Electric Fields In Vacuum

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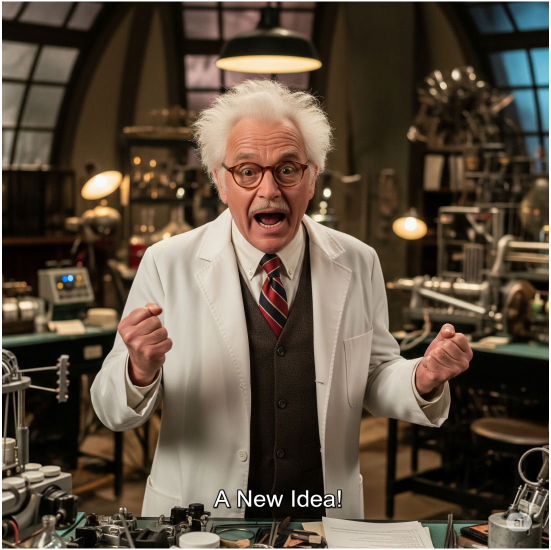
Action at Distance:

In the first class, we commented that

Coulomb's Law=Action at Distance

Is troublesome since it brings in some conceptual challenges

Indeed, in our current 21st-century view, it violates Einstein's Special Relativity, which states that nothing can travel faster than the speed of light. Instantaneous action violates causality. In essence, the problem isn't with the concept of action at a distance itself, but rather with the implications of its potential for instantaneous transmission of information, which can clash with our understanding of causality and the limits of the speed of light.


A New Idea!

The British scientist Michael Faraday (22 September 1791 – 25 August 1867) introduced the concept of "lines of force" to visualize and understand electric and magnetic interactions.

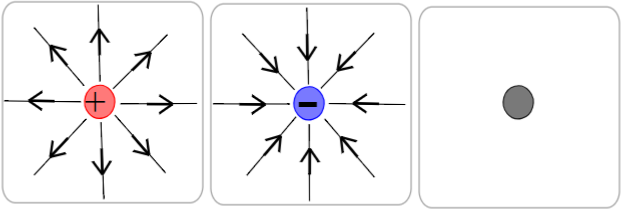


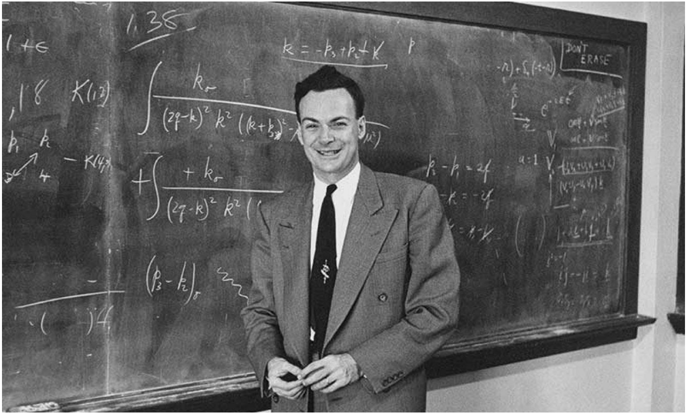
Figure: Electric Field Lines of a Positive, Negative, and Discharged Point Particle. Figure authored by By Nein Arimasen

Instead of action at a distance, Faraday pictured the space around a charged particle as filled with lines of force that affected any *test charge* sitting on them.

Faraday viewed these lines not just as a visual aid but as representing a real physical property of space, implying that the space surrounding a magnet or electric charge was no longer considered empty. Faraday believed that these lines represented the direction and magnitude of the force at each point in space.

In a word, Faraday (and Maxwell) gave birth to the modern concept of FIELD

In Feynman's Lectures on Physics, a **field** is defined as any physical quantity that takes on different values at different points in space. Essentially, a field is a function that assigns a specific value to each location in space.



- A Scalar Field is characterized by a single number (a scalar) at each point in space, according to The Feynman Lectures on Physics. Examples include temperature, where the temperature at each point is a single value.
- A vector field assigns a vector (a magnitude and direction) to each point in space. Examples include the electric or magnetic fields, where both magnitude and direction vary at each point in space.
- Time Dependence Fields can also be time-dependent, meaning their values change over time as well as with position. For example, the velocity field of a flowing liquid would vary both with location and time.

The contemporary view of the field concept is that physical fields are essentially local properties of space. The clearest example of this is the gravitational field, which is represented by a mathematical entity called the Einstein tensor, describing how spacetime warps due to mass-energy.

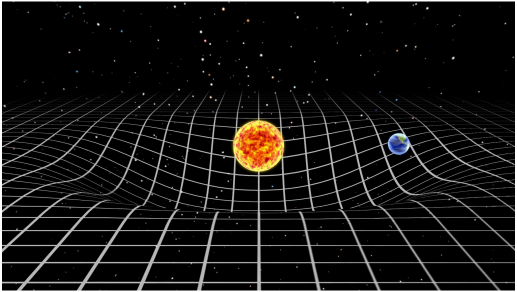


FIGURE: Gravity is not a force, is a warping of the space time itself whose source is mass-energy

The Electric Field

Definition

When a charged (test) particle with charge q is located in a region of space where there is an electric field, the charge is acted upon by a force $\mathbf{F}$ *related to the electric field by*

$$\mathbf{F} = q\,\mathbf{E} \tag{1}$$

Comment

The above definition is completely operational and tell us how to measure an electric field

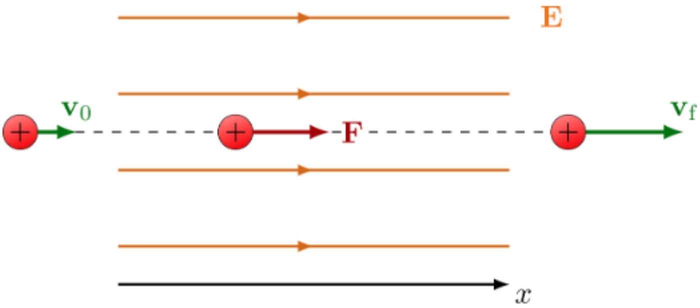


FIGURE: Never forget, the electric fields exerts force on test particles, since $\mathbf{F} = q\mathbf{E}$, electric fields do work on charges.

Elements of the Electric Field

- A mathematical entity called a vector field (see any calculus book)
- The units of electric field are: $[\mathbf{E}] = Nw/C$
- The point at which a measure of the electric field at some moment is performed, is called the Field Point

Comment

Think of $\mathbf{E}$ as a set of infinite arrows filling the space. Placing a test charge on a point in the space, makes the arrow of electric field at that point to push the charge with a force $\mathbf{F} = q\mathbf{E}$

What are field lines?

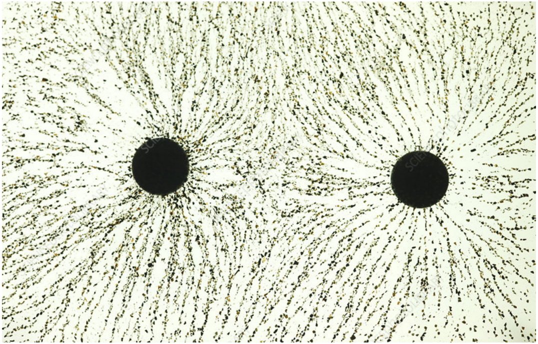
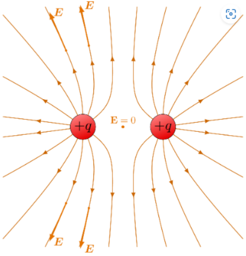


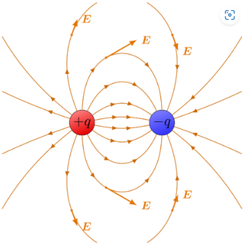
Figure: Caption Electric field lines are shown by using pepper in cooking oil. The two center electrodes are charged up to $30,000\ v$ with an electrostatic generator. The two central electrodes have the same charge and the field lines radiate out Credit.

- Electric field lines constitute a way to represent the electric field.
- At every point in space, the field vector at that point is tangent to the field line at that same point. For the mathematical inclined mind, field lines are the integral curves of the electric field.
- Electric field lines either originate on positive charges or come in from infinity, and either terminate on negative charges or extend out to infinity.
- The number of field lines originating or terminating at a charge is proportional to the magnitude of that charge. A charge of $2q$ will have twice as many lines as a charge of q.

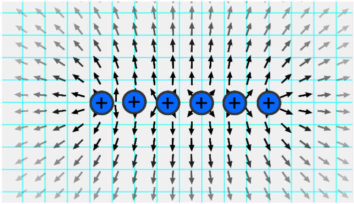
- The field line density at any point in space is proportional to (and therefore is representative of) the magnitude of the field at that point in space.
- Field lines can never cross. Since a field line represents the direction of the field at a given point, if two field lines crossed at some point, that would imply that the electric field was pointing in two different directions at a single point. This in turn would suggest that the (net) force on a test charge placed at that point would point in two different directions. Since this is obviously impossible, it follows that field lines must never cross.

Could you give a heuristic argument justifying the following sketches for the electric fields produced by a set of two charges? Credit





Why do aligned charges produce the following electric field lines?



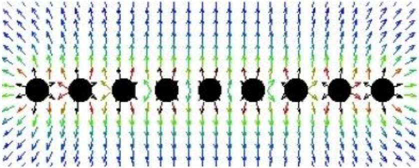


Figure: Credit


EXAMPLES

Our first example is the Coulomb Field, according to Coulomb's Law and the definition of electric field, the field produced at a point located at $\mathbf{r}$ by a point charge q located at position $\mathbf{r}'$ is

$$\boxed{\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}} \tag{2}$$

To make sense of this formula, think of an origin $\mathcal{O}$, the position of the field point $\mathbf{r}$, the position of the charge $\mathbf{r}'$ and the vector joining the position of the charge and the field point $\mathbf{R}$. Certainly:

$$\mathbf{r} = \mathbf{r}' + \mathbf{R}$$

and write the Coulomb force on a test charge Q as

$$\mathbf{F} = \frac{qQ}{4\pi\epsilon_0}\frac{\hat{\mathbf{R}}}{|\mathbf{R}|^2} = \frac{qQ}{4\pi\epsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} \tag{3}$$

As a second example, let us study the motion of a charge q placed (originally at rest) in a region where a uniform electric field is present.
To begin with, we need to state that uniform means the vector field is constant (the electric field is the same at any field point).

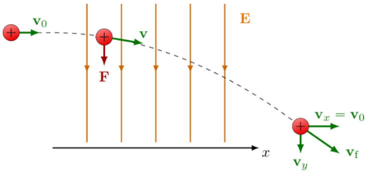


Figure: Credit

Newton's second law allows us to write the acceleration of the charged particle as

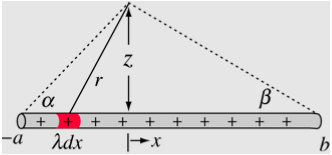
$$\mathbf{a} = \frac{q}{m}\mathbf{E} = \text{constant vector}$$

Knowledge of Newtonian mechanics allows us to find the position vector of the particle with the initial condition $\mathbf{r}(t=0) = \mathbf{r}_0$, $\mathbf{v}(t=0) = \mathbf{v}_0$:

$$\mathbf{r}(t) = \frac{q}{2m}\mathbf{E}t^2 + \mathbf{v}_0 t + \mathbf{r}_0$$

¡This is the ballistic motion with the electric field times the charge and divided by the mass taking the place of the gravity acceleration $\mathbf{g}$!

As an interesting exercise, **we want to calculate the vertical (z) component of the electric field** associated with the uniformly charged rod at the field point shown in the figure.



The calculation begins by noting that at the field point, the contribution to the electric field ($d\mathbf{E}$) caused by the infinitesimal element of charge $dq = \lambda dx$ is parallel to the line joining dq and the field point and has magnitude

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}$$

In order to calculate the vertical component dE_z of $d\mathbf{E}$ we must project over the z axis, that is, we must multiply dE by the cosine of the appropiate angle

$$dE_z = dE\, cos\theta = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2}\, cos\theta \tag{4}$$

In formula 4

$$cos\theta = \frac{z}{r}$$

so

$$dE_z = \frac{1}{4\pi\epsilon_0} \frac{dq}{r^2} \frac{z}{r} \tag{5}$$

The charge is distributed, that is, there is a linear charge density λ which in the notation of the figure is defined by[1]:

$$\lambda = \frac{dq}{dx} \quad \text{so} \quad dq = \lambda dx$$

In addition, $r = \sqrt{x^2 + z^2}$. Consequently,

$$dE_z = \frac{1}{4\pi\epsilon_0} \frac{\lambda\, dx}{(x^2 + z^2)} \frac{z}{\sqrt{x^2 + z^2}}$$

or, grouping the constant terms together,

$$dE_z = \frac{\lambda\, z}{4\pi\epsilon_0} \frac{dx}{(x^2 + z^2)^{3/2}} \tag{6}$$

[1] Think of population density

All that is still missing is adding over the elements of length, this amounts to integrating from x from $x = -a$ to $x = b$, which leaves us with

$$E_z = \frac{\lambda\, z}{4\pi\epsilon_0} \int_{-a}^{b} \frac{dx}{(x^2 + z^2)^{3/2}} \tag{7}$$

We might integrate using a table, or some symbolic manipulation software or even take a little risk and try a generative AI. But I (Mario, your Prof) rather calculate by hand in order to have control over all details.

For the integration, change variables as $x = z\, tan(\theta)$ so $dx = z\, sec^2\theta\, d\theta$ and $(z^2 + x^2)^{3/2} = z^3(1 + tan^2\theta)^{3/2} = z^3\, sec^3\theta$, these convert 7 into

$$E_z = \frac{\lambda}{4\pi\epsilon_0\, z} \int_{-atan(a/z)}^{atan(z/b)} cos\theta\, d\theta\,, \quad \text{yielding}$$

$$\boxed{E_z = \frac{\lambda}{4\pi\epsilon_0\, z}\left[\frac{b}{\sqrt{b^2+z^2}} + \frac{a}{\sqrt{a^2+z^2}}\right]}$$

There are some final comments regarding this example.

- The change of variables in the integral is natural given the geometry of the problem
- If we take $a = b$ the field point will be located in the line that bisects the charged segment and

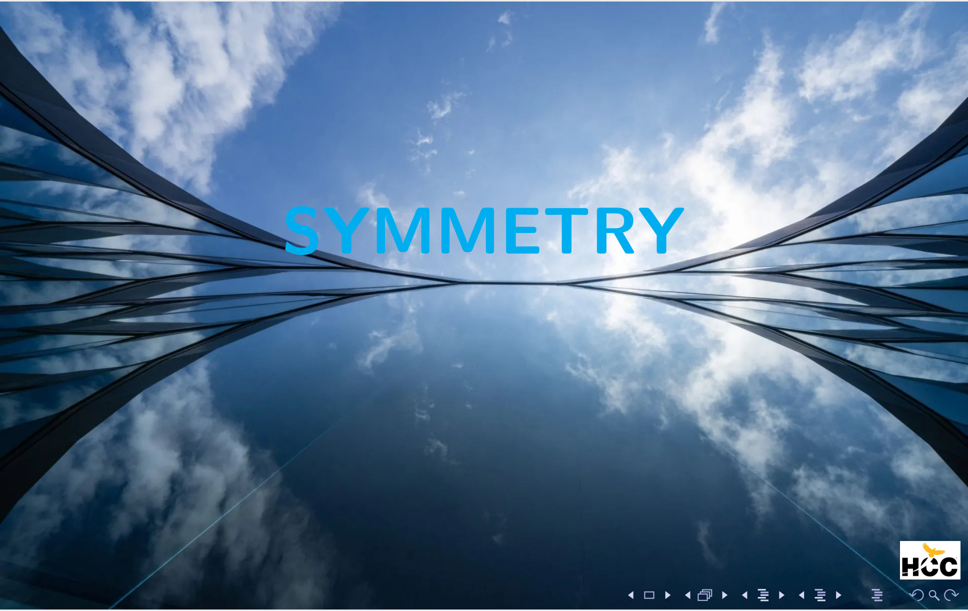
$$\boxed{E_z = \frac{2\lambda}{4\pi\epsilon_0\, z}\, \frac{b}{\sqrt{b^2 + z^2}}} \tag{8}$$

- If we take the field point very far form the charged segment in the In the symmetric case

$$\boxed{E_z = \frac{2\lambda\, b}{4\pi\epsilon_0\, z}\,\frac{1}{\sqrt{b^2+z^2}} \approx \frac{2\lambda\, b}{4\pi\epsilon_0\, z^2} = \frac{Q}{4\pi\epsilon_0\, z^2}} \tag{9}$$

which is what we would expect for a point charge (why?)

- Finally: what happens if $a = b \to \infty$?



SYMMETRY

Symmetry in physics is nature's way of being lazy – in the best possible way!

Think about it: if you have a situation where multiple parts look identical or are arranged in a regular pattern, nature doesn't waste time making each part behave differently. Instead, it treats identical parts identically, and it applies the same fundamental rules regardless of where you are, when you are, or how you're oriented.

This "laziness" isn't about cutting corners; it's about incredible efficiency and elegance in the universe's underlying structure.

To truly grasp this *laziness*

Consider what we call the "Mirror Test." If, in the analysis of a physical situation or experiment, you can draw a line (or imagine a plane) such that everything on one side is a perfect, identical reflection of the other, then nature, in its elegant efficiency, will ensure the physical outcome on both sides is also perfectly symmetrical, i.e. the same. It simply wouldn't make sense for identical setups to yield different results!

For instance

If you have two identical charges positioned equally far from a specific point, symmetry immediately tells you their individual forces at that point must have the exact same magnitude – you don't even need to pick up your calculator.

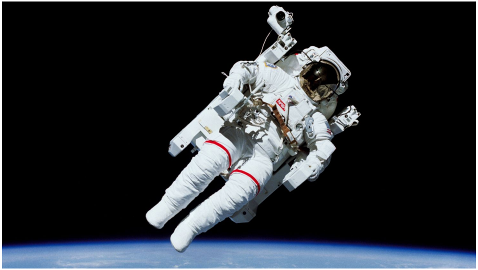


FIGURE: In the absence of gravity there is no distinction between up down, left right. That is complete spherical symmetry, if gravity is restored, then the symmetry is broken and reduced to azimuthal symmetry.

For a **perfectly uniform sphere**, symmetry means that no matter where you are at a given distance from its center, the gravitational field will always point directly inward with the same strength.

Now, for The Power Move: When you spot symmetry in a physical problem, you unlock incredible shortcuts. You can often:

- Predict results without calculating: Like knowing forces will be equal in magnitude.
- Cancel out components: Understanding that perpendicular components will perfectly cancel in symmetric arrangements.
- Solve half the problem: And confidently use symmetry to instantly get the other half.

Bottom line: Symmetry is your physics superpower!

Symmetry lets you see fundamental patterns that make even the most complex problems much simpler. Instead of grinding through messy math, you can say, "These two parts are identical, so they must behave identically," and literally cut your work in half!

This deep-seated "laziness" or consistency in nature's laws is precisely what we call symmetry. And here's the compelling part: these symmetries aren't just abstract concepts; they directly lead to the most fundamental conservation laws we know.

As Incredible As This Might Read

Because the laws of physics are the same everywhere in space (a spatial symmetry), we get the conservation of momentum. Because the laws don't change over time (a temporal symmetry), we get the conservation of energy. So, when nature is "lazy" and consistent, it gives us powerful, unchanging truths about the universe.

EmmyNoether

Even not direclty relevant for this class, we must mention Amalie Emmy Noether (23 March 1882 – 14 April 1935) was a German mathematician who made many important contributions to abstract algebra. She also proved Noether's first and second theorems, which are fundamental in mathematical physics. Noether was described by Pavel Alexandrov, Albert Einstein, Jean Dieudonné, Hermann Weyl and Norbert Wiener as the most important woman in the history of mathematics. As one of the leading mathematicians of her time, she developed theories of rings, fields, and algebras. In physics, Noether's theorem explains the connection between symmetry and conservation laws.

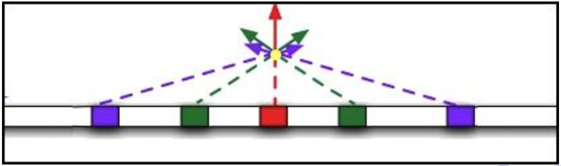
Cancellation by Symmetry: A First Look

The obvious symmetry of the figure, ensures the complete cancellation of the horizontal components of the fields, leaving the sum of the vertical components to produce a net vertical electric field only. In equations:

$$\mathbf{E} = [\sum_i E_{xi}]\hat{\mathbf{e}}_x + [\sum E_{yi}]\hat{\mathbf{e}}_y =$$

$$= \mathcal{E}_y \hat{\mathbf{e}}_y$$

where: $\mathcal{E}_y = \sum E_{yi}$, no horizontal component



Mathematical Tool:Cylindrical Coordinates

In situations, where axial symmetry is present, it is convenient to introduce cylindrical coordinates

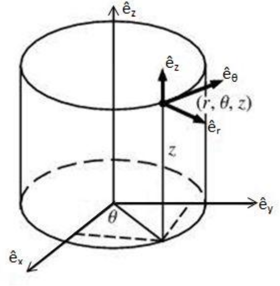
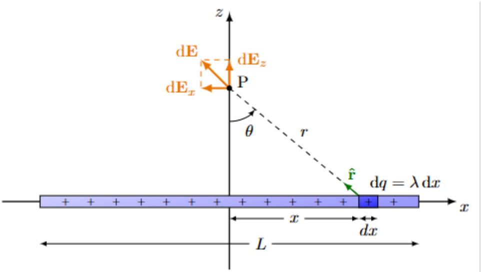


Figure: Cylindrical unit vectors. In these classes,to avoid confusion, we will use ρ for the radial coordinate and $\hat{\mathbf{e}}_\rho$ for the corresponding unit vector.

A generalization of the example we just studied is the calculation of the resulting electric field $\mathbf{E}_{BP}$ produced in any point along **the** bisection plane of a uniformly charged rod of finite length L. Symmetry considerations demonstrate that, field lines at the field points under consideration must be perpendicular to the rod.



Fortunately, we already possess the all the calculational tools to solve our problem.

We just need to make some adjustments to formula 8. Considering the axial symmetry of the problem we will use cylindrical coordinates to get

$$\boxed{\mathbf{E}_{BP} = E_\rho(\rho)\,\hat{\mathbf{e}}_\rho} \tag{10}$$

with

$$\boxed{E_\rho(\rho) = \frac{\lambda}{2\pi\epsilon_0\,\rho}\,\frac{L}{\sqrt{L^2+\rho^2}}} \tag{11}$$

Every course on electrostatics presents several interesting examples of field calculations that heavily utilize symmetry.

A classic example comes from an obvious extension of our previous problem, namely, extending the length to make the uniformly charged rod to become infinite in length.

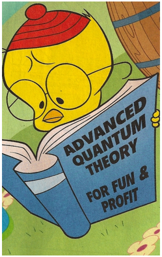
This modification does not spoil the axial symmetry but introduces a translational symmetry along the rod which makes any plane perpendicular to the rod a “bisecting” plane.

In brief

1. Due to its translational symmetry, every point along an infinite rod is equivalent, and thus any plane perpendicular to it is a plane of symmetry.

2. Therefore, for a positively charged rod, the electric field lines are all perpendicular to the rod and extend radially outwards from it. [what happens if the rod is negatively charged?]

Plan



1. Choose cylindrical coordinates so $\mathbf{E} = E(\rho)\,\hat{\mathbf{e}}_\rho$,
2. Define

$$E_\rho(\rho) = \lim_{L\to\infty} \frac{\lambda}{2\pi\epsilon_0\,\rho}\,\frac{L}{\sqrt{L^2+\rho^2}}$$

Upon executing our plan [do it!] we get

$$\boxed{\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0\,\rho}\,\hat{\mathbf{e}}_\rho} \tag{12}$$

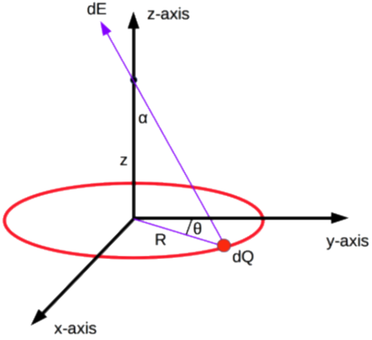
The precise meaning of this formula is:

- The field is radial (parallel to the unit vector $\hat{\mathbf{e}}_\rho$, accordingly, the field lines are perpendicular to the rod.
- The field intensity E_ρ depends on the distance to the rod only
- The dependence with the distance to the rod goes like ρ^{-1}

Students are urged to:

1. Review the entire calculation of the finite charged line.
2. Understand that the integration limits for the infinite line must be $\theta = -\pi/2$ and $\theta = \pi/2$.
3. Perform the calculation.

For our next example, we want to calculate the electric field produced by an uniformly charged ring (of charge density λ) along its central axis



Symmetry tells us that the field will be along the z axis [cancellation of horizontal components (why?)]

From our previous experience, it should not be hard to see that the contribution to the magnitude of the elementary field ($d\mathbf{E}$) at the field point due to the charge element dQ is

$$dE = \frac{1}{4\pi\epsilon_0}\frac{dQ}{R^2 + z^2}$$

To find the vertical component dE_z, we simply project over the z axis using

$$\cos\alpha = \frac{z}{\sqrt{R^2 + z^2}}$$

So,

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{z\,dQ}{(R^2 + z^2)^{3/2}} \tag{13}$$

Using the charge density, we know that $dQ = \lambda d(length)$, and recalling that a tiny arch has length $d(length) = R\,d\phi$, where ϕ is the arch' s angle.
In contrast to what happened with the line of charge, nothing depends on the angle making the integration trivial,

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{\lambda z R}{(R^2+z^2)^{3/2}}\int_0^{2\pi} d\phi = \frac{1}{4\pi\epsilon_0}\frac{\lambda R\, z}{(R^2+z^2)^{3/2}} 2\pi$$

Yielding

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{(2\pi\lambda R)z}{(R^2+z^2)^{3/2}} \tag{14}$$

If we:

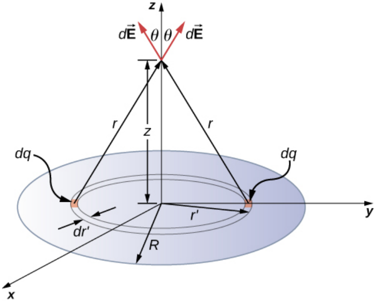
- Note that $2\pi R$ is the total length of the ring, $2\pi R\lambda = Q$ is the total charge of the ring.
- Introduce the unit vector $\hat{\mathbf{e}}_z$ pointing along the z axis, and
- combine the above with formula 14

we may write the final result, i.e., the field along the central axis of the ring of charge, as:

$$\boxed{\mathbf{E} = \begin{cases} \frac{Q}{4\pi\epsilon_0}\frac{z}{(R^2+z^2)^{3/2}}\hat{\mathbf{e}}_z, & z > 0 \\ -\frac{Q}{4\pi\epsilon_0}\frac{z}{(R^2+z^2)^{3/2}}\hat{\mathbf{e}}_z, & z < 0 \end{cases}} \tag{15}$$

Discussion: Why the signs?

Ready for a new challenge? We're going to figure out the electric field along the axis of a **disc** this time, and it's got a neat twist!



[The disc carries a uniform surface charge density σ]

Good news

There's a really intuitive way to approach this. We can think of the disc as being made up of a bunch of **infinitesimally thin rings** (picture them as incredibly tiny loops of charge!). This approach simplifies things a lot.

If we go along with this idea, the small contribution (dE_z) from just one of these rings to the field's vertical component looks like this (slightly modified from formula 15):

$$dE_z = \frac{1}{4\pi\epsilon_0}\frac{z\,dq_{\text{ring}}}{(r^2+z^2)^{3/2}} \tag{16}$$

In this formula, dq_{ring} represents the total charge on that single, thin ring.

- In this situation, we use **area charge density** (σ). This just means how much charge is spread out over a certain area. So, the tiny bit of charge on one of our rings (dq_{ring}) is equal to σ multiplied by the ring's tiny bit of area, or $dq_{\text{ring}} = \sigma\, d(\text{area}_{\text{ring}})$.
- Now, for the **area of one of those rings**, it's found by $d(\text{area}_{\text{ring}}) = 2\pi r\, dr$ where r is the radius. (Think about it: if you cut the ring and unroll it, it's pretty much a long, thin rectangle with length $2\pi r$ and width dr. Pretty neat, right?) We then plug this into formula 16.

After putting all the pieces together, we get the contribution from just one of those thin rings to the vertical component of the field produced by the whole disc. It looks like this:

$$\boxed{dE_z = \frac{\sigma z}{2\epsilon_0} \frac{r dr}{(r^2 + z^2)^{3/2}}} \tag{17}$$

Formula 17 looks like it depends on the radius (r) in a bit of a tricky way. This means we'll have to be super careful with our integration! But don't worry, figuring out the integration limits is actually pretty straightforward. We'll be integrating from the center of the disc ($r = 0$) all the way to its outer edge ($r = R$):

$$E_z = \frac{\sigma z}{2\epsilon_0} \int_0^R \frac{r dr}{(r^2 + z^2)^{3/2}} \tag{18}$$

To solve this integral, we can use a clever trick called **u-substitution**. It works like this:
Let $u = r^2 + z^2$. Then, if we take the derivative of u with respect to r, we get $du = 2r\,dr$. This little change helps transform our integral into something much simpler. Remember, when we change the variable from r to u, we also need to change the integration limits:

- When $r = 0$, $u = 0^2 + z^2 = z^2$.
- When $r = R$, $u = R^2 + z^2$.

So, our integral now looks like this:

$$E_z = \frac{2\pi\sigma z}{4\pi\epsilon_0}\frac{1}{2}\int_{z^2}^{R^2+z^2}\frac{du}{u^{3/2}} = -2\frac{\pi\sigma z}{4\pi\epsilon_0}u^{-1/2}\Big|_{z^2}^{R^2+z^2} \qquad (19)$$

And, after working through the calculus and plugging in our new limits, here's what we find for the electric field of the disc along its axis:

$$\boxed{E_z = \frac{\sigma}{2\epsilon_0}\left[\mathrm{sign}(z) - \frac{z}{\sqrt{R^2+z^2}}\right]} \tag{20}$$

The students are invited to discuss this formula and give the final vector expression of the result

A Fundamental Digression About The Charged Disk

Imagine yourself as tiny, very, very small and picture you very close to the disc we just studied. Argue this:
If you were measuring the electric field you would find

$$\boxed{E(z) = \begin{cases} \frac{\sigma}{2\epsilon_0} & z > 0 \\ -\frac{\sigma}{2\epsilon_0} & z < 0 \end{cases}} \tag{21}$$

From the mathematical point of view, $E(z)$ is discontinuous, and in fact the jump (value of the discontinuity) amounts to

$$\lim_{z \to 0} (E_+ - E_-) = \frac{\sigma}{\epsilon_0}$$

Under the conditions of this digression, what are you approximately describing?

PROBLEM: The electric field due to a uniformly charged plane can be calculated using a simple limit procedure on formula 20.

1. What's the simple procedure you'd use?
2. Can you give an intuitive argument (a 'common-sense' reason) for why this trick works?
3. What do you get as the final answer for the electric field (and remember to give it as a vector!)?

Hint: Most of the solution is is already in the slides


LECTURE SUMMARY
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- The electric field ($\mathbf{E}$) at a point in space is defined as the electric force ($\mathbf{F}$) experienced by a small positive test charge (q) placed at that point, divided by the magnitude of the test charge:

$$\mathbf{E} = \frac{\mathbf{F}}{q}$$

- Electric Fields observe the superposition Principle
- Fields may graphically been represented by field lines.
- The arrowheads on the field lines indicate the direction of the electric field (and thus the direction of the force on a positive test charge
- Field lines start from a positive charge and end on a negative charge

- Electric field lines never cross each other. If they did, it would imply that the electric field has two different directions at the same point, which is physically impossible
- The density of electric field lines (how close together they are) is proportional to the magnitude of the electric field. Where the lines are closer, the field is stronger; where they are farther apart, the field is weaker.