Problem Set 2: Exchange with production

Exercise 1: Exchange economy with production 1

We consider an exchange and production economy with two consumer goods X and Y, a production factor L, two consumers A and B and two firms 1 and 2. Firm 1 produces good X from factor L and firm 2 produces good Y from factor L. We note :

- X_i the quantity of good X consumed by $i = A, B$.
- Y_i the quantity of good Y consumed by $i = A, B$.
- L_j the quantity of L consumed by $j = 1, 2$.
- x_1 the quantity of good X produced by firm 1.
- y_2 the quantity of good Y produced by firm 2.
- $U_A = \sqrt{X_A} + Y_A$ the utility of A.
- $U_B = \sqrt{X_B Y_B}$ the utility of B.
- $x_1 = F_1(L_1) = 2\sqrt{L_1}$ the production function of 1.
- $y_2 = F_2(L_2) = L_2$ the production function of 2.
- p_X the price of X.
- p_Y the price of Y.
- w the price of L.

Let's suppose that A is the owner of firm 1 and B is the owner of firm 2. Each consumer has 4 units of L. The firms do not initially have any stock of L. All markets are assumed to be competitive.

1. Determine the total demand function for each good X and Y.
2. Specify the supply function for good X.
3. Show that the market for Y can only be in equilibrium if $p_Y = w$.
4. Compute the incomes R^A and R^B of consumers A and B.
5. We pose X as the numeraire good, meaning that $p_X = 1$. Write the condition of equality between total supply and demand in the market of X. What is the condition for a general equilibrium to exist in this economy?
6. Now assume that $U_A(X_A, Y_A) = \sqrt{X_A Y_A}$ and that $p_X = 1$. Show that $(p_X, p_Y, w) = (1, \sqrt{3/8}, \sqrt{3/8})$ is a vector of equilibrium prices.
7. Recall the definition of a Pareto optimum and show that the allocation associated with the present competitive equilibrium is Pareto optimal.

Exercise 2: Exchange economy with production 2

Consider an economy with two consumers ($i = 1, 2$), two consumption goods ($h = 1, 2$), one production factor (labor) and two firms ($j = 1, 2$). We note x_h^i the consumption of good h by individual i. The preferences of the consumers are represented by utility functions: $U^i(x_1^i, x_2^i) = \sqrt{x_1^i x_2^i}, x_1^i > 0, x_2^i > 0$ Each consumer i offers a fixed quantity of work, equal to 1 unit for $i = 1$ and 2 units for $i = 2$. The firm $j = 1$ produces the consumer good $h = 1$ with labor, with a constant returns to scale technology represented by the production function : $y_1 = l_1$ where y_1 and l_1 represent respectively the production of firm 1 and the amount of labor it uses. The firm $j = 2$ produces the consumption good $h = 2$ also with labor and with a constant return technology; we assume : $y_2 = 1/2l_2$ where y_2 and l_2 represent respectively the output of firm 2 and the amount of labor used. The state of the economy is a vector $(x_1^1, x_2^1, x_1^2, x_2^2, y_1, l_1, y_2, l_2)$ which summarizes the behavior of consumers and firms.

1. Characterize the states of the economy that can be achieved (which take into account the use-resource constraints and the technical conditions defined by the production functions).
2. Determine the Pareto optimum that is most advantageous for consumer 1 and the one that is most advantageous for consumer 2. We will note respectively u_*^1 and u_*^2 the maximum utilities of consumers 1 and 2.
3. Let u^1 be a level of consumer 1, between 0 and u_*^1. Determine the Pareto optimum that assigns this level of utility to consumer 1. Find the relation between the utility level of the two consumers, u_1 and u_2, for the set of Pareto optima.
4. The prices of consumer goods 1 and 2 are denoted by p_1 and p_2 respectively, and it is assumed that the price of labor factor w is 1. Show that in a general market equilibrium, we have $p_1 = 1$ and $p_2 = 2$. Determine the state of the economy that corresponds to this general equilibrium. Verify that this is a Pareto optimum.