BIOELECTRICAL SIGNAL PROCESSING

IN CARDIAC AND NEUROLOGICAL APPLICATIONS

SOLUTIONS MANUAL

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Preface

This manual offers detailed solutions to the problems contained in our book "Bioelectrical Signal Processing in Cardiac and Neurological Applications" (Elsevier/Academic Press, 2005). We would be grateful to receive any comments, suggestions, or corrections that the reader may have.

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Solutions Manual

Chapter 3

3.1 Calculating the required values we obtain

$$\begin{aligned}\overline{x}_1 &= 0\\ \sigma^2_{x_1} &= A^2\\ \gamma_s(x_1) &= 0;\\ \gamma_k(x_1) &= -2;\end{aligned}$$

and

$$\begin{aligned}\overline{x}_2 &= 0\\ \sigma^2_{x_2} &= A^2\\ \gamma_s(x_2) &= 0;\\ \gamma_k(x_2) &= -0.3125;\end{aligned}$$

Analyzing these values we realize that:

1. The mean is the same (i.e., zero) for both signals, evident from the PDF analysis.
2. The variance is also the same in both cases, coming from the fact that the higher contribution of larger peaks in x_2 are compensated by the lower contribution of the smaller peaks and the results is as for x_1 where the peaks are more evenly distributed.
3. The skewness γ_s is the same in both cases and equal to zero, meaning that the PDF is symmetric for positive and negative values.
4. The kurtosis γ_k differs (from –2 to –1.4375), and is the only difference between the two signals. This means that the kurtosis becomes larger

as the dispersion of the PDF increases, and then becomes a good estimate of how a random process moves from a Gaussian to a Laplacian PDF (larger tails).

Note that x_1 does not have a Gaussian PDF (which would imply $\gamma_k = 0$). Rather, it has a probability density function $P_{x_1}(x_1)$ with only values at $x_1 = \pm A$,

3.2 In order to derive (3.77) we should differentiate the error

$$\mathcal{E}'_{\mathbf{h}} = \mathcal{E}_{\mathbf{h}} + \nu \sum_{i=1}^{M} (\mathbf{h}_i - \overline{\mathbf{h}}_i)^T (\mathbf{h}_i - \overline{\mathbf{h}}_i) \tag{3.1}$$

with respect to $\mathbf{h}_j$ by making

$$\nabla_{\mathbf{h}_j} \mathcal{E}'_{\mathbf{h}} = 0, \quad j = 1, \ldots, M.$$

The gradient will be composed of the term already solved in (3.72), resulting from differentiation of $\mathcal{E}_{\mathbf{h}}$, and a new term that results from differentiation of the second part in (3.1). Together, we obtain

$$E\left[\tilde{\mathbf{v}}_j(n)\left(x(n) - \sum_{i=1}^{M} \mathbf{h}_i^T \tilde{\mathbf{v}}_i(n)\right)\right] + \nu(\mathbf{h}_i - \overline{\mathbf{h}}_i) = 0, \quad j = 1, \ldots, M.$$

Solving as for (3.73) but now also considering the new term, we obtain the desired equation

$$\left(\begin{bmatrix} \mathbf{R}_{\mathbf{v_1 v_1}} & \ldots & \mathbf{R}_{\mathbf{v_1 v_M}} \\ \vdots & \ddots & \vdots \\ \mathbf{R}_{\mathbf{v_M v_1}} & \ldots & \mathbf{R}_{\mathbf{v_M v_M}} \end{bmatrix} - 2\nu\mathbf{I}\right) \begin{bmatrix} \mathbf{h}_1 \\ \vdots \\ \mathbf{h}_M \end{bmatrix} = \begin{bmatrix} \mathbf{r}_{xv_1} - 2\nu\overline{\mathbf{h}}_1 \\ \vdots \\ \mathbf{r}_{xv_M} - 2\nu\overline{\mathbf{h}}_M \end{bmatrix}.$$

3.3 Recognizing that $\hat{r}_x(k)$ is the only random variable on the right hand side of (3.79), it is straightforward to show that

$$E\left[\hat{S}_x(e^{\jmath\omega})\right] = \cdots = \sum_{k=-N+1}^{N-1} E[\hat{r}_x(k)]e^{-\jmath\omega k}.$$

The remaining part of the result in (3.82) is found by studying the estimated correlation function (3.78) and realizing that it is a biased estimator. When calculating the spectrum, the estimate $\hat{r}_x(k)$, $-N+1 \leq k \leq N+1$, is used, where $\hat{r}_x(-k) = \hat{r}_x(k)$. Thus, since $r_x(k) = E[x(n+k)x(n)]$, we have that

$$E\left[\hat{S}_x(e^{\jmath\omega})\right] = \sum_{k=-N+1}^{N-1} \frac{N-|k|}{N} r_x(k)e^{-\jmath\omega k},$$

where

$$w_B(k) = \frac{N - |k|}{N}, \quad -N \le n \le N,$$

denotes the Bartlett window, i.e., triangular weighting caused by the biased estimate of the correlation function in (3.78).

We may instead use the unbiased estimate of the correlation function

$$\hat{r}_x(k) = \frac{1}{N-k} \sum_{n=0}^{N-1-k} x(n+k)x(n), \quad 0 \le k < N.$$

However, the unbiased property comes at expense of a very large variance for large lags k due to that fewer and fewer product terms are included in the summation.

3.4 Find that value of b which minimizes the quadratic cost function

$$J(b) = \int_{-\pi}^{\pi} (\log S_x(e^{\jmath\omega}) - b|\omega|)^2 d\omega.$$

Differentiation yields

$$\begin{aligned}
\frac{dJ}{db} &= \frac{d}{db} \int_{-\pi}^{\pi} (\log S_x(e^{\jmath\omega}) - b|\omega|)^2 d\omega \\
&= \int_{-\pi}^{\pi} \frac{d}{db} (\log S_x(e^{\jmath\omega}) - b|\omega|)^2 d\omega \\
&= \int_{-\pi}^{\pi} 2(-|\omega|)(\log S_x(e^{\jmath\omega}) - b|\omega|) d\omega.
\end{aligned}$$

Finally, by setting

$$\frac{dJ}{db} = 0$$

and rearranging the expression, we obtain the following estimate

$$\hat{b} = \frac{\displaystyle\int_{-\pi}^{\pi} |\omega| \log S_x(e^{\jmath\omega}) d\omega}{\displaystyle\int_{-\pi}^{\pi} \omega^2 d\omega} = \frac{3}{2\pi^3} \int_{-\pi}^{\pi} |\omega| \log S_x(e^{\jmath\omega}) d\omega.$$

3.5 The spectral moments are defined as

$$\overline{\omega}_n = \int_{-\pi}^{\pi} \omega^n S_x(e^{\jmath\omega}) d\omega,$$

and can be related to the “analog” frequency Ω through the sampling period T_s and the following relations

$$\begin{aligned} \Omega &= F_s 2\pi f = \frac{1}{T_s}\omega, \\ d\Omega &= \frac{1}{T_s} d\omega, \end{aligned}$$

and

$$S_x(\Omega) = \begin{cases} T_s S_x(e^{\jmath\omega})\lfloor_{\omega=\Omega T_s} & |\Omega| \le 2\pi F_s/2 \\ \\ 0, & |\Omega| > 2\pi F_s/2, \end{cases}$$

where the band-limited spectrum $S_x(e^{\jmath\omega})$ for the discrete-time signal is multiplied by T_s in order to preserve the spectral energy. Using these three relations, the spectral moments of the continuous-time case can be expressed as

$$\begin{aligned} \overline{\Omega}_n &= \frac{\overline{\omega}_n}{T_s^n} \\ &= \frac{1}{T_s^n}\int_{-\infty}^{\infty} (\Omega T_s)^n \frac{1}{T_s} S_x(\Omega) T_s d\Omega \\ &= \int_{-\infty}^{\infty} \Omega^n S_x(\Omega) d\Omega \end{aligned} \tag{3.2}$$

Since $S_x(\Omega)$ is an even function of Ω, i.e., $S_x(\Omega) = S_x(-\Omega)$, only even-valued spectral moments ($n = 0, 2, \dots$) are non-zero. Therefore, $\overline{\Omega}_n$ can be expressed as

$$\overline{\Omega}_n = \int_{-\infty}^{\infty} (-1)^{n/2} (\jmath\Omega)^n S_x(\Omega) d\Omega, \quad n = 0, 2, 4, ...$$

A useful observation when evaluating (3.2) is that the integrand can by viewed as the power spectrum following one or several differentiators defined by the transfer function $H(\Omega) = \jmath\Omega$. This observation is based on the general result from filtering of a stationary, stochastic process, stating that the power spectrum $S_y(\Omega)$ of the output signal $y(t)$ is related to the power spectrum $S_x(\Omega)$ of the input signal $x(t)$ through

$$S_y(\Omega) = |H(\Omega)|^2 S_x(\Omega).$$

Hence, for $n = 2$, we have that

$$S_{x'}(\Omega) = |\jmath\Omega|^2 S_x(\Omega) = \Omega^2 S_x(\Omega),$$

where the prime in $S_{x'}(\Omega)$ denotes that the signal has been differentiated once. Then, we obtain

$$\begin{aligned}\overline{\Omega}_2 &= \int_{-\infty}^{\infty} (-1)(-\Omega)^2 S_x(\Omega) d\Omega \\ &= \int_{-\infty}^{\infty} \Omega^2 S_x(\Omega) d\Omega \\ &= \int_{-\infty}^{\infty} S_{x'}(\Omega) d\Omega.\end{aligned}$$

The filtering interpretation of this operation is presented in Figure 3.1 for the case when $n = 4$.

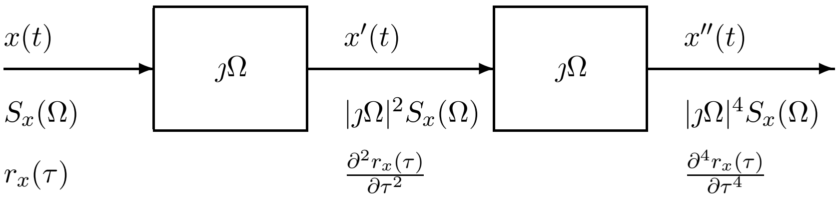


Figure 3.1: Differentiation of the input signal $x(t)$ and the corresponding effect on the related autocorrelation function $r_x(\tau)$ and power spectrum $S_x(\Omega)$, respectively.

We recall that the autocorrelation function $r_x(\tau)$ and the power spectrum $S_x(\Omega)$ are related to each other by the inverse Fourier transform,

$$r_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\Omega) e^{\jmath\Omega\tau} d\Omega.$$

The effect of differentiation on the autocorrelation function is described by the Fourier transform pair

$$(\jmath\Omega)^n S_x(\Omega) \xleftrightarrow{\mathcal{F}} \frac{\partial^n r_x(\tau)}{\partial\tau^n}.$$

Based on the above observations, the time domain expressions of the spectral moments are obtained as

$$\begin{aligned}
\overline{\omega}_0 = T_s^0\overline{\Omega}_0 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_x(\Omega)e^{j\Omega\tau}d\Omega\bigg|_{\tau=0} = 2\pi r_x(\tau)|_{\tau=0} = 2\pi E\left[x(t)^2\right], \\
\overline{\omega}_2 = T_s^2\overline{\Omega}_2 &= -2\pi T_s^2\frac{\partial^2 r_x(\tau)}{\partial\tau^2}\bigg|_{\tau=0} = 2\pi T_s^2 E\left[(x'(t))^2\right], \\
\overline{\omega}_4 = T_s^4\overline{\Omega}_4 &= 2\pi T_s^4\frac{\partial^4 r_x(\tau)}{\partial\tau^4}\bigg|_{\tau=0} = 2\pi T_s^4 E\left[(x''(t))^2\right].
\end{aligned}$$

where the stationarity property has been used to obtain

$$\begin{aligned}
\frac{\partial^2 r_x(\tau)}{\partial\tau^2} &= \frac{\partial^2 E[x(t)x(t+\tau)]}{\partial\tau^2} \\
&= \frac{\partial E[x(t)x'(t+\tau)]}{\partial\tau} \\
&= \frac{\partial E[x(t-\tau)x'(t)]}{\partial\tau} \\
&= -E[x'(t-\tau)x'(t)]
\end{aligned}$$

and similar for higher orders of even value. From this result, it is obvious that the spectral moments can be computed directly from the signal and its derivatives without having to first calculate the power spectrum!

3.6

a) Let us assume that the EEG signal is modeled by the power spectrum shown in Figure 3.2 (for simplicity, we assume that $T_s = 1$).

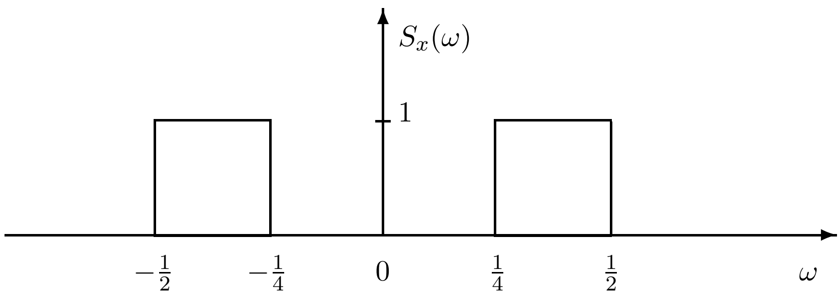


Figure 3.2: Power spectrum $S_x(\omega)$ representing the power spectrum of an EEG signal.

We can compute the $\mathcal{H}_1 = \sqrt{\frac{\overline{\omega}_2}{\overline{\omega}_0}}$ from the expressions

$$\overline{\omega}_2 = \int_{-1/2}^{-1/4} \omega^2 d\omega + \int_{1/4}^{1/2} \omega^2 d\omega = \frac{7}{96} = 0.0729$$

and

$$\overline{\omega}_0 = 2\int_{1/4}^{1/2} d\omega = \frac{1}{2},$$

which yields

$$\mathcal{H}_1 = \sqrt{\frac{\overline{\omega}_2}{\overline{\omega}_0}} \approx 0.3818.$$

This value is a very good approximation of the mean value at $\omega = 0.375$ which constitutes the center point of the power spectrum (and easily computed directly from the diagram!).

b) To compute the *complexity* Hjorth descriptor we need the fourth order moment

$$\overline{\omega}_4 = \int_{-1/2}^{-1/4} \omega^4 d\omega + \int_{1/4}^{1/2} \omega^4 d\omega = \frac{31}{2560} = 0.0121$$

and thus

$$\mathcal{H}_2 = \sqrt{\frac{\overline{\omega}_4}{\overline{\omega}_2} - \frac{\overline{\omega}_2}{\overline{\omega}_0}} \approx 0.1423.$$

This value is also a very good approximation of half the bandwidth which is $\Delta\omega/2 = 0.125$, constituting half the width of the power spectrum (and which easily computed directly from the diagram!).

c) The spectral purity index Γ_{SPI} can now be calculated by

$$\Gamma_{\text{SPI}} = \frac{\overline{\omega}_2^2}{\overline{\omega}_0 \overline{\omega}_4} = 0.8781,$$

suggesting that the power spectrum is relatively well-concentrated (band-limited) in the spectral domain.

3.7 Since only two spectral components are found, the modulating signal must be a constant amplitude sinusoid. The amplitude modulated signal may be expressed as

$$\begin{aligned} s(t) &= A\cos(2\pi F_{AM}t)\cos(2\pi 9t) \\ &= \frac{A}{2}[\cos(2\pi(9 - F_{AM})t) + \cos(2\pi(9 + F_{AM})t)], \end{aligned}$$

from which it can be concluded that the frequency of the modulating signal is $F_{AM} = 1$ Hz.

Hence, the Fourier transform is a linear tool which cannot unveil non-time-invariant processes such as the amplitude modulated signal.

3.8 The Yule-Walker equations for *the model* AR(p) are

$$\begin{bmatrix} r_x(0) & r_x(-1) & \dots & r_x(-p) \\ r_x(1) & r_x(0) & \dots & r_x(-p+1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(p) & r_x(p-1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_{e_p}^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For the real-valued case, the Yule-Walker equations correspond to the normal equations of linear prediction with the *prediction model* order $p' = p$. Now, assumed $p' > p$ is used in the linear prediction. Then, the normal equations would be

$$\begin{bmatrix} r_x(0) & r_x(1) & \dots & r_x(p') \\ r_x(-1) & r_x(0) & \dots & r_x(p'-1) \\ \vdots & \vdots & & \vdots \\ r_x(-p') & r_x(-p'+1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1^{(p')} \\ \vdots \\ a_{p'}^{(p')} \end{bmatrix} = \begin{bmatrix} \sigma_{e_{p'}}^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The correlation matrix has full rank, and thus the solution to these normal equations is unique. If the AR(p) model is extended into a virtual AR(p') model by adding the coefficients $a_{p+1} = a_{p+2} = \cdots = a_{p'} = 0$, the Yule-Walker equations become

$$\begin{bmatrix} r_x(0) & r_x(-1) & \dots & r_x(-p) & r_x(-p-1) & \dots & r_x(-p') \\ r_x(1) & r_x(0) & \dots & r_x(-p+1) & r_x(-p) & \dots & r_x(-p'+1) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ r_x(p) & r_x(p-1) & \dots & r_x(0) & r_x(-1) & \dots & r_x(-p'+p) \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ r_x(p') & r_x(p'-1) & \dots & r_x(p'-p) & r_x(p'-p-1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_{e_p}^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The correlation matrix for the AR(p') model corresponds to the one of the normal equations for the linear prediction order p'. Hence, since the coefficient vector and the right hand side of the Yule-Walker equations is a valid solution to the normal equations, it is concluded that

$$a_l^{(p')} = \begin{cases} a_l^p, & 1 \le l \le p' \\ 0, & p+1 \le l \le p', \end{cases}$$

and

$$\sigma_{e_{p'}^2} = \sigma_{e_p}^2.$$

3.9 The backward prediction filter of length p is

$$\hat{x}(n-p) = -b_1 x(n-p+1) - b_2 x(n-p+2) - \cdots - b_p x(n).$$

Introduce the backward prediction error

$$e_p^-(n) = x(n-p) - \hat{x}(n-p) = \mathbf{b}_p^T \mathbf{x}(n),$$

where the backward filter coefficient vector is $\mathbf{b}_p = [1\ b_1\ b_2 \cdots\ b_p]^T$ and the sample vector is $\mathbf{x}(n) = [x(n-p)\ x(n-p+1)\ \cdots\ x(n)]^T$, and then use the orthogonality principle:

$$E[\mathbf{x}(n) e_p^-(n)] = E[\mathbf{x}(n)\mathbf{b}_p^T \mathbf{x}(n)] = E[\mathbf{x}(n)\mathbf{x}^T(n)]\mathbf{b}_p = \mathbf{R}\mathbf{b}_p = \sigma_{e_p^-}^2 \mathbf{i}.$$

Since $\mathbf{R} = \tilde{\mathbf{R}}$ for real-valued stationary stochastic processes, and $\mathbf{R}$ has full rank which implies uniqueness of the solution to $\mathbf{R}\mathbf{b}_p = \sigma_{e_p^-}^2 \mathbf{i}$, it is evident that $\mathbf{b_p} = \mathbf{a_p}$, and thus $\sigma_{e_p}^2 = \sigma_{e_p^-}^2$.

3.10 The sampling rate determines the resolution of the autocorrelation function. The plots displayed in Figure 3.3 show a continuous-time autocorrelation function sampled at 5 Hz ($T_1 = 0.2$) and 10 Hz ($T_2 = 0.1$), respectively.

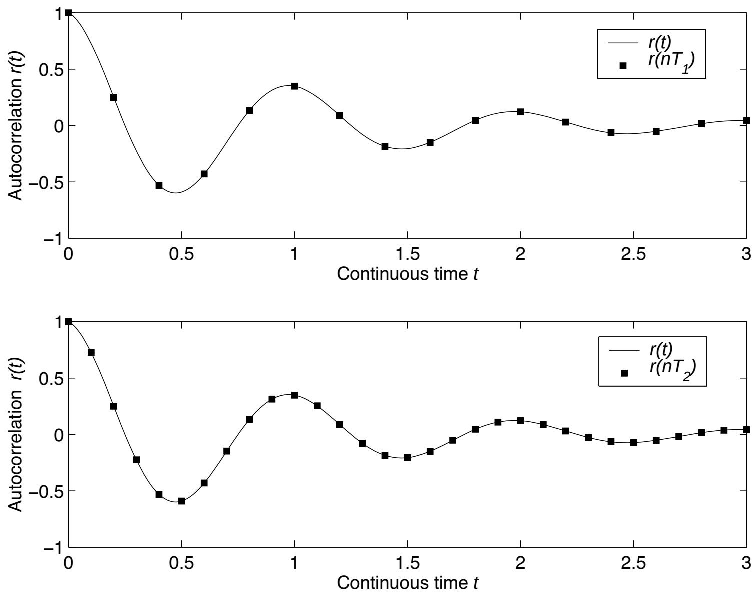


Figure 3.3: Sampling of an autocorrelation function at two different rates.

Comparing the two sampled correlation functions, it is obvious that under the requirement of maintaining the information contained in a specific time lag interval of $r(t)$, doubling the sampling frequency calls for doubling of the discrete-time AR model order.

3.11 The general normal equation

$$\tilde{\mathbf{R}}_x \mathbf{a}_p = \sigma_e^2 \mathbf{i}$$

can be rewritten as

$$\begin{bmatrix} r_x(0) & r_x(1) & \dots & r_x(p) \\ r_x(1) & r_x(0) & \dots & r_x(p-1) \\ \vdots & \vdots & \ddots & \vdots \\ r_x(p) & r_x(p-1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

From the first row multiplication we obtain

$$\sigma_e^2 = r_x(0) + \sum_{i=1}^{p} a_i r_x(i).$$

and for the remaining rows

$$\begin{bmatrix} r_x(1) \\ r_x(2) \\ \vdots \\ r_x(p) \end{bmatrix} + \begin{bmatrix} r_x(0) & \dots & r_x(p-1) \\ r_x(1) & \dots & r_x(p-2) \\ \vdots & \ddots & \vdots \\ r_x(p-1) & \dots & r_x(0) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

which thus yields the desired result

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} r_x(0) & \dots & r_x(p-1) \\ r_x(1) & \dots & r_x(p-2) \\ \vdots & \ddots & \vdots \\ r_x(p-1) & \dots & r_x(0) \end{bmatrix}^{-1} \begin{bmatrix} -r_x(1) \\ -r_x(2) \\ \vdots \\ -r_x(p) \end{bmatrix}.$$

3.12 We know that

$$c_i = (1 - d_i z^{-1}) H(z)\big|_{z=d_i}, \quad i = 1, \dots, p.$$

Then

$$c_{2i} = (1 - d_{2i} z^{-1}) H(z)\big|_{z=d_{2i}}$$

and

$$c_{2i-1} = (1 - d_{2i-1} z^{-1}) H(z)\big|_{z=d_{2i-1}}$$

using that $d_{2i-1} = d_{2i}^*$

$$c_{2i-1} = (1 - d_{2i}^* z^{-1}) H(z)\big|_{z=d_{2i}^*} = c_{2i}^*.$$

3.13 The power for each component in the parametric EEG analysis with the SPA decomposition can be obtained by evaluating

$$\begin{aligned} r_{x_i}(0) &= \frac{1}{2\pi\jmath}\oint_C S_{x_i}(z)z^{-1}dz \\ &= \sum_{l=0}^{1} \text{Res}\left[S_{x_i}(z)z^{-1}, d_{2i-l}\right]. \end{aligned}$$

with the residual value in

$$\text{Res}\left[S_{x_i}(z)z^{-1}, d_{2i-l}\right] = \lim_{z\to d_{2i-l}} (z - d_{2i-l})S_{x_i}(z)z^{-1}.$$

By substituting we obtain

$$\begin{aligned} P_i = r_{x_i}(0) = \sigma_v^2 &\frac{\left(2\Re(c_{2i}) - 2\Re(c_{2i}d_{2i}^*)d_{2i}^{-1}\right)\left(2\Re(c_{2i}) - 2\Re(c_{2i}d_{2i}^*)d_{2i}\right)}{\left(1 - d_{2i-1}d_{2i}^{-1}\right)\left(1 - d_{2i-1}d_{2i}\right)\left(1 - d_{2i}d_{2i}\right)} \\ + \sigma_v^2 &\frac{\left(2\Re(c_{2i}) - 2\Re(c_{2i}d_{2i}^*)d_{2i-1}^{-1}\right)\left(2\Re(c_{2i}) - 2\Re(c_{2i}d_{2i}^*)d_{2i-1}\right)}{\left(1 - d_{2i}d_{2i-1}^{-1}\right)\left(1 - d_{2i-1}d_{2i-1}\right)\left(1 - d_{2i}d_{2i-1}\right)} \end{aligned}$$

and using the fact that $d_{2i-1} = d_{2i}^*$ we have

$$P_i = \frac{8\sigma_v^2}{(1 - d_{2i}d_{2i-1})}\left[\Re\left(\frac{\left(\Re(c_{2i}) - \Re(c_{2i}d_{2i}^*)d_{2i}^{-1}\right)\left(\Re(c_{2i}) - \Re(c_{2i}d_{2i}^*)d_{2i}\right)}{\left(1 - d_{2i}^*d_{2i}^{-1}\right)\left(1 - d_{2i}^2\right)}\right)\right]$$

$$P_i = \frac{8\sigma_v^2}{1 - |d_{2i}|^2}\left[\Re\left(\frac{\Re^2(c_{2i}) + \Re^2(c_{2i}d_{2i}^*) - \Re(c_{2i})\Re(c_{2i}d_{2i}^*)(d_{2i} + d_{2i}^{-1})}{1 + |d_{2i}|^2 - d_{2i}^2 - d_{2i}^*d_{2i}^{-1}}\right)\right].$$

3.14 From the decomposition presented in Section 3.4.5 we can express the power spectrum corresponding to i^{th} peak as

$$S_{x_i}(e^{\jmath\omega}) = \left|\frac{2\Re(c_{2i}) - 2\Re(c_{2i}r_ie^{-\jmath\phi_i})e^{-\jmath\omega}}{(1 - r_ie^{-\jmath\phi_i}e^{-\jmath\omega})(1 - r_ie^{\jmath\phi_i}e^{-\jmath\omega})}\right|^2 \sigma_v^2 = \frac{N\left(e^{\jmath\omega}\right)}{D\left(e^{\jmath\omega}\right)}.$$

It is obvious that the term $S_{x_i}(e^{\jmath\omega})$ is defined by a ratio with a numerator $N\left(e^{\jmath\omega}\right)$ and a denominator $D\left(e^{\jmath\omega}\right)$. To search for the frequency peak ω_i we should differentiate the fraction expression with respect to ω and solve for zero value, i.e.,

$$\frac{dS_{x_i}\left(e^{\jmath\omega}\right)}{d\omega} = \frac{\frac{dN(e^{\jmath\omega})}{d\omega}D\left(e^{\jmath\omega}\right) - N\left(e^{\jmath\omega}\right)\frac{dD(e^{\jmath\omega})}{dw}}{D^2\left(e^{\jmath\omega}\right)} = 0. \tag{3.3}$$

Keeping in mind that the nominator $N(e^{\jmath\omega})$ has real-valued zeros and that the poles are located near the unit circle, this implies that around the maximum ω_i the variation of $N(e^{\jmath\omega})$ with ω will be small as will $D(e^{\jmath\omega})$. As a result, the product of both terms will be very small and to obtain a zero in (3.3) the second part in the numerator $N(e^{\jmath\omega})\frac{dD(e^{\jmath\omega})}{d\omega}$ needs to be small. $N(e^{\jmath\omega})$ is large (distance to the zeros) so the only solution is that $\frac{dD(e^{\jmath\omega})}{dw}$ is approximately equal to zero. This is equivalent to finding the minimum of the resonator represented by the denominator.

By evaluating the denominator we get

$$\begin{aligned}
D(e^{\jmath\omega}) &= |(1 - r_i e^{-\jmath\phi_i} e^{-\jmath\omega})(1 - r_i e^{\jmath\phi_i} e^{-\jmath\omega})|^2 \\
&= |(1 - 2r_i \cos\phi_i e^{-\jmath\omega} + r_i^2 e^{-\jmath 2\omega})|^2 \\
&= 1 + 4r_i^2\cos^2\phi_i + r_i^4 - 4r_i(1 + r_i^2)\cos\phi_i\cos\omega + 2r_i^2\cos(2\omega).
\end{aligned}$$

Differentiation with respect to ω yields

$$\frac{dD(e^{\jmath\omega})}{d\omega} = 4r_i(1 + r_i^2)\cos\phi_i\sin\omega - 4r_i^2\sin(2\omega),$$

which when set to zero becomes

$$(1 + r_i^2)\cos\phi_i\sin\omega_i - r_i\sin(2\omega_i) = 0.$$

After some trigonometric manipulations, we obtain the solution

$$\omega_i = \arccos\left(\frac{1 + r_i^2}{2r_i}\cos\phi_i\right).$$

which is the maximum frequency of the peak. Note that as $r_i \to 1$ then $\omega_i \to \phi_i$ giving the maximum at the pole angle.

3.15 a) the residue γ_j is express as

$$\begin{aligned}
\gamma_j &= \text{Res}\left[S_x(z)z^{-1}, d_j\right] \\
&= \lim_{z \to d_j}(z - d_j)S_x(z)z^{-1} \\
&= \frac{\sigma^2}{\prod\limits_{\substack{i=1 \\ i \neq j}}^{p}(1 - d_i d_j^{-1}) \prod\limits_{k=1}^{p}(1 - d_k^* d_j)}
\end{aligned} \tag{3.4}$$

b) The power spectrum can be express as:

$$\begin{aligned}
S_x(z) &= \sum_{k=-\infty}^{\infty} r_x(k) z^{-k} \\
&= \sum_{k=-\infty}^{\infty} \sum_{j=1}^{p} \gamma_j d_j^{|k|} z^{-k}; \\
&= \sum_{j=1}^{p} \sum_{k=-\infty}^{\infty} \gamma_j d_j^{|k|} z^{-k}; \\
&= \sum_{j=1}^{p} S_{x_j}(z)
\end{aligned}$$

with

$$\begin{aligned}
S_{x_j}(z) &= \sum_{k=-\infty}^{\infty} \gamma_j d_j^{|k|} z^{-k} \\
&= \frac{\gamma_j z}{z - d_j} - \gamma_j + \frac{\gamma_j z^{-1}}{z^{-1} - d_j} \\
&= \frac{\gamma_j d_j}{z - d_j} + \frac{\gamma_j d_j}{z^{-1} - d_j} + \gamma_j
\end{aligned} \tag{3.5}$$

However, this is a decomposition that does not need to preserve the positive characteristic of a real power spectral decomposition, so if poles are close together some terms γ_j can even be negative (not implying there is any pole with negative contribution). The decomposition in this problem, where poles are well-separated, is equivalent to the one presented in the text.

3.16 The solution is the same $\Delta_2(n)$ but taking out the square of the denominator, and introducing the reference window power σ_e^2 to independent from original power.

$$\Delta_2'(n) = \frac{\frac{1}{2\pi}\int_{-\pi}^{\pi} \left(S_e(e^{\jmath\omega}; n) - \sigma_e^2\right)^2 d\omega}{\frac{\sigma_e^2}{2\pi}\int_{-\pi}^{\pi} S_e(e^{\jmath\omega}; n) d\omega}$$

When $S_e(e^{\jmath\omega}; n) = \alpha\sigma_e^2$ it can be shown that $\Delta_2(n) = \frac{(\alpha-1)^2}{\alpha^2}$, and when $S_e(e^{\jmath\omega}; n) = \frac{1}{\alpha}\sigma_e^2$ then $\Delta_2(n) = (\alpha - 1)^2$. However in both cases

$$\Delta_2'(n) = \frac{(\alpha-1)^2}{\alpha}$$

making this a better suited error criteria than the one defined by $\Delta_2(n)$. Note that the factor σ_e^2 in the denominator of $\Delta_2'(n)$ is introduced in order to make the error independent of the power. Without this factor, the measure is still insensitive for power increases or decreases but yes to original reference window power σ_e^2 that is not a nice feature.

3.17 Proceeding in terms analogous to (3.203) we obtain

$$\Delta_2'(n) = \left(\frac{r_e(0;n)}{r_e(0;0)} + \frac{r_e(0;0)}{r_e(0;n)} - 2\right) + \frac{2}{r_e(0;n)r_e(0;0)}\sum_{k=1}^{\infty} r_e^2(k;n)$$

where the first term evidently is symmetric with respect to power changes. However, the second is only symmetric for power changes that maintain the white noise properties. In other words, the redefinition of $\Delta_2'(n)$ in the previous problem alleviates the symmetry problem with respect to power but does not solve it completely. The effects of variation in shape of the spectrum can be less pronounced than those because the power change (second term respect first term). When this is no longer true, in this time domain a new error measure can be proposed, $\Delta_2''(n)$, that still corrects for this remaining dependency by

$$\Delta_2''(n) = \left(\frac{r_e(0;n)}{r_e(0;0)} + \frac{r_e(0;0)}{r_e(0;n)} - 2\right) + \frac{2}{r_e^2(0;0)}\sum_{k=1}^{\infty} r_e^2(k;n).$$

3.18 We start by recalling the relationship between discrete and continuous-time Fourier transforms when the signal is sampled without aliasing,

$$X(e^{\jmath\omega}) = \frac{1}{T_s} X(\Omega)\lfloor_{\Omega=\omega/T_s}; \qquad |\omega| \le \pi. \tag{3.6}$$

With this relation and Parseval's relation we can write

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} x^2(n) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{\jmath\omega})|^2 d\omega \\
&= \frac{1}{2\pi}\int_{-\Omega_s/2}^{\Omega_s/2} \frac{|X(\Omega)|^2}{T_s^2} d\Omega T_s \\
&= \frac{1}{T_s}\int_{-\infty}^{\infty} x^2(t) dt
\end{aligned}$$

Using this relation, the continuous-time, temporal duration Δ_t can be expressed as:

$$\Delta_t^2 = \frac{\displaystyle\int_{-\infty}^{\infty} (t-t_0)^2 x^2(t)dt}{\displaystyle\int_{-\infty}^{\infty} x^2(t)dt} = \frac{T_s \displaystyle\sum_{n=-\infty}^{\infty} \left((n-n_0)T_s\right)^2 x^2(n)}{T_s \displaystyle\sum_{n=-\infty}^{\infty} x^2(n)} = T_s^2 \Delta_n^2$$

and the continuous-time frequency duration Δ_Ω as,

$$\Delta_\Omega^2 = \frac{\dfrac{1}{2\pi}\displaystyle\int_0^{\infty} (\Omega-\Omega_0)^2 |X(\Omega)|^2 d\Omega}{\dfrac{1}{2\pi}\displaystyle\int_0^{\infty} |X(\Omega)|^2 d\Omega} = \frac{\dfrac{1}{2\pi}\displaystyle\int_0^{\pi} \frac{(\omega-\omega_0)^2}{T_s^2} T_s^2 |X(e^{\jmath\omega})|^2 \frac{d\omega}{T_s}}{\dfrac{1}{2\pi}\displaystyle\int_0^{\pi} T_s^2 |X(e^{\jmath\omega})|^2 \frac{d\omega}{T_s}} = \frac{\Delta_\omega^2}{T_s^2}$$

So, the same uncertainty holds for the discrete-time defined widths

$$\Delta_\Omega \Delta_t = \Delta_\omega \Delta_n \geq \frac{1}{2} \tag{3.7}$$

3.19 The windowed STFT is defined by

$$X(t,\Omega) = \int_{-\infty}^{\infty} x(\tau) w(\tau - t) e^{-\jmath\Omega\tau} d\tau. \tag{3.8}$$

In order to realize that the STFT can be interpreted as a linear filter, the definition can be rewritten as

$$X(t,\Omega) = \int_{-\infty}^{\infty} x(\tau) e^{-\jmath\Omega\tau} h(t-\tau) d\tau, \tag{3.9}$$

where

$$h(t) = w(-t), \tag{3.10}$$

i.e., a linear filter applied to the modulated signal $x(\tau)e^{-\jmath\Omega\tau}$.

When we discretize the frequency as multiples of some reference Ω_0, $\Omega = m\Omega_0$, and sample in time, $t = nT_s$, we have a modulated filter bank (the signal is "demodulated" by Ω and filtered by $h(t)$). If $h(t)$ is a narrow band low-pass filter, then the filter bank interpretation of the STFT results.

3.20 By making a variable change we can express the Wigner-Ville distribution as

$$W_x(t,\Omega) = 2\int_{-\infty}^{\infty} x^*(t-\tau)\, x(t+\tau)\, e^{-\jmath 2\Omega\tau} d\tau, \tag{3.11}$$

which can be expressed as a function of the Fourier transform of the function $g(t,\tau)$

$$g(t,\tau) = x^*\left(t-\tau\right)x\left(t+\tau\right) \tag{3.12}$$

with respect to the variable τ

$$G(t,\Omega) = \mathcal{FT}\{g(t,\tau)\}, \tag{3.13}$$

thus giving

$$W_x(t,\Omega) = 2G(t,\Omega')\rfloor_{\Omega'=2\Omega}. \tag{3.14}$$

If the original signal spectrum $X(\Omega)$ has a bandwidth of B Hz, then the spectrum $G(t,\Omega)$ has a bandwidth of $2B$ Hz since it comes from the multiplication of $x(t)$ with itself, giving a convolution spectrum whose bandwidth extends twice that of the original, see Figure 3.4(a). If we sample the sig-

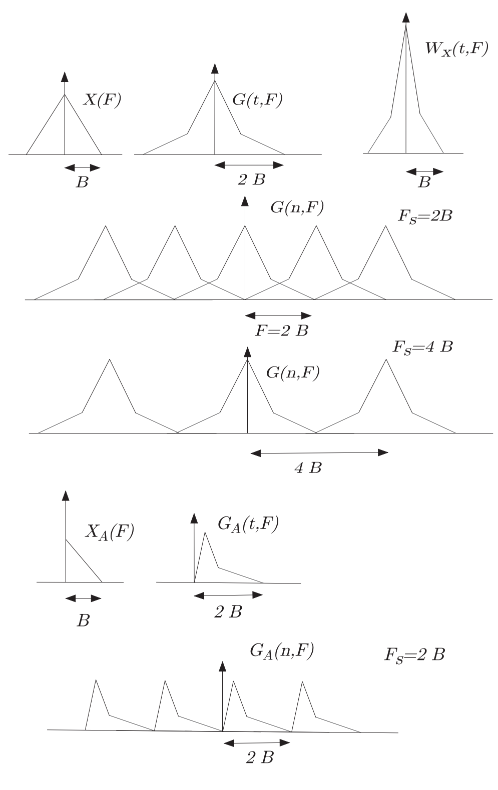


Figure 3.4: Wigner-Ville sampling phenomenon.

nal $x(t)$ at $F_s = 2B$ it is obvious that when computing the discrete Fourier transform of

$$g(n,k) = x^*(n-k)x(n+k) \tag{3.15}$$

it will appear aliasing at $G(n,\omega)$ since we are sampling $g(t,\tau)$ at $F_s = 2B$ which is half their Nyquist rate 4B, see Figure 3.4(b).

$$W_x(n,\omega) = 2G(n,\omega')\rfloor_{\omega'=2\omega} \tag{3.16}$$

Since aliasing is introduced at $F_s/4$, rather than at $F_s/2$, in the Wigner-Ville representation, it is necessary to sample with at least at $F_s = 4B$, i.e., twice that of the Nyquist rate of $x(t)$. Accordingly, when analyzing on the $G(n,\omega)$ one has to go up to π or when analyzing $W_x(n,\omega)$ go only up to $\pi/2$. See Figure 3.4c.

Another alternative is to estimate the Wigner-Ville distribution from the analytic signal $x_A(t)$ rather than from $x(t)$. The spectrum has only contributions for positive frequencies and therefore aliasing is avoided even when sampling at $F_s = 2B$, since the folding has no contributions from the negative spectrum, see Figures 3.4(d) and (e). Note that the same analysis can be done for the the ambiguity function.

3.21 By recalling the definition of $\overline{\Omega}(t)$,

$$\overline{\Omega}(t) = \frac{\displaystyle\int_{-\infty}^{\infty} \Omega W_{x_c}(t,\Omega)d\Omega}{\displaystyle\int_{-\infty}^{\infty} W_{x_c}(t,\Omega)d\Omega},$$

and introducing the Fourier transform $Y(\Omega)$ of a signal $y(\tau)$ and the time marginal condition,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\Omega)d\Omega = y(\tau)|_{\tau=0}$$

we have that

$$\overline{\Omega}(t) = \frac{\left[\displaystyle\int_{-\infty}^{\infty} (\Omega W_{x_c}(t,\Omega))e^{j\Omega\tau}d\Omega\right]_{\tau=0}}{2\pi|x_c(t)|^2}$$

Moreover, introducing the Fourier pair relation

$$\Omega Y(\Omega) \longleftrightarrow \frac{1}{j}\frac{dy(\tau)}{d\tau}$$

we have that

$$\begin{aligned}\overline{\Omega}(t) &= \frac{\frac{2\pi}{\jmath}\frac{d}{d\tau}\left[x_c^*\left(t-\frac{\tau}{2}\right)x_c\left(t+\frac{\tau}{2}\right)\right]_{\tau=0}}{2\pi|x_c(t)|^2} \\ &= \frac{x_c^*(t)x_c'(t)-x_c(t)x_c'^*(t)}{2\jmath|x_c(t)|^2} \\ &= \frac{\Im\left(x_c^*(t)x_c'(t)\right)}{|x_c(t)|^2}.\end{aligned}$$

Since

$$x_c(t) = s(t)e^{\jmath\varphi(t)}$$

it is immediately clear that

$$\overline{\Omega}(t) = \varphi'(t)$$

Chapter 4

4.1 The estimate of the deterministic signal $\mathbf{s}$ from the noisy response $\mathbf{x}_i$ is obtained by ensemble averaging,

$$\hat{\mathbf{s}}_a = \frac{1}{M}\sum_{i=1}^{M}\mathbf{x}_i = \mathbf{s} + \frac{1}{M}\sum_{i=1}^{M}\mathbf{v}_i.$$

For only one response (the first), the SNR is defined by

$$\mathrm{SNR}_{a,1} = 10\log\frac{\mathbf{s}^T\mathbf{s}}{E[\mathbf{v}_1^T\mathbf{v}_1]} = -5 \text{ dB}.$$

Averaging over M responses yields the following SNR

$$\mathrm{SNR}_{a,M} = 10\log\frac{\mathbf{s}^T\mathbf{s}}{E\left[\left(\frac{1}{M}\sum_{i=1}^{M}\mathbf{v}_i\right)^T\left(\frac{1}{M}\sum_{j=1}^{M}\mathbf{v}_j\right)\right]}.$$

Since the noise vectors $\mathbf{v}_i$ of different responses are assumed to be uncorrelated, that is, $E[\mathbf{v}_i^T\mathbf{v}_j] = 0, i \neq j$, we can simplify this expression to

$$\begin{aligned}
\mathrm{SNR}_{a,M} &= 10\log\frac{\mathbf{s}^T\mathbf{s}}{\frac{1}{M^2}\sum_{i=1}^{M}E[\mathbf{v}_i^T\mathbf{v}_i]} \\
&= 10\log M\frac{\mathbf{s}^T\mathbf{s}}{E[\mathbf{v}_1^T\mathbf{v}_1]} \\
&= 10\log M + \mathrm{SNR}_{a,1},
\end{aligned}$$

where the next last equality is due to that the noise variance is assumed to be identical in all responses. In order to obtain an SNR of 10 dB, we therefore require that

$$10\log M - 5 \geq 10$$

$$M \geq 10^{15/10} = 31.6$$

and thus $M = 32$ responses are needed.

4.2 From the ensemble of M responses, subaverages over even- and odd-numbered responses, respectively, are calculated by

$$\hat{s}_{a_0}(n) = \frac{2}{M}\sum_{i=1}^{M/2} x_{2i}(n)$$
$$\hat{s}_{a_1}(n) = \frac{2}{M}\sum_{i=1}^{M/2} x_{2i-1}(n).$$

The signal is deterministic, and the noise is zero-mean and uncorrelated between different responses. Thus

$$\begin{aligned} V[\hat{s}_{a_0}(n) - \hat{s}_{a_1}(n)] &= V\left[\frac{2}{M}\sum_{i=1}^{M/2}(v_{2i}(n) - v_{2i-1}(n))\right] \\ &= \left(\frac{2}{M}\right)^2 E\left[\sum_{i=1}^{M/2}\sum_{k=1}^{M/2}(v_{2i}(n) - v_{2i-1}(n))(v_{2k}(n) - v_{2k-1}(n))\right] \\ &= \left(\frac{2}{M}\right)^2 \frac{M2\sigma_v^2}{2} \\ &= \frac{4\sigma_v^2}{M}. \end{aligned}$$

4.3 Taking the expression of the variance estimator for a number of M recurrences

$$\hat{\sigma}^2_{v,M}(n) = \frac{1}{M}\sum_{i=1}^{M}\left(x_i(n) - \hat{s}_{a,M}(n)\right)^2,$$

it can be rewritten by grouping the terms related to the $\hat{\sigma}^2_{v,M-1}(n)$ estimator as:

$$\hat{\sigma}^2_{v,M}(n) = \frac{1}{M}\sum_{i=1}^{M-1}\left(x_i(n) - \hat{s}_{a,M}(n)\right)^2 + \frac{1}{M}\left(x_M(n) - \hat{s}_{a,M}(n)\right)^2,$$

and assuming $\hat{s}_{a,M}(n) = \hat{s}_{a,M-1}(n)$, it can be rewritten to

$$\begin{aligned} \hat{\sigma}^2_{v,M}(n) = &\left(\frac{1}{M-1} - \frac{1}{M(M-1)}\right)\sum_{i=1}^{M-1}\left(x_i(n) - \hat{s}_{a,M-1}(n)\right)^2 \\ &+ \frac{1}{M}\left(x_M(n) - \hat{s}_{a,M}(n)\right)^2, \end{aligned} \tag{4.17}$$

resulting finally in the recursive expression

$$\hat{\sigma}^2_{v,M}(n) = \hat{\sigma}^2_{v,M-1}(n) + \frac{1}{M}\left[(x_M(n) - \hat{s}_{a,M}(n))^2 - \hat{\sigma}^2_{v,M-1}(n)\right].$$

Note: If at the original estimator we would introduce a factor $\frac{1}{M-1}$ rather than $\frac{1}{M}$ to have an unbiased estimate, the recursive estimate would differ just by replacing the same factors in recursive expression.

4.4 Assume that M concatenated responses $\mathbf{x}_i$ are available, each with a length N. Initializing the recursion with $\hat{\mathbf{s}}_{e,0} = \mathbf{0}$, we get

$$\begin{aligned}
\hat{\mathbf{s}}_{e,1} &= (1-\alpha)\hat{\mathbf{s}}_{e,0} + \alpha\mathbf{x}_1 = \alpha\mathbf{x}_1\\
\hat{\mathbf{s}}_{e,2} &= (1-\alpha)\hat{\mathbf{s}}_{e,1} + \alpha\mathbf{x}_2 = \alpha(1-\alpha)\mathbf{x}_1 + \alpha\mathbf{x}_2\\
\hat{\mathbf{s}}_{e,3} &= (1-\alpha)\hat{\mathbf{s}}_{e,2} + \alpha\mathbf{x}_3 = \alpha(1-\alpha)^2\mathbf{x}_1 + \alpha(1-\alpha)\mathbf{x}_2 + \alpha\mathbf{x}_3\\
&\vdots\\
\hat{\mathbf{s}}_{e,M} &= \sum_{m=0}^{M-1} \alpha(1-\alpha)^m \mathbf{x}_{M-m}.
\end{aligned}$$

Now, by concatenating all available responses into one long piled vector, i.e.,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x_1} \\ \mathbf{x_2} \\ \vdots \\ \mathbf{x_M} \end{bmatrix},$$

we get

$$y(n) = \hat{s}_{e,(\lfloor \frac{n}{N} \rfloor + 1)}\left(n - \left\lfloor \frac{n}{N} \right\rfloor N\right) = \sum_{m=0}^{M-1} \alpha(1-\alpha)^m x(n - mN).$$

Performing the same calculations as in (4.26), i.e.,

$$\begin{aligned}
x(n) &= \sum_{l=-\infty}^{\infty} x(l)\delta(n-l),\\
x(n-mN) &= \sum_{l=-\infty}^{\infty} x(l)\delta(n-mN-l),\\
\Rightarrow \quad y(n) &= \sum_{m=0}^{M-1} \alpha(1-\alpha)^m \sum_{l=-\infty}^{\infty} x(l)\delta(n-mN-l),\\
&= \sum_{l=-\infty}^{\infty} x(l) \sum_{m=0}^{M-1} \alpha(1-\alpha)^m \delta(n-mN-l),
\end{aligned}$$

the impulse response is given by

$$h(n) = \sum_{m=0}^{M-1} \alpha(1-\alpha)^m \delta(n - mN),$$

and is plotted in Figure 4.5. From this figure it is evident that older responses in the ensemble are weighted less and less.

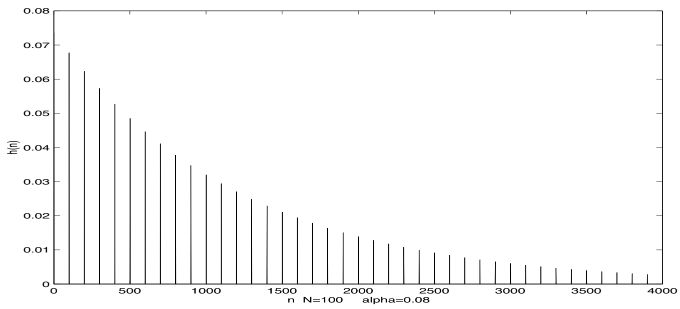


Figure 4.5: The impulse response of the exponential averager for $\alpha = 0.08$ and $N = 100$.

4.5 Following Problem 4.4 but with $\hat{\mathbf{s}}_{e,0} = \mathbf{x}_1$, it is found that

$$\hat{\mathbf{s}}_{e,M} = (1-\alpha)^{M-1}\mathbf{x}_1 + \sum_{m=0}^{M-2} \alpha(1-\alpha)^m \mathbf{x}_{M-m}.$$

Taking the expectation of $\hat{s}_{e,M}(n)$ yields

$$\begin{aligned} E[\hat{s}_{e,M}(n)] &= (1-\alpha)^{M-1}E[x_1(n)] + \alpha \sum_{m=0}^{M-2} (1-\alpha)^m E[x_{M-m}(n)] \\ &= \left((1-\alpha)^{M-1} + \alpha \frac{1-(1-\alpha)^{M-1}}{1-(1-\alpha)} \right) s(n) \\ &= s(n), \end{aligned}$$

which shows that $\hat{s}_{e,M}(n)$ is an unbiased estimate when initiated by $\hat{\mathbf{s}}_{e,0} = \mathbf{x}_1$; this is in contrast to the asymptotically unbiased which was initiated by $\hat{\mathbf{s}}_{e,0} = \mathbf{0}$.

The variance can also be calculated by the following calculations

$$\begin{aligned}
V[\hat{s}_{e,M}(n)] &= E[(\hat{s}_{e,M}(n) - E[\hat{s}_{e,M}(n)])^2] \\
&= E\left[\left((1-\alpha)^{M-1}x_1(n) + \sum_{m=0}^{M-2}\alpha(1-\alpha)^m x_{M-m}(n) - s(n)\right)^2\right] \\
&= E\left[\left((1-\alpha)^{M-1}v_1(n) + \sum_{m=0}^{M-2}\alpha(1-\alpha)^m v_{M-m}(n)\right)^2\right] \\
&= (1-\alpha)^{2(M-1)}E[v_1^2(n)] + \sum_{m=0}^{M-2}\alpha(1-\alpha)^m E[v_{M-m}^2(n)] + 0,
\end{aligned}$$

where all the cross-terms are equal to zero. Thus,

$$V\left[\hat{s}_{e,M}(n)\right] = \sigma_v^2(1-\alpha)^{2(M-1)} + \sigma_v^2\alpha\frac{1-(1-\alpha)^{2(M-1)}}{2-\alpha},$$

and the asymptotic variance is

$$\lim_{M\to\infty} V\left[\hat{s}_{e,M}(n)\right] = \sigma_v^2\frac{\alpha}{2-\alpha},$$

which is identical to the asymptotic variance when the initialization is done with $\hat{s}_{e,0}(n) = 0$.

4.6

a) From (4.36) and from Problem 4.5 it is evident that initialization with $\hat{\mathbf{s}}_{e,0} = \mathbf{0}$ produces a biased estimate. On the other hand, when using $\hat{\mathbf{s}}_{e,0} = \mathbf{x}_1$, the estimator is unbiased but has a larger variance, especially for small values of M. However, as the number of responses M approaches infinity, the bias decreases such that both estimators are *asymptotically unbiased* and have equal variances.

b) Using any of the two previously considered initializations, i.e., $\hat{\mathbf{s}}_{e,0} = \mathbf{0}$ or $\hat{\mathbf{s}}_{e,0} = \mathbf{x}_1$, we have an asymptotic variance of

$$\lim_{M\to\infty} V[\hat{s}_{e,M}] = \sigma_v^2\frac{\alpha}{2-\alpha}.$$

By setting

$$\alpha = \frac{2}{M-1}$$

we obtain a variance that is asymptotically equal to the ensemble averager.

4.7 From Problem 4.4, we have that

$$\hat{s}_{e,M}(n) = \alpha \sum_{m=0}^{M-1} (1-\alpha)^m x_{M-m}(n),$$

whose expected value is

$$E[\hat{s}_{e,M}(n)] = \alpha \sum_{m=0}^{M-1} (1-\alpha)^m s(n).$$

Thus,

$$\begin{aligned}
V[\hat{s}_{e,M}(n)] &= E[(\hat{s}_{e,M}(n) - E[\hat{s}_{e,M}(n)])^2] \\
&= \alpha^2 E\left[\left(\sum_{m=0}^{M-1} (1-\alpha)^m (s(n) + v_{M-m}(n) - s(n))\right)^2\right] \\
&= \alpha^2 E\left[\left(\sum_{m=0}^{M-1} (1-\alpha)^m v_{M-m}(n)\right)^2\right] \\
&= \alpha^2 \sum_{m=0}^{M-1} \sum_{k=0}^{M-1} (1-\alpha)^{m+k} \underbrace{E[v_{M-m}(n) v_{M-k}(n)]}_{\sigma_v^2,\ k=m;\ 0,\ k\neq m} \\
&= \alpha^2 \sum_{m=0}^{M-1} (1-\alpha)^{2m} \sigma_v^2 \\
&= \alpha^2 \frac{1-(1-\alpha)^{2M}}{1-(1-\alpha)^2} \sigma_v^2,
\end{aligned}$$

which is equal to the expression given in (4.38).

4.8 *Ensemble averager:*

To estimate the -3 dB cut-off frequency we should find that ω_c for which $|H_a(e^{\jmath\omega_c})| = 1/\sqrt{2}$. In the text, the ensemble average was shown to have the transfer function

$$H_a(e^{\jmath\omega}) = \frac{\sin(\omega NM/2)}{M\sin(\omega N/2)} e^{-\jmath\omega N(M-1)/2}.$$

Since the transfer function is a ratio between two sinusoids, this expression is best studied either numerically or with some kind of approximation. We will first find the first zero after a lobe maximum. We will perform the calculation at the DC lobe, i.e., $\omega = 0$. The first zero is reached at frequency

ω_0 when the sinusoid in the numerator becomes zero, i.e., $\omega_0 NM/2 = \pi$. Then

$$\omega_0 = \frac{2\pi}{NM},$$

When approximating ω_c we realize that $\omega_c < \omega_0 = \frac{2\pi}{NM}$ then $\omega_c N/2 < \frac{\pi}{M} \ll 1$. This implies the sinusoid in the denominator of $|H_a(e^{\jmath\omega_c})|$ can be approximated by the argument

$$\sin(\omega_c N/2) \approx \omega_c N/2.$$

However for the sinusoid in the numerator we have that

$$\omega_c \frac{NM}{2} < \frac{2\pi}{NM}\frac{NM}{2} = \pi,$$

which is a far too crude approximation since this frequency is not far smaller than one. Therefore a higher order approximation is used:

$$\sin(\omega_c NM/2) \approx \frac{\omega_c NM/2}{1!} - \frac{(\omega_c NM/2)^3}{3!}.$$

To estimate ω_c we should solve

$$\frac{\omega_c NM/2 - (\omega_c NM/2)^3/3!}{\omega_c MN/2} = \frac{1}{\sqrt{2}},$$

which gives

$$\omega_c = \frac{\left(1 - 1/\sqrt{2}\right)^{1/2} 3^{1/2} 2^{3/2}}{NM} = \frac{2.6513}{NM}.$$

Exponential averager:
For this estimate the transfer function is

$$H_e(e^{\jmath\omega}) = \frac{\alpha}{1 + (\alpha - 1)e^{-\jmath\omega N}},$$

The frequency ω_c at which $|H_e(e^{\jmath\omega_c})| = 1/\sqrt{2}$ can be calculated from the maximum ($\omega = 2\pi n/N$) for any of the lobes. We will again do this calculation from the DC lobe ($\omega = 0$). Then, $\omega_c \ll 2\pi/N$ and the transfer function can be approximated by

$$H_e(e^{\jmath\omega}) = \frac{\alpha}{1 + (\alpha - 1)\cos(\omega N) - \jmath(\alpha - 1)\sin(\omega N)} \approx \frac{\alpha}{\alpha - \jmath(\alpha - 1)\omega N}.$$

So the -3 dB cut-off frequency ω_c can be calculated as

$$\omega_c N(1 - \alpha) = \alpha$$

and

$$\omega_c = \frac{\alpha}{N(1-\alpha)} \approx \frac{\alpha}{N}.$$

An alternative approach to obtain this result is to study the poles of the transfer function. The poles can be calculated as

$$\begin{aligned} 1 + (\alpha - 1)z^{-N} = 0 \Rightarrow z^N &= (1-\alpha) \\ z_k &= \underbrace{(1-\alpha)^{1/N}}_{\text{pole radius}} e^{\jmath \frac{2\pi}{N} k}, \quad k = 0, \ldots, N-1. \end{aligned}$$

An approximation of the -3 dB bandwidth is given in (3.189),

$$\Delta\omega_{3\text{ dB}} \approx 2(1-r),$$

where r is the pole radius which is assumed to be close to one. Hence,

$$\omega_c \approx \frac{2\left(1 - \sqrt[N]{1-\alpha}\right)}{2} = 1 - \sqrt[N]{1-\alpha}$$

for the exponential averager. Using the Taylor series expansion for $\alpha \ll 1$ we obtain the earlier result

$$\omega_c \approx \frac{\alpha}{N}.$$

4.9 The transfer function for the exponential averager is given by (4.44),

$$H_e(e^{\jmath\omega}) = \frac{\alpha}{1 + (\alpha-1)e^{-\jmath\omega N}},$$

where it should be observed that the maximum gain of each peak is unity.

$$\begin{aligned} |H(e^{\jmath\omega})|^2 &= \frac{\alpha^2}{(1+(\alpha-1)e^{-\jmath\omega N})(1+(\alpha-1)e^{\jmath\omega N})} \\ &= \frac{\alpha^2}{1+(\alpha-1)^2+2(\alpha-1)\cos(\omega N)} \\ |H(e^{\jmath\omega})|^2 = 10^{-3/10} &\Rightarrow \cos(\omega N) = \frac{10^{3/10}\alpha^2 - 1 - (\alpha-1)^2}{2(\alpha-1)} \\ \omega N &= \arccos\left(\frac{10^{3/10}\alpha^2 - 1 - (\alpha-1)^2}{2(\alpha-1)}\right) + 2\pi k, \quad k = 0, 1, \ldots N-1 \\ \omega_k &= \frac{1}{N}\arccos\left(\frac{10^{3/10}\alpha^2 - 1 - (\alpha-1)^2}{2(\alpha-1)}\right) + \frac{2\pi}{N}k. \end{aligned}$$

The magnitude function is -3 dB at the frequencies $\pm\omega_0, \ldots, \pm\omega_{N-1}$ since the cosine is an even function. It is easily verified that the magnitude is unity at $\omega = 0$. Thus, due to symmetry, $\Delta\omega_{3\text{dB}} = 2\omega_0$.

4.10 The transfer function for the exponential averager is given by (4.44),

$$H_e(e^{\jmath\omega}) = \frac{\alpha}{1 + (\alpha - 1)e^{-\jmath\omega N}},$$

The magnitude and phase are plotted in Figure 4.6. The phase function is given by

$$\begin{aligned}\Phi(e^{\jmath\omega}) &= \text{phase}(H_e(e^{\jmath\omega})) = \text{phase}(1 + (\alpha - 1)e^{\jmath\omega N}) \\ &= \arctan\left(\frac{(\alpha - 1)\sin(\omega N)}{1 + (\alpha - 1)\cos(\omega N)}\right) \approx \arctan\left(\frac{-\sin(\omega N)}{1 - \cos(\omega N)}\right) \\ &= \arctan\left(\frac{-\sin(\omega N)}{2\sin^2(\omega N/2)}\right)\end{aligned}$$

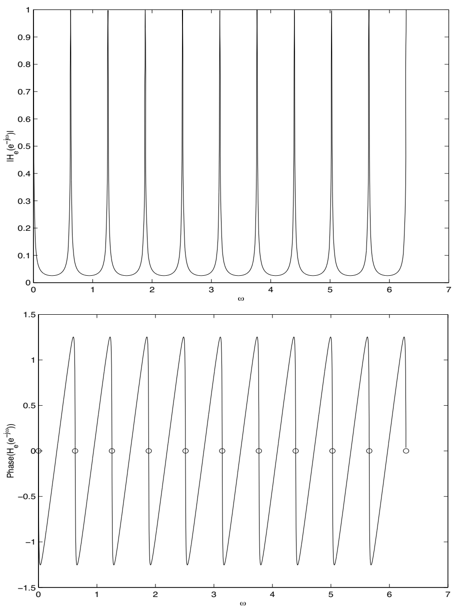


Figure 4.6: $\alpha = 0.05, N = 10$

Since the phase response is nonlinear it seems reasonable to assume that the response will be distorted in shape. However, the response $s(n)$ is assumed to repeat itself in every response (period of N) such that the frequencies of interest are constituted by the DC level, the fundamental frequency at $\omega = 2\pi/N$ and its harmonics. At other frequencies, only the noise will be affected since it is not periodic and will then be distributed at all frequencies. A careful look at the phase response shows that at $\omega = 2\pi/N$ and its multiples the phase is always equal to zero. Hence, the components of $s(n)$ are unaffected due to the zero phase and consequently undistorted.

Note the linear behavior in between every period $2\pi/N$ of the spectrum which can be explained by graphically analyzing the transfer function in the z-plane.

$$\begin{aligned}\Phi(e^{\jmath\omega}) &= \text{phase}(H_e(e^{\jmath\omega}) \\ &= \text{phase}(1 + (\alpha - 1)e^{\jmath\omega N}) \\ &\approx -\frac{\pi}{2} + \frac{(\omega - \frac{2\pi k}{N})N}{2} \qquad 0 \le k < N-1; \qquad \frac{2\pi k}{N} < \omega < \frac{2\pi(k+1)}{N}\end{aligned}$$

This approximation is perhaps not evident from a mathematical point of view but is suggested by Figure 4.7 where the z-plane of $1 + (\alpha - 1)e^{\jmath\omega N}$ is shown. As far as $\alpha \ll 1$, the triangle is approximately equilateral and then the above phase approximation becomes evident.

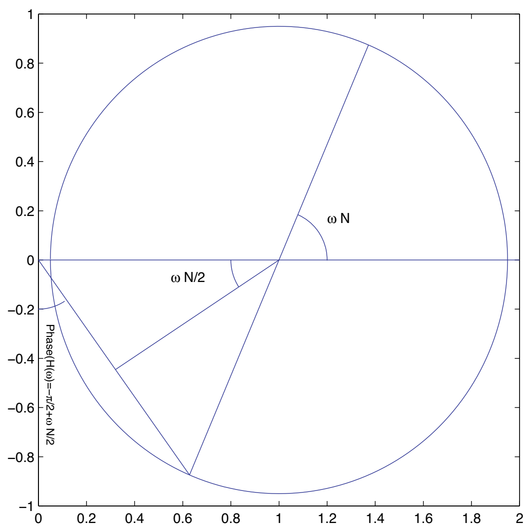


Figure 4.7: z-plane

4.11 Assume that the amplitude drops from 0.6 μV to 0.2 μV after the M^{th} response at the peak in n_0, and that k responses with lowered amplitude are collected. Thus, the ensemble average is based on $M + k$ responses.

a) Ensemble averaging.

$$E[\hat{s}_a(n_0)] = E\left[\frac{1}{M+k}\left(\sum_{i=1}^{M} x_i(n_0) + \sum_{i=M+1}^{M+k} x_i(n_0)\right)\right]$$
$$= \frac{M \cdot 0.6 + k \cdot 0.2}{M+k} < 0.3 \Rightarrow k > 3M.$$

b) The initial condition $\hat{s}_{e,0}(n) = 0$ is used for exponential averaging. The first M recursions yield

$$E[\hat{s}_{e,M}(n_0)] = E\left[\sum_{m=0}^{M-1} \alpha(1-\alpha)^m x_{M-m}(n_0)\right]$$
$$= \sum_{m=0}^{M-1} \alpha(1-\alpha)^m 0.6.$$

For the following k recursions, $E[\hat{s}_{e,M}(n_0)]$ acts as an initial value. Hence,

$$\begin{aligned}
E[\hat{s}_{e,M+k}(n_0)] &= (1-\alpha)^k \sum_{m=0}^{M-1} \alpha(1-\alpha)^m 0.6 \\
&\quad + E\left[\sum_{m=0}^{k-1} \alpha(1-\alpha)^m x_{M+k-m}(n_0)\right] \\
&= (1-\alpha)^k \sum_{m=0}^{M-1} \alpha(1-\alpha)^m 0.6 + \sum_{m=0}^{k-1} \alpha(1-\alpha)^m 0.2 \\
&= 0.6(1-\alpha)^k - 0.6(1-\alpha)^{M+k} + 0.2 - 0.2(1-\alpha)^k \\
&= 0.4(1-\alpha)^k - 0.6(1-\alpha)^{M+k} + 0.2.
\end{aligned}$$

We now have to solve

$$0.4(1-\alpha)^k - 0.6(1-\alpha)^{M+k} + 0.2 < 0.3$$

and

$$(1-\alpha)^k(0.4 - 0.6(1-\alpha)^M) < 0.1 \Rightarrow$$

$$k > \frac{\log(0.1/(0.4 - 0.6(1-\alpha)^M))}{\log(1-\alpha)}$$

Assuming that M is large, we may approximate it by

$$0.4(1-\alpha)^k < 0.1 \Rightarrow$$

$$k > \frac{\log(0.1/0.4)}{\log(1-\alpha)} = \frac{-0.6020}{\log(1-\alpha)} \approx \frac{0.6020}{\alpha}.$$

(The assumption that M is large corresponds to the assumption that the average has reached 0.6μV before the change occurs.)

4.12 The weighted average for the case of varying noise variance but constant signal amplitudes is given by

$$\hat{s}_{w,M}(n) = \frac{\displaystyle\sum_{i=1}^{M} \frac{x_i(n)}{\sigma_{v_i}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}}.$$

It is now desired to express the weighted average $\hat{s}_{w,M}(n)$ as an update of $\hat{s}_{w,M-1}(n)$:

$$\hat{s}_{w,M}(n) = \hat{s}_{w,M-1}(n) + \alpha_M(x_M(n) - \hat{s}_{w,M-1}(n)),$$

where α_M is the weighting function to determine:

$$\begin{aligned}
\hat{s}_{w,M}(n) &= \frac{\displaystyle\sum_{i=1}^{M} \frac{x_i(n)}{\sigma_{v_i}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} = \frac{\displaystyle\sum_{i=1}^{M-1} \frac{x_i(n)}{\sigma_{v_i}^2} + \frac{x_M(n)}{\sigma_{v_M}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} \\
&= \frac{\left(\displaystyle\sum_{i=1}^{M-1} \frac{1}{\sigma_{v_i}^2}\right) \hat{s}_{w,M-1}(n) + \frac{x_M(n)}{\sigma_{v_M}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} \\
&= \hat{s}_{w,M-1}(n) - \frac{\frac{1}{\sigma_{v_M}^2}\hat{s}_{w,M-1}(n) + \frac{1}{\sigma_{v_M}^2} x_M(n)}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} \\
&= \hat{s}_{w,M-1}(n) + \frac{\frac{1}{\sigma_{v_M}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}} (x_M(n) - \hat{s}_{w,M-1}(n)).
\end{aligned}$$

Hence,

$$\alpha_M = \frac{\frac{1}{\sigma^2_{v_M}}}{\sum_{i=1}^{M} \frac{1}{\sigma^2_{v_i}}}.$$

When the noise samples have the same variance σ_v^2 for all responses, the weighting factor reduces to $\alpha_M = 1/M$, i.e., the weighted average becomes similar to the ensemble average.

4.13 The PDF of the signal $\mathbf{x}$ in that interval

$$\mathbf{x} = [x(-N) \ldots x(-1)]^T \tag{4.18}$$

is

$$p_x(\mathbf{x}) = \frac{1}{(2\pi\sigma_v^2)^{\frac{N}{2}}} \exp \cancel{x} \left[-\frac{(\mathbf{x} - \mathbf{1}m_v)^T(\mathbf{x} - \mathbf{1}m_v)}{2\sigma_v^2} \right], \tag{4.19}$$

Then, the log function becomes

$$\ln p_x(\mathbf{x}) = \text{constant} - \frac{N}{2} \ln \sigma_v^2 - \frac{1}{2\sigma_v^2}(\mathbf{x} - \mathbf{1}m_v)^T(\mathbf{x} - \mathbf{1}m_v), \tag{4.20}$$

which differentiated respect to σ_v^2 and equaling to zero gives the equation for the ML estimate, $\hat{\sigma}_v^2$:

$$-\frac{N/2}{\hat{\sigma}_v^2} + \frac{1}{2\hat{\sigma}_v^4}(\mathbf{x} - \mathbf{1}m_v)^T(\mathbf{x} - \mathbf{1}m_v) = 0, \tag{4.21}$$

that results in

$$\hat{\sigma}_v^2 = \frac{(\mathbf{x} - \mathbf{1}m_v)^T(\mathbf{x} - \mathbf{1}m_v)}{N}. \tag{4.22}$$

This estimates requires to know m_v that usually is also estimates as $\hat{m}_v = \frac{\mathbf{1}^T\mathbf{x}}{N}$, which lead to an approximate ML estimate of the variance $\breve{\sigma}_v^2$

$$\breve{\sigma}_v^2 = \frac{(\mathbf{x} - \mathbf{1}\hat{m}_v)^T(\mathbf{x} - \mathbf{1}\hat{m}_v)}{N}. \tag{4.23}$$

4.14 The idea is to express $V[\hat{s}_{w,M}(n)]$ in $V[\hat{s}_{w,M-1}(n)]$ such that an update equation of the form $V[\hat{s}_{w,M}(n)] = (1 - g_M)V[\hat{s}_{w,M-1}(n)]$ is obtained. We start this derivation by first determining the variance of $s_{w,M}(n)$

$$
\begin{aligned}
V[\hat{s}_{w,M}(n)] &= V\left[\sum_{m=1}^{M} x_m(n)\frac{\frac{1}{\sigma_m^2}}{\sum_{i=1}^{M}\frac{1}{\sigma_i^2}}\right] \\
&= V\left[s(n)\frac{\sum_{m=1}^{M}\frac{1}{\sigma_m^2}}{\sum_{i=1}^{M}\frac{1}{\sigma_i^2}} + \sum_{m=1}^{M} v_m(n)\frac{\frac{1}{\sigma_m^2}}{\sum_{i=1}^{M}\frac{1}{\sigma_i^2}}\right] \\
&= E\left[\sum_{k=1}^{M}\sum_{m=1}^{M} v_k(n)v_m(n)\frac{\frac{1}{\sigma_k^2}\frac{1}{\sigma_m^2}}{\left(\sum_{i=1}^{M}\frac{1}{\sigma_i^2}\right)^2}\right] \\
&= \frac{1}{\displaystyle\sum_{i=1}^{M}\frac{1}{\sigma_i^2}}.
\end{aligned}
$$

The recursion is then obtained by

$$
\begin{aligned}
V[\hat{s}_{w,M}(n)] &= \frac{1}{\displaystyle\sum_{j=1}^{M}\frac{1}{\sigma_{v_j}^2}} \\
&= \frac{1}{\displaystyle\sum_{j=1}^{M-1}\frac{1}{\sigma_{v_j}^2} + \frac{1}{\sigma_{v_M}^2}} \\
&= \frac{1}{\dfrac{1}{V[\hat{s}_{w,M-1}(n)]} + \dfrac{1}{\sigma_{v_M}^2}} \\
&= \frac{\sigma_{v_M}^2}{V[\hat{s}_{w,M-1}(n)] + \sigma_{v_M}^2} V[\hat{s}_{w,M-1}(n)] \\
&= \left(1 - \underbrace{\left(\frac{V[\hat{s}_{w,M-1}(n)]}{V[\hat{s}_{w,M-1}(n)] + \sigma_{v_M}^2}\right)}_{g_M}\right) V[\hat{s}_{w,M-1}(n)].
\end{aligned}
$$

4.15 The weighted average is given by, $\hat{\mathbf{s}}_w = \mathbf{X}\mathbf{w}$, where $\mathbf{X} = \mathbf{s}\mathbf{a}^T + \mathbf{V}$ is the model of the signals in the ensemble, with $\mathbf{s}$ denoting a deterministic waveform, $\mathbf{a}$ a vector with possibly varying amplitudes, and $\mathbf{V}$ column vectors with noise realizations. In this exercise, the *case 2* scenario with varying signal amplitudes and constant noise variance is considered. Thus the optimal weights are

$$\mathbf{w} = \frac{\mathbf{a}}{\mathbf{a}^T\mathbf{a}}.$$

It is assumed that the noise is white, and that the number of signals in the ensemble is M.

$$\begin{aligned}
E[\hat{\mathbf{s}}_w] &= \begin{bmatrix} E[\hat{s}_w(0)] \\ E[\hat{s}_w(1)] \\ \vdots \\ E[\hat{s}_w(N-1)] \end{bmatrix} \\
&= E[\mathbf{X}\mathbf{w}] \\
&= \mathbf{s}E[\mathbf{a}^T]\mathbf{w} + E[\mathbf{V}]\mathbf{w} \\
&= \mathbf{s}\mathbf{a}^T\mathbf{a}\frac{1}{\mathbf{a}^T\mathbf{a}} = \mathbf{s}.
\end{aligned}$$

The variance is

$$\begin{aligned}
V[\hat{s}_w(n)] &= V\left[\sum_{m=1}^{M} x_m(n)\frac{a_m}{\sum_{i=1}^{M} a_i^2}\right] \\
&= V\left[s(n)\frac{\sum_{m=1}^{M} a_m^2}{\sum_{i=1}^{M} a_i^2} + \sum_{m=1}^{M} v_m(n)\frac{a_m}{\sum_{i=1}^{M} a_i^2}\right] \\
&= E\left[\sum_{k=1}^{M}\sum_{m=1}^{M} v_k(n)v_m(n)\frac{a_k a_m}{\left(\sum_{i=1}^{M} a_i^2\right)^2}\right] \\
&= \sigma_v^2\frac{\sum_{m=1}^{M} a_m^2}{\left(\sum_{i=1}^{M} a_i^2\right)^2} \\
&= \sigma_v^2\frac{1}{\sum_{i=1}^{M} a_i^2}.
\end{aligned}$$

Provided that the weights are accurate, the weighted average is unbiased. Furthermore, it is consistent since

$$\lim_{M\to\infty} V[\hat{s}_w(n)] = \lim_{M\to\infty} \sigma_v^2\frac{1}{\sum\limits_{i=1}^{M} a_i^2} = 0.$$

4.16

a. We should find those weights w_i which minimize the mean square error

$$\mathcal{E} = E\left[\left(s(n) - \sum_{i=1}^{M} w_i x_i(n)\right)^2\right].$$

This is done by differentiating $\mathcal{E}$ with respect to w_i and setting the result to zero, i.e.,

$$\frac{\partial \mathcal{E}}{\partial w_j} = 0, \quad j = 1, \ldots, M,$$

which yield the following system of linear equations,

$$E\left[\left(s(n) - \sum_{i=1}^{M} w_i x_i(n)\right) x_j(n)\right] = 0, \quad j = 1, \ldots, M.$$

With the signal model $x_i(n) = s(n) + v_i(n)$ and the knowledge that the noise variance differs from response to response, i.e., $E[v_i^2(n)] = \sigma_{v_i}^2$, the equation system can be rewritten as

$$s^2(n) = \sum_{i=1}^{M} w_i s^2(n) + w_j E\left[v_j^2(n)\right], \quad j = 1, \ldots, M.$$

which can be rewritten as

$$w_j \sigma_{v_j}^2 = \left(1 - \sum_{i=1}^{M} w_i\right) s^2(n), \quad j = 1, \ldots, M.$$

By subtracting two equations in this system, we obtain that

$$w_i = \frac{\sigma_{v_j}^2}{\sigma_{v_i}^2} w_j,$$

and then

$$w_j \sigma_{v_j}^2 = \left(1 - w_j \sigma_{v_j}^2 \sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}\right) s^2(n).$$

we obtain that

$$w_j = \frac{\dfrac{1}{\sigma_{v_j}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2} + \frac{1}{s^2(n)}}.$$

This expression is "Wiener-like" and is time dependent since the weights depend on $s(n)$.

b. Now we will introduce the property that the estimate should be unbiased by requiring that the sum of weights should equal one,

$$\sum_{i=1}^{M} w_i = 1,$$

and thus

$$w_M = 1 - \sum_{i=1}^{M-1} w_i.$$

With this restriction we obtain the error

$$\epsilon = E\left[\left(s(n) - \sum_{i=1}^{M-1} w_i x_i(n) - \left(1 - \sum_{i=1}^{M-1} w_i\right) x_M(n)\right)^2\right]$$

and the linear equation system

$$E\left[\left(s(n) - \sum_{i=1}^{M-1} w_i x_i(n) - \left(1 - \sum_{i=1}^{M-1} w_i\right) x_M(n)\right)(-x_j(n) + x_M(n))\right] = 0$$

for $j = 1, \ldots, M$, that with the same signal model as before arrives to

$$w_j \sigma_{v_j}^2 + \sigma_{v_M}^2 \left(\sum_{i=1}^{M-1} w_i - 1\right), \quad j = 1, \ldots, M.$$

Subtracting again the following relationship between weights is obtained

$$w_i = \frac{\sigma_{v_j}^2}{\sigma_{v_i}^2} w_j,$$

and then

$$w_j \sigma_{v_j}^2 \left(1 + \sigma_{v_M}^2 \sum_{i=1}^{M-1} \frac{1}{\sigma_{v_i}^2}\right) = \sigma_{v_M}^2$$

which gives

$$w_j = \frac{\dfrac{1}{\sigma_{v_j}^2}}{\displaystyle\sum_{i=1}^{M} \frac{1}{\sigma_{v_i}^2}}.$$

This solution is identical to the one obtained when the SNR was maximized also with the restriction on unbiased estimate.

4.17 Since both the amplitude and the noise variance vary from response to response, we have that

$$\mathbf{a} = \begin{bmatrix} a_1 & a_2 & \dots & a_M \end{bmatrix}^T$$
$$\mathbf{R}_V = N \begin{bmatrix} \sigma_{v_1}^2 & 0 & \dots & 0 \\ 0 & \sigma_{v_2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_{v_M}^2 \end{bmatrix}.$$

Using (4.61), the maximization of the SNR (or $\mathcal{L}$) leads to the following generalized eigenvalue problem,

$$\mathbf{R}_V^{-1}\mathbf{a}\mathbf{a}^T\mathbf{w} = \lambda\mathbf{w}.$$

Since $\mathbf{R}_V^{-1}\mathbf{a}\mathbf{a}^T$ is of rank one, $\lambda_{max} = \mathbf{a}^T\mathbf{R}^{-1}\mathbf{a}$, cf. (4.64). This implies that the optimal weight vector is given by the eigenvector

$$\mathbf{w} = c_w\mathbf{R}_V^{-1}\mathbf{a} = c_w \begin{bmatrix} \frac{a_1}{\sigma_{v_1}^2} \\ \frac{a_2}{\sigma_{v_2}^2} \\ \vdots \\ \frac{a_M}{\sigma_{v_M}^2} \end{bmatrix},$$

which may be verified by insertion of λ_{max} and $\mathbf{w} = c_w\mathbf{R}_V^{-1}\mathbf{a}$ into (4.61),

$$c_w\mathbf{R}_V^{-1}\mathbf{a}\underbrace{\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a}}_{\text{scalar}} = c_w\underbrace{\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a}}_{\text{scalar}}\mathbf{R}_V^{-1}\mathbf{a}.$$

The final step is to determine the factor c_w such that unbiasedness, $E[\hat{\mathbf{s}}_w] = \mathbf{s}$, is obtained:

$$\begin{aligned} E[\hat{\mathbf{s}}_w] &= E[\mathbf{X}\mathbf{w}] = c_wE[\mathbf{X}\mathbf{R}_V^{-1}\mathbf{a}] = c_wE[\mathbf{s}\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a}] + c_wE[\mathbf{V}\mathbf{R}_V^{-1}\mathbf{a}] \\ &= c_w\mathbf{s}(\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a}) + c_wE[\mathbf{V}]\mathbf{R}_V^{-1}\mathbf{a} = c_w(\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a})\mathbf{s}. \end{aligned}$$

Hence, $c_w = (\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a})^{-1}$ leads to unbiasedness. Thus, the optimal weight vector is

$$\mathbf{w} = \frac{1}{\mathbf{a}^T\mathbf{R}_V^{-1}\mathbf{a}}\mathbf{R}_V^{-1}\mathbf{a}.$$

4.18 In cases with high SNR, considering as noise power estimate the total single-trial power is not adequate, rather we need to exclude the deterministic signal component $\mathbf{s}$ from the observation before to estimate. One way to do that will be to first subtract some estimate of $\mathbf{s}$ that do not require the weighted averaging. One such option is to subtract the ensemble average before power estimation

$$\hat{\sigma}_{v_i}^2 = \frac{1}{N}(\mathbf{x}_i - \hat{\mathbf{s}}_a)^T(\mathbf{x}_i - \hat{\mathbf{s}}_a), \tag{4.24}$$

This estimate can, however, introduce some errors when the noise variability include outliers, making the estimate, $\hat{\mathbf{s}}_a$, unreliable and then propagating that to every trial estimate of $\hat{\sigma}_{v_i}^2$. To avoid that it can be considered the median rather than the mean and come with the noise variance estimate

$$\hat{\sigma}_{v_i}^2 = \frac{1}{N}(\mathbf{x}_i - \hat{\mathbf{s}}_{\text{med}})^T(\mathbf{x}_i - \hat{\mathbf{s}}_{\text{med}}), \tag{4.25}$$

4.19 In the proposed system based on the adaptive linear combiner, we know that, for the stationary signals case, the optimum weights follow the following relation

$$\mathbf{w}^o = \mathbf{R}_x^{-1}\mathbf{r}_{\hat{s}_a x} \tag{4.26}$$

with

$$\begin{aligned}
\mathbf{R}_x(n) &= E\left[\mathbf{x}(n)\mathbf{x}^T(n)\right] && (4.27)\\
&= \begin{bmatrix} r_{x_1x_1}(n) & r_{x_1x_2}(n) & \dots & r_{x_1x_M}(n) \\ r_{x_2x_1}(n) & r_{x_2x_2}(n) & \dots & r_{x_2x_M}(n) \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_Mx_1}(n) & r_{x_Mx_2}(n) & \dots & r_{x_Mx_M}(n) \end{bmatrix} && (4.28)\\
&= s^2(n)\mathbf{1}\mathbf{1}^T + \begin{bmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_M^2 \end{bmatrix}, && (4.29)
\end{aligned}$$

where it has been assumed that noise is uncorrelated between different realizations and stationary within each realization, and

$$\mathbf{r}_{\hat{s}_a x}(n) = E[\hat{s}_a(n)\mathbf{x}(n)] = s^2(n)\mathbf{1}, \tag{4.30}$$

that has also involve assumption of un-correlation between residual noise in $\hat{s}_a$ and noise at every realization. Using the matrix inversion lemma that states: if $\mathbf{A}$ and $\mathbf{B}$ are two positive definite M-by-M matrices related by

$$\mathbf{A} = \mathbf{B}^{-1} + \mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T$$

where $\mathbf{D}$ is a positive definite N-by-N matrix, and $\mathbf{C}$ is an M-by-N matrix. Then the inverse of $\mathbf{A}$ may be expressed as

$$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{BC}(\mathbf{D} + \mathbf{C}^T\mathbf{BC})^{-1}\mathbf{C}^T\mathbf{B}.$$

Identify the following matrices:

$$\begin{aligned}
\mathbf{A} &= \mathbf{R}_x(n) \\
\mathbf{B} &= \begin{bmatrix} \frac{1}{\sigma_1^2} & 0 & \dots & 0 \\ 0 & \frac{1}{\sigma_2^2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{\sigma_M^2} \end{bmatrix} \\
\mathbf{C} &= \mathbf{1} \\
\mathbf{D} &= \frac{1}{s^2(n)}
\end{aligned}$$

we obtain

$$\mathbf{w}^o(n) = s^2(n)\mathbf{R}_x^{-1}(n)\mathbf{1} = \frac{s^2(n)}{1 + \sum_{i=1}^{M} \frac{s^2(n)}{\sigma_i^2}} \begin{bmatrix} \frac{1}{\sigma_1^2} \\ \frac{1}{\sigma_2^2} \\ \vdots \\ \frac{1}{\sigma_M^2} \end{bmatrix} \tag{4.31}$$

To solve the problem of the bias in the estimate we can introduce the MSE, $\mathcal{E}_{\mathbf{w}}$, with the constrain, Lagrange multiplier, that the weights are unbiased, $\mathbf{w}^T\mathbf{1} = 1$,

$$\mathcal{E}_{\mathbf{w}} = E\left[\left(\hat{s}_a(n) - \mathbf{w}^T\mathbf{x}(n)\right)^2\right] + \lambda(\mathbf{w}^T\mathbf{1} - 1). \tag{4.32}$$

Taking the gradient respect to $\mathbf{w}$ we obtain

$$\begin{aligned}
\nabla_{\mathbf{w}}\mathcal{E}_{\mathbf{w}} &= \nabla_{\mathbf{w}}\left(E\left[\hat{s}_a^2(n)\right] + \mathbf{w}^T\mathbf{R}_x(n)\mathbf{w} - 2\mathbf{w}^T\mathbf{r}_{\hat{s}_a x}(n) + \lambda(\mathbf{w}^T\mathbf{1} - 1)\right) \\
&= 2\mathbf{R}_x(n)\mathbf{w} - 2\mathbf{r}_{\hat{s}_a x}(n) + \lambda\mathbf{1}.
\end{aligned} \tag{4.33}$$

which gives and optimum constrained solution $\mathbf{w}_c^o$

$$\mathbf{w}_c^o = \mathbf{R}_x^{-1}(n)\mathbf{r}_{\hat{s}_a x}(n) - \frac{\lambda}{2}\mathbf{R}_x^{-1}(n)\mathbf{1} \tag{4.34}$$

and imposing that $\mathbf{1}^T\mathbf{w}_c^o = 1$, we obtain the value of λ

$$\lambda = \frac{2\left(\mathbf{1}^T\mathbf{R}_x^{-1}(n)\mathbf{r}_{\hat{s}_a x}(n) - 1\right)}{\mathbf{1}^T\mathbf{R}_x^{-1}(n)\mathbf{1}} = -\frac{2}{\sum_{i=1}^{M} \frac{1}{\sigma_i^2}} \tag{4.35}$$

and

$$\mathbf{w}_c^o = \frac{1}{\sum_{i=1}^{M} \frac{1}{\sigma_i^2}} \begin{bmatrix} \frac{1}{\sigma_1^2} \\ \frac{1}{\sigma_2^2} \\ \vdots \\ \frac{1}{\sigma_M^2} \end{bmatrix} \tag{4.36}$$

which is the unbiased solution that now does not depend on time n. A constrained LMS can be derived from the *steepest descent* method

$$\mathbf{w}(n+1) = \mathbf{w}(n) - \frac{1}{2}\mu \nabla_{\mathbf{w}} \mathcal{E}_{\mathbf{w}}(n) \tag{4.37}$$

$$= \mathbf{w}(n) + \mu E[e(n)\mathbf{x}(n)] - \frac{\mu}{2}\lambda \mathbf{1} \tag{4.38}$$

and by taking the LMS approximation, $E[e(n)\mathbf{x}(n)] \approx e(n)\mathbf{x}(n)$, and forcing that $\mathbf{w}(n+1)$ is also subject to the constrain $\mathbf{1}^T\mathbf{w}(n+1) = 1$ we obtain that

$$\lambda = \frac{2(\mathbf{1}^T\mathbf{w}(n) - 1)}{\mu M} + \frac{2e(n)\mathbf{1}^T\mathbf{x}(n)}{M} \tag{4.39}$$

and the resulting LMS constrained algorithm is

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu \left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{M} \right) e(n)\mathbf{x}(n) + \frac{\mathbf{1}}{M} \left(1 - \mathbf{1}^T\mathbf{w}(n) \right). \tag{4.40}$$

If the algorithm is correctly initialize so $\mathbf{1}^T\mathbf{w}(0) = 1$, e.g.

$$\mathbf{w}(0) = \left[\frac{1}{M} \frac{1}{M} \cdots \frac{1}{M} \right]^T, \tag{4.41}$$

then it will always be $\mathbf{1}^T\mathbf{w}(n) = 1$ and the algorithm can simply to

$$\mathbf{w}(n+1) = \mathbf{w}(n) + \mu \left(\mathbf{I} - \frac{\mathbf{1}\mathbf{1}^T}{M} \right) e(n)\mathbf{x}(n). \tag{4.42}$$

To allow the algorithm to converge, and under the assumption of stationarity, we can take the weight values at the end of the recurrences $n = N$, and use the weights $\mathbf{w} = \mathbf{w}(N)$, in the averaging. If noise variance even change within each realization during time, then the weight adaptive estimator output can be used as signal estimate.

4.20 The model of the signals in the ensemble is

$$\mathbf{X} = \mathbf{s}\mathbf{a}^T + \mathbf{V}.$$

Hence,

$$\begin{aligned} E[\mathrm{tr}(\mathbf{X}^T\mathbf{X})] &= E[\mathrm{tr}((\mathbf{s}\mathbf{a}^T + \mathbf{V})^T(\mathbf{s}\mathbf{a}^T + \mathbf{V}))] \\ &= \mathrm{tr}(\mathbf{a}\,\underbrace{\mathbf{s}^T\mathbf{s}}_{=1}\,\mathbf{a}^T) + \mathrm{tr}(E[\mathbf{V}^T\mathbf{V}]) \\ &= \mathbf{a}^T\mathbf{a} + N\sum_{m=1}^{M}\sigma_{v_m}^2, \end{aligned}$$

where N is the length of each response, M is the number of responses in the ensemble, and $\sigma_{v_m}^2$ is the variance of the noise samples in the m^{th} response. It is evident that the estimate is severely biased by the noise.

4.21 Solving the ML as was done for the stationary case, we will have now the following likelihood function

$$p_v(x_1(n),\ldots,x_M(n);s(n)) = \prod_{i=1}^{M}\frac{1}{\sqrt{2\sigma_{v_i}^2}}\exp\left[-\sqrt{\frac{2}{\sigma_{v_i}^2}}|x_i(n)-s(n)|\right], \tag{4.43}$$

and by maximizing their log function with respect $s(n)$ we will have

$$\frac{\partial \ln p_v\left(x_1(n),\ldots,x_M(n);s(n)\right)}{\partial s(n)} = -\frac{\partial}{\partial s(n)}\left(\sum_{i=1}^{M}\sqrt{\frac{2}{\sigma_{v_i}^2}}|x_i(n)-s(n)|\right) = 0. \tag{4.44}$$

The function to be maximized is now $J(s(n))$

$$J(s(n)) = \sum_{i=1}^{M}\frac{\sqrt{\left(x_i(n)-s(n)\right)^2}}{\sigma_{v_i}} \tag{4.45}$$

which, when differentiated, yields

$$\frac{\partial J(s(n))}{\partial s(n)} = \sum_{i=1}^{M}\frac{\mathrm{sgn}(x_i(n)-s(n))}{\sigma_{v_i}} = 0. \tag{4.46}$$

To make sure that the sum in (4.46) is equal to zero, we must choose $s(n)$ such that the number of sample values greater than $s(n)$, all summed with a weighted by $1/\sigma_{v_i}$, equals the sum of the number of sample values smaller than $s(n)$, also weighted by $1/\sigma_{v_i}$; this procedure can be denoted as the *weighted median*.

The procedure for computing this weighted median consists of sorting the sequence of samples,

$$\{x_1(n), x_2(n), \ldots, x_M(n)\} \xrightarrow{\text{sort}} \{x_{(1)}(n), x_{(2)}(n), \ldots, x_{(M)}(n)\}, \tag{4.47}$$

where the subscript parenthesis indicates that the samples have been ordered in increasing order, $x_{(1)}(n) < x_{(2)}(n) < \ldots < x_{(M)}(n)$, and followed by computation of the K point such that

$$\sum_{(i)=1}^{K-1} \frac{1}{\sigma_{v_{(i)}}} = \sum_{(i)=K+1}^{M} \frac{1}{\sigma_{v_{(i)}}} \tag{4.48}$$

Since it will be difficult that the equality holds, we can define the "weighed median" as

$$\hat{s}_{\text{weight med}}(n) = x_{(K)}(n) \tag{4.49}$$

such that

$$K = \arg\min_K \left| \sum_{(i)=1}^{K-1} \frac{1}{\sigma_{v_{(i)}}} - \sum_{(i)=K+1}^{M} \frac{1}{\sigma_{v_{(i)}}} \right|. \tag{4.50}$$

This proceeds to estimate the weighted median will require estimation the noise variance at each observation. This can be done for any of the procedures considered for weighted average. Also note that this technique, in contrast to the non weighted median, will take information no only of the actual value $x_i(n)$ of the sample, but of the total observation noise information σ_{v_i} and so can result in a better median estimate when noise is highly non stationary.

4.22

$$\hat{s}_M(n) = \hat{s}_{M-1}(n) + \alpha_M \psi(x_M(n) - \hat{s}_{M-1}(n)) \tag{4.51}$$

$$\psi(x) = \eta \cdot \text{sgn}(x)$$

If we initialize the $\hat{s}_0(n) = 0$, and assume that $s(n) > 0$ and the noise is zero-mean (reverse argument for negative), then initially $x_1(n) - \hat{s}_0(n) = x_1(n) - 0 = v_1(n) + s(n)$ is more likely to be positive than negative since $v_i(n)$ is zero mean. This implies that it is more likely we add in (4.51) a value η and then the $\hat{s}_i(n)$ will overall increase. This will happen until the probability that $(x_i(n) - \hat{s}_{i-1}(n)) > 0$ was the same as the one for being < 0. In this moment it will be equal (in mean) number of times $x_i(n) > \hat{s}_{i-1}(n)$ than the reverse and this happen when $\hat{s}_{i-1}(n)$ is the median of $x_i(n)$ (same number of values larger than lower) and then it is demonstrated.

4.23 The cut-off frequency Ω_c can be estimated by solving

$$P_\tau(\Omega_c) = \frac{\sin \frac{1}{2}\Omega_c T}{\frac{1}{2}\Omega_c T} = \frac{1}{\sqrt{2}} \tag{4.52}$$

This transfer function has a DC lobe with the first zero at $\Omega = \frac{2\pi}{T}$. This implies that the -3dB cut-off frequency will be much lower than this.

$$\Omega_c << \frac{2\pi}{T}$$

$$\Omega_c T/2 << \pi$$

and the sinusoid in (4.52) can be approximated in the area of the cut-off frequency as

$$P_\tau(\Omega_c) \approx 1 - (\Omega_c T/2)^2/6 = \frac{1}{\sqrt{2}}$$

and then

$$F_c \approx \frac{\sqrt{(1-1/\sqrt{2})6}}{\pi T} = \frac{0,422}{T}$$

Compare with Figure 4.20(b) to see that this expression of F_c is a good estimate.

4.24 The "discretized" Gaussian PDF of a time-discrete jitter θ is

$$P_\theta(\theta) = \frac{1}{\sqrt{2\pi\sigma_\theta^2}} e^{-\frac{\theta^2}{2\sigma_\theta^2}},$$

where σ_θ determines the dispersion of θ (σ_θ does not have to be an integer). This discrete-time PDF can be interpreted as a sampled continuous-time function $P_t(t)$

$$P_t(t) = \frac{T_s}{\sqrt{2\pi\sigma_t^2}} e^{-\frac{t^2}{2\sigma_t^2}},$$

where $\sigma_t = \sigma_\theta T_s$. Since $P_\theta(\theta) = P_t(\theta T_s)$, we can use the well-known relation between the continuous- and the discrete-time Fourier transforms

$$P_\theta(e^{-\jmath\omega}) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} P_t(\Omega - 2\pi n/T_s)|_{\Omega=\omega/T_s}.$$

Since

$$P_t(\Omega) = T_s e^{-\frac{1}{2}\Omega^2\sigma_t^2}$$

then

$$P_\theta(e^{-\jmath\omega}) = \sum_{n=-\infty}^{\infty} e^{-\frac{1}{2}(\omega-2\pi n)^2\sigma_\theta^2}. \tag{4.53}$$

To estimate the -3 dB cut-off frequency ω_c we can make the assumption that the width of the Gaussian in (4.53), i.e., $1/\sigma_\theta$, is small compared with the 2π repetition spectrum and then estimate the cut-off frequency ω_c as it would have been done from a single Gaussian. Then

$$e^{-\frac{1}{2}\sigma_\theta^2\omega_c^2} = 1/\sqrt{2} \Rightarrow \omega_c = \frac{\sqrt{\ln 2}}{\sigma_\theta}$$

Another alternative is to study the behavior of the estimate of discrete-time signal but with continuous-time jitter. This situation is the one that model the real situation since the jitter often originates from an "analog" signal acquisition process, so the jitter can be viewed as a continuous quantity according to

$$E\left[\hat{s}_a(n)\right] = \int_{-\infty}^{\infty} s(n-\tau)p_\tau(\tau)d\tau = s(t) * p_\tau(t)|_{t=nT_s}$$

Thus, this means that it represents a continuous-time filtering followed by sampling. Then the effect will be studied in continuous-time as present in (4.110) and (4.112) for cut-off frequency in case of Gaussian distributed jitter.

4.25 One could obtain a latency estimate with better time resolution by either

- interpolate the signal for increasing the sampling rate, or
- by using a frequency domain formulation of the convolution sum to achieve a finer resolution of τ.

The former approach is straightforward to do and may be implemented by using `interp` in Matlab.

The latter approach is based on the Parseval's formula which states that

$$\sum_{n=-\infty}^{\infty} x(n)y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\jmath\omega})Y^*(e^{\jmath\omega})d\omega.$$

For our model, we have that

$$\sum_{n=n_0}^{n_0+M-1} x(n)s(n-n_0) = \sum_{n=-\infty}^{\infty} x(n)s(n-n_0)$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\jmath\omega})\left[S(e^{\jmath\omega})e^{-\jmath\omega n_0}\right]^* d\omega.$$

Thus, a frequency domain formulation of the latency estimation is

$$\hat{\tau} = \arg \max_{n_0\in[0,N-M]} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{\jmath\omega})S^*(e^{\jmath\omega})e^{\jmath\omega n_0} d\omega.$$

Since $X(e^{\jmath\omega})$ and $S(e^{\jmath\omega})$ are continuous functions, it is possible to examine non-integer delays n_0.

4.26 Arranging the N samples of signal and noise into the vectors

$$\mathbf{s} = \begin{bmatrix} s(0) \\ s(1) \\ \vdots \\ s(N-1) \end{bmatrix}^T \quad \mathbf{v} = \begin{bmatrix} v(0) \\ v(1) \\ \vdots \\ v(N-1) \end{bmatrix}^T$$

respectively, and the N filter coefficients into $\mathbf{h} = [h(0)\ h(1)\ldots h(N-1)]^T$, at time instant $n = N-1$ the filtered signal is

$$y(N-1) = \mathbf{h}^T\tilde{\mathbf{s}} + \mathbf{h}^T\tilde{\mathbf{v}},$$

where $\tilde{\cdot}$ denotes reversal of the vectors. The energy of $\mathbf{y}$ is

$$E[y(N-1)^2] = \mathbf{h}^T\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T\mathbf{h} + E[\mathbf{h}^T\tilde{\mathbf{v}}\tilde{\mathbf{v}}^T\mathbf{h}]$$
$$= \mathbf{h}^T\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T\mathbf{h} + \mathbf{h}^T\mathbf{R}_v\mathbf{h},$$

and thus the SNR is

$$\text{SNR} = \frac{\mathbf{h}^T\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T\mathbf{h}}{\mathbf{h}^T\mathbf{R}_v\mathbf{h}}.$$

Since the $\mathbf{h}$ that maximizes the SNR is not unique (if $\mathbf{h}$ maximizes it, so does $c\mathbf{h}$, $c \in \mathcal{R}$), a constraint $\mathbf{h}^T\mathbf{R}_v\mathbf{h} = 1$ is imposed. Using the method of Lagrangian multipliers, the quantity

$$\mathcal{L} = \mathbf{h}^T\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T\mathbf{h} + \lambda(1 - \mathbf{h}^T\mathbf{R}_v\mathbf{h})$$

is to be minimized. This results in a generalized eigenvalue problem

$$(\tilde{\mathbf{s}}\tilde{\mathbf{s}}^T)\mathbf{h} = \lambda\mathbf{R}_v\mathbf{h},$$

which is similar to the one studied in Problem 3.14. The filter is thus

$$\mathbf{h} = c_h \mathbf{R}_v^{-1} \tilde{\mathbf{s}}$$

where the positive factor c_h is to be determined such that the constraint on $\mathbf{h}^T \mathbf{R}_v \mathbf{h}$ is satisfied. Then

$$\begin{aligned}
\mathbf{h}^T \mathbf{R}_v \mathbf{h}_{|\mathbf{h}=c_h \mathbf{R}_v^{-1}\tilde{\mathbf{s}}} &= (c_h^2 \mathbf{R}_v^{-1} \tilde{\mathbf{s}})^T \mathbf{R}_v (\mathbf{R}_v^{-1} \tilde{\mathbf{s}}) \\
&= c_h^2 \tilde{\mathbf{s}}^T \mathbf{R}_v^{-1} \tilde{\mathbf{s}} = 1 \Rightarrow \\
c_h &= \frac{1}{\sqrt{\tilde{\mathbf{s}}^T \mathbf{R}_v^{-1} \tilde{\mathbf{s}}}}
\end{aligned}$$

Hence, the optimal filter is

$$\mathbf{h} = \frac{1}{\sqrt{\tilde{\mathbf{s}}^T \mathbf{R}_v^{-1} \tilde{\mathbf{s}}}} \mathbf{R}_v^{-1} \tilde{\mathbf{s}}.$$

4.27 The evoked potential is now model by

$$\mathbf{x}_i = \mathbf{s}_{\theta_i} + \mathbf{v}_i \tag{4.54}$$

with

$$\mathbf{s}_{\theta_i} = \begin{bmatrix} \mathbf{0}_{\theta_i} \\ \mathbf{s}æ \\ \mathbf{0}_{N-D-\theta_i} \end{bmatrix}, \tag{4.55}$$

where $\mathbf{0}_i$ denotes a column vector with i zeros. With this notation the PDF of $\mathbf{x}_i$ can be written as

$$p(\mathbf{x}_i; \theta_i) = \frac{1}{(2\pi)^{\frac{N}{2}} |\mathbf{R}_v|^{\frac{1}{2}}} \exp æ \left[-\frac{1}{2} (\mathbf{x}_i - \mathbf{s}_{\theta_i})^T \mathbf{R}_v^{-1} (\mathbf{x}_i - \mathbf{s}_{\theta_i}) \right], \tag{4.56}$$

that after taken the logarithm, and using that $\mathbf{R}_v^{-1}$ is symmetric, results in

$$\ln p(\mathbf{x}_i; \theta_i) = \text{constant} - \frac{1}{2} \mathbf{s}_{\theta_i}^T \mathbf{R}_v^{-1} \mathbf{s}_{\theta_i} + \mathbf{x}_i^T \mathbf{R}_v^{-1} \mathbf{s}_{\theta_i} \tag{4.57}$$

The second term does depend of θ_i but if we consider the data record length, N, much larger than the correlation time d for $v(n)$ ($r_v(k) = 0$ for $k \geq d$), the so call *asymptotic Gaussian PDF* assumption, then the matrix $\mathbf{R}_v$ can be decompose as (Kay detection book page 34)

$$\mathbf{R}_v = \sum_{i=0}^{N-1} \lambda_i \varphi_i \varphi_i^T \tag{4.58}$$

with

$$\varphi_i = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 \\ e^{\jmath 2\pi f_i} \\ e^{\jmath 4\pi f_i} \\ \vdots \\ e^{\jmath 2\pi (N-1) f_i} \end{bmatrix} \tag{4.59}$$

and $f_i = i/N$. The inverse matrix

$$\mathbf{R}_v^{-1} = \sum_{i=0}^{N-1} \frac{1}{\lambda_i} \varphi_i \varphi_i^T \tag{4.60}$$

and the second term in 4.57 becomes

$$-\frac{1}{2} \mathbf{s}_{\theta_i}^T \mathbf{R}_v^{-1} \mathbf{s}_{\theta_i} = -\frac{1}{2} \sum_{i=0}^{N-1} \frac{1}{\lambda_i} |\varphi_i^T \mathbf{s}_{\theta_i}|^2 \tag{4.61}$$

with

$$|\varphi_i^T \mathbf{s}_{\theta_i}|^2 = \frac{1}{N} \left| \sum_{n=0}^{D-1} s(n) e^{-\jmath 2\pi f_i (n+\theta_i)} \right|^2 \tag{4.62}$$

that is the periodogram of the, shifted by θ_i, $s(n)$ which is not affected by the phase factor θ_i and then does not depend on θ_i implying that R_v^{-1} is Toeplitz under the assumption of short correlation lag compared with the observation interval, *asymptotic Gaussian PDF*. Maximization of the log-PDF function implies that the ML estimate of θ_i becomes

$$\hat{\theta}_i = \arg\max_{\theta_i} (\mathbf{x}_i^T \mathbf{R}_v^{-1} \mathbf{s}_{\theta_i}). \tag{4.63}$$

meaning that the matched filter is now

$$\mathbf{h} = \widetilde{\mathbf{R}_v^{-1}} \mathbf{s} \tag{4.64}$$

4.28 The channel weights β_i should be taken such that channels with more reliable estimates are emphasized. Since the matched filter has been selected by minimizing an MSE error criteria that is equivalent to maximize the SNR, we should used those weights β_i which are related to the particular SNR of each channel. If we assume that the SNR_i at each response is very low we can estimate the noise in each response as the total power and then

$$\hat{\beta}_i = \frac{1}{\mathbf{x_i}^T \mathbf{x_i}}.$$

When this later condition is not satisfied still we can obtain an estimate of β_i after having subtracted the mean value, i.e.,

$$\hat{\beta}_i = \frac{1}{\left(\mathbf{x}_i - \tilde{\mathbf{h}}_i\right)^T \left(\mathbf{x_i} - \tilde{\mathbf{h}}_i\right)}.$$

4.29 To find $\mathbf{R}_x^{-1}$, the matrix inversion lemma will be used. It states that:

> Let $\mathbf{A}$ and $\mathbf{B}$ be two positive definite M-by-M matrices related by
>
> $$\mathbf{A} = \mathbf{B}^{-1} + \mathbf{C}\mathbf{D}^{-1}\mathbf{C}^T$$
>
> where $\mathbf{D}$ is a positive definite N-by-N matrix, and $\mathbf{C}$ is an M-by-N matrix. Then the inverse of $\mathbf{A}$ may be expressed as
>
> $$\mathbf{A}^{-1} = \mathbf{B} - \mathbf{B}\mathbf{C}(\mathbf{D} + \mathbf{C}^T\mathbf{B}\mathbf{C})^{-1}\mathbf{C}^T\mathbf{B}.$$

Identify the following matrices:

$$\begin{aligned}
\mathbf{A} &= \mathbf{R}_x \\
\mathbf{B}^{-1} &= (1-\rho(n))\mathbf{I} \\
\mathbf{C} &= \mathbf{1} \\
\mathbf{D}^{-1} &= \rho(n)
\end{aligned}$$

Thus,

$$\begin{aligned}
\mathbf{R}_x^{-1} &= \frac{1}{1-\rho(n)}\mathbf{I} - \frac{1}{1-\rho(n)}\mathbf{I}\mathbf{1}\left(\frac{1}{\rho(n)} + \frac{1}{1-\rho(n)}\mathbf{1}^T\mathbf{1}\right)^{-1}\mathbf{1}^T\frac{1}{1-\rho(n)}\mathbf{I} \\
&= \frac{1}{1-\rho(n)}\mathbf{I} - \frac{1}{(1-\rho(n))^2}\cdot\frac{\rho(n)(1-\rho(n))}{1+(M-1)\rho(n)}\mathbf{1}\mathbf{1}^T \\
&= \frac{1}{(1-\rho(n))(1+(M-1)\rho(n))}((1+(M-1)\rho(n))\mathbf{I} - \rho(n)\mathbf{1}\mathbf{1}^T) \\
&= \frac{1}{(1-\rho(n))(1+(M-1)\rho(n))}\begin{bmatrix} 1+(M-2)\rho(n) & -\rho(n) & \cdots & -\rho(n) \\ -\rho(n) & 1+(M-2)\rho(n) & \cdots & -\rho(n) \\ \vdots & \vdots & \ddots & \vdots \\ -\rho(n) & -\rho(n) & \cdots & 1+(M-2)\rho(n) \end{bmatrix}.
\end{aligned}$$

4.30 a. Minimizing (4.141) with the modeling that $s(n)$ is deterministic will lead us to solve

$$2E[\hat{s}_a(n)(s(n) - w(n)\hat{s}_a(n))] = 0 \tag{4.65}$$

which result in

$$w(n) = \frac{s^2(n)}{s^2(n) + \frac{\sigma_v^2}{M}}$$

b. To estimate these weights we can used the ML estimate of $s(n)$ that we already have shown is the ensemble average $\hat{s}_a(n)$ and an estimate of σ_v^2 that can be computed by (4.17). Then the new estimate results in

$$\breve{s}(n) = \frac{\hat{s}_a^2(n)}{\hat{s}_a^2(n) + \frac{\hat{\sigma}_v^2}{M}} \hat{s}_a(n)$$

4.31 Differentiation of the MSE with respect to $\mathbf{w}_i$ results in

$$\boldsymbol{\nabla}_{\mathbf{w}_i} \mathcal{E}_{\mathbf{w}} = -2\boldsymbol{\Phi} E[x_i] + 2\boldsymbol{\Phi}^T \boldsymbol{\Phi} \mathbf{w}_i,$$

which, when set to zero to find the stationary points, yields the minimum MSE provided that the MSE is a convex function. Albeit it is well-known that the MSE is convex, the task here is to show it by exercising some math. In the scalar case, a minimum point is characterized by that the first derivative is zero, and the second derivative is positive. When carried over to the multivariate case, the requirement on the stationary points is that the *Hessian* matrix, defined as

$$[\mathbf{H}]_{(k,l)} = \frac{\partial^2 f(\mathbf{x})}{\partial x_k \partial x_l},$$

is positive definite. In this exercise, the Hessian is

$$[\mathbf{H}]_{(k,l)} = \frac{\partial^2}{\partial w_{i,k} \partial w_{i,l}} E[||\mathbf{x}_i - \boldsymbol{\Phi}\mathbf{w}_i||^2] \Rightarrow \mathbf{H} = 2\boldsymbol{\Phi}^T \boldsymbol{\Phi} = 2\mathbf{I}.$$

Since $\mathbf{H} = 2\mathbf{I}$ is positive definite, we can conclude that the solution yields the minimum MSE.

4.32 Exploiting the assumption that signal and noise are uncorrelated and taking benefit of un-correlation between different realization we can propose and estimate that crosses info from distinct realizations as

$$\hat{\mathbf{R}}_s = \frac{1}{M(M-1)} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} \mathbf{x}_i \mathbf{x}_j^T \tag{4.66}$$

Computing the expected value of the estimate

$$E[\hat{\mathbf{R}}_s] = \frac{1}{M(M-1)} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} \mathbf{R}_s = \mathbf{R}_s \tag{4.67}$$

we see it is a unbiased estimate of the signal correlation matrix. The variance can be computed element by element,$\hat{r}_{s_{l,r}}$, in the matrix $\hat{\mathbf{R}}_s$ as

$$\begin{aligned}
&E[(\hat{r}_{s_{l,r}} - E[\hat{r}_{s_{l,r}}])^2] \\
=&E\left[\left(\frac{1}{M(M-1)} \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} x_i(l)x_j(r) - s(l)s(r)\right)^2\right] \\
=&\frac{E\left[\left(\sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} (s(l)v_j(r) + v_i(l)s(r) + v_i(l)v_j(r))\right)^2\right]}{(M(M-1))^2} \\
=&\frac{E\left[\left((M-1)\left(s(l)\sum_{j=1}^{M} v_j(r) + s(r)\sum_{i=1}^{M} v_i(l)\right) + \sum_{i=1}^{M} \sum_{\substack{j=1 \\ i \neq j}}^{M} v_i(l)v_j(r)\right)^2\right]}{(M(M-1))^2} \\
=&\frac{(M-1)^2\left(s^2(l)M\sigma_v^2 + s^2(r)M\sigma_v^2\right) + 2M(M-1)\sigma_v^4 + 2M(M-1)^2 s(l)s(r)\sigma_v^2}{(M(M-1))^2} \\
=&\frac{\sigma_v^2\left(s(l)+s(r)\right)^2}{M} + \frac{2\sigma_v^4}{M(M-1)}
\end{aligned} \tag{4.68}$$

which appear to be dependent on the signal

4.33 The minimization of the error

$$\mathcal{E}(\mathbf{w}_i) = ||\mathbf{x}_i - \mathbf{\Phi}\mathbf{w}_i||^2$$

can be made in a parallel way to what was done to arrive to (4.202) from (4.199) with the only difference in that the expectation are taken out. So the optimum in this case will be

$$\mathbf{w}_i^o = \mathbf{\Phi}^T\mathbf{x}_i$$

And off course a very good estimate of this optimum is itself since both $\mathbf{\Phi}^T$ and $\mathbf{s}_i$ are known.

$$\hat{\mathbf{w}}_i = \mathbf{w}_i^o = \mathbf{\Phi}^T\mathbf{x}_i$$

that is the same estimate as presented in (4.203) when derived from the mean.

4.34 The steady-state behavior of $E[\mathbf{w}(n)]$ is more easily investigated for the case when $\mathbf{\Phi}_s$ is composed of all N basis functions and thus no truncation occurs. Then, since $\mathbf{\Phi}_s\mathbf{\Phi}_s^T = \mathbf{\Phi}_s^T\mathbf{\Phi}_s = \mathbf{I}$, we can simplify $\mathbf{F}_m(n)$ to

$$\mathbf{F}_m(n) = \mathbf{I} - \mu \sum_{j=m}^{n} \boldsymbol{\varphi}_s(j)\boldsymbol{\varphi}_s^T(j)$$

since all cross-terms $\boldsymbol{\varphi}_s(j)\boldsymbol{\varphi}_s^T(j)\boldsymbol{\varphi}_s(k)\boldsymbol{\varphi}_s^T(k)$ in (4.283) for $j \neq k$ will be equal to zero. A natural time instant for studying the behavior of $\mathbf{w}(n)$ is at the end of a potential. For the first EP, this means that (4.282) for $n = N$ becomes

$$\mathbf{w}(N) = (1-\mu)\mathbf{w}(0) + \mu \sum_{m=0}^{N-1} x(m)\boldsymbol{\varphi}_s(m),$$

where $\mathbf{F}_0(N-1) = (1-\mu)\mathbf{I}$ and $\mathbf{F}_{m+1}(N-1) = \mathbf{I}$. By applying this equation iteratively, we obtain an expression for the weight vector at the end of the i^{th} potential,

$$\mathbf{w}(iN) = (1-\mu)^i\mathbf{w}(0) + \mu \sum_{l=0}^{i-1}(1-\mu)^i \sum_{m=0}^{N-1} x(lN+m)\boldsymbol{\varphi}_s(m).$$

In order to make the terms $(1-\mu)^i$ to decay to zero, the adaptation parameter μ must be chosen such that $0 < \mu < 2$. Taking the expected value of $\mathbf{w}(iN)$ and skipping the first term which vanishes because of the convergence condition, we obtain

$$\begin{aligned}\lim_{i\to\infty} E[\mathbf{w}(iN)] &= \lim_{i\to\infty}\left(\mu \sum_{l=0}^{i-1}(1-\mu)^i \sum_{m=0}^{N-1} E[x(lN+m)]\boldsymbol{\varphi}_s(m)\right)\\ &= \sum_{m=0}^{N-1} s(m)\boldsymbol{\varphi}_s(m) = \mathbf{\Phi}^T\mathbf{s} = \mathbf{w}^{\mathrm{o}}\end{aligned}$$

which is identical to the optimal solution given in (4.202) and thus an unbiased solution. When the basis functions are not complete but still provide a good representation of the $s(n)$ the above result remains a good approximation and implies that the bias is negligible.

4.35 Each scale operation has now the same form than before except replacing the decimation part by the interpolated filter (Fig. 4.8).

$$d_j(n) = h^u_\psi(-n) * c_{j+1}(n)$$
$$c_j(n) = h^u_\varphi(-n) * c_{j+1}(n)$$

with

$$h^u_\varphi(n) = \begin{cases} h_\varphi(n) & n \text{ even} \\ 0, & n \text{ odd,} \end{cases}$$

and $h^u_\psi(n)$ analogous. So each detail coefficient series $d_j(n)$ has been subject to one filtering by $h_\psi(n)$ and J-j-1 filtering stages ($j = j_0, ..., J-1$) by $h_\varphi(n)$ filters, so the transfer function becomes (Fig. 4.9)

$$D_j(e^{\jmath\omega}) = \begin{cases} H^*_\psi(e^{\jmath\omega}) & j = J-1 \\ H^*_\psi(e^{\jmath 2^{(J-j-1)}\omega}) \prod_{i=j+2}^{J} H^*_\varphi(e^{\jmath 2^{(J-i)}\omega}), & j = j_0, ..., J-2, \end{cases}$$

These filters are band-pass filters that cover the high frequency part of the signal spectrum. The coarse coefficients, $c_{j_0}(n)$, has been subject in and analogous way to J-j_0 filtering stages ($j = j_0, ..., J-1$) by $h_\varphi(n)$ filter.

$$C_{j_0}(e^{\jmath\omega}) = \prod_{i=j_0+1}^{J} H^*_\varphi(e^{\jmath 2^{(J-i)}\omega}),$$

that is a low-pass filters that cover the remaining part of the spectrum.

Since the filter $H_\varphi(e^{\jmath\omega})$, for orthogonal wavelets, is not symmetric the output has not linear phase that is not good for many biomedical application. This can be solved by using biorthogonal wavelet that still allows dyadic sampling with the filter bank implementation (even different length in the $h_\varphi(n)$ and $h_\psi(n)$ filters) and allows to interpret the wavelet coefficients series as a filter bank over the signal.

4.36 Starting by the refinement equation applied to $\varphi(t)$ we have

$$\varphi(t) = \sqrt{2} \sum_{n=0}^{N_\varphi - 1} h_\varphi(n)\varphi(2t-n) \tag{4.69}$$

and taken the Fourier transform in both sides

$$\int_{-\infty}^{\infty} \varphi(t)e^{-\jmath\Omega t}dt = \sqrt{2} \sum_{n=0}^{N_\varphi-1} h_\varphi(n) \int_{-\infty}^{\infty} \varphi(2t-n)e^{-\jmath\Omega t}dt \tag{4.70}$$

$$= \frac{1}{\sqrt{2}} \sum_{n=0}^{N_\varphi-1} h_\varphi(n)e^{-\jmath\frac{\Omega n}{2}} \int_{-\infty}^{\infty} \varphi(t)e^{-\jmath\frac{\Omega t}{2}}dt \tag{4.71}$$

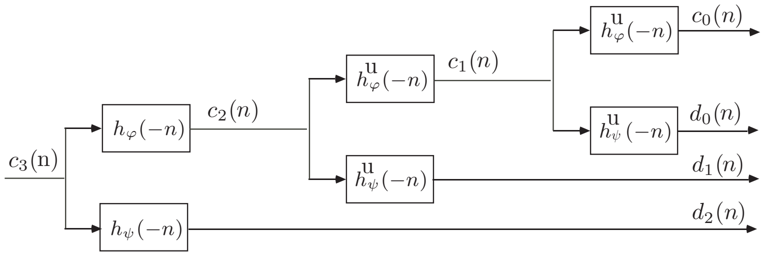


Figure 4.8: Decomposition filter bank without decimation (*algorithme a trous*)

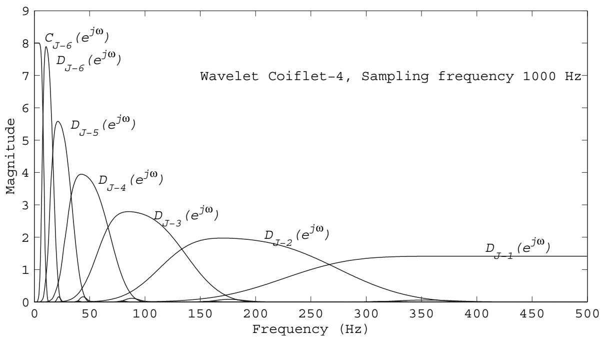


Figure 4.9: Analysis filter bank for six details scales, $D_j(e^{\jmath\omega})$, and their corresponding coarse scale, $C_{J-6}(e^{\jmath\omega})$ for Coiflet-4 wavelet, applied on a signal sampled a 1 kHz, analyzing the six finest scales plus the coarse approximation

we obtain

$$\Phi(\Omega) = \frac{1}{\sqrt{2}} H_\varphi\left(e^{\jmath\Omega/2}\right)\Phi(\Omega/2) \tag{4.72}$$

Where $H_\varphi\left(e^{\jmath\Omega}\right)$ is the discrete Fourier transform of $h_\varphi(n)$, which is periodic in frequency. Continuing to carry out this decomposition then

$$\Phi(\Omega) = \Phi(\Omega/2^l)\prod_{i=1}^{l}\frac{1}{\sqrt{2}} H_\varphi\left(e^{\jmath\Omega/2^i}\right) \tag{4.73}$$

and by letting the iteration number $l \to \infty$

$$\Phi(\Omega) = \Phi(0)\prod_{i=1}^{\infty}\frac{1}{\sqrt{2}} H_\varphi\left(e^{\jmath\Omega/2^i}\right) \tag{4.74}$$

which, without loss of generality, can be normalize so

$$\Phi(0) = \int_{-\infty}^{\infty} \varphi(t)dt = 1, \tag{4.75}$$

and then

$$\Phi(\Omega) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2}} H_\varphi \left(e^{\jmath\Omega/2^i}\right) \tag{4.76}$$

which only depend of the filter coefficients, we can think in a iterative algorithm *cascade algorithm* that estimates the scaling function at iteration $i+1$ from the scale at iteration i as

$$\varphi^{(i+1)}(t) = \sqrt{2} \sum_{n=0}^{N_\varphi - 1} h_\varphi(n)\varphi^{(i)}(2t-n) \tag{4.77}$$

Latter we will see how to initiate the recursion $\varphi^{(0)}(t)$. Taken the Fourier transform in both sides, proceeding as before and continuing to carry out this decomposition l times, from $i = 1$ to l, then

$$\Phi^l(\Omega) = \Phi^0(\Omega/2^l) \prod_{i=1}^{l} \frac{1}{\sqrt{2}} H_\varphi \left(e^{\jmath\Omega/2^i}\right) \tag{4.78}$$

and by letting the iteration number $l \to \infty$

$$\Phi^\infty(\Omega) = \Phi^0(0) \prod_{i=1}^{\infty} \frac{1}{\sqrt{2}} H_\varphi \left(e^{\jmath\Omega/2^i}\right) \tag{4.79}$$

which, again without loss of generality, can be normalize so

$$\Phi^0(0) = \int_{-\infty}^{\infty} \varphi^0(t)dt = 1, \tag{4.80}$$

and then

$$\Phi^\infty(\Omega) = \prod_{i=1}^{\infty} \frac{1}{\sqrt{2}} H_\varphi \left(e^{\jmath\Omega/2^i}\right) \tag{4.81}$$

becomes a good estimation of the scale function Fourier transform. which again only depend of the filter coefficients, and not on the starting shape of the $\varphi^0(t)$ function. If, rather than taking up to ∞, we cat the development at l, then we have a computationally accessible l order approximation to the scaling function

$$\Phi^{(l)}(\Omega) = \Phi^0(\Omega/2^l) \prod_{i=1}^{l} \frac{1}{\sqrt{2}} H_\varphi \left(e^{\jmath\Omega/2^i}\right) \tag{4.82}$$

than in some cases converge, even for small number of l [1]. This only make sense if the $\lim_{l\to\infty} \Phi(\Omega/2^l)$ is well defined as when it is continuous at $\Omega = 0$.

Chapter 5

5.1 a) The MSE of the estimate is

$$\begin{aligned}\mathrm{mse}(\hat{\sigma}_v^2) &= E[(\hat{\sigma}_v^2 - \sigma_v^2)^2] \\ &= E[\hat{\sigma}_v^4] - \sigma_v^4 \end{aligned} \tag{5.83}$$

Computing $E[\hat{\sigma}_v^4]$ we have

$$\begin{aligned} E[\hat{\sigma}_v^4] &= E\left[\frac{1}{N}\sum_{n=0}^{N-1} y^2(n)\frac{1}{N}\sum_{m=0}^{N-1} y^2(m)\right] \\ &= \frac{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1} E[y^2(n)y^2(m)]}{N^2} \\ &= \frac{\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\left(\sigma_v^4 + 2r_y^2(m-n)\right)}{N^2} \\ &= \sigma_v^4 + \frac{\sum_{n=0}^{N-1}\sum_{m'=-n}^{N-1-n} 2r_y^2(m')}{N^2}. \end{aligned} \tag{5.84}$$

Assuming that the significant correlation lags are much shorter than the observation interval N, we can approximate

$$E[\hat{\sigma}_v^4] =\approx \sigma_v^4 + \frac{\sum_{n=-N+1}^{N-1} 2r_y^2(n)}{N}, \tag{5.85}$$

and then

$$\mathrm{mse}(\hat{\sigma}_v^2) \approx \frac{2\sum_{n=-N+1}^{N-1} r_y^2(n)}{N}. \tag{5.86}$$

b) Applying the Parseval theorem

$$\begin{aligned}\mathrm{mse}(\hat{\sigma}_v^2) &\approx \frac{2\frac{1}{2\pi}\int_{-\pi}^{\pi} R_y^2(e^{\jmath\omega})d\omega}{N} \\ &= \frac{2\sum_{k=0}^{L-1} R_y^2(\omega_k)}{NL} \end{aligned} \tag{5.87}$$

were $R_y(\omega_k)$ is the L-point DFT of $r_y(n)$. Minimization of this MSE with the restriction that the power of the process $y(n)$ does not change with the coloring filter, i.e.,

$$\frac{1}{L}\sum_{k=0}^{L-1} R_y(\omega_k) = \sigma_v^2, \tag{5.88}$$

can be done by minimizing the Lagrange function,

$$\mathcal{L}(\mathbf{R}_y, \sigma_v^2) = \sum_{k=0}^{L-1} R_y^2(\omega_k) + \lambda\left(\frac{1}{L}\sum_{k=0}^{L-1} R_y(\omega_k) - \sigma_v^2)\right) \tag{5.89}$$

were $\mathbf{R}_y = [R_y(\omega_0), \ldots, R_y(\omega_{L-1})]^T$. Differentiating and equaling to zero appears

$$\frac{\partial \mathcal{L}(\mathbf{R}_y, \sigma_v^2)}{\partial R_y(\omega_k)} = 2R_y(\omega_k) + \frac{\lambda}{L} = 0, \tag{5.90}$$

giving that

$$R_y(\omega_k) = -\frac{\lambda}{2L} \tag{5.91}$$

This results implies that the spectrum of $y(n)$ should be flat (white noise) and again the whitening filter appears as the one which result in the lower variance for the amplitude estimate. Note that forcing the Lagrange condition to be satisfied

$$\frac{1}{L}\sum_{k=0}^{L-1} R_y(\omega_k) = \frac{1}{L}\sum_{k=0}^{L-1} -\frac{\lambda}{2L} = \sigma_v^2, \tag{5.92}$$

results in

$$\lambda = -2L\sigma_v^2, \tag{5.93}$$

and then

$$R_y(\omega_k) = \sigma_v^2 \tag{5.94}$$

Other way to see it is at (5.86 or 5.84) impose the restriction that $r_y(0) = \sigma_v^2$ and minimizing the variance in 5.86 which implies $r_y(n) = 0$ for $n \neq 0$ and again it is shown that $y(n)$ need to be white to have the lower possible variance

5.2 The only difference with the derivation of the ML estimate for σ will be that the PDF will include the explicit dependence with $\mathcal{F}$ as $g(\mathcal{F})$ so the differentiation will be

$$\frac{\partial lnp(\mathbf{x}; g(\mathcal{F}))}{\partial \mathcal{F}} = \left.\frac{dg(\mathcal{F})}{d\mathcal{F}}\right|_{\mathcal{F}=\hat{\mathcal{F}}} \left(-\frac{N}{g(\hat{\mathcal{F}})} + \frac{1}{g^3(\hat{\mathcal{F}})} (\mathbf{H}^{-1}\mathbf{x})^T (\mathbf{H}^{-1}\mathbf{x}) \right) = 0 \tag{5.95}$$

That, since $g(\mathcal{F})$ is a monotonic function with $\mathcal{F}$, result in

$$\hat{\mathcal{F}} = g^{-1} \left(\sqrt{\frac{1}{N} (\mathbf{H}^{-1}\mathbf{x})^T (\mathbf{H}^{-1}\mathbf{x})} \right) \tag{5.96}$$

5.3 a) We have a problem of estimating the mean of a random variable $\hat{\mathcal{F}}$ which is a function of other random variable $\xi = \sum_{n=0}^{N-1} y^2(n)$ through the estimate of σ, $\hat{\sigma}$.

$$\mathcal{F} = \left(\frac{\sigma}{k}\right)^{1/a} \tag{5.97}$$

For that we need to know the PDF of ξ which comes from the summation of N squared random variables $y(n)$ with Gaussian PDF (zero mean and variance σ) and independent. The PDF of this random variable is

$$p(\xi, \sigma(\mathcal{F})) = \frac{1}{(2\sigma^2)^{N/2} \Gamma(N/2)} \xi^{(N/2-1)} e^{\frac{-\xi}{2\sigma^2}} \tag{5.98}$$

and the $E[\hat{\mathcal{F}}]$ is

$$\begin{aligned} E[\hat{\mathcal{F}}] &= E\left[\left(\frac{1}{k} \left(\frac{\xi}{N} \right)^{1/2} \right)^{1/a} \right] \\ &= E\left[\left(\frac{\xi}{Nk^2} \right)^{1/2a} \right] \\ &= \int_0^{\infty} \left(\frac{\xi}{Nk^2} \right)^{1/2a} \frac{\xi^{(N/2-1)} e^{-\frac{\xi}{2k^2\mathcal{F}^{2a}}}}{(2k^2\mathcal{F}^{2a})^{N/2}\Gamma(N/2)} d\xi \\ &= \frac{\Gamma(N/2 + 1/(2a))}{\Gamma(N/2)} \left(\frac{2}{N} \right)^{1/2a} \mathcal{F} \end{aligned} \tag{5.99}$$

b) To compute the SNR we need to compute $E[\hat{\mathcal{F}}^2]$ that proceeding in a parallel mode than before we obtain

$$E[\hat{\mathcal{F}}^2] = \int_0^\infty \left(\frac{\xi}{Nk^2}\right)^{1/a} \frac{\xi^{(N/2-1)} e^{-\frac{\xi}{2k^2\mathcal{F}^{2a}}}}{(2k^2\mathcal{F}^{2a})^{N/2}\Gamma(N/2)} d\xi$$
$$= \frac{\Gamma(N/2+1/a)}{\Gamma(N/2)} \left(\frac{2}{N}\right)^{1/a} \mathcal{F}^2 \tag{5.100}$$

and the SNR is

$$SNR = \frac{E[\hat{\mathcal{F}}]^2}{E[(\hat{\mathcal{F}} - E[\hat{\mathcal{F}}])^2]}$$
$$= \frac{E[\hat{\mathcal{F}}]^2}{E[\hat{\mathcal{F}}^2] - E[\hat{\mathcal{F}}]^2}$$
$$= \left(\frac{\Gamma(N/2+1/a)\Gamma(N/2)}{\Gamma^2(N/2+1/(2a))} - 1\right)^{-1} \tag{5.101}$$

which result independent of the force $\mathcal{F}$. That means that the bigger the force $\mathcal{F}$ the bigger the standard deviation of their estimate which is proportional to the force.

Making use of the Γ function property that $\Gamma(n+1) = n\Gamma(n)$ and for the particular case were $\sigma = k\mathcal{F}$ ($a = 1$, linear relation between force and amplitude) the expression can be rewrite to

$$SNR = \left(\frac{N}{2}\frac{\Gamma^2(N/2)}{\Gamma^2(N/2+1/2)} - 1\right)^{-1} \tag{5.102}$$

and using the $\Gamma(J)$ approximation for large J, $\Gamma(J+1/2)/\Gamma(J) \approx \sqrt{J}\left(1 - \frac{1}{8J}\right)$, we obtain

$$SNR \approx \left(\frac{N}{2}\left(\frac{1}{\sqrt{\frac{N}{2}}(1-\frac{1}{4N})}\right)^2 - 1\right)^{-1} \approx 2N \tag{5.103}$$

which result in the expected behavior that the bigger the number of observations N the higher the SNR

5.4 According to these considerations, and considering that the noise will affect equally at any projection if considered white from channel to channel the terms with smaller λ_i in

$$\hat{\sigma}^2 = \frac{1}{NM}\sum_{n=0}^{N-1}\sum_{m=1}^{M}\frac{1}{\lambda_m}z_m^2(n) \tag{5.104}$$

will be more noise affected, lower SNR ratio, and then other estimate can be done by truncation of the sum

$$\hat{\sigma}^2 = \frac{1}{NM_1} \sum_{n=0}^{N-1} \sum_{m=1}^{M_1} \frac{1}{\lambda_m} z_m^2(n) \tag{5.105}$$

where M_1 accounts for the $M_1 < M$ higher eigenvalues of the spatial covariance matrix

5.5 The observed signal, $x(n) = s(n) + v(n)$, has a power spectral density, assuming un-correlation between $s(n)$ and $v(n)$, of

$$S_x(e^{\jmath\omega}) = S_s(e^{\jmath\omega}) + S_v(e^{\jmath\omega}) \tag{5.106}$$

and to develop the pre-whitening filter of the EMG part we need to estimate $S_s(e^{\jmath\omega})$. Since at 0% MVC ($\sigma_s = 0$) just the noise will be recorded , an estimate of this power spectrum can be derived as

$$\hat{S}_v(e^{\jmath\omega}) = \hat{S}_x(e^{\jmath\omega}; \sigma_s = 0) \tag{5.107}$$

which also allows to have an estimate of the noise power, $\hat{P}_v$. When in the calibration recording with an amplitude level $\sigma_{s_{cal}}$ we can have and estimate of the observed signal power spectrum $\hat{S}_x(e^{\jmath\omega}; \sigma_{s_{cal}})$ and their total power $\hat{P}_{x_{cal}}$ and derive and estimate of the underlying EMG signal power spectrum as

$$\hat{S}_s(e^{\jmath\omega}, \sigma_{s_{cal}}) = \hat{S}_x(e^{\jmath\omega}; \sigma_{s_{cal}}) - \hat{S}_v(e^{\jmath\omega}), \tag{5.108}$$

from where the whitening filter $H^{-1}(e^{\jmath\omega})$ can be designed as the inverse of this power spectrum after normalization with the estimate of $\sigma_{s_{cal}}$:

$$\hat{\sigma}^2_{s_{cal}} = \hat{P}_{x_{cal}} - \hat{P}_v \tag{5.109}$$

and then

$$|H^{-1}(e^{\jmath\omega})|^2 = \frac{\hat{\sigma}^2_{s_{cal}}}{\hat{S}_s(e^{\jmath\omega}; \sigma_{s_{cal}})} = \frac{\hat{P}_{x_{cal}} - \hat{P}_v}{\hat{S}_x(e^{\jmath\omega}; \sigma_{s_{cal}}) - \hat{S}_v(e^{\jmath\omega})} \tag{5.110}$$

b) if we call the signal after the whitening filter $x_w(n) = s_w(n) + v_w(n)$ we realize that $s_w(n)$ will be white, $S_{s_w}(e^{\jmath\omega}) = \sigma_s^2$, but not $v_w(n)$ that will have the power spectrum

$$S_{v_w}(e^{\jmath\omega}) = S_v(e^{\jmath\omega}) |H^{-1}(e^{\jmath\omega})|^2 \tag{5.111}$$

that will contribute to the amplitude estimate with a bias equal to the squared root of noise power $P_v^{1/2}$. To reduce this noise we know that the optimum linear filter is the Winner filter which has the form

$$H^*(e^{\jmath\omega}) = \alpha \frac{S_{s_w}(e^{\jmath\omega})}{S_{s_w}(e^{\jmath\omega}) + S_{v_w}(e^{\jmath\omega})} = \alpha \frac{\sigma_s^2}{\sigma_s^2 + S_{v_w}(e^{\jmath\omega})} \tag{5.112}$$

where the parameter α is scale parameter. This filter need to have and estimate of σ_s and $S_{v_w}(e^{\jmath\omega})$ before it is implemented, situation that is problematic since those values are unknown. Assuming that the noise level remains fix during the recording $S_{v_w}(e^{\jmath\omega})$ can be estimated from the previous recording at 0% MVC so

$$\hat{S}_{v_w}(e^{\jmath\omega}) = \hat{S}_v(e^{\jmath\omega})|H^{-1}(e^{\jmath\omega})|^2. \tag{5.113}$$

The signal amplitude of the filter σ_s^2 can be initialize to some value and then actualize adaptively with the previous amplitude estimate of the filter output, so if the amplitude increases the filter moves to become more all pass and if the force reduces and so the amplitude, the filter attenuates more the noise contribution and makes a stronger filtering. The parameter α can be estimate adaptively so to guaranty that the signal power after the two filter stages is the same than the power of the signal input, circumstances that is not satisfied just with the pre-whitening since the noise can be amplified by it.

5.6 If we have a power spectral density $S_x(e^{\jmath\omega})$ corresponding to an EMG signal $x(n) = x_c(nT)$ and at some point the signal, was scaled by a factor ν becoming $y(n) = x_c(\nu nT)$ the PSD of the $y(n)$ becomes $S_y(e^{\jmath\omega}) = \frac{1}{\nu} S_x\left(e^{\jmath\omega/\nu}\right)$. The mean frequency for signal $x(n)$, ω_{MNF_x} is

$$\omega_{\mathrm{MNF}_x} = \frac{\displaystyle\int_0^\pi \omega\ S_x(e^{\jmath\omega})d\omega}{\displaystyle\int_0^\pi S_x(e^{\jmath\omega})d\omega} \tag{5.114}$$

and for signal $y(t)$, ω_{MNF_y}

$$\omega_{\mathrm{MNF}_y} = \frac{\displaystyle\int_0^\pi \omega\ S_y(e^{\jmath\omega})d\omega}{\displaystyle\int_0^\pi S_y(e^{\jmath\omega})d\omega} = \frac{\displaystyle\int_0^\pi \omega\ \frac{1}{\nu} S_x(e^{\jmath\omega/\nu})d\omega}{\displaystyle\int_0^\pi \frac{1}{\nu} S_x(e^{\jmath\omega/\nu})d\omega} \tag{5.115}$$

by doing the change $\omega' = \omega/\nu$

$$\omega_{\mathrm{MNF}_y} = \frac{\displaystyle\int_0^{\pi/\nu} \nu\omega' \; S_x(e^{\jmath\omega'})d\omega'}{\displaystyle\int_0^{\pi/\nu} S_x(e^{\jmath\omega'})d\omega'} = \nu\omega_{\mathrm{MNF}_x} \tag{5.116}$$

where the last equality holds since the spectrum of $S_x(e^{\jmath\omega})$ is suppose to by band limited in frequency at least to $[0,\, \pi/\nu]$ so that $S_y(e^{\jmath\omega})$ does not suffer aliasing.

Doing parallel analysis for the median frequency

$$\int_0^{\omega_{\mathrm{MDF}_x}} S_x(e^{\jmath\omega})d\omega = \int_{\omega_{\mathrm{MDF}_x}}^{\pi} S_x(e^{\jmath\omega})d\omega \tag{5.117}$$

and

$$\int_0^{\omega_{\mathrm{MDF}_y}} S_y(e^{\jmath\omega})d\omega = \int_{\omega_{\mathrm{MDF}_y}}^{\pi} S_y(e^{\jmath\omega})d\omega \tag{5.118}$$

$$\int_0^{\omega_{\mathrm{MDF}_y}} \frac{1}{\nu} S_x(e^{\jmath\omega/\nu})d\omega = \int_{\omega_{\mathrm{MDF}_y}}^{\pi} \frac{1}{\nu} S_x(e^{\jmath\omega/\nu})d\omega \tag{5.119}$$

again doing the change $\omega' = \omega/\nu$ we have

$$\int_0^{\omega_{\mathrm{MDF}_y}/\nu} S_x(e^{\jmath\omega'})d\omega' = \int_{\omega_{\mathrm{MDF}_y}/\nu}^{\pi/\nu} S_x(e^{\jmath\omega'})d\omega' \tag{5.120}$$

and from here it follows that $\omega_{\mathrm{MDF}_y} = \nu\omega_{\mathrm{MDF}_x}$ which is the same dependency than for the mean frequency and so $\frac{\omega_{\mathrm{MDF}_y}}{\omega_{\mathrm{MNF}_y}} = \frac{\omega_{\mathrm{MDF}_x}}{\omega_{\mathrm{MNF}_x}}$

5.7 The power spectrum of the observed signal to deal with is

$$\begin{aligned} S_x(e^{\jmath\omega}) &= S_s(e^{\jmath\omega}) + S_v(e^{\jmath\omega}) \\ &= S_s(e^{\jmath\omega}) + \sigma_v^2 \end{aligned} \tag{5.121}$$

so the expected value of the mean frequency is:

$$\begin{aligned} E[\hat{\omega}_{\mathrm{MNF}}] &= \frac{\displaystyle\int_0^{\pi} \omega \left(S_s(e^{\jmath\omega}) + \sigma_v^2\right) d\omega}{\displaystyle\int_0^{\pi} \left(S_s(e^{\jmath\omega}) + \sigma_v^2\right) d\omega} \\ &= \frac{\displaystyle\int_0^{\pi} \omega S_s(e^{\jmath\omega})d\omega}{\displaystyle\int_0^{\pi} S_s(e^{\jmath\omega})d\omega + \pi\sigma_v^2} + \frac{(\pi^2\sigma_v^2)/2}{\displaystyle\int_0^{\pi} S_s(e^{\jmath\omega})d\omega + \pi\sigma_v^2} \end{aligned} \tag{5.122}$$

and considering high SNR, $\int_0^\pi S_s(e^{\jmath\omega})d\omega \gg \pi\sigma_v^2$, it results

$$E[\hat{\omega}_{\mathrm{MNF}}] = \omega_{\mathrm{MNF}} + \frac{(\pi^2\sigma_v^2)/2}{\int_0^\pi S_s(e^{\jmath\omega})d\omega} \tag{5.123}$$

so the bias for the mean, b_{MNF}, is

$$b_{\mathrm{MNF}} = \frac{(\pi^2\sigma_v^2)/2}{\int_0^\pi S_s(e^{\jmath\omega})d\omega} = \frac{\pi}{2\mathrm{SNR}} \tag{5.124}$$

Proceeding now with the median w_{MDF}

$$\int_0^{\hat{\omega}_{\mathrm{MDF}}} \left(\hat{S}_s(e^{\jmath\omega}) + \hat{S}_v(e^{\jmath\omega})\right) d\omega = \int_{\hat{\omega}_{\mathrm{MDF}}}^{\pi} \left(\hat{S}_s(e^{\jmath\omega}) + \hat{S}_v(e^{\jmath\omega})\right) d\omega \tag{5.125}$$

Assuming the $\hat{S}_s(e^{\jmath\omega})$ not bias and white $E[\hat{S}_s(e^{\jmath\omega})] = \sigma_v^2$ we can write

$$\int_0^{\hat{\omega}_{\mathrm{MDF}}} \hat{S}_s(e^{\jmath\omega})d\omega + \sigma_v^2\hat{\omega}_{\mathrm{MDF}} = \int_{\hat{\omega}_{\mathrm{MDF}}}^{\pi} \hat{S}_s(e^{\jmath\omega})d\omega + \sigma_v^2(\pi - \hat{\omega}_{\mathrm{MDF}}) \tag{5.126}$$

assuming that the estimate of the EMG spectrum does not introduce error, or that this is negligible respect to that introduced by the noise we can write

$$\int_{\omega_{\mathrm{MDF}}}^{\hat{\omega}_{\mathrm{MDF}}} \hat{S}_s(e^{\jmath\omega})d\omega + \sigma_v^2\hat{\omega}_{\mathrm{MDF}} = -\int_{\omega_{\mathrm{MDF}}}^{\hat{\omega}_{\mathrm{MDF}}} \hat{S}_s(e^{\jmath\omega})d\omega + \sigma_v^2(\pi - \hat{\omega}_{\mathrm{MDF}}) \tag{5.127}$$

Assuming now high SNR so $\hat{S}_s(e^{\jmath\omega}) \approx \hat{S}_s(e^{\jmath\omega_{\mathrm{MDF}}})$ in the interval $[\omega_{\mathrm{MDF}}, \hat{\omega}_{\mathrm{MDF}}]$ we can write

$$2(\hat{\omega}_{\mathrm{MDF}} - \omega_{\mathrm{MDF}})\hat{S}_s(e^{\jmath\omega_{\mathrm{MDF}}}) = \sigma_v^2(\pi - 2\hat{\omega}_{\mathrm{MDF}}) \tag{5.128}$$

Taken the expected value and assuming $\hat{\omega}_{\mathrm{MDF}}$ and $\hat{S}_s(e^{\jmath\omega_{\mathrm{MDF}}})$ uncorrelated

$$2(E[\hat{\omega}_{\mathrm{MDF}}] - \omega_{\mathrm{MDF}})S_s(e^{\jmath\omega_{\mathrm{MDF}}}) = \sigma_v^2(\pi - 2E[\hat{\omega}_{\mathrm{MDF}}]) \tag{5.129}$$

so

$$2b_{\mathrm{MDF}}S_s(e^{\jmath\omega_{\mathrm{MDF}}}) = \sigma_v^2\pi - 2\sigma_v^2(\omega_{\mathrm{MDF}} + b_{\mathrm{MDF}}) \tag{5.130}$$

and so the bias,b_{MDF}, is

$$\begin{aligned} b_{\mathrm{MDF}} &= \frac{(\pi - 2\omega_{\mathrm{MDF}})\sigma_v^2}{2S_s(e^{\jmath\omega_{\mathrm{MDF}}}) + 2\sigma_v^2} \\ &\approx \frac{(\pi - 2\omega_{\mathrm{MDF}})\sigma_v^2}{2S_s(e^{\jmath\omega_{\mathrm{MDF}}})} \end{aligned}$$

If we accept that the ratio $\frac{\sigma_v^2}{S_s(e^{\jmath\omega_{\mathrm{MDF}}})}$ is of the order of the inverse SNR then we can write

$$b_{\mathrm{MDF}} \approx \left(\frac{\pi - 2\omega_{\mathrm{MDF}}}{2\mathrm{SNR}}\right)$$

which will usually be smaller than for the b_{MNF} case since the mean frequency rarely will be at the band extreme. Also the value at $S_s(e^{\jmath\omega_{\mathrm{MDF}}})$ is usually larger than the mean spectrum since the spectrum then to have their maximum values around the median so the bias even smaller than this expression. Then if noise contamination is appreciable this bias contribution will control the variance of the estimate in both cases and will made the median a better estimate to characterize the dominant frequency.

The bias can certainly be reduced by filtering the noise so the factor π in both expressions will reduce to the bandwidth, ω_B, of the noise resulting from a low pass filter with that frequency

$$b_{\mathrm{MDF}} \approx \left(\frac{\omega_B}{2} - \omega_{\mathrm{MDF}}\right)\frac{1}{\mathrm{SNR}}$$
$$b_{\mathrm{MNF}} \approx \left(\frac{\omega_B}{2}\right)\frac{1}{\mathrm{SNR}}$$

5.8 a) Doing parallel analysis as the one done in problem 5.6 for the median frequency

$$\int_0^{\omega^c_{\mathrm{MDF}_x}} S_x(e^{\jmath\omega})d\omega = c\int_0^{\pi} S_x(e^{\jmath\omega})d\omega \tag{5.131}$$

and

$$\int_0^{\omega^c_{\mathrm{MDF}_y}} S_y(e^{\jmath\omega})d\omega = c\int_0^{\pi} S_y(e^{\jmath\omega})d\omega \tag{5.132}$$

$$\int_0^{\omega^c_{\mathrm{MDF}_y}} \frac{1}{\nu}S_x(e^{\jmath\omega/\nu})d\omega = c\int_0^{\pi} \frac{1}{\nu}S_x(e^{\jmath\omega/\nu})d\omega \tag{5.133}$$

doing the variable change $\omega' = \omega/\nu$ we have

$$\int_0^{\omega^c_{\mathrm{MDF}_y}/\nu} S_x(e^{\jmath\omega'})d\omega' = c\int_0^{\pi/\nu} S_x(e^{\jmath\omega'})d\omega' \tag{5.134}$$

and from here it follows that $\omega^c_{\mathrm{MDF}_y} = \nu\omega^c_{\mathrm{MDF}_x}$

b) From previous observation it seems reasonable to estimate the so call percentile median frequency ω^c_{MDF} at a set of different percentiles c in the range [0,1]. From them we can estimate the least squared regression line fit between $\omega^c_{\mathrm{MDF}_y}$ and $\omega^c_{\mathrm{MDF}_x}$ which slope will be an estimate of ν. This estimate will be less affected by the noise than estimates from a individual frequencies $\omega^c_{\mathrm{MDF}_y}$ as far as the noisy estimates at different percentiles have some degree of un-correlation.

5.9 From the definition we have for the EMG signal $x(n)$

$$\mathcal{H}_{1_x} = \sqrt{\frac{\displaystyle\int_{-\pi}^{\pi} \omega^2 S_x(e^{\jmath\omega}) d\omega}{\displaystyle\int_{-\pi}^{\pi} S_x(e^{\jmath\omega}) d\omega}} \tag{5.135}$$

and for the EMG signal $y(n) = x(\nu n)$

$$\begin{aligned}
\mathcal{H}_{1_y} &= \sqrt{\frac{\displaystyle\int_{-\pi}^{\pi} \omega^2 S_y(e^{\jmath\omega}) d\omega}{\displaystyle\int_{-\pi}^{\pi} S_y(e^{\jmath\omega}) d\omega}} = \sqrt{\frac{\displaystyle\int_{-\pi}^{\pi} \omega^2 \frac{1}{\nu} S_x(e^{\jmath\omega/\nu}) d\omega}{\displaystyle\int_{-\pi}^{\pi} \frac{1}{\nu} S_x(e^{\jmath\omega/\nu}) d\omega}} \\
&= \nu \sqrt{\frac{\displaystyle\int_{-\pi/\nu}^{\pi/\nu} \omega'^2 S_x(e^{\jmath\omega'}) d\omega'}{\displaystyle\int_{-\pi/\nu}^{\pi/\nu} S_x(e^{\jmath\omega'}) d\omega'}} = \nu \mathcal{H}_{1_x}
\end{aligned} \tag{5.136}$$

In Problem 3.6 we see that for a bandpass spectrum the mean frequency results a bit overestimated by the $\mathcal{H}_1$ but here we show that the excursion of this parameter with the scaled spectrum generated by fatigue is equally good than in the other frequency related parameters considered ω_{MNF} and ω_{MDF}.

5.10 From the definition of ω_{MNF_x} we can proceed as:

$$\begin{aligned}
\omega_{\mathrm{MNF}_x} &= \frac{\displaystyle\int_{0}^{\pi} \omega \; S_x(e^{\jmath\omega}) d\omega}{\displaystyle\int_{0}^{\pi} S_x(e^{\jmath\omega}) d\omega} \\
&= \frac{1}{\jmath} \frac{\displaystyle\int_{0}^{\pi} \jmath\omega \; S_x(e^{\jmath\omega}) d\omega}{\displaystyle\int_{0}^{\pi} S_x(e^{\jmath\omega}) d\omega}
\end{aligned} \tag{5.137}$$

where, apart form a factor 2, the numerator is the inverse Fourier transform of the analytic function of the autocorrelation derivative, $(r'_x)_A(n)$, evaluated at $n = 0$, so we can express it as

$$\begin{aligned}\omega_{\mathrm{MNF}_x} &= \frac{1}{j}\frac{\pi\left(r'_x(m) + j\check{r}'_x(m)\right)|_{m=0}}{\pi r_x(0)} \\ &= \frac{\check{r}'_x(m)|_{m=0}}{r_x(0)}\end{aligned} \tag{5.138}$$

and introducing the autocorrelation definition, using the stationarity condition, and denoting as $\check{E}$ the Hilbert transform of the autocorrelation when express in expectation terms

$$\begin{aligned}\omega_{\mathrm{MNF}_x} &= \frac{\check{E}[x(n)x'(n-m)]|_{m=0}}{r_x(0)} \\ &= \frac{\check{E}[x(n+m)x'(n)]|_{m=0}}{r_x(0)} \\ &= \frac{E[\check{x}(n+m)x'(n)]|_{m=0}}{r_x(0)} \\ &= \frac{E[\check{x}(n)x'(n)]}{r_x(0)}\end{aligned} \tag{5.139}$$

and then an estimate of the frequency can be obtained by

$$\hat{\omega}_{\mathrm{MNF}_x} = \frac{\displaystyle\sum_{n=0}^{N-1} \check{x}(n)x'(n)}{\displaystyle\sum_{n=0}^{N-1} x^2(n)} \tag{5.140}$$

This implies to compute the derivative of the signal $x'_x(n)$ and the Hilbert Transform $\check{x}(n)$ and normalize by the signal power. All these operations can be done from the discrete-time signal and then implemented without need to use the Fourier transform. Care should be taken in that both the differentiator filter and the Hilbert transformed had the same phase delay, otherwise the result will be not correct. The differentiate signal with zero delay can be estimated by

$$x'(n) = \frac{1}{2}(x(n+1) - x(n-1)) \tag{5.141}$$

and the Hilbert transform by (see Section 7.4 on QRS detection)

$$\check{x}(n) = \frac{2}{\pi}(x(n-1) - x(n+1)) \tag{5.142}$$

5.11 a) The error in the velocity will be driven by the error in the delay that in the best case will be driven by the sampling interval. Since the relation between the two magnitudes is $\hat{\nu} = d/\hat{\theta}$ the relation between the errors will be

$$\Delta\nu = \frac{\partial \nu}{\partial \theta}\Delta\theta = \frac{d}{\theta^2}\Delta\theta = \frac{\nu^2}{d}\Delta\theta \tag{5.143}$$

That considering the maximum estimation error, for sampling interval T, as $\Delta\theta = \frac{T}{2}$ and a sampling rate of 1 kHz, $\Delta_\theta = 0.5$ ms. This result gives a maximum possible value of $\Delta\nu = 0.8$ m/s (relative error of 20%), which is not acceptable in clinical applications

b) This problem can be referred to the one already consider in problem 4.25 for time delay estimate of EP. Here again the problem is in estimating the delay θ and the we saw in Problem 4.25 That a finer resolution in the estimated delay can be achieved by using a frequency domain formulation of the latency estimation from the matched filter:

$$\hat{\theta} = \arg\max_{n_0} \sum_{n=-\infty}^{\infty} x_2(n)s(n-\theta) = \arg\max_{\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{\jmath\omega})S^*(e^{\jmath\omega})e^{\jmath\omega\theta}d\omega.$$

Since $X_2(e^{\jmath\omega})$ and $S(e^{\jmath\omega})$ are continuous functions, it is possible to examine noninteger delays θ.

To show that this is equivalent to an interpolation we will consider θ with finer, non integer, resolution than one sampling rate. We also denote the smaller integer number θ' such that $\theta' = D\theta$. Making a variable change $D\omega' = \omega$, and using the relation between the Fourier transform of a signal $X(e^{\jmath\omega})$ and that of their interpolated one $X^i(e^{\jmath\omega})$,

$$X^i(e^{\jmath\omega}) = \begin{cases} D\ X(e^{\jmath D\omega}) & |\omega| < \frac{\pi}{D} \\ 0 & |\omega| > \frac{\pi}{D} \end{cases} \tag{5.144}$$

we have that

$$\begin{aligned}
\hat{\theta} &= \arg\max_{\theta} \frac{1}{2\pi}\int_{-\pi/D}^{\pi/D} X(e^{\jmath D\omega'})S^*(e^{\jmath D\omega'})e^{\jmath D\omega'\theta}Dd\omega' \\
&= \arg\max_{\theta} \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{D}X^i(e^{\jmath\omega'})S^{i^*}(e^{\jmath\omega'})e^{\jmath D\omega'\theta}d\omega' \\
&= \arg\max_{\theta'} \frac{1}{D}\sum_{n=-\infty}^{\infty} x^i(n)s^i(n-\theta')
\end{aligned} \tag{5.145}$$

where $x^i(n)$ is the signal $x(n)$ interpolated by a factor D and same with $s(n)$ and their respective Fourier transforms. The operation then represents the matched filter after interpolation by a factor D.

c) considering that a estimate of $s(n)$ can be the first signal recording $x_1(n)$ we can compute the maximization as

$$\arg\max_{\theta}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X_2(e^{\jmath\omega})S^*(e^{\jmath\omega})e^{\jmath\omega\theta}d\omega\right) \approx \arg\max_{\theta}\left(\frac{1}{N}\sum_{k=0}^{N-1} X_2(k)X_1^*(k)e^{\jmath 2\pi\theta k/N}\right) \tag{5.146}$$

so we can apply the iterative Newton method for minimization to the performance surface

$$\mathcal{J}(\theta) = -\sum_{k=0}^{N-1}\frac{1}{N}X_2(k)X_1^*(k)e^{\jmath 2\pi\theta k/N} \tag{5.147}$$

$$\theta^{(i+1)} = \theta^{(i)} - \alpha\frac{\left.\frac{d\mathcal{J}(\theta)}{d\theta}\right|_{\theta=\theta^{(i)}}}{\left.\frac{d^2\mathcal{J}(\theta)}{d\theta^2}\right|_{\theta=\theta^{(i)}}} \tag{5.148}$$

where

$$\begin{aligned}\frac{d\mathcal{J}(\theta)}{d\theta} &= \frac{2}{N}\sum_{k=0}^{N/2-1}\left(\frac{2\pi k}{N}\right)\Im\left[X_2(k)X_1^*(k)e^{\jmath 2\pi\theta k/N}\right]\\ \frac{d^2\mathcal{J}(\theta)}{d\theta^2} &= \frac{2}{N}\sum_{k=0}^{N/2-1}\left(\frac{2\pi k}{N}\right)^2\Re\left[X_2(k)X_1^*(k)e^{\jmath 2\pi\theta k/N}\right]\end{aligned} \tag{5.149}$$

and $\alpha < 1$ is an updating parameter speed. The iteration can follow like in the Woody method until the difference between different iteration estimated delay is lower than a threshold. The initialization can be done by the estimate form matched filter without interpolation.

5.12 Starting in the expression

$$\begin{aligned}\hat{\theta} = \arg\max_{\theta}\Bigg(\sum_{m=1}^{M}\sum_{n=0}^{N-1}\Bigg(&\frac{2}{M}\sum_{l=1}^{M}x_m(n)x_l(n+(l-m)\theta)\\ &-\frac{1}{M^2}\left(\sum_{l=1}^{M}x_l(n+(l-m)\theta)\right)^2\Bigg)\Bigg)\end{aligned} \tag{5.150}$$

and decomposing the second term

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{m=1}^{M} \sum_{n=0}^{N-1} \left(\frac{2}{M} \sum_{l=1}^{M} x_m(n) x_l(n + (l-m)\theta) \right.\right.$$
$$\left.\left. - \frac{1}{M^2} \sum_{l=1}^{M} \sum_{l'=1}^{M} x_l(n + (l-m)\theta) x_{l'}(n + (l'-m)\theta) \right)\right) \tag{5.151}$$

Noting that the second term does not really depend on m since the delay affecting x_l and $x_{l'}$ in both cases includes the factor $m\theta$, it can be rewritten as

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{n=0}^{N-1} \sum_{m=1}^{M} \frac{2}{M} \sum_{l=1}^{M} x_m(n) x_l(n + (l-m)\theta) \right.$$
$$\left. - \frac{1}{M} \sum_{n=0}^{N-1} \sum_{l=1}^{M} \sum_{l'=1}^{M} x_l(n + l\theta) x_{l'}(n + l'\theta) \right) \tag{5.152}$$

we can proceed

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{n=0}^{N-1} \sum_{m=1}^{M} \frac{2}{M} \sum_{l=1}^{M} x_m(n) x_l(n + (l-m)\theta) \right.$$
$$\left. - \frac{1}{M} \sum_{n=0}^{N-1} \sum_{l=1}^{M} \sum_{l'=1}^{M} x_l(n) x_{l'}(n + (l'-l)\theta) \right) \tag{5.153}$$

and then

$$\hat{\theta} = \arg\max_{\theta} \left(\frac{1}{M} \sum_{m=1}^{M} \sum_{l=1}^{M} \sum_{n=0}^{N-1} x_m(n) x_l(n + (l-m)\theta) \right) \tag{5.154}$$

and since when $m = l$ the cross product does not depend on θ it can also be express as

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{m=1}^{M} \sum_{\substack{l=1 \\ l \neq m}}^{M} \sum_{n=0}^{N-1} x_m(n) x_l(n + (l-m)\theta) \right) \tag{5.155}$$

and considering that all cross product are counted twice it can also be further reduced to

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \sum_{n=0}^{N-1} x_m(n) x_l(n + (l-m)\theta) \right) \tag{5.156}$$

5.13 We know that the estimate is

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \frac{1}{2\pi} \int_{-\pi}^{\pi} X_m(e^{\jmath\omega}) X_l^*(e^{\jmath\omega}) e^{\jmath(m-l)\theta\omega} d\omega \right) \quad (5.157)$$

which can be compute from the DFT as

$$\hat{\theta} = \arg\max_{\theta} \left(\sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \frac{1}{N} \sum_{k=0}^{N-1} X_m(k) X_l^*(k) e^{\jmath(m-l)\theta 2\pi k/N} \right). \quad (5.158)$$

So we can search the minimum of the performance surface

$$\mathcal{J}(\theta) = -\sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \left(\frac{1}{N} \sum_{k=0}^{N-1} X_m(k) X_l^*(k) e^{\jmath(m-l)\theta 2\pi k/N} \right) \quad (5.159)$$

and apply the iterative algorithm as in previous problem

$$\theta^{(i+1)} = \theta^{(i)} - \alpha \frac{\left.\frac{d\mathcal{J}(\theta)}{d\theta}\right|_{\theta=\theta^{(i)}}}{\left.\frac{d^2\mathcal{J}(\theta)}{d\theta^2}\right|_{\theta=\theta^{(i)}}} \quad (5.160)$$

where now

$$\begin{aligned}
\frac{d\mathcal{J}(\theta)}{d\theta} &= \sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \left(\frac{2}{N} \sum_{k=0}^{N/2-1} \left(\frac{2\pi k(m-l)}{N} \right) \Im \left[X_m(k) X_l^*(k) e^{\jmath 2\pi\theta k(m-l)/N} \right] \right) \\
\frac{d^2\mathcal{J}(\theta)}{d\theta^2} &= \sum_{m=1}^{M-1} \sum_{l=m+1}^{M} \left(\frac{2}{N} \sum_{k=0}^{N/2-1} \left(\frac{2\pi k(m-l)}{N} \right)^2 \Re \left[X_2(k) X_1^*(k) e^{\jmath 2\pi\theta k(m-l)/N} \right] \right)
\end{aligned} \quad (5.161)$$

5.14 a) Assuming no interaction between MUAPT the autocorrelation function can be express as

$$\begin{aligned}
r_x(\tau) &= E[x(t)x(t-\tau)] \\
&= E\left[\left(\left(\sum_{l=1}^{L} u_l(t) + v(t) \right) \left(\sum_{j=1}^{L} u_j(t-\tau) + v(t-\tau) \right) \right) \right] \\
&= \sum_{l=1}^{L} r_{u_l}(\tau) + \sum_{l=1}^{L} \sum_{\substack{j=1 \\ j\neq l}}^{L} E\left[\sum_{k=1}^{K} h_l(t-t_k) \right] E\left[\sum_{k'=1}^{K} h_j(t-t_{k'}-\tau) \right] + r_v(\tau) \\
&= \sum_{l=1}^{L} r_{u_l}(\tau) + \lambda^2 \sum_{l=1}^{L} \sum_{\substack{j=1 \\ j\neq l}}^{L} \left(\int_{-\infty}^{\infty} h_l(t)dt \right) \left(\int_{-\infty}^{\infty} h_j(t)dt \right) + r_v(\tau).
\end{aligned} \quad (5.162)$$

We see that it results in the summation of the corresponding MUAPT spectra plus a DC level plus the noise power spectrum. Taking the Fourier transform

$$S_x(\Omega) = \sum_{l=1}^{L} S_{u_l}(\Omega) + \lambda_r^2 \sum_{l=1}^{L} \sum_{\substack{j=1 \\ j\neq l}}^{L} \left(\int_{-\infty}^{\infty} h_l(t)dt \right) \left(\int_{-\infty}^{\infty} h_j(t)dt \right) \delta(\Omega) + S_v(\Omega) \tag{5.163}$$

b) If we now assume that the statistical properties of the inter-pulse interval are equal for all MUAP trend then we can use $P_{r_l}(\Omega) = P_r(\Omega)$ and then

$$\begin{aligned} S_x(\Omega) &= \lambda_r \left(1 + \frac{P_r(\Omega)}{1 - P_r(\Omega)} + \frac{P_r^*(\Omega)}{1 - P_r^*(\Omega)} \right) \sum_{l=1}^{L} |H_l(\Omega)|^2 \\ &\quad + \lambda_r^2 \sum_{l=1}^{L} \sum_{\substack{j=1 \\ j\neq l}}^{L} \left(\int_{-\infty}^{\infty} h_l(t)dt \right) \left(\int_{-\infty}^{\infty} h_j(t)dt \right) \delta(\Omega) + S_v(\Omega) \end{aligned} \tag{5.164}$$

So if the assumptions are satisfied it is possible to estimate the firing rate from the lowest spectrum peak.

Even not close to the physiology in general one can also explore the expression for invariant MUAP shape and $S_x(\Omega)$ can be simplified to

$$\begin{aligned} S_x(\Omega) &= \lambda_r L \left(1 + \frac{P_r(\Omega)}{1 - P_r(\Omega)} + \frac{P_r^*(\Omega)}{1 - P_r^*(\Omega)} \right) |H(\Omega)|^2 \\ &\quad + \lambda_r^2 L(L-1) \left(\int_{-\infty}^{\infty} h(t)dt \right)^2 \delta(\Omega) + S_v(\Omega) \end{aligned} \tag{5.165}$$

were it is evident that the EMG spectrum shape is basically controlled by the MUAP shape. When the EMG goes away form this assumption, then the shape can no longer be associated to the MUAP shape but the the average over the shapes of the MUAP. Also if the MUAP firing is not correlated only the DC level suffer from the iteration between MUAP trends that has no relevance for the frequency domain EMG analysis

Chapter 7

7.1 Assume that the ECG signal is described by the following Fourier series,

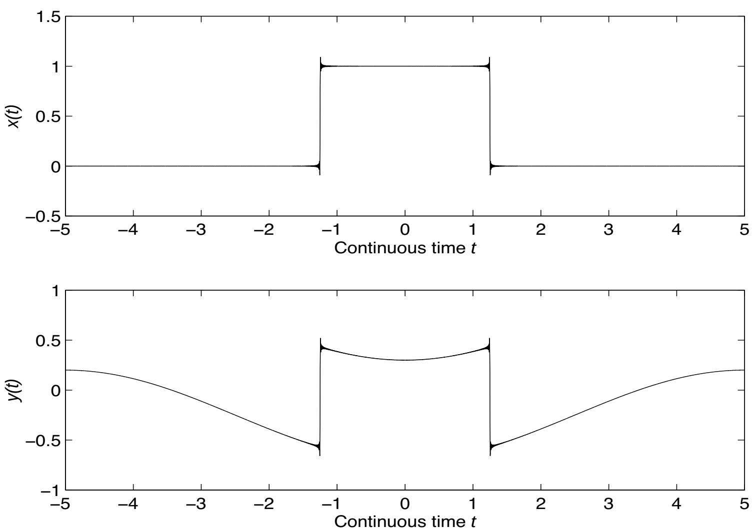
$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)).$$

The angular frequency of the baseline wander component is less than Ω_0, the heart rate, and thus to reject the baseline wandering without distorting the ECG waveform, the cut-off frequency of the highpass filter must be lower than Ω_0. Should one accidentally use a too high cut-off frequency, higher than Ω_0 but lower than $2\Omega_0$, the fundamental frequency of the ECG signal is lost:

$$\begin{aligned} y(t) &= \sum_{n=2}^{\infty}(a_n \cos(n\Omega_0 t) + b_n \sin(n\Omega_0 t)) \\ &= x(t) - \frac{a_0}{2} - a_1 \cos(\Omega_0 t) - b_1 \sin(\Omega_0 t). \end{aligned}$$

This means that after the filtering, the ECG signal will appear as it is originally but with an added oscillatory component with frequency Ω_0 (the heart rate).

The following plots show the effect on a square wave with repetition rate $1/10$ ($\Omega_0 = 2\pi/10$).



7.2 For the FIR filter, the convolution sum

$$y(n) = \sum_{k=n-N+1}^{n} x(k)h(n-k) \tag{7.166}$$

can be compacted using the symmetry properties of $h(n)$ as:

$$y(n) = \begin{cases} \displaystyle\sum_{k=n-(N-1)/2+1}^{n} \left(x(k) + x(2n-k-N+1)\right) h(n-k) & \\ \qquad +x(n-(N-1)/2)h((N-1)/2) & \text{for N odd} \\ \displaystyle\sum_{k=n-N/2+1}^{n} (x(k) + x(2n-k-N+1))h(n-k) & \text{for N even} \end{cases}$$

The total number of operations are counted in table 7.1:

Filter	Sums	Products
Symmetric FIR (order N odd)	$N-1$	$(N-1)/2+1$
Symmetric FIR (order N even)	$N-1$	$N/2$
Forward/Backwards IIR (order N)	2 N	2 (N+1)
Decimation by D, even; Interpolation filter order M, even (FIR order N/D, even)	$\frac{N+2M}{D}-3$	$\frac{N+2M}{2D}$

Table 7.1: Computational load for different filters structures

In the case of forward/backward filtering using an Nth order IIR filter we have that the filter equation can be express as two times the IIR filter equation:

$$y(n) = b_0 x(n) + \sum_{k=1}^{N} a_k y(n-k) \tag{7.167}$$

which gives the total number of operation express in table 7.1, that becomes efficient when we get same amplitude transfer function for N/2 order that is usually the case.

In case of sampling rate decimation by a factor D, and interpolation filters (decimator and interpolator filters of order M) we have that we will have a reduction in the FIR filter order of D so the equivalent order will be N/D to which we should add the antialiasing filter of order M, but only computed once every D samples, and the interpolation filter that we can assume the same filter and then order M, but since there are D-1 zeros for each non-zero sample in the series each apply, it need again only a number of operations like if were order M/D. the total number of operations will be the described in table 7.1 that compared with the FIR symmetric case will represent some improvement in number of products when

$$D > \frac{N+2M}{N+2} \tag{7.168}$$

and in number of summations when

$$D > \frac{N+2M}{N} \tag{7.169}$$

which gives an order of magnitude of the filter needs to be computationally efficient the decimation/interpolation approach. When N=1564 (-30 dB stop band attenuation for sampling rate 250 Hz, see Table 7.1), the anti-aliasing filter has order M=500, with decimation D=20, we have that number of sums reduce from 1563 to 126, and the number of multiplication reduces from 783 to 65, that in both cases represent a factor around 12 of reduction in computation load, which is very remarkable.

7.3 If the length of the filter is N, then a buffer of length $2N$ should store the $2N$ recent samples, then the output at time $-N$ can be computed both for the forward and backwards filters and generate the output. This implies a delay of at least N samples (assuming that computation time can neglected in comparison to the sampling period).

7.4 We will show this property for a IIR filter with a single pair of complex conjugate poles $d_0 = re^{\jmath\omega_0}$, $r < 1$, which transfer function is:

$$H(z) = \frac{z^2}{(z-d_0)(z-d_0^*)} \tag{7.170}$$

The frequency transfer function of this filter can by computed by replacing $z = e^{\jmath\omega}$, and computed the module that will result in the ratio between the product of distances from the unit circle location of ω to the zeros divided to the same product but to the poles.

$$H(e^{\jmath\omega}) = \frac{1}{|e^{\jmath\omega} - re^{\jmath\omega_0}||e^{\jmath\omega} - re^{-\jmath\omega_0}|} \tag{7.171}$$

for r close to one, and the poles far from the origin, $z = 1$, the maximum transfer function of this filter is located at $\omega_m \approx \omega_0$ with a value $|H(e^{\jmath\omega_m})| \approx \frac{1}{(1-r)2\sin(\omega_0)}$ (implies give a fix value to the influence of the second pole when

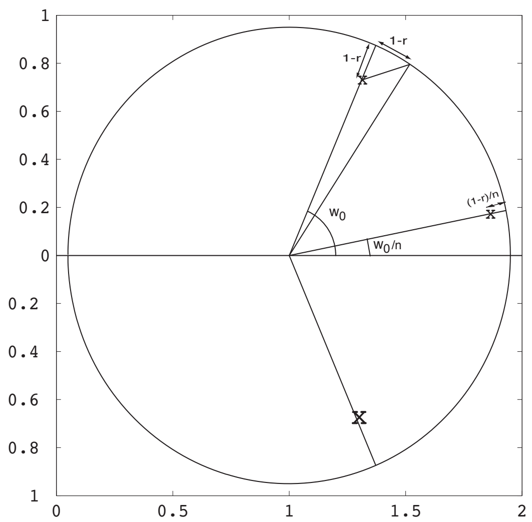


Figure 7.10: z-plane pole locations

evaluating $H(e^{\jmath\omega})$ close to the first one), and the -3 dB cut-off frequency will be located at a frequency ω_c that satisfies $|H(e^{\jmath\omega_c})| \approx \frac{1}{\sqrt{2}(1-r)2\sin(\omega_0)}$, which under the same approximation becomes, $\omega_c = \omega_m \pm (1-r)$, equivalent in the un-normalized frequency to $\Omega_c = \Omega_m \pm (1-r)F_s$.

Increasing the sampling rate, F_s, by a factor n, and still requiring the same un-normalized frequency behavior will, by necessity, imply some changes in the filter poles location that we can investigate in the following way:

To maintain the un-normalized frequency value, $\Omega_m = \omega_m F_s$, of the filter frequency response maximum unchanged, we need to move the normalized frequency to $\omega'_m = \omega_m/n$. Using same approximations as before, this can be obtained by rotating the poles a factor $1/n$; $d'_0 = re^{\jmath\omega_0/n}$.

The cut-off frequency of this filter, in the normalized domain, will have again the expression $\omega'_c = \omega'_m \pm (1-r)$ which result in the un-normalized frequency $\Omega'_c = \Omega_m \pm (1-r)nF_s$. This frequency is not the same as it was in the previous case since it is a factor n farther away from the maximum frequency Ω_m. To solve this, we can also move the poles closer to the unit circle, $d''_0 = r'e^{\jmath\omega_0/n}$, with $(1-r') = (1-r)/n$, and with them we obtain the same cut-off frequency in the un-normalized domain.

This solution, can lead to instabilities when the poles are so close to the unit circle that round-off error made them cross to outside the circle and made the filter unstable. For filter with pole number the approximations are less evident, but the overall behavior also forces the poles closer to the unit circle, increasing the risk for instabilities because round-off errors, see figure 7.11.

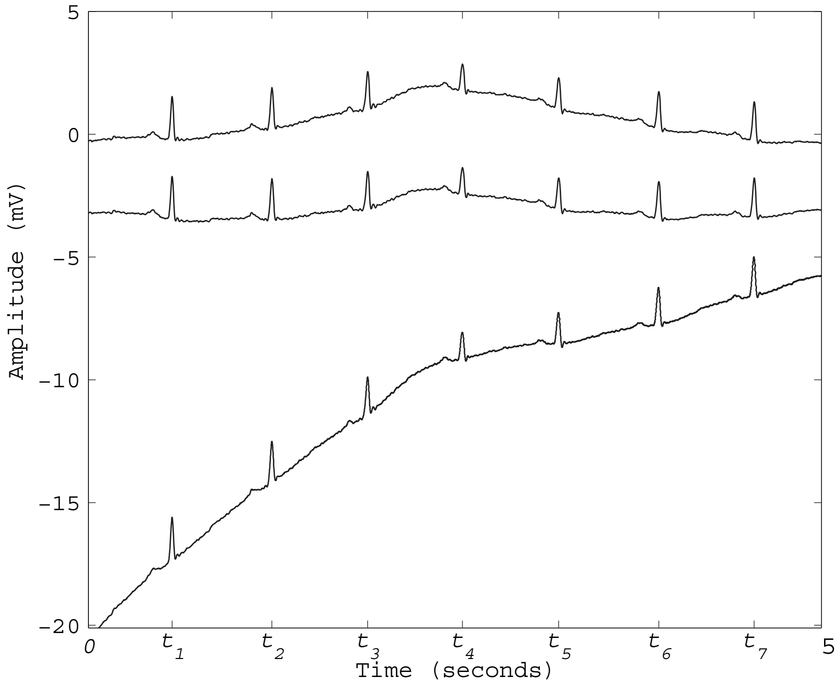


Figure 7.11: a) original ECG sampled at 360 Hz, b) forward/backwards filtered ECG with a order 5 Butterworth high-pass filter with cut-off at 0.721 Hz (poles at distance from origin $r_1 = 0.9990$, $r_2 = 0.9975$ and $r_3 = 0.9969$) c) Same than in b) but after up-sampling to 1500 Hz and redesigning the filters for same order and same un-normalized cut-off frequency (poles at distance from origin $r_1 = 0.9998$, $r_2 = 0.9994$ and $r_3 = 0.9992$). See that the output has become unstable

7.5 The ideal low-pass frequency response of a filter with cut-off frequency ω_{c_0} is

$$H_0(e^{\jmath\omega}) = \begin{cases} 1 & \text{for } 0 \leq \omega \leq \omega_{c_0} \\ 0 & \text{for } \omega_{c_0} < \omega \leq \pi \end{cases}$$

which taking the inverse Discrete-Time Fourier transform result in the following ideal filter impulse response $h_{IL_0}(n)$:

$$h_0(n) = \frac{\sin(\omega_{c_0} n)}{\pi n} \tag{7.172}$$

This filter has dependency on the cut-off frequency only in the numerator, so we can design a filter for a general cut-off frequency ω_c still keeping $h_0(n)$ as

$$\begin{aligned} h(n) &= \frac{\sin(\omega_{c_0} n)}{\pi n} \frac{\sin(\omega_c n)}{\sin(\omega_{c_0} n)} \\ &= \frac{h_0(n)}{\sin(\omega_{c_0} n)} \sin(\omega_c n) \\ &= c(n) \sin(\omega_c n) \end{aligned} \tag{7.173}$$

and truncated to obtain the filter FIR impulse response $h_T(n)$ of length 2N+1

$$h_T(n) = \begin{cases} c(0) & \text{for } n = 0 \\ c(n) \sin(\omega_c n) & 1 \le |n| \le N \end{cases} \tag{7.174}$$

with

$$c(n) = \begin{cases} \frac{1}{\pi} & \text{for } n = 0 \\ \frac{h_0(n)}{\sin(\omega_{c_0} n)} & 1 \le |n| \le N \end{cases}$$

The filter design obtained just from truncation is well know not to be the best design in terms of leakage, so some other more sophisticated methods like windows method or Remez exchange algorithm [2] can be used. With the obtained filter structure we can design just once the prototype filter $h_0(n)$ and use equation (7.174) to obtain the time-varying filter once the frequency ω_c has been decided.

With the filter structure as function of ω_c, the next step is the determination of this ω_c as function of the observed ECG and their time-varying properties, $\omega_c(k)$, denoting by k the time index of the observed signal, to obtain the time-varying impulse response $h_T(k,n)$. Already in the book it was proposed the ratio

$$\omega_c(k) = 2\pi f_c(k) \sim \frac{2\pi}{r(k)}. \tag{7.175}$$

as an estimate the heart rate dominant frequency, with

$$r(k) = r_i + \frac{r_{i+1} - r_i}{\theta_{i+1} - \theta_i}(k - \theta_i), \quad \theta_i \le k \le \theta_{i+1}. \tag{7.176}$$

To restrict the cut-off frequencies variability to a reasonable interval value, we can force $\omega_c(k)$ to take discrete values in the interval between ω_{c_1} and ω_{c_2} in a grid of L different possible frequencies

$$\omega_c(k) = \omega_{c_1} + l(k) \left(\frac{\omega_{c_2} - \omega_{c_1}}{L} \right) \qquad L > l(k) > 0 \tag{7.177}$$

and the problem reduces to match the observed heart rate to the integer value $l(k)$. One possibility can be [3]

$$l(k) = \text{int}\left[\frac{1}{\Delta\omega}\left(\frac{2\pi}{r(k)F_s} - \omega_{c_1}\right)\right] + \gamma \tag{7.178}$$

with $\Delta\omega = \frac{\omega_{c_2}-\omega_{c_1}}{L}$ the frequency grid step, and γ an integer accounting for the offset we want to introduce from the heart rate frequency to the cut-off frequency ω_c. Other alternatives could be to estimate the $l(k)$ as a function of the RMS baseline estimate differences for $l(k) = L$ (best estimation, with ECG distortion) and a running $l(k) < L$. If the difference is small it means that the baseline is still well estimated with a reduced $\omega_c(k)$ and then preferable to avoid ECG distortion. When the RMS increases it implies that higher $l(k)$ will be preferable. The interested reader for such a implementation can find further details on [3]

7.6 a) The discrete-time Fourier transform $H(\omega)$ of the triangular shaped $h(n)$ is given by

$$H(\omega) = \frac{1}{L}\left(\frac{\sin(\omega L/2)}{\sin(\omega/2)}\right)^2.$$

Other interpolator of higher order will involve more samples in the interpolation process and then $h(n)$ will extend more than the $2L$ samples, L being the interpolation factor. For example, a cubic interpolation will involve $2\cdot 3L$ samples.

b) The transfer function $H(\omega)$ is a lowpass filter which a cut-off frequency ω_c at -3 dB can be estimated in a way analogous to Problem 4.8. Since the cut-off frequency ω_c will be small compared with π we can approximate by

$$\begin{aligned}
H(\omega_c) = 0.7079L =& \frac{1}{L}\left(\frac{\sin(\omega_c L/2)}{\sin(\omega_c/2)}\right)^2 \\
\approx& \frac{1}{L}\frac{\left(\omega_c L/2 - \frac{(\omega_c L/2)^3}{3!}\right)^2}{(\omega_c/2)^2} \\
\approx& \frac{1}{L}\frac{(\omega_c L/2)^2 - 2\frac{(\omega_c L/2)^4}{3!}}{(\omega_c/2)^2} \\
=& L\left(1 - \frac{L^2\omega_c^2}{12}\right)
\end{aligned}$$

obtaining

$$\omega_c = \frac{1.8722}{L}$$

c) To evaluate the equivalent analog frequency F_c we obtain

$$F_c = \frac{\omega_c}{2\pi}\frac{L}{T} = \frac{0.30}{T}$$

which for a heart rate of 60 bpm (T=1s) implies $F_c = 0.3Hz$. A lower heart rate at 50 bpm results in $F_c = 0.25Hz$ making the low-pass effect more pronounced and therefore distorting the heart rate variability information over this cut-off frequency.

The sin approximation introduced to arrive at this result is rather crude, especially for low values of L. Solving the cut-off frequency with numerical techniques we obtain $F_c = 0.36/T\ Hz$ for small L up to $F_c = 0.32/THz$ for large L values. Also when interpolating a signal that is sampled at the beat locations the interpolation interval, T, differs from beat to beat, thus becoming a time-varying linear filter.

The lowpass filtering effect is illustrated by the figure below:

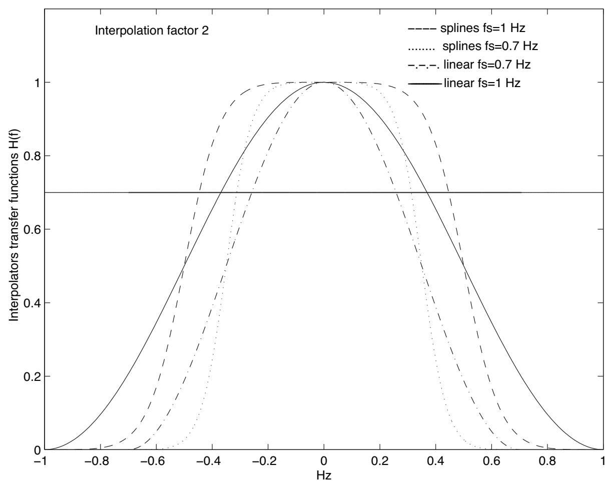


Figure 7.12: Transfer function of the interpolating filters for linear and cubic spline interpolation. The functions are calculated for an interpolation factor of $L = 2$, assuming that $F_s = 1/T = 1$ and 0.7 Hz. Note that the lowpass filtering is more drastic when the period between samples increases (F_s decreases). Also, it is evident that cubic spline interpolation exhibits a better behavior than does linear interpolation.

7.7 a) The Taylor series of $y(t)$ around t_0 is

$$y(t) = \sum_{l=0}^{\infty} \frac{(t-t_0)^l}{l!} y^{(l)}(t_0)$$
$$= y(t_0) + (t-t_0)y^{(1)}(t_0) + \frac{(t-t_0)^2}{2} y^{(2)}(t_0) + \dots .$$

Now, assume that there is a function $x(t) = y^{(k)}(t)$. Then the Taylor series of $x(t)$ around t_0 is

$$x(t) = y^{(k)}(t) = \sum_{l=0}^{\infty} \frac{(t-t_0)^l}{l!} y^{(l+k)}(t_0)$$
$$= y^{(k)}(t_0) + (t-t_0)y^{(k+1)}(t_0) + \frac{(t-t_0)^2}{2} y^{(k+2)}(t_0) + \dots .$$

Hence, the matrix $\mathbf{A}$ has the following structure,

$$\mathbf{A} = \begin{bmatrix} 1 & (t-t_0) & (t-t_0)^2/2 & (t-t_0)^3/6 & \dots \\ 0 & 1 & (t-t_0) & (t-t_0)^2/2 & \dots \\ 0 & 0 & 1 & (t-t_0) & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

b) Inserting the origin at $t_0 = 0$ and $t = T = 1$ (T the sampling period) into $\mathbf{A}$ given in (a), or alternatively considering $(t-t_0) = T = 1$, yields

$$\mathbf{A}\Big|_{\substack{t=T=1,\, t_0=0 \\ or \\ (t-t_0)=T=1}} = \begin{bmatrix} 1 & 1 & 1/2 & 1/6 & \dots \\ 0 & 1 & 1 & 1/2 & \dots \\ 0 & 0 & 1 & 1 & \dots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Thus, from this matrix it can be recursively compute the estimate of the baseline to be subtracted as:

$$\begin{bmatrix} y(n) \\ y^{(1)}(n) \\ y^{(2)}(n) \\ y^{(3)}(n) \\ y^{(4)}(n) \end{bmatrix} = \mathbf{A} \begin{bmatrix} y(n-1) \\ y^{(1)}(n-1) \\ y^{(2)}(n-1) \\ y^{(3)}(n-1) \\ y^{(4)}(n-1) \end{bmatrix}$$

which implies to start with $n_o T = t_0$ and assume the sampling interval T=1, so the elements of $\mathbf{A}$ can be estimated as mentioned. In practice, this implies a time units change that does not affect the result. It should be paid attention when initiating the recursion with the values of $y^{(k)}(n_0)$, where Δt_{i_j} need to be included in the renormalized units.

7.8 To find the transfer function of the IIR filter which appears when the non-linear 50/60 Hz removal filter in (7.45) is replaced by the linear one in (7.44) we proceed in the following way: The equation which subtracts the estimated powerline interference $\hat{v}(n)$ from the observed ECG signal $x(n)$ is

$$y(n) = x(n) - \hat{v}(n),$$

where the sinusoidal interference $\hat{v}(n)$ is estimated as a linear update from the error $e'(n)$

$$\hat{v}(n) = v(n) + \alpha e'(n).$$

Expressing the filter as function of the $\hat{v}(n)$ recursion and the $e'(n)$ definition in (6.38) we have that

$$\begin{aligned} y(n) &= x(n) - v(n) - \alpha e'(n) \\ &= x(n) - v(n) - \alpha(x(n) - x(n-1) - v(n) + v(n-1)) \\ &= (1-\alpha)(x(n) - v(n)) + \alpha y(n-1). \end{aligned}$$

Note that $(x(n-1) - v(n-1))$ can be replaced by $y(n-1)$ since $v(n)$ is replaced by $\hat{v}(n)$ after each iteration and then $v(n-1)$ is $\hat{v}(n-1)$ at iteration $(n-1)$ and so it can be replaced. This step cannot be done with $(x(n)-v(n))$ because the replacement will take place after $y(n)$ has been obtained and before starting calculations of the sample at time $(n-1)$. So we should replace $v(n)$ by the recursive estimation equation used to obtain it in (7.37),

$$\begin{aligned} y(n) &= (1-\alpha)(x(n) - v(n)) + \alpha y(n-1) \\ &= (1-\alpha)(x(n) - 2\cos\omega_0 v(n-1) + v(n-2)) + \alpha y(n-1). \end{aligned}$$

One way to eliminate $v(n-1)$ and $v(n-2)$ is by adding and subtracting the necessary $x(n-1)$ and $x(n-2)$ which can be replaced by $y(n-1)$ and $y(n-2)$ since, again, $v(n-1)$ and $v(n-2)$ equal the estimated $\hat{v}(n-1)$ and $\hat{v}(n-2)$ at time instants $(n-1)$ and $(n-2)$, respectively. In doing so, we obtain

$$\begin{aligned} y(n) =& (1-\alpha)\left[x(n) - 2\cos\omega_0(x(n-1) - y(n-1)) + x(n-2) - y(n-2)\right] \\ &+ \alpha y(n-1). \end{aligned}$$

The corresponding transfer function is given by

$$H(z) = \frac{(1-\alpha)\left(1 - 2\cos\omega_0 z^{-1} + z^{-2}\right)}{1 - (\alpha + 2(1-\alpha)\cos\omega_0)z^{-1} + (1-\alpha)z^{-2}}.$$

Note that the poles are not located at ω_0 but rather at ω_0'

$$\cos(\omega_0') = \frac{\alpha + 2(1-\alpha)\cos(\omega_0)}{2\sqrt{1-\alpha}}.$$

This result can be obtained by identification from (6.34). Note that if $\alpha << 1$, as is usually the case, $\omega_0' \approx \omega_0$.

7.9 The 3 dB and 10 dB notch bandwidths of the filters are found by solving the following equation,

$$|H(e^{\jmath\omega})|^2 = 10^{-x/10} \cdot \max_{\omega} |H(e^{\jmath\omega})|^2,$$

where x is the attenuation in dB, and $\max_\omega |H(e^{\jmath\omega})|^2$ is the maximum gain of the filter. Both filters have maximum gain at $\omega = 0$ for $\omega_0 > \pi/2$, and at $\omega = \pi$ for $\omega_0 < \pi/2$. For $\omega_0 = \pi/2$, $|H(e^{\jmath 0})|^2 = |H(e^{\jmath\pi})|^2$.

The second order FIR filter:

$$\begin{aligned}
H(z) &= 1 - 2\cos(\omega_0)z^{-1} + z^{-2} \\
|H(e^{\jmath\omega})|^2 &= H(z)H(z^{-1})_{|z=e^{\jmath\omega}} = 4(\cos^2(\omega) - 2\cos(\omega)\cos(\omega_0) + \cos^2(\omega_0)) \\
&= 4\left(\cos(\omega) - \cos(\omega_0)\right)^2 \\
|H(e^{\jmath\omega})|^2_{|\omega=0} &= 4(1-\cos(\omega_0))^2
\end{aligned}$$

Thus,

$$\begin{aligned}
\cos(\omega) - \cos(\omega_0) &= \pm 10^{-x/20}(1 - \cos(\omega_0)) \\
\omega_{1,2} &= \arccos(\cos(\omega_0) \pm 10^{-x/20}(1-\cos(\omega_0))) \\
\Delta\omega_{x \text{ dB}} &= \omega_2 - \omega_1 \quad (\omega_2 > \omega_1)
\end{aligned}$$

For $\omega_0 = \pi/2$, $\Delta\omega_{3 \text{ dB}} \approx 0.5008\pi$ and $\Delta\omega_{10 \text{ dB}} \approx 0.2048\pi$.

The second order IIR filter (notch):

$$\begin{aligned}
H(z) &= \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 - 2r\cos(\omega_0)z^{-1} + r^2 z^{-2}} \\
|H(e^{\jmath\omega})|^2 &= H(z)H(z^{-1})_{|z=e^{\jmath\omega}} \\
&= \frac{4(\cos^2(\omega) - 2\cos(\omega)\cos(\omega_0) + \cos^2(\omega_0))}{(1-r^2)^2 + 4r(r\cos^2(\omega) - (1+r^2)\cos(\omega)\cos(\omega_0) + r\cos^2(\omega_0))}
\end{aligned}$$

This expression will obviously be difficult to use. Inserting $\omega_0 = \pi/2$ yields

$$\begin{aligned}
|H(e^{\jmath\omega})|^2_{|\omega_0=\pi/2} &= \frac{4\cos^2(\omega)}{(1-r^2)^2 + 4r^2\cos^2(\omega)} \\
|H(e^{\jmath\omega})|^2_{|\omega=0,\ \omega_0=\pi/2} &= \frac{4}{(1-r^2)^2 + 4r^2} = \frac{4}{(1+r^2)^2}
\end{aligned}$$

Thus,

$$
\begin{aligned}
\frac{\cos^2(\omega)}{(1-r^2)^2+4r^2\cos^2(\omega)} &= 10^{-x/10}\frac{1}{(1+r^2)^2} \\
\cos^2(\omega) &= \frac{(1-r^2)^2}{10^{x/10}(1+r^2)^2-4r^2} \\
\cos(\omega) &= \pm\sqrt{\frac{(1-r^2)^2}{10^{x/10}(1+r^2)^2-4r^2}} \\
\omega_{1,2} &= \arccos\left(\pm\sqrt{\frac{(1-r^2)^2}{10^{x/10}(1+r^2)^2-4r^2}}\right) \\
\Delta\omega_{x\ \mathrm{dB}} &= \omega_2-\omega_1 \quad (\omega_2>\omega_1)
\end{aligned}
$$

For $\omega_0 = \pi/2$ and $r = 0.95$, $\Delta\omega_{3\ \mathrm{dB}} \approx 0.0327\pi$ and $\Delta\omega_{10\ \mathrm{dB}} \approx 0.0109\pi$. Note that the value for $3dB$ agrees with the one that can be estimated from the approximation in [4, p. 342] $\Delta\omega_{3\ \mathrm{dB}} \approx 2(1-r)$.

When $\omega_0 \neq \pi/2$, the magnitude at $\omega = 0$ will differ from the one at $\omega = \pi$. If this difference is larger than, e.g., 3 dB, $\Delta\omega_{3\ \mathrm{dB}}$ is no longer well-defined, and indicates that a very large part of the spectrum is distorted by the filter. While this effect is observed for the IIR filter only when ω_0 is close to 0 or π, relatively small deviations from $\omega_0 = \pi/2$ will have a dramatic impact on the FIR filter. This phenomenon is best understood by studying a pole-zero plot, recalling how poles and zeros interact in forming the spectrum.

7.10 First, using the linear combiner depicted in Figure 7.15, the estimated interference is obtained by adjusting the weights w_1 and w_2 for $\cos(\omega_0 n)$ and $\sin(\omega_0 n)$, respectively, using the LMS algorithm. Matching of the amplitude and phase of the disturbance $A\cos(\omega_0 n - \phi)$ can be achieved since

$$w_1\cos(\omega_0 n) + w_2\sin(\omega_0 n) = \sqrt{w_1^2+w_2^2}\cos\left(\omega_0 n - \arctan\frac{w_2}{w_1}\right).$$

Now, assume that instead of $\cos(\omega_0 n)$ and $\sin(\omega_0 n)$, the reference samples $\cos(\omega_0 n)$ and $\cos(\omega_0(n-1))$ are available, corresponding to the two most recent samples of a signal oscillating with angular frequency ω_0. Then by using the trigonometric relation

$$\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

we obtain that

$$
\begin{aligned}
w_1\cos(\omega_0 n) + w_2\cos(\omega_0(n-1)) &= (w_1 + w_2\cos(\omega_0))\cos(\omega_0 n) \\
&\quad + w_2\sin(\omega_0)\sin(\omega_0 n) \\
&= A'\cos(\omega_0 n - \phi')
\end{aligned}
$$

where

$$A' = \sqrt{w_1^2 + 2w_1 w_2 \cos(\omega_0) + w_2^2}$$
$$\phi' = \arctan \frac{w_2 \sin(\omega_0)}{w_1 + w_2 \cos(\omega_0)}.$$

Since A' and ϕ' can be adjusted to any amplitude and phase by tuning w_1 and w_2, it is clear that the linear combiner may be implemented as a first-order FIR filter. When $\omega_0 = \pi/2$, the two approaches are equivalent.

7.11 The structure of $\mathbf{\Phi}$ in $\mathbf{H} = \mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T$ is

$$\mathbf{\Phi} = [\boldsymbol{\varphi}_1 \quad \boldsymbol{\varphi}_2] = \sqrt{\frac{2}{N}} \begin{bmatrix} 1 & 0 \\ \cos(\omega_0) & \sin(\omega_0) \\ \vdots & \vdots \\ \cos(\omega_0(N-1)) & \sin(\omega_0(N-1)) \end{bmatrix}.$$

The matrix product $\mathbf{\Phi}\mathbf{\Phi}^T = \boldsymbol{\varphi}_1\boldsymbol{\varphi}_1^T + \boldsymbol{\varphi}_2\boldsymbol{\varphi}_2^T$ is of dimension $N \times N$ elements. The element (i, j) is

$$\begin{aligned}[\mathbf{\Phi}\mathbf{\Phi}^T]_{(i,j)} &= \frac{2}{N}\cos(\omega_0(i-1))\cos(\omega_0(j-1)) + \frac{2}{N}\sin(\omega_0(i-1))\sin(\omega_0(j-1)) \\ &= \frac{2}{N}\cos(\omega_0(i-j)), \quad i = 1,\ 2, \ldots, N, \quad j = 1,\ 2, \ldots, N.\end{aligned}$$

Hence, it is evident that

$$\begin{aligned}[\mathbf{H}]_{(i,j)} &= \left[\mathbf{I} - \mathbf{\Phi}\mathbf{\Phi}^T\right]_{ij} \\ &= \begin{cases} -\frac{2}{N}\cos(\omega_0(i-j)), & i \neq j \\ 1 - \frac{2}{N}, & i = j \end{cases}.\end{aligned}$$

7.12 This filter is such that the impulse responses $\mathbf{H}_i^T$ (rows in $\mathbf{H}$)

$$\mathbf{H}_i = -\frac{2}{N} \begin{bmatrix} \cos((i-1)\omega_0) \\ \cos((i-2)\omega_0) \\ \vdots \\ 1 - \frac{N}{2} \\ \vdots \\ \cos((N-i)\omega_0) \end{bmatrix} \quad i = 1,\ 2, \ldots, N. \tag{7.179}$$

are circularly shifted versions of each other. Since every impulse response acts on the input vector $\mathbf{x}$

$$\mathbf{y} = \mathbf{H}\mathbf{x} \tag{7.180}$$

the output signal $y(i)$ is obtained as the convolution of the impulse response with the periodic extension of the vector $\mathbf{x}$

$$y(i) = \mathbf{H}_i^T \mathbf{x}, \quad i = 1, 2, \ldots, N. \tag{7.181}$$

This filter is a noncausal, a property which does not impose any problems since the processing is performed off-line. It should be noted that $x(i)$ is always multiplied by $(1 - 2/N)$ in estimating $y(i)$, as is done with $\mathbf{x}(i + 1)$ which always multiplied by $(-2)\cos\omega_0/N$, and so on. This can alternatively be expressed by generating the periodic extension $x_c(n)$, that is,

$$x_c(n) = x(n - \left\lfloor \frac{n}{N} \right\rfloor N) \tag{7.182}$$

and then

$$y(i) = \sum_{n=1}^{N} H_{(N+1)/2}(n) x_c(i - (N-1)/2 + n), \quad i = 1, 2, \ldots, N. \tag{7.183}$$

where $H_{(N+1)/2}(n)$ is a symmetric impulse response and, hence, with linear phase.

7.13 The transfer function is

$$H(z) = \frac{B(z)}{A(z)} = \frac{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}{1 - 2(1 - \mu C^2)\cos(\omega_0)z^{-1} + (1 - 2\mu C^2)z^{-2}}.$$

The zeros of the numerator are

$$\begin{aligned} B(z) &= 1 - 2\cos(\omega_0)z^{-1} + z^{-2} = 1 - (e^{\jmath\omega_0} + e^{-\jmath\omega_0})z^{-1} + z^{-2} \\ &= (1 - e^{\jmath\omega_0}z^{-1})(1 - e^{-\jmath\omega_0}z^{-1}) \Rightarrow e^{\pm\jmath\omega_0}. \end{aligned}$$

which are located on the unit circle at the angles $\pm\omega_0$. When $\mu C << 1$, the poles of the denominator are approximately given by

$$\begin{aligned} A(z) &= 1 - 2(1 - \mu C^2)\cos(\omega_0)z^{-1} + (1 - 2\mu C^2)z^{-2} \\ &\approx 1 - (1 - \mu C^2)(e^{\jmath\omega_0} + e^{-\jmath\omega_0})z^{-1} + (1 - \mu C^2)^2 z^{-2} \\ &= (1 - (1 - \mu C^2)e^{\jmath\omega_0}z^{-1})(1 - (1 - \mu C^2)e^{-\jmath\omega_0}z^{-1}) \\ &\Rightarrow (1 - \mu C^2)e^{\pm\jmath\omega_0}. \end{aligned}$$

Another way to arrive to this value is by identifying the terms in (7.35) such that the poles $p_{1,2} = re^{\pm\jmath\omega_0'}$ are defined by

$$r = \sqrt{(1 - 2\mu C^2)} \tag{7.184}$$

and

$$\cos(\omega_0') = \frac{(1 - \mu C^2)\cos(\omega_0)}{\sqrt{(1 - 2\mu C^2)}} \tag{7.185}$$

Again, when $\mu C^2 << 1$, the previous derivation will result. The pole are located on the inside of the unit circle, with radius $(1-\mu C^2)$, and at the same angles as are the zeros at ω_0. The approximation is that a quadratic term, which is assumed to be zero, is introduced: $1 - 2\mu C^2 \approx 1 - 2\mu C^2 + \mu^2 C^4 = (1 - \mu C^2)^2$. It is valid when $\mu C^2 << 1$, i.e., when the adaptation rate is slow.

7.14 The filter output will be

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k,n) \tag{7.186}$$

that to be express as sum of $x(n)$ plus other term, it seems to be reasonable to expand $x(n+k)$ around n with a Taylor series expansion that, assuming unit sampling period, is:

$$x(n+k) = x(n) + \sum_{m=1}^{\infty} \frac{k^m x^{(m)}(n)}{m!} \tag{7.187}$$

which inserted into the convolution sum with the impulse response gives:

$$y(n) = \left(\frac{\beta(n)}{\pi}\right)^{1/2} \left(x(n) \sum_{k=-\infty}^{\infty} e^{-\beta(n)k^2} \right. \tag{7.188}$$

$$\left. + \sum_{m=1}^{\infty} \frac{x^{(m)}(n)}{m!} \left(\sum_{k=-\infty}^{\infty} (-1)^m k^m e^{-\beta(n)k^2} \right) \right) \tag{7.189}$$

which after introducing the normalized impulse response sum, and the fact that for odd orders of m, the sum in k is of a antisymmetric function and equals zero, can be simplified to

$$y(n) = x(n) + \left(\frac{\beta(n)}{\pi}\right)^{1/2} \sum_{m=1}^{\infty} \frac{x^{(2m)}(n)}{(2m)!} \left(\sum_{k=-\infty}^{\infty} k^{2m} e^{-\beta(n)k^2} \right). \tag{7.190}$$

The sum in k, can be approximated by the the integral of the continuous-time function (sampling period fine enough)

$$\sum_{k=-\infty}^{\infty} k^{2m} e^{-\beta(n)k^2} = \int_{-\infty}^{\infty} t^{2m} e^{-\beta(n)t^2}\, dt \tag{7.191}$$

$$= \frac{1 \cdot 3 \cdot 5 \cdots (2m-1)}{2^m} \left(\frac{\pi}{\beta(n)^{2m+1}}\right)^{1/2} \tag{7.192}$$

which results in

$$y(n) = x(n) + \sum_{m=1}^{\infty} \frac{x^{(2m)}(n)}{(2\beta(n))^m \prod_{p=1}^{m} 2p}. \tag{7.193}$$

then the difference, ϵ, between observed, $x(n)$, and filtered $y(n)$, signal is

$$\epsilon = \left| \sum_{m=1}^{\infty} \frac{x^{(2m)}(n)}{(2\beta(n))^m \prod_{p=1}^{m} 2p} \right| \tag{7.194}$$

from this expression we have a relation between ϵ and $\beta(n)$ and the observed signal (its derivatives) $x(n)$. If we assume that the signal $x(n)$, around n, is adequately represented by a polynomial up to third order, then

$$\beta(n) = \left| \frac{x^{(2)}(n)}{4\epsilon} \right| \tag{7.195}$$

which allows to, once estimated the noise variance at some area of the ECG like the TP interval, give a value to $\beta(n)$ from a estimate of the second derivative of the signal. Note that for high values of $x^{(2)}(n)$ it is obtained high $\beta(n)$ so narrow $h(k,n)$ which implies higher low-pass cut-off frequency, as expected for high values of the derivatives implying high frequency content. By other hand small ϵ again implies high cut-off frequency and the reverse, meaning that higher filtering is obtained by higher noise contamination as one would find natural.

For higher order polynomial, like up to fifth order we need to solve the relation

$$\epsilon = \left| \frac{x^{(2)}(n)}{4\beta(n)} + \frac{x^{(4)}(n)}{32\beta(n)^2} \right|. \tag{7.196}$$

At areas with very high frequency like at the QRS peaks, still this approximation can fail, and some *ad hoc* rules could help in improving the filter performance [5].

For real filter implementation, also truncation need to be introduced in the impulse response resulting in:

$$y(n) = \sum_{k=-m}^{m} x(n-k) \frac{e^{-\beta(n)k^2}}{C(n)} \tag{7.197}$$

with $C(n) = \sum_{k=-m}^{m} e^{-\beta(n)k^2}$

7.15 The orthonormal basis set Φ taken into consideration should be that ones that made the filter characteristics adapted to the time-varying properties of the signal, so it can be selected from a basis that resemble the beat morphology, as the Hermite polynomials [6], even more adapted to the signal shape from a truncated KL-based set of basis obtained from a training set. The training set can be obtained form a universal purpose data set or from the initial part of each recording, so better fit to the analyzing data will be obtained.

Then the issue of number of basis, $K < N$, used in the truncation need to be taken in consideration to obtain the projection matrix Φ_s, and this can be designed from the behavior illustrated in figure 4.33 where one possibility is to design the number of basis, K, as those that give a power of the residual difference between the observed $\mathbf{x}_i$ ith beat and filtered one $\mathbf{y}_i = \Phi_s \Phi_s^T \mathbf{x}_i$ equal to the *a priori* estimated noise power in the observed signal σ_i^2 at the ith beat (as usually estimated at the at TP interval).

$$\left(\mathbf{x}_i - \Phi_s \Phi_s^T \mathbf{x}_i\right)^T \left(\mathbf{x}_i - \Phi_s \Phi_s^T \mathbf{x}_i\right) < N\sigma_i^2 \tag{7.198}$$

Finally, if continuous filtered signal is required, we need to deal with the segmentation gaps generated by no-uniform RR intervals. Since this gaps usually are at the TP interval they can be filled by a connecting line between recurrences, always keeping in main that no useful info is provided in that area.

7.16 Arrange the recorded noise-contaminated signal and the known waveform into $\mathbf{x}$ and $\mathbf{s}$, respectively:

$$\mathbf{x} = \begin{bmatrix} x(0) & x(1) & \dots & x(N-1) \end{bmatrix}^T,$$
$$\mathbf{s} = \begin{bmatrix} s(0) & s(1) & \dots & s(N-1) \end{bmatrix}^T.$$

Now the probability density functions and the hypothesis testing may be formulated as

$$\begin{aligned}
p(\mathbf{x};\mathcal{H}_0) &= \frac{1}{(2\pi\sigma_v^2)^{N/2}} e^{-\frac{1}{2\sigma_v^2}\mathbf{x}^T\mathbf{x}} \\
p(\mathbf{x};\mathcal{H}_1) &= \frac{1}{(2\pi\sigma_v^2)^{N/2}} e^{-\frac{1}{2\sigma_v^2}(\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s})} \\
L(\mathbf{x}) &= \frac{p(\mathbf{x};\mathcal{H}_1)}{p(\mathbf{x};\mathcal{H}_0)} = e^{-\frac{1}{2\sigma_v^2}\left((\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s})-\mathbf{x}^T\mathbf{x}\right)} \overset{\mathcal{H}_1}{>} \eta \\
\ln L(\mathbf{x}) &= -\frac{1}{2\sigma_v^2}\left((\mathbf{x}-\mathbf{s})^T(\mathbf{x}-\mathbf{s})-\mathbf{x}^T\mathbf{x}\right) \overset{\mathcal{H}_1}{>} \ln\eta \\
\ln L(\mathbf{x}) &= \frac{1}{\sigma_v^2}\left(\mathbf{x}^T\mathbf{s} - \frac{1}{2}\underbrace{\mathbf{s}^T\mathbf{s}}_{\text{Constant}}\right) \overset{\mathcal{H}_1}{>} \ln\eta.
\end{aligned}$$

The constant term is known *a priori*. Hence, the test statistics is

$$\mathbf{x}^T\mathbf{s} \overset{\mathcal{H}_1}{>} \eta',$$

where $\eta' = \sigma_v^2 \ln \eta + \frac{1}{2}\mathbf{s}^T\mathbf{s}$. We recognize this as the **matched filter** approach.

7.17 The filter is described by

$$\begin{aligned}
H(z)_{|z=e^{j\omega}} &= (1-e^{-j\omega})^K(1+e^{-j\omega})^{L-K} \\
&= e^{-j\frac{\omega}{2}K}\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right)^K e^{-j\frac{\omega}{2}(L-K)}\left(e^{j\frac{\omega}{2}}+e^{-j\frac{\omega}{2}}\right)^{L-K} \\
&= e^{-j\frac{\omega}{2}L}\left(2j\sin\left(\frac{\omega}{2}\right)\right)^K\left(2\cos\left(\frac{\omega}{2}\right)\right)^{L-K} \\
&= 4^L e^{-j\left(\frac{\omega}{2}L-\frac{\pi}{2}K\right)}\sin^K\left(\frac{\omega}{2}\right)\cos^{L-K}\left(\frac{\omega}{2}\right) \\
&= 4^L e^{-j\left(\frac{\omega}{2}L-\frac{\pi}{2}K\right)}\tan^K\left(\frac{\omega}{2}\right)\cos^{L}\left(\frac{\omega}{2}\right)
\end{aligned}$$

Observing that $(1-z^{-1})^K(1+z^{-1})^{L-K}$ places K zeros at $z=1$ (DC component) and $L-K$ zeros at $z=-1$ ($\omega=\pi$), it is clear that a lowpass filter is obtained when $K=0,\ L\neq K$, and a highpass filter when $L=K$. Any other combination of K and L will yield a bandpass filter, with damping at $\omega=0$ and $\omega=\pi$. The degree of damping at each end of the spectrum, and hence the passband, is determined by the specific combination of K and L.

7.18 The squared deviation signal $\sigma^2(n)$ is

$$\begin{aligned}
\sigma^2(n) &= E\left[(x_i(n)-E[x_i(n)])^2\right] \\
&= E\left[(x_i(n)-s(n))^2\right] \\
&= E\left[v_i^2(n)\right]
\end{aligned} \tag{7.199}$$

So it can be estimated by

$$\hat{\sigma}^2(n) = \frac{\sum_{i=1}^{M} v_i^2(n)}{M} \tag{7.200}$$

When misalignment is introduced in the model, we will temporary move to the continuous-time domain to account for continuous delay τ_i. Within this framework

$$\sigma^2(t) = E\left[(x_i(t)-E[x_i(t)])^2\right] \tag{7.201}$$

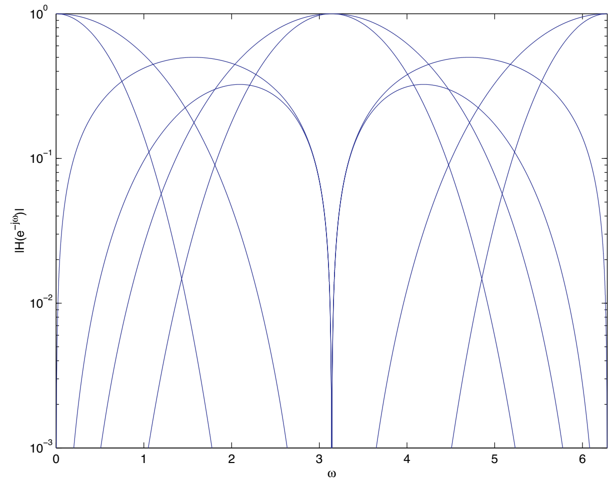


Figure 7.13: Examples of magnitude functions of the binomial-Hermite family of filters for different values of K and L, that is, (K, L)=(0,5), (0,15), (5,5), (10,10), (1,2, and (3,4). Note that the normalized frequency ranges from 0 to 2π.

with

$$E[x_i(t)] = E[s(t - \tau_i) + v_i(t)] \tag{7.202}$$
$$= E\left[s(t) - \tau_i \frac{ds(t)}{dt}\right] \tag{7.203}$$
$$= s(t) \tag{7.204}$$

and then

$$\sigma^2(t) = E\left[(x_i(t) - s(t))^2\right] \tag{7.205}$$
$$= E\left[\left(\tau_i \frac{ds(t)}{dt} + v_i(t)\right)^2\right] \tag{7.206}$$
$$= E[\tau_i^2]\left(\frac{ds(t)}{dt}\right)^2 + E[v_i^2(t)] \tag{7.207}$$

that when sampled and estimated results in

$$\hat{\sigma}^2(n) = \sigma_\tau^2 s'^2(n) + \frac{\sum_{i=1}^{M} v_i^2(n)}{M} \tag{7.208}$$
$$\tag{7.209}$$

This result shows that the misalignment introduce errors but not only dependent of its statistics (σ_τ^2) if not also of the signal properties $s'(n)$, being more relevant as the derivative is bigger, in other words for components of higher frequency.

If now we want to keep the contribution of the misalignment to $\mathcal{P}$ lower than 10% of the $\mathcal{P}_v$ contribution, knowing that power of $v_i(n)$, $\mathcal{P}_v$, is 30 dB lower than that of $\mathcal{P}_s$, $\mathcal{P}_v = 10^{-3}$, we should impose that

$$\sigma_\tau^2 \mathcal{P}_{s'} < \frac{\mathcal{P}_v}{10} \tag{7.210}$$

$$\frac{\mathcal{P}_{s'}}{12F_s^2} < \frac{\mathcal{P}_v}{10} \tag{7.211}$$

$$F_s > \sqrt{\frac{\mathcal{P}_{s'}\mathcal{P}_v}{120}} = 4\text{kHz} \tag{7.212}$$

7.19 The QRS complex always acts as a big step impulse for the filter, and then it spreads ringing after the QRS that will eventually overlap the late potentials being impossible to distinguish this ringing from the real clinically relevant high frequency contents. This is similar to the ringing introduced by the linear phase filter used for 50/60 Hz attenuation.

One way to solve this is with a bidirectional filter which filters the rightmost half of the QRS part in backward direction. This will made the ringing to spread into the first part of the QRS complex rather than into the last part, so avoiding the overlap with the late potentials. The first leftmost part is filtered forward just in case QRS width duration is to be measured so also avoiding ringing into the "early" potential area and being possible to estimate both onset and end positions of the filtered QRS waveform. If the filter is IIR then the effect is even better marked since the phase response is highly non-linear and frequencies lower than those at cut-off frequency are even more delayed than those on the pass band so further separating the high frequency components from the attenuated low frequency from the QRS waveform.

7.20 a) Since the error τ is uniformly distributed in the sampling interval $[-T/2, T/2]$ the PDF function is $P_\tau(\tau) = 1/T$ on that interval and the misalignment error variance, σ_τ^2, is

$$\begin{aligned}\sigma_\tau^2 &= \int_{-T/2}^{T/2} \frac{1}{T}\tau^2 d\tau \\ &= \frac{T^2}{12}\end{aligned} \tag{7.213}$$

which give a standard deviation $\sigma_\tau = 0.144$ ms

b) The cut-off frequency can be compute from the result already obtained in problem 4.23 as $F_c = 0.422/T$ resulting in $F_c = 844\ Hz$

c) the bigger restriction will be the averaging since impose a cut-off frequency of 844 Hz whereas the sampling limits to half the sampling rate, being then 1000 Hz

7.21 Let us assume that the $\max\limits_{n_0>m>n} g(m, -\varepsilon_0) = g(c, -\varepsilon_0)$ occurs at sample c, and the $\min\limits_{n_0>m>n} g(m, \varepsilon_0) = g(n, \varepsilon_0)$ occurs at the sample n just candidate to be a vertex since next one $n+1$ already satisfies 7.21. In this situation the reconstruction error e_m at sample m will be (see Fig. 7.14)

$$e_m = \left| x(m) - x(n_0) - (m - n_0)\frac{x(n) - x(n_0)}{n - n_0} \right| \tag{7.214}$$

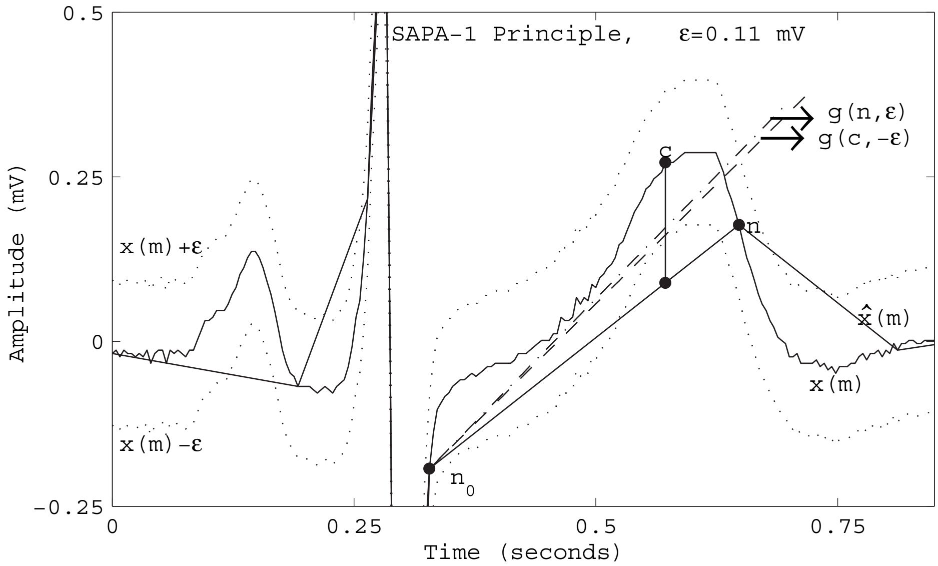


Figure 7.14: Plot for SAPA-1 principle

For any value m it is satisfied

$$|x(m) - x(n_0) - (m - n_0)g(n, \varepsilon)| \leq \varepsilon \tag{7.215}$$

and also

$$\left| x(n_0) + (m - n_0)\frac{x(n) - x(n_0)}{n - n_0} - x(n_0) - (m - n_0)g(n, \varepsilon) \right| \leq \varepsilon \tag{7.216}$$

by combining both equations

$$e_m = \left| x(m) - x(n_0) - (m - n_0) \frac{x(n) - x(n_0)}{n - n_0} \right| \le 2\varepsilon \tag{7.217}$$

7.22 With the model assumptions the running average beat $\hat{s}_i(t) = s(t)$, and the residual $y_i(t)$ signal will be

$$y_i(t) = x_i(t) - \hat{s}_i(t) \tag{7.218}$$
$$y_i(t) = s(t) + v_i(t) - s(t - \tau_i) + v_{i-1}(t) \tag{7.219}$$
$$y_i(t) = r_{ecg_i}(t) + v_i(t) + v_{i-1}(t) \tag{7.220}$$

where τ_i is the misalignment error at beat ith, and $r_{ecg_i}(t) = s(t) - s(t - \tau_i)$. We made the study in continuous-time since the misalignment refer to a continuous value respect to the perfect alignment, the sampling will be done at the end. The power spectrum of $y(t)$, $S_y(\Omega)$, will include two terms

$$S_y(\Omega) = E[S_{r_{ecg_i}}(\Omega)] + 2S_v(\Omega) \tag{7.221}$$

where the second term comes from the consideration that the noise is stationary across beats and uncorrelated from beat-to-beat, and the first term in (7.221) can be express as

$$E\left[S_{r_{ecg_i}}(\Omega)\right] = S_s(\Omega) E\left[\left|1 - e^{-\jmath\Omega\tau_i}\right|^2\right] \tag{7.222}$$
$$= S_s(\Omega) E\left[(2 - 2\cos\Omega\tau_i)\right]. \tag{7.223}$$

Making the expected value for τ_i from $-0.5T_m$ to $0.5T_m$ we obtain

$$E[S_{r_{ecg_i}}(\Omega)] = 2S_s(\Omega)\left(1 - \frac{2\sin(\Omega T_m/2)}{\Omega T_m}\right), \tag{7.224}$$

so the power spectral density of the residual is

$$S_y(\Omega) = 2S_s(\Omega)\left(1 - \frac{2\sin(\Omega T_m/2)}{\Omega T_m}\right) + 2S_v(\Omega), \tag{7.225}$$

which has a multiplicative factor with the ECG power spectrum $S_s(\Omega)$ that increases with Ω and with the misalignment T_m. See that this factor, for $\Omega << 2/T_m$, takes a quadratic dependence

$$2\left(1 - \frac{2\sin(\Omega T_m/2)}{\Omega T_m}\right) \approx \Omega^2 T_m^2/4. \tag{7.226}$$

This implies this problem is more relevant at the QRS area, higher frequency components, than at the P and T wave areas, see Figure 7.39, where the residual energy increment with misalignment at the QRS area is much higher and at the rest, as predicted by this result. In the best alignment situation, $T_m = T$, with T the sampling period, still this problems have a relevant effect, mainly for low sampling rates. When discrete-time signal is considered, sampling rate $F_s = 1/T$, the sampling theorem should be applied to equation (7.225) obtaining

$$S_y(e^{\jmath\omega}) = F_s \sum_{i=-\infty}^{\infty} S_y(\Omega - 2\pi i F_s)\lfloor_{\Omega=\omega F_s} \tag{7.227}$$

and same analysis can be done.

7.23 In the strategy a), the quantized signal, $x_{q_i}(n) = Q_c(x_i(n))$, i the beat number, will be

$$x_{q_i}(n) = s(n) + v_i(n) - e_{q_i}(n). \tag{7.228}$$

The averager output is

$$\hat{s}_{q_i}(n) = Q_c\left(\frac{1}{N}\sum_{j=i-N+1}^{i} x_{q_j}(n)\right), \tag{7.229}$$

where a new roundoff error, $e'_{q_i}(n)$, appears because quantization of the average to be able to posterior subtraction from the $x_{q_i}(n)$ signal. $\hat{s}_{q_i}(n)$ becomes

$$\hat{s}_{q_i}(n) = s(n) + \sum_{j=i-N+1}^{i} \frac{v_j(n) - e_{q_j}(n)}{N} - e'_{q_i}(n) \tag{7.230}$$

and so the residual signal $y_{q_i}(n)$ takes the value

$$y_{q_i}(n) = v_i(n) - e_{q_i}(n) - \sum_{j=i-N}^{i-1} \frac{v_j(n) - e_{q_j}(n)}{N} + e'_{q_i}(n). \tag{7.231}$$

Assuming roundoff error after the averager, $e'_{q_i}(n)$, uncorrelated with the running quantization error, $e_{q_i}(n)$, and neglecting the averaged noise power, results in a power of

$$P_{y_q} = \sigma_v^2 + 2\sigma_{e_q}^2 + \frac{\sigma_v^2 + \sigma_{e_q}^2}{N} \approx \sigma_v^2 + 2\frac{\Delta_c^2}{12}. \tag{7.232}$$

For the second strategy b), the analysis is a bit different, with two different quantizer, "f" for fine and "c" for coarse, take place.

$$x_{q_i}(n) = s(n) + v_i(n) - e^f_{q_i}(n). \tag{7.233}$$

At the output of the averager, and already neglecting the averaged noise component, we will have

$$\hat{s}_{q_i}(n) = s(n) - e^{f\prime}_{q_i}(n), \tag{7.234}$$

and the residual $y_{q_i}(n)$, after the coarse quantizer, is

$$y^c_{q_i}(n) = v_i(n) - e^f_{q_i}(n) + e^{f\prime}_{q_i}(n) - e^c_{q_i}(n), \tag{7.235}$$

which lead to the power expression

$$P_{y_q} = \sigma_v^2 + 2\sigma^2_{e^f_q} + \sigma^2_{e^c_q} = \sigma_v^2 + 2\frac{\Delta_f^2}{12} + \frac{\Delta_c^2}{12}. \tag{7.236}$$

Comparing with strategy a) the residual power is $\left(\frac{\Delta_c^2}{12} - 2\frac{\Delta_f^2}{12}\right)$ times lower in the case b). For fine quantization with 12 bits, dynamic range 4 mV, and coarse quantization with 4 bits, dynamic range 0.5 mV the difference is of 44.7 μV^2

7.24 To minimize E_i we should differentiate it with respect to the coefficients a_k to be estimated, and set the result to zero.

$$\frac{\partial E_i}{\partial a_j} = \sum_{n=-N_1}^{N_2-1} 2 \sum_{k=-p}^{p} a_k x(\hat{\theta}_{i-1} + n + k)x(\hat{\theta}_{i-1} + n + j) \tag{7.237}$$

$$- \sum_{n=-N_1}^{N_2-1} 2x(\hat{\theta}_i + n)x(\hat{\theta}_{i-1} + n + j) \tag{7.238}$$

by denoting

$$r(k,j) = \sum_{n=-N_1}^{N_2-1} x(\hat{\theta}_i + n - k)x(\hat{\theta}_i + n - j) \tag{7.239}$$

and $\alpha_i = \hat{\theta}_i - \hat{\theta}_{i-1}$ the estimated RR interval at beat ith, we have the set of equations

$$\sum_{k=-p}^{p} a_k r(\alpha_i - k, \alpha_i - j) = r(0, \alpha_i - j); \qquad j = -p, \dots, p \tag{7.240}$$

that in matrix form becomes

$$\mathbf{R}_i \mathbf{a} = \mathbf{r}_i \tag{7.241}$$

where

$$\mathbf{R}_i = \begin{bmatrix} r(\alpha_i+p,\alpha_i+p) & \dots & r(\alpha_i+p,\alpha_i-p) \\ r(\alpha_i+p-1,\alpha_i+p) & & r(\alpha_i+p-1,\alpha_i-p) \\ \vdots & \ddots & \vdots \\ r(\alpha_i-p,\alpha_i+p) & & r(\alpha_i-p,\alpha_i-p) \end{bmatrix}$$

and

$$\mathbf{a}^T = [a_{-p}, a_{-p+1}, \dots, a_p] \tag{7.242}$$
$$\mathbf{r}_i^T = [r(0,\alpha_i+p), \dots, r(0,\alpha_i-p)]. \tag{7.243}$$

Then

$$\mathbf{a} = \mathbf{R}_i^{-1}\mathbf{r}_i \tag{7.244}$$

Matrix $\mathbf{R}_i$ is symmetric, positive semi-define and can be inverted using matrices algebra. Note that $\mathbf{R}_i$ is basically the auto-correlation of beat $i-1$ and $\mathbf{r}_i$ is the cross-correlation between observed beat i and previous one $i-1$, and even it has been omitted the estimated parameters $\mathbf{a}$ depend on the beat under consideration. A straight forward extension of the technique will be to use an averaged beat to estimated $\mathbf{R}_i$ and $\mathbf{r}_i$ rather than the previous beat, so noise will be attenuated in the parameter estimation

7.25 The truncated transform coding, in noise free signals where every beat block of N samples can be model by $\mathbf{x} = \mathbf{s}$, gives a reconstruction error, $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} = \hat{\mathbf{s}} - \mathbf{s}$, signal that can be properly quantified by their $\mathcal{P}_{RMS}$ value,

$$\mathcal{P}_{RMS} = \sqrt{\frac{\mathbf{e}^T\mathbf{e}}{N}}. \tag{7.245}$$

However, when noise is present in the signal, $\mathbf{x} = \mathbf{s}+\mathbf{v}$, any coder which tries to minimize the $\mathcal{P}_{RMS}$ value what in fact is doing is to code the noise. This can not always be a desirable situation, and rather we will like to truncate the transform when the signal of interest, $\mathbf{s}$, is fully coded, rather than the observed signal $\mathbf{x}$.

If we assume the conditions at the formulation, with noise uncorrelated with the signal and white, it will be equally spread in the transform domain and then with the transform also select to have high compression performance for low truncation value K ($K << N$), we can use the approximation proposed at the formulation obtaining that $\hat{\mathbf{x}}$, is basically an approximation, $\hat{\mathbf{s}}$, of the desired signal

$$\hat{\mathbf{x}} = \hat{\mathbf{s}} + \hat{\mathbf{v}} \approx \hat{\mathbf{s}}. \tag{7.246}$$

Then the error signal

$$\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x} \approx \hat{\mathbf{s}} - \mathbf{s} - \mathbf{v}, \tag{7.247}$$

and the $\mathcal{P}_{RMS}$ value, considering that $\hat{\mathbf{s}} - \mathbf{s}$ is uncorrelated with the noise $\mathbf{v}$ is

$$\mathcal{P}_{RMS} \approx \sqrt{\frac{(\hat{\mathbf{s}} - \mathbf{s})^T(\hat{\mathbf{s}} - \mathbf{s})}{N}} + \sigma_v. \tag{7.248}$$

So a much better performance criteria of the fit between the reconstructed and the desired signals could be

$$\mathcal{P}'_{RMS} = \sqrt{\frac{\mathbf{e}^T\mathbf{e}}{N}} - \hat{\sigma}_v. \tag{7.249}$$

where the estimate of the noise power $\hat{\sigma}_v^2$ can be obtained at the TP interval, or by some alternative technique like average beat subtraction and computing the power of the residual, etc.

7.26 The relevant error to be computed can now be refereed to the inter-observer variability in such a way that errors of the same order of magnitude that the inter-observer variability should be given equal relevance in the index. For that a new normalized error can be introduced as

$$\Delta\beta'_k = \frac{|\beta_k - \tilde{\beta}_k|}{V_r} \tag{7.250}$$

where values around one will imply acceptable performance and as the values increases from one, the performance deteriorates progressively. Also a non-quadratic norm can be introduced to better account for linear interpretation between the performance index and the real distance from the inter-observer variability values.

$$\mathcal{P}'_{WDD} = \frac{\sum_{k=1}^{P} \alpha_k \Delta\beta'_k}{\sum_{k=1}^{P} \alpha_k} \tag{7.251}$$

This index again should be read taken one as the the error that will be made between two different expert measuring manually, according to the data provide by the V_k values.

Chapter 8

8.1 In order to study this influence we will start by modeling the misestimation by the error

$$t_k = \theta_k - \xi_k$$

where ξ_k can be consider as a zero-mean white noise sequence. If we assume that ξ_k is solely given by the ECG sampling frequency $1/T$ effect, ξ_k will be a white random variable which is uniformly distributed in $[-T/2, T/2]$ whose variance σ_ξ^2 is given by

$$\sigma_\xi^2 = E[\xi_k^2] = \frac{T^2}{12}.$$

Now we will study the effect on the different HRV representations

- *Interval Tachogram*

  The estimation $\hat{d}_{IT}(k)$ from the observed QRS detector mark will be

$$\begin{aligned}\hat{d}_{IT}(k) &= \theta_k - \theta_{k-1} \\ &= t_k - t_{k-1} + \xi_k - \xi_{k-1} \\ &= d_{\mathrm{IT}}(k) + \varepsilon(k)\end{aligned}$$

  where the error $\varepsilon(k) = \xi_k - \xi_{k-1}$ is independent of the signal $d_{\mathrm{IT}}(k)$ and has a power spectral density that can be calculated from their autocorrelation function and the fact that

$$\begin{aligned}r_\varepsilon(k) &= E[\varepsilon(n)\varepsilon(n-k)] \\ &= E[(\xi_n - \xi_{n-1})(\xi_{n-k} - \xi_{n-1-k})] \\ &= 2r_\xi(k) - r_\xi(k+1) - r_\xi(k-1) \\ &= \frac{T^2}{6}\delta(k) - \frac{T^2}{12}(\delta(k+1) + \delta(k-1))\end{aligned}$$

  then the power spectral density of the error ε is

$$S_\varepsilon(e^{\jmath\omega}) = \frac{T^2}{6}(1 - \cos(\omega))$$

  where ω is the normalized frequency in cycles-by-interval. The power spectral density under consideration $S_{d_{\mathrm{IT}}}(e^{\jmath\omega})$ is then

$$\begin{aligned}\hat{S}_{d_{\mathrm{IT}}}(e^{\jmath\omega}) &= S_{d_{\mathrm{IT}}}(e^{\jmath\omega}) + S_\varepsilon(e^{\jmath\omega}) \\ &= S_{d_{\mathrm{IT}}}(e^{\jmath\omega}) + \frac{T^2}{6}(1 - \cos(\omega))\end{aligned}$$

Note that this term is a new bias term in the already biased $S_{d_{\mathrm{IT}}}(e^{\jmath\omega})$ when used to estimate the modulating signal spectrum $S_m(e^{\jmath\omega})$ according to the IPFM model. The bias is proportional to the sampling period T.

- *Inverse Interval Tachogram*

  Now, the estimation, $\hat{d}_{IIT}(k)$, from the observed QRS detector mark will be

$$\begin{aligned}\hat{d}_{IIT}(k) &= \frac{1}{\theta_k - \theta_{k-1}} \\ &= \frac{1}{t_k - t_{k-1} + \varepsilon(k)}.\end{aligned}$$

  Considering that usually $\varepsilon(k) << (t_k - t_{k-1})$ because the sampling interval T is much much smaller that the interbeat RR interval, we can approximate

$$\begin{aligned}\hat{d}_{IIT}(k) &\approx \frac{1}{t_k - t_{k-1}}\left(1 - \frac{\varepsilon(k)}{t_k - t_{k-1}}\right) \\ &= d_{\mathrm{IIT}}(k) - \frac{\varepsilon(k)}{(t_k - t_{k-1})^2}\end{aligned}$$

  and by noting that $(t_k - t_{k-1}) = T_0 + \Delta T_{0_k}$ with T_0 being the mean heart period with usually $\Delta T_{0_k} << T_0$ we can write

$$\hat{d}_{IIT}(k) \approx d_{\mathrm{IIT}}(k) - \frac{\varepsilon(k)}{T_0^2}$$

  and then

$$\begin{aligned}\hat{S}_{d_{\mathrm{IIT}}}(e^{\jmath\omega}) &= S_{d_{\mathrm{IIT}}}(e^{\jmath\omega}) + \frac{S_\varepsilon(e^{\jmath\omega})}{T_0^4} \\ &= S_{d_{\mathrm{IIT}}}(e^{\jmath\omega}) + \frac{T^2}{6T_0^4}\left(1 - \cos(\omega)\right)\end{aligned}$$

  From this result it could erroneously be concluded that $d_{\mathrm{IIF}}(k)$ is less sensitive to the sampling rate that is the $d_{\mathrm{IF}}(k)$ because the factor T_0 in the denominator. However this is not true because to be able to compare the HRV from this two different measures we need to generate a dimensionless signal [7] as $T_0 \cdot d_{\mathrm{IIF}}(k)$ and $d_{\mathrm{IF}}(k)/T_0$. When comparing this two signal the sampling effect becomes the same.

- *Event series*

Now, the estimation, $\hat{d}_E(t)$, from the observed QRS detector mark will be

$$\hat{d}^u_E(t) = \sum_{k=-\infty}^{\infty} \delta(t - \theta_k)$$

and its Fourier transform can be expressed as

$$\begin{aligned}\hat{D}^u_E(\Omega) &= \sum_{k=-\infty}^{\infty} e^{-\jmath\Omega\theta_k} \\ &= \sum_{k=-\infty}^{\infty} e^{-\jmath\Omega t_k} e^{\jmath\Omega\xi_k}\end{aligned}$$

Since $\Omega < 2\pi 0.5$ in HRV signals, and $\xi_k < T/2$ with T typically having values $T < 0.004$ (sampling rates bigger than 250 Hz) we have that $e^{\Omega\xi_k} < 2\pi 0.001 << 1$ and then it can be approximated

$$\begin{aligned}\hat{D}^u_E(\Omega) &\approx \sum_{k=-\infty}^{\infty} e^{-\jmath\Omega t_k}(1 + \jmath\Omega\xi_k) \\ &= D^u_E(\Omega) + \jmath\Omega \sum_{k=-\infty}^{\infty} \xi_k e^{-\jmath\Omega t_k}\end{aligned}$$

When estimating the power spectrum by truncating to $k = 0, ..., N$, squaring, taking the expectation respect to ξ_k, and making the limit $N \to \infty$ we have

$$\hat{S}^u_{d_E}(\Omega) \approx S^u_{d_E}(\Omega) + \frac{\Omega^2\sigma^2_\xi}{T_0}$$

where it has been taken into account that ξ_k is zero-mean and white.

- *Heart timing*

  Now, the estimation, $\hat{d}_{HT}(t)$, from the observed QRS detector mark will be

$$\begin{aligned}\hat{d}^u_{HT}(t) &= \sum_{k=-\infty}^{\infty} (kT_0 - \theta_k)\delta(t - \theta_k) \\ &= \sum_{k=-\infty}^{\infty} (kT_0 - t_k + \xi_k)\delta(t - \theta_k)\end{aligned}$$

assuming that $d_{HT}(t_k) = d_{HT}(\theta_k)$,which is reasonable since d_{HT} is band-limited to 0.5 Hz and t_k will only differ from θ_k in several ms compared with 500 ms that is the fast period of change in $d_{HT}(t)$.

$$\hat{d}^u_{HT}(t) = d_{HT}(t) \sum_{k=-\infty}^{\infty} \delta(t-\theta_k) + \sum_{k=-\infty}^{\infty} \xi_k \delta(t-\theta_k)$$

We have then two term. The first one is the convolution of the $d_{HT}(t)$ with the previously solved term for the event series. The second one is a white noise term represented by ξ_k. So we can write.

$$\hat{D}^u_{HT}(\Omega) = D^u_{HT}(\Omega) + D_{HT}(\Omega) * \jmath\Omega \sum_{k=-\infty}^{\infty} \xi_k e^{-\jmath\Omega t_k} + \sum_{k=-\infty}^{\infty} \xi_k e^{-\jmath\Omega\theta_k}$$

where the first term is the value to estimate, the second is the effect of the error in the time locations, and the third is the effect of the error in the signal amplitude estimation.

The second term can be neglected respect to the third because we have a noise spectrum (same term than in the third) multiplied by $\Omega < 0.5$, then resulting in a noise with lower power. This is then convolved with a spectrum $D_{HT}(\Omega)$ whose area is usually much lower than one since $m(t) < 1$, zero-mean and band-limited, and $d_{HT}(t)$ is the integral of $m(t)$. The convolution results is then even lower so the dominant term will be the third. This implies that the effect on time location is much reduced that the effect in amplitude modification. Then estimating $\hat{D}_m(\Omega) = \jmath\Omega\hat{D}_{HT}(\Omega)$

$$\hat{D}^u_M(\Omega) = D^u_M(\Omega) + \jmath\Omega \sum_{k=-\infty}^{\infty} \xi_k e^{-\jmath\Omega\theta_k}$$

and then

$$\hat{S}^u_M(\Omega) = S^u_M(\Omega) + \frac{\Omega^2\sigma_\xi^2}{T_0}$$

- *Interval function*

Now, the estimation, $\hat{d}_{IF}(t)$, from the observed QRS detector mark, and following parallel procedure as in $d_{\text{IT}}(k)$ and $d_{HT}(\theta_k)$ it will be

$$\hat{d}^u_{IF}(t) = d_{\text{IF}}(t) \sum_{k=-\infty}^{\infty} \delta(t-\theta_k) + \sum_{k=-\infty}^{\infty} \varepsilon_k \delta(t-\theta_k)$$

and then

$$\hat{D}^u_{IF}(\Omega) \approx D^u_{IF}(\Omega) + \sum_{k=-\infty}^{\infty} \varepsilon_k e^{-\jmath\Omega\theta_k}$$

The difference now is that ε_k is not longer white so

$$\hat{S}^u_{IF}(\Omega) = S^u_{IF}(\Omega) + \frac{2\sigma_\xi^2}{T_0} - \lim_{N\to\infty} \sum_{k=-N}^{N} \frac{\sigma_\xi^2}{2NT_0}\left(e^{-\jmath\Omega(\theta_k-\theta_{k+1})} + e^{-\jmath\Omega(\theta_k-\theta_{k-1})}\right)$$

since we are calculating the term due to the amplitude misestimation, we can very well estimate it for the case of uniform sampling in the $\theta_k = kT_0$ so

$$\begin{aligned}\hat{S}^u_{IF}(\Omega) &= S^u_{IF}(\Omega) + \frac{2\sigma_\xi^2}{T_0} - 2\frac{\sigma_\xi^2}{T_0}\cos(\Omega T_0)\\ &= S^u_{IF}(\Omega) + \frac{2\sigma_\xi^2}{T_0}(1-\cos(\Omega T_0)) \qquad (8.252)\end{aligned}$$

that is equivalent to the case of interval tachogram

- *Inverse Interval function*

  Just proceeding as with Interval function and the Interval functionInverse interval tachogram we will get

$$\hat{S}^u_{IIF}(\Omega) = S^u_{IIF}(\Omega) + \frac{2\sigma_\xi^2}{T_0^5}(1-\cos(\Omega T_0)) \qquad (8.253)$$

Finally note that to compare d_{HT} with d_{IF} and d_{IIF} the former should be divided by T_0 and the later multiplied by the same quantity, so the results in the three cases are equal for low Ω and for the case of d_{HT} this effect is more pronounced with increased frequency.

8.2 a) The problem with the I_T index comes from the definition, since given a recording the number M is fix, but the maximum P_r^m will depend on the precision at which the RR histogram is computed. The maximum precision will be the sampling interval, that can change from experiment to experiment, but also computation of the histogram can be made at lower RR resolution that the sampling interval. e.g in Fig. 8.2 the resolution that was used is 10 ms. With this situation the index can only be used and compared if given together with the resolution that is not a nice property.

b) The second index does not suffers from this problem since the result is directly express in ms. One way to estimate the r_o and r_e will be to made a LS fit of a triangle, $q(r)$, to the estimated histogram $\hat{P}_r(r)$ forcing the triangle to have the peak at the maximum position (r_m, P_r^m). See figure in formulation. This can be obtained by minimization of the error ε

$$\varepsilon = \int_0^\infty (\hat{P}_r(r) - q(r))^2 dr \tag{8.254}$$

with respect to r_o and r_e being included in the triangle approximation $q(r)$ as.

$$q(r) = \begin{cases} 0 & \text{for } r \leq r_o \\ \frac{P_r^m}{r_m - r_o} r - \frac{P_r^m r_o}{r_m - r_o} & \text{for } r_o \leq r \leq r_m \\ -\frac{P_r^m}{r_e - r_m} r + \frac{P_r^m r_e}{r_e - r_m} & \text{for } r_m \leq r \leq r_e \\ 0 & \text{for } r \geq r_e \end{cases} \tag{8.255}$$

The minimization can be done separately in the upwards slope of the triangle to obtain r_o and in the downwards slope to get the r_e value.

8.3

a) The first thing that we need to obtain is the PDF $p_{IIT}(x)$ to be able to estimate from it the variance. For that we first estimate the probability distribution function

$$P_{IT}(x) = \text{Probability}(d_{\text{IT}}(k) \leq x). \tag{8.256}$$

since $d_{\text{IIT}}(k) = 1/d_{\text{IT}}(k)$ we can easily relate one to the other by

$$\begin{aligned} P_{IIT}(x) &= \text{Probability}(d_{\text{IIT}}(k) \leq x). \\ &= \text{Probability}\left(d_{\text{IT}}(k) \geq \frac{1}{x}\right) \\ &= 1 - \text{Probability}\left(d_{\text{IT}}(k) < \frac{1}{x}\right) \\ &= 1 - P_{IT}\left(\frac{1}{x}\right) \end{aligned} \tag{8.257}$$

where we have assumed $P_{IT}(x)$ is a continuous function, Probability($d_{\text{IT}}(k)$=1/x)=0. Since the probability density functions, PDF, are the differentiated of the probability distribution function, F, we have

$$p_{IIT}(x) = \frac{p_{IT}(x)}{x^2} = \begin{cases} \dfrac{1}{2Ax^2} & \frac{1}{m_{d_{\text{IT}}}+A} \leq x \leq \frac{1}{m_{d_{\text{IT}}}-A} \\ 0 & \text{otherwise.} \end{cases}$$

First observation is that this density distribution is no longer uniform even if the $p_{IT}(x)$ is. We can now estimate the mean and variances of the two PDF. for the d_{IT} the calculation are very straight forward giving $E[d_{\text{IT}}]=m_{d_{\text{IT}}}$ and $Var[d_{\text{IT}}] = A^2/3$.

For the d_{IIT} we need to compute

$$\begin{aligned} m_{d_{\text{IIT}}} = E[d_{\text{IIT}}] &= \int_{\frac{1}{m_{d_{\text{IT}}}+A}}^{\frac{1}{m_{d_{\text{IT}}}-A}} x\frac{1}{2Ax^2}dx \\ &= \frac{1}{2A}\ln\left(\frac{m_{d_{\text{IT}}}+A}{m_{d_{\text{IT}}}-A}\right) \end{aligned}$$

and the variance

$$\begin{aligned} Var[d_{\text{IIT}}(k)] &= \int_{\frac{1}{m_{d_{\text{IT}}}+A}}^{\frac{1}{m_{d_{\text{IT}}}-A}} (x-m_{d_{\text{IIT}}})^2\frac{1}{2Ax^2}dx \\ &= \frac{1}{m_{d_{\text{IT}}}^2 - A^2} - m_{d_{\text{IIT}}}^2 \end{aligned}$$

b) Next table shows the results for values $m_{d_{\text{IT}}}$=1s and A=0.2 and those when moving to $m_{d_{\text{IT}}}$=707s and A=0.1414.

$m_{d_{\text{IT}}}$	A	Var[$m_{d_{\text{IT}}}$]	$m_{d_{\text{IIT}}}$	Var[$m_{d_{\text{IIT}}}$]
1	0,2	0,0133	1,0136	0,0143
0,707	0,1414	0,0066	1,4541	0,0602

We see that Var[$m_{d_{\text{IT}}}$] is reduced by a 50% whereas Var[$m_{d_{\text{IIT}}}$] is increased by 300% . So this illustrate how care should be taken when interpreting HRV from different quantification indices.

8.4 If the modulating signal $m(t)$ is compose of a single tone

$$m(t) = m_1 \cos(2\pi F_1 t)$$

we can use equation (6.147)

$$D_E^u(\Omega) = \left(\frac{\delta(\Omega) + M(\Omega)}{T_I}\right) * \left[\delta(\Omega) + \sum_{k=1}^{\infty} D_{FM-HT_k}(\Omega)\right]$$

and estimate the frequency modulating term

$$D_{FM-HT_k}(\Omega) = \mathcal{F}\left\{2\cos\left(\frac{2\pi k}{T_I}(t + d_{HT}(t))\right)\right\}$$

now $d_{HT}(t)$ is

$$d_{HT}(t) = \int_{-\infty}^{t} m(\tau)d\tau = -\frac{m_1}{2\pi F_1}\sin(2\pi F_1 t)$$

so the frequency modulating term will be the know expression for tone modulation [8]

$$D_{FM-HT_k}(\Omega) = \mathcal{F}\left\{2\cos\left(\frac{2\pi k}{T_I}\left(t - \frac{m_1}{2\pi F_1}\sin(2\pi F_1 t)\right)\right)\right\}$$

$$D_{FM-HT_k}(\Omega) = \mathcal{F}\left\{2\sum_{l=-\infty}^{\infty} J_l\left(\frac{m_1 k}{F_1 T_I}\right)\cos\left(\frac{2\pi k}{T_I}t + 2\pi l F_1 t\right)\right\}$$

and taking the inverse Fourier transform we obtain

$$\begin{aligned} d_E(t) = \frac{1}{T_I} + \frac{m_1}{T_I}\cos(2\pi F_1 t) + \frac{2}{T_I}\sum_{k=1}^{\infty}\sum_{l=-\infty}^{\infty} J_l\left(\frac{m_1 k}{F_1 T_I}\right)\cos\left(\frac{2\pi k}{T_I}t + 2\pi l F_1 t\right) \\ + \frac{2m_1}{T_I}\sum_{k=1}^{\infty}\sum_{l=-\infty}^{\infty}\frac{J_{l-1}\left(\frac{m_1 k}{F_1 T_I}\right) + J_{l+1}\left(\frac{m_1 k}{F_1 T_I}\right)}{2}\cos\left(\frac{2\pi k}{T_I}t + 2\pi l F_1 t\right) \end{aligned}$$

and using the property of the Bessel functions

$$J_{l+1}(x) + J_{l-1}(x) = \frac{2l}{x}J_l(x) \tag{8.258}$$

$$\begin{aligned} d_E(t) &= \frac{1}{T_I} + \frac{m_1}{T_I}\cos(2\pi F_1 t) \\ &+ \frac{2}{T_I}\sum_{k=1}^{\infty}\sum_{l=-\infty}^{\infty}\left(1 + \frac{l F_1 T_I}{k}\right) J_l\left(\frac{m_1 k}{F_1 T_I}\right)\cos\left(\frac{2\pi k}{T_I}t + 2\pi l F_1 t\right) \end{aligned} \tag{8.259}$$

8.5 a) The impulse response $h(t)$ of a ideal low-pass filter with cutoff frequency F_c is:

$$h(t) = \frac{\sin(2\pi F_c t)}{\pi t}, \quad -\infty < t < \infty, \tag{8.260}$$

and then $d_{LE}(t)$ can be expressed according to 8.17 as

$$\begin{aligned} d_{LE}(t) &= \int_{-\infty}^{\infty} h(t-\tau) d_E^u(\tau) d\tau \\ &= \sum_{k=0}^{K} h(t-t_k) \\ &= \sum_{k=0}^{K} \frac{\sin(2\pi F_c(t-t_k))}{\pi(t-t_k)}. \end{aligned} \quad (8.261)$$

Sampling this $d_{LE}(t)$ function at the Nyquist rate, $2F_c$, will give the $d_{LE}(n)$ when substituting t by $n/(2F_c)$ so,

$$\begin{aligned} d_{LE}(n) &= \sum_{k=0}^{K} \frac{\sin(2\pi F_c(\frac{n}{2F_c} - t_k))}{\pi(\frac{n}{2F_c} - t_k)} \\ &= 2F_c \sum_{k=0}^{K} \frac{\sin(\pi n)\cos(2\pi F_c t_k) - \cos(\pi n)\sin(2\pi F_c t_k)}{\pi n - 2\pi F_c t_k} \\ &= 2F_c \sum_{k=0}^{K} \frac{(-1)^{(n+1)} \sin(2\pi F_c t_k)}{\pi n - 2\pi F_c t_k} \end{aligned} \quad (8.262)$$

If we had used a different sampling rate $F_s > F_c/2$, then the term $\sin(\pi n)$ that has appeared in the numerator will be $\sin(2\pi n F_c/F_s)$ and will no longer be zero, also the term $(-1)^{(n+1)}$ will become $\cos(2\pi n F_c/F_s)$ which is more involve from a computational point of view.

b) In practice previous result can be argued that, since the available data in newer infinity the filter will never be ideal and the sampling at exactly the Nyquist frequency will introduce some attenuation and aliasing errors [9]. To avoid the aliasing and still have nice computational properties, in [9] was presented the algorithm for sampling at two times the Nyquist rate, 4 F_c. Preceding as before

$$
\begin{aligned}
d_{LE}(n) &= \sum_{k=0}^{K} \frac{\sin(2\pi F_c(\frac{n}{4F_c} - t_k))}{\pi(\frac{n}{4F_c} - t_k)} \\
&= 2F_c \sum_{k=0}^{K} \frac{\sin(\frac{\pi n}{2})\cos(2\pi F_c t_k) - \cos(\frac{\pi n}{2})\sin(2\pi F_c t_k)}{\frac{\pi n}{2} - 2\pi F_c t_k} \\
&= \begin{cases} 2F_c \displaystyle\sum_{k=0}^{K} \frac{(-1)^{\frac{n+2}{2}} \sin(2\pi F_c t_k)}{\pi\left(\frac{n}{2} - 2F_c t_k\right)} & n \text{ even} \\ \\ 2F_c \displaystyle\sum_{k=0}^{K} \frac{(-1)^{\frac{n+3}{2}} \cos(2\pi F_c t_k)}{\pi\left(\frac{n}{2} - 2F_c t_k\right)} & n \text{ odd.} \end{cases}
\end{aligned}
\tag{8.263}
$$

8.6 a) The double integral IPFM model can be express in mathematical terms as

$$\int_0^{t_k} \left(\int_0^{t} (1 + m(\tau))d\tau \right) dt = kT_I \tag{8.264}$$

being k the beat index. Integrating the expression we obtain

$$\frac{t_k^2}{2} + \int_0^{t_k} \left(\int_0^{t} m(\tau)d\tau \right) dt = kT_I \tag{8.265}$$

and defining a new heart timing as

$$
\begin{aligned}
d_{HT}(t) &= kT_I - \frac{t^2}{2} \\
&= \int_0^{t} \left(\int_0^{t'} m(\tau)d\tau \right) dt'
\end{aligned}
\tag{8.266}
$$

we have that this signal can be estimated at the event times

$$d_{HT}(t_k) = kT_I - \frac{t_k^2}{2}$$

b) An estimate of the spectrum of the modulating signal $M(\Omega)$ can be obtained from an estimate of the spectrum of the heart timing signal $D_{HT}(\Omega)$, assuming $m(t)$ zero-mean and causal, trough the relation

$$M(\Omega) = (\jmath\Omega)^2 D_{HT}(\Omega) \tag{8.267}$$

8.7 Respect to the maximum heart period, in problem 7.6 we already shown that a linear interpolator has a -3 dB cut-off frequency that can be estimated from the expression.

$$F_c = \frac{0.30}{T} \tag{8.268}$$

being T the interval gap between samples under interpolation. Then, to have always $F_c > 0.4Hz$ we need to be sure that always

$$T < 0.3/0.4 = 0.75$$

implying that the maximum RR interval should have 750 ms. In other words the heart rate need to be always higher than 80 bpm.

8.8 Taylor series expansion of $g(t)$ around $t = \tau$ gives

$$g(t) = g(\tau) + \left.\frac{\partial g(t)}{\partial t}\right|_{t=\tau} (t - \tau) + \ldots \tag{8.269}$$

Inserting the condition that $g(\tau) = 0$ we obtain

$$g(t) = \left.\frac{\partial g(t)}{\partial t}\right|_{t=\tau} (t - \tau) + \ldots \tag{8.270}$$

To estimate $\delta(g(t))$ we need to care about the t values for which $g(t) = 0$ that is when $t = \tau$ so we can use the first order approximation of $g(t)$ around $t = \tau$ and made the equality

$$\begin{aligned} \delta(g(t)) &= \delta\left(\left.\frac{\partial g(t)}{\partial t}\right|_{t=\tau} (t - \tau) \right) \\ &= \frac{\delta(t-\tau)}{\left|\frac{\partial g(t)}{\partial t}\right|_{t=\tau}}. \end{aligned} \tag{8.271}$$

Then, we obtain the desired result

$$\begin{aligned} \delta(t-\tau) &= \left|\frac{\partial g(t)}{\partial t}\right|_{t=\tau} \delta(g(t)) \\ &= \left|\frac{\partial g(t)}{\partial t}\right| \delta(g(t)) \end{aligned} \tag{8.272}$$

8.9 In order to speed the computations of the Lomb periodogram, it is proposed to shift the sin and cos basis function by a factor τ so to get the condition express in (8.68)

$$\mathbf{h}_{1,\tau}^T \mathbf{h}_{2,\tau} = \sum_{k=0}^{K} \cos(\Omega(t_k - \tau)) \sin(\Omega(t_k - \tau)) = 0$$

The value of τ satisfying this condition can be obtained by doing some trigonometric transformations, and expressing

$$\begin{aligned}\mathbf{h}_{1,\tau}^T\mathbf{h}_{2,\tau} &= \sum_{k=0}^{K}\frac{1}{2}\sin(2\Omega(t_k-\tau))\\ &= \frac{1}{2}\left[\cos(2\Omega\tau)\sum_{k=0}^{K}\sin(2\Omega t_k) - \sin(2\Omega\tau)\sum_{k=0}^{K}\cos(2\Omega t_k)\right] = 0.\end{aligned}$$

From here we obtain

$$\tan(2\Omega\tau) = \frac{\sum_{k=0}^{K}\sin(2\Omega t_k)}{\sum_{k=0}^{K}\cos(2\Omega t_k)},$$

and the result for τ in (8.68) appears,

$$\tau = \frac{1}{2\Omega}\arctan\left(\frac{\sum_{k=0}^{K}\sin(2\Omega t_k)}{\sum_{k=0}^{K}\cos(2\Omega t_k)}\right),$$

8.10 Under the IPFM model we assume that the signal $m(t)$ is zero mean, so

$$\int_0^{t_K} m(\tau)d\tau = 0$$

then

$$\begin{aligned}d_{HT}(t_K) &= \int_0^{t_K} m(\tau)d\tau\\ KT_I - t_K &= 0\end{aligned}$$

and one estimate of T_I, in ectopic beat absence is (6.125)

$$\hat{T}_I = \frac{t_K}{K}.$$

When there is one ectopic beat the

$$\begin{aligned}d_{HT}(t_K) &= \int_0^{t_K} m(\tau)d\tau\\ (K+s)T_I - t_K &= 0\end{aligned}$$

and the estimate of T_I in (8.89) can be obtained from a estimate of s as in (8.88)

$$\hat{T}_I = \frac{t_K}{K+\hat{s}}.$$

When there is more than isolated ectopic beat, we will have associated to each jth ectopic a s_j parameter, so

$$d_{HT}(t_K) = \int_0^{t_K} m(\tau)d\tau$$
$$(K + \sum_j s_j)T_I - t_K = 0$$

and the estimate of T_I is then

$$\hat{T}_I = \frac{t_K}{K + \sum_j \hat{s}_j}.$$

8.11 In the HRV analysis by the $d_{LE}(t)$ representation the ectopics will introduce two types of errors. First the ectopic will generate a spike in a position that does not correspond with the regular series coming from the SA node. In addition the spike sequences after the ectopic will be shifted because the resetting of the SA node generating the so call compensatory pause. The first problem can be address like with any other method, identify the ectopic beat and reject the spike associated with it.

From the "cleaned" spike series $d_E(t)$ we can propose a first strategy by just low-pass filtering this even series to obtain $d_{LE}(t)$. This strategy has to interpolate a large gap in the ectopic neighborhood and will result in low-pass filtering of the estimated signal.

Other alternative can be to estimate also the $\hat{s}$ associated with the ectopic as introduced with the $d_{HT}(t)$ signal. Once s is estimated, we can backwards shift the spikes after the ectopic by one amount, $\hat{s}\,\hat{\bar{T}}_0$, that is the responsible for the compensatory pause after the ectopic.

$$d_E(t) = \sum_{k=0}^{k_e} \delta(t - t_k) + \sum_{k=k_e+1}^{K} \delta(t - t_k + \hat{s}\hat{\bar{T}}_0).$$

From this even series we can proceed to generate the $d_{LE}(t)$ as a estimate of $m(t)$.

8.12 From the definition of $S_{xy}(e^{\jmath\omega})$ we see that

$$r_{xy}(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xy}(e^{\jmath\omega})d\omega \tag{8.273}$$

Introducing the definition of the $\Gamma_{xy}(e^{\jmath\omega})$ we obtain that

$$S_{xy}(e^{\jmath\omega}) = \Gamma_{xy}(e^{\jmath\omega})\sqrt{S_{xx}(e^{\jmath\omega})}\sqrt{S_{yy}(e^{\jmath\omega})} \tag{8.274}$$

and integrating and realizing that

$$r_{xy}(0) = \frac{1}{N} E[\mathbf{x}^T \mathbf{y}] \tag{8.275}$$

it appear the result

$$\rho = \frac{\frac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} \Gamma_{xy}(e^{\jmath\omega}) \sqrt{S_{xx}(e^{\jmath\omega})} \sqrt{S_{yy}(e^{\jmath\omega})} d\omega}{\sqrt{E[\mathbf{x}^T \mathbf{x}]} \sqrt{E[\mathbf{y}^T \mathbf{y}]}}. \tag{8.276}$$

8.13 The solution is obtained by solving equation 8.105 (ref??') to obtain the elements of (1.104) as function of the element in (1.103).

$$S_{x_1x_1}(e^{\jmath\omega}) = \sigma_{v_1}^2 |H_{11}(e^{\jmath\omega})|^2 + \sigma_{v_2}^2 |G_{12}(e^{\jmath\omega}) H_{22}(e^{\jmath\omega})|^2 \tag{8.277}$$

$$S_{x_2x_2}(e^{\jmath\omega}) = \sigma_{v_1}^2 |G_{21}(e^{\jmath\omega}) H_{11}(e^{\jmath\omega})|^2 + \sigma_{v_2}^2 |H_{22}(e^{\jmath\omega})|^2 \tag{8.278}$$

$$S_{x_1x_2}(e^{\jmath\omega}) = \sigma_{v_1}^2 G_{21}(e^{\jmath\omega}) |H_{11}(e^{\jmath\omega})|^2 + \sigma_{v_2}^2 G_{12}^*(e^{\jmath\omega}) |H_{22}(e^{\jmath\omega})|^2 \tag{8.279}$$

and using the definition

$$\Gamma_{xy}(e^{\jmath\omega}) = \left. \frac{\sigma_{v_1}^2 G_{21}(z)|H_{11}(z)|^2 + \sigma_{v_2}^2 G_{12}^*(z)|H_{22}(z)|^2}{\left(\sigma_{v_1}^4 |H_{11}(z)|^4 |G_{21}(z)|^2 + \sigma_{v_2}^4 |H_{22}(z)|^4 |G_{12}(z)|^2 + \sigma_{v_1}^2 \sigma_{v_2}^2 |H_{11}(z) H_{22}(z)|^2 (1 + |G_{12}(z) G_{21}(z)|^2)\right)^{1/2}} \right|_{z=e^{\jmath\omega}}. \tag{8.280}$$

with

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