Background. The power spectrum is defined as

$$S_x(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} r_x(k)e^{-\jmath\omega k}, \tag{1}$$

where the autocorrelation $r_x(k) = E[x(n+k)x(n)]$ is a symmetric function,

$$r_x(k) = r_x(-k). \tag{2}$$

Since the autocorrelation function is rarely known, it has to be estimated from the N observed samples $x(0), \dots, x(N-1)$ using the *sample autocorrelation*:

$$\hat{r}_x(k) = \frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x(n), \tag{3}$$

which is a symmetric function

$$\hat{r}_x(k) = \hat{r}_x(-k), \quad -(N-1) \leq k \leq (N-1). \tag{4}$$

The *periodogram* is defined as

$$\hat{S}_x(e^{\jmath\omega}) = \sum_{k=-(N-1)}^{N-1} \hat{r}_x(k)e^{-\jmath\omega k}, \tag{5}$$

Task. Let us try to find an expression which relates $\hat{S}_x(e^{\jmath\omega})$ directly to $x(n)$. As often is the case in digital signal processing, it is helpful to use the z-transform. The z-transform counterpart to the power spectrum in (1) is

$$S_x(z) = \sum_{k=-\infty}^{\infty} r_x(k)z^{-k}. \tag{6}$$

Since the z-transform requires that the signal extends from $-\infty$ to ∞, we simply assume that the signal $x(n)$ is zero before $n = 0$ and after $n = N-1$. This "zero extension" of $x(n)$ implies that

$$\hat{r}_x(k) = 0, \quad |k| > N-1. \tag{7}$$

For simplicity, the same notation is used for the extended signal as for the original N-length signal. Combining (6) and (3), we obtain

$$\hat{S}_x(z) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x(n) \right) z^{-k} = \mathcal{Z} \left(\frac{1}{N} \sum_{n=0}^{N-1-k} x(n+k)x(n) \right). \tag{8}$$

From this expression we note that computing $\hat{S}_x(z)$ boils down to computing the z-transform of the autocorrelation—an operation which bears striking resemblance to convolution! We know from the fundamental course in signal processing that the convolution of $x(n)$ with $h(n)$ is defined as

$$y(k) = \sum_n x(k-n)h(n) = \sum_n x(-n+k)h(n), \tag{9}$$

which corresponds to multiplication in z-domain,

$$Y(z) = X(z)H(z). \tag{10}$$

When comparing autocorrelation with convolution, we notice that time is reversed in autocorrelation since $x(n+k)x(n)$ is computed rather than $x(-n+k)x(n)$. Thus, we make use of the result that a time reversed signal $x(-n)$ has the z-transform $X(z^{-1})$, see table in textbook. Thus,

$$\hat{S}_x(z) = \mathcal{Z}\left(\frac{1}{N}\sum_{n=0}^{N-1-k} x(n+k)x(n)\right) \tag{11}$$

$$= \frac{1}{N}\mathcal{Z}\left(x(n) * x(-n)\right) \tag{12}$$

$$= \frac{1}{N}X(z)X(z^{-1}) \tag{13}$$

In order to arrive at an expression which relates $\hat{S}_x(e^{\jmath\omega})$ directly to $x(n)$, we evaluate $\hat{S}_x(z)$ on the unit circle, i.e., $z = e^{\jmath\omega}$,

$$\hat{S}_x(z = e^{\jmath\omega}) = \frac{1}{N}X(e^{\jmath\omega})X(e^{-\jmath\omega}). \tag{14}$$

Since $X(e^{-\jmath\omega}) = X^*(e^{\jmath\omega})$ (see note below), we have that

$$\hat{S}_x(e^{\jmath\omega}) = \frac{1}{N}|X(e^{\jmath\omega})|^2 = \frac{1}{N}\left|\sum_{n=0}^{N-1} x(n)e^{-\jmath\omega n}\right|^2, \tag{15}$$

and thus we have accomplished our task! The right-hand side is obtained by simply computing the FFT of $x(n)$, squaring its magnitude, and finally dividing by N.

Note:

$$X(e^{-\jmath\omega}) = \sum_{n=0}^{N-1} x(n)e^{\jmath\omega n} \tag{16}$$

$$X^*(e^{\jmath\omega}) = \left(\sum_{n=0}^{N-1} x(n)e^{-\jmath\omega n}\right)^* = \text{(recall that } x(n) \text{ is real-valued)} \tag{17}$$

$$= \sum_{n=0}^{N-1} x(n)e^{\jmath\omega n} = X(e^{-\jmath\omega}) \tag{18}$$

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