1. $f(x)=\begin{cases} e^{-x}, & \text{for } x>0 \\ 0, & \text{o.w.} \end{cases}$

(1) $Z=\max(X,Y)$

$$F_Z(z)=P(Z\le z)$$
$$=P(\max(X,Y)\le z)$$
$$=P((X\le z)\text{ and }(Y\le z))$$
$$=P(X\le z)P(Y\le z)$$
$$=\int_0^z e^{-x}dx\int_0^z e^{-y}dy$$
$$=(-e^{-x}\big|_0^z)^2=(-e^{-z}+1)^2$$
$$=1-2e^{-z}+e^{-2z}$$

$$\underline{f_Z(z)=2e^{-z}-2e^{-2z}}\ \#$$

(2) $W=\min(X,Y)$

$$f_W(w)=\frac{d}{dw}\left(P(\min(X,Y)\le w)\right)$$
$$=\frac{d}{dw}\left(1-P(\min(X,Y)>w)\right)$$
$$=-\frac{d}{dw}\left(P(X>w)P(Y>w)\right)$$
$$=-\frac{d}{dw}(1-F_X(w))(1-F_Y(w))$$
$$=f_X(w)(1-F_Y(w))+(1-F_X(w))f_Y(w)$$
$$=e^{-w}(1-(1-e^{-w}))+(1-(1-e^{-w}))e^{-w}$$
$$=2e^{-w}(e^{-w})=\underline{2e^{-2w}}\ \#$$

2.

(1) $x,y,z\in X$

① $f(x,y)\ge 0$

② $f(x,y)=0$ if and only if $x=y$

③ $f(x,y)=f(y,x)$

④ $f(x,z)\le f(x,y)+f(y,z)$

(2) $F(x)=\|x\|^2$

$$D_F(x,y)=\|x-y\|^2$$
$$=\langle x-y,x-y\rangle$$
$$=\underline{\|x\|^2-\|y\|^2-\langle 2y,x-y\rangle}\ \#$$

(3) $H(X,f(X))=H(X)+H(f(X)|X)$

$$=H(f(X))+H(X|f(X))$$
$$\Rightarrow H(X)=H(f(X))+H(X|f(X))$$
$$\Rightarrow \underline{H(f(X))\le H(X)}\ \#$$

3、(1)

$$E(x) = \frac{0+\theta}{2} = \frac{X_1+X_2+\cdots+X_n}{n} = \bar{X}$$

$$\hat{\theta} = 2\bar{X} \quad \#$$

(2) $f_X(x) = \frac{1}{\theta},\ 0 \le x \le \theta$

$L(x_1, x_2, \ldots, x_n, \theta) = \frac{1}{\theta^n} \Rightarrow$ decreasing function

$$\hat{\theta}_{ML} = \max(X_1, X_2, \ldots, X_n) \quad \#$$

(3) $f(x/\theta) = \frac{1}{6}$

$$u(x,\theta) = h(\theta)\, f(x/\theta) = \frac{1}{\theta}$$

$$g(x) = \int_x^1 \frac{1}{\theta}\, d\theta = -\ln x$$

$$k(\theta/x) = \frac{u(x,\theta)}{g(x)} = -\frac{1}{\theta \ln x}$$

$$\hat{\theta} = E[\theta/x] = \int_x^1 \theta\, k(\theta/x)\, d\theta = \int_x^1 \theta \frac{-1}{\theta \ln x}\, d\theta = \frac{x-1}{\ln x} \qquad \hat{\theta} = \frac{x-1}{\ln x} \quad \#$$

4、If $c_1 + c_2 = 1 \Rightarrow E(c_1\hat{\theta}_1 + c_2\hat{\theta}_2) = c_1\theta + c_2\theta = \theta$

(1) $c_2 = 1 - c_1$

$Var(c_1\hat{\theta}_1 + c_2\hat{\theta}_2)$

$= c_1^2 Var(\hat{\theta}_1) + c_2^2 Var(\hat{\theta}_2) + 2c_1c_2 Cov(\hat{\theta}_1, \hat{\theta}_2)$

$= c_1^2 + 2c_2^2 + 2c_1c_2 \cdot \frac{1}{4}$

$= c_1^2 + 2c_2^2 + \frac{1}{2}c_1c_2$

$= c_1^2 + 2(1-c_1)^2 + \frac{1}{2}c_1(1-c_1)$

$= \frac{5}{2}c_1^2 - \frac{7}{2}c_1 + 2$

$= \frac{5}{2}(c_1^2 - \frac{7}{10})^2 + \frac{31}{40}$

A: $\frac{7}{10}\hat{\theta}_1 + \frac{3}{10}\hat{\theta}_2$

A: $\frac{31}{40}$

(2) $Var(c_3\hat{\theta}_3 + c_4\hat{\theta}_4)$

$= c_3^2 + 2c_4^2 + 2c_1c_2 \cdot \frac{3}{4}$

$= \frac{3}{2}(c_3 - \frac{5}{6})^2 + \frac{23}{24}$

A: $\frac{5}{6}\hat{\theta}_3 + \frac{1}{6}\hat{\theta}_4$

A: $\frac{23}{24}$

5、Interval Estimation

$X_1, X_2, \dots X_n$

$f(x;\theta) = e^{-(x-\theta)}$ $\theta$: unknown parameter

(1) $Q = X_{(1)} - \theta$

pdf of $Q = g(q) = ne^{-nq}$

Q is a pivotal quantity #

(2) $1-\alpha = P\left(\frac{2}{2-\alpha} \le e^{-nq} \le \frac{2}{\alpha}\right)$

$= P\left(\ln\left(\frac{2}{2-\alpha}\right) \le -nq \le \ln\left(\frac{2}{\alpha}\right)\right)$

$= P\left(\frac{-1}{n}\ln\left(\frac{2}{\alpha}\right) \le q \le \frac{-1}{n}\ln\left(\frac{2}{2-\alpha}\right)\right)$

$= P\left(\frac{-1}{n}\ln\left(\frac{2}{\alpha}\right) \le X_{(1)} - \theta \le \frac{-1}{n}\ln\left(\frac{2}{2-\alpha}\right)\right) = P\left(X_{(1)} - \frac{1}{n}\ln\left(\frac{2}{\alpha}\right) < \theta < X_{(1)} - \frac{1}{n}\ln\frac{2}{2-\alpha}\right)$

$\left[X_{(1)} - \frac{1}{n}\ln\left(\frac{2}{\alpha}\right), X_{(1)} - \frac{1}{n}\ln\left(\frac{2}{2-\alpha}\right)\right)$ #

$\left[X_{(1)} - \frac{1}{n}\ln\left(\frac{2}{\alpha}\right), X_{(1)} - \frac{1}{n}\ln\left(\frac{2}{2-\alpha}\right)\right]$

$\frac{2}{2-\alpha} < e^{n(-\theta + X_{(1)})} < \frac{2}{\alpha}$ $\alpha \Rightarrow$