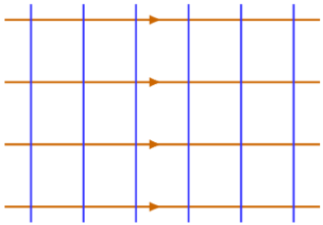
Electric field lines (red) are all radially outgoing and repel each other, they become denser near each charge (high field)

Equipotential lines are spheres and therefore, strictly

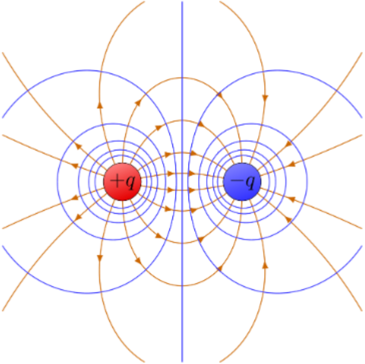
perpendicular to the electric field lines, the zero potential line is now at infinity



Uniform Elecric Field. Another Simple Example.



Electric field lines (red) are uniformly spaced and parallel, indicating a constant field.
Equipotential lines (blue) are also parallel and equally spaced, but they are **perpendicular** to the electric field lines.
Since electric field lines point in the direction of decreasing potential, the potential in this diagram is **decreasing towards the right**



Electric field lines go from positive to negative charge, they become denser near the charges (high field)

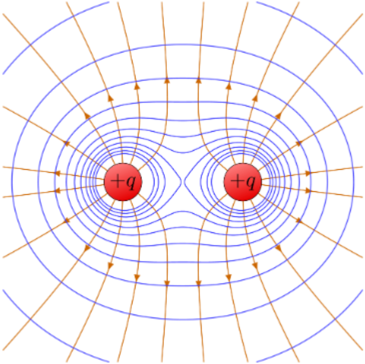
Equipotential lines are strictly **perpendicular** to the electric field lines, note the straight line (plane) of zero potential.

The potential in this diagram is **decreasing** in the direction of the negative charge.

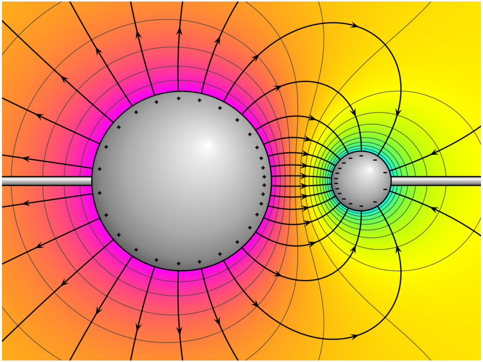
Electric field lines are all outgoing and repel each other, they become denser near each charge (high field)

Equipotential lines are strictly **perpendicular** to the electric field lines, the zero potential line is now at infinity

As distance increases the electric field lines and potentials tend to resemble those of a point charge of magnitude $+2q$



Electric potential around two oppositely charged conducting spheres. Purple represents the highest potential, yellow zero, and cyan the lowest potential. The electric field lines are shown leaving perpendicularly to the surface of each sphere.



- **Electric Potential Energy (U):** Energy a charge possesses due to its position in an electric field ($\Delta U = -W$).
- **Electric Potential (V):** Potential energy per unit charge ($\Delta V = \Delta U / q_0$). It's a scalar field.
- **Voltage (ΔV):** Another name for potential difference, which is the key quantity in circuits.
- **Point Charge Potential:** $V = k\frac{q}{r}$
- **Superposition for Potential:** Add potentials algebraically (scalar sum).
- **Equipotential Surfaces:** Surfaces of constant potential, perpendicular to electric field lines.
- **E and V Relationship:** **E** is the negative gradient of **V**, indicating **E** points towards decreasing potential.

Electric field and potential are two different ways to describe the same underlying electric influence, and they are mathematically related.

- The electric field is the **negative gradient** of the electric potential. This means:
  - The electric field points in the direction where the potential decreases most rapidly.
  - Its magnitude is equal to the rate of decrease of potential with distance.
- In one dimension (e.g., along the x-axis):

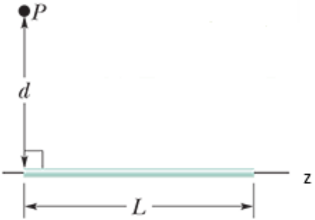
$$E_x = -\frac{dV}{dx}$$

- In three dimensions, this generalizes to the gradient operator:

$$\mathbf{E} = -\nabla V$$

- This relationship allows us to find the electric field if we know the potential, and vice-versa (via integration).

For this example we want to calculate the electric potential caused by the uniformly charged (linear charge density λ) rod of finite length L shown in the figure. We limit our calculation to the field point P located in the xy-plane at a distance $\rho = d$ from the extreme of the rod.



Our calculation of the potential $V(\rho)$ will strictly follow the superposition principle. To that end, we sum the contributions from the elements of charge $dq = \lambda dz$ located at position z along the rod, namely

$$dV(\rho) = \frac{1}{4\pi\epsilon_0}\frac{dq}{R_{\text{dist}}}$$

Here R_{dist} the distance from dq to the field point which we take to be in a plane perpendicular to the rod at its $z = 0$. Taking ρ to be the radius form the rod axis to the field point,

$$R_{\text{dist}} = \sqrt{z^2 + \rho^2}$$

therefore

$$dV(\rho) = \frac{1}{4\pi\epsilon_0}\frac{dq}{\sqrt{z^2 + \rho^2}}$$

The potential is found by adding the contributions from $z = 0$ to $z = L$:

$$V(\rho) = \frac{\lambda}{4\pi\epsilon_0} \int_0^L \frac{dz}{\sqrt{z^2 + \rho^2}}$$

Using the standard integral $\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$

$$V(\rho) = \frac{\lambda}{4\pi\epsilon_0} \left[\ln\left(z + \sqrt{z^2 + \rho^2}\right)\right]_0^L$$

Evaluating the limits:

$$V(\rho) = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(L + \sqrt{L^2+\rho^2}\right) - \ln\left(0 + \sqrt{0^2+\rho^2}\right)\right]$$

$$V(\rho) = \frac{\lambda}{4\pi\epsilon_0}\left[\ln\left(L + \sqrt{L^2+\rho^2}\right) - \ln(\rho)\right]$$

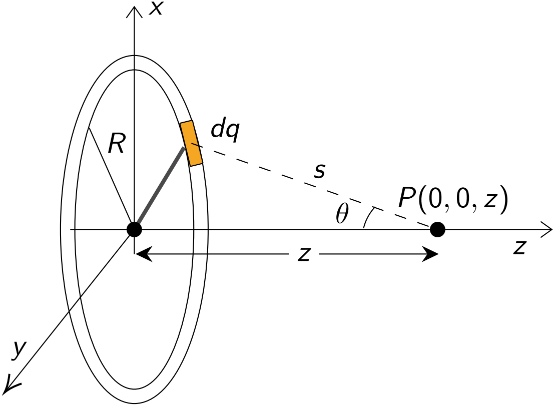
Since $\ln A - \ln B = \ln(A/B)$:

$$\boxed{V(\rho) = \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{L + \sqrt{L^2+\rho^2}}{\rho}\right)}$$

This is the explicit potential for a finite rod at a point in the plane perpendicular to one of its extremes.

According to the figure, the contribution to the potential due to the element of charge (dq) is:

$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{s}$$



If we let Q be the total charge of the ring, and ϕ an azimuthal angle measured in the $x - y$ plane,

$$\lambda = \frac{Q}{2\pi R}, \quad \text{and} \;\; dq = \lambda R d\phi$$

so

$$dV = \frac{1}{4\pi\epsilon_0} \frac{\lambda R \, d\phi}{\sqrt{R^2 + z^2}}$$

a trivial integration gives us the potemtial created at P by the ring

$$V = \frac{1}{4\pi\epsilon_0} \frac{\lambda R}{\sqrt{R^2 + z^2}} \int_0^{2\pi} d\phi = \frac{\lambda}{2\epsilon_0} \frac{R}{\sqrt{R^2 + z^2}}$$

The formula -which we have just found- for the potential due to an uniformly charged ring at any point of an axis passing by its center in the direction perpendicular to the plain where the ring is located, namely

$$\boxed{V = \frac{\lambda}{2\epsilon_0} \frac{R}{\sqrt{R^2 + z^2}}} \tag{6}$$

suggests two interesting exercises for the student:

1. Show that under the right conditions, this indeed reduces to the potential due to a point charge Q
2. Take the gradient and change the sign to see if you get the correct electric field (you should, shouldn't you?)

In the lecture on electric fields, we learned that the ring we just studied might be used as a step towards a more complex problem, namely, the uniformly charged disk.

The key idea was to use the ring as a contributor to the field of the whole disk. We will now take that idea and apply it to the potential of the disk.

Our first step consists of writing the potential of the ring (we will take the radius as ρ) in the form)

$$V_{ring} = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{\rho^2 + z^2}} \tag{7}$$

Our second step is to substitute the ring by a of a very thin circular strip of radius ρ and with $d\rho$ carrying a uniform surface density. This makes the strip to carry an infinitesimal:

$$dQ = \sigma \times \textit{ area of strip} = 2\pi\sigma\rho d\rho$$

consequently, the contribution

$$dV = \frac{1}{4\pi\epsilon_0}\frac{2\pi\sigma\rho\, d\rho}{\sqrt{\rho^2 + z^2}} \tag{8}$$

To find V all is needed is integration

Taking the radius of the disk to be R, we write

$$V = \frac{1}{4\pi\epsilon_0}\int_0^R \frac{2\pi\sigma\rho\, d\rho}{\sqrt{\rho^2 + z^2}}\,, \tag{9}$$

since $d(\rho^2 + z^2) = 2\rho d\rho$,

$$\begin{aligned} V &= \frac{2\pi\sigma}{4\pi\epsilon_0}\sqrt{\rho^2 + z^2}|_0^R = \\ &= \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2 + z^2} - abs(z)\right]\,. \end{aligned} \tag{10}$$

Taking the gradient (exercise) gives the correct formula for the electric field [see formula 11 in the next slide].

In the class on electric field, we found that the electric field on the axis of an uniformly charged disk was given by the formula

$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}\left[\mathrm{sign}(z) - \frac{z}{\sqrt{R^2 + z^2}}\right]} \tag{11}$$

In that class, we reached this formula by direct integration of the vertical component of the electric field.
We have also found found formula 11 by taking the gradient of formula 10 giving the potential of the disk, which is a nice proof of consistency.

Now, if we take the limit $R \to \infty$ in formula 11 we get the electric field for the infinite plane that we got from symmetry and Gauss's theorem

Our previous developments give rise to a natural question:

Is it possible to get the electric field of an infinite plane by calculating the potential of a disk, taking the limit of the radius of the disk going to infinity and then taking the gradient (and change the sign)?

There is only one way to answer: give it a try

Our previous developments give rise to a natural question:

Is it possible to get the electric field of an infinite plane by calculating the potential of a disk, taking the limit of the radius of the disk going to infinity and then taking the gradient (and change the sign)?

There is only one way to answer: give it a try

The potential of the disk of radius R is given by

$$V = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2 + z^2} - abs(z)\right] . \tag{12}$$

this formula is horribly divergent when $R \to \infty$, nevertheless, if we take the gradient first and the limit, the correct result

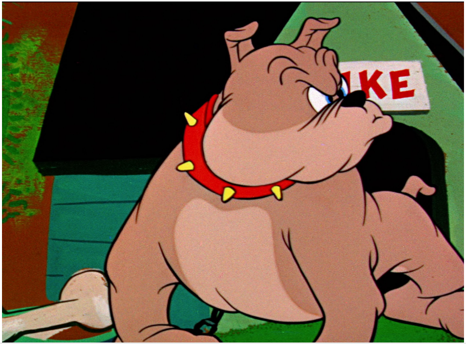
$$\boxed{E(z) = \begin{cases} \frac{\sigma}{2\epsilon_0} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} & z < 0 \end{cases}} \tag{13}$$

is obtained.

This curious behavior is not so rare, it is found in unbounded charge distributions such as the plane or the infinite rod.

Beware

When dealing with systems having charges located at infinity (like the infinite bar), there is always the potential of finding divergent integrals when calculating the potential, this is why in such cases it is better to appeal to symmetry and Gauss's law to find the electric field and then integrate the field to get the potential



We have previously found the electric field due to an infinite line of charge using Gauss's Law:

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}$$

where $\hat{\boldsymbol{\rho}}$ is the radial unit vector perpendicular to the rod.
To find the potential $V(\rho)$, we integrate $\mathbf{E}$ from a reference point ρ_0 to ρ:

$$V(\rho) - V(\rho_0) = -\int_{\rho_0}^{\rho} \mathbf{E} \cdot d\mathbf{r}$$

Choosing a radial path $d\mathbf{r} = d\rho'\hat{\boldsymbol{\rho}}$:

$$V(\rho) - 0 = -\int_{\rho_0}^{\rho}\left(\frac{\lambda}{2\pi\epsilon_0\rho'}\hat{\boldsymbol{\rho}}\right)\cdot(d\rho'\hat{\boldsymbol{\rho}})$$

$$V(\rho) = -\frac{\lambda}{2\pi\epsilon_0}\int_{\rho_0}^{\rho}\frac{1}{\rho'}d\rho'$$

Performing the integral:

$$V(\rho) = -\frac{\lambda}{2\pi\epsilon_0}\left[\ln(\rho')\right]_{\rho_0}^{\rho}$$

Evaluating the limits:

$$V(\rho) = -\frac{\lambda}{2\pi\epsilon_0}(\ln(\rho) - \ln(\rho_0))$$

$$\boxed{V(\rho) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\rho_0}{\rho}\right)}$$

We now want to calculate the electric potential generated by an infinite plane carrying a uniform surface charge density σ (Coulombs per square meter).

- We had already used Gauss's to find that the electric field due to such charge configuration is uniform and perpendicular to the plane:

$$\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit vector pointing away from the plane. The field is constant everywhere outside the plane.

To find the potential $V(z)$, we integrate $\mathbf{E}$ from our chosen reference point z_0 to z:

$$V(z) - V(z_0) = -\int_{z_0}^{z} \mathbf{E} \cdot d\mathbf{l}$$

We take the position of the plane to be $z = 0$. For field points in the positive z semispace ($\hat{\mathbf{n}} = \hat{\mathbf{z}}$), and $\mathbf{E} = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}$. Choose de path element $d\mathbf{l} = dz\hat{\mathbf{z}}$:

$$V(z) - 0 = -\int_{z_0}^{z} \left(\frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}\right) \cdot (dz\hat{\mathbf{z}})$$

Giving

$$V(z) = -\frac{\sigma}{2\epsilon_0}[z']_{z_0}^{z}$$

so

$$\boxed{V(z) = -\frac{\sigma}{2\epsilon_0}(z - z_0)}$$

This result shows that the potential changes linearly with distance

The physically meaningful quantity is the **potential difference (ΔV)** between two distinct points, say at "heights" z_A and z_B from the plane.
Since

$$V(z) = -\frac{\sigma}{2\epsilon_0}(z - z_0)$$

The potential difference $\Delta V = V(z_B) - V(z_A)$ is:

$$V(z_B) - V(z_A) = \left[-\frac{\sigma}{2\epsilon_0}(z_B - z_0)\right] - \left[-\frac{\sigma}{2\epsilon_0}(z_A - z_0)\right]$$

$$V(z_B) - V(z_A) = -\frac{\sigma}{2\epsilon_0}(z_B - z_0 - z_A + z_0)$$

$$\boxed{\Delta V = V(z_B) - V(z_A) = -\frac{\sigma}{2\epsilon_0}(z_B - z_A)}$$

A result is independent of the arbitrary reference point z_0, confirming that potential differences are the truly physical quantities, especially for infinite charge distributions.


LECTURE SUMMARY
- PHIESOPHENT LIKE YOU PHILSOMENS NEVET ARE PHEDANT
- PEGROBES YOUR SACKEST FLUNT
- COOR INY ANGHANDAY THE
- COPFRY
PHILESHPERT

Key Point for Infinite Geometries (Revisited):

- Just like with the infinite line of charge, the potential for an infinite plane also diverges if we try to set $V = 0$ at infinity.
- Therefore, we must choose a finite reference point, say z_0, where we define the potential to be zero, i.e., $V(z_0) = 0$.

In Electrostatics

$$V(P) = -\int_{arb}^{P} \mathbf{E}.d\mathbf{l} \Leftrightarrow \mathbf{E} = -(\text{gradient of the potential})$$

- *arb* usually taken to be ∞.
  - Take care with infinite charge distributions.
- Field lines perpendicular to equipotential surfaces
- Conductors are equipotential
- Field lines perpendicular to conducting surfaces
- Not usually taught at this level. When crossing a surface form side 1 to side 2
$$(\mathbf{E}_2 - \mathbf{E}_1) \cdot \hat{\mathbf{n}}_{12} = \frac{\sigma}{\epsilon_0}$$