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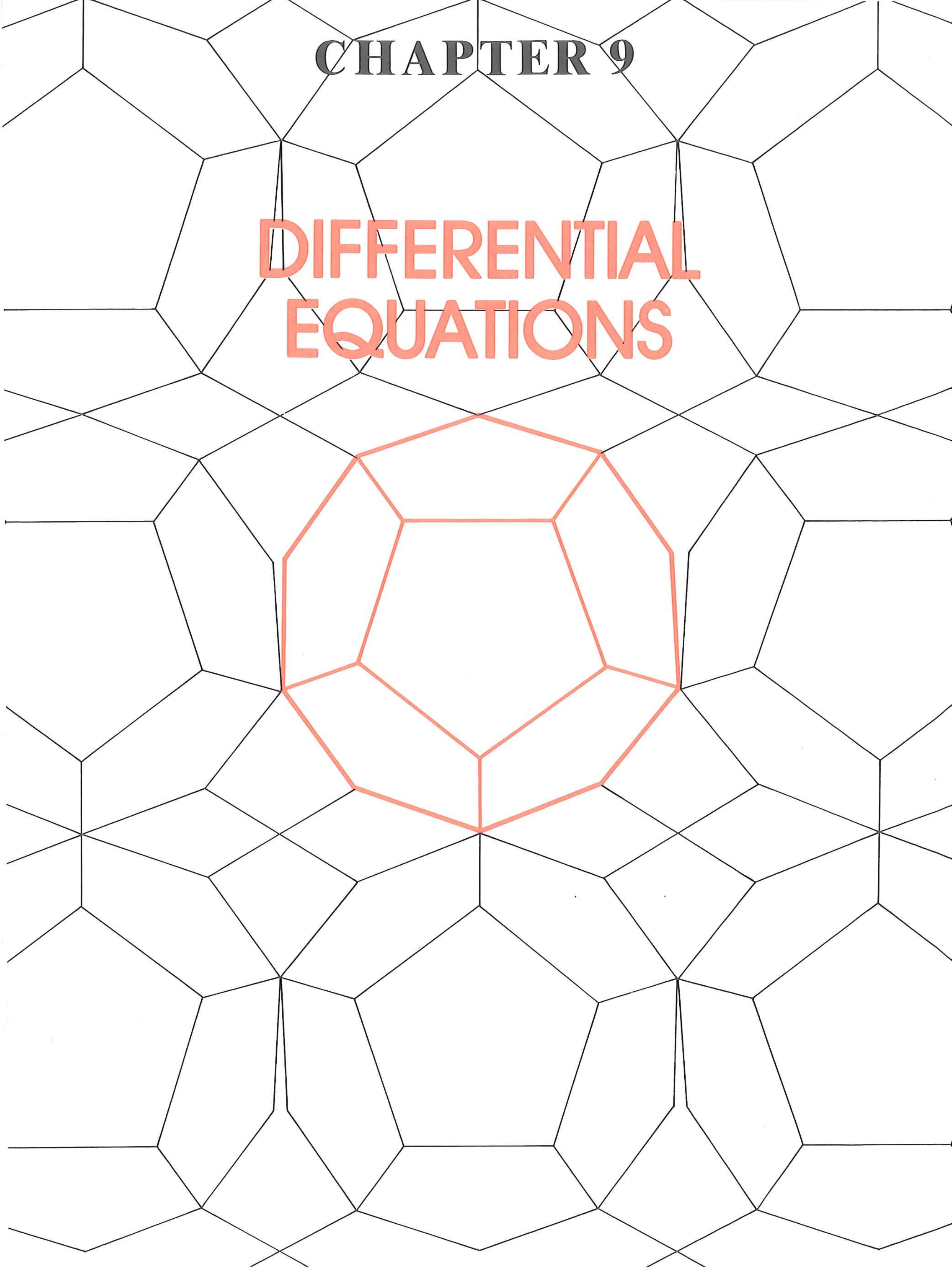
CHAPTER 9

DIFFERENTIAL EQUATIONS



REVIEW AND PREVIEW TO CHAPTER 9

Differentiation

EXERCISE 1

1. Find F'.

(a) $F(x) = 1.6x^{10} - 1.8x^5 + 1.5x^2 + \pi$

(b) $F(x) = 2.6x^{1.5} - 3.7x^{1.9} - 1$

(c) $F(x) = -3 \ln x - \dfrac{5}{x} + 6$

(d) $F(x) = \ln(2x + 7) + \sqrt{x - 3} + 11$

(e) $F(x) = \dfrac{6}{x^2} - \dfrac{5}{x} + 3 \ln 4x^2 - 2$

(f) $F(x) = -\frac{1}{2} \cos 2x + 2 \sin x + 8$

(g) $F(x) = 7 \sin 3x - 11 \cos 7x + 13$

(h) $F(x) = -4 \sin(x + 2) + 5 \cos(3x - 7) + 6$

(i) $F(x) = \frac{1}{2}e^{2x} - \frac{1}{3}e^{-3x} + \frac{1}{4}e^{4x}$

(j) $F(x) = -5e^{8x} + 2e^{-6x} - 37.1$

(k) $F(x) = \sqrt{x} + \sqrt{1 - x}$

(l) $F(x) = \ln(x - x^2)$

(m) $F(x) = \ln\left(\dfrac{x^4}{(1 - x)^5}\right)$

(n) $F(x) = 2e^{x^2} - 3e^{2x^2}$

(o) $F(x) = \sin x \cos x$

(p) $F(x) = \sqrt{x^2 + 1}$

(q) $F(x) = \ln(x^3 + 6x + 7)$

(r) $F(x) = \ln \cos x$

2. Find y''.

(a) $y = 3 \cos 4x - 5 \sin 4x$ (b) $y = 2 \cos x + 7 \sin x$

(c) $y = \cos \sqrt{2}x + 3 \sin \sqrt{2}x$ (d) $y = a \cos \sqrt{k}x + b \sin \sqrt{k}x$

INTRODUCTION

Many of the general laws of nature find their most useful form in equations that involve rates of change. These equations are called **differential equations** because they contain functions and their differential quotients. Some examples of differential equations are:

$$\frac{dy}{dx} = 2x \qquad (1)$$

$$\frac{dv}{dt} = -9.8 \qquad (2)$$

$$P' = 3P \qquad (3)$$

In these equations we have to solve

Equation 1	Equation 2	Equation 3
for y in terms of x,	for v in terms of t,	for P in terms of t.

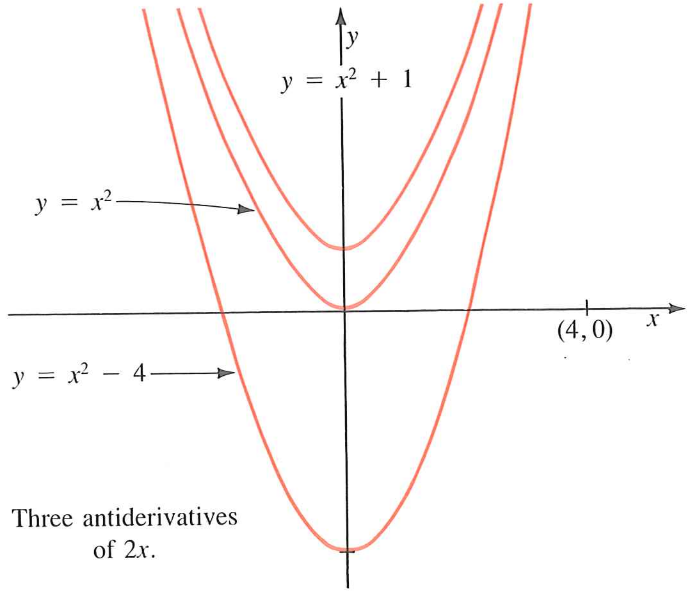
In the first three sections we deal with differential equations of the form $y' = f(x)$; solutions to these equations are called **antiderivatives** of $f(x)$. So we talk of finding antiderivatives as well as, or instead of, solving differential equations of the form $y' = f(x)$.

In the later sections we discuss differential equations that involve both the function and its derivative; Equation 3 is an example of this type and was seen previously in Section 8.5. The equations in these sections are used to solve cooling/warming problems, mixing problems, population processes, and cyclical (periodic) phenomena.

9.1 ANTIDERIVATIVES

A function F is an **antiderivative** of f on an interval if $F'(x) = f(x)$ for all x in that interval.

We consider the problem of finding all the antiderivatives of $f(x) = 2x$. Certainly $F(x) = x^2$ is an antiderivative of $2x$ since $\dfrac{dx^2}{dx} = 2x$, by the Power Rule. But $G(x) = x^2 + 1$, and $H(x) = x^2 - 4$ are also antiderivatives of $2x$. Indeed, for any constant C, $F(x) = x^2 + C$ is an antiderivative of $f(x) = 2x$.



Three antiderivatives of $2x$.

In fact, every antiderivative of $2x$ is of the form $F(x) = x^2 + C$, for some constant C. For if $F'(x) = 2x$ then

$$y = F(x) - x^2$$

satisfies

$$\begin{aligned} y' &= F'(x) - 2x \\ &= 2x - 2x. \end{aligned}$$

So $\qquad y' = 0$

on an interval. It can be shown that in these circumstances

$$y = C$$

Hence, we have shown that

$$\begin{aligned} F(x) - x^2 &= C \\ F(x) &= x^2 + C \end{aligned}$$

This shows that any two antiderivatives of $2x$ differ by a constant. This result is true in general.

> If F is an antiderivative of f on an interval, then the most general antiderivative of f on that interval is
>
> $$F(x) + C$$
>
> where C is an arbitrary constant.

Example 1 Find the (most general) antiderivative of $f(x) = 4x^3 - 6x^2 + 11$ on the interval $(-\infty, \infty)$.

Solution The result above tells us we need only find a single antiderivative of $4x^3 - 6x^2 + 11$; we can find all by adding an arbitrary constant.

Because the derivative of a sum is the sum of the derivatives, we look at each term in $4x^3 - 6x^2 + 11$ in order to find an antiderivative.

Now the derivative of x^4 is $4x^3$, the derivative of x^3 is $3x^2$, and the derivative of x is 1. So

$$\begin{aligned} x^4 &\text{ is an antiderivative of } 4x^3 \\ -2x^3 &\text{ is an antiderivative of } -6x^2 \\ 11x &\text{ is an antiderivative of } 11 \end{aligned}$$

Thus

$$F(x) = x^4 - 2x^3 + 11x + C$$

is the most general antiderivative of $f(x) = 4x^3 - 6x^2 + 11$ on the interval $(-\infty, \infty)$.

We see from this example that to find antiderivatives we use our knowledge of derivatives. In effect, we use differentiation tables backwards. Here is a table of commonly encountered antiderivatives.

Function	**Particular antiderivative**
0	1
1	x
$x^n \; (n \neq -1)$	$\dfrac{1}{n+1}x^{n+1}$
$\dfrac{1}{x}$	$\ln\lvert x \rvert$
$e^{kx} \; (k \neq 0)$	$\dfrac{1}{k}e^{kx}$
$\cos kx \; (k \neq 0)$	$\dfrac{1}{k}\sin kx$
$\sin kx \; (k \neq 0)$	$-\dfrac{1}{k}\cos kx$

Example 2 Find the antiderivative of f.

(a) $f(x) = 2x^2 - x + 7$ (b) $f(x) = \cos x - \sin x$

(c) $f(x) = -3e^{-x} + 6e^{2x}$

Solution We use the table above.

(a)
$$\begin{aligned} F(x) &= 2\left(\tfrac{1}{3}x^3\right) - \left(\tfrac{1}{2}x^2\right) + 7(x) + C \\ &= \tfrac{2}{3}x^3 - \tfrac{1}{2}x^2 + 7x + C \end{aligned}$$

(b)
$$\begin{aligned} F(x) &= \sin x - (-\cos x) + C \\ &= \sin x + \cos x + C \end{aligned}$$

(c)
$$\begin{aligned} F(x) &= -3\left(\frac{1}{-1}e^{-x}\right) + 6\left(\frac{1}{2}e^{2x}\right) + C \\ &= 3e^{-x} + 3e^{2x} + C \end{aligned}$$

Example 3 Find the antiderivative of f on the interval $(0, \infty)$.

(a) $f(x) = \dfrac{2}{x^2} - \dfrac{5}{x} + x$ (b) $f(x) = \sin x + \dfrac{1}{x^3}$

Solution (a) We rewrite f in a form suitable for using the table.

$$f(x) = 2x^{-2} - 5\left(\frac{1}{x}\right) + x$$

So
$$\begin{aligned} F(x) &= 2\left(\frac{1}{-2+1}x^{-2+1}\right) - 5 \ln |x| + \tfrac{1}{2}x^2 + C \\ &= -2x^{-1} - 5 \ln |x| + \tfrac{1}{2}x^2 + C \end{aligned}$$

Since we are considering the problem on the interval $(0, \infty)$, we have $|x| = x$, so

$$F(x) = -\frac{2}{x} - 5 \ln x + \tfrac{1}{2}x^2 + C$$

is the most general antiderivative of f on $(0, \infty)$.

(b) Since $f(x) = \sin x + \dfrac{1}{x^3}$ we can write $f(x) = \sin x + x^{-3}$. So the antiderivative of f is

$$\begin{aligned} F(x) &= -\cos x + \frac{1}{-3+1}x^{-3+1} + C \\ &= -\cos x - \frac{1}{2x^2} + C \end{aligned}$$

We end this section by considering an example where the antiderivative occurs in many equivalent forms.

Example 4 Find the most general antiderivative of $f(x) = \sin x \cos x$.

Solution 1 Since $\cos x = \dfrac{d}{dx} \sin x$ we see that

$$f(x) = \sin x \frac{d}{dx} \sin x$$

Hence,

$$F(x) = \tfrac{1}{2} \sin^2 x + C_1$$

as is easily verified. Here C_1 is an arbitrary constant.

Solution 2 Since $\sin x = -\dfrac{d}{dx}\cos x$ we see that

$$f(x) = -\cos x \frac{d}{dx}\cos x$$

Hence,

$$F(x) = -\tfrac{1}{2}\cos^2 x + C_2$$

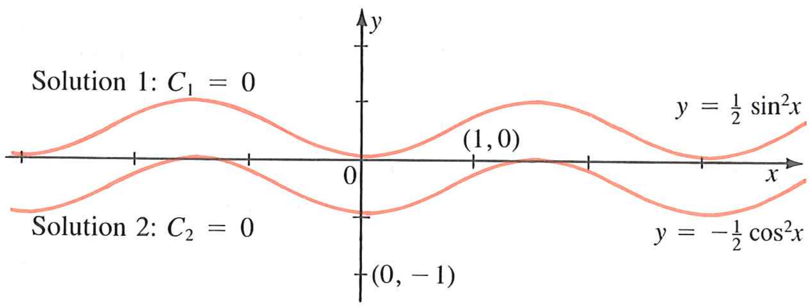
is the most general antiderivative on $(-\infty, \infty)$.

Solution 3 Since $\sin x \cos x = \frac{1}{2}\sin 2x$, we have from the table that

$$F(x) = \tfrac{1}{2}\left(-\tfrac{1}{2}\cos 2x\right) + C_3$$

is the most general antiderivative of f.

Let us look more closely at these seemingly different answers. First we sketch the antiderivatives corresponding to $C_1 = C_2 = 0$ in Solutions 1 and 2.



We see that the graphs are a constant vertical distance apart. So the curves have parallel tangent lines, and thus have identical slopes at points whose x-coordinates are the same.

Algebraically, we see that

$$\begin{aligned}\tfrac{1}{2}\sin^2 x + C_1 &= \tfrac{1}{2}(1 - \cos^2 x) + C_1 \\ &= -\tfrac{1}{2}\cos^2 x + \tfrac{1}{2} + C_1\end{aligned}$$

So the arbitrary constants in Solutions 1 and 2 are related by

$$\tfrac{1}{2} + C_1 = C_2$$

Similarly, the constants in Solutions 2 and 3 are related by

$$-\tfrac{1}{4} + C_2 = C_3$$

(A proof of this is requested in the Exercise.)

EXERCISE 9.1

B **1.** Find the most general antiderivative of the given function on the interval $(-\infty, \infty)$.

(a) $f(x) = 2x + 1$ (b) $f(x) = 4x^3 - 11$

(c) $f(x) = 16x^9 - 9x^4 + 3x$ (d) $f(x) = x^7 + x^5 + x^3 + x$

2. Find the antiderivative of f on $(0, \infty)$.

(a) $f(x) = \dfrac{2}{x^7} + \dfrac{x^5}{2}$ (b) $f(x) = \sqrt{x} + \sqrt[3]{x}$

(c) $f(x) = \dfrac{-3}{x} + \dfrac{5}{x^2}$ (d) $f(x) = \dfrac{1}{x^7} + \dfrac{1}{x^5} + \dfrac{1}{x^3} + \dfrac{1}{x}$

3. Find the antiderivative of f on $(-\infty, 0)$.

(a) $f(x) = \dfrac{1}{x}$ (b) $f(x) = \dfrac{2}{x^3} - \dfrac{3}{x^2}$

(c) $f(x) = \sqrt{-x}$ (d) $f(x) = \dfrac{1}{x^4} + x^3 + \dfrac{1}{x^2}$

4. Find the antiderivative of f on $(-\infty, \infty)$.

(a) $f(x) = \sin 2x + 2 \cos x$ (b) $f(x) = -3 \cos 5x + 8 \sin x$

(c) $f(x) = 7 \cos x - 11 \sin 11x$ (d) $f(x) = -4 \cos(x + 2)$

5. Find the most general antiderivative of f on $(-\infty, \infty)$.

(a) $f(x) = e^x + e^{-x}$ (b) $f(x) = e^x - e^{-x}$

(c) $f(x) = 4e^{2x} - 6e^{-3x}$ (d) $f(x) = e^x - e^{-2x} + e^{3x}$

6. Find the antiderivative of f on $(0, 1)$.

(a) $f(x) = \sqrt{x} - \sqrt{1 - x}$ (b) $f(x) = \dfrac{1}{x} - \dfrac{1}{1 - x}$

(c) $f(x) = \dfrac{1}{\sqrt{1 - x}} + \dfrac{1}{\sqrt{x}}$ (d) $f(x) = \dfrac{4}{x} + \dfrac{5}{1 - x}$

7. Find the antiderivative of f on $(-\infty, \infty)$.

(a) $f(x) = xe^{x^2}$ (b) $f(x) = \sin^2 x \cos x$

(c) $f(x) = \dfrac{2x}{x^2 + 1}$ (d) $f(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

C **8.** Find the most general antiderivative of f on the indicated interval.

(a) $f(x) = \dfrac{1}{x^2 + 1}$ on $(-\infty, \infty)$

(b) $f(x) = -\tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(c) $f(x) = \sec x \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(d) $f(x) = e^{\ln x}$ on $(0, \infty)$

9. Prove that the constants C_2 of Solution 2 and C_3 of Solution 3 in Example 4 are related by $C_3 = C_2 - \frac{1}{4}$.

10. Let $F(x) = \begin{cases} \ln|x| + 3 & \text{if } x > 0 \\ \ln|x| - 7 & \text{if } x < 0 \end{cases}$

(a) Show that $F'(x) = \dfrac{1}{x}$ for all $x \neq 0$.

(b) Show that there is no constant C such that $F(x) = \ln|x| + C$ for all $x \neq 0$.

(c) Deduce that it is not true that the most general antiderivative of $F'(x) = f(x)$ is of the form $F(x) + C$ for an arbitrary constant.

(d) Show that part (c) does not contradict our statement about the most general antiderivative being of the form $F(x) + C$.

9.2 DIFFERENTIAL EQUATIONS WITH INITIAL CONDITIONS

In the applications of differential equations in this chapter, we often deal with time-dependent processes where information is available at specific instants, usually the initial instant, $t = 0$.

Example 1 Solve $\dfrac{ds}{dt} = 2t$, with the initial condition: $s = 3$ when $t = 0$.

Solution We see that $s = t^2 + C$ is the most general antiderivative of $2t$, using the results of Section 9.1. We want the particular solution that has the value 3 when $t = 0$. So we set

$$3 = 0^2 + C$$

Thus, $C = 3$ and

$$s = t^2 + 3$$

is the desired solution.

This example is typical in that the initial condition specifies a particular value of the arbitrary constant in the most general antiderivative.

Sometimes conditions on differential equations are specified geometrically.

Example 2 Find the curve $y = F(x)$ that passes through $(-1,0)$ and satisfies $\dfrac{dy}{dx} = 6x^2 + 6x$.

Solution Since $\dfrac{dy}{dx} = F'(x)$, we see that

$$F'(x) = 6x^2 + 6x,$$

and the most general antiderivative is

$$F(x) = 2x^3 + 3x^2 + C$$

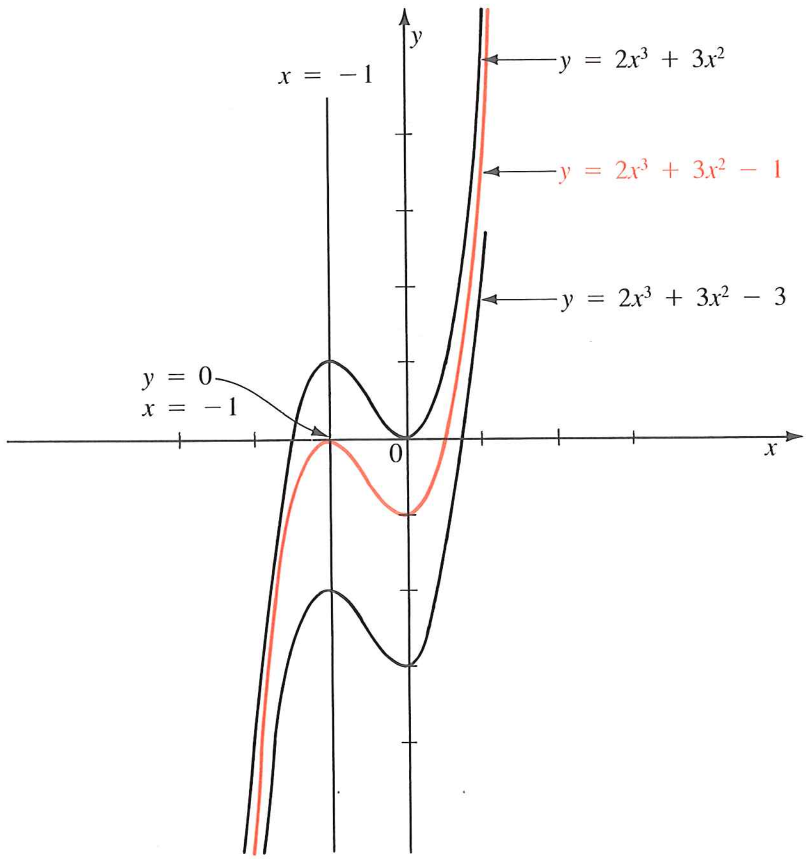
Now $F(-1) = 0$ since $(-1, 0)$ is on $y = F(x)$, so

$$0 = 2(-1)^3 + 3(-1)^2 + C$$

Thus $C = -1$ and $y = 2x^3 + 3x^2 - 1$

is the desired curve.

The diagram shows three curves satisfying $\dfrac{dy}{dx} = 6x^2 + 6x$ with the particular curve that passes through $(-1, 0)$ shown in red.



We say that the graphs $y = F(x)$ and $y = G(x)$ are **parallel** if there is a constant vertical distance between points with the same x-coordinate: in symbols

$$F(x) - G(x) = C$$

for all x.

If graphs are parallel, their tangent lines are parallel because they have the same slope.

Example 3 Suppose for the graph G that at every point

$$\frac{dy}{dx} = e^{-x}$$

Find the equation of a graph parallel to G that passes through the origin.

Solution From $\frac{dy}{dx} = e^{-x}$, we deduce that

$$y = -e^{-x} + C$$

is the most general antiderivative. So every graph parallel to G has an equation of the form $y = -e^{-x} + C$ for a suitable constant C. The graph we seek passes through $(0, 0)$, so substituting $x = 0$, $y = 0$ we obtain

$$0 = -e^0 + C$$

Since $e^0 = 1$, we have $C = 1$ and the desired graph has the equation $y = 1 - e^{-x}$.

EXERCISE 9.2

B **1.** Solve the differential equation $\frac{dy}{dx} = 4x - 3$ with the initial condition

(a) $y = 0$ when $x = 0$ (b) $y = -1$ when $x = 0$
(c) $y = 2$ when $x = -1$ (d) $y = 0$ when $x = 3$

2. Solve the differential equation with the initial condition $s = 0$ when $t = 0$.

(a) $\frac{ds}{dt} = 9.8t$ (b) $\frac{ds}{dt} = t^3 - t$
(c) $\frac{ds}{dt} = \sin t$ (d) $\frac{ds}{dt} = e^{0.1t}$

3. Find the function F given that the point $(2, 3)$ is on the graph $y = F(x)$.

(a) $F'(x) = 3x^2 - 2x + 6$ (b) $F'(x) = 3\sqrt{2x}$
(c) $F'(x) = 2e^{\frac{x}{2}}$ (d) $F'(x) = \sqrt{x} - \sqrt{4 - x}$

4. The slope of the tangent to the graph G is given at each point. Find the equation of a graph parallel to G that passes through the origin.

(a) $\frac{dy}{dx} = \cos x + \sin x$ (b) $\frac{dy}{dx} = e^x + e^{-x}$
(c) $\frac{dy}{dx} = \frac{1}{\sqrt{x + 1}}$ (d) $\frac{dy}{dx} = x(x^2 + 1)$

5. The line $x + y = 0$ is tangent to the graph of $y = F(x)$. Find $F(x)$ if

(a) $F'(x) = x$ (b) $F'(x) = x^3$

(c) $F'(x) = -x^5$ (d) $F'(x) = -1$

6. Find an equation for a graph parallel to $y = e^x$ that has $y = 4$ as a horizontal asymptote.

C **7.** The lines $x + y = 0$ and $x + y = \frac{4}{3}$ are tangent to the graph of $y = F(x)$ where F is an antiderivative of $-x^2$. Find F.

8. Using double angle formulas solve:

(a) $\dfrac{ds}{dt} = \cos^2 t - \sin^2 t$; $s = 1$ when $t = \dfrac{\pi}{4}$

(b) $\dfrac{ds}{dt} = \sin^2 t$; $s = 0$ when $t = \dfrac{\pi}{2}$

9.3 PROBLEMS INVOLVING MOTION

In this section we study the problem of determining the motion of a particle along a straight line path under the action of predetermined forces.

We use the variables t for time, s for displacement, and v for velocity, as we did in Chapter 3.

Example 1 If a body moves so that its velocity is proportional to the time elapsed, then $v = at$ where a is constant. Show that $v^2 = 2as$.

Solution Since $v = \dfrac{ds}{dt}$, and $v = at$ we have the differential equation

$$\frac{ds}{dt} = at$$

Thus, s is an antiderivative of at and so

$$s = \tfrac{1}{2}at^2 + C$$

for some constant C.

Since $s = 0$ when $t = 0$, by the way s, t are defined, we have

$$0 = \tfrac{1}{2}a(0)^2 + C$$

Hence $C = 0$, and we deduce that

$$s = \tfrac{1}{2}at^2$$

Multiplying this equation by $2a$ we obtain the desired form:

$$\begin{aligned} 2as &= 2a\left(\tfrac{1}{2}at^2\right) \\ 2as &= (at)^2 \\ 2as &= v^2 \end{aligned}$$

Example 2 A particle is accelerated in a line so that its velocity in metres per second is equal to the square root of the time elapsed, measured in seconds. How far does the particle travel in the first hour?

Solution We are told that

$$v = \sqrt{t}$$

So $$\frac{ds}{dt} = \sqrt{t} = t^{\frac{1}{2}}$$

and $$s = \tfrac{2}{3}t^{\frac{3}{2}} + C$$

When $t = 0$ we have $s = 0$, so substituting, we obtain

$$0 = \tfrac{2}{3}(0) + C$$

Thus $C = 0$ and it follows that

$$s = \tfrac{2}{3}t^{\frac{3}{2}}$$

Since $\dfrac{ds}{dt} > 0$, distance = displacement

We are asked to find s when $t = 1$ h $= 3600$ s, so we substitute

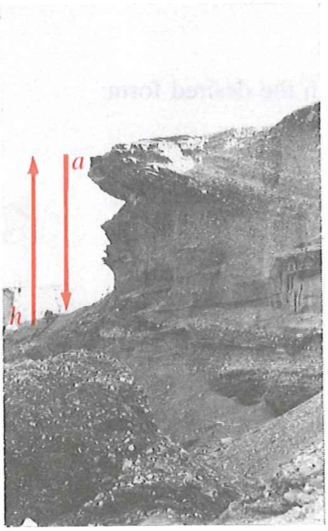
$$s = \tfrac{2}{3}(3600)^{\frac{3}{2}} = \tfrac{2}{3}60^3 = 144\,000$$

The particle travels 144 000 m, that is, 144 km, in the first hour.

Next we consider a problem where we do not choose the standard variables s, t.

Example 3 A stone is tossed upward with a velocity of 8 m/s from the edge of a cliff 63 m high. How long will it take the stone to hit the ground at the foot of the cliff?

Solution We let h denote the height, in metres, of the stone above the ground at time t, and we let $v = \dfrac{dh}{dt}$ denote its velocity.



Since gravity is the force acting on the stone we have

$$\frac{dv}{dt} = -9.8$$

where 9.8 m/s^2 is the gravitational constant near the surface of the earth. We see that v is an antiderivative of -9.8 so

$$v = -9.8t + C_1$$

for some constant C_1. Since we are told that the stone is tossed upward at the initial instant with a velocity of 8 m/s, we can substitute $v = 8$, $t = 0$ to deduce

$$8 = -9.8(0) + C_1$$

This shows that $C_1 = 8$, and consequently

$$v = -9.8t + 8$$

We rewrite this expression for v as a differential equation for h:

$$\frac{dh}{dt} = -9.8t + 8$$

From this we deduce that h is an antiderivative of $-9.8t + 8$, so

$$h = -4.9t^2 + 8t + C_2$$

for a suitable constant C_2. Since $h = 63$ when $t = 0$,

$$63 = -4.9(0^2) + 8(0) + C_2$$

Thus $C_2 = 63$ and so

$$h = -4.9t^2 + 8t + 63$$

We can now answer the question. The stone hits the ground when t is such that $h = 0$. So we solve the quadratic equation

$$-4.9t^2 + 8t + 63 = 0$$

From the quadratic formula we obtain the roots

$$t = \frac{-8 + \sqrt{8^2 - 4(-4.9)(63)}}{-9.8} \doteq -2.86$$

and $$t = \frac{-8 - \sqrt{1298.8}}{-9.8} \doteq 4.49$$

Because the negative root has no physical significance in this problem, we see that about 4.5 s after being tossed up the stone hits the ground.

EXERCISE 9.3

B 1. An object moves in a straight line with velocity $v = 6t - 3t^2$, where v is measured in metres per second.
   (a) How far does the object move in the first second?
   (b) How far does the object move in the first two seconds?
   (c) The object is back where it started when $t = 3$. How far did it travel to get there?

2. A canister is dropped from a helicopter hovering 500 m above the ground. Unfortunately its parachute does not open. It has been designed to withstand an impact velocity of 100 m/s. Will it burst or not?

3. A stone is tossed down, with a speed of 8 m/s, from the edge of a cliff 63 m high. How long will it take the stone to hit the ground at the foot of the cliff?

4. A pebble is tossed upward at 5 m/s from the roof of a building 80 m high. When will it hit the ground?

5. Margaret has found that by a proper choice of gears she can steadily increase her speed on her bicycle. One day she sets out and, ten minutes later, she achieves her cruising speed of 30 km/h by increasing her speed steadily. How far did she travel in that ten minutes?

6. Since raindrops grow as they fall, their surface area increases and therefore the resistance to their falling increases. A raindrop has an initial downward speed of 10 m/s and its downward acceleration a is given by

$$a = \begin{cases} 9 - 0.9t & 0 \leq t \leq 10 \\ 0 & t > 10 \end{cases}$$

   (a) How far does the raindrop fall in the first 10 seconds?
   (b) What is the velocity of the raindrop after 1 s?
   (c) If the raindrop is initially 600 m above the ground, how long does it take to fall?

C 7. A cliff is h_0 m high. A stone is tossed off the cliff with a velocity of v_0 m/s. Thus if the stone is tossed up, v_0 is positive. How long does it take the stone to reach the ground?

9.4 THE LAW OF NATURAL GROWTH

We saw in Section 8.5 that the differential equation

$$\frac{dy}{dx} = ky$$

corresponds to exponential growth if k is positive, and exponential decay if k is negative. We also saw that the solutions to the differential equation are easily described.

> The most general solution of
>
> $$\frac{dy}{dx} = ky$$
>
> is given by
>
> $$y = Ce^{kx}$$
>
> where C is an arbitrary constant.

Example 1 The population of bacteria grown in a culture follows the law of natural growth, with a growth rate of 15% per hour. If there are 10 000 bacteria present initially, how many will there be after four hours?

Solution Let P denote the number of bacteria at time t. Then

$$\frac{dP}{dt} = kP$$

since we are assuming the law of natural growth. Now $k = \dfrac{1}{P}\dfrac{dP}{dt}$ is the per capita growth rate, and we are told that $k = 15\% = 0.15$, so

$$\frac{dP}{dt} = 0.15P$$

The general solution of this differential equation is

$$P = Ce^{0.15t}$$

for a suitable constant C.

Initially, when $t = 0$, $P = 10\ 000$, so

$$10\ 000 = Ce^{0}$$

Hence $C = 10\ 000$ and we see that

$$P = 10\ 000e^{0.15t}$$

We can now answer the question as to how many bacteria there are after four hours by substituting $t = 4$ in the expresson for P:

$$P = 10\,000e^{0.15(4)} \doteq 18\,000$$

There will be approximately 18 000 bacteria present after four hours.

The law of natural growth occurs in many contexts; here is one.

Newton's Law of Cooling

The rate at which a hot body cools to the temperature of its surroundings is proportional to the temperature difference between the body and its surroundings.

This law says, for example, that a cup of tea at 60° above its surroundings cools degree by degree twice as rapidly as when it is 30° above its surroundings.

Example 2 Suppose a hot body is initially at temperature T_0, while the surroundings are at temperature A. Using Newton's Law of Cooling, find a formula for the temperature T, at an arbitrary instant, in terms of the time t and the constants T_0 and A.

Solution Newton's Law of Cooling can be stated as a differential equation:

$$\frac{dT}{dt} = k(T - A)$$

This is almost the law of natural growth, except that we have T related to $T - A$. But A is constant, so $\frac{dA}{dt} = 0$.

Hence
$$\begin{aligned}\frac{d(T - A)}{dt} &= \frac{dT}{dt} - \frac{dA}{dt}\\ &= \frac{dT}{dt} - 0\\ \frac{d(T - A)}{dt} &= k(T - A)\end{aligned}$$

Thus, the variable $T - A$ obeys the law of natural growth. Consequently, there is a constant C such that

$$T - A = Ce^{kt}$$

In order to evaluate C, we use the fact that $T = T_0$ when $t = 0$:

$$T_0 - A = Ce^0$$

Hence $C = T_0 - A$ and

$$T - A = (T_0 - A)e^{kt}$$

So the formula is

$$\boxed{T = A + (T_0 - A)e^{kt}}$$

We observe that in the formula just given there are *three* constants:

A is the temperature of the surroundings
T_0 is the initial temperature of the body
k is a characteristic of the material of the body

We consider a numerical example to show how A, T_0, and k are determined from the given information.

Example 3 In a steel mill, rod steel at 900°C is cooled by forced air at a temperature of 20°C. The temperature of the steel after one second is 400°C. When will the steel reach a temperature of 40°C?

Solution Let T denote the temperature in °C of the steel t seconds after encountering the forced air. Then

$$T = A + (T_0 - A)e^{kt}$$

by Newton's Law of Cooling. Since A is the temperature of the surroundings, in this case the forced air, we have $A = 20$. Since T_0 is the initial temperature of the steel, we have $T_0 = 900$. Hence,

$$T = 20 + (900 - 20)e^{kt}$$
$$T = 20 + 880e^{kt}$$

We only need to determine k: this is done by using the fact that $T = 400$ when $t = 1$. Substituting we have

$$400 = 20 + 880e^{k(1)}$$
$$380 = 880e^{k}$$
$$e^k = \frac{380}{880}$$

Taking natural logarithms we have

$$k = \ln\left(\frac{380}{880}\right) \doteq -0.84.$$

Hence

$$T \doteq 20 + 880e^{-0.84t}$$

We can now answer the question: find t when $T = 40$. Now we have

$$40 \doteq 20 + 880e^{-0.84t}$$

so $$e^{-0.84t} \doteq \frac{40 - 20}{880}$$

Taking natural logarithms we obtain

$$\begin{aligned} -0.84t &\doteq \ln\left(\frac{20}{880}\right) \\ -0.84t &\doteq -3.78 \\ t &\doteq 4.5 \end{aligned}$$

Thus, in 4.5 s the temperature of the steel is about 40°C.

In our last example we see how to use three temperatures of the body, at distinct times, to find the temperature of the surroundings.

Example 4 On a hot day a thermometer is taken outside from an air-conditioned room where the temperature is 20°C. After one minute it reads 26°C and after two minutes it reads 29°C. What is the outdoor temperature?

Solution Let T denote the reading on the thermometer t minutes after it has been taken outdoors. Then, by Newton's Law of Cooling (in this case *Warming*)

$$T = A + (T_0 - A)e^{kt}$$

We know that the initial reading T_0 is 20, so

$$T = A + (20 - A)e^{kt}$$

Next we use our knowledge of T when $t = 1$ and $t = 2$:

$$\begin{aligned} 26 &= A + (20 - A)e^{k(1)} \\ 29 &= A + (20 - A)e^{k(2)} \end{aligned}$$

Therefore

$$26 - A = (20 - A)e^{k} \qquad (1)$$
$$29 - A = (20 - A)e^{2k} \qquad (2)$$

Keeping in mind that we want to determine A, we eliminate k from these equations by squaring Equation 1 to obtain the term e^{2k} that occurs in Equation 2. Thus

$$(26 - A)^2 = (20 - A)^2e^{2k} \qquad (3)$$

Now dividing Equation 3 by Equation 2 gives

$$\frac{(26 - A)^2}{29 - A} = \frac{(20 - A)^2 e^{2k}}{(20 - A)e^{2k}}$$

$$\frac{(26 - A)^2}{29 - A} = 20 - A$$

$$(26 - A)^2 = (29 - A)(20 - A)$$

$$26^2 - 52A + A^2 = 29(20) - 49A + A^2$$

$$-3A = 580 - 676$$

$$A = 32$$

Hence the outdoor temperature is 32°C.

EXERCISE 9.4

B

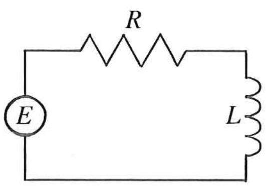
1. Annette buys an investment certificate for \$1000 that yields 6.25% (continuously compounded) per year. For how much will she redeem this certificate five years from now?
2. The population of a town is following the law of natural growth, with a growth rate of 1.6% per annum. A new sewage treatment plant must be in place before the population doubles. How long does the town have to build the plant?
3. In a certain forest there are two species of moth, grey and black. Presently there are twice as many grey as black, but the intrinsic growth rate of the black moth is 4% per month, while that of the grey is only 3% per month. In how many months will there be twice as many black as grey?
4. A metal ball is heated to a temperature of 100°C and then immersed in water that is maintained at 10°C. After one second the temperature of the ball is 25°C.
   (a) How long after immersion did the ball have a temperature of 50°C?
   (b) When will the ball's temperature be 12°C?
5. A thermometer reading −7°C is brought into a room kept at 23°C. Half a minute later the thermometer reads 8°C. What is the temperature reading of the thermometer after three minutes?
6. A pie is removed from a 175°C oven. The room temperature is 24°C. How long will it take the pie to cool to 37°C (body temperature) if it cooled 60° in the first four minutes?
7. On a hot day (35°C) a thermometer is taken from a cool room. After one minute outdoors it reads 29°C and after a further minute it reads 32°C. What is the temperature of the room?

C **8.** (The learning curve.) Psychologists have found that people initially learn a new subject rapidly and then slow down. A possible model for this can be formulated as follows. Suppose that M is the total amount of new knowledge that a person can learn and A is the amount learned up to time t. Then the rate of change of A is proportional to the amount still left to learn.

(a) State a differential equation that expresses the relationship stated.

(b) Solve this differential equation with the initial condition that $A = 0$ when $t = 0$.

(c) Suppose Jim can memorize 40 syllables in a row in an hour but he cannot learn 41 syllables in a row even if given a whole day. Suppose further that Jim learns 13 syllables in the first five minutes. How long does it take him to learn 36 syllables?



9. The varying current I in a circuit containing only a constant resistance R and an inductance L (also constant) in series with a constant voltage E is governed by $L\dfrac{dI}{dt} + RI = E$. Solve this differential equation if $I = 0$ when $t = 0$.

9.5 MIXING PROBLEMS

In this section we see how to formulate a differential equation for a **mixing process**.

Example 1 A tank contains 100 L of salt water, at a salt concentration of 15 g/L. Water containing 23 g/L of salt is pumped in at a rate of 3 L/min and the mixture, being steadily stirred, is pumped out of the tank at the same rate.

(a) Write a differential equation relating the amount of salt in the tank to the time.

(b) Find the general solution of this differental equation.

(c) Find a formula for the amount of salt in the tank at time t.

(d) Determine how much salt is in the tank after one hour.

Solution (a) We introduce variables to describe the process. Let A denote the amount (in grams) of salt present t minutes after the process began. We want to express A as a function of t.

In order to write a differential equation, we need to make a statement about what is happening *instantaneously*. So we see what changes occur over a short time interval, then take the limit.

Let ΔA denote the change in the amount of salt in the tank during the brief time interval Δt. Now

$$\Delta A = [\text{salt in}] - [\text{salt out}]$$

so to estimate ΔA we need to estimate the quantities [salt in] and [salt out].

First,

$$\begin{aligned}\text{[salt in]} &= \text{[volume of water in]} \times \text{[concentration of salt]}\\ &= \text{[rate of water in]} \times \Delta t \times \text{[concentration of salt]}\\ &= 3 \times \Delta t \times 23\end{aligned}$$

Before going on, we check units:

$$\mathrm{g} = \frac{\mathrm{L}}{\mathrm{min}} \times \mathrm{min} \times \frac{\mathrm{g}}{\mathrm{L}}$$

Next,

$$\begin{aligned}\text{[salt out]} &= \text{[volume of water out]} \times \text{[concentration of salt]}\\ &= \text{[rate of water out]} \times \Delta t \times \text{[concentration of salt]}\\ &= 3 \times \Delta t \times \text{[concentration of salt going out]}\end{aligned}$$

Now, we do not know the concentration of the salt leaving the tank because it depends on the amount of salt in the tank; and that is what we are trying to determine. However, we can say that the amount of salt present during the brief time interval Δt does not change much from the amount A present at the beginning of the interval. Thus,

$$\text{[concentration of salt going out]} \doteq \frac{A}{100}$$

where 100 L is the volume of the tank.

We have shown that

$$\text{[salt out]} \doteq 3 \times \Delta t \times \frac{A}{100}$$

Once again we check units:

$$\mathrm{g} = \frac{\mathrm{L}}{\mathrm{min}} \times \mathrm{min} \times \frac{\mathrm{g}}{\mathrm{L}}$$

Returning to ΔA we have

$$\Delta A \doteq 3 \times \Delta t \times 23 - 3 \times \Delta t \times \frac{A}{100}$$

$$\frac{\Delta A}{\Delta t} \doteq 69 - \frac{3}{100}A$$

In the limit as Δt tends to 0 we get

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$$

and $$\lim_{\Delta t \to 0} \left(69 - \frac{3A}{100}\right) = 69 - \frac{3A}{100}$$

Thus we arrive at the differential equation

$$\frac{dA}{dt} = 69 - \frac{3A}{100} \qquad (1)$$

(b) In order to solve this differential equation we write it in a more easily solved form, motivated by our approach to the equation arising in Newton's Law of Cooling in Section 9.4.

From $$\frac{dA}{dt} = \frac{-3}{100}(A - 2300)$$

we deduce that $$\frac{d(A - 2300)}{dt} = \frac{-3}{100}(A - 2300)$$

Hence, there is a constant C such that

$$A - 2300 = Ce^{-\frac{3t}{100}} \qquad (2)$$

is the general solution of Equation 1.

(c) To find a formula for A we determine the constant C. At time $t = 0$ the tank contains

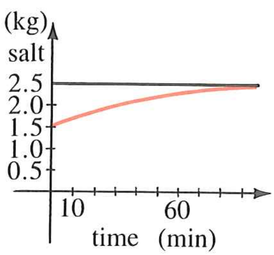
$$100 \text{ L} \times 15 \frac{\text{g}}{\text{L}} = 1500 \text{ g}$$

of salt. Substituting these values in Equation 2 we have

$$1500 - 2300 = Ce^0$$

so $C = -800$. Thus the formula is

$$A - 2300 = -800e^{-0.03t}$$
$$A = 2300 - 800e^{-0.03t} \qquad (3)$$



(d) To find out how much salt is in the tank after one hour we substitute $t = 60$ in Equation 3:

$$\begin{aligned} A &= 2300 - 800e^{-0.03(60)} \\ &\doteq 2300 - 800(0.165) \\ &\doteq 2170 \end{aligned}$$

There are approximately 2170 g of salt in the tank after one hour.

Next we look at a different type of situation that still fits a "mixing" treatment.

Example 2 George has a savings account with a balance of \$4500. This account earns 6% interest a year, compounded daily. He plans to withdraw \$30 each week for the three years he attends college. What will his bank balance be when he graduates?

Solution The processes here are not continuous: the interest is added daily and the withdrawals are weekly. However, we can get a very good approximation by assuming continuity.

Let B denote the balance (in \$) of George's account at time t (in years). Then $\frac{dB}{dt}$ = [rate money in] − [rate money out]. Now the rate at which money comes in depends on the balance B. Since we can approximate daily interest by continuously compounded interest, we have

$$[\text{rate money in}] \doteq 0.06B$$

Next,

$$[\text{rate money out}] = 52 \times 30 = 1560$$

Thus we have shown that

$$\frac{dB}{dt} = 0.06B - 1560 \qquad (1)$$

is a reasonable differential equation for this process.

In order to solve this equation we rewrite it as

$$\begin{aligned} \frac{dB}{dt} &= 0.06(B - 26\ 000) \\ \frac{d(B - 26\ 000)}{dt} &= 0.06(B - 26\ 000) \\ B - 26\ 000 &= Ce^{0.06t} \\ B &= 26\ 000 + Ce^{0.06t} \qquad (2) \end{aligned}$$

Now $B = 4500$ when $t = 0$, so we can determine C:

$$4500 = 26\ 000 + Ce^0$$

Hence $C = -21\ 500$, and we deduce that

$$B = 26\ 000 - 21\ 500e^{0.06t} \qquad (3)$$

When George graduates $t = 3$, and his bank balance is found by substituting $t = 3$ in Equation 3.

$$\begin{aligned} B &= 26\ 000 - 21\ 500e^{0.06(3)} \\ &\doteq 26\ 000 - 21\ 500(1.197) \\ &\doteq 260 \end{aligned}$$

When he graduates, George will have about \$260 in his bank account.

EXERCISE 9.5

B **1.** Twenty kilograms of salt is dissolved in a tank holding 1000 L of water. A brine solution is pumped into the tank at a rate of eight litres per minute and the well-stirred solution flows out at the same rate. If the brine entering the tank has a concentration of 100 g/L determine

(a) the amount of salt present after one hour, and

(b) the time needed for there to be 80 kg of salt in the tank.

2. Suppose that in Question 1 the flow rate is 20 L/min. Answer (a) and (b).

3. Suppose that in Question 1 the initial amount of salt is 5 kg instead of 20 kg. Answer (a) and (b).

4. Suppose that in Question 1 the volume of the tank is 500 L instead of 1000 L. Answer (a).

5. When a quantity of sugar is placed in a container of water the sugar dissolves at a rate proportional to the amount of undissolved sugar. Suppose that 30 g of sugar is placed in 1 L of water and that 5 min later the concentration of sugar dissolved in the water is 25 g/L.

(a) Let A denote the amount in grams of sugar dissolved at time t. Find a formula relating A and t.

(b) When was half the sugar dissolved?

(c) How long will it take for 29 g to be dissolved?

6. A man, initially having a mass of 95 kg, decides to lose weight through a scientific program of diet and exercise. He controls his intake to 2600 calories each day, of which 1320 go to basal metabolism. So he knows that his daily intake contributing to weight gain is actually 1280 ($= 2600 - 1320$) calories.

He exercises daily so that he expends 16 calories for each kilogram of his body mass. Assume that the storage of fat is 100% efficient and that one kilogram of fat contains ten thousand calories.

(a) Find how his mass varies with time.

(b) What mass is he aiming for with this program?

(c) When will he be halfway to his goal?

C **7.** The rate of growth of the mass m of a falling raindrop is km for some positive constant k. Newton's Law of Motion, applied to the raindrop, is $\frac{d(mv)}{dt} = gm$, where $g = 9.8 \text{ m/s}^2$ is the acceleration due to gravity and v is the velocity of the raindrop, directed down.

Show that $\lim_{t \to \infty} v$ exists and determine it. This limit is the *terminal velocity* of the raindrop.

8. There are V litres of water with A_0 kilograms of dissolved salt in a tank into which brine, with a concentration of c kilograms per litre, is entering at a rate of r litres per minute. The well-stirred solution leaves the tank by an overflow mechanism at the same rate of r litres per minute.

Let A denote the amount, in kilograms, of salt present at time t. Find A as a function of t.

9.6 THE LOGISTIC EQUATION

The law of natural growth

$$\frac{1}{P}\frac{dP}{dt} = k$$

is used to model population growth when populations are small relative to the capacity of the environment to support them.

A more realistic model for population growth is obtained by recognizing that a given environment can carry only a limited number K of individuals, and that as the population approaches this number, the unit rate of growth decreases toward zero.

A differential equation that has these more realistic properties is the **logistic differential equation**.

$$\boxed{\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{K}\right)}$$

where k and K are positive constants.

We can solve the logistic differential equation by considering a new variable Q defined by

$$Q = \frac{P}{K - P}$$

and showing that

$$\frac{1}{Q}\frac{dQ}{dt} = k \qquad (1)$$

as follows:

$$\frac{d(K-P)}{dt} = -\frac{dP}{dt}$$
since K is a constant

$$\frac{1}{Q}\frac{dQ}{dt} = \frac{1}{Q}\,\frac{(K-P)\dfrac{dP}{dt} - P\dfrac{d(K-P)}{dt}}{(K-P)^2}$$
$$= \frac{1}{Q(K-P)^2}\left[(K-P)\frac{dP}{dt} + P\frac{dP}{dt}\right]$$
$$= \frac{1}{Q}\frac{K}{(K-P)^2}\frac{dP}{dt}$$
$$= \frac{K-P}{P} \times \frac{K}{(K-P)^2} \times \frac{dP}{dt}$$
$$= \frac{K}{K-P} \times \frac{1}{P}\frac{dP}{dt}$$
$$= \frac{K}{K-P}\,k\left(1-\frac{P}{K}\right)$$
$$= k\left(\frac{K}{K-P}\right)\left(\frac{K-P}{K}\right)$$
$$= k$$

From Equation 1 we deduce that there is a constant c such that

$$Q = ce^{kt}$$

Now $Q = \dfrac{P}{K-P}$ gives $P = \dfrac{KQ}{Q+1} = \dfrac{K}{1+Q^{-1}}$ so

$$P = \frac{K}{1 + c^{-1}e^{-kt}}$$

since $Q^{-1} = c^{-1}e^{-kt}$

Letting $C = c^{-1}$, we have proved:

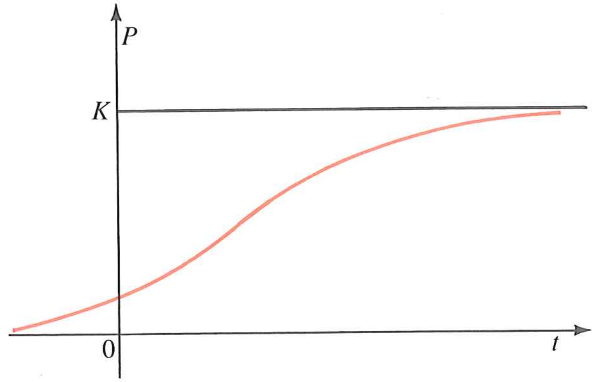
General Solution of Logistic Equation

If $\dfrac{1}{P}\dfrac{dP}{dt} = k\left(1-\dfrac{P}{K}\right)$

then $P = \dfrac{K}{1 + Ce^{-kt}}$

for an arbitrary constant C.

The function $P = \dfrac{K}{1 + Ce^{-kt}}$ is called a **logistic growth function.** The diagram shows the graph of a typical logistic growth function: it has a characteristic S shape and for this reason is called an *S-curve* or a *sigmoid curve*.



Example 1 Fish biologists put 200 fish into a lake whose carrying capacity for that species is estimated to be 10 000. The number of fish quadruples in the first year.

(a) Obtain a formula relating P, the number of fish in the lake, with t, the number of years since the fish were introduced.

(b) How many years will it take for there to be 5000 fish in the lake?

(c) When there are 5000 fish in the lake fishermen are allowed to catch 20% of them. How long will it take for the population to increase to 5000 again?

(d) When there are 7000 fish in the lake a 20% catch is allowed again. How long will it take for the population to return to 7000 fish?

Solution (a) Since we are dealing with a population in a limited environment, we assume that the logistic differential equation adequately describes what is happening. So, we see that P and t are related by

$$P = \frac{K}{1 + Ce^{-kt}}$$

for suitable constants K, C, and k.

We have identified K as the carrying capacity, so $K = 10\ 000$, and thus

$$P = \frac{10\ 000}{1 + Ce^{-kt}}$$

To determine C we use the initial condition that $P = 200$ when $t = 0$:

$$200 = \frac{10\ 000}{1 + Ce^{0}}$$

Thus $1 + C = \dfrac{10\ 000}{200}$, so $C = 49$. Now we have

$$P = \frac{10\ 000}{1 + 49e^{-kt}}$$

To determine k we use the fact that the population quadrupled in the first year; that is, $P = 800$ when $t = 1$.

$$\begin{aligned} 800 &= \frac{10\ 000}{1 + 49e^{-k(1)}} \\ 1 + 49e^{-k} &= \frac{10\ 000}{800} \\ 49e^{-k} &= 11.5 \\ e^{-k} &= \frac{11.5}{49} \end{aligned}$$

Taking natural logarithms we obtain

$$-k = \ln\left(\frac{11.5}{49}\right) \doteq -1.45$$

so $k \doteq 1.45$. Therefore, a formula relating P and t is

$$P = \frac{10\ 000}{1 + 49e^{-1.45t}}$$

This is the formula we use to answer the remaining questions. All our answers are approximate, since the carrying capacity is estimated, and surely the quadrupling of the numbers in the first year is only a close figure.

(b) We find t such that $P = 5000$.

$$\begin{aligned} \text{If} \qquad 5000 &= \frac{10\ 000}{1 + 49e^{-1.45t}} \\ \text{then} \quad 1 + 49e^{-1.45t} &= 2 \\ \text{so} \qquad e^{1.45t} &= 49 \\ \text{and} \qquad 1.45t &= \ln 49. \end{aligned}$$

Hence $t \doteq 2.7$. Therefore in about 32 months (2.7 a) the population will number 5000.

(c) We need to find how long the population takes to increase from 4000 (=5000 less 20%) to 5000. So we find t when $P = 4000$.

$$\begin{aligned} 4000 &= \frac{10\ 000}{1 + 49e^{-1.45t}} \\ 1 + 49e^{-1.45t} &= 2.5 \\ e^{1.45t} &= \frac{49}{1.5} \\ 1.45t &= \ln\left(\frac{49}{1.5}\right) \end{aligned}$$

Thus $t \doteq 2.4$, and the length of time needed to increase from 4000 to 5000 is $2.7 - 2.4 = 0.3$ a, which is about 4 months.

(d) We need to find how long the population takes to increase from 5600 (=7000 less 20%) to 7000. So we find the respective values of t from the formula and take their difference. We solve for t algebraically.

$$\begin{aligned} P &= \frac{10\,000}{1 + 49e^{-kt}} \\ 1 + 49e^{-kt} &= \frac{10\,000}{P} \\ 49e^{-kt} &= \frac{10\,000}{P} - 1 \\ e^{kt} &= \frac{49}{\dfrac{10\,000}{P} - 1} \end{aligned}$$

Thus $$t = \frac{1}{k}\ln\left(\frac{49}{\dfrac{10\,000}{P} - 1}\right)$$

$$t = \frac{1}{1.45}\ln\left(\frac{49}{\dfrac{10\,000}{P} - 1}\right)$$

So when $P = 5600$, $t \doteq \dfrac{1}{1.45}\ln(62.36) \doteq 2.85$

and when $P = 7000$, $t \doteq \dfrac{1}{1.45}\ln(114) \doteq 3.27$

The length of time needed to increase from 5600 to 7000 is $3.27 - 2.85 = 0.42$ a, which is about 5 months.

We see from this example that the larger population takes longer to recover from a 20% harvest than the smaller does. This type of analysis, using logistic growth, helps in deciding when to harvest certain species, such as salmon. For this purpose, accurate population levels must be determined.

Example 2 (Spread of an epidemic.) One law suggested for explaining the spread of an epidemic is that the rate of spread is jointly proportional to the fraction that is infected and the fraction that is uninfected.

In a small town of 5000, 160 people have a disease at the beginning of the week and 1200 have it at the end of the week.

(a) Find a formula relating the fraction F of infected people in the town and the number t of weeks since the disease was noticed.

(b) Show that in 11 days over half the town is infected.

(c) How long does it take for 4000 people to be infected?

Solution (a) The differential equation expressing the law is

$$\frac{dF}{dt} = kF(1 - F)$$

We see that this is the logistic differential equation with $K = 1$, as no more than 100% of the town can be infected. From our result on the general solution to the logistic differential equation, we deduce that

$$F = \frac{1}{1 + Ce^{-kt}}$$

We have to determine C and k. Initially, at $t = 0$, $F = \dfrac{160}{5000} = 0.032$, so

$$0.032 = \frac{1}{1 + Ce^{0}}$$

Hence $C = 30.25$, and therefore

$$F = \frac{1}{1 + 30.25e^{-kt}}$$

To determine k we use the fact that when $t = 1$, $F = \dfrac{1200}{5000} = 0.24$. So

$$0.24 = \frac{1}{1 + 30.25e^{-k(1)}}$$

$$1 + 30.25e^{-k} = \frac{1}{0.24}$$

$$e^{k} \doteq 9.55$$

Therefore $k \doteq \ln(9.55) \doteq 2.26$, and hence

$$F = \frac{1}{1 + 30.25e^{-2.26t}}$$

This is the desired formula relating F and t. We use it to answer the remaining questions.

(b) In 11 d, $t = \dfrac{11}{7}$ weeks, so

$$F = \frac{1}{1 + 30.25e^{-2.26\left(\frac{11}{7}\right)}} \doteq 0.54$$

Hence about 54% of the population is infected after 11 d.

(c) We find t such that

$$\begin{aligned} 0.8 &= \frac{1}{1 + 30.25e^{-2.26t}} \\ 30.25e^{-2.26t} &= 0.25 \\ e^{2.26t} &= \frac{30.25}{0.25} \end{aligned}$$

Thus $t = \dfrac{1}{2.26} \ln\left(\dfrac{30.25}{0.25}\right) \doteq 2.12.$

We conclude that in about 15 d 80% of the population will be infected.

EXERCISE 9.6

B 1. Three hundred fish are put into a lake whose carrying capacity is six thousand. The number of fish doubles in the first year.
   (a) Obtain a formula relating P, the number of fish in the lake, to the number t of years since the fish were introduced.
   (b) How many fish are in the lake after four years?
   (c) How long does it take for there to be 4800 fish in the lake?

2. Yeast is grown in a laboratory under conditions that ensure a limiting population of 700 cells. If the initial number of cells is 10 and after nine hours there are 500, when were 350 cells present?

3. In an isolated town of 2000 people, the disease Rottenich creates an epidemic. The initial number of infected is 12 but by the end of the first week 100 people have caught the disease.
   (a) How long does it take for half the town to be infected?
   (b) How many new cases appear in the week following the time when half were infected?

4. (a) Use a logistic growth model to predict the carrying capacity of Canada given the census figures following.

Year	Population (in millions)
1950	14.0
1960	18.2
1970	21.6

   (b) Use your resulting formula to predict the population of Canada in 1980. (The actual population was 24.04 million.)
   (c) From your model, predict when the population will pass 25 million. (Our population actually passed that mark in 1984.)

5. Cigarette consumption in Canada increased from 50 per capita in 1900 to 3900 per capita in 1960. Assume that the growth in cigarette consumption is governed by the logistic equation with limiting consumption being 4000 per capita. Estimate consumption of cigarettes in the years 1910, 1920, 1930, 1940, 1950, and 1970.

(Per capita consumption of cigarettes began to decrease in the 1970s due, in part, to the publicity given to health hazards, so the logistic equation no longer applies.)

6. Graph $y = \dfrac{10}{1 + e^{-x}}$.

C **7.** Suppose that P is positive and less than K and that there is a positive constant k, such that $\dfrac{dP}{dt} = kP - \dfrac{k}{K}P^2$ for all t. Set $Q = \dfrac{dP}{dt}$ and show that Q reaches a maximum when $P = \dfrac{K}{2}$ by evaluating $\dfrac{dQ}{dt} = \dfrac{dQ}{dP}\dfrac{dP}{dt}$. This shows that a population growing according to the logistic equation has maximum rate of growth when it reaches half the carrying capacity. So a sustainable resource is best harvested then.

*9.7 A SECOND ORDER DIFFERENTIAL EQUATION

The differential equation we consider in this section is

$$y'' + ky = 0$$

where k is a positive constant. This equation arises where the acceleration (or trend) is opposite to the state. These situations occur in many periodic phenomena. The swinging pendulum and the oscillating spring are examples from mechanics. However, this process is also recognized implicitly in the principle that prices are lowest when supplies are highest. We observe, too, that people on a roller-coaster scream after the falling is over. Finally, we recall the saying: it is always darkest before the dawn.

$y'' = \dfrac{d^2y}{dx^2}$

Example 1 Show that $y'' + 25y = 0$ if $y = -2 \cos 5x + 7 \sin 5x$.

Solution

$$y' = -2(-5) \sin 5x + 7(5) \cos 5x$$

So, $$y' = 10 \sin 5x + 35 \cos 5x$$

Hence $$\begin{aligned} y'' &= 10(5) \cos 5x + 35(-5) \sin 5x \\ &= 50 \cos 5x - 175 \sin 5x \\ y'' + 25y &= 50 \cos 5x - 175 \sin 5x - 50 \cos 5x + 175 \sin 5x \\ &= 0 \end{aligned}$$

We see that the periodic functions $\cos 5x$ and $\sin 5x$ are indeed involved in the solution of $y'' + 25y = 0$. In fact, this sort of result generally holds.

Theorem

If $y'' + ky = 0$, where k is a positive constant, then there are constants A, B such that

$$y = A\cos(\sqrt{k}x) + B\sin(\sqrt{k}x)$$

We note that the equation $y'' + ky = 0$ involves the *second* derivative of y and that the solution depends on *two* arbitrary constants (denoted A, B above).

Example 2 Solve $\dfrac{d^2s}{dt^2} + 4s = 0$ with $s = 0$ and $\dfrac{ds}{dt} = 1$ when $t = 0$.

Solution From the theorem we deduce that there are constants A, B such that

$$s = A\cos(\sqrt{4}t) + B\sin(\sqrt{4}t)$$
$$s = A\cos 2t + B\sin 2t$$

Now $s = 0$ when $t = 0$ so

$$0 = A\cos 0 + B\sin 0$$

Hence $A = 0$ and thus

$$s = B\sin 2t$$

Next, we differentiate to obtain

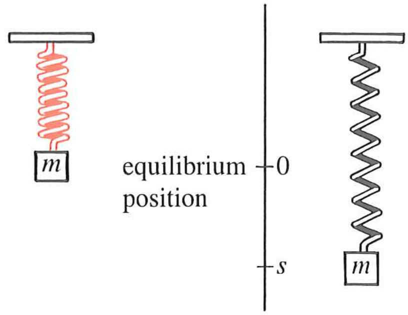
$$\frac{ds}{dt} = 2B\cos 2t$$

Substituting $t = 0$, $\dfrac{ds}{dt} = 1$ we obtain

$$1 = 2B\cos 0$$

Hence $B = \frac{1}{2}$, and thus we have shown that $s = \frac{1}{2}\sin 2t$ is the solution.

We end this introduction to periodic motion by considering the motion of a mass on the end of a spring.



There are two laws to be aware of here. First, Hooke's Law, which states that the force $F(s)$ required to stretch a spring s units beyond its natural length is proportional to s.

$$F(s) = ks,\ k > 0.$$

Second, Newton's Law of Motion, which states

$$F = ma$$

where m is the mass of the object in motion and a is its acceleration. In the situation of a mass on a spring, the acceleration, $a = \dfrac{d^2s}{dt^2}$, is a *restoring force* so it acts opposite to the direction of the displacement. Hence

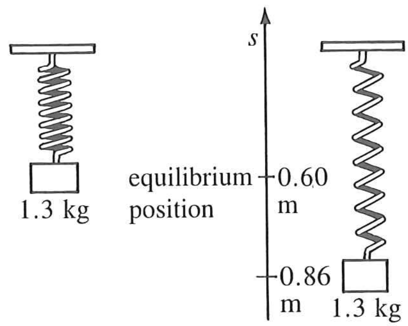
$$ks = -m\frac{d^2s}{dt^2}$$

and so s satisfies the differential equation

$$\frac{d^2s}{dt^2} + \frac{k}{m}s = 0$$

where $\dfrac{k}{m}$ is a positive constant depending on the system.

Example 3 A spring with mass 1.3 kg has natural length 0.60 m. A force of 20.8 N is required to stretch it to a length of 1.00 m. If the spring is stretched to a length of 0.86 m then released from rest, find the displacement s of the mass at time t.



Solution We first determine k, the positive constant of proportionality, in Hooke's Law. The force required to stretch the spring $1.00 - 0.60 = 0.4$ m is 20.8 N. Thus

$$20.8 = k(0.40)$$

and therefore $k = 52$.

Since $m = 1.3$ the equation for s is

$$\frac{d^2s}{dt^2} + \frac{52}{1.3}s = 0$$

or

$$\frac{d^2s}{dt^2} + 40s = 0$$

From the theorem we deduce that

$$s = A\cos(\sqrt{40}t) + B\sin(\sqrt{40}t)$$

We know that when $t = 0$, $s = 0.60 - 0.86 = -0.26$. Substituting, we get

$$-0.26 = A\cos 0 + B\sin 0$$

So $A = -0.26$. After differentiating, we have

$$\frac{ds}{dt} = A(-\sqrt{40})\sin(\sqrt{40}t) + B\sqrt{40}\cos(\sqrt{40}t)$$

Because the mass is at rest when $t = 0$, $\dfrac{ds}{dt} = 0$. So

$$0 = A(-\sqrt{40})\sin 0 + B\sqrt{40}\cos 0$$

Hence $B = 0$. We can now state that

$$s = -0.26\cos(\sqrt{40}t)$$

is a formula for the displacement.

Since $\cos(\sqrt{40}t)$ varies regularly between 1 and -1, we see that s in Example 3 varies regularly between -0.26 m and 0.26 m. Of course a real spring only behaves like this for a short time. Eventually, *damping* takes over, and the motion slows down. There is a differential equation that describes this very well, but that story must wait.

EXERCISE 9.7

B **1.** Solve the differential equation $\frac{d^2s}{dt^2} + 4s = 0$ with the indicated initial conditions.

(a) When $t = 0$, $s = 0$ and $\frac{ds}{dt} = 0$.

(b) When $t = 0$, $s = 1$ and $\frac{ds}{dt} = 0$.

(c) When $t = 0$, $s = -1$ and $\frac{ds}{dt} = 2$.

(d) When $t = 0$, $s = 3$ and $\frac{ds}{dt} = -5$.

2. Solve.

(a) $y'' + y = 0$ (b) $y'' + 9y = 0$
(c) $4y'' + 9y = 0$ (d) $y'' + 2y = 0$

3. Find $f(x)$ given that the point (0, 1) is on the graph $y = f(x)$, and that the line $2x + y = 1$ is tangent to $y = f(x)$ at the point (0, 1). Moreover f satisfies the differential equation.

(a) $f'' + f = 0$ (b) $f'' + 4f = 0$
(c) $4f'' + 9f = 0$ (d) $f'' + 2f = 0$

4. A spring with mass 1.3 kg has natural length 0.60 m. A force of 20.8 N is required to stretch it to a length of 1.00 m. If the spring is stretched or compressed to the length given then released from rest, find the displacement s at time t.

(a) 0.70 m (b) 0.42 m
(c) 1.00 m (d) 0.21 m

5. A spring with mass 1.3 kg has natural length 0.60 m. A force of 20.8 N is required to stretch it to a length of 1.00 m. If the spring is stretched to a length of 0.86 m then released with the given velocity, find the displacement s at time t.

(a) 1 m/s (b) -2 m/s
(c) 3.7 m/s (d) -4.1 m/s

6. Find the maximum value of $f(x)$.
(a) $f(x) = \cos x + \sin x$ (b) $f(x) = \cos x + \sqrt{3} \sin x$
(c) $f(x) = 3 \cos x - 4 \sin x$ (d) $f(x) = -2 \cos 3x + \sin 3x$

C 7. A spring with mass 1.0 kg has a natural length of 0.53 m. A force of 4.25 N is required to stretch it to a length of 0.70 m. If the spring is stretched to a length of 0.66 m then released with a downward speed of 2.1 m/s, find the maximum displacement of the mass.

9.8 REVIEW EXERCISE

1. Find the antiderivative of f on $(-\infty, \infty)$.
(a) $f(x) = 3x - \pi$ (b) $f(x) = e \sin x + \sqrt{2} \cos x$
(c) $f(x) = 4e^{\sqrt{2}x} - \frac{1}{7}e^{-\pi x}$ (d) $f(x) = \dfrac{4x^3}{x^4 + 1}$

2. Find the antiderivative of f on $(0, \infty)$.
(a) $f(x) = \dfrac{1}{10x} + \dfrac{\sqrt{2}}{x^2}$ (b) $f(x) = 4x^{1.5} - 3x^{2.7}$
(c) $f(x) = \sqrt{x} + \sqrt{2x} + \sqrt{3x}$ (d) $f(x) = -\dfrac{1}{x^2}e^{\frac{1}{x}}$

3. Find the function F given that the point $(-1, 4)$ is on the graph of $y = F(x)$ and that
(a) $F'(x) = 2x^2 - 3x$ (b) $F'(x) = e^x - e^{-2x}$
(c) $F'(x) = \sin x - \cos x$ (d) $F'(x) = \sqrt{3 + 2x}$

4. A pebble is tossed upward at 30 m/s from the edge of a bridge 210 m above the river below. How many seconds elapse between toss and splash?

5. A raindrop has an initial downward speed of 13 m/s and its acceleration a downward is given by
$$a = \begin{cases} 8.4 - 0.7t & 0 \leqslant t \leqslant 12 \\ 0 & t > 12 \end{cases}$$
(a) How far does the raindrop fall in the first 12 s?
(b) What is the velocity of the raindrop after 12 s?
(c) If the raindrop is initially 1 km above the ground, how long does it take to fall?

6. A metal ball is heated to 105°C and then immersed in water that is maintained at 17°C. After one second the temperature of the ball is 37°C. What will the ball's temperature be after one more second?

7. Ten kilograms of salt is dissolved in 800 L of water in a large tank. A brine solution, having a salt concentration of 75 g/L, enters the tank at 24 L/min and the well-stirred solution overflows at the same rate.

(a) Find the amount of salt present after one hour.

(b) When will there be 35 kg of salt in the tank?

8. The species of protozoa called *Paramecium caudatum* has been used to verify that the logistic growth model is applicable in some situations. Suppose 20 Paramecia are placed in a small test tube containing enough nutrient that the carrying capacity is 420 individuals. Suppose that by the end of the first day the number of individuals present increases to 160.

(a) Find a formula relating P, the number of Paramecia present, to t, the number of days since the experiment began.

(b) Use this formula to find how many Paramecia were present after three days.

(c) How long does it take for the population to reach 220?

9. Solve $y'' + 25y = 0$ with the initial conditions:

(a) When $x = 0$, $y = 0$, $y' = -3$.

(b) When $x = 0$, $y = 2$, $y' = 1$.

(c) When $x = \pi$, $y = -1$, $y' = 0$.

(d) When $x = 2\pi$, $y = 3$, $y' = 3$.

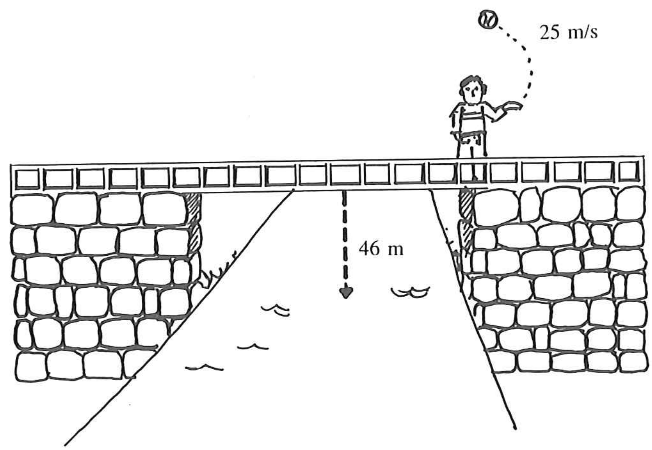
10. A particle moves in a straight line path in such a way that $\dfrac{d^2s}{dt^2} = -3s$, where s is the displacement. Suppose that $s = 0$ when $t = 0$, and $\dfrac{ds}{dt} = 4$ when $t = 0$.

(a) Find s as a function of t.

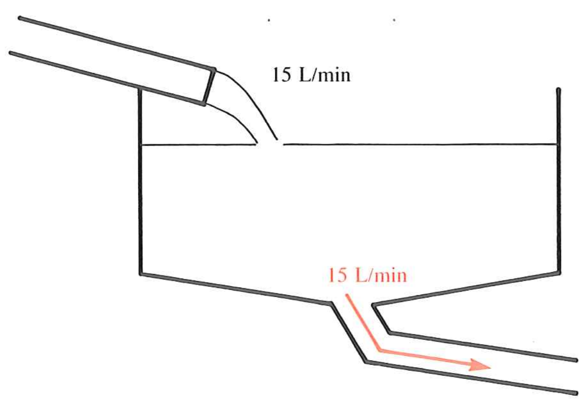
(b) What is the maximum value of s?

9.9 CHAPTER 9 TEST

1. (a) Define the expression "F is an antiderivative of f" where f is a given function, continuous on an interval.
   (b) Find the most general antiderivative of
   $$f(x) = x^2 + 3e^{-x} + 4\sin x$$
2. Suppose $F'(x) = \sqrt{2x} + 6$ and the point $(2, 5)$ is on $y = F(x)$. Find F.
3. A pebble is tossed upward at 25 m/s from the edge of a bridge 46 m above the lake below. How many seconds elapse between toss and splash?



4. A cup of coffee is at 80°C when first brought into the classroom where the temperature is 22°C. After one minute the temperature of the coffee is 60°C. How much longer does it take for the coffee to reach 33°C?
5. Fresh water enters a tank, containing 100 L brine, at a rate of 15 L/min. If the mixture is stirred and leaves the tank at 15 L/min, how long will it take for the solution to be half as salty, given that the initial concentration is 10 g/L?



6. A good model for the spread of rumours by word of mouth in a town is that the rate of spread of the rumour is jointly proportional to the number of people who have heard the rumour and the number of those who have not yet heard it.
   (a) Write down a differential equation expressing this model.
   (b) Solve this differential equation using the extra information that the town has 1500 rumour-prone inhabitants, 6 people knew the rumour initially, and after three days half the people had heard the rumour.

7. Solve $9y'' + 4y = 0$ where $y = 2$, $y' = -3$ at $x = 0$.

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ANSWERS

CHAPTER 9 DIFFERENTIAL EQUATIONS

REVIEW AND PREVIEW TO CHAPTER 9

EXERCISE 1

1. (a) $F'(x) = 16x^9 - 9x^4 + 3x$

(b) $F'(x) = 3.9x^{0.5} - 7.03x^{0.9}$

(c) $F'(x) = -\dfrac{3}{x} + \dfrac{5}{x^2}$

(d) $F'(x) = \dfrac{2}{2x + 7} + \dfrac{1}{2\sqrt{x - 3}}$

(e) $F'(x) = -\dfrac{12}{x^3} + \dfrac{5}{x^2} + \dfrac{6}{x}$

(f) $F'(x) = \sin 2x + 2 \cos x$

(g) $F'(x) = 21 \cos 3x + 77 \sin 7x$

(h) $F'(x) = -4 \cos(x + 2) - 15 \sin(3x - 7)$

(i) $F'(x) = e^{2x} + e^{-3x} + e^{4x}$

(j) $F'(x) = -40e^{8x} - 12e^{-6x}$

(k) $F'(x) = \dfrac{1}{2\sqrt{x}} - \dfrac{1}{2\sqrt{1 - x}}$

(l) $F'(x) = \dfrac{1}{x} - \dfrac{1}{1 - x}$

(m) $F'(x) = \dfrac{4}{x} + \dfrac{5}{1 - x}$

(n) $F'(x) = 4xe^{x^2} - 12xe^{2x^2}$

(o) $F'(x) = \cos^2 x - \sin^2 x (= \cos 2x)$

(p) $F'(x) = \dfrac{x}{\sqrt{x^2 + 1}}$

(q) $F'(x) = \dfrac{3(x^2 + 2)}{x^3 + 6x + 7}$

(r) $F'(x) = -\tan x$

EXERCISE 2

2. (a) $-16y$ (b) $-y$ (c) $-2y$ (d) $-ky$

EXERCISE 9.1

1. (a) $F(x) = x^2 + x + C$
(b) $F(x) = x^4 - 11x + C$
(c) $F(x) = 1.6x^{10} - 1.8x^5 + 1.5x^2 + C$
(d) $F(x) = \frac{1}{8}x^8 + \frac{1}{6}x^6 + \frac{1}{4}x^4 + \frac{1}{2}x^2 + C$

2. (a) $F(x) = -\dfrac{1}{3x^6} + \frac{1}{12}x^6 + C$
(b) $F(x) = \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{4}x^{\frac{4}{3}} + C$
(c) $F(x) = -3 \ln x - \dfrac{5}{x} + C$
(d) $F(x) = -\frac{1}{6}x^{-6} - \frac{1}{4}x^{-4} - \frac{1}{2}x^{-2} + \ln x + C$

3. (a) $F(x) = \ln|x| + C$
(b) $F(x) = -\dfrac{1}{x^2} + \dfrac{3}{x} + C$
(c) $F(x) = -\frac{2}{3}(-x)^{\frac{3}{2}} + C$
(d) $F(x) = -\dfrac{1}{3x^3} + \frac{1}{4}x^4 - \dfrac{1}{x} + C$

4. (a) $F(x) = -\frac{1}{2}\cos 2x + 2\sin x + C$
(b) $F(x) = -\frac{3}{5}\sin 5x - 8\cos x + C$
(c) $F(x) = 7\sin x + \cos 11x + C$
(d) $F(x) = -4\sin(x+2) + C$

5. (a) $F(x) = e^x - e^{-x} + C$
(b) $F(x) = e^x + e^{-x} + C$
(c) $F(x) = 2e^{2x} + 2e^{-3x} + C$
(d) $F(x) = e^x + \frac{1}{2}e^{-2x} + \frac{1}{3}e^{3x} + C$

6. (a) $F(x) = \frac{2}{3}\left(x^{\frac{3}{2}} + (1-x)^{\frac{3}{2}}\right) + C$
(b) $F(x) = \ln(x - x^2) + C$
(c) $F(x) = 2(\sqrt{x} - \sqrt{1-x}) + C$
(d) $F(x) = 4\ln x - 5\ln(1-x) + C$

7. (a) $F(x) = \frac{1}{2}e^{x^2} + C$ (b) $F(x) = \frac{1}{3}\sin^3 x + C$
(c) $F(x) = \ln(x^2+1) + C$
(d) $F(x) = \sqrt{x^2+1} + C$

8. (a) $F(x) = \arctan x + C$
(b) $F(x) = \ln\cos x + C$
(c) $F(x) = \sec x + C$ (d) $F(x) = \frac{1}{2}x^2 + C$

EXERCISE 9.2

1. (a) $y = 2x^2 - 3x$ (b) $y = 2x^2 - 3x - 1$
(c) $y = 2x^2 - 3x - 3$ (d) $y = 2x^2 - 3x - 9$

2. (a) $s = 4.9t^2$ (b) $s = 0.25t^4 - 0.5t^2$
(c) $s = 1 - \cos t$ (d) $s = 10(e^{0.1t} - 1)$

3. (a) $F(x) = x^3 - \frac{3}{2}x^2 + 6x - 13$
(b) $F(x) = (2x)^{\frac{3}{2}} - 5$
(c) $F(x) = 4e^{\frac{x}{2}} + 3 - 4e$
(d) $F(x) = \frac{2}{3}(x^{1.5} + (4-x)^{1.5}) + 3 - \dfrac{8\sqrt{2}}{3}$

4. (a) $y = \sin x - \cos x + 1$ (b) $y = e^x - e^{-x}$
(c) $y = 2(\sqrt{x+1} - 1)$
(d) $y = 0.25x^4 + 0.5x^2$

5. (a) $F(x) = \frac{1}{2}(x^2+1)$ (b) $F(x) = \frac{1}{4}(x^4+3)$
(c) $F(x) = -\frac{1}{6}(x^6+5)$ (d) $F(x) = -x$

6. $y = e^x + 4$ **7.** $F(x) = \frac{1}{3}(2 - x^3)$

8. (a) $s = \frac{1}{2}(1 + \sin 2t)$
(b) $s = 0.5t - 0.25\sin 2t - \dfrac{\pi}{4}$

EXERCISE 9.3

1. (a) 2 m (b) 4 m (c) 8 m

2. The canister lands with a velocity of 99 m/s, so it probably will not burst.

3. 2.9 s **4.** 4.6 s **5.** 2.5 km

6. (a) 400 m (b) 55 m/s (c) About 13.6 s

7. $\frac{1}{9.8}(v_0 + \sqrt{v_0^2 + 19.6\,h_0})$ seconds

EXERCISE 9.4

1. \$1366.84 **2.** 43 a **3.** 11.5 a

4. (a) 0.45 s (b) 2.1 s **5.** 22.5°C

6. 19 min **7.** 23°C

8. (a) $\dfrac{dA}{dt} = k(M - A)$ (b) $A = M(1 - e^{-kt})$
(c) 29 min

9. $I = \dfrac{E}{R}(1 - e^{-\frac{Rt}{L}})$

EXERCISE 9.5

1. (a) 50.5 kg (b) 173 min

2. (a) 75.9 kg (b) 69 min

3. (a) 41.2 kg (b) 195 min

4. (a) 38.5 kg

5. (a) $A = 30(1 - e^{-0.36t})$ (b) After two minutes
(c) 10 min

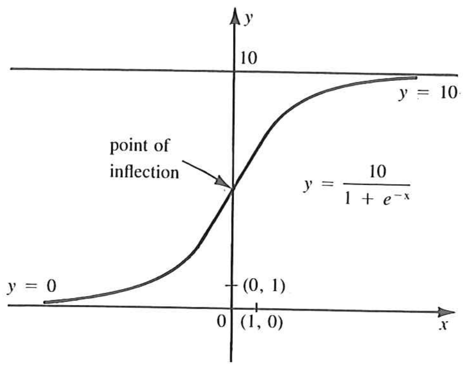
6. (a) $M = 80 + 15e^{-0.0016t}$ (b) 80 kg (c) 433 d

7. $\dfrac{g}{k}$ metres per second

8. $A = cV + (A_0 - cV)e^{-\frac{rt}{V}}$

EXERCISE 9.6

1. (a) $P = \dfrac{6000}{1 + 19e^{-0.75t}}$ (b) 3100 (c) 5.8 a
2. 7.4 h
3. (a) 16 d (b) About 800
4. (a) 27.2 million (b) 23.9 million (c) 1987
5. 185, 620, 1650, 2910, 3640, and 3980.
6.



EXERCISE 9.7

1. (a) $s = 0$ (b) $s = \cos 2t$
   (c) $s = -\cos 2t + \sin 2t$
   (d) $s = 3\cos 2t - 2.5\sin 2t$
2. (a) $y = A\cos x + B\sin x$
   (b) $y = A\cos 3x + B\sin 3x$
   (c) $y = A\cos 1.5x + B\sin 1.5x$
   (d) $y = A\cos\sqrt{2}x + B\sin\sqrt{2}x$
3. (a) $f(x) = \cos x - 2\sin x$
   (b) $f(x) = \cos 2x - \sin 2x$
   (c) $f(x) = \cos 1.5x - \dfrac{2}{1.5}\sin 1.5x$
   (d) $f(x) = \cos\sqrt{2}x - \sqrt{2}\sin\sqrt{2}x$
4. (a) $s = -0.1\cos\sqrt{40}t$
   (b) $s = 0.18\cos\sqrt{40}t$
   (c) $s = -0.40\cos\sqrt{40}t$
   (d) $s = 0.39\cos\sqrt{40}t$
5. (a) $s = -0.26\cos\sqrt{40}t + \dfrac{1}{\sqrt{40}}\sin\sqrt{40}t$
   (b) $s = -0.26\cos\sqrt{40}t - \dfrac{2}{\sqrt{40}}\sin\sqrt{40}t$
   (c) $s = -0.26\cos\sqrt{40}t + \dfrac{3.7}{\sqrt{40}}\sin\sqrt{40}t$
   (d) $s = -0.26\cos\sqrt{40}t - \dfrac{4.1}{\sqrt{40}}\sin\sqrt{40}t$
6. (a) $\sqrt{2}$ (b) 2 (c) 5 (d) $\sqrt{5}$
7. 0.44 m

9.8 REVIEW EXERCISE

1. (a) $F(x) = 1.5x^2 - \pi x + C$
   (b) $F(x) = -e\cos x + \sqrt{2}\sin x + C$
   (c) $F(x) = 2\sqrt{2}\,e^{\sqrt{2}x} + \dfrac{1}{7\pi}e^{-\pi x} + C$
   (d) $F(x) = \ln(x^4 + 1) + C$
2. (a) $F(x) = 0.1\ln x - \sqrt{2}x^{-1} + C$
   (b) $F(x) = 1.6\,x^{2.5} - \dfrac{3}{3.7}x^{3.7} + C$
   (c) $F(x) = \frac{2}{3}(1 + \sqrt{2} + \sqrt{3})x^{\frac{3}{2}} + C$
   (d) $F(x) = e^{\frac{1}{x}} + C$
3. (a) $F(x) = \frac{2}{3}x^3 - \frac{3}{2}x^2 + \frac{37}{6}$
   (b) $F(x) = e^x + \frac{1}{2}e^{-2x} + 4 - e^{-1} - \frac{1}{2}e^2$
   (c) $F(x) = -\cos x - \sin x + 4 + \cos 1 - \sin 1$
   (d) $F(x) = \frac{1}{3}[(3 + 2x)^{1.5} + 11]$
4. 103 s
5. (a) 560 m (b) 63.4 m/s (c) 19 s
6. 22°C
7. (a) 52 kg (b) 23 min
8. (a) $P = \dfrac{420}{1 + 20e^{-2.5t}}$ (b) 415 (c) 30 h
9. (a) $y = -0.6\sin 5x$
   (b) $y = 2\cos 5x + 0.2\sin 5x$
   (c) $y = \cos 5x$
   (d) $y = 3\cos 5x + 0.6\sin 5x$
10. (a) $s = \dfrac{4}{\sqrt{3}}\sin\sqrt{3}t$ (b) $\dfrac{4}{\sqrt{3}}$

9.9 CHAPTER 9 TEST

1. (b) $F(x) = \frac{1}{3}x^3 - 3e^{-x} - 4\cos x + C$
2. $F(x) = \frac{1}{3}(2\sqrt{2}x^{\frac{3}{2}} + 18x - 29)$
3. 6.5 s
4. 3 min
5. 4.6 min
6. If N is the number who have heard the rumour after t days, then $N = \dfrac{1500}{1 + 249e^{-1.8t}}$
7. $y = 2\cos\left(\frac{2x}{3}\right) - 4.5\sin\left(\frac{2x}{3}\right)$