$$\text{Stationary Stochastic Processes}$$
$$\text{Table of Formulas}$$

Estimation

- Estimation of expected value:

$$\hat{m}_n = \frac{1}{n}\sum_{t=1}^{n} X_t$$

$$V[\hat{m}_n] = \frac{1}{n^2}\sum_{\tau=-n+1}^{n-1}(n-|\tau|)r_X(\tau)$$

$$V[\hat{m}_n] \approx \frac{1}{n}\sum_{\tau=-\infty}^{\infty} r_X(\tau) \qquad \text{for large n}$$

- If $\hat{m}_n \in N(m, V[\hat{m}_n])$, the confidence interval for m is

$$I_m: \quad \{\hat{m}_n - \lambda_{\alpha/2}\sqrt{V[\hat{m}_n]},\ \hat{m}_n + \lambda_{\alpha/2}\sqrt{V[\hat{m}_n]}\}$$

with confidence level $1-\alpha$. For confidence level 0.95, $\alpha = 0.05$ and $\lambda_{\alpha/2} = 1.96$.

- Estimation of covariance function:

$$\hat{r}_n(\tau) = \frac{1}{n}\sum_{t=1}^{n-\tau}(X_t - m_X)(X_{t+\tau} - m_X) \quad \text{for} \quad \tau \geq 0$$

where m_X is replaced by $\hat{m}_n$ if m_X is unknown.

Spectral representations

- Relations between covariance function $r_X(\tau)$ and spectral density $R_X(f)$:

Continuous time	Discrete time
$r_X(\tau) = \int_{-\infty}^{\infty} R_X(f)e^{i2\pi f\tau}\,df$	$r_X(\tau) = \int_{-1/2}^{1/2} R_X(f)e^{i2\pi f\tau}\,df$
$R_X(f) = \int_{-\infty}^{\infty} r_X(\tau)e^{-i2\pi f\tau}\,d\tau$	$R_X(f) = \sum_{\tau=-\infty}^{\infty} r_X(\tau)e^{-i2\pi f\tau}$

- Folding (aliasing): Let $\{Z_t, t = 0, \pm d, \pm 2d, \dots\}$ be the continuous time process $Y(t)$ sampled with time interval d and sampling frequency $f_s = 1/d$:

$$R_Z(f) = \sum_{k=-\infty}^{\infty} R_Y(f + kf_s) \quad \text{for} \quad -f_s/2 < f \leq f_s/2$$

- Sum of harmonic components with random phase and amplitude:

$$X(t) = A_0 + \sum_{k=1}^{n} A_k \cos(2\pi f_k t + \varphi_k)$$

where $\varphi_k \in \text{Rect}(0, \ 2\pi)$, A_k, $k = 0, \ \ldots, \ n$, are independent and $E[A_0] = 0$.

  * Covariance function:

$$r_X(\tau) = \sigma_0^2 + \sum_{k=1}^{n} \sigma_k^2 \cos 2\pi f_k \tau$$

  where $\sigma_0^2 = E\left[A_0^2\right]$ and $\sigma_k^2 = E\left[A_k^2\right]/2$.

  * Spectral density:

$$R_X(f) = \sum_{k=-n}^{n} b_k \delta_{f_k}(f),$$

  where $b_0 = \sigma_0^2 = E\left[A_0^2\right]$, and $b_k = \sigma_k^2/2 = E\left[A_k^2\right]/4$.

Linear filters - general theory

- Impulse response $h(u)$:

$$Y(t) = \begin{cases} \int_{-\infty}^{\infty} h(u)X(t-u)\,du & \text{(continuous time)} \\[2mm] \sum_{u=-\infty}^{\infty} h(u)X(t-u) & \text{(discrete time)} \end{cases}$$

- Relation between covariance functions:

$$r_Y(\tau) = \begin{cases} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(u)h(v)\,r_X(\tau+u-v)\,du\,dv & \text{(continuous time)} \\[2mm] \sum_{u=-\infty}^{\infty} \sum_{v=-\infty}^{\infty} h(u)h(v)\,r_X(\tau+u-v) & \text{(discrete time)} \end{cases}$$

- Frequency function $H(f)$ and impulse response $h(t)$:

Continuous time	Discrete time
$h(t) = \int_{-\infty}^{\infty} H(f)e^{i2\pi ft}\,df$	$h(t) = \int_{-1/2}^{1/2} H(f)e^{i2\pi ft}\,df$
$H(f) = \int_{-\infty}^{\infty} h(t)e^{-i2\pi ft}\,dt$	$H(f) = \sum_{t=-\infty}^{\infty} h(t)e^{-i2\pi ft}$

- Relation between spectral densities:

$$R_Y(f) = |H(f)|^2 R_X(f).$$

- Differentiation: $X'(t)$ exists (in quadratic mean) if $r_X''(t)$ exists. This is equivalent to $\int_{-\infty}^{\infty} (2\pi f)^2 R(f) df < \infty$. If $X'(t)$ exists, the following relations hold:

$$r_{X'}(\tau) = -r_X''(\tau)$$

$$R_{X'}(f) = (2\pi f)^2 R_X(f)$$

$$r_{X^{(j)}, X^{(k)}}(\tau) = (-1)^j r_X^{(j+k)}(\tau)$$

- Integration:

$$E\left[\int g(s)X(s)\,ds\right] = \int g(s)E[X(s)]\,ds$$

$$C\left[\int g(s)X(s)\,ds, \int h(t)Y(t)\,dt\right] = \int\int g(s)h(t)\,C[X(s), Y(t)]\,ds\,dt$$

- Cross-covariance and cross-spectrum:

$$r_{X,Y}(\tau) = C[X(t), Y(t+\tau)] = \int e^{i2\pi f\tau}R_{X,Y}(f)\,df$$

$$R_{X,Y}(f) = H(f)R_X(f) = A_{X,Y}(f)e^{i\Phi_{X,Y}(f)}$$

where $A_{X,Y}(f)$ is the amplitude spectrum and $\Phi_{X,Y}(f)$ the phase spectrum. The squared coherence spectrum is

$$\varkappa_{X,Y}^2(f) = \frac{A_{X,Y}^2(f)}{R_X(f)R_Y(f)}$$

AR- MA- and ARMA-models

- AR(p)-process: $(a_0 = 1)$

$$X_t + a_1 X_{t-1} + a_2 X_{t-2} + \ldots + a_p X_{t-p} = e_t$$

where $\{e_t,\ t = 0, \pm 1, \ldots\}$, is a white noise process with $E[e_t] = 0$ and $V[e_t] = \sigma^2$.

⋆ Yule-Walker equations:

$$r_X(k) + a_1 r_X(k-1) + \ldots + a_p r_X(k-p) = \begin{cases} \sigma^2 & \text{for } k = 0 \\ 0 & \text{for } k = 1, 2, \ldots \end{cases}$$

⋆ Spectral density:

$$R_X(f) = \frac{1}{\left|\sum_{k=0}^{p} a_k e^{-i2\pi fk}\right|^2}\sigma^2$$

- MA(q)-process: $(b_0 = 1)$

$$X_t = e_t + b_1 e_{t-1} + b_2 e_{t-2} + \ldots + b_q e_{t-q}$$

  ★ Covariance function:

$$r_X(\tau) = \begin{cases} \sigma^2 \sum_{j-k=\tau} b_j b_k & \text{for } |\tau| \le q \\ 0 & \text{for } |\tau| > q \end{cases}$$

  ★ Spectral density:

$$R_X(f) = |\sum_{k=0}^{q} b_k e^{-i2\pi fk}|^2 \sigma^2$$

Matched filter and Wiener filter

- Matched filter for white noise disturbance:

$$\begin{aligned} h(t) &= s(T-t), \quad 0 \le t \le T \\ \text{SNR}_{opt} &= \frac{\int_0^T s^2(T-u)\,du}{\sigma_N^2} \end{aligned}$$

- Wiener filter:

$$\begin{aligned} H(f) &= \frac{R_S(f)}{R_S(f) + R_N(f)} \\ \text{SNR} &= \frac{\int R_S(f)\,df}{\int \frac{R_S(f)R_N(f)}{R_S(f)+R_N(f)}\,df} \end{aligned}$$

Spectral estimation

Periodogram of the sequence $\{x(t), t = 0, 1, 2, \ldots n-1\}$ is defined as

$$\hat{R}_x(f) = \frac{1}{n} \left| \sum_{t=0}^{n-1} x(t) e^{-i2\pi ft} \right|^2$$

with

$$E\left[\hat{R}_x(f)\right] = \int_{-1/2}^{1/2} K_n(f-u) R_X(u) du$$

where $K_n(f) = \sum_{\tau=-n+1}^{n-1} (1 - \frac{|\tau|}{n}) e^{-i2\pi f\tau}$, and

$$V\left[\hat{R}_x(f)\right] \approx \begin{cases} R_X^2(f) & \text{for } 0 < |f| < 1/2 \\ 2R_X^2(f) & \text{for } f = 0, \pm 1/2 \end{cases}$$

Fourier transforms

$g(\tau) \qquad (\alpha > 0)$	$G(f) = \int_{-\infty}^{\infty} e^{-i2\pi f \tau} g(\tau)\, d\tau$
$e^{-\alpha|\tau|}$	$\frac{2\alpha}{\alpha^2 + (2\pi f)^2}$
$\frac{1}{\alpha^2 + \tau^2}$	$\frac{\pi}{\alpha} e^{-2\pi\alpha|f|}$
$|\tau| e^{-\alpha|\tau|}$	$2\frac{(\alpha^2 - (2\pi f)^2)}{(\alpha^2 + (2\pi f)^2)^2}$
$|\tau|^k e^{-\alpha|\tau|}$	$\frac{k!}{(\alpha^2 + (2\pi f)^2)^{k+1}}\left\{(\alpha + i2\pi f)^{k+1} + (\alpha - i2\pi f)^{k+1}\right\}$
$e^{-\alpha\tau^2}$	$\sqrt{\pi/\alpha}\exp\left(-\frac{(2\pi f)^2}{4\alpha}\right)$
$e^{-\alpha|\tau|}\cos(2\pi f_0\tau)$	$\frac{\alpha}{\alpha^2 + (2\pi f_0 - 2\pi f)^2} + \frac{\alpha}{\alpha^2 + (2\pi f_0 + 2\pi f)^2}$
$e^{-\alpha|\tau|}\sin(2\pi f_0\tau)$	$\frac{2\pi f_0 - 2\pi f}{\alpha^2 + (2\pi f_0 - 2\pi f)^2} + \frac{2\pi f_0 + 2\pi f}{\alpha^2 + (2\pi f_0 + 2\pi f)^2}$
$\begin{cases} \alpha & \text{if} \quad \tau = 0 \\ \frac{\sin(2\pi\alpha\tau)}{2\pi\tau} & \text{if} \quad \tau \neq 0 \end{cases}$	$\begin{cases} 1/2 & \text{if} \quad |f| \leq \alpha \\ 0 & \text{if} \quad |f| > \alpha \end{cases}$
$\begin{cases} 1 - \alpha|\tau| & \text{if} \quad |\tau| \leq \frac{1}{\alpha} \\ 0 & \text{if} \quad |\tau| > \frac{1}{\alpha} \end{cases}$	$\begin{cases} \frac{1}{\alpha} & \text{if} \quad f = 0 \\ \frac{2\alpha}{(2\pi f)^2}\left(1 - \cos\left(\frac{2\pi f}{\alpha}\right)\right) & \text{if} \quad f \neq 0 \end{cases}$
$g(\tau)h(\tau)$	$G(f) * H(f) = \int G(\nu)H(f - \nu)d\nu$
$g(\tau) * h(\tau) = \int g(t)h(\tau - t)dt$	$G(f)H(f)$
$g'(\tau)$	$i2\pi f\, G(f)$
$g(\alpha\tau)$	$\frac{1}{\alpha}G(\frac{f}{\alpha})$
$\frac{1}{\alpha}g(\frac{\tau}{\alpha})$	$G(\alpha f)$
$g(\tau - \tau_0)$	$G(f)e^{-i2\pi f \tau_0}$
$g(\tau)e^{i2\pi f_0\tau}$	$G(f - f_0)$

Gaussian distribution table

$$F(x) = \Phi(x)$$

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950
3.3	0.99952	0.99953	0.99955	0.99957	0.99958	0.99960	0.99961	0.99962	0.99964	0.99965
3.4	0.99966	0.99968	0.99969	0.99970	0.99971	0.99972	0.99973	0.99974	0.99975	0.99976
3.5	0.99977	0.99978	0.99978	0.99979	0.99980	0.99981	0.99981	0.99982	0.99983	0.99983
3.6	0.99984	0.99985	0.99985	0.99986	0.99986	0.99987	0.99987	0.99988	0.99988	0.99989
3.7	0.99989	0.99990	0.99990	0.99990	0.99991	0.99991	0.99992	0.99992	0.99992	0.99992
3.8	0.99993	0.99993	0.99993	0.99994	0.99994	0.99994	0.99994	0.99995	0.99995	0.99995
3.9	0.99995	0.99995	0.99996	0.99996	0.99996	0.99996	0.99996	0.99996	0.99997	0.99997
4.0	0.99997	0.99997	0.99997	0.99997	0.99997	0.99997	0.99998	0.99998	0.99998	0.99998