Problem Set 4: Joint distributions

Exercise 1: Counters in a supermarket
A local supermarket has three checkout counters. Two customers arrive at the counters at different times when the counters are serving no other customers. Each customer chooses a counter at random, independently of the other. Let X denote the number of customers who choose counter A and Y, the number who select counter B.

1. Find the joint probability function of X and Y. (Hint: you can display it in a table)
2. Find F(-1,2), F(1.5,2) and F(5,7)

Exercise 2: Location of a particle
Suppose that a radioactive particle is randomly located in a square with sides of unit length. That is, if two regions within the unit square and of equal area are considered, the particle is equally likely to be in either region. Let Y_1 and Y_2 denote the coordinates of the particle's location. A reasonable model for the relative frequency histogram for Y_1 and Y_2 is the bivariate analogue of the univariate uniform density function:

$$f(y_1, y_2) = \begin{cases} 1, & \text{if } 0 \leq y_1 \leq 1 \text{ and } 0 \leq y_2 \leq 1, \\ 0, & \text{elsewhere.} \end{cases}$$

1. Find $F(0.2, 0.4)$.
2. Find $P(0.1 \leq Y_1 \leq 0.3, 0 \leq Y_2 \leq 0.5)$.

Exercise 3: Coins
Three balanced coins are tossed independently. One of the variables of interest is X, the number of heads. Let Y denote the amount of money won on a side bet in the following manner. If the first head occurs on the first toss, you win \$1. If the first head occurs on toss 2 or on toss 3, you win \$2 or \$3, respectively. If no heads appear, you lose \$1 (that is, win -\$1).

1. Find the joint probability function for X and Y. Follow the following steps:
   (a) Compute the marginal probability of X
   (b) Find the conditional probability $P(Y|X)$
   (c) Find $P(x, y)$
2. What is the probability that fewer than three heads will occur and you will win \$1 or less? [That is, find $F(2, 1)$.]
3. Derive the marginal probability distribution for your winnings on the side bet.
4. What is the probability that you obtained three heads, given that you won \$1 on the side bet?

Exercise 4: Independence
Determine whether X and Y in each of the following PDFs are independent.

1. PDF illustrated in Table 1
2. $$f(x, y) = 2 \qquad \text{if } 0 \leq x \leq y; 0 \leq y \leq 1$$

Table 1: PDF of X and Y

		Y		
		0	1	2
	0	1/9	2/9	1/9
X	1	2/9	2/9	0
	2	1/9	0	0

3.
$$f(x,y) = 6(1-y) \qquad \text{if } 0 \leq x \leq y \leq 1$$

4.
$$f(x,y) = 4xy, \qquad \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

Exercise 5: Covariance
Find the covariance between X and Y in Table 1. Find the Cov(X,Y) of pfd 4.

Exercise 6: Linear functions of random variables
Assume that Y_1, Y_2, and Y_3 are random variables, with the following characteristics:

$$\begin{aligned}
E(Y_1) &= 2,\\
E(Y_2) &= -1,\\
E(Y_3) &= 4,\\
V(Y_1) &= 4,\\
V(Y_2) &= 6,\\
V(Y_3) &= 8,\\
\text{Cov}(Y_1, Y_2) &= 1,\\
\text{Cov}(Y_1, Y_3) &= -1,\\
\text{Cov}(Y_2, Y_3) &= 0.
\end{aligned}$$

Find:
$$E(3Y_1 + 4Y_2 - 6Y_3)$$
and
$$V(3Y_1 + 4Y_2 - 6Y_3).$$