Capacitors: Storing Energy and Shaping Circuits

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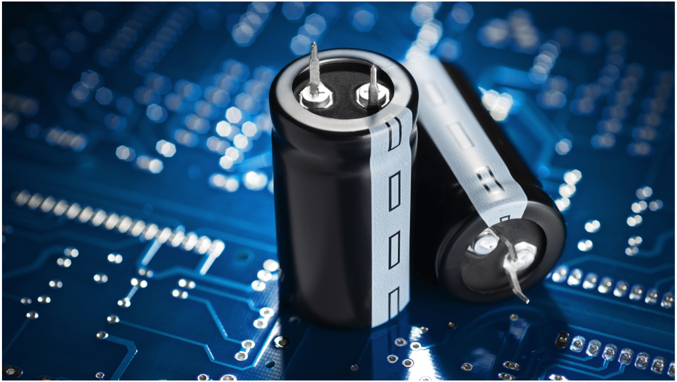
From Electrostatics to Technology: Storing Charge and Energy



- We have rigorously explored the fundamental laws governing **Electrostatics**:
  - Coulomb's Law: The force between stationary charges.
  - Electric Field: The influence of charges on space.
  - Gauss's Law: A powerful tool for calculating fields.
  - Electric Potential: The energy landscape for charges.
  - Conductors and Insulators: How materials behave in electric fields.
- These laws describe the "why" and "how" of static charge interactions.

- Now, we transition from pure theory to a concept that directly enables us to **manipulate electric phenomena** for practical applications:
  - We will explore devices designed to **store electric charge** and **electric potential energy**.
  - This fundamental understanding will directly bridge our knowledge to the exciting world of **modern electronics and technology**.

System of Conductors: Capacitance and Induction Coefficients

In electrostatic equilibrium, a system of n **conductors** (with charges $Q_1, Q_2, \ldots, Q_n$) in an otherwise empty space **will each be an equipotential object, possessing its own potential V_i.** Furthermore, the charge on any given conductor i can be expressed as a linear combination of the potentials of all conductors in the system:

$$Q_i = \sum_{j=1}^{n} C_{ij} V_j$$

The diagonal coefficients, C_{ii}, are called **coefficients of capacitance** (or self-capacitance), and they are always positive. The off-diagonal coefficients, C_{ij} where $i \neq j$, are called **coefficients of induction**

The capacitance of a conductor is therefore the total charge on the conductor when it is maintained at unit potential, all other conductors being held at zero potential.

Sometimes the capacitance of a system of conductors is also defined. For example, the capacitance of two conductors carrying equal and opposite charges in the presence of other grounded conductors is defined as the ratio of the charge on one conductor ,to the potential difference between them.

What is Capacitance?

We've learned how charges create electric fields and potentials. Now, let's explore a device designed to **store electric charge and electric potential energy**: the **capacitor**.

- A capacitor typically consists of two separated conducting plates or surfaces.
- When a potential difference (voltage) is applied across these conductors, charge moves from one conductor to the other, building up equal and opposite charges on the two conductors.
- Capacitors are fundamental components in nearly all electronic circuits, used for:
  - Energy storage (e.g., camera flashes, defibrillators)
  - Filtering out unwanted signals
  - Timing circuits
  - Smoothing power supplies

Defining Capacitance (C)

For a given capacitor, the amount of charge (Q) it stores is directly proportional to the potential difference (V) applied across its plates.

- This proportionality constant is called **Capacitance (C)**.
- We define capacitance as the ratio of the magnitude of the charge on either conductor to the magnitude of the potential difference between them:

$$\boxed{C = \frac{Q}{V}}$$

- **Key points:**
  - Q is the magnitude of the charge on *one* of the plates (the other has $-Q$).
  - V is the magnitude of the potential difference (voltage) between the plates.
  - Capacitance C is always a **positive scalar quantity**.
  - Capacitance depends only on the **geometry** of the conductors and the **insulating material** between them, not on Q or V.

Units of Capacitance: The Farad

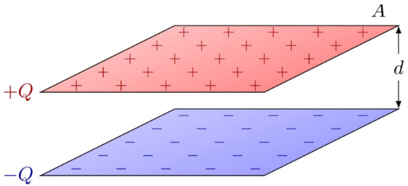
The SI unit for capacitance is the **Farad (F)**, named after Michael Faraday.

- From the definition $C = Q/V$, we can see that:

$$1 \text{ Farad (F)} = 1 \text{ Coulomb (C) / Volt (V)}$$

- The Farad is a very large unit of capacitance. Typical capacitors found in circuits have capacitances in the range of:
  - Microfarads (μF $= 10^{-6}$ F)
  - Nanofarads (nF $= 10^{-9}$ F)
  - Picofarads (pF $= 10^{-12}$ F)
- A capacitor with a capacitance of 1 F would be enormous.

The Parallel-Plate Capacitor



A parallel plate capacitor is nothing but two very thin conducting plates carrying charges $\sigma = \pm Q/A$ [figure]
The electric field between the plates is

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{\epsilon_0 A}$$

Very little extra work shows that the magnitude of the potencial difference between the plates is

$$V = Ed = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}$$

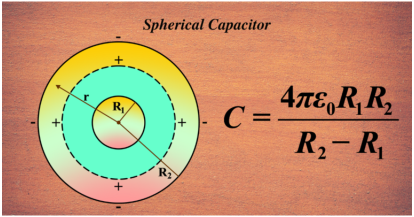
Using the definition (C=Q/V), we finally get

$$\boxed{C = \frac{\epsilon_0 A}{d}}$$

This formula teaches us the elements that determine capacitance: Area, distance; both geometric factors and Permittivity of Free Space (ϵ_0): This fundamental constant reflects the ability of a vacuum to permit electric fields. If an insulating material (a dielectric) is placed between the plates, this will change, and the capacitance will increase (more on this later).

Spherical Capacitor

This is also a simple device consisting of two concentric conducting spherical shells of internal and external radii R_{in} and R_{out} respectively. The inner sphere carries a charge $+Q$ and the outer shell carries a charge $-Q$.



We have already gained enough experience calculating electric fields as to be able to ensure that

$$\mathbf{E} = \begin{cases} 0 & r < R_{in} \\ \frac{Q}{4\pi\epsilon_0\, r^2}\,\hat{\mathbf{e}}_r & R_{in} < r < R_{out} \\ 0 & R_{out} < \end{cases} \tag{1}$$

The potential difference between the plates is

$$V_1 - V_2 = -\int_{R_2}^{R_1} \mathbf{E}\cdot d\mathbf{l}\,, \quad d\mathbf{l} = dr\hat{\mathbf{e}}_r$$

$$V_1 - V_2 = \int_{R_2}^{R_1} E dr = \int_{R_2}^{R_1} \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} dr \frac{Q}{4\pi\epsilon_0}\int_{R_2}^{R_1} r^{-2} dr = \frac{Q}{4\pi\epsilon_0}\left[-\frac{1}{r}\right]_{R_2}^{R_1}$$

$$V_1 - V_2 = \frac{Q}{4\pi\epsilon_0}\left(-\frac{1}{R_1} - (-\frac{1}{R_2})\right) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_2} - \frac{1}{R_1}\right)$$

We are really interested in V, the magnitude of the potential difference.

$$V = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

Using the definition to give a final expression

$$C = \frac{Q}{V} = \frac{Q}{\frac{Q}{4\pi\epsilon_0}\frac{R_2 - R_1}{R_1 R_2}}$$

The charge Q cancels out, leaving us with:

$$\boxed{C = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}}$$

This formula depends only on the **geometry** of the spheres (R_1, R_2) and the **permittivity of free space** (ϵ_0), as expected.

Special Case

If the outer sphere is sent to infinity ($R_2 \to \infty$), we get the capacitance of an isolated spherical conductor [take the absolute value]:

$$C_{isolated} = \left| 4\pi\epsilon_0 \frac{R_1 \cdot \infty}{\infty - R_1} \right| = 4\pi\epsilon_0 R_1$$

This is consistent with earlier results for the potential of an isolated sphere ($V = Q/(4\pi\epsilon_0 R)$).

The question of scale: How Big is 1 Farad

We will use our recent result of the spherical capacitor to address this question. Let's imagine a single, isolated spherical conductor that has a capacitance of exactly 1 Farad (1 F). What would its radius R be?

- We can rearrange the formula to solve for R:

$$R = \frac{C}{4\pi\epsilon_0}$$

- We know: $C = 1$ F, $\epsilon_0 \approx 8.854 \times 10^{-12}$ F/m so $1/(4\pi\epsilon_0) \approx 9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$ (Coulomb's constant, k_e)

Now, let's plug in the values:

$$R = (1 \text{ F}) \times (9 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2)$$

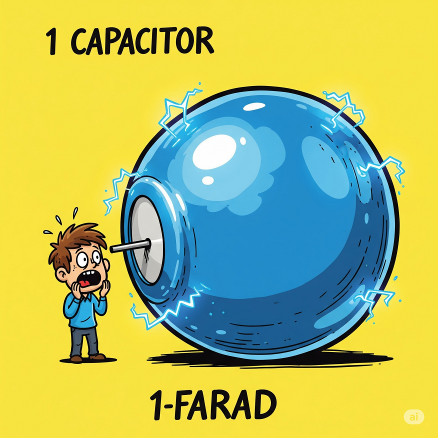
Wait, the units don't directly cancel in that form. Let's use ϵ_0 explicitly:

$$R = \frac{1 \text{ F}}{4\pi(8.854 \times 10^{-12} \text{ F/m})}$$

$$R \approx \frac{1}{1.112 \times 10^{-10} \text{ m}^{-1}} \approx 8.987 \times 10^{9} \text{ m}$$

The calculated radius for a 1-Farad spherical capacitor is approximately ...

$$\boxed{R_{1\mathrm{F}} \approx 8.987 \times 10^9 \ \text{meters}}$$



Why the Shock?

$\boxed{R_{1\mathrm{F}} \approx 8.987 \times 10^{9} \text{ meters}}$ is approximately 9 million kilometers. Now, the average radius of the Earth (R_{Earth}) is approximately $R_{\mathrm{Earth}} \approx 6.371 \times 10^{6}$ meters (or about 6,371 kilometers), meaning that

$$\boxed{\frac{R_{1\mathrm{F}}}{R_{\mathrm{Earth}}} \approx 1410}$$

A spherical capacitor with a capacitance of just 1 Farad would have a radius approximately 1410 times larger than the radius of the Earth!

Dealing With Scale, The Magic Of Dielectrics

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.. any electronic device houses lots and lots of capacitors, how is that possible if a Farad is such a massive unit?

Reading electronics schema one learns that practical capacitors are almost always measured in microfarads, nanofarads, or picofarads

....how can that be if even a spherical capacitor of $C = 1\,\mu F = 1 \times 10^{-6}\,F$ has a radius $R \approx 8987.55\ m$ which we might say is Everest size?.

The solution lies in dielectrics(insulators). Indeed, when engineers fill the internal "empty space" of a capacitor, with a dielectric, its capacity changes by a factor

$$C = \kappa C_0$$

where C_0 is the original capacitance and κ is referred to as the relative permittivity (or dielectric constant) of the dielectric being used.

Let's dive a little bit more into this tech challenge. Asking ourselves whether is there a material allowing to bckto our microfarad spherical ... would some material do the trick of shrinking the massive Everest size $1\mu F$ spherical capacitor to, say, a 50 cm dog size capacitor with the same capacitance?
We recall that

$$C = \kappa C_0$$

this means that to get the same capacitance in a smaller radius, you need a dielectric and the radii will be related by

$$R_{with\, dielectric} = \frac{R_{vacuum}}{\kappa}$$

The required dielectric constant κ would need to be:

$$\kappa = \frac{R_{vacuum}}{R_{target\ dog\ \ size}} = \frac{8990\, m}{0.5\, m} \approx 17980$$

Design Requirements

A dielectric material with a dielectric constant of approximately 18,000 is required to make a 1 μF spherical capacitor fit into a "dog-sized" object!

Typical Dielectric Constants (κ)

Material	Dielectric Constant (κ)
Air	≈ 1
Paper	$2 - 3.5$
Glass	$4 - 10$
Mica	$3 - 6$
Water (distilled)	≈ 80 (at room temperature)
Titanium Dioxide (TiO_2, Rutile)	$\approx 86 - 173$
Strontium Titanate ($SrTiO_3$)	≈ 310
Barium Titanate ($BaTiO_3$)	$1,200 - 10,000+$ (highly variable, T-dependent)
Calcium Copper Titanate (CCTO)	$> 250,000$ (reported, specialized conditions)

Findings

The table in the previous frame shows there are materials with the required κ.

However, these materials often come with challenges like high dielectric loss (energy dissipation) or strong temperature dependence, making them difficult for general purpose capacitors.

Material Science

Even though κ of 18,000 is extremely high, it's theoretically achievable with specialized ceramic materials like certain perovskites (e.g., Barium Titanate compounds, or even CCTO). So yes, with cutting-edge materials,a 1 μ F spherical capacitor that's roughly dog-sized could potentially created!

$E\upsilon\rho\eta\kappa\alpha$!

Dielectrics are the unsung heroes that make compact capacitors possible.

CCTO, Calcium Copper Titanate ($CaCu_3Ti_4O_{12}$) is a ceramic material that has gained significant attention in the field of dielectrics and advanced ceramics due to its "colossal permittivity" or "giant dielectric constant."

Energy of a System of Charges

Recall that **electric potential energy (U)** is associated with the configuration of a system of charges. Work must be done to assemble a system of charges.

- If we bring a first charge into empty space, no work is done ($U = 0$).
- To bring a second charge near the first, work is done against (or by) the electric field created by the first charge. This work becomes the potential energy of the two-charge system.

- For multiple charges, the total potential energy of the system is the sum of the work required to bring each charge from infinity to its final position, in the presence of all previously placed charges.

Where is the energy stored?

The stored potential energy is not "in" any single charge, but "in" the configuration of the entire system.

How energy is Saved

Capacitors are explicitly designed (and manufactured) to store electric potential energy. When a capacitor is charged, charge is moved from one plate to the other against the electric field, which requires work. This work is then saved in the capacitor from where it may be latter recovered

Energy Stored in a Capacitor

- Consider moving a small amount of charge dq from the negative plate to the positive plate.
- If the potential difference across the plates at that instant is V', the work done dW is $V'dq$.
- Since $V' = q/C$, the work done to add dq is:

$$dW = \frac{q}{C} dq$$

- To charge the capacitor from 0 to a total charge Q, the total work done is the integral:

$$W = \int_0^Q \frac{q}{C} dq = \frac{1}{C}\left[\frac{q^2}{2}\right]_0^Q = \frac{Q^2}{2C}$$

This work done is stored as the electric potential energy (U) in the capacitor:

$$\boxed{U = \frac{Q^2}{2C}}$$

The energy stored in a capacitor can be expressed in different, equivalent forms by using the definition of capacitance $C = Q/V$.

- Starting from $U = \frac{Q^2}{2C}$:
  - Substitute $Q = CV$:

$$U = \frac{(CV)^2}{2C} = \frac{C^2V^2}{2C} = \boxed{\frac{1}{2}CV^2}$$

  - This form is often most useful when C and V are known.

- Starting from $U = \frac{1}{2}CV^2$:
  - Substitute $V = Q/C$:

$$U = \frac{1}{2}C\left(\frac{Q}{C}\right)^2 = \frac{1}{2}C\frac{Q^2}{C^2} = \boxed{\frac{1}{2}QV}$$

  - This form is intuitive as it looks like "average voltage times total charge."

All three forms represent the same stored electric potential energy.

Capacitors in Circuits: From Storage to Analysis

We've now developed a solid understanding of:

- What a capacitor is: A device designed to store electric charge and potential energy.
- How its capacitance is determined by geometry and dielectric material.

These fundamental concepts are essential, and now we will transition to how we mathematically incorporate and analyze these energy-storing devices or combinations of them as fundamental elements within larger electric circuits. This understanding will provide the framework for analyzing circuit behavior.

Capacitors in Parallel Connection

When two capacitors are connected in parallel, their corresponding plates are joined.

- The positive plate of one of the capacitors is connected to the positive plate of the other. Similarly, the negative plates of both capacitors are connected
- These connections are done by conducting cables, which guarantees that the connected positive plates form one equipotential surface, and the negative plates form another equipotential surface.
- Consequently,both capacitors have the same potential difference (V) between them.

- The total charge (Q_{total}) stored in the parallel combination is the sum of the charges stored on each individual capacitor .

A New Capacitor

The connection between the two capacitors we just described has a practical consequence, we now have two conductors with some total charge and a potential difference between them, that is, a new capacitor of unknown capacity.

Our task is to find the capacity of this new device.

The original charges on the two capacitors satisfy

$$Q_1 = C_1 V \quad Q_2 = C_2 V$$

.

The total charge Q_{total} is the sum

$$Q_{total} = Q_1 + Q_2$$

which we may express as

$$Q_{total} = C_1 V + C_2 V = (C_1 + C_2) V$$

Meaning that the new configuration has an equivalent capacitance (C_{eq}) given by

$$\boxed{C_{eq} = C_1 + C_2}$$

The generalization is clear, if several capacitors are parallel connected, their equivalent capacity will be the sum of the individual capacities

Capacitors in Series Connection

When two or more capacitors are connected in series, they are connected end-to-end along a single path, without any branching points between them.

- When a series combination of capacitors is charged, the charge (Q) is the same on each capacitor.
  - This is because charge moved from one plate of a capacitor must displace an equal amount of charge from its other plate, which then moves to the next capacitor in the series, and so on.
  - Effectively, the "middle" plates (those connected only to each other in series) simply redistribute charge; no net charge is lost or gained on them.

- The total potential difference (V_{total}) across the entire series combination is the sum of the potential differences across each individual capacitor ($V_1, V_2, \ldots$).

A New Capacitor (again!)

Just like with parallel connections, a series combination of capacitors also acts as a single, new capacitor with some total charge and a total potential difference across its ends.

Our task -as happened with capacitors connected in parallel- is to find the capacity of this new equivalent device.

Thinking of three capacitors, let C_1, C_2, C_3 be the their individual capacitances.

- For each capacitor, $V = Q/C$. So, for the individual capacitors (remembering that Q is the same for all):

$$V_1 = \frac{Q}{C_1}, \quad V_2 = \frac{Q}{C_2}, \quad V_3 = \frac{Q}{C_3}$$

- The total potential difference V_{total} across the combination, on the other hand, equal the sum of the individual potential differences:

$$V_{total} = V_1 + V_2 + V_3$$

- or:

$$V_{total} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}$$

$$V_{total} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

- The equivalent capacitance (C_{eq}) is the one associated with that of a single capacitor that stores the same total charge Q with the same total potential difference V_{total}:

$$V_{total} = \frac{Q}{C_{eq}}$$

- so

$$\frac{Q}{C_{eq}} = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right)$$

Conclusion

The reciprocal of the equivalent capacitance of capacitors in series is the sum of the reciprocals of their individual capacitances.

$$\boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$

Various Capacitors


LECTURE SUMMARY
- PHIESOPHENT LIKE YOU PHILSOMENS
NEVET ARE PHEDANT
- PEGROBES YOUR SACKEST FLUNT
- COOR INY ANGHANDAY THE
- COPFRY
PHILESHPERT

- Capacitors are energy storing devices
- Picture a capacitor as a two conductor device with opposite charges in each conductor and voltage V across them.
- Charge and voltage in a capacitor are related by the formula

$$V = \frac{Q}{C}$$

- The energy stored in a capacitor is found by the formulas

$$U = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$$