Gambler's ruin approach to the "Optiver Prove It: Perpetual Children"challenge

Mario Veca

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1 Introduction

This document presents a solution to the "Optiver Prove It: Perpetual Children"challenge, a probability problem designed to test analytical thinking and mathematical reasoning. The problem describes a simple but intriguing scenario: a couple starts by having a first child, then continues to have children until the number of male and female children is equal. We assume that each child is independently male or female with probability $\frac{1}{2}$ at each birth. Of course, we are simplifying the real-world case by neglecting biological constraints, such as the impossibility of arbitrarily high numbers of childbirths. The goal of the challenge is to prove that this process will eventually terminate, even though the expected number of children (or the expected time for termination) is infinite.

In this document, we use the gambler's ruin problem as a framework, modeling the process as a random walk with an absorbing state to establish its eventual termination. This analogy allows us to rigorously derive the probability of eventual stopping and prove that it must be equal to 1.

The original problem can be found on the Optiver Prove It: Perpetual Children challenge page.

2 Proof

Proving that the process of child generation will eventually end is analogous to a typical "gambler's ruin"problem within a fair game. Suppose, without loss of generality, that the first child born is male (the case of a female firstborn is analogous). We call k the difference between the number of male children and female children and define P_k^{STOP} as the probability that the process will eventually end if we start with k more males than females. We therefore need to compute $P_{k=1}^{\text{STOP}}$, knowing the trivial stopping condition $P_{k=0}^{\text{STOP}} = 1$.
Since each new child is independently male or female with probability $\frac{1}{2}$, we can express the probability of eventual stopping recursively as:

$$P_k^{\text{STOP}} = \frac{1}{2} P_{k-1}^{\text{STOP}} + \frac{1}{2} P_{k+1}^{\text{STOP}} \ . \tag{1}$$

Rewriting the above equation:

$$\frac{1}{2}\left(P_{k+1}^{\text{STOP}} - P_k^{\text{STOP}}\right) = \frac{1}{2}\left(P_k^{\text{STOP}} - P_{k-1}^{\text{STOP}}\right) \ . \tag{2}$$

Defining $a_k = \left(P_k^{\text{STOP}} - P_{k-1}^{\text{STOP}}\right)$, we find the recurrence relation:

$$a_k = a_{k-1} \ , \tag{3}$$

which has the general solution $a_k = a_0$ for all k, with a_0 to be determined.
We can now show that the only possible value that a_0 can take, consistently with the condition that P_k^{STOP} is a probability for every k, is $a_0 = 0$. The above solution, indeed, implies that

$$P_{k+1}^{\text{STOP}} - P_k^{\text{STOP}} = a_0 \ . \tag{4}$$

Summing from $k = 0$ to $k = N - 1$ for an arbitrarily large natural number N, we find:

$$P_N^{\text{STOP}} - P_0^{\text{STOP}} = N a_0 \ , \tag{5}$$

which simplifies to:

$$P_N^{\mathrm{STOP}} = Na_0 + 1 \ . \tag{6}$$

Since P_N^{STOP} must remain a probability (i.e., between 0 and 1), it follows that $a_0 \leq 0$. However, if a_0 were negative, then for sufficiently large N, P_N^{STOP} would become negative, which is impossible. Thus, $a_0 = 0$, and from Eq. (4) evaluated at $k = 0$, we conclude:

$$P_1^{\mathrm{STOP}} = 1 \ , \tag{7}$$

which proves that the process will always terminate. By symmetry, the case where the firstborn child is female follows the same reasoning, requiring us to show that $P_{k=-1}^{\mathrm{STOP}} = 1$, which follows directly from Eq. (4) evaluated at $k = -1$. Thus, regardless of whether the first child is male or female, the process will always eventually terminate.