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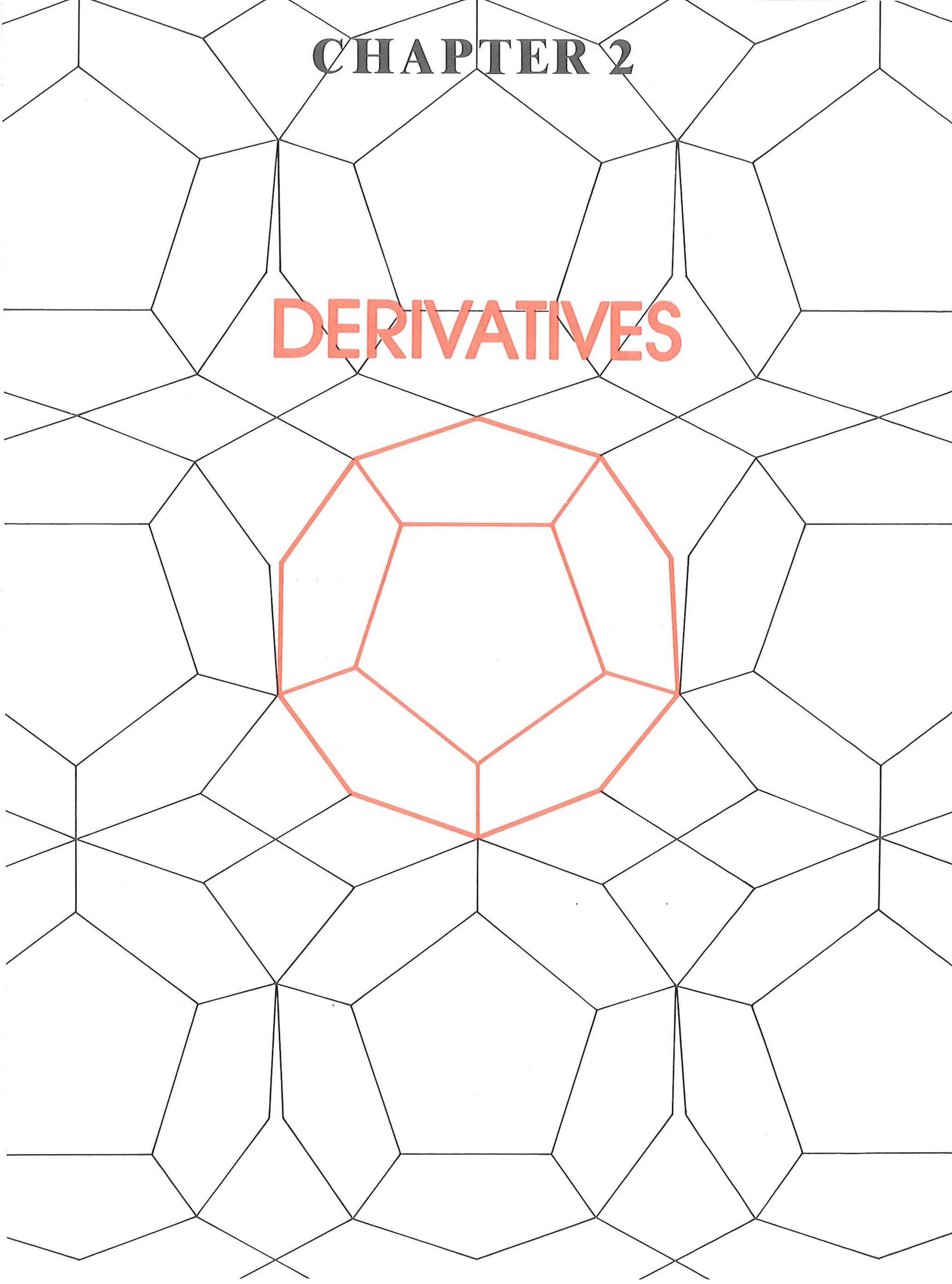
CHAPTER 2

DERIVATIVES



REVIEW AND PREVIEW TO CHAPTER 2

The Domain of a Function

If a formula is given for $f(x)$, but the domain is not given, then the domain of f is assumed to be the set of all real values of x for which the given expression is meaningful. For example, an expression will not be meaningful when it contains a zero denominator or the square root of a negative number.

EXERCISE 1

1. Find the domains of the following functions.

(a) $f(x) = 1 - 18x$

(b) $g(x) = x^4 - x^2 + 15x$

(c) $h(x) = \sqrt{x - 5}$

(d) $F(x) = \sqrt[4]{-x}$

(e) $G(x) = \sqrt{1 - x^2}$

(f) $H(x) = \sqrt{x^2 - 2}$

(g) $y = \dfrac{3 + x}{3 - x}$

(h) $y = \dfrac{x^2}{x^2 + 4x - 5}$

(i) $y = \dfrac{1}{\sqrt{t^2 + 5}}$

(j) $y = \dfrac{t}{\sqrt{t^2 - 5t + 6}}$

(k) $f(x) = \sqrt{x} + \sqrt{4 - x}$

(l) $f(x) = \sqrt{2 - \sqrt{4 - x}}$

Composition of Functions

The **composition**, or **composite**, of f and g is the function $f \circ g$ defined by

$$(f \circ g)(x) = f(g(x))$$

Example. If $f(x) = 2 - 3x$ and $g(x) = 5x^2 + x$, find the functions $f \circ g$ and $g \circ f$.

Solution

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(5x^2 + x) \\ &= 2 - 3(5x^2 + x) \\ &= 2 - 15x^2 - 3x\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(2 - 3x) \\ &= 5(2 - 3x)^2 + (2 - 3x) \\ &= 5(4 - 12x + 9x^2) + 2 - 3x \\ &= 22 - 63x + 45x^2\end{aligned}$$

EXERCISE 2

1. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$.

(a) $f(x) = 2x - 1$, $g(x) = 4 - 3x$

(b) $f(x) = x^2$, $g(x) = x + 1$

(c) $f(x) = 1 - x^2$, $g(x) = 5$

(d) $f(x) = \sqrt{x}$, $g(x) = x^2 - 4$

(e) $f(x) = 3x - 5$, $g(x) = \dfrac{1}{x}$

(f) $f(x) = \dfrac{1}{1 - x}$, $g(x) = \dfrac{x - 2}{x + 2}$

(g) $f(x) = \sqrt{x}$, $g(x) = \sqrt{1 + x}$

2. Find functions f and g such that $h(x) = f(g(x))$.

(a) $h(x) = (2x + 1)^9$

(b) $h(x) = 1 + 2x^2 + 3x^4$

(c) $h(x) = \dfrac{1}{x^2 - 7}$

(d) $h(x) = \sqrt{6 + x}$

INTRODUCTION

One of the main concepts in calculus is the *derivative*, which is defined as a limit and arises when finding slopes of tangents and rates of change. In this chapter, we learn rules for computing derivatives so that we can apply them to curve-sketching and maximum and minimum problems in the following chapters.

2.1 DERIVATIVES

In Section 1.4 we saw that limits of the form

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

occur as slopes of tangents and in Section 1.5 we saw that this type of limit also arises in computing velocities. In fact it occurs as a rate of change in all branches of science and engineering. Since this type of limit occurs so widely, we give it a special name and notation.

> The **derivative of a function f at a number a** is
>
> $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
>
> if this limit exists.

An alternative way of writing the definition of derivative is as follows.

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

In fact we also used this expression in Section 1.4 to calculate slopes of tangents.

Example 1 If $f(x) = 2x^2 - 5x + 6$, find $f'(4)$, the derivative of f at 4.

Solution The calculation resembles the computation of slopes of tangents and velocities in Chapter 1. According to the definition of derivative, we have

$$
\begin{aligned}
f'(4) &= \lim_{h\to 0} \frac{f(4+h) - f(4)}{h} \\
&= \lim_{h\to 0} \frac{[2(4+h)^2 - 5(4+h) + 6] - [2(4)^2 - 5(4) + 6]}{h} \\
&= \lim_{h\to 0} \frac{32 + 16h + 2h^2 - 20 - 5h + 6 - 18}{h} \\
&= \lim_{h\to 0} \frac{11h + 2h^2}{h} \\
&= \lim_{h\to 0} (11 + 2h) \\
&= 11
\end{aligned}
$$

The derivative of f at 4 is 11.

By comparing the definition of a derivative with the definitions in Chapter 1, we have the following:

Interpretations of the Derivative

1. **As the slope of a tangent.** The tangent line to the curve $y = f(x)$ at the point $(a, f(a))$ is the line through $(a, f(a))$ with slope $f'(a)$.
2. **As a rate of change.** The (instantaneous) rate of change of $y = f(x)$ with respect to x when $x = a$ is equal to $f'(a)$. In particular, if $s = f(t)$ is the position function of a particle, then $v = f'(a)$ is the velocity of the particle at time $t = a$.

Example 2 Find the derivative of $f(x) = x^2 - 3x$ at any number a. Then use it to find the slopes of the tangents to parabola $y = x^2 - 3x$ when $x = 1, 2, 3, 4$.

Solution

$$
\begin{aligned}
f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h\to 0} \frac{[(a+h)^2 - 3(a+h)] - (a^2 - 3a)}{h} \\
&= \lim_{h\to 0} \frac{a^2 + 2ah + h^2 - 3a - 3h - a^2 + 3a}{h} \\
&= \lim_{h\to 0} \frac{(2a - 3)h + h^2}{h} \\
&= \lim_{h\to 0} (2a - 3 + h) \\
&= 2a - 3
\end{aligned}
$$

The derivative at a is $f'(a) = 2a - 3$.

The slopes of the tangents are obtained by putting $a = 1, 2, 3$, and 4:

$$f'(1) = 2(1) - 3 = -1 \qquad f'(2) = 2(2) - 3 = 1$$
$$f'(3) = 2(3) - 3 = 3 \qquad f'(4) = 2(4) - 3 = 5$$

The Derivative as a Function

The derivative of a function f at a is a number $f'(a)$. But if we let a vary over the domain of f, we can change our point of view and regard f' as a function.

Given a function f, the **derivative** of f is the function f' defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The domain of this new function f' is the set of all numbers x for which the limit exists. Since $f(x)$ occurs in the expression for $f'(x)$, the domain of f' will always be a subset of the domain of f.

Example 3 Find the derivative of the function $f(x) = x^2$.

Solution In computing the limit that defines $f'(x)$, we must remember that the variable is h and regard x temporarily as a constant.

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2}{h} \\ &= \lim_{h \to 0} (2x + h) \\ &= 2x \end{aligned}$$

The derivative is the function f' given by $f'(x) = 2x$.

Notice that in Example 3 the domains of f and f' are both R, the set of all real numbers. The next example shows that the domain of f' can be smaller than the domain of f.

Example 4 If $f(x) = \sqrt{x+2}$, find f' and state the domains of f and f'.

Solution

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{x+h+2} - \sqrt{x+2}}{h}\left(\frac{\sqrt{x+h+2} + \sqrt{x+2}}{\sqrt{x+h+2} + \sqrt{x+2}}\right) \\ &= \lim_{h\to 0} \frac{(x+h+2) - (x+2)}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\ &= \lim_{h\to 0} \frac{h}{h(\sqrt{x+h+2} + \sqrt{x+2})} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{x+h+2} + \sqrt{x+2}} \\ &= \frac{1}{\sqrt{x+2} + \sqrt{x+2}} \\ &= \frac{1}{2\sqrt{x+2}} \end{aligned}$$

Rationalize the numerator.

The domain of f is $\{x \mid x+2 \geqslant 0\} = \{x \mid x \geqslant -2\}$.
The domain of f' is $\{x \mid x+2 > 0\} = \{x \mid x > -2\}$.

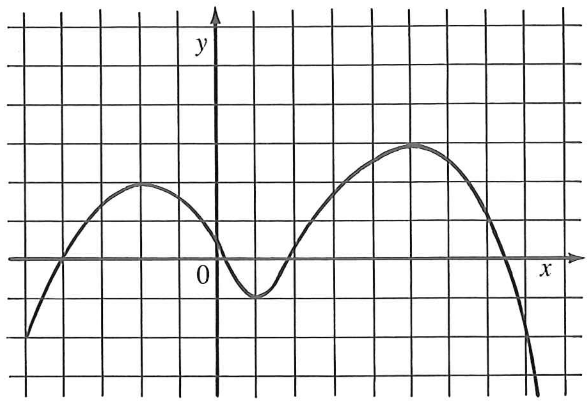
Example 5 Find f' if $f(x) = \dfrac{x+1}{3x-2}$.

Solution

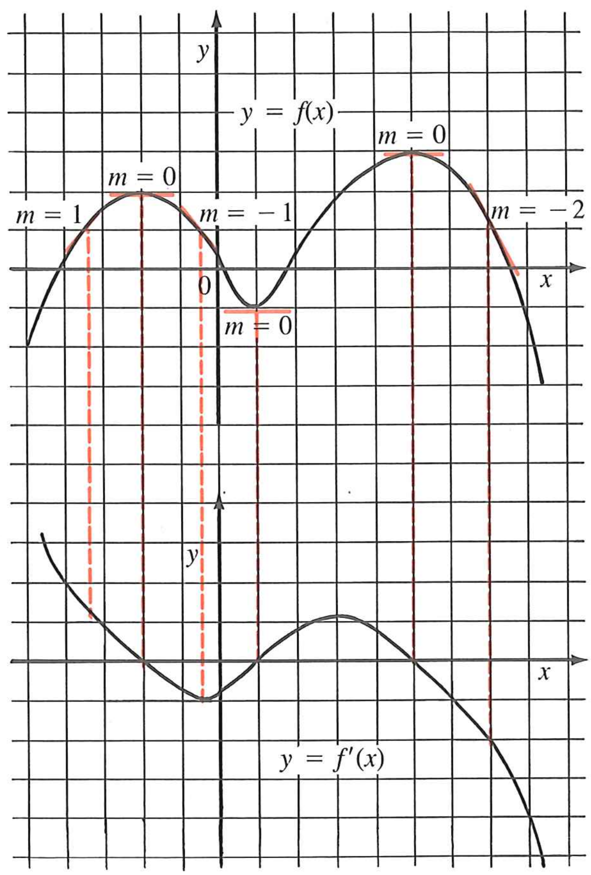
$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(x+h)+1}{3(x+h)-2} - \dfrac{x+1}{3x-2}}{h} \\ &= \lim_{h\to 0} \frac{\dfrac{(x+h+1)(3x-2) - (x+1)(3x+3h-2)}{(3x+3h-2)(3x-2)}}{h} \\ &= \lim_{h\to 0} \frac{(x+h+1)(3x-2) - (x+1)(3x+3h-2)}{h(3x+3h-2)(3x-2)} \\ &= \lim_{h\to 0} \frac{(3x^2+3xh+3x-2x-2h-2) - (3x^2+3xh-2x+3x+3h-2)}{h(3x+3h-2)(3x-2)} \\ &= \lim_{h\to 0} \frac{-5h}{h(3x+3h-2)(3x-2)} \\ &= \lim_{h\to 0} \frac{-5}{(3x+3h-2)(3x-2)} \\ &= \frac{-5}{(3x-2)^2} \end{aligned}$$

We know that the value of the derivative of f at x, $f'(x)$, can be interpreted geometrically as the slope of the tangent line to the graph of f at the point $(x, f(x))$. This enables us to give a rough sketch of the graph of f' if we already have the graph of f.

Example 6 Use the given graph of f to sketch the graph of f'.



Solution



We estimate the slopes of the tangents at various points on the curve $y = f(x)$. These give the values of f' and we plot them directly beneath the graph of f. In particular, we get the x-intercepts of f' from the fact that horizontal lines have slope 0.

Other Notations

Another notation for the derivative was introduced by the German mathematician Gottfried Leibniz:

$$\text{If } y = f(x), \text{ we write } \frac{dy}{dx} = f'(x).$$

In this notation, the results of Examples 3, 4, and 5 can be expressed as follows.

$$\text{If } y = x^2, \text{ then } \frac{dy}{dx} = 2x.$$

$$\text{If } y = \sqrt{x+2}, \text{ then } \frac{dy}{dx} = \frac{1}{2\sqrt{x+2}}.$$

$$\text{If } y = \frac{x+1}{3x-2}, \text{ then } \frac{dy}{dx} = -\frac{5}{(3x-2)^2}.$$

Leibniz used this notation as a reminder of the procedure for finding a derivative:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

For now, the symbol $\frac{dy}{dx}$ should not be regarded as a ratio; it is just a synonym for $f'(x)$. The Leibniz notation has the advantage that both the independent variable x and the dependent variable y are indicated. For instance, if the displacement s of a particle is given as function of the time t, then the velocity is expressed as

$$v = \frac{ds}{dt}$$

A slight variation of the Leibniz notation occurs when we think of the process of finding the derivative of a function as an operation, called **differentiation**, which is performed on f to produce a new function f'. Then we write

$$\frac{dy}{dx} = \frac{d}{dx} f(x)$$

and think of $\frac{d}{dx}$ as a differentiation operator. Thus we could write

$$\frac{d}{dx}(x^2) = 2x \quad \text{and} \quad \frac{d}{dx}\sqrt{x+2} = \frac{1}{2\sqrt{x+2}}$$

Sometimes the symbols D or D_x are also used as differentiation operators. Thus we have the following notations for the derivative of $y = f(x)$:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x y$$

If we want to indicate the value of a derivative $\frac{dy}{dx}$ in Leibniz notation at a specific number a, we use the notation

$$\left.\frac{dy}{dx}\right]_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{x=a}$$

which is a synonym for $f'(a)$.

Differentiable Functions

A function f is said to be **differentiable** at a if $f'(a)$ exists. It is called **differentiable on an interval** if it is differentiable at every number in the interval. In Example 3 we saw that $f(x) = x^2$ is differentiable on R and in Example 4 we found that $f(x) = \sqrt{x+2}$ is differentiable for $x > -2$.

Example 7 Show that the function $f(x) = |x|$ is not differentiable at 0.

Solution We must show that $f'(0)$ does not exist. Now

$$f'(0) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h} = \lim_{h\to 0}\frac{f(h) - 0}{h} = \lim_{h\to 0}\frac{|h|}{h}$$

To show that this limit does not exist we compute the right and left limits separately. Since $|h| = h$ if $h > 0$, we have

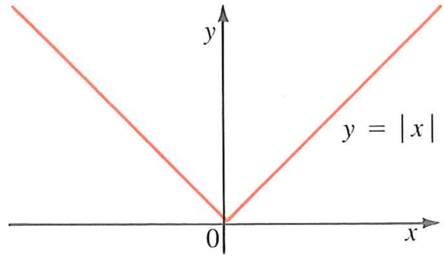
$$\lim_{h\to 0^+}\frac{|h|}{h} = \lim_{h\to 0^+}\frac{h}{h} = \lim_{h\to 0^+} 1 = 1$$

Since $|h| = -h$ if $h < 0$, we have

$$\lim_{h\to 0^-}\frac{|h|}{h} = \lim_{h\to 0^-}\frac{-h}{h} = \lim_{h\to 0^-}(-1) = -1$$

These one-sided limits are different, so $f'(0)$ does not exist. Therefore f is not differentiable at 0.

The geometric significance of Example 7 can be seen from the graph of $f(x) = |x|$ in the figure. The curve does not have a tangent line at $(0, 0)$. In general, functions whose graphs have "corners" or "kinks" are not differentiable there.



EXERCISE 2.1

A **1.** Each of the following limits represents the derivative of some function f at some number a. State f and a in each case.

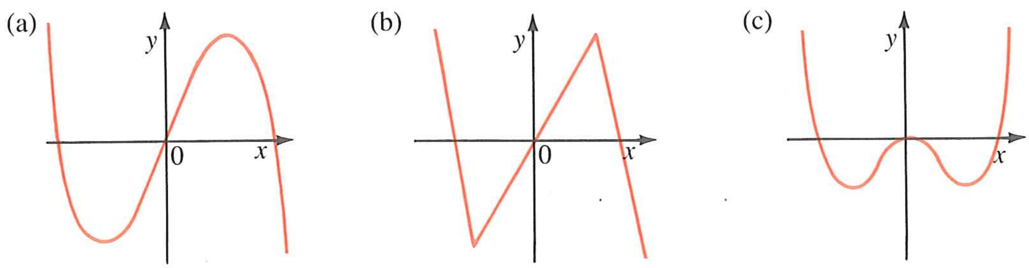
(a) $\displaystyle\lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$ (b) $\displaystyle\lim_{h\to 0} \frac{(2+h)^3 - 8}{h}$

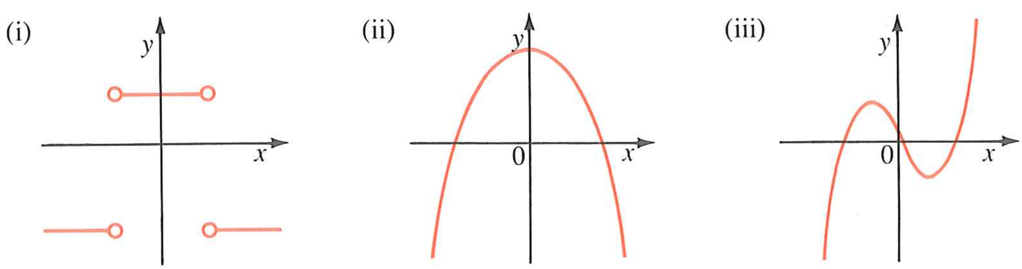
(c) $\displaystyle\lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h}$

(d) $\displaystyle\lim_{h\to 0} \frac{[(1+h)^4 + 3(1+h)] - 4}{h}$

(e) $\displaystyle\lim_{h\to 0} \frac{2^{1+h} - 2}{h}$ (f) $\displaystyle\lim_{x\to 1} \frac{x^5 - 1}{x - 1}$

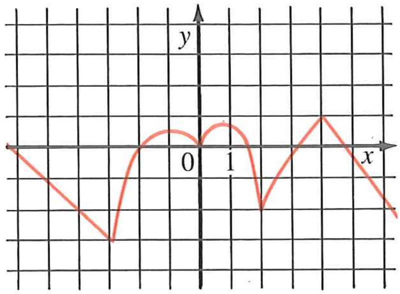
2. The graph of f is given. Match it with the graph of its derivative.



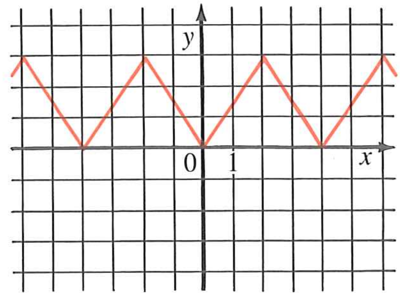


3. At what values of x are the functions not differentiable?

(a)



(b)



B 4. If $f(x) = x^2 + 7x$, find $f'(3)$.

5. If $g(x) = 15 - 3x^2$, find $g'(-1)$.

6. If $f(x) = \frac{1}{x}$, find $f'(3)$ and use it to find the equation of the tangent to the curve $y = \frac{1}{x}$ at the point $\left(3, \frac{1}{3}\right)$.

7. If $f(x) = x^3$, find $f'(a)$ and use it to find the slopes of the tangent lines to the cubic curve $y = x^3$ at the points $(-1, -1)$, $(0, 0)$, $(1, 1)$ and $(2, 8)$. Illustrate by sketching the curve and these tangents.

8. Find $f'(a)$ for each of the following functions.

(a) $f(x) = 7x - x^2$ (b) $f(x) = 2x^3 + 5$

(c) $f(x) = \dfrac{1 + 2x}{1 + x}$ (d) $f(x) = \sqrt{x}$

9. The position function of a particle moving along a line is given by $s = f(t) = 5t^2 - 2t + 6$, where t is measured in seconds and s in metres. Find $f'(a)$ and use it to find the velocity of the particle after 1 s, 2 s, and 3 s.

10. Find the derivative $f'(x)$ of each function.

(a) $f(x) = 3x^2 + 2x - 4$ (b) $f(x) = x^2 - x^3$

(c) $f(x) = x^4$ (d) $f(x) = \dfrac{x}{5x - 1}$

11. Find the derivative of each function. Find the domains of both the function and its derivative.

(a) $f(x) = \sqrt{2x - 1}$ (b) $g(x) = \dfrac{1}{\sqrt{x}}$

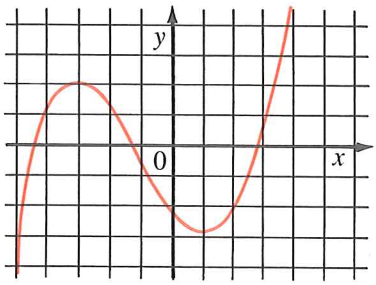
(c) $F(x) = \dfrac{3 - 2x}{4 + x}$ (d) $f(t) = \dfrac{2}{t^2 - 1}$

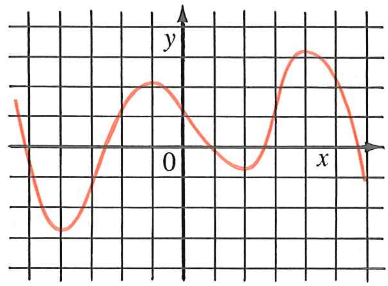
12. Find the derivative $\dfrac{dy}{dx}$.

(a) $y = 7 - 3x$ (b) $y = 3x^3 + 2x$

(c) $y = x + \dfrac{1}{x}$ (d) $y = \dfrac{1}{x^2}$

13. Use the given graph of f to sketch the graph of f'.

(a)


(b)


C **14.** (a) Sketch the graph of the cube root function $f(x) = \sqrt[3]{x}$.
(b) Show that f is not differentiable at 0.
(c) If $a \neq 0$, find $f'(a)$.

15. (a) Show that the function $f(x) = x^{\frac{2}{3}}$ is not differentiable at 0.
(b) Sketch the curve $y = x^{\frac{2}{3}}$.

16. A function f is defined by the following conditions:

$$f(x) = |x| \text{ if } -1 \leq x \leq 1$$
$$f(x + 2) = f(x) \text{ for all values of } x$$

(a) Sketch the graph of f.
(b) For what values of x is f not differentiable?

2.2 THE POWER RULE

It would be time-consuming and tedious if we always had to compute derivatives directly from the definition of a derivative, as we did in the preceding section. Fortunately, there are several rules that greatly simplify the task of differentiation.

The first rule tells us how to find the derivative of a constant function.

Constant Rule

If f is a constant function, $f(x) = c$, then $f'(x) = 0$.

In Leibniz notation: $\dfrac{d}{dx}(c) = 0$

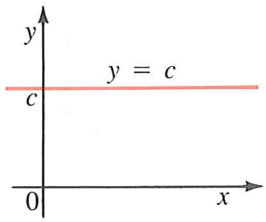
Proof

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= \lim_{h \to 0} 0$$

$$= 0$$



The figure illustrates the constant rule geometrically. The graph of a constant function $f(x) = c$ is the horizontal line $y = c$; the tangent line at any point on this line is the line itself. Since a horizontal line has slope 0, the slope of the tangent line is 0.

Example 1 (a) If $f(x) = 7$, then $f'(x) = 0$.
(b) If $y = \pi$, then $y' = 0$.
(c) $\dfrac{d}{dx}(-4.5) = 0$

The next rule gives a formula for differentiating the power function $f(x) = x^n$. We have already computed derivatives of special cases of the power function. In Example 3 of Section 2.1 we showed that

$$\frac{d}{dx} x^2 = 2x$$

Then in the exercises of that section you were asked to show that

$$\frac{d}{dx} x^3 = 3x^2 \quad \text{and} \quad \frac{d}{dx} x^4 = 4x^3$$

Therefore it seems reasonable to make the guess that $\dfrac{d}{dx}(x^n) = nx^{n-1}$. In fact, this is true and is called the Power Rule.

Power Rule

If $f(x) = x^n$, where n is a positive integer, then

$$f'(x) = nx^{n-1}$$

In Leibniz notation: $$\frac{d}{dx} x^n = nx^{n-1}$$

Proof

We use the formula

$$x^n - a^n = (x - a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})$$

which can be verified by multiplying out the right side and cancelling all but two of the terms. (See the Appendix.) Thus we have

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \frac{x^n - a^n}{x - a} \\ &= \lim_{x \to a} \frac{(x - a)(x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1})}{x - a} \\ &= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \dots + xa^{n-2} + a^{n-1}) \\ &= a^{n-1} + a^{n-2}a + \dots + aa^{n-2} + a^{n-1} \\ &= na^{n-1} \end{aligned}$$

Replacing a by x, we have $f'(x) = nx^{n-1}$.

Another proof, using the Binomial Theorem, is given in the Appendix.

Example 2 (a) If $f(x) = x^7$, then $f'(x) = 7x^6$.

(b) If $y = x^{100}$, then $y' = 100x^{99}$.

(c) If $y = t^5$, then $\dfrac{dy}{dt} = 5t^4$.

(d) $\dfrac{d}{du}(u^9) = 9u^8$

Example 3 Find the equation of the tangent line to the curve $y = x^6$ at the point $(-2, 64)$.

Solution The curve is the graph of the function $f(x) = x^6$ and we know that the slope of the tangent line at $(-2, 64)$ is the derivative $f'(-2)$. From the Power Rule,

we have $\quad f'(x) = 6x^5$

so $\quad f'(-2) = 6(-2)^5 = -192$

Therefore the equation of the tangent line at $(-2, 64)$ is

$$y - 64 = -192(x + 2)$$

or $\quad 192x + y + 320 = 0$

Although we have proved the Power Rule when the exponent n is a positive integer, it turns out that it is true for any real number n. We will prove this fact in Chapter 8, but in the meantime we use it in the examples and exercises.

General Power Rule

If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example 4 Differentiate: (a) $f(x) = \dfrac{1}{x^3}$ (b) $y = \sqrt{x}$

Solution (a) We use a negative exponent to rewrite the function as

$$f(x) = \frac{1}{x^3} = x^{-3}$$

Then the Power Rule gives

$$f'(x) = (-3)x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$$

(b) Here we use a fractional exponent:

$$y = \sqrt{x} = x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \tfrac{1}{2}x^{\frac{1}{2}-1} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

The next rule says that *the derivative of a constant times a function is the constant times the derivative of the function.*

Constant Multiple Rule

If $g(x) = cf(x)$, then $g'(x) = cf'(x)$.

In Leibniz notation: $\dfrac{d}{dx}[cf(x)] = c\dfrac{d}{dx}f(x)$

Proof

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{cf(x+h) - cf(x)}{h} \\
&= \lim_{h\to 0} c\left[\frac{f(x+h) - f(x)}{h}\right] \\
&= c \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \qquad \text{(by Property 3 of limits)} \\
&= cf'(x)
\end{aligned}$$

Example 5 Differentiate: (a) $f(x) = 8x^3$ (b) $y = 6x^{\frac{8}{3}}$

Solution (a) $f(x) = 8x^3$

$$f'(x) = 8\frac{d}{dx}(x^3) = 8(3x^2) = 24x^2$$

(b) $y = 6x^{\frac{8}{3}}$

$$\frac{dy}{dx} = 6\frac{d}{dx}x^{\frac{8}{3}} = 6\left(\tfrac{8}{3}x^{\frac{8}{3}-1}\right) = 6\left(\tfrac{8}{3}x^{\frac{5}{3}}\right) = 16x^{\frac{5}{3}}$$

Example 6 At what points on the hyperbola $xy = 12$ is the tangent line parallel to the line $3x + y = 0$?

Solution Since $xy = 12$ can be written as $y = \dfrac{12}{x}$, we have

$$\frac{dy}{dx} = 12\frac{d}{dx}(x^{-1}) = 12(-x^{-2}) = -\frac{12}{x^2}$$

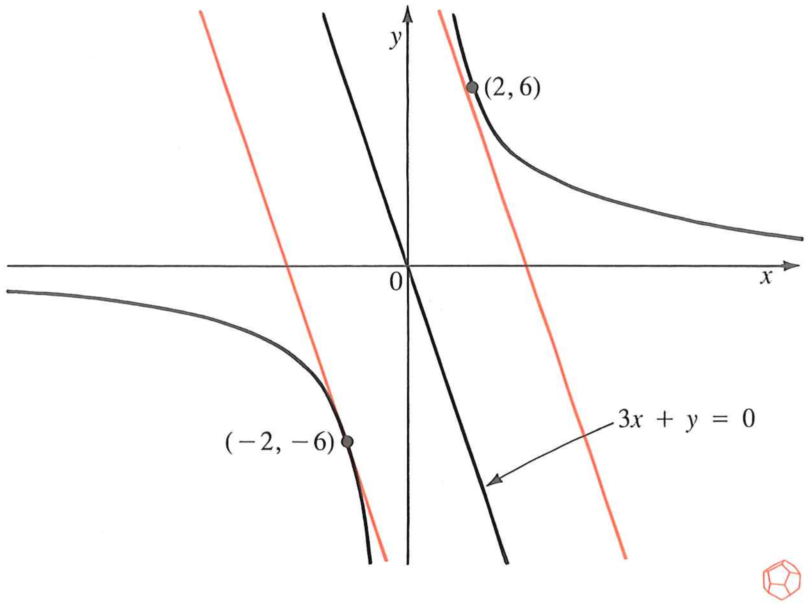
Let the x-coordinate of a required point be a. Then the slope of the tangent line at that point is

$$-\frac{12}{a^2}$$

This tangent line will be parallel to the line $3x + y = 0$, or $y = -3x$, if it has the same slope, that is, -3. Equating slopes we get

$$-\frac{12}{a^2} = -3 \quad \text{or} \quad a^2 = 4 \quad \text{or} \quad a = \pm 2$$

Therefore the required points are $(2, 6)$ and $(-2, -6)$. The hyperbola and the tangents are shown in the figure.



Example 7 A ball is dropped from the upper observation deck of the CN Tower. How fast is the ball falling after 3 s?

Solution We solved this problem in Example 1 of Section 1.5, but now we can give a simpler solution using the rules of differentiation. The distance fallen, in metres, after t seconds is

$$s = 4.9t^2$$

and we know that the derivative of this function is the velocity of the ball. Since

$$\frac{ds}{dt} = 4.9\frac{d}{dt}(t^2) = 4.9(2t) = 9.8t$$

we have

$$\left.\frac{ds}{dt}\right]_{t=3} = 9.8(3) = 29.4$$

The velocity after 3 s is 29.4 m/s.

EXERCISE 2.2

A **1.** State the derivative of each function.

(a) $f(x) = 32$ (b) $f(x) = x^4$

(c) $y = x^{12}$ (d) $y = -3.724$

(e) $f(x) = x$ (f) $f(x) = x^{\pi}$

(g) $f(x) = x^{43}$ (h) $f(x) = 2^5$

(i) $g(x) = x^{-2}$ (j) $g(x) = x^{\frac{3}{2}}$

B **2.** Differentiate.

(a) $f(x) = 8x^{12}$ (b) $f(x) = -3x^9$

(c) $f(t) = 3t^{\frac{4}{3}}$ (d) $g(t) = 8t^{-\frac{3}{4}}$

(e) $y = \dfrac{1}{x^4}$ (f) $y = \dfrac{2}{x^2}$

(g) $g(t) = (2t)^3$ (h) $h(y) = \left(\dfrac{y}{3}\right)^2$

(i) $f(x) = \sqrt[3]{x}$ (j) $f(x) = \sqrt[3]{x^2}$

(k) $y = \dfrac{1}{\sqrt{x}}$ (l) $y = \dfrac{3}{\sqrt[4]{x}}$

(m) $y = \sqrt{3}x^{\sqrt{2}}$ (n) $y = (x^3)^4$

3. Find the slope of the tangent line to the graph of the given function at the point whose x-coordinate is given.

(a) $f(x) = 2x^3$, $x = \frac{1}{3}$ (b) $f(x) = x^{1.4}$, $x = 1$

(c) $g(x) = x^{-3}$, $x = -1$ (d) $g(x) = \sqrt[5]{x}$, $x = 32$

(e) $y = \sqrt{x^3}$, $x = 8$ (f) $y = \dfrac{6}{x}$, $x = -3$

4. Find the equation of the tangent line to the curve at the given point.

(a) $y = x^5$, $(2, 32)$ (b) $y = 2\sqrt{x}$, $(9, 6)$

(c) $xy = 1$, $\left(5, \frac{1}{5}\right)$ (d) $y = \sqrt[3]{x}$, $(-8, -2)$

5. Use the definition of derivative to show that

$$\text{if } f(x) = \frac{1}{x}, \text{ then } f'(x) = -\frac{1}{x^2}$$

(This proves the Power Rule for the case $n = -1$.)

6. Use the definition of derivative to show that

$$\text{if } f(x) = \sqrt{x}, \text{ then } f'(x) = \frac{1}{2\sqrt{x}}$$

(This proves the Power Rule for the case $n = \frac{1}{2}$.)

7. At what point on the parabola $y = 3x^2$ is the slope of the tangent line equal to 24?

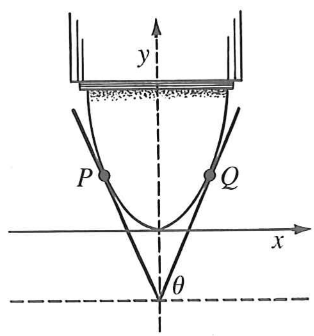
8. Find the point on the curve $y = x\sqrt{x}$ where the tangent line is parallel to the line $6x - y = 4$.

9. At what point on the curve $y = -2x^4$ is the tangent line perpendicular to the line $x - y + 1 = 0$?

10. Find the points on the curve $y = 1 - \dfrac{1}{x}$ where the tangent line is perpendicular to the line $y = 1 - 4x$.

C 11. Draw a diagram to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -5)$. Find the coordinates of the points where these tangent lines meet the parabola.

12. A manufacturer of cartridges for stereo systems has designed a stylus with a parabolic cross-section as shown in the figure. The equation of the parabola is $y = 8x^2$, where x and y are measured in millimetres. If the stylus sits in a record groove whose sides make an angle of θ with the horizontal direction, where $\tan \theta = 2.5$, find the points of contact P and Q of the stylus with the groove.



2.3 THE SUM AND DIFFERENCE RULES

The Sum Rule gives a simple rule for finding the derivative of a sum of two functions if we know the derivatives of the two functions. It says that *the derivative of a sum is the sum of the derivatives.*

Sum Rule

If both f and g are differentiable, then so is $f + g$ and

$$(f + g)' = f' + g'$$

In Leibniz notation:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof

Let $F = f + g$, that is, $F(x) = (f + g)(x) = f(x) + g(x)$. Then

$$\begin{aligned}
F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\
&= \lim_{h\to 0} \frac{[f(x+h) + g(x+h)] - [f(x) + g(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h}\right] && \text{(by rearranging terms)} \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} && \text{(by Property 1 of limits)} \\
&= f'(x) + g'(x)
\end{aligned}$$

This shows that $(f + g)' = f' + g'$.

The Sum Rule can be extended to a sum of any number of functions. For instance, using the rule twice we get

$$\begin{aligned}
(f + g + h)' &= [(f + g) + h]' \\
&= (f + g)' + h' \\
&= f' + g' + h'
\end{aligned}$$

The corresponding rule for differences says that *the derivative of a difference is the difference of the derivatives*. It is proved in a similar way.

Difference Rule

$$(f - g)' = f' - g'$$

Example 1 Find the derivatives of the following functions.

(a) $f(x) = 2x^4 + \sqrt{x}$ (b) $g(x) = 6x^4 - 5x^3 - 2x + 17$

Solution We combine the sum and difference rules with the power rule and the constant multiple rule.

$$\begin{aligned}
\text{(a)}\quad f'(x) &= \frac{d}{dx}(2x^4 + \sqrt{x}) \\
&= 2\frac{d}{dx}x^4 + \frac{d}{dx}x^{\frac{1}{2}} \\
&= 2(4x^3) + \tfrac{1}{2}x^{-\frac{1}{2}} \\
&= 8x^3 + \frac{1}{2\sqrt{x}}
\end{aligned}$$

$$\begin{aligned}\text{(b)}\quad \frac{d}{dx}g(x) &= \frac{d}{dx}(6x^4 - 5x^3 - 2x + 17)\\ &= 6\frac{d}{dx}x^4 - 5\frac{d}{dx}x^3 - 2\frac{d}{dx}x + \frac{d}{dx}17\\ &= 6(4x^3) - 5(3x^2) - 2(1) + 0\\ &= 24x^3 - 15x^2 - 2\end{aligned}$$

With practice, it is possible to use these rules mentally and simply write down the answer.

Example 2 Differentiate $y = \left(x - \dfrac{2}{\sqrt{x}}\right)^2$.

Solution We first simplify the function.

$$\begin{aligned} y &= \left(x - \frac{2}{\sqrt{x}}\right)^2\\ &= x^2 - 2x\left(\frac{2}{\sqrt{x}}\right) + \left(\frac{2}{\sqrt{x}}\right)^2\\ &= x^2 - 4\sqrt{x} + \frac{4}{x}\\ &= x^2 - 4x^{\frac{1}{2}} + 4x^{-1}\end{aligned}$$

Now it is easy to differentiate each term.

$$\begin{aligned} y' &= 2x - 4\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right) + 4(-1)x^{-2}\\ &= 2x - \frac{2}{\sqrt{x}} - \frac{4}{x^2}\end{aligned}$$

Example 3 Find the equations of both lines that pass through the point $P(2, 9)$ and are tangent to the parabola $y = 2x - x^2$. Sketch the parabola and the tangents.

Solution We are not given the coordinates of the points where the tangents touch the parabola. So we let the x-coordinate of such a point be a. Then the point is $Q(a, 2a - a^2)$. We determine the values of a by expressing the slope of the tangent line PQ in two ways. Using the formula for slope, we have

$$m_{PQ} = \frac{2a - a^2 - 9}{a - 2}$$

But, on the other hand, we know that the slope of the tangent at Q is $f'(a)$, where $f(x) = 2x - x^2$. The differentiation rules give

$$f'(x) = 2 - 2x$$

so the equation $m_{PQ} = f'(a)$ becomes

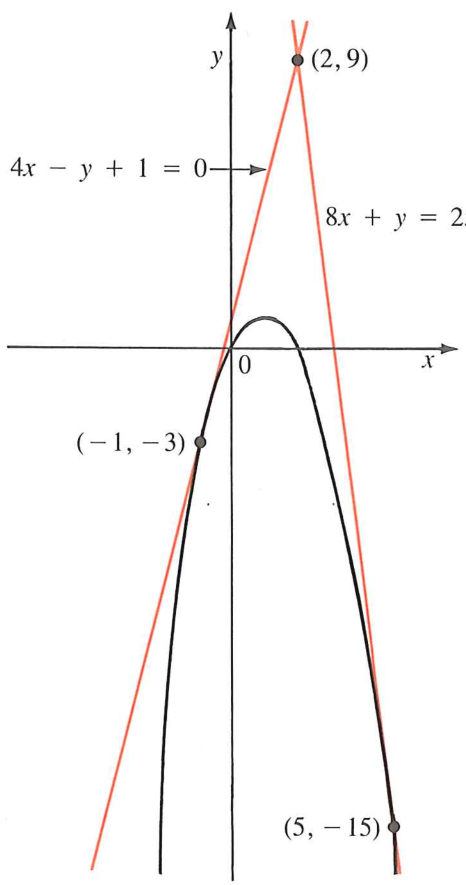
$$\begin{aligned}
\frac{2a - a^2 - 9}{a - 2} &= 2 - 2a \\
2a - a^2 - 9 &= (2 - 2a)(a - 2) \\
2a - a^2 - 9 &= -2a^2 + 6a - 4 \\
a^2 - 4a - 5 &= 0 \\
(a - 5)(a + 1) &= 0 \\
a &= 5, -1
\end{aligned}$$

Also
$$\begin{aligned}
f(5) &= 2(5) - 5^2 = -15 \\
f(-1) &= 2(-1) - (-1)^2 = -3
\end{aligned}$$

The points of contact are $(5, -15)$ and $(-1, -3)$ and the slopes of the tangents at these points are $f'(5) = -8$ and $f'(-1) = 4$. The equations of the tangents at these points are

$$y + 15 = -8(x - 5) \qquad \text{and} \qquad y + 3 = 4(x + 1)$$

or
$$8x + y - 25 = 0 \qquad \text{and} \qquad 4x - y + 1 = 0$$



EXERCISE 2.3

B **1.** Differentiate the following functions.

(a) $f(x) = x^2 + 4x$

(b) $f(x) = 3x^5 - 6x^4 + 2$

(c) $g(x) = x^{10} + 25x^5 - 50$

(d) $g(x) = x^2 - \dfrac{2}{x^2}$

(e) $h(x) = \sqrt{x} - 5x^4$

(f) $h(x) = (x - 1)(x + 6)$

(g) $y = \dfrac{x + 1}{\sqrt{x}}$

(h) $y = t^5 - 6t^{-5}$

(i) $f(t) = (1 + t)^3$

(j) $F(x) = \sqrt{x} + \sqrt[3]{x} + \sqrt[4]{x}$

(k) $u(t) = a + \dfrac{b}{t} + \dfrac{c}{t^2}$

(l) $v(r) = \sqrt{r}(2 + 3r)$

2. Find $f'(x)$ and state the domains of f and f'.

(a) $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4$

(b) $f(x) = 4x - \sqrt[4]{x}$

(c) $f(x) = x + \dfrac{\sqrt{10}}{x^5}$

(d) $f(x) = \sqrt{x} + \dfrac{2}{\sqrt{x}}$

3. Find the equation of the tangent line to the curve at the given point.

(a) $y = x^3 - x^2 + x - 1$, $(1, 0)$

(b) $y = 7\sqrt{x} - 3x$, $(1, 4)$

(c) $y = x + \dfrac{6}{x}$, $(2, 5)$

(d) $y = (x^2 + 1)^2$, $(-1, 4)$

4. If a ball is thrown upward with a velocity of 40 m/s, its height in metres after t seconds is

$$h = 40t - 5t^2$$

Find the velocity of the ball after 2 s, 4 s, and 5 s.

5. The displacement in metres of a particle moving in a straight line is given by $s = 8t^2 - 5t + 6$, where t is measured in seconds. Find the velocity of the particle after 1 s, 2 s, and 5 s.

6. At what point on the curve $y = x^4 - 25x + 2$ is the tangent parallel to the line $7x - y = 2$?

7. At what points does the curve $y = x^3 + 3x^2 - 24x + 1$ have a horizontal tangent?

8. Show that the curve $y = 10x^3 + 4x + 2$ has no tangent lines with slope 3.

9. Find the equations of both lines that pass through the origin and are tangent to the parabola $y = 1 + x^2$.

10. Find the equations of the tangent lines to the parabola $y = x^2 + x$ that pass through the point $(2, -3)$. Sketch the curve and the tangents.

11. Find the x-coordinates of the points on the hyperbola $xy = 1$ where the tangents from the point $(1, -1)$ intersect the curve.

C **12.** Let

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leqslant x \leqslant 1 \\ 3 - 2x & \text{if } x > 1 \end{cases}$$

(a) Where is f differentiable?
(b) Find an expression for f' and sketch the graphs of f and f'.

13. (a) Sketch the graph of $f(x) = |x^2 - 4|$.
(b) For what values of x is f not differentiable?
(c) Find a formula for f' and sketch its graph.

PROBLEMS PLUS

Suppose that the tangent line at a point P on the curve $y = x^3$ intersects the curve again at a point Q. Show that the slope of the tangent at Q is four times the slope of the tangent at P.

2.4 THE PRODUCT RULE

In this section we develop a formula for the derivative of a product of two functions. It is tempting to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let

$$f(x) = x \qquad\qquad g(x) = x^2$$

Then $\quad f'(x) = 1 \qquad\qquad g'(x) = 2x$

so $\quad f'(x)g'(x) = 2x$

But $(fg)(x) = f(x)g(x) = x(x^2) = x^3$, so

$$(fg)'(x) = 3x^2$$

Thus, in general,

$$(fg)' \neq f'g'$$

The correct formula is called the Product Rule and was discovered by Leibniz (soon after his false start).

Product Rule

If both f and g are differentiable, then so is fg and

$$(fg)' = fg' + f'g$$

In Leibniz notation: $\dfrac{d}{dx}[f(x)g(x)] = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Proof

Let $F = fg$, that is, $F(x) = (fg)(x) = f(x)g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

To evaluate this limit we would like to separate the functions f and g, as in the proof of the Sum Rule. To achieve this separation, we add and subtract the term $f(x + h)g(x)$ in the numerator. This allows us to factor as follows:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right] \\ &= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

Notice that

$$\lim_{h\to 0} g(x) = g(x)$$

since $g(x)$ is a constant with respect to h. The reason that

$$\lim_{h\to 0} f(x+h) = f(x)$$

is that f is continuous. (Differentiable functions are continuous. See the Appendix.)

Example 1 Find $\dfrac{dy}{dx}$ if $y = (2x^3 + 5)(3x^2 - x)$.

Solution According to the Product Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= (2x^3 + 5)\frac{d}{dx}(3x^2 - x) + (3x^2 - x)\frac{d}{dx}(2x^3 + 5)\\ &= (2x^3 + 5)(6x - 1) + (3x^2 - x)(6x^2)\end{aligned}$$

If desired, this expression could be simplified as follows:

$$\begin{aligned}\frac{dy}{dx} &= 12x^4 - 2x^3 + 30x - 5 + 18x^4 - 6x^3\\ &= 30x^4 - 8x^3 + 30x - 5\end{aligned}$$

Example 2 Differentiate $f(x) = \sqrt{x}(2 - 3x)$ and simplify.

Solution

$$\begin{aligned}f'(x) &= \sqrt{x}\frac{d}{dx}(2 - 3x) + (2 - 3x)\frac{d}{dx}\sqrt{x}\\ &= \sqrt{x}\,(-3) + (2 - 3x)\left(\frac{1}{2\sqrt{x}}\right)\\ &= -3\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}\\ &= \frac{1}{\sqrt{x}} - \frac{9}{2}\sqrt{x}\end{aligned}$$

Notice that we do not actually need the Product Rule to differentiate the functions in Examples 1 and 2. We could have multiplied the factors and proceeded as in Section 2.2. (In fact this is often easier.) But we will later meet functions such as $y = x^2 2^x$ for which the Product Rule must be used.

Example 3 Find the slope of the tangent to the graph of the function $f(x) = (3x^2 + 2)(2x^3 - 1)$ at the point $(1, 5)$.

Solution The Product Rule gives

$$\begin{aligned}f'(x) &= (3x^2 + 2)\frac{d}{dx}(2x^3 - 1) + (2x^3 - 1)\frac{d}{dx}(3x^2 + 2)\\ &= (3x^2 + 2)(6x^2) + (2x^3 - 1)(6x)\end{aligned}$$

There is no need to simplify before substituting $x = 1$.

$$f'(1) = (5)(6) + (1)(6) = 36$$

The slope of the tangent line at $(1, 5)$ is 36.

EXERCISE 2.4

B **1.** Use the Product Rule to find the derivative. Do not simplify your answer.

(a) $f(x) = (2x - 1)(x^2 + 1)$ (b) $f(x) = x(3x - 8)$
(c) $y = x^2(1 + x - 3x^2)$ (d) $y = (x^3 + x^2 + 1)(x^2 + 2)$
(e) $f(t) = (t^4 + t^2 - 1)(t^2 - 2)$ (f) $f(t) = \sqrt[3]{t}(1 - t)$
(g) $F(y) = \sqrt{y}(y - 2\sqrt{y} + 2)$ (h) $G(y) = (y - y^2)(2y - y^{\frac{4}{3}})$

2. Use the Product Rule to differentiate each function. Simplify your answer.

(a) $y = x^3(x^2 + 2x + 3)$ (b) $y = x^{-2}(x^3 - 3x^2 + 6)$
(c) $f(x) = (1 - x^2)(2 - x^3)$ (d) $f(x) = (3x^3 + 4)(1 - 2x^3)$
(e) $f(t) = (6 + t^{-2})(8t^{10} - 5t^3)$ (f) $f(t) = (at + b)(ct^2 - d)$
(g) $g(u) = \sqrt{u}(2 - u^2 + 5u^4)$ (h) $g(v) = (v - \sqrt{v})(v^2 + \sqrt{v})$

3. Find the slope of the tangent to the given curve at the point whose x-coordinate is given.

(a) $y = (1 - 2x)(3x - 4)$, $x = 2$
(b) $y = (1 - x + x^2)(x - 2)$, $x = 1$
(c) $y = x^4(4x^3 + 2)$, $x = -1$
(d) $y = (1 + x - 2x^2)(3x^3 + x - 1)$, $x = 1$
(e) $y = x^{-5}(1 + x^{-1})$, $x = 1$
(f) $y = (2 - 3\sqrt{x})(4 - \sqrt{x})$, $x = 4$

4. If $f(x) = (6x^4 - 3x^2 + 1)(2 - x^3)$, find $f'(1)$ by two methods:

(a) by using the Product Rule;
(b) by expanding $f(x)$ first.

5. Find the equation of the tangent line to the curve $y = (2 - \sqrt{x})(1 + \sqrt{x} + 3x)$ at the point $(1, 5)$.

6. If $f(2) = 3$, $f'(2) = 5$, $g(2) = -1$, and $g'(2) = -4$, find $(fg)'(2)$.

7. If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $g(x) = xf(x)$ (b) $h(x) = \sqrt{x}f(x)$ (c) $F(x) = x^c f(x)$

8. (a) Use the Product Rule with $g = f$ to show that if f is differentiable, then

$$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$$

(b) Use part (a) to differentiate $y = (2 + 5x - x^3)^2$.

C **9.** (a) Use the Product Rule twice to show that if f, g, and h are differentiable, then

$$(fgh)' = f'gh + fg'h + fgh'$$

(b) Use part (a) to differentiate $y = \sqrt{x}(3x + 5)(6x^2 - 5x + 1)$.

10. (a) Taking $f = g = h$ in Question 9, show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(b) Use part (a) to differentiate $y = (1 + x^3 + x^6)^3$.

11. Use the Principle of Mathematical Induction and the Product Rule to prove the Power Rule

$$\frac{d}{dx} x^n = nx^{n-1}$$

when n is a positive integer.

2.5 THE QUOTIENT RULE

In this section we present a formula for the derivative of a quotient of two functions. In particular, this will enable us to differentiate any rational function (ratio of two polynomials) such as

$$F(x) = \frac{x^2 + 2x - 3}{x^3 + 1}$$

Quotient Rule

If both f and g are differentiable, then so is the quotient

$$F(x) = \frac{f(x)}{g(x)} \text{ and}$$

$$F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The Quotient Rule can be proved by a method similar to the one used in proving the Product Rule. But if we make the assumption that F is differentiable, then we can use the following simpler method.

Proof Since

$$F(x) = \frac{f(x)}{g(x)}$$

we have

$$f(x) = F(x)g(x)$$

So, by the Product Rule,

$$f'(x) = F(x)g'(x) + F'(x)g(x)$$

Now we solve for $F'(x)$:

$$
\begin{aligned}
F'(x)g(x) &= f'(x) - F(x)g'(x) \\
&= f'(x) - \frac{f(x)}{g(x)} g'(x) \\
F'(x) &= \frac{f'(x) - \dfrac{f(x)}{g(x)} g'(x)}{g(x)} \\
&= \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

In Leibniz notation, the Quotient Rule can be written as follows:

$$
\boxed{\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\dfrac{d}{dx}f(x) - f(x)\dfrac{d}{dx}g(x)}{[g(x)]^2}}
$$

We must be careful to remember the order of the terms in this formula because of the minus sign in the numerator. In words, the Quotient Rule says that *the derivative of a quotient is the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, all divided by the square of the denominator.*

Example 1 Differentiate $F(x) = \dfrac{x^2 + 2x - 3}{x^3 + 1}$.

Solution By the Quotient Rule, we have

$$
\begin{aligned}
F'(x) &= \frac{(x^3 + 1)\dfrac{d}{dx}(x^2 + 2x - 3) - (x^2 + 2x - 3)\dfrac{d}{dx}(x^3 + 1)}{(x^3 + 1)^2} \\
&= \frac{(x^3 + 1)(2x + 2) - (x^2 + 2x - 3)(3x^2)}{(x^3 + 1)^2} \\
&= \frac{(2x^4 + 2x^3 + 2x + 2) - (3x^4 + 6x^3 - 9x^2)}{(x^3 + 1)^2} \\
&= \frac{-x^4 - 4x^3 + 9x^2 + 2x + 2}{(x^3 + 1)^2}
\end{aligned}
$$

After using the Quotient Rule, it is usually worthwhile to simplify the resulting expression.

Example 2 Find $\dfrac{dy}{dx}$ if $y = \dfrac{\sqrt{x}}{1 + 2x}$.

Solution

$$\frac{dy}{dx} = \frac{(1 + 2x)\dfrac{d}{dx}\sqrt{x} - \sqrt{x}\dfrac{d}{dx}(1 + 2x)}{(1 + 2x)^2}$$

$$= \frac{(1 + 2x)\dfrac{1}{2\sqrt{x}} - \sqrt{x}(2)}{(1 + 2x)^2}$$

Now we multiply the numerator and denominator by $2\sqrt{x}$:

$$\frac{dy}{dx} = \frac{1 + 2x - (2\sqrt{x})(2\sqrt{x})}{2\sqrt{x}(1 + 2x)^2} = \frac{1 - 2x}{2\sqrt{x}(1 + 2x)^2}$$

EXERCISE 2.5

B **1.** Differentiate.

(a) $f(x) = \dfrac{x - 1}{x + 1}$

(b) $f(x) = \dfrac{2x - 1}{x^2 + 1}$

(c) $g(x) = \dfrac{x}{x^2 + 2x - 1}$

(d) $g(x) = \dfrac{x^3 - 1}{x^2 + x + 1}$

(e) $y = \dfrac{\sqrt{x}}{x^2 + 1}$

(f) $y = \dfrac{\sqrt{x} + 2}{\sqrt{x} - 2}$

(g) $f(t) = \dfrac{2t + 1}{t^2 - 3t + 4}$

(h) $g(t) = \dfrac{2t^2 + 3t + 1}{t - 1}$

(i) $f(x) = \dfrac{1}{x^4 - x^2 + 1}$

(j) $f(x) = \dfrac{ax + b}{cx + d}$

(k) $f(x) = \dfrac{x^6}{x^5 - 10}$

(l) $f(x) = \dfrac{1 - \dfrac{1}{x}}{x + 1}$

2. Find the domain of f and compute its derivative.

(a) $f(x) = \dfrac{2 + x}{1 - 2x}$

(b) $f(x) = \dfrac{x}{x^2 - 1}$

(c) $f(x) = \dfrac{1}{(x + 1)(2x - 3)}$

(d) $f(x) = \dfrac{2x + 1}{x^2 + 2x - 3}$

(e) $f(x) = \dfrac{x^2 + 2x}{x^4 - 1}$

(f) $f(x) = \dfrac{x^2}{\sqrt{x} - 3}$

3. Find an equation of the tangent line to the curve at the given point.

(a) $y = \dfrac{x}{x-2}$, $(4, 2)$ (b) $y = \dfrac{1+3x}{2-3x}$, $(1, -4)$

(c) $y = \dfrac{1}{x^2+1}$, $\left(-2, \frac{1}{5}\right)$ (d) $y = \dfrac{x^3-1}{1+2x^2}$, $(1, 0)$

4. If $f(2) = 3$, $f'(2) = 5$, $g(2) = -1$, and $g'(2) = -4$, find $\left(\dfrac{f}{g}\right)'(2)$.

5. Show that there are no tangents to the curve $y = \dfrac{x+2}{3x+4}$ with positive slope.

6. At what points on the curve $y = \dfrac{x^2}{2x+5}$ is the tangent line horizontal?

7. Find the points on the curve $y = \dfrac{x}{x-1}$ where the tangent line is parallel to the line $x + 4y = 1$.

8. If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \dfrac{f(x)}{x}$ (c) $y = \dfrac{x}{f(x)}$

C **9.** In Section 2.2 we proved the Power Rule for positive integer exponents. Use the Quotient Rule to deduce the Power Rule for the case of negative integer exponents; that is, prove that

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

when n is a positive integer.

2.6 THE CHAIN RULE

Although we have learned to differentiate a variety of functions, our differentiation rules still do not enable us to find the derivative of the function

$$F(x) = \sqrt{2x^2 + 3}$$

Notice that F is a composite function; it can be built up from simpler functions. If we let

$$y = f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = 2x^2 + 3$$

then $\quad f(g(x)) = f(2x^2 + 3) = \sqrt{2x^2 + 3} = F(x)$

that is, $F = f \circ g$. The Chain Rule tells us how to compute the derivative of a composite function $F = f \circ g$ in terms of the derivatives of f and g.

If we interpret derivatives as rates of change, then we can guess what the rule says. Regard $\frac{du}{dx}$ as the rate of change of u with respect to x, $\frac{dy}{du}$ as the rate of change of y with respect to u, and $\frac{dy}{dx}$ as the rate of change of y with respect to x. If u changes twice as fast as x and y changes three times as fast as u, then it seems reasonable that y changes six times as fast as x, and so we expect that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

The Chain Rule

If the derivatives $g'(x)$ and $f'(g(x))$ both exist and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then $F'(x)$ exists and is given by the product $F'(x) = f'(g(x))g'(x)$; that is

$$\frac{d}{dx} f(g(x)) = f'(g(x))\, g'(x) \tag{1}$$

Thus the Chain Rule says that we differentiate a composite function $f(g(x))$ by working from the outside to the inside. We first differentiate the outer function f, but we evaluate it at the inner function $g(x)$. Then we multiply by the derivative of the inner function g.

The Chain Rule in Leibniz Notation

If $y = f(u)$, where $u = g(x)$, and f and g are differentiable, then y is a differentiable function of x and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{2}$$

Equation 2 is easy to remember because if $\frac{dy}{du}$ and $\frac{du}{dx}$ were quotients, then we could cancel the du's. Remember, however, that du has not been defined and $\frac{du}{dx}$ should not be thought of as an actual quotient.

To indicate why the Chain Rule is true we use increment notation. If x changes by an amount Δx, then u changes by an amount

$$\Delta u = g(x + \Delta x) - g(x)$$

and the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u)$$

If we assume that $\Delta u \neq 0$ whenever Δx is small and $\Delta x \neq 0$, then we can write

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} && \text{(definition of derivative)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} && \text{(multiply and divide by } \Delta u\text{)} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} && \text{(Property 4 of limits)} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} && (\Delta u \to 0 \text{ as } \Delta x \to 0 \text{ since } g \text{ is continuous}) \\
&= \frac{dy}{du}\frac{du}{dx}
\end{aligned}$$

Our assumption about Δu is true for most functions g, but there are some functions for which it is false. A proof of the Chain Rule that is valid for all differentiable functions is given in more advanced courses.

Example 1 Find $F'(x)$ if $F(x) = \sqrt{2x^2 + 3}$.

Solution 1 At the beginning of this section we expressed F as $F(x) = f(g(x))$, where the outer function is $f(u) = \sqrt{u}$ and the inner function is $g(x) = 2x^2 + 3$. Since

$$f'(u) = \tfrac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}} \quad \text{and} \quad g'(x) = 4x$$

Equation 1 gives

$$\begin{aligned}
F'(x) &= f'(g(x))g'(x) \\
&= \frac{1}{2\sqrt{2x^2 + 3}}(4x) \\
&= \frac{2x}{\sqrt{2x^2 + 3}}
\end{aligned}$$

Solution 2 If $u = 2x^2 + 3$ and $y = \sqrt{u}$, then Equation 2 gives

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\
&= \frac{1}{2\sqrt{u}}(4x) \\
&= \frac{1}{2\sqrt{2x^2 + 3}}(4x) \\
&= \frac{2x}{\sqrt{2x^2 + 3}}
\end{aligned}$$

Example 2 If $y = u^{10} + u^5 + 2$, where $u = 1 - 3x^2$, find $\left.\dfrac{dy}{dx}\right]_{x=1}$.

Solution Using the Chain Rule with Leibniz notation, we have

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (10u^9 + 5u^4)(-6x)$$

It is not necessary to write this expression entirely in terms of x. We note that when $x = 1$ we have $u = 1 - 3(1)^2 = -2$ and so

$$\left.\frac{dy}{dx}\right]_{x=1} = [10(-2)^9 + 5(-2)^4][(-6)(1)] = (-5040)(-6) = 30\ 240$$

An important special case of the Chain Rule occurs when the outer function f is a power function. Suppose that $y = f(u) = u^n$, where $u = g(x)$. If we use the Power Rule and then the Chain Rule, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}\,g'(x)$$

Power Rule Combined with Chain Rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

or

$$\frac{d}{dx}\,[g(x)]^n = n[g(x)]^{n-1}\,g'(x)$$

Special cases of this rule ($n = 2$ and 3) were developed in Exercise 2.4 using the Product Rule.

Example 3 If $y = (x^2 - x + 2)^8$, find $\dfrac{dy}{dx}$.

Solution Taking $u = g(x) = x^2 - x + 2$ and $n = 8$, we have

$$\begin{aligned}\frac{dy}{dx} &= 8(x^2 - x + 2)^7\,\frac{d}{dx}\,(x^2 - x + 2)\\ &= 8(x^2 - x + 2)^7(2x - 1)\end{aligned}$$

Example 4 Find $f'(x)$ if $f(x) = \dfrac{1}{\sqrt[3]{1 - x^4}}$.

Solution First we write the function in the form

$$f(x) = (1 - x^4)^{-\frac{1}{3}}$$

Then we have

$$\begin{aligned} f'(x) &= -\tfrac{1}{3}(1 - x^4)^{-\frac{4}{3}} \frac{d}{dx}(1 - x^4) \\ &= -\frac{1}{3(1 - x^4)^{\frac{4}{3}}}(-4x^3) \\ &= \frac{4x^3}{3(1 - x^4)^{\frac{4}{3}}} \end{aligned}$$

Example 5 Differentiate $s = \left(\dfrac{2t - 1}{t + 2}\right)^6$.

Solution Here we combine the Power Rule, Chain Rule, and Quotient Rule:

$$\begin{aligned} \frac{ds}{dt} &= 6\left(\frac{2t - 1}{t + 2}\right)^5 \frac{d}{dt}\left(\frac{2t - 1}{t + 2}\right) \\ &= 6\left(\frac{2t - 1}{t + 2}\right)^5 \frac{(t + 2)(2) - (2t - 1)(1)}{(t + 2)^2} \\ &= 6\left(\frac{2t - 1}{t + 2}\right)^5 \frac{5}{(t + 2)^2} \\ &= \frac{30(2t - 1)^5}{(t + 2)^7} \end{aligned}$$

Example 6 Find the derivative of the function $f(x) = (x^2 + 1)^3(2 - 3x)^4$.

Solution We use the Product Rule before using the Chain Rule:

$$\begin{aligned} f'(x) &= (x^2 + 1)^3 \frac{d}{dx}(2 - 3x)^4 + (2 - 3x)^4 \frac{d}{dx}(x^2 + 1)^3 \\ &= (x^2 + 1)^3(4)(2 - 3x)^3(-3) + (2 - 3x)^4(3)(x^2 + 1)^2(2x) \\ &= -12(x^2 + 1)^3(2 - 3x)^3 + 6x(2 - 3x)^4(x^2 + 1)^2 \end{aligned}$$

We will see in Chapter 4 that, for some purposes, it is useful to solve an equation of the form $f'(x) = 0$ and this is made easier by writing $f'(x)$ in factored form. For this reason it is usually preferable to simplify the derivative using common factors as follows.

$$\begin{aligned} f'(x) &= -6(x^2 + 1)^2(2 - 3x)^3[2(x^2 + 1) - x(2 - 3x)] \\ &= -6(x^2 + 1)^2(2 - 3x)^3(5x^2 - 2x + 2) \end{aligned}$$

Example 7 If h is a differentiable function find the derivatives of the following functions.

(a) $F(x) = [h(x)]^3$ (b) $G(x) = h(x^3)$

Solution (a) Here h is the inner function and the outer function is $y = u^3$, so, by the Chain Rule,

$$F'(x) = \frac{d}{dx}[h(x)]^3 = 3[h(x)]^2 h'(x)$$

(b) Here h is the outer function and the inner function is $y = x^3$, so the Chain Rule gives

$$G'(x) = \frac{d}{dx}h(x^3) = h'(x^3)\frac{d}{dx}(x^3) = 3x^2h'(x^3)$$

In the next example the Chain Rule is used twice.

Example 8 Find y' if $y = \sqrt{x + \sqrt{x^2 + 1}}$.

Solution

$$y = [x + (x^2 + 1)^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\begin{aligned} y' &= \tfrac{1}{2}[x + (x^2 + 1)^{\frac{1}{2}}]^{-\frac{1}{2}}\frac{d}{dx}[x + (x^2 + 1)^{\frac{1}{2}}] \\ &= \frac{1}{2\sqrt{x + \sqrt{x^2 + 1}}}[1 + \tfrac{1}{2}(x^2 + 1)^{-\frac{1}{2}}\frac{d}{dx}(x^2 + 1)] \\ &= \frac{1}{2\sqrt{x + \sqrt{x^2 + 1}}}\left[1 + \frac{1}{2\sqrt{x^2 + 1}}(2x)\right] \\ &= \frac{1}{2\sqrt{x + \sqrt{x^2 + 1}}}\left[1 + \frac{x}{\sqrt{x^2 + 1}}\right] \end{aligned}$$

This expression could be further simplified as follows:

$$\begin{aligned} y' &= \frac{1}{2\sqrt{x + \sqrt{x^2 + 1}}}\left[\frac{\sqrt{x^2 + 1} + x}{\sqrt{x^2 + 1}}\right] \\ &= \frac{\sqrt{x + \sqrt{x^2 + 1}}}{2\sqrt{x^2 + 1}} \end{aligned}$$

Now that we discussed all the rules that are necessary to differentiate any algebraic function, we list them here for your convenience. It is wise to memorize them.

Differentiation Rules

Constant Rule:	$\frac{d}{dx}(c) = 0$
Constant Multiple Rule:	$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$
Sum Rule:	$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$
Difference Rule:	$\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$
Product Rule:	$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x)$
Quotient Rule:	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}f(x) - f(x)\frac{d}{dx}g(x)}{[g(x)]^2}$
Chain Rule:	$\frac{d}{dx}[f(g(x))] = f'(g(x))\frac{d}{dx}g(x)$
Power Rule:	$\frac{d}{dx}(x^n) = nx^{n-1}$
Power and Chain Rules:	$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}g'(x)$

PROBLEMS PLUS

Graph the function $f(x) = |x-2| + |x-5|$. Where is f discontinuous? Where is it not differentiable?

EXERCISE 2.6

B **1.** Find the derivatives of the following functions.

(a) $F(x) = (5 - 3x)^7$ (b) $F(x) = (2x^2 + 1)^{20}$

(c) $G(x) = (x^3 + x^2 - 2)^{\frac{3}{4}}$ (d) $G(x) = \sqrt{x^4 - x + 1}$

(e) $y = \sqrt[4]{x^2 + x}$ (f) $y = (1 + 3x + 4x^2)^{-3}$

(g) $y = \dfrac{1}{(x^3 + 2x^2 + 1)^2}$ (h) $y = \dfrac{4}{\sqrt{9 - x^2}}$

(i) $y = (1 + 2\sqrt{x})^6$ (j) $y = \sqrt{x + \sqrt{x}}$

(k) $y = x - \sqrt[5]{1 + x^5 - 6x^{10}}$ (l) $y = x^2 + (x^2 - 1)^5$

2. If $y = u^4 + 5u^2$, where $u = x^5 + 2x^2 + 1$, find $\frac{dy}{dx}$. Leave your answer in terms of u and x.

3. Find $\left.\dfrac{dy}{dx}\right]_{x=4}$ if $y = u^2 - 2u^5$ and $u = x - \sqrt{x}$.

4. Find $\left.\dfrac{dy}{dt}\right]_{t=1}$ if $y = \sqrt{1 + r^2}$ and $r = \dfrac{t+1}{2t+1}$.

5. Find $\left.\dfrac{ds}{dt}\right]_{t=4}$ if $s = v + \dfrac{50}{v}$ and $v = 3t - \sqrt{t}$.

6. Differentiate:

(a) $F(x) = x\sqrt{x^2 + 1}$

(b) $F(x) = (2x + 1)(4x - 1)^5$

(c) $G(x) = (x^2 - 1)^4(2 - 3x)$

(d) $G(x) = (x^4 - x + 1)^2(x^2 - 2)^3$

(e) $F(x) = \dfrac{x}{\sqrt{2x + 3}}$

(f) $f(t) = \dfrac{(1 + 2t)^5}{(3t^2 - 5)^2}$

(g) $g(x) = \left(\dfrac{x + 2}{x - 2}\right)^3$

(h) $h(t) = \left(\dfrac{t^2 + 1}{t + 1}\right)^{10}$

(i) $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$

(j) $y = \dfrac{(2x + 3)^3}{\sqrt{4x - 7}}$

(k) $y = 3\sqrt{x}(2x + 1)^5 + \sqrt{4x - 3}$

(l) $y = \sqrt{1 + \sqrt[3]{x}}$

(m) $y = (t + \sqrt[3]{t + t^2})^{20}$

(n) $y = \sqrt{x + \sqrt{x + \sqrt{x}}}$

7. Find the equation of the tangent line to the curve $y = (x^2 - 3)^8$ at the point $(2, 1)$.

8. Find the equation of the tangent line to the curve $y = \dfrac{1}{\sqrt{20 - x^4}}$ at the point $\left(2, \frac{1}{2}\right)$.

9. If $F(x) = f(g(x))$, where $g(2) = 4$, $g'(2) = 3$, and $f'(4) = 5$, find $F'(2)$.

10. If $G(x) = h(p(x))$, where $h(5) = 1$, $h'(5) = 2$, $h'(1) = 3$, $p(1) = 5$, and $p'(1) = 7$, find $G'(1)$.

C **11.** If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $F(x) = f(x^4)$

(b) $G(x) = [f(x)]^4$

(c) $H(x) = f(\sqrt{x})$

(d) $P(x) = \sqrt{f(x)}$

(e) $y = f(f(x))$

(f) $y = \sqrt{1 + [f(x)]^2}$

(g) $y = [f(x^2)]^2$

(h) $y = f([f(x)]^3)$

12. (a) Use the Chain Rule and the fact that $|x| = \sqrt{x^2}$ to show that

$$\frac{d}{dx}|x| = \frac{x}{|x|}$$

(b) Sketch the graphs of the function $f(x) = |x|$ and its derivative.

(c) Use the result of part (a) to differentiate the function $g(x) = x|x|$.

2.7 IMPLICIT DIFFERENTIATION

So far we have described functions by expressing one variable explicitly in terms of another variable; for example,

$$y = x^2 \quad \text{or} \quad y = \frac{\sqrt{4 - x^2}}{x + 1}$$

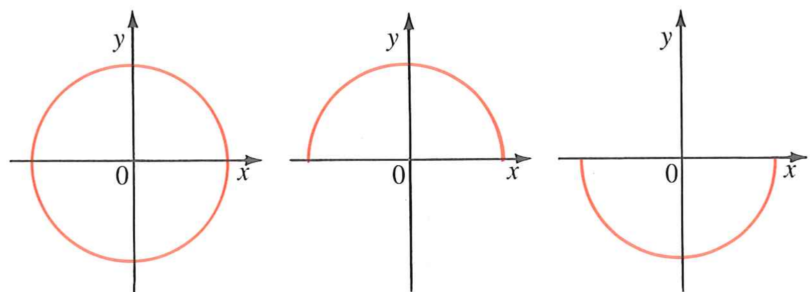
or, in general, $y = f(x)$. But other functions are defined implicitly by a relation between x and y such as

$$x^2 + y^2 = 25$$

In this case it is possible to solve the equation for y to get $y = \pm\sqrt{25 - x^2}$ and so two functions defined by the implicit equations are

$$f(x) = \sqrt{25 - x^2} \quad \text{and} \quad g(x) = -\sqrt{25 - x^2}$$

The graphs of f and g are the upper and lower semicircles of the circle $x^2 + y^2 = 25$.



$x^2 + y^2 = 25$ $\quad$ $f(x) = \sqrt{25 - x^2}$ $\quad$ $g(x) = -\sqrt{25 - x^2}$

Thus we could find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-4, 3)$ by differentiating the function $f(x) = \sqrt{25 - x^2}$ and substituting $x = -4$. An easier method, called **implicit differentiation**, is illustrated in the following example. In using this method, we differentiate both sides of the equation with respect to x and then we solve the resulting equation for y'.

Example 1 (a) If $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$.

(b) Find the equation of the tangent line to the circle $x^2 + y^2 = 25$ at the point $(-4, 3)$.

Solution (a) We differentiate both sides of the equation $x^2 + y^2 = 25$ with respect to x:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \qquad (1)$$

To differentiate y^2 we use the Chain Rule and keep in mind that y is a function of x:

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

So, from Equation 1, we have

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for the required derivative:

$$\frac{dy}{dx} = -\frac{x}{y}$$

(b) The expression for the derivative in part (a) involves both x and y, but this is not a disadvantage. At the point $(-4, 3)$ we have $x = -4$ and $y = 3$, so

$$\frac{dy}{dx} = -\frac{(-4)}{3} = \frac{4}{3}$$

This is the slope of the tangent at $(-4, 3)$, so the equation is

$$y - 3 = \tfrac{4}{3}(x + 4)$$

or $\qquad 4x - 3y + 25 = 0$

We have seen that the problem in Example 1 could be solved either by implicit differentiation or by first solving the given equation for y. In the next example, however, it is impossible to solve the equation for y as an explicit function of x. Here the method of implicit differentiation is not just the most convenient method for finding y'; it is the *only* method.

Example 2 (a) Find $\dfrac{dy}{dx}$ if $2x^5 + x^4y + y^5 = 36$.

(b) Find the slope of the tangent to the curve $2x^5 + x^4y + y^5 = 36$ at the point $(1, 2)$.

Solution (a) In differentiating the second term we have to regard y as a function of x and so we use the Product Rule:

$$\frac{d}{dx}(x^4y) = x^4\frac{d}{dx}(y) + y\frac{d}{dx}(x^4) = x^4\frac{dy}{dx} + (4x^3)y$$

In differentiating the third term we use the Chain Rule:

$$\frac{d}{dx}(y^5) = \frac{d}{dy}(y^5)\frac{dy}{dx} = 5y^4\frac{dy}{dx}$$

Thus differentiating both sides of the given equation, we have

$$10x^4 + x^4\frac{dy}{dx} + 4x^3y + 5y^4\frac{dy}{dx} = 0$$

Then, solving for $\dfrac{dy}{dx}$, we get

$$\frac{dy}{dx} = -\frac{4x^3y + 10x^4}{x^4 + 5y^4}$$

(b) When $x = 1$ and $y = 2$,

$$\frac{dy}{dx} = -\frac{4(1)^3(2) + 10(1)^4}{1^4 + 5(2)^4} = -\frac{18}{81} = -\frac{2}{9}$$

The slope of the tangent line at $(1, 2)$ is $-\frac{2}{9}$.

Example 3 Find y' if $x^2 + \sqrt{y} = x^2y^3 + 5$.

Solution Differentiate both sides with respect to x:

$$2x + \frac{d}{dy}(\sqrt{y})\frac{dy}{dx} = x^2\frac{d}{dx}(y^3) + y^3\frac{d}{dx}(x^2)$$

$$2x + \frac{1}{2\sqrt{y}}y' = x^2(3y^2)y' + 2xy^3$$

$$2x - 2xy^3 = 3x^2y^2y' - \frac{1}{2\sqrt{y}}y'$$

$$y' = \frac{2x(1 - y^3)}{3x^2y^2 - \dfrac{1}{2\sqrt{y}}}$$

EXERCISE 2.7

B **1.** Use implicit differentiation to find $\frac{dy}{dx}$.

(a) $x^2 - y^2 = 1$ (b) $x^3 + y^3 = 6$
(c) $xy = 4$ (d) $x^2 + xy + y^2 = 1$
(e) $x^3 + y^3 = 6xy$ (f) $2xy^2 - y^3 = x^2$
(g) $\sqrt{x} + \sqrt{y} = 1$ (h) $\frac{2x}{x + y} = y$

2. Find the slope of the tangent line to the curve at the given point.

(a) $x^2 + 4y^2 = 5$, $(1, -1)$ (b) $x^4 + y^4 = 17$, $(2, 1)$
(c) $x^2 + x^3y^2 - y^3 = 13$, $(1, -2)$
(d) $y^2 = 2xy - 3$, $(2, 3)$
(e) $\sqrt{x + y} + \sqrt{xy} = 4$, $(2, 2)$ (f) $\frac{1}{x} + \frac{1}{y} = 1$, $\left(\frac{3}{2}, 3\right)$

3. Find the equation of the tangent line to the curve at the given point.

(a) $2x^2 - y^2 = 1$, $(-1, -1)$ (b) $x^3 + y^3 = 9$, $(2, 1)$
(c) $y^5 + x^2y^3 = 10$, $(-3, 1)$ (d) $(x + y)^3 = x^3 + y^3$, $(-1, 1)$

4. (a) Use implicit differentiation to find the slope of the tangent line to the ellipse $9x^2 + 4y^2 = 36$ at the point $\left(\sqrt{2}, \frac{3}{2}\sqrt{2}\right)$.

(b) Find the slope in part (a) by first solving for y explicitly as a function of x.

(c) Find the equation of the tangent line.

(d) Sketch the ellipse and the tangent line.

5. (a) Find an equation of the tangent line to the circle $x^2 + y^2 + 2x - 4y - 20 = 0$ at the point $(2, -2)$.

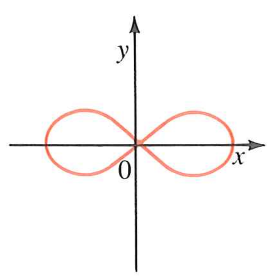
(b) Sketch the circle and the tangent line.

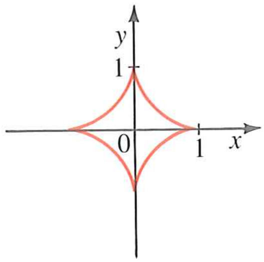
6. The curve with equation $2(x^2 + y^2)^2 = 25(x^2 - y^2)$ is called a *lemniscate* and is shown in the figure.

(a) Find y'.

(b) Find the equation of the tangent line to the lemniscate at the point $(-3, 1)$.

(c) Find the points on the lemniscate where the tangent line is horizontal.





7. The curve with equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ is called an *astroid* and is shown in the figure.

(a) Find y'.

(b) Find the equation of the tangent line to the astroid at the point $\left(\frac{1}{8}, \frac{3\sqrt{3}}{8}\right)$.

(c) Find the points on the astroid where the tangent line has slope 1.

8. Use implicit differentiation to show that an equation of the tangent line to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

C **9.** Suppose f is a function such that $x[f(x)]^3 + x^2 f(x) = 3$ and $f(2) = 1$. Find $f'(2)$.

10. Use implicit differentiation to show that any tangent line at a point P to a circle with centre C is perpendicular to the radius CP.

11. Use implicit differentiation to show that, whenever a hyperbola with equation $x^2 - y^2 = k$ intersects a hyperbola with equation $xy = c$, the tangent lines at the points of intersection are perpendicular.

2.8 HIGHER DERIVATIVES

Since the derivative of a function f is itself a function f', we can take its derivative $(f')'$. The result is a function called the **second derivative** of f and denoted by f''.

If $y = f(x)$ and we use Leibniz notation, then

$$f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

and we abbreviate this as

$$\frac{d^2y}{dx^2}$$

If we use D-notation, the symbol D^2 indicates that the operation of differentiation is performed twice. Thus we have the following notations for the second derivative:

$$y'' = f''(x) = \frac{d^2y}{dx^2} = D^2 f(x) = D_x^2 f(x)$$

Example 1 Find $\dfrac{d^2y}{dx^2}$ if $y = x^6$.

Solution

$$\frac{dy}{dx} = 6x^5$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(6x^5) = 30x^4$$

Example 2 Find the second derivative of $f(x) = 5x^2 + \sqrt{x}$.

Solution

$$f(x) = 5x^2 + x^{\frac{1}{2}}$$

$$f'(x) = 10x + \tfrac{1}{2}x^{-\frac{1}{2}}$$

$$f''(x) = 10 + \left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)x^{-\frac{3}{2}} = 10 - \tfrac{1}{4}x^{-\frac{3}{2}}$$

Example 3 Find $f''(1)$ if $f(x) = (2 - x^2)^{10}$.

Solution

$$\begin{aligned} f'(x) &= 10(2 - x^2)^9(-2x) = -20x(2 - x^2)^9 \\ f''(x) &= (-20x)9(2 - x^2)^8(-2x) - 20(2 - x^2)^9 \\ &= 360x^2(2 - x^2)^8 - 20(2 - x^2)^9 \\ f''(1) &= (360)(1)^2(1)^8 - 20(1)^9 = 340 \end{aligned}$$

Since the first derivative of a function can be interpreted either as the slope of a tangent line or as a rate of change, the second derivative can be interpreted as the rate of change of the slope of the tangent line. This idea will be pursued in Chapter 5 where the second derivative gives valuable information about the shape of the graph. Another application occurs in Chapter 3 where the second derivative of a position function represents acceleration.

Higher derivatives can also be defined. The **third derivative** is the derivative of the second derivative: $f''' = (f'')'$. Other notations are as follows:

$$y''' = f'''(x) = \frac{d^3y}{dx^3} = D^3f(x) = D_x^3f(x)$$

Beyond the third derivative we usually do not use the prime notation. For instance, the fourth derivative is denoted by $f^{(4)}$ instead of f''''. In general the ***n*th derivative** of f is denoted by $f^{(n)}$ and is obtained by differentiating n times. We write

$$y^{(n)}(x) = f^{(n)} = \frac{d^ny}{dx^n} = D^nf(x) = D_x^nf(x)$$

Example 4 Find the first five derivatives of $y = x^4 + 2x^3 - 5x^2 + 3x - 6$.

Solution

$$\begin{aligned} y' &= 4x^3 + 6x^2 - 10x + 3 \\ y'' &= 12x^2 + 12x - 10 \\ y''' &= 24x + 12 \\ y^{(4)} &= 24 \\ y^{(5)} &= 0 \end{aligned}$$

Example 5 If $x^3 + y^3 = 5$, use implicit differentiaton to find y''.

Solution Differentiating the equation with respect to x, we get

$$3x^2 + 3y^2y' = 0$$

Solving for y', we have

$$y' = -\frac{x^2}{y^2} \tag{1}$$

To find y'' we differentiate this expression using the Quotient Rule and remembering that y is a function of x:

$$\begin{aligned} y'' &= \frac{d}{dx}\left(-\frac{x^2}{y^2}\right) \\ &= -\frac{y^2 \dfrac{d}{dx}(x^2) - x^2 \dfrac{d}{dx}(y^2)}{(y^2)^2} \\ &= -\frac{y^2(2x) - x^2(2yy')}{y^4} \end{aligned}$$

Now we substitute Equation 1 into this expression and obtain

$$\begin{aligned} y'' &= -\frac{2xy^2 - 2x^2y\left(-\dfrac{x^2}{y^2}\right)}{y^4} \\ &= -\frac{2xy^2 + \dfrac{2x^4}{y}}{y^4} \\ &= -\frac{2xy^3 + 2x^4}{y^5} \qquad \text{(multiply numerator and denominator by } y\text{)} \\ &= -\frac{2x(y^3 + x^3)}{y^5} \end{aligned}$$

But the values of x and y must satisfy the original equation $x^3 + y^3 = 5$ and so the expression simplifies as follows:

$$y'' = -\frac{2x(5)}{y^5} = -\frac{10x}{y^5}$$

EXERCISE 2.8

B **1.** Find the first and second derivatives of the given functions.
(a) $f(x) = x^5 - 4x^2 + 1$
(b) $g(x) = 7x^4 + 12x^3 - 4x + 8$
(c) $f(t) = 2t - \dfrac{1}{t + 1}$ (d) $g(t) = \dfrac{4}{\sqrt{t}}$
(e) $y = (2x + 1)^8$ (f) $y = t^3 + \dfrac{1}{t^3}$
(g) $y = \sqrt{x^2 + 1}$ (h) $y = \dfrac{t}{t - 1}$

2. Find the third derivative.
(a) $f(x) = 1 - 12x + 4x^2 - x^3$ (b) $f(x) = \dfrac{1}{x^5}$
(c) $y = \dfrac{3}{(4 - x)^2}$ (d) $y = \sqrt{1 + 2x}$

3. Find the first six derivatives of the function $y = x^5 + x^4 + x^3 + x^2 + x + 1$.

4. If $f(x) = \sqrt{1 + x^3}$, find $f''(2)$.

5. If $g(x) = \dfrac{1}{\sqrt{3x + 4}}$, find $g'''(4)$.

6. If $f(x) = x^n$, find $f^{(n)}(x)$.

7. Find y'' by implicit differentiation.
(a) $x^4 + y^4 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^3 + y^3 = 6xy$

8. Find a quadratic function f such that $f(3) = 33$, $f'(3) = 22$, and $f''(3) = 8$.

C **9.** Suppose that $f(x) = g(x)h(x)$.
(a) Express f'' in terms of g, g', g'', h, h', h''.
(b) Find a similar expression for f'''.

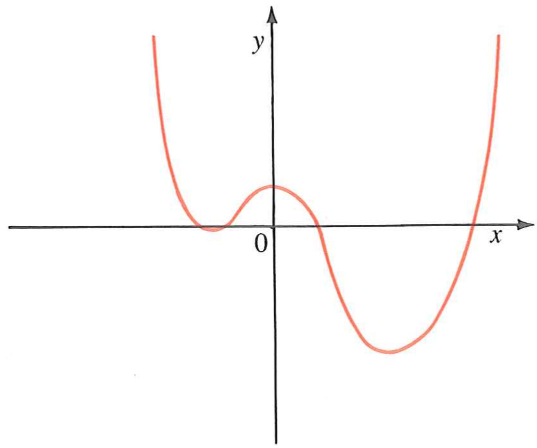
10. (a) If $f(x) = |x^2 - 1|$, find f' and f'' and state their domains.
(b) Sketch the graphs of f, f', and f''.

2.9 REVIEW EXERCISE

1. Find $f'(x)$ from first principles, that is, directly from the definition of a derivative.
   (a) $f(x) = 1 - 2x + 3x^2$
   (b) $f(x) = x^3 + 4x$
   (c) $f(x) = \dfrac{x}{1 - x}$
   (d) $f(x) = \sqrt{2x + 1}$

2. The limit
$$\lim_{h \to 0} \frac{(1 + h)^4 - 1}{h}$$
is equal to $f'(a)$ for some function f and some number a. State the value of a and give a formula for the function f.

3. Use the given graph of f to sketch the graph of f'.



4. Differentiate the given functions.
   (a) $y = 12x^3 + 8x - 1$
   (b) $y = 2x^{\pi + 1}$
   (c) $y = 2x - \dfrac{3}{x}$
   (d) $y = \sqrt[5]{x^6}$
   (e) $y = \sqrt{x}(5 - \sqrt{x})$
   (f) $y = \dfrac{x^2 - 2x}{\sqrt{x}}$
   (g) $y = \dfrac{2x - 1}{1 + 3x}$
   (h) $y = (2x^3 - 1)^7$
   (i) $f(x) = (x^2 + x)\sqrt{1 - x^2}$
   (j) $g(x) = \dfrac{3x^2 + 1}{2 - x}$
   (k) $h(x) = \dfrac{1}{\sqrt[3]{2x^4 - 1}}$
   (l) $F(x) = (x^4 + 1)^3(1 - 2x)$
   (m) $f(t) = \dfrac{t}{\sqrt{1 + 2t}}$
   (n) $g(t) = \left(\dfrac{t + 1}{t + 2}\right)^4$

(o) $R(u) = \sqrt[4]{u+1} - \dfrac{2}{u^2}$ (p) $S(v) = \sqrt{v - (v^2 - 8)^5}$

(q) $M(z) = \sqrt{\dfrac{1+z}{1+z^2}}$ (r) $F(y) = \dfrac{1}{2 + \dfrac{3}{y}}$

5. Find f' and state the domains of f and f'.

(a) $f(x) = \dfrac{2x-1}{x^2-5}$ (b) $f(x) = \sqrt{x^2 - x - 6}$

6. Find $\left[\dfrac{dy}{dx}\right]_{x=1}$ if $y = u^2 - u^3 + 2u^4$ and $u = \dfrac{x}{2x-1}$.

7. Find $\dfrac{dy}{dx}$.

(a) $x^4 + y^4 = 1$ (b) $x^2 - x^2y + y^2 = 1$

(c) $2x^2y^2 = x^3 + y^3$ (d) $y\sqrt{x-1} + x\sqrt{y-1} = xy$

8. Find y''.

(a) $y = 4x^5 - \frac{1}{2}x^4 + 3x^2$ (b) $y = \sqrt{3x+1}$

(c) $y = \dfrac{t-1}{t+1}$ (d) $x^2 + y^2 = 16$

9. Find the equation of the tangent line to the curve at the given point.

(a) $y = x^2 - 2x + 5$, $(-1, 8)$ (b) $y = \dfrac{2}{1-x}$, $(2, -2)$

(c) $y = \dfrac{1}{\sqrt{x^5}}$, $\left(2, \dfrac{1}{4\sqrt{2}}\right)$ (d) $y = x\sqrt{x^2+5}$, $(-2, -6)$

(e) $(x-1)^2 + (y+2)^2 = 25$, $(-2, 2)$

(f) $x^3 + y^3 = 9xy$, $(2, 4)$

10. If a ball is dropped from the top of the CN Tower, 550 m above the ground, then its height in metres after t seconds is $h = 550 - 5t^2$. Find the velocity of the ball after 1 s, 2 s, and 5 s.

11. Find the point on the parabola $y = 2x^2 - 3x + 6$ where the tangent line is parallel to the line $7x + y = 1$.

12. Find the points on the curve $y = \dfrac{1}{2x-1}$ where the tangent line is perpendicular to the line $x - 2y = 1$.

13. Find the equations of both lines that pass through the point $(2, -3)$ and are tangent to the parabola $y = x^2 + x$.

14. Suppose $f(3) = 4$, $f'(3) = -1$, $f'(6) = 5$, $g(3) = 6$, and $g'(3) = 2$. Find

(a) $(fg)'(3)$ (b) $\left(\dfrac{f}{g}\right)'(3)$ (c) $(f \circ g)'(3)$

15. If g is a differentiable function, find expressions for f' in terms of g'.

(a) $f(x) = x^2 g(x)$ (b) $f(x) = \dfrac{g(x)}{\sqrt{x}}$

(c) $f(x) = g\left(\dfrac{1}{x}\right)$ (d) $f(x) = \sqrt{g(\sqrt{x})}$

16. If g is a differentiable function and $f(x) = g(g(x))$, find an expression for $f''(x)$.

17. Let

$$f(x) = \begin{cases} 2x - x^2 & \text{if } x \leqslant 0 \\ 2x & \text{if } 0 < x \leqslant 1 \\ x + 1 & \text{if } x \geqslant 1 \end{cases}$$

(a) Where is f not differentiable?
(b) Find an expression for $f'(x)$ and sketch the graphs of f and f'.

PROBLEMS PLUS

Draw a diagram to show that there are two lines tangent to both of the parabolas $y = -x^2$ and $y = 4 + x^2$. Find the coordinates of the four points at which these tangents touch the parabolas.

2.10 CHAPTER 2 TEST

1. (a) Give the definition of the derivative $f'(x)$ as a limit.
   (b) Use your definition in part (a) to find the derivatives of the following functions:

   (i) $f(x) = x^2 - 7x + 4$ (ii) $f(x) = \dfrac{1}{2x + 1}$

2. Find each derivative.
   (a) $f(x) = \sqrt[3]{x^2}$
   (b) $f(x) = \dfrac{x^2 + 3}{2x - 1}$
   (c) $f(x) = (x^2 - 1)^4(2x + 1)^3$
   (d) $f(x) = (x + \sqrt{x^4 - 2x + 1})^7$

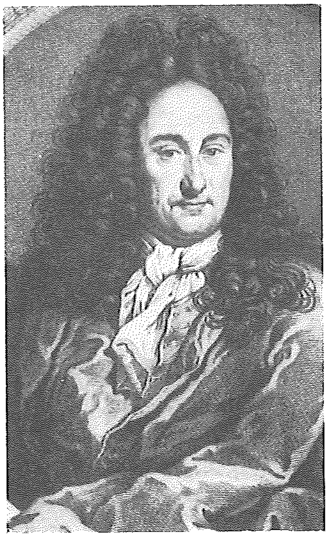
3. A curve is given by the equation $3xy = x^3 + y^3$.
   (a) Find $\dfrac{dy}{dx}$.
   (b) Find the equation of the tangent line to the curve at the point $\left(\frac{2}{3}, \frac{4}{3}\right)$.

4. Find y''' if $y = \dfrac{1}{(3 - 2x)^2}$.

5. Find the point on the curve $y = \sqrt{2x - 1}$ where the tangent line is parallel to the line $x - 3y = 16$.

6. If f is a differentiable function, find expressions for the derivatives of the following functions.
   (a) $g(x) = f(x^6)$
   (b) $h(x) = [f(x)]^6$
   (c) $F(x) = \dfrac{x^2}{f(x)}$

FOUNDERS OF CALCULUS



Gottfried Wilhelm Leibniz (1646–1716) was born in Leipzig, Germany, entered the university there at age fifteen, and earned his bachelor's degree at age seventeen. He studied logic, philosophy, mathematics, and law, and he is sometimes considered the last scholar to achieve universal knowledge. At the age of twenty he received his doctorate in law. Although he was offered a professorship, he declined and entered the diplomatic service. As a governmental representative, he travelled widely and on visits to Paris and London became interested in research in calculus.

The main contribution of Leibniz to mathematics was his development of calculus, which was published in 1684. His theory and notation were quite different from those of Newton, but they led to the same results. Today we often use the notation of Leibniz: $\frac{dy}{dx}$ for a derivative, dx for a differential, and $\int y\, dx$ for an integral. (See Chapter 11.)

It is often said that calculus was invented independently by Newton and Leibniz. However, integral calculus (the problem of areas) goes back to the ancient Greeks in about 500 B.C., and differential calculus (stemming from the problem of tangents) was started by Fermat and others in the 1630s. Newton's teacher at Cambridge, Isaac Barrow, saw the connection between the two branches of calculus. What Newton and Leibniz did was to exploit this connection and organize calculus into a systematic and powerful method.

Unfortunately, a dispute arose between the followers of Newton and the followers of Leibniz as to who had discovered the method of calculus first. Later, Newton and Leibniz themselves were drawn into the priority controversy. It seems clear now that Newton had invented the method first but Leibniz arrived at his results independently and was the first to publish the method. In spite of the controversy, Leibniz fully recognized Newton's genius. Leibniz said, "Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half."

ANSWERS

CHAPTER 2 DERIVATIVES

REVIEW AND PREVIEW TO CHAPTER 2

EXERCISE 1

1. (a) R (b) R (c) $\{x|x \geq 5\}$ (d) $\{x|x \leq 0\}$
(e) $\{x|-1 \leq x \leq 1\}$
(f) $\{x|x \geq \sqrt{2} \text{ or } x \leq -\sqrt{2}\}$
(g) $\{x|x \neq 3\}$ (h) $\{x|x \neq -5, 1\}$ (i) R
(j) $\{t|t > 3 \text{ or } t < 2\}$
(k) $\{x|0 \leq x \leq 4\}$ (l) $\{x|0 \leq x \leq 4\}$

EXERCISE 2

1. (a) $(f \circ g)(x) = 7 - 6x$, $(g \circ f)(x) = 7 - 6x$, $(f \circ f)(x) = 4x - 3$, $(g \circ g)(x) = 9x - 8$
(b) $(f \circ g)(x) = x^2 + 2x + 1$, $(g \circ f)(x) = x^2 + 1$, $(f \circ f)(x) = x^4$, $(g \circ g)(x) = x + 2$
(c) $(f \circ g)(x) = -24$, $(g \circ f)(x) = 5$, $(f \circ f)(x) = 2x^2 - x^4$, $(g \circ g)(x) = 5$
(d) $(f \circ g)(x) = \sqrt{x^2 - 4}$, $(g \circ f)(x) = x - 4$, $(f \circ f)(x) = \sqrt[4]{x}$, $(g \circ g)(x) = x^4 - 8x^2 + 12$
(e) $(f \circ g)(x) = \dfrac{3}{x} - 5$, $(g \circ f)(x) = \dfrac{1}{3x - 5}$, $(f \circ f)(x) = 9x - 20$, $(g \circ g)(x) = x$
(f) $(f \circ g)(x) = \dfrac{x + 2}{4}$, $(g \circ f)(x) = \dfrac{2x - 1}{3 - 2x}$, $(f \circ f)(x) = \dfrac{x - 1}{x}$, $(g \circ g)(x) = \dfrac{-x - 6}{3x + 2}$ (g) $(f \circ g)(x) = \sqrt[4]{1 + x}$, $(g \circ f)(x) = \sqrt{1 + \sqrt{x}}$, $(f \circ f)(x) = \sqrt[4]{x}$, $(g \circ g))(x) = \sqrt{1 + \sqrt{1 + x}}$

2. (a) $f(x = x^9$, $g(x) = 2x + 1$
(b) $f(x) = 1 + 2x + 3x^2$, $g(x) = x^2$
(c) $f(x) = \dfrac{1}{x}$, $g(x) = x^2 - 7$
(d) $f(x) = \sqrt{x}$, $g(x) = 6 + x$

EXERCISE 2.1

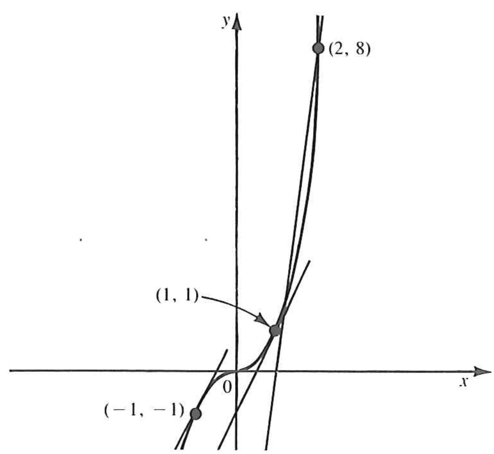
1. (a) $f(x) = x^2$, $a = 3$ [or $g(x) = (3 + x)^2$, $a = 0$] (b) $f(x) = x^3$, $a = 2$
(c) $f(x) = \sqrt{x}$, $a = 4$
(d) $f(x) = x^4 + 3x$, $a = 1$
(e) $f(x) = 2^x$, $a = 1$ (f) $f(x) = x^5$, $a = 1$

2. (a) (ii) (b) (i) (c) (iii)

3. (a) $-3, 0, 2, 4$ (b) $-4, -2, 0, 2, 4, 6$

4. 13 **5.** 6 **6.** $-\frac{1}{9}$, $x + 9y - 6 = 0$

7. $3a^2$; 3, 0, 3, 12



8. (a) $7 - 2a$ (b) $6a^2$ (c) $\dfrac{1}{(1 + a)^2}$ (d) $\dfrac{1}{2\sqrt{a}}$

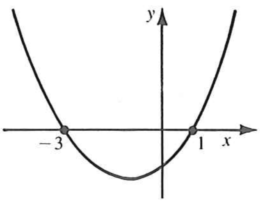
9. $10a - 2$, 8 m/s, 18 m/s, 28 m/s

10. (a) $f'(x) = 6x + 2$ (b) $f'(x) = 2x - 3x^2$
(c) $f'(x) = 4x^3$ (d) $f'(x) = -\dfrac{1}{(5x-1)^2}$

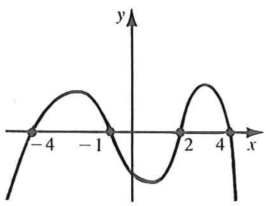
11. (a) $f'(x) = \dfrac{1}{\sqrt{2x-1}}$, $\text{dom}(f) = \{x|x \geqslant \frac{1}{2}\}$,
$\text{dom}(f') = \{x|x > \frac{1}{2}\}$
(b) $g'(x) = -\dfrac{1}{2\sqrt{x^3}}$, $\text{dom}(g) = \text{dom}(g') = \{x|x > 0\}$
(c) $F'(x) = -\dfrac{11}{(4+x)^2}$, $\text{dom}(F) = \text{dom}(F') = \{x|x \neq -4\}$
(d) $f'(t) = -\dfrac{4t}{(t^2-1)^2}$, $\text{dom}(f) = \text{dom}(f') = \{t|t \neq \pm 1\}$

12. (a) -3 (b) $9x^2 + 2$ (c) $1 - \dfrac{1}{x^2}$ (d) $-\dfrac{2}{x^3}$

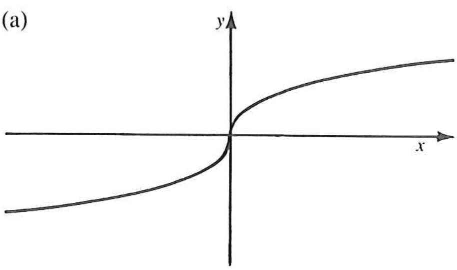
13.
(a)



(b)

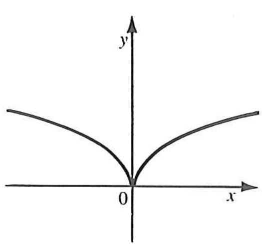


14. (a)

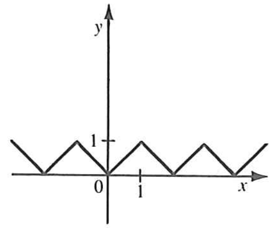


(c) $\dfrac{1}{3\sqrt[3]{a^2}}$

15. (b)



16. (a)



(b) Not differentiable at any integer

EXERCISE 2.2

1. (a) $f'(x) = 0$ (b) $f'(x) = 4x^3$ (c) $y' = 12x^{11}$
(d) $y' = 0$ (e) $f'(x) = 1$ (f) $f'(x) = \pi x^{\pi - 1}$
(g) $f'(x) = 43x^{42}$ (h) $f'(x) = 0$
(i) $g'(x) = -2x^{-3}$ (j) $g'(x) = \frac{3}{2}x^{\frac{1}{2}}$

2. (a) $f'(x) = 96x^{11}$ (b) $f'(x) = -27x^8$
(c) $f'(t) = 4t^{\frac{1}{3}}$ (d) $g'(t) = -6t^{-\frac{7}{4}}$
(e) $y' = -\dfrac{4}{x^5}$ (f) $y' = -\dfrac{4}{x^3}$
(g) $g'(t) = 24t^2$ (h) $h'(y) = \dfrac{2y}{9}$
(i) $f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$ (j) $f'(x) = \dfrac{2}{3}x^{-\frac{1}{3}}$
(k) $y' = -\frac{1}{2}x^{-\frac{3}{2}}$ (l) $y' = -\frac{3}{4}x^{-\frac{5}{4}}$
(m) $y' = \sqrt{6}x^{\sqrt{2}-1}$ (n) $y' = 12x^{11}$

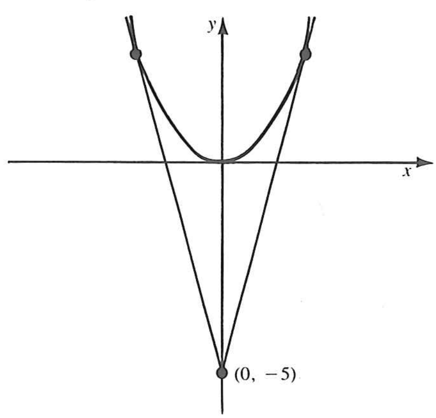
3. (a) $\frac{2}{3}$ (b) 1.4 (c) -3 (d) $\frac{1}{80}$ (e) $3\sqrt{2}$
(f) $-\frac{2}{3}$

4. (a) $80x - y - 128 = 0$
(b) $x - 3y + 9 = 0$ (c) $x + 25y - 10 = 0$
(d) $x - 12y - 16 = 0$

7. $(4, 48)$ **8.** $(16, 64)$

9. $\left(\frac{1}{2}, -\frac{1}{8}\right)$ **10.** $\left(2, \frac{1}{2}\right), \left(-2, \frac{3}{2}\right)$

11. $(\pm\sqrt{5}, 5)$
$2\sqrt{5}x - y - 5 = 0$
$2\sqrt{5}x + y + 5 = 0$



12. $\left(\pm\frac{5}{32}, \frac{25}{128}\right)$

EXERCISE 2.3

1. (a) $f'(x) = 2x + 4$ (b) $f'(x) = 15x^4 - 24x^3$
(c) $g'(x) = 10x^9 + 125x^4$
(d) $g'(x) = 2x + \dfrac{4}{x^3}$
(e) $h'(x) = \dfrac{1}{2\sqrt{x}} - 20x^3$ (f) $h'(x) = 2x + 5$
(g) $y' = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{2}}$ (h) $y' = 5t^4 + 30t^{-6}$
(i) $f'(t) = 3t^2 + 6t + 3$
(j) $F'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{4}x^{-\frac{3}{4}}$
(k) $u'(t) = -\dfrac{b}{t^2} - \dfrac{2c}{t^3}$ (l) $v'(r) = \dfrac{1}{\sqrt{r}} + \frac{9}{2}\sqrt{r}$

2. (a) $f'(x) = 1 + x + x^2 + x^3$,
$\text{dom}(f) = \text{dom}(f') = R$
(b) $f'(x) = 4 - \frac{1}{4}x^{-\frac{3}{4}}$,
$\text{dom}(f) = \{x \mid x \geq 0\}$, $\text{dom}(f') = \{x \mid x > 0\}$
(c) $f'(x) = 1 - \dfrac{5\sqrt{10}}{x^6}$,
$\text{dom}(f) = \text{dom}(f') = \{x \mid x \neq 0\}$
(d) $f'(x) = \frac{1}{2}x^{-\frac{1}{2}} - x^{-\frac{3}{2}}$,
$\text{dom}(f) = \text{dom}(f') = \{x \mid x > 0\}$

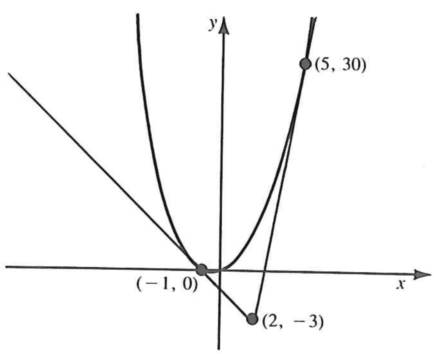
3. (a) $2x - y - 2 = 0$ (b) $x - 2y + 7 = 0$
(c) $x + 2y - 12 = 0$ (d) $8x + y + 4 = 0$

4. 20 m/s, 0 m/s, -10 m/s

5. 11 m/s, 27 m/s, 75 m/s **6.** $(2, -32)$

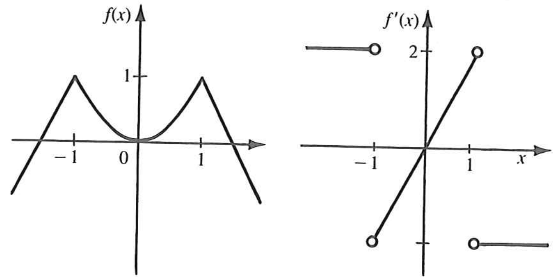
7. $(-4, 81)$, $(2, -27)$ **9.** $y = 2x$, $y = -2x$

10. $x + y + 1 = 0$, $11x - y - 25 = 0$



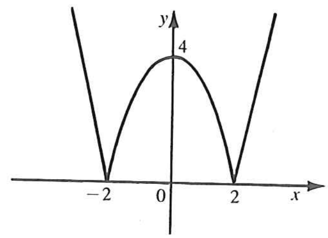
11. $-1 \pm \sqrt{2}$

12. (a) everywhere except at $x = -1, 1$

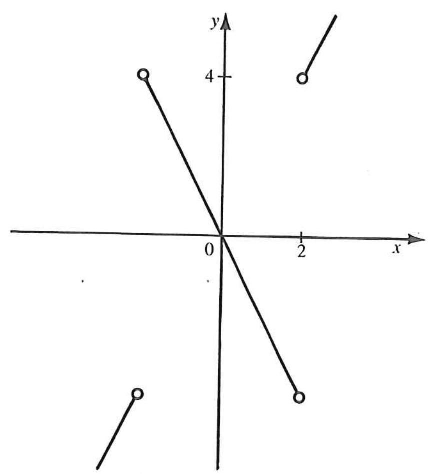


(b)
$$f'(x) = \begin{cases} 2 & \text{if } x < -1 \\ 2x & \text{if } -1 < x < 1 \\ -2 & \text{if } x > 1 \end{cases}$$

13. (a)



(b) 2, -2
(c)
$$f'(x) = \begin{cases} 2x & \text{if } x < -2 \\ -2x & \text{if } -2 < x < 2 \\ 2x & \text{if } x > 2 \end{cases}$$



EXERCISE 2.4

1. (a) $f'(x) = (2x - 1)(2x) + 2(x^2 + 1)$
(b) $f'(x) = 3x + (3x - 8)$
(c) $y' = x^2(1 - 6x) + 2x(1 + x - 3x^2)$
(d) $y' = (x^3 + x^2 + 1)(2x) +$
$(x^2 + 2)(3x^2 + 2x)$

(e) $f'(t) = (t^4 + t^2 - 1)(2t) + (t^2 - 2)(4t^3 + 2t)$

(f) $f'(t) = -\sqrt[3]{t} + \frac{1}{3}t^{-\frac{2}{3}}(1 - t)$

(g) $F'(y) = \sqrt{y}\left[1 - \frac{1}{\sqrt{y}}\right] + (y - 2\sqrt{y} + 2)\frac{1}{2\sqrt{y}}$

(h) $G'(y) = (y - y^2)(2 - \frac{4}{3}y^{\frac{1}{3}}) + (2y - y^{\frac{4}{3}})(1 - 2y)$

2. (a) $y' = 5x^4 + 8x^3 + 9x^2$
(b) $y' = 1 - 12x^{-3}$
(c) $f'(x) = 5x^4 - 3x^2 - 4x$
(d) $f'(x) = -36x^5 - 15x^2$
(e) $f'(t) = 480t^9 + 64t^7 - 90t^2 - 5$
(f) $f'(t) = 3act^2 + 2bct - ad$
(g) $g'(u) = \frac{45}{2}u^{\frac{7}{2}} - \frac{5}{2}u^{\frac{3}{2}} + u^{-\frac{1}{2}}$
(h) $g'(v) = 3v^2 - \frac{5}{2}v^{\frac{3}{2}} + \frac{3}{2}\sqrt{v} - 1$

3. (a) -13 (b) 0 (c) 20 (d) -9 (e) -11 (f) $-\frac{1}{2}$ **4.** 6

5. $x - y + 4 = 0$ **6.** -17

7. (a) $g'(x) = xf'(x) + f(x)$
(b) $h'(x) = \sqrt{x}f'(x) + \frac{1}{2\sqrt{x}}f(x)$
(c) $F'(x) = x^c f'(x) + cx^{c-1}f(x)$

8. (b) $y' = 2(2 + 5x - x^3)(5 - 3x^2)$

9. (b) $y' = \sqrt{x}\,(3x + 5)(12x - 5) + 3\sqrt{x}(6x^2 - 5x + 1) + \frac{1}{2\sqrt{x}}(3x + 5)(6x^2 - 5x + 1)$

10. (b) $y' = 3(1 + x^3 + x^6)^2\frac{d}{dx}(1 + x^3 + x^6) = 3(3x^2 + 6x^5)(1 + x^3 + x^6)^2$

EXERCISE 2.5

1. (a) $f'(x) = \frac{2}{(x + 1)^2}$
(b) $f'(x) = \frac{-2x^2 + 2x + 2}{(x^2 + 1)^2}$
(c) $g'(x) = \frac{-x^2 - 1}{(x^2 + 2x - 1)^2}$ (d) $g'(x) = 1$
(e) $y' = \frac{1 - 3x^2}{2\sqrt{x}(x^2 + 1)^2}$
(f) $y' = -\frac{2}{\sqrt{x}(\sqrt{x} - 2)^2}$
(g) $f'(t) = \frac{-2t^2 - 2t + 11}{(t^2 - 3t + 4)^2}$
(h) $g'(t) = \frac{2t^2 - 4t - 4}{(t - 1)^2}$
(i) $f'(x) = \frac{-4x^3 + 2x}{(x^4 - 2x^2 + 1)^2}$
(j) $f'(x) = \frac{ad - bc}{(cx + d)^2}$ (k) $f'(x) = \frac{x^{10} - 60x^5}{(x^5 - 10)^2}$
(l) $f'(x) = \frac{1 + 2x - x^2}{x^2(x + 1)^2}$

2. (a) $\{x|x \neq \frac{1}{2}\}, f'(x) = \frac{5}{(1 - 2x)^2}$
(b) $\{x|x \neq \pm 1\}, f'(x) = \frac{-x^2 - 1}{(x^2 - 1)^2}$
(c) $\{x|x \neq -1, x \neq \frac{3}{2}\}$, $f'(x) = \frac{1 - 4x}{(x + 1)^2(2x - 3)^2}$
(d) $\{x|x \neq -3, x \neq 1\}$, $f'(x) = -\frac{2(x^2 + x + 4)}{(x^2 + 2x - 3)^2}$
(e) $\{x|x \neq 1, x \neq -1\}$, $f'(x) = \frac{-2x^5 - 6x^4 - 2x - 2}{(x^4 - 1)^2}$
(f) $\{x|x \geq 0, x \neq 9\}, f'(x) = \frac{3x\sqrt{x} - 12x}{2(\sqrt{x} - 3)^2}$

3. (a) $x + 2y - 8 = 0$ (b) $9x - y - 13 = 0$
(c) $4x - 25y + 13 = 0$ (d) $x - y - 1 = 0$

4. 7 **6.** $(0, 0), (-5, 5)$ **7.** $\left(3, \frac{3}{2}\right), \left(-1, \frac{1}{2}\right)$

8. (a) $y' = -\frac{f'(x)}{[f(x)]^2}$ (b) $y' = \frac{xf'(x) - f(x)}{x^2}$
(c) $y' = \frac{f(x) - xf'(x)}{[f(x)]^2}$

EXERCISE 2.6

1. (a) $F'(x) = -21(5 - 3x)^6$
(b) $F'(x) = 80x(2x^2 + 1)^{19}$
(c) $G'(x) = \frac{9x^2 + 6x}{4\sqrt[4]{x^3 + x^2 - 2}}$
(d) $G'(x) = \frac{4x^3 - 1}{2\sqrt{x^4 - x + 1}}$
(e) $y' = \frac{2x + 1}{4(x^2 + x)^{\frac{3}{4}}}$
(f) $y' = -\frac{3(3 + 8x)}{(1 + 3x + 4x^2)^4}$
(g) $y' = -\frac{2(3x^2 + 4x)}{(x^3 + 2x^2 + 1)^3}$
(h) $y' = \frac{4x}{(9 - x^2)^{\frac{3}{2}}}$

(i) $y' = \dfrac{6(1 + 2\sqrt{x})^5}{\sqrt{x}}$

(j) $y' = \dfrac{2\sqrt{x} + 1}{4\sqrt{x}\sqrt{x + \sqrt{x}}}$

(k) $y' = 1 - \dfrac{x^4 - 12x^9}{(1 + x^5 - 6x^{10})^{\frac{4}{5}}}$

(l) $y' = 2x + 10x(x^2 - 1)^4$

2. $(4u^3 + 10u)(5x^4 + 4x)$ **3.** -117

4. $-\dfrac{2}{9\sqrt{13}}$ **5.** $\frac{11}{8}$

6. (a) $F'(x) = \dfrac{2x^2 + 1}{\sqrt{x^2 + 1}}$

(b) $F'(x) = 6(8x + 3)(4x - 1)^4$

(c) $G'(x) = (3 + 16x - 27x^2)(x^2 - 1)^3$

(d) $G'(x) = 2(x^4 - x + 1)(x^2 - 2)^2 \times (7x^5 - 8x^3 - 4x^2 + 3x + 2)$

(e) $F'(x) = \dfrac{x + 3}{(2x + 3)^{\frac{3}{2}}}$

(f) $f'(t) = \dfrac{2(1 + 2t)^4(3t^2 - 6t - 25)}{(3t^2 - 5)^3}$

(g) $g'(x) = \dfrac{-12(x + 2)^2}{(x - 2)^4}$

(h) $h'(t) = \dfrac{10(t^2 + 1)^9(t^2 + 2t - 1)}{(t + 1)^{11}}$

(i) $y' = \dfrac{2x}{(x^2 + 1)^{\frac{3}{2}}\sqrt{x^2 - 1}}$

(j) $y' = \dfrac{4(2x + 3)^2(5x - 12)}{(4x - 7)^{\frac{3}{2}}}$

(k) $y' = \dfrac{3(2x + 1)^4[22x + 1]}{2\sqrt{x}} + \dfrac{2}{\sqrt{4x - 3}}$

(l) $y' = \dfrac{1}{6x^{\frac{2}{3}}\sqrt{1 + \sqrt[3]{x}}}$

(m) $y' = 20(t + \sqrt[3]{t + t^2})^{19}\left(1 + \dfrac{1 + 2t}{3(t + t^2)^{\frac{2}{3}}}\right)$

(n) $y' = \dfrac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\left(1 + \dfrac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}\right)$

7. $32x - y - 63 = 0$ **8.** $4x - 2y - 7 = 0$

9. 15 **10.** 14

11. (a) $F'(x) = 4x^3f'(x^4)$ (b) $G'(x) = 4[f(x)]^3f'(x)$

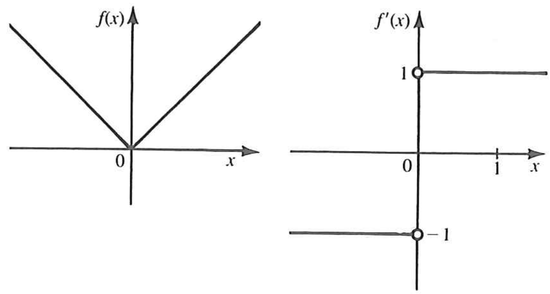
(c) $H'(x) = \dfrac{1}{2\sqrt{x}}f'(\sqrt{x})$

(d) $P'(x) = \dfrac{f'(x)}{2\sqrt{f(x)}}$ (e) $y' = f'(f(x))f'(x)$

(f) $y' = \dfrac{f(x)f'(x)}{\sqrt{1 + [f(x)]^2}}$ (g) $y' = 4xf(x^2)f'(x^2)$

(h) $y' = 3f'(x)[f(x)]^2f'([f(x)]^3)$

12. (b)



(c) $g'(x) = 2|x|$

EXERCISE 2.7

1. (a) $\dfrac{x}{y}$ (b) $-\dfrac{x^2}{y^2}$ (c) $-\dfrac{y}{x}$ (d) $-\dfrac{y + 2x}{x + 2y}$

(e) $\dfrac{2y - x^2}{y^2 - 2x}$ (f) $\dfrac{2x - 2y^2}{4xy - 3y^2}$ (g) $-\dfrac{\sqrt{y}}{\sqrt{x}}$

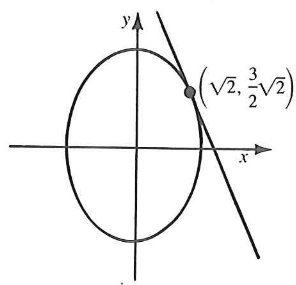
(h) $\dfrac{2y}{(x + y)^2 + 2x}$

2. (a) $\frac{1}{4}$ (b) -8 (c) $\frac{7}{8}$ (d) 3 (e) -1 (f) -4

3. (a) $2x - y + 1 = 0$ (b) $4x + y - 9 = 0$
(c) $3x - 16y + 25 = 0$ (d) $x + y = 0$

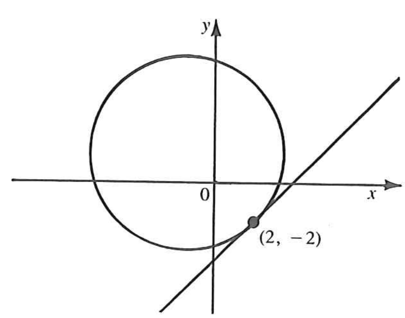
4. (a) $-\frac{3}{2}$ (c) $3x + 2y - 6\sqrt{2} = 0$

(d)



5. (a) $3x - 4y - 14 = 0$

(b)



6. (a) $\dfrac{dy}{dx} = \dfrac{x[25 - 4(x^2 + y^2)]}{y[25 + 4(x^2 + y^2)]}$

(b) $9x - 13y + 40 = 0$ (c) $\left(\pm\frac{5}{4}\sqrt{3}, \pm\frac{5}{4}\right)$

7. (a) $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$ (b) $2\sqrt{3}x + 2y - \sqrt{3} = 0$

(c) $\left(\dfrac{\sqrt{2}}{4}, -\dfrac{\sqrt{2}}{4}\right)$ and $\left(-\dfrac{\sqrt{2}}{4}, \dfrac{\sqrt{2}}{4}\right)$ **9.** $-\frac{1}{2}$

EXERCISE 2.8

1. (a) $f'(x) = 5x^4 - 8x, f''(x) = 20x^3 - 8$

(b) $f'(x) = 28x^3 + 36x^2 - 4$,
$f''(x) = 84x^2 + 72x$

(c) $f'(t) = 2 + \dfrac{1}{(t + 1)^2}; f''(t) = -\dfrac{2}{(1 + t)^3}$

(d) $g'(t) = -2t^{-\frac{3}{2}}; g''(t) = 3t^{-\frac{5}{2}}$

(e) $y' = 16(2x + 1)^7, y'' = 224(2x + 1)^6$

(f) $y' = 3t^2 - 3t^{-4}, y'' = 6t + 12t^{-5}$

(g) $y' = x(x^2 + 1)^{-\frac{1}{2}}, y'' = (x^2 + 1)^{-\frac{3}{2}}$

(h) $y' = -\dfrac{1}{(t - 1)^2}, y'' = \dfrac{2}{(t - 1)^3}$

2. (a) $f'''(x) = -6$ (b) $f'''(x) = -\dfrac{210}{x^8}$

(c) $y''' = \dfrac{72}{(4 - x)^5}$ (d) $y''' = 3(1 + 2x)^{-\frac{5}{2}}$

3. $y' = 5x^4 + 4x^3 + 3x^2 + 2x + 1, y'' = 20x^3 + 12x^2 + 6x + 2, y''' = 60x^2 + 24x + 6, y^{(4)} = 120x + 24, y^{(5)} = 120, y^{(6)} = 0$

4. $\frac{2}{3}$ **5.** $-\frac{405}{131\,072}$ **6.** n!

7. (a) $-\dfrac{3x^2}{y^7}$ (b) $-\dfrac{1}{y^3}$ (c) $\dfrac{16xy}{(2x - y^2)^3}$

8. $f(x) = 4x^2 - 2x + 3$

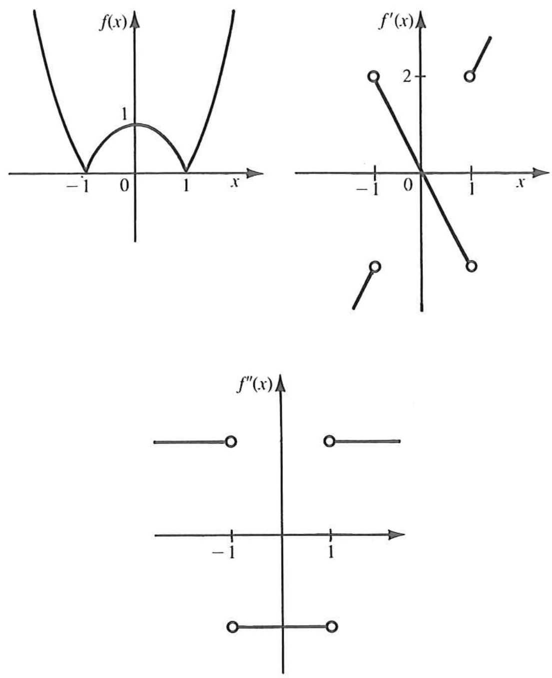
9. (a) $f'' = g''h + 2g'h' + gh''$

(b) $f''' = g'''h + 3g''h' + 3g'h'' + gh'''$

10. (a)
$$f'(x) = \begin{cases} 2x & \text{if } |x| > 1 \\ -2x & \text{if } |x| < 1 \end{cases}$$
$\text{dom}(f') = \{x | x \neq \pm 1\}$

$$f''(x) = \begin{cases} 2 & \text{if } |x| > 1 \\ -2 & \text{if } |x| < 1 \end{cases}$$
$\text{dom}(f'') = \{x | x \neq \pm 1\}$

(b)



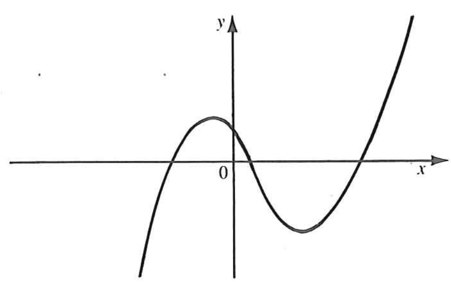
2.9 REVIEW EXERCISE

1. (a) $6x - 2$ (b) $3x^2 + 4$ (c) $\dfrac{1}{(1 - x)^2}$

(d) $\dfrac{1}{\sqrt{2x + 1}}$

2. $a = 1, f(x) = x^4$

3.



4. (a) $y' = 36x^2 + 8$ (b) $y' = 2(\pi + 1)x^\pi$
(c) $y' = 2 + \dfrac{3}{x^2}$ (d) $y' = \frac{6}{5}\sqrt[5]{x}$
(e) $y' = \dfrac{5}{2\sqrt{x}} - 1$ (f) $y' = \dfrac{3x - 2}{2\sqrt{x}}$
(g) $y' = \dfrac{5}{(1 + 3x)^2}$ (h) $y' = 42x^2(2x^3 - 1)^6$
(i) $f'(x) = \dfrac{-3x^3 - 2x^2 + 2x + 1}{\sqrt{1 - x^2}}$
(j) $g'(x) = \dfrac{-3x^2 + 12x + 1}{(2 - x)^2}$
(k) $h'(x) = \dfrac{-8x^3}{3\sqrt[3]{(2x^4 - 1)^4}}$
(l) $F'(x) = 2(x^4 + 1)^2(-13x^4 + 6x^3 - 1)$
(m) $f'(t) = \dfrac{1 + t}{\sqrt{(1 + 2t)^3}}$ (n) $g'(t) = \dfrac{4(t + 1)^3}{(t + 2)^5}$
(o) $R'(u) = \dfrac{1}{4\sqrt[4]{(u + 1)^3}} + \dfrac{4}{u^3}$
(p) $S'(v) = \dfrac{1 - 10v(v^2 - 8)^4}{2\sqrt{v - (v^2 - 8)^5}}$
(q) $M'(z) = \dfrac{1 - 2z - z^2}{2\sqrt{1 + z}\,\sqrt{(1 + z^2)^3}}$
(r) $F'(y) = \dfrac{3}{(2y + 3)^2}$

5. (a) $f'(x) = \dfrac{-2x^2 + 2x - 10}{(x^2 - 5)^2}$,
$\mathrm{dom}(f) = \mathrm{dom}(f') = \{x \mid x \neq \pm\sqrt{5}\}$
(b) $f'(x) = \dfrac{2x - 1}{2\sqrt{x^2 - x - 6}}$,
$\mathrm{dom}(f) = \{x \mid x \geqslant 3 \text{ or } x \leqslant -2\}$,
$\mathrm{dom}(f') = \{x \mid x > 3 \text{ or } x < -2\}$

6. -7

7. (a) $-\dfrac{x^3}{y^3}$ (b) $\dfrac{2xy - 2x}{2y - x^2}$ (c) $\dfrac{3x^2 - 4xy^2}{4x^2y - 3y^2}$
(d) $\dfrac{y - \dfrac{y}{2\sqrt{x - 1}} - \sqrt{y - 1}}{\sqrt{x - 1} + \dfrac{x}{2\sqrt{y - 1}} - x}$

8. (a) $80x^3 - 6x^2 + 6$ (b) $\dfrac{-9}{4\sqrt{(3x + 1)^3}}$
(c) $\dfrac{-4}{(t + 1)^3}$ (d) $-\dfrac{16}{y^3}$

9. (a) $4x + y - 4 = 0$ (b) $2x - y - 6 = 0$
(c) $5\sqrt{2}x + 32y - 14\sqrt{2} = 0$
(d) $13x - 3y + 8 = 0$
(e) $3x - 4y + 14 = 0$
(f) $4x - 5y + 12 = 0$

10. -10 m/s, -20 m/s, -50 m/s

11. $(-1, 11)$ **12.** $(0, -1)$, $(1, 1)$

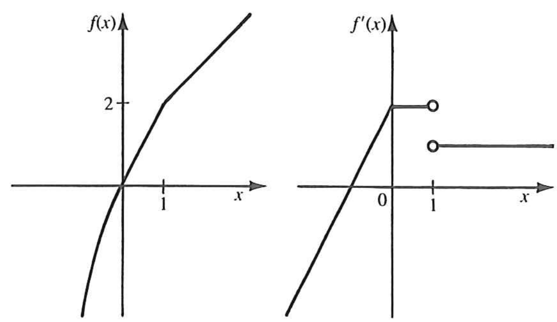
13. $11x - y - 25 = 0$, $x + y + 1 = 0$

14. (a) 2 (b) $-\frac{7}{18}$ (c) 10

15. (a) $f'(x) = x^2g'(x) + 2xg(x)$
(b) $f'(x) = \dfrac{2xg'(x) - g(x)}{2x\sqrt{x}}$
(c) $f'(x) = -\dfrac{1}{x^2}\,g'\!\left(\dfrac{1}{x}\right)$
(d) $f'(x) = \dfrac{g'(\sqrt{x})}{4\sqrt{xg(\sqrt{x})}}$

16. $f''(x) = g'(g(x))g''(x) + g''(g(x))[g'(x)]^2$

17. (a) 1
(b) $f(x) = \begin{cases} 2 - 2x & \text{if } x \leqslant 0 \\ 2 & \text{if } 0 < x < 1 \\ 1 & \text{if } x > 1 \end{cases}$



2.10 CHAPTER 2 TEST

1. (a) $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$
(b) (i) $f'(x) = 2x - 7$ (ii) $f'(x) = \dfrac{-2}{(2x + 1)^2}$

2. (a) $f'(x) = \dfrac{2}{3\sqrt[3]{x}}$ (b) $f'(x) = \dfrac{2x^2 - 2x - 6}{(2x - 1)^2}$

(c) $f'(x) = 2(x^2 - 1)^3(2x + 1)^2 \times (11x^2 + 4x - 3)$
(d) $f'(x) = 7(x + \sqrt{x^4 - 2x + 1})^6 \times \left(1 + \dfrac{2x^3 - 1}{\sqrt{x^4 - 2x + 1}}\right)$

3. (a) $\dfrac{x^2 - y}{x - y^2}$ (b) $4x - 5y + 4 = 0$

4. $y''' = \dfrac{192}{(3 - 2x)^5}$ **5.** $(5, 3)$

6. (a) $g'(x) = 6x^5 f'(x^6)$ (b) $h'(x) = 6f'(x)[f(x)]^5$
(c) $F'(x) = \dfrac{2xf(x) - x^2 f'(x)}{[f(x)]^2}$