Formulary
Engineering Mechanics
FHLA05

Division of Solid Mechanics
Lund University
January 7, 2021

Center of Gravity, Center of Mass, Centroids

Center of Forces

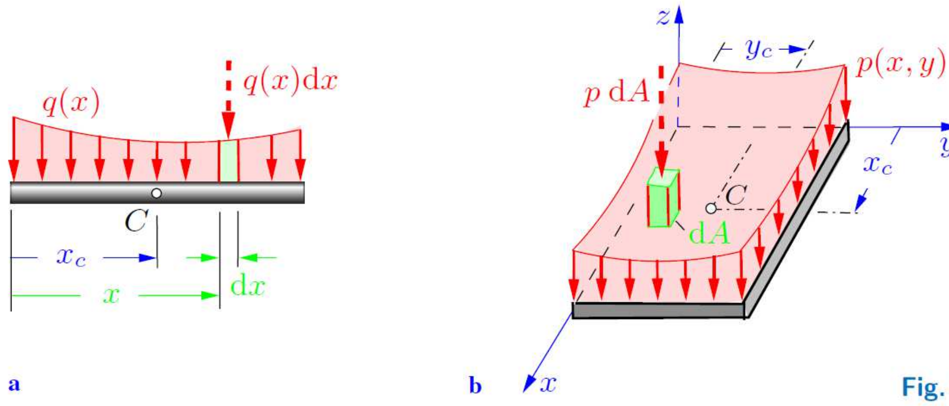


Fig. 4.2

$$x_c = \frac{\int x\, q(x)\,\mathrm{d}x}{\int q(x)\,\mathrm{d}x} \qquad x_c = \frac{\int x\, q(x,y)\,\mathrm{d}x\mathrm{d}y}{\int q(x,y)\,\mathrm{d}x\mathrm{d}y}$$

$$y_c = \frac{\int y\, q(x,y)\,\mathrm{d}x\mathrm{d}y}{\int q(x,y)\,\mathrm{d}x\mathrm{d}y}$$

Centroid of an area

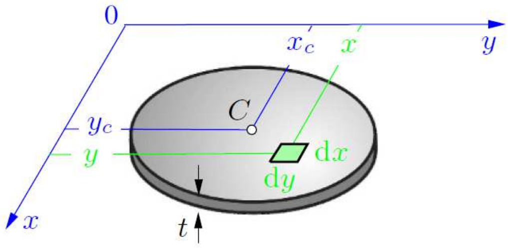


Fig. 4.6

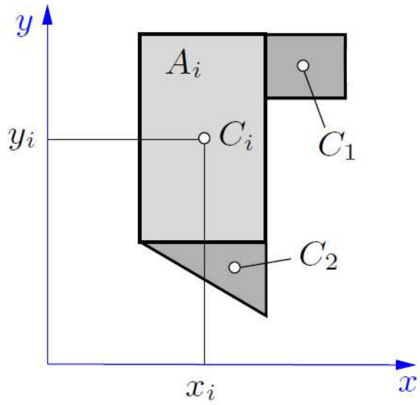
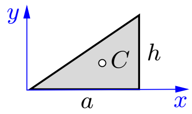
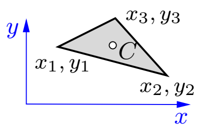
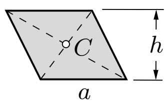
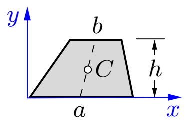
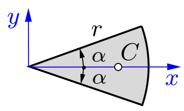
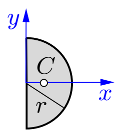
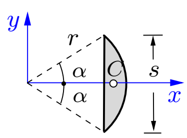
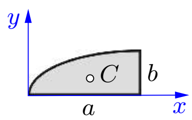


Fig. 4.8

$$x_c = \frac{1}{A}\int x\,\mathrm{d}A \qquad y_c = \frac{1}{A}\int y\,\mathrm{d}A \qquad\qquad x_c = \frac{\sum x_i A_i}{\sum A_i} \qquad y_c = \frac{\sum y_i A_i}{\sum A_i}$$

Table 4.1 Location of Centroids

Area		Location of Centroid
Rectangular triangle 	$A = \frac{1}{2}\, ah$	$x_c = \frac{2}{3}\, a, \;\; y_c = \frac{h}{3}$
Arbitrary triangle 	$A = \frac{1}{2}[(x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)]$	$x_c = \frac{1}{3}(x_1 + x_2 + x_3)$ $y_c = \frac{1}{3}(y_1 + y_2 + y_3)$
Parallelogram 	$A = a\, h$	C is determined by the intersection of the diagonals
Trapezium 	$A = \frac{h}{2}(a + b)$	C is located at the median line $y_c = \frac{h}{3}\, \frac{a + 2\, b}{a + b}$
Circular sector 	$A = \alpha\, r^2$	$x_c = \frac{2}{3}\, r\, \frac{\sin\alpha}{\alpha}$
Semicircle 	$A = \frac{\pi}{2}\, r^2$	$x_c = \frac{4\, r}{3\, \pi}$
Circular segment 	$A = \frac{1}{2}\, r^2 (2\, \alpha - \sin 2\, \alpha)$	$x_c = \frac{s^3}{12\, A} = \frac{4}{3}\, r\, \frac{\sin^3 \alpha}{2\, \alpha - \sin 2\, \alpha}$
Quadratic parabola 	$A = \frac{2}{3}\, a\, b$	$x_c = \frac{3}{5}\, a$ $y_c = \frac{3}{8}\, b$

Stress tensor – plane stress state

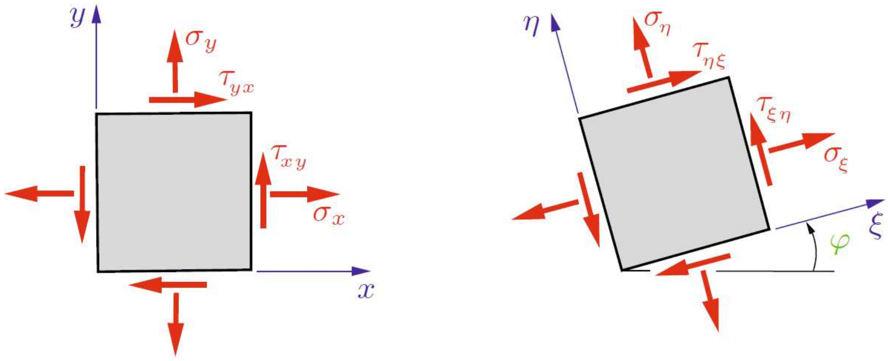


Fig. 2.5

Transformation relations

$$\begin{aligned}
\sigma_\xi &= \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi + \tau_{xy}\sin 2\varphi\,, \\
\sigma_\eta &= \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos 2\varphi - \tau_{xy}\sin 2\varphi\,, \\
\tau_{\xi\eta} &= -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin 2\varphi + \tau_{xy}\cos 2\varphi\,.
\end{aligned}$$

Principal stresses and directions

$$\sigma_{1,2} = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \sqrt{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}\,,$$

$$\tan 2\varphi^* = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad \rightarrow \quad \varphi_1^*,\ \varphi_2^* = \varphi_1^* \pm \pi/2\,.$$

Maximum shear stress and directions

$$\tau_{\max} = \sqrt{\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2}\,, \qquad \varphi^{**} = \varphi^* \pm \pi/4\,.$$

Strain tensor – plane strain state

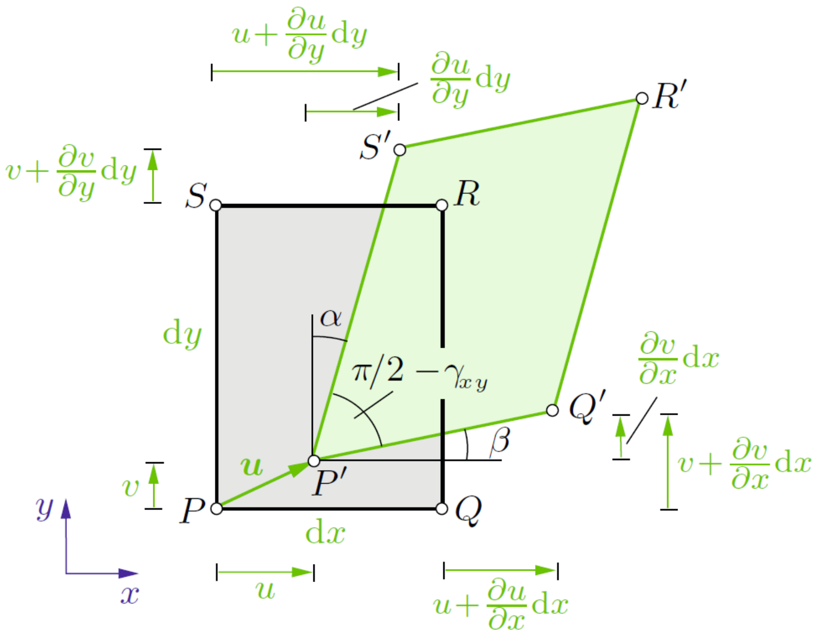


Fig. 3.2

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Transformation relations

$$\varepsilon_\xi = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) + \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\varphi + \tfrac{1}{2}\gamma_{xy}\sin 2\varphi\,,$$

$$\varepsilon_\eta = \tfrac{1}{2}(\varepsilon_x + \varepsilon_y) - \tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\cos 2\varphi - \tfrac{1}{2}\gamma_{xy}\sin 2\varphi\,,$$

$$\tfrac{1}{2}\gamma_{\xi\eta} = -\tfrac{1}{2}(\varepsilon_x - \varepsilon_y)\sin 2\varphi + \tfrac{1}{2}\gamma_{xy}\cos 2\varphi\,.$$

Principal direction and principal strains

$$\tan 2\,\varphi^* = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y}$$

$$\varepsilon_{1,2} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{1}{2}\gamma_{xy}\right)^2}$$

Hooke's law – plane stress state

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu\sigma_y] \qquad \sigma_x = \frac{E}{1-\nu^2}[\varepsilon_x + \nu\varepsilon_y]$$
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu\sigma_x] \qquad \sigma_y = \frac{E}{1-\nu^2}[\varepsilon_y + \nu\varepsilon_x]$$
$$\gamma_{xy} = \frac{1}{G}\tau_{xy} \qquad \tau_{xy} = G\,\gamma_{xy}$$

$G = \dfrac{E}{2[1+\nu]}$ $\quad$ G shear modulus $\quad$ E Young's modulus $\quad$ ν Poisson's ratio

Strength hypotheses – plane stress state

$\sigma_e \leq \sigma_{allow}$ $\quad$ σ_e equivalent stress $\quad$ σ_{allow} allowable stress

1. Maximum-normal-stress hypothesis

$$\sigma_e = \sigma_1$$

2. Maximum-shear-stress hypothesis

$$\sigma_e = \sigma_1 - \sigma_2 = \sqrt{[\sigma_x - \sigma_y]^2 + 4\tau_{xy}^2}$$

3. von Mises hypothesis (maximum-distortion-energy hypothesis)

$$\sigma_e = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}$$

Tension and compression in bars

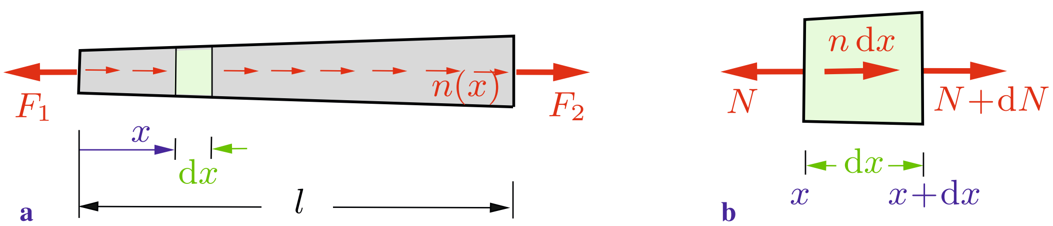


Fig. 1.8

$$\frac{dN}{dx} + n = 0 \qquad \sigma = \frac{N}{A} \qquad \varepsilon = \frac{du}{dx} \qquad \varepsilon = \frac{\sigma}{E} + \alpha_T \Delta T$$

Beams

Stress resultants

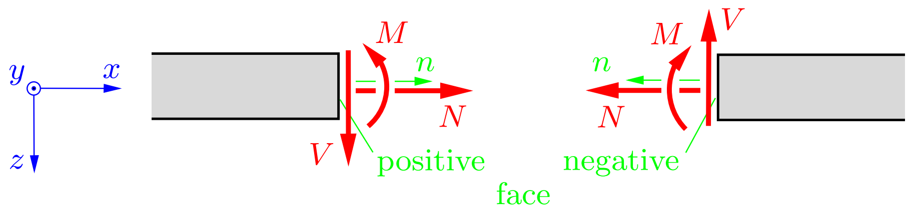


Fig. 7.3

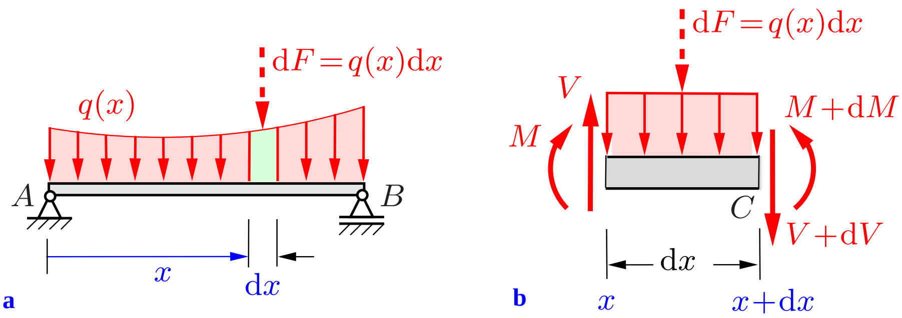


Fig. 7.10

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -q \quad \text{and} \quad \frac{\mathrm{d}M}{\mathrm{d}x} = V$$

Normal stress due to a bending moment

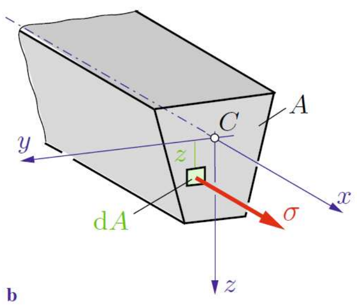
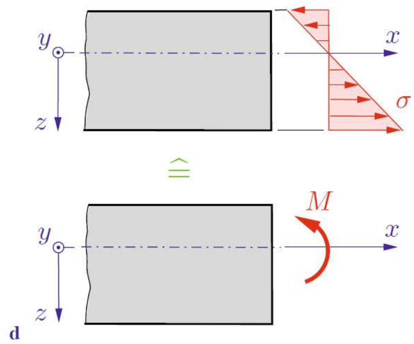
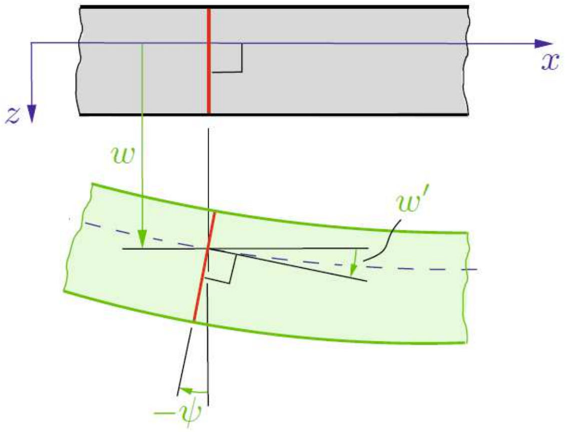


Fig. 4.1



$\sigma = \dfrac{M}{I_y} z$	with I_y rectangular moment of inertia
$W = \dfrac{I_y}{\lvert z \rvert_{max}}$	section modulus
$\sigma_{max} = \dfrac{M}{W}$	maximum tensile or compressive stress

Deflection curve – Bernoulli beam



$$w'' = -\frac{M}{EI_y}$$
$$[EI_y w'']'' = q$$

Fig. 4.17

Second moments of area

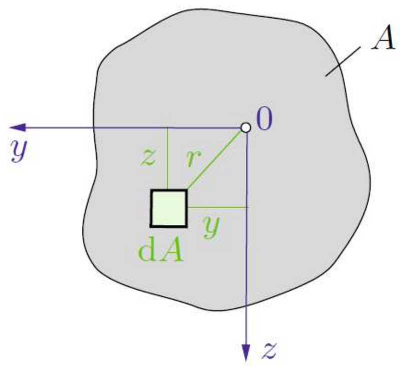


Fig. 4.2

rectangular moments of inertia

$$I_y = \int z^2\,\mathrm{d}A \qquad I_z = \int y^2\,\mathrm{d}A$$

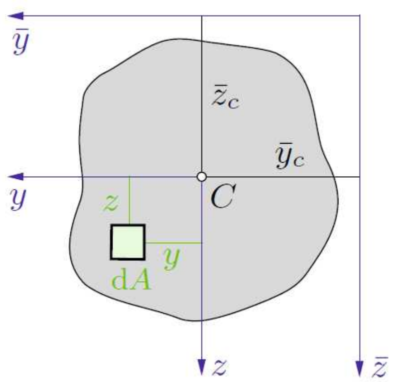
product of inertia

$$I_{yz} = I_{zy} = -\int yz\,\mathrm{d}A$$

polar moment of inertia

$$I_p = \int r^2\,\mathrm{d}A = \int y^2 + z^2\,\mathrm{d}A$$

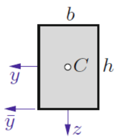
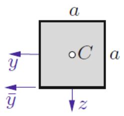
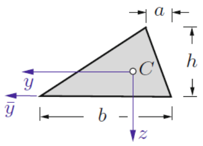
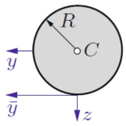
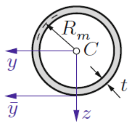
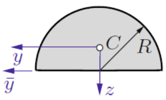
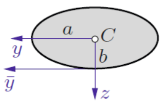
Parallel-axis theorem



$$I_{\bar{y}} = I_y + \bar{z}_c^2 A$$
$$I_{\bar{z}} = I_z + \bar{y}_c^2 A$$
$$I_{\bar{y}\bar{z}} = I_{yz} - \bar{y}_c \bar{z}_c A$$

Fig. 4.7

Table 4.1. Moments of Inertia

Area	I_y	I_z	I_{yz}	I_p	$I_{\bar{y}}$
Rectangle 	$\frac{b\,h^3}{12}$	$\frac{h\,b^3}{12}$	0	$\frac{b\,h}{12}(h^2+b^2)$	$\frac{b\,h^3}{3}$
Square 	$\frac{a^4}{12}$	$\frac{a^4}{12}$	0	$\frac{a^4}{6}$	$\frac{a^4}{3}$
Triangle 	$\frac{b\,h^3}{36}$	$\frac{b\,h}{36}(b^2-b\,a+a^2)$	$-\frac{b\,h^2}{72}(b-2\,a)$	$\frac{b\,h}{36}(h^2+b^2-b\,a+a^2)$	$\frac{b\,h^3}{12}$
Circle 	$\frac{\pi\,R^4}{4}$	$\frac{\pi\,R^4}{4}$	0	$\frac{\pi\,R^4}{2}$	$\frac{5\pi}{4}R^4$
Thin Circular Ring $t \ll R_m$ 	$\pi\,R_m^3\,t$	$\pi\,R_m^3\,t$	0	$2\,\pi\,R_m^3\,t$	$3\,\pi\,R_m^3\,t$
Semi-Circle 	$\frac{R^4}{72\,\pi}(9\,\pi^2-64)$	$\frac{\pi\,R^4}{8}$	0	$\frac{R^4}{36\,\pi}(9\,\pi^2-32)$	$\frac{\pi\,R^4}{8}$
Ellipse 	$\frac{\pi}{4}a\,b^3$	$\frac{\pi}{4}b\,a^3$	0	$\frac{\pi\,a\,b}{4}(a^2+b^2)$	$\frac{5\pi}{4}a\,b^3$

Torsion

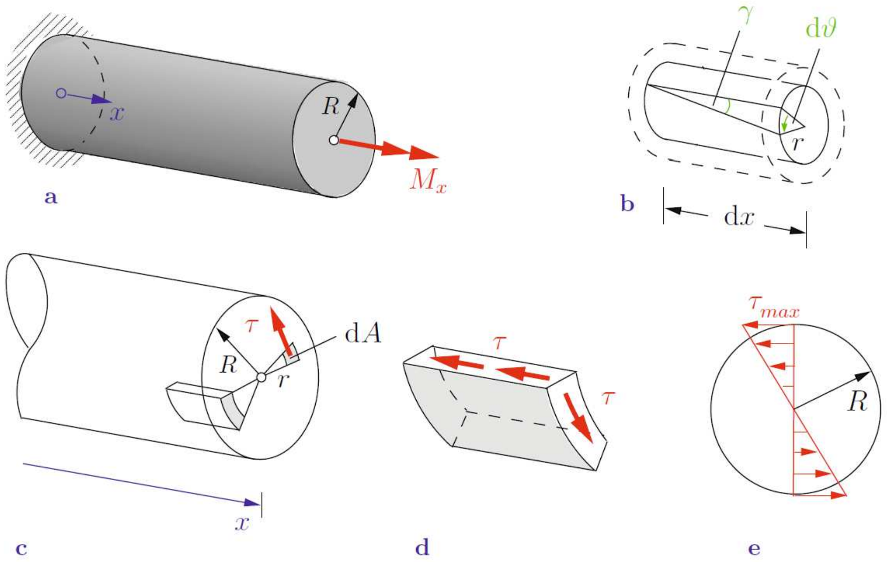


Fig. 5.2

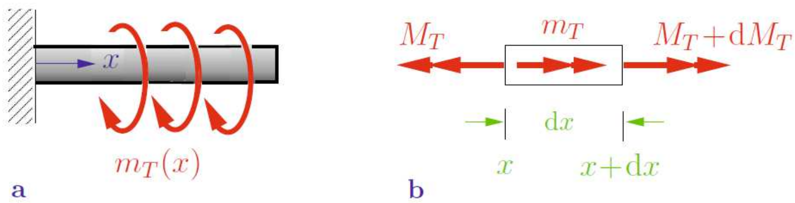
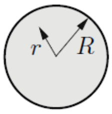
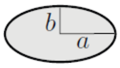
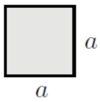
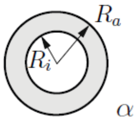
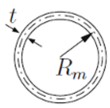


Fig. 5.3

$$GI_T\vartheta' = M_T$$

$$[GI_T\vartheta']' = -m_T$$

$$\tau_{max} = \frac{M_T}{W_T}$$

Cross section	W_T	I_T	Remarks
solid circle 	$\dfrac{\pi R^3}{2}$	$\dfrac{\pi R^4}{2}$	$\tau(r) = \dfrac{M_T}{I_T}\, r$ Maximum shear stress at boundary $r = R$
solid ellipse 	$\dfrac{\pi ab^2}{2}$	$\dfrac{\pi a^3 b^3}{a^2 + b^2}$	Maximum shear stress at end points of minor semi axis
solid square 	$0,208\, a^3$	$0,141\, a^4$	Maximum shear stress in the middle of boundary edges
thick-walled circular tube  $\alpha = \dfrac{R_i}{R_a}$	$\dfrac{\pi R_a^3}{2}(1-\alpha^4)$	$\dfrac{\pi R_a^4}{2}(1-\alpha^4)$	Maximum shear stress at outer boundary R_a
thin-walled circular tube $t = $ const 	$2\pi R_m^2\, t$	$2\pi R_m^3\, t$	

Motion of a point mass

Cartesian coordinates

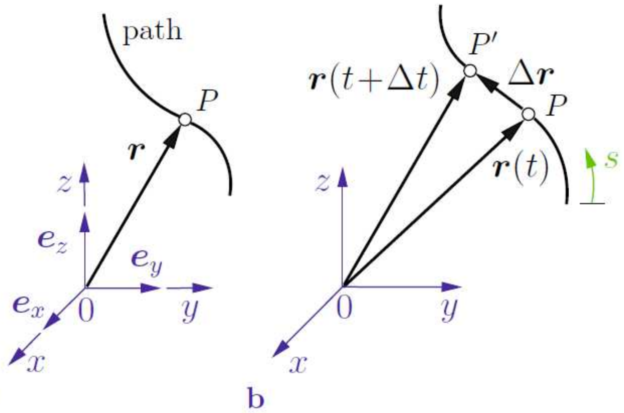
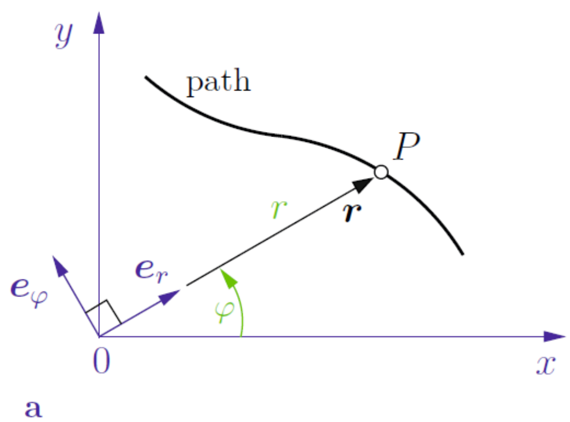


Fig. 1.1

$$
\begin{aligned}
\boldsymbol{r}(t) &= x(t)\boldsymbol{e}_x + y(t)\boldsymbol{e}_y + z(t)\boldsymbol{e}_z \\
\boldsymbol{v} &= \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{e}_x + \dot{y}\boldsymbol{e}_y + \dot{z}\boldsymbol{e}_z \\
\boldsymbol{a} &= \dot{\boldsymbol{v}} = \ddot{\boldsymbol{r}} = \ddot{x}\boldsymbol{e}_x + \ddot{y}\boldsymbol{e}_y + \ddot{z}\boldsymbol{e}_z
\end{aligned}
$$

Planar motion, polar coordinates



$$\boldsymbol{r} = r\,\boldsymbol{e}_r,$$

$$\boldsymbol{v} = v_r\,\boldsymbol{e}_r + v_\varphi\,\boldsymbol{e}_\varphi = \dot{r}\,\boldsymbol{e}_r + r\dot{\varphi}\,\boldsymbol{e}_\varphi,$$

$$\boldsymbol{a} = a_r\,\boldsymbol{e}_r + a_\varphi\,\boldsymbol{e}_\varphi = (\ddot{r} - r\dot{\varphi}^2)\,\boldsymbol{e}_r + (r\ddot{\varphi} + 2\,\dot{r}\dot{\varphi})\,\boldsymbol{e}_\varphi\,.$$

Impulse law

$$m\boldsymbol{v} - m\boldsymbol{v}_0 = \int_{t_0}^{t} \boldsymbol{F}\,\mathrm{d}\bar{t}$$

Angular momentum theorem

$\boldsymbol{L}^{(0)} = \boldsymbol{r} \times m\boldsymbol{v}$	angular momentum vector
$\dfrac{\mathrm{d}\boldsymbol{L}^{(0)}}{\mathrm{d}t} = \boldsymbol{M}^{(0)}$	angular momentum theorem

Work-energy theorem

$T = \dfrac{1}{2}mv^2$	kinetic energy
$U = \displaystyle\int_{\boldsymbol{r}_0}^{\boldsymbol{r}_1} \boldsymbol{F}^{(a)} \cdot \mathrm{d}\boldsymbol{r}$	work integral
$U = T_1 - T_0$	work-energy theorem

Central impact

$m_1 v_1 + m_2 v_2 = m_1 \bar{v}_1 + m_2 \bar{v}_2$	conservation of linear momentum
$e = -\dfrac{\bar{v}_1 - \bar{v}_2}{v_1 - v_2}$	coefficient of restitution

Rigid body

Work-energy theorem

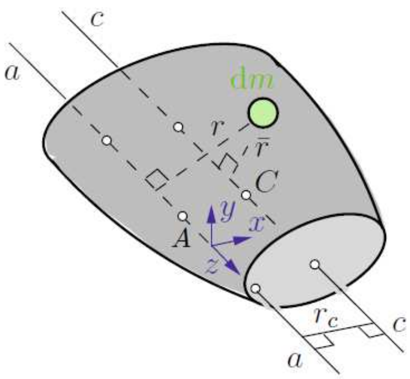
$T = \dfrac{1}{2}mv^2 + \dfrac{1}{2}\Theta_C\,\omega^2$	kinetic energy
$U = T_1 - T_0$	work-energy theorem

Mass moment of inertia

Definition

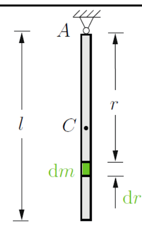
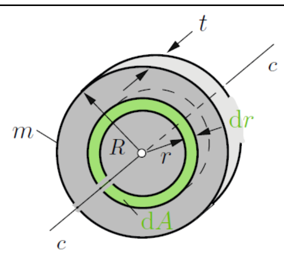
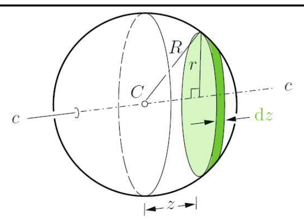
$$\Theta_a = \int r^2 \mathrm{d}m$$

Parallel-axis theorem



$$\Theta_a = \Theta_c + r_c^2 m$$

Examples for homogeneous bodies

slender rod		$\Theta_C = \frac{1}{12} m l^2$
solid cylinder		$\Theta_c = \frac{1}{2} m R^2$
solid sphere		$\Theta_c = \frac{2}{5} m R^2$