Introduction to Travelling Waves

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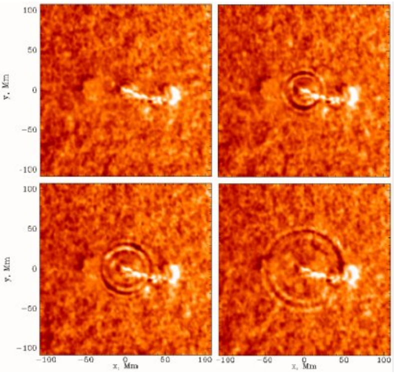
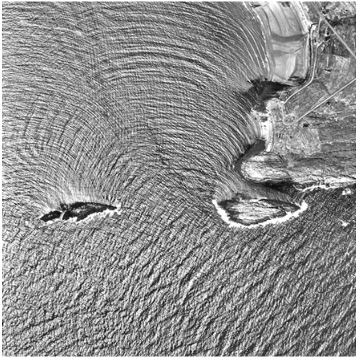
Waves are everywhere

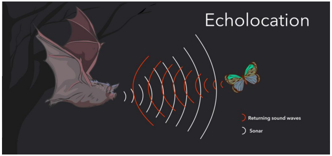


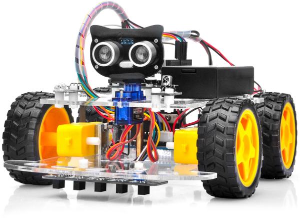


We Hear Them

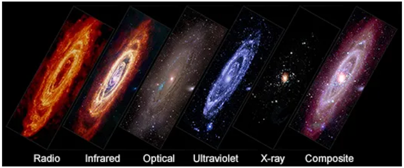
We play with them

Nature makes use of them .. and so we do





They are windows to the universe



They can show us hidden things

... and, the beauty, undeniable beauty

Sumarizing

There is great interest in waves because

- Wave behavior extends to many realms
- There are lots of interesting physical phenomena associated with waves themselves
- Teere is tech interest in waves, and ...
- Were those reasons not enough, waves are fundamental for quantum mechanics

Definition

A wave is any recognizable signal that is transferred from on part of the medium to another with a recognizable velocity of propagation.

G. B. Whitham, Linear and Nonlinear Waves, Wiley Interscience, ISBN 0471359424

Remark

Linear waves transport energy and momemtum, they do not transport matter.

Definition

A wave is a disturbance (signal) that propagates while maintaining certain relatively well-defined characteristics.

Remark

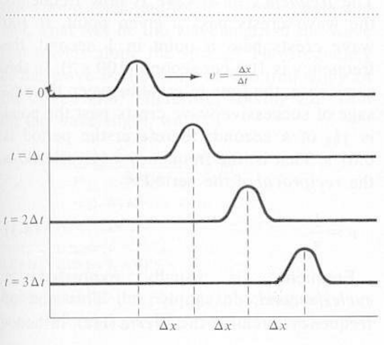
This way of saying things does not involve the need for any medium.

Remark

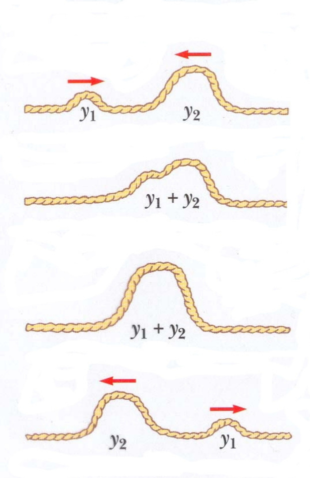
Electromagnetic and gravitational waves (March 17, 2014) are capable of propagating in a vacuum, requiring no medium for their support.

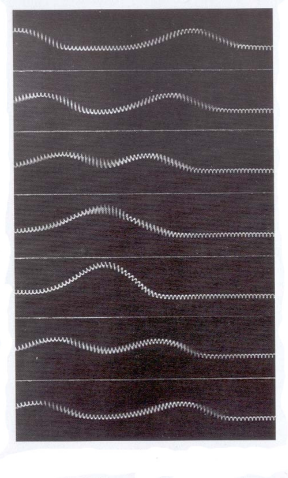
In D=1+1 dimensions, a disturbance traveling at velocity v can be represented using a real-valued function ($f : \Re \to \Re$, $f : x \in \Re \to f(x)$) via the formula

$$u(x,t) = f(x \pm v\,t) \tag{1}$$



Sometimes, waves can be added

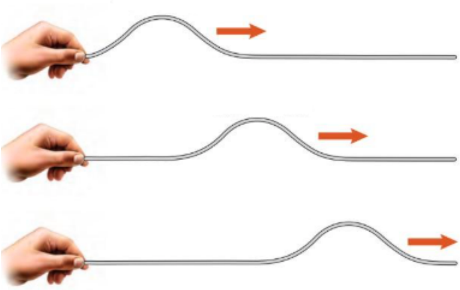




Sometimes, they can't

1. We want a mathematical model for traveling waves.
2. This model should work at least in $1+1$ (motion along a line)
3. We demand our model to include superposition as a property of waves
4. We would really like the model to be extensible for higher space dimensions

For positive times (t) a function like $f(x - vt)$ represents a signal traveling to the positive (usually right) region of the x axis, with velocity v.
This is a right moving wave or right mover as physicists usually call them.



It is clear that if $df(u)/du = F(u)$ then

$$\frac{\partial f(x-vt)}{\partial x} = F(x-vt)\,, \text{ and } \frac{\partial f(x-vt)}{\partial t} = -vF(x-vt)$$

This in turn means that a right mover satisfies the equation

$$\boxed{\frac{\partial f(x-vt)}{\partial x} + \frac{1}{v}\frac{\partial f(x-vt)}{\partial t} = 0} \tag{2}$$

Similarly, left movers $(f(x+vt))$ satisfy the equation

$$\boxed{\frac{\partial f(x+vt)}{\partial x} - \frac{1}{v}\frac{\partial f(x+vt)}{\partial t} = 0} \tag{3}$$

- The equations for right and left movers are called one way equations
- One way equations satisfy the superposition principle, this means that the sum of two rightmovers is a right mover and equally for left movers.
- A natural question arises is it possible to have an equation that allows both left and right movers? or stated differently, is there an equation describing traveling ways along any direction along the x axis?

A possible answer

- When beginning calculus many of us learnded that taking the second derivative of a function was the same as applying the product of two first derivative operators D to a function.
- The idea of the D operator and products of it was powerful to say the least.
- In fact, it allowed the introduction of a full algebra of operators related to differential equations.
- What if we attempt something similar?

- Intoduce ∂_L as the operator

$$\partial_L = \frac{\partial}{\partial x} - \frac{1}{v}\frac{\partial}{\partial t}$$

- We can ensure that the solutions of $\partial_L f(x,t) = 0$ are all left-movers.
- Similarly, the solutions to $\partial_R f = 0$ where

$$\partial_R = \frac{\partial}{\partial x} + \frac{1}{v}\frac{\partial}{\partial t}$$

are right movers

The fantastic observation is that the D'Alambert operator

$$\Box \equiv \partial_L \partial_R = \frac{\partial^2}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}$$

is such that the most general solution of

$$\Box f(x,t) = 0$$

is the superposition of a rightmover and a left mover, i.e.

$$f(x,t) = u(x - vt) + w(x + vt)$$

Proposition

Equation

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0 \tag{4}$$

satisfies our requirements for a wave propagation model

What the Wave Equation Tells Us

The one-dimensional wave equation, $\frac{\partial^2 u}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2} = 0$, is a cornerstone of physics.

- The term $\frac{\partial^2 u}{\partial x^2}$ describes the curvature of the wave shape in space. Think about how sharply the wave is bending or curving at a given point.
- The term $\frac{\partial^2 u}{\partial t^2}$ describes the acceleration of the wave at a given point in time. It tells us how rapidly the disturbance is changing its rate of change.
- The wave equation essentially states that the spatial curvature of the wave is directly proportional to its temporal acceleration. This constant of proportionality is related to the wave's speed, v.
- This fundamental relationship allows disturbances to propagate through space without changing their basic form.

Understanding the "Acceleration" of a Wave

When we see the term

$$\frac{\partial^2 u}{\partial t^2}$$

in the wave equation, it's easy to get confused. Let's clarify its meaning:

- This term represents the acceleration of the displacement or disturbance (u) at a fixed point in space (x).
- Imagine a single point on a vibrating string: its vertical position $u(x_0, t)$ changes over time, and $\frac{\partial^2 u}{\partial t^2}$ describes how that point is accelerating up or down.
- It does NOT mean the wave itself is speeding up or slowing down. The wave's propagation velocity (v) in the traveling wave $u(x, t) = f(x \pm vt)$, is constant.

Think of it like this: A car is driving down the road at a constant speed (v). But a passenger inside the car might be accelerating forward or backward in their seat (their "displacement" u relative to the seat is changing with its own acceleration).
So,

$$\frac{\partial^2 u}{\partial t^2}$$

refers to the acceleration of the oscillation of the medium (or field value) at a specific location, not the acceleration of the wave's overall movement.

Introducing the Harmonic Wave: Spatial Properties

A common and very useful type of traveling wave is the harmonic monochromatic wave, described by a cosine (or sine) function. For a wave traveling in the positive x-direction, its disturbance u can be written as:

$$u(x,t) = A\cos(kx - \omega t + \phi)$$

Here:

- A: The amplitude of the wave (the maximum displacement or strength of the disturbance).
- x: Position along the direction of propagation.
- t: Time.
- ϕ: The phase constant (determines the value of u at $x = 0, t = 0$).

Let's define the terms related to its spatial characteristics:

- Wave Number (k): This describes how rapidly the wave oscillates in space. It's related to the wavelength by:

  $$\boxed{k = \frac{2\pi}{\lambda}}$$

  Its units are radians per meter (rad/m) or inverse meters (m^{-1}).

- Wavelength (λ): This is the spatial period of the wave; the distance over which the wave's shape repeats itself. Its units are meters (m).

Introducing the Harmonic Wave: Temporal Properties

Now let's look at how the wave changes over time at a fixed point in space.

- Angular Frequency (ω): This describes how rapidly the wave oscillates in time. It's the temporal equivalent of the wave number.

$$\boxed{\omega = \frac{2\pi}{T}}$$

  Its units are radians per second (rad/s).

- Period (T): This is the temporal period of the wave; the time it takes for one complete oscillation to pass a fixed point in space. Its units are seconds (s).

- Frequency (f): This is the number of complete oscillations per unit time. It's the inverse of the period:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi}}$$

Its units are Hertz (Hz), where 1 Hz = 1 oscillation per second.

Visualizing Waves: The Beach Analogy

Imagine yourself at the beach, looking out at the ocean on a day when the waves are very regular.

- **Wavelength (λ): A "Snapshot" of Space**
  - If you were to **take a photograph of the ocean** at a single moment in time, the waves would look like a series of repeating peaks and troughs.
  - The **wavelength (λ)** is simply the **distance between two consecutive peaks** (or any two identical points on adjacent wave cycles).
  - It tells you how long one complete wave "segment" is in space.


λ

Visualizing Waves: Experiencing Time

Now, let's consider how the wave changes over time at a single location:

- **Frequency (f) and Period (T): Experiencing Time**
  - Picture yourself **floating on an inner tube** at a fixed spot in the water. As the waves pass by, you'll **bob up and down**.
  - The **period (T)** is the **time it takes for you to go through one complete up-and-down cycle**, returning to the exact same condition (e.g., from peak, down to trough, and back up to the next peak).
  - The **frequency (f)** is how often this happens. It's the **number of complete cycles you experience per second** (or per unit of time).
  - So, if your period is T seconds per cycle, your frequency is $f = 1/T$ cycles per second.

These two perspectives

A snapshot in space (wavelength) and a continuous observation in time (frequency/period) are fundamental to understanding all types of waves!

The Dispersion Relation: Connecting Space and Time

For many types of waves, including electromagnetic waves, the wave number (k) and angular frequency (ω) are related to the wave's speed (v) by a function $\omega = \omega(k)$ called the dispersion relation.
In this course, and unless told differently, the dispersion relation will be

$$\boxed{\omega = v\,k}$$

From the dispersion relation, we can connect all the properties we have been discussing

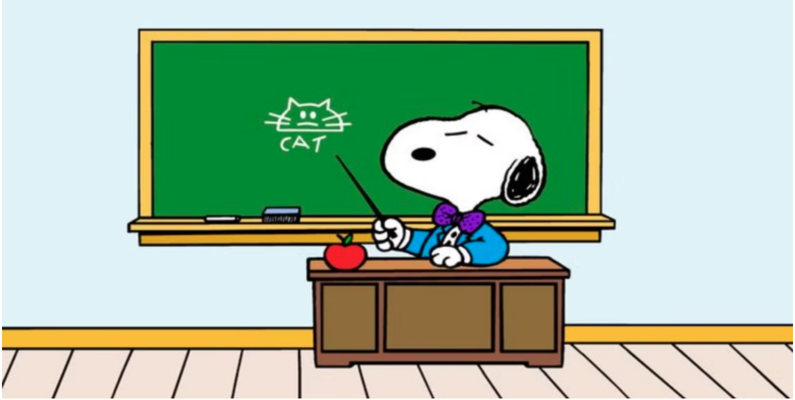
- Since $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$:

$$\frac{2\pi}{\lambda} = \frac{2\pi f}{v}$$

- This directly implies the famous relationship for the wave's speed:

$$\boxed{v = \lambda f}$$

This means that for a wave to travel at a constant speed v, its spatial (wavelength) and temporal (frequency) characteristics must be in a specific proportion.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Example 1: A Wave on a String

Let's consider a **transverse wave on a long string** described by the equation:

$$u(x,t) = 0.05\cos(2\pi x - 4\pi t)$$

Here, u is the vertical displacement of the string in meters, x is in meters, and t is in seconds. The phase constant $\phi = 0$.

Let's interpret the quantities:

- **Amplitude (A):**
  - From the equation, $A = 0.05$.
  - **Interpretation:** The maximum vertical displacement of any point on the string from its equilibrium position is **0.05 meters** (or 5 cm).

- **Wave Number (k):**
  - From the equation, $k = 2\pi$ rad/m.
  - **Interpretation:** For every meter you move along the string, the wave completes 2π radians of its oscillation cycle.
- **Wavelength (λ):**
  - Using $k = \frac{2\pi}{\lambda}$, we get $2\pi = \frac{2\pi}{\lambda} \Rightarrow \lambda = 1$ m.
  - **Interpretation:** The spatial length of one complete wave cycle is **1 meter**.
- **Angular Frequency (ω):**
  - From the equation, $\omega = 4\pi$ rad/s.
  - **Interpretation:** At any fixed point on the string, the oscillation completes 4π radians of its cycle every second.
- **Frequency (f):**
  - Using $f = \frac{\omega}{2\pi}$, we get $f = \frac{4\pi}{2\pi} = 2$ Hz.
  - **Interpretation:** Two complete wave cycles pass a fixed point on the string every second.

- **Period (T):**
  - Using $T = \frac{1}{f}$, we get $T = \frac{1}{2} = 0.5$ s.
  - **Interpretation:** It takes **0.5 seconds** for one complete wave cycle to pass a fixed point.
- **Wave Speed (v):**
  - Using $v = \frac{\omega}{k}$, we get $v = \frac{4\pi}{2\pi} = 2$ m/s.
  - **Interpretation:** The wave propagates along the string at a constant speed of **2 meters per second**.

Example 2: A Radio Wave

Let's look at a **radio wave** (an electromagnetic wave) traveling through space, described by its electric field component:

$$E(x,t) = 10\cos(0.5x - 1.5 \times 10^8 t)$$

Here, E is the electric field strength in Volts per meter (V/m), x is in meters, and t is in seconds. We've set $\phi = 0$.

Let's interpret the quantities:

- **Amplitude (A or $\mathcal{E}$):**
  - From the equation, $A = 10$.
  - **Interpretation:** The maximum strength of the electric field oscillation is **10 Volts per meter**.
- **Wave Number (k):**
  - From the equation, $k = 0.5$ rad/m.
  - **Interpretation:** The wave completes 0.5 radians of its oscillation for every meter it travels in space.

- **Wavelength (λ):**
  - Using $k = \frac{2\pi}{\lambda}$, we get $0.5 = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{0.5} \approx 12.57$ m.
  - **Interpretation:** The spatial length of one complete wave cycle is approximately **12.57 meters**. This is a typical wavelength for radio waves.
- **Angular Frequency (ω):**
  - From the equation, $\omega = 1.5 \times 10^8$ rad/s.
  - **Interpretation:** At any fixed point in space, the electric field oscillation completes 1.5×10^8 radians of its cycle every second.
- **Frequency (f):**
  - Using $f = \frac{\omega}{2\pi}$, we get $f = \frac{1.5\times10^8}{2\pi} \approx 2.39 \times 10^7$ Hz or **23.9 MHz**.
  - **Interpretation:** The wave oscillates approximately **23.9 million times per second**. This falls within the FM radio band.

- **Period (T):**
  - Using $T = \frac{1}{f}$, we get $T = \frac{1}{2.39 \times 10^7} \approx 4.18 \times 10^{-8}$ s.
  - **Interpretation:** It takes a tiny **41.8 nanoseconds** for one complete wave cycle to pass a fixed point.
- **Wave Speed (v):**
  - Using $v = \frac{\omega}{k}$, we get $v = \frac{1.5 \times 10^8}{0.5} = 3.0 \times 10^8$ m/s.
  - **Interpretation:** The radio wave propagates through space at a speed of 3.0×10^8 **meters per second**, which is the speed of light (c).