TABLE OF CONTENTS

3 APPLICATIONS OF DERIVATIVES 117

4 EXTREME VALUES 161

5 CURVE SKETCHING 203

6 TRIGONOMETRIC FUNCTIONS 249

7 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS 297

8 EXPONENTIAL AND LOGARITHMIC FUNCTIONS 349

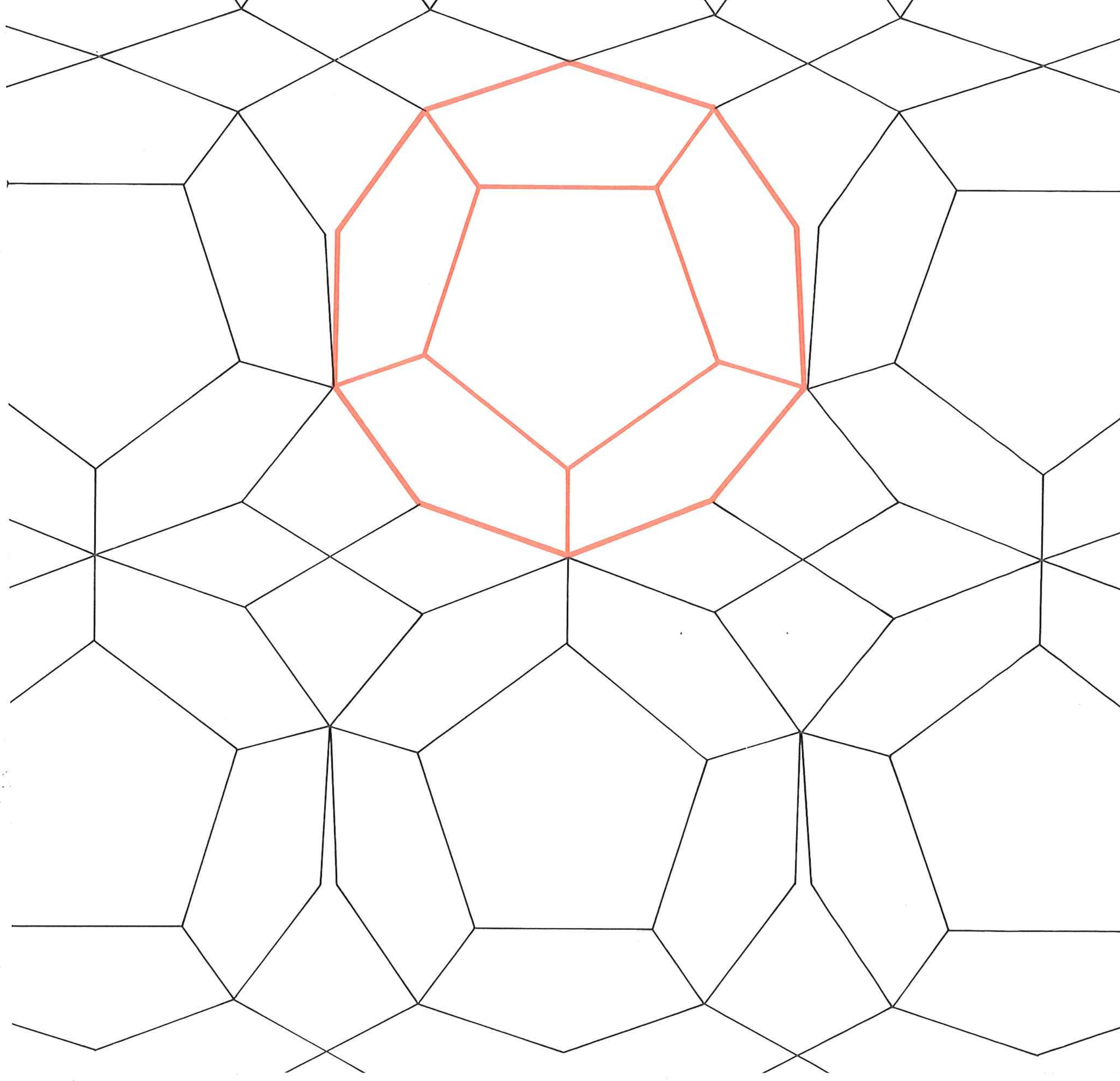
9 DIFFERENTIAL EQUATIONS 401

10 AREA 443

11 INTEGRALS 491

CHAPTER 7

DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

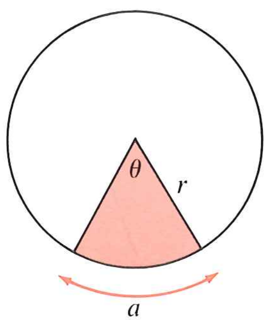


REVIEW AND PREVIEW TO CHAPTER 7

Area of a Sector

The region of a circle cut off by two radii is called a **sector**. The angle contained by the radii at the centre of the circle is the **sector angle**. In this chapter, the formula for the area of a sector is required to develop a very important limit involving the sine function.

A simple proportion enables us to find the area of a sector given the sector angle in radians. Suppose we have a circle with radius r containing a sector with sector angle θ. We compare the area of the sector to the area of the circle.



Sector area
$= \frac{1}{2} r^2\theta$
$= \frac{1}{2} ar$

$$\frac{\text{Area of sector}}{\text{Area of circle}} = \frac{\text{sector angle}}{1 \text{ revolution}} = \frac{\theta}{2\pi}$$

$$\frac{\text{Area of sector}}{\pi r^2} = \frac{\theta}{2\pi}$$

Therefore

$$\boxed{\text{Sector area} = \tfrac{1}{2}r^2\theta,\ \theta \text{ in radians}}$$

Since $\theta = \frac{a}{r}$, we can express the sector area in terms of its arc length.

Similar to the formula for the area of a triangle $\frac{1}{2}bh$

$$\boxed{\text{Sector area} = \tfrac{1}{2}ar}$$

Example 1 Calculate the area of the sector having sector angle 2 if the arc length is 3.5 cm.

Remember: 2 means 2 rad

Solution The radius is required to calculate the area.

$$r = \frac{a}{\theta} = \frac{3.5}{2} = 1.75$$

$$\text{Sector area} = \tfrac{1}{2}ar = \tfrac{1}{2}(3.5)(1.75) = 3.0625$$

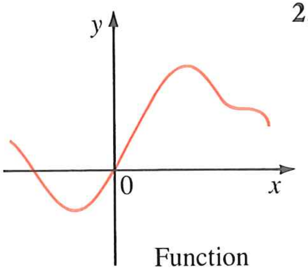
The area is 3.0625 cm^2.

EXERCISE 1

1. The radius of a circle is 10 cm. Calculate:
   (a) the arc length of the sector with sector angle 2.5.
   (b) the sector angle of the sector with arc length 12 cm.
   (c) the area of the sector with arc length 20 cm.
   (d) the area of the sector with sector angle $\frac{2}{3}\pi$.

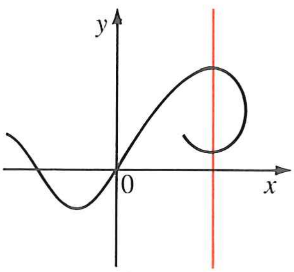
2. A sector with area π cm^2 is contained in a circle with radius 6 cm. Find the arc length of the sector and the sector angle.

Inverse of a Function



Function

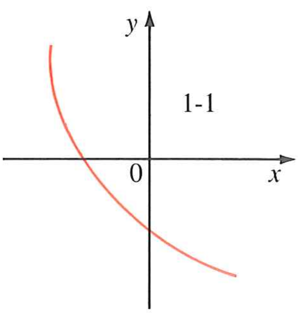
A **function** f is a rule that assigns to each element a in a set A exactly one element, called $f(a)$, in a set B. The set A is called the **domain** of the function and the set of all possible values of $f(a)$ is called the **range**. In order to decide whether a graph is the graph of a function, we use the *Vertical Line Test*.



Not a function

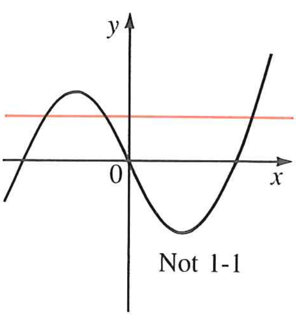
> **Vertical Line Test**
>
> If each vertical line intersects a figure in at most one point, then the figure is the graph of a function. If there is at least one vertical line that intersects the graph in more than one point, then the figure is not the graph of a function.

Some functions are *one-to-one*.



> A function f with domain A and range B is called a **one-to-one** (or 1–1) function if no two elements of A have the same image; that is, every element of B is the image of only one element of A. In symbols
>
> $$f(a_1) \neq f(a_2) \text{ whenever } a_1 \neq a_2.$$

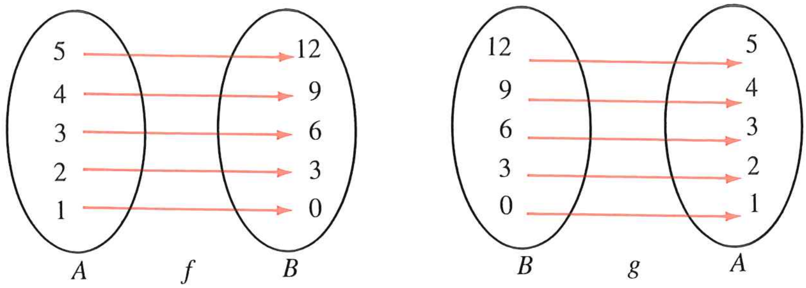
The *Horizontal Line Test* applied to the graph of a function determines if the function is 1–1.



Not 1-1

> **Horizontal Line Test**
>
> If each horizontal line intersects the graph of a function f in at most one point, then f is 1–1. If there is at least one horizontal line that intersects the graph of f in more than one point then f is not 1–1.

In the following arrow diagram, notice that f is a 1–1 function. If we reverse the direction of the arrows, we get a new function g.



g is called the *inverse function* of f and is usually denoted by f^{-1}. (If f were not 1–1, then by reversing the arrows we would not get a function.)

Notice that

$$\begin{aligned} f(1) &= 0 \quad \text{and} \quad f^{-1}(0) = 1 \\ f(2) &= 3 \quad \text{and} \quad f^{-1}(3) = 2 \\ f(3) &= 6 \quad \text{and} \quad f^{-1}(6) = 3 \end{aligned}$$

In general,

$$f(a) = b \quad \text{and} \quad f^{-1}(b) = a$$

If f maps a to b, then f^{-1} maps b back into a. In other words, the inverse function f^{-1} undoes what f does.

Notice also that

$$\begin{aligned} \text{domain of } f = A &= \text{range of } f^{-1} \\ \text{range of } f = B &= \text{domain of } f^{-1} \end{aligned}$$

If f is a 1–1 function with domain A and range B then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b$$

for any $b \in B$.

The defining equation for f^{-1} is found by first solving the equation $y = f(x)$ for x and then interchanging x and y.

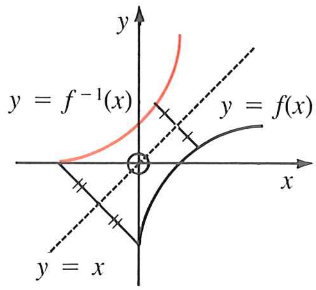
Example 1 Find the inverse of the function $y = \dfrac{1}{x-1}$.

Solution Solve the equation for x.

$$\begin{aligned} y &= \frac{1}{x-1} \\ x - 1 &= \frac{1}{y} \\ x &= 1 + \frac{1}{y} \end{aligned}$$

Now interchange x and y. The equation of the inverse is

$$y = 1 + \frac{1}{x}$$



In Example 1, we performed the transformation $(x, y) \rightarrow (y, x)$. This is a reflection in the line $y = x$.

> The graph of f^{-1} is the reflection of the graph of f in the line $y = x$.

EXERCISE 2

1. Sketch the graph and determine if y is a function of x.
   (a) $x = y^2$ (b) $x = y^2 - 2y$ (c) $y = x^2 + x$
   (d) $x + 2y = 3$ (e) $2x + 5y < 1$ (f) $3y - x \geqslant 6$
   (g) $x = \dfrac{2 - y}{7}$ (h) $x^2 + y^2 + 2x = 0$

2. Which of the following functions are 1–1?
   (a) $f(x) = x + 1$ (b) $g(x) = |x|$ (c) $y = 3 - 2x$
   (d) $h(x) = \dfrac{1}{x}$ (e) $F(x) = \dfrac{1}{x^2}$ (f) $y = 1 - x^2$
   (g) $f(t) = -t^3$ (h) $f(t) = t^4$ (i) $y = \sqrt{x}$
   (j) $f(x) = \dfrac{1}{x^2}, x < 0$

3. Find the inverse function.
   (a) $y = \frac{1}{2}(x - 7)$ (b) $y = \frac{1}{5}(36 - x)$ (c) $y = 5x^3 - 6$
   (d) $y = \sqrt{x}$ (e) $y = \sqrt{x - 3}$ (f) $y = 1 + \dfrac{1}{x}$
   (g) $y = \dfrac{1}{1 + x}$ (h) $y = \dfrac{1 - x}{1 + x}$ (i) $y = \dfrac{4x - 1}{3x + 2}$
   (j) $y = \dfrac{\pi - 3x}{x}$ (k) $y = x^4, x \geqslant 0$
   (l) $y = 3(x - 1)^2, x \geqslant 1$ (m) $y = \sqrt{x^2 + 9}, x \geqslant 0$
   (n) $y = \sqrt{25 - x^2}, x \leqslant 0$

4. Draw the graph of f^{-1} by reflecting the graph of f in the line $y = x$. Find an expression for $f^{-1}(x)$.
   (a) $f(x) = 2x + 1$ (b) $f(x) = x^2 + 2, x \geqslant 0$
   (c) $f(x) = x^3$ (d) $f(x) = -\dfrac{1}{x}$

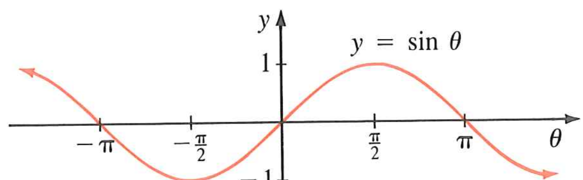
INTRODUCTION

In this chapter we develop the derivatives of the trigonometric functions and the inverse trigonometric functions. These derivatives will be used to sketch curves and solve problems involving optimization or related rates.

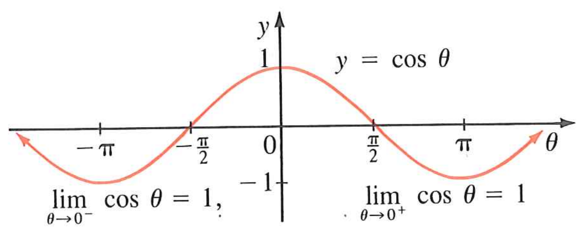
7.1 LIMITS OF TRIGONOMETRIC FUNCTIONS

In order to find the derivatives of the trigonometric functions, we must first evaluate some special limits involving trigonometric functions. It is important to note that the arguments of the trigonometric functions in this section are expressed in radians.

Inspection of the graphs of $y = \sin\theta$ and $y = \cos\theta$ gives us our first two limits.



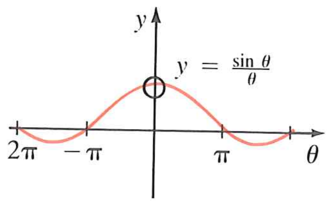
$\lim_{\theta\to 0^-} \sin\theta = 0,\qquad \lim_{\theta\to 0^+} \sin\theta = 0$



$\lim_{\theta\to 0^-} \cos\theta = 1,\qquad \lim_{\theta\to 0^+} \cos\theta = 1$

Therefore $\boxed{\lim_{\theta\to 0} \sin\theta = 0}$ $\qquad$ $\boxed{\lim_{\theta\to 0} \cos\theta = 1}$

The most important limit involving trigonometric functions is $\lim_{\theta\to 0} \dfrac{\sin\theta}{\theta}$. We examine the value of $\dfrac{\sin\theta}{\theta}$ for values of θ close to 0.



Approaching 0 From the Right

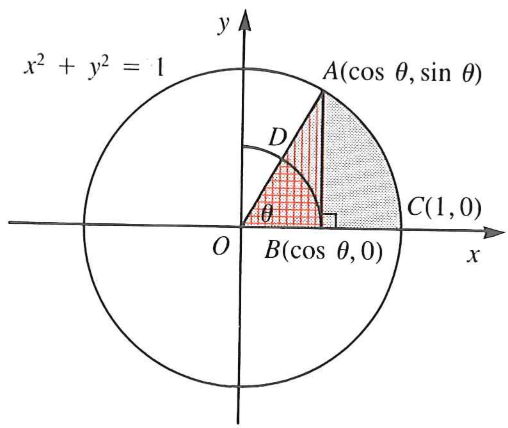
θ	$\frac{\sin \theta}{\theta}$
0.3	0.985 067
0.2	0.993 347
0.1	0.998 334
0.05	0.999 583
0.02	0.999 933
0.01	0.999 983

Approaching 0 From the Left

θ	$\frac{\sin \theta}{\theta}$
−0.3	0.985 067
−0.2	0.993 347
−0.1	0.998 334
−0.05	0.999 583
−0.02	0.999 933
−0.01	0.999 983

The trend of the values of $\frac{\sin \theta}{\theta}$ in the tables suggests that $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$. A proof follows.

In the diagram, point A is on the unit circle $x^2 + y^2 = 1$. Point A determines angle θ, $0 < \theta < \frac{\pi}{2}$. The perpendicular drawn from point A meets the x-axis at B. The circle, radius OB, meets line segment OA at D.



Sector area $= \frac{1}{2}r^2\theta$

Area of Triangle $= \frac{1}{2}bh$

Now Area sector $OBD \leqslant$ Area triangle $OAB \leqslant$ Area sector OAC

Therefore $\frac{1}{2}(OB)^2\,(\theta) \leqslant \frac{1}{2}(OB)(BA) \leqslant \frac{1}{2}(OC)^2(\theta)$

$$\left(\frac{1}{2}\cos^2 \theta\right)(\theta) \leqslant \frac{1}{2}\cos \theta \sin \theta \leqslant \frac{1}{2}(1)^2(\theta)$$

$$\theta \cos^2 \theta \leqslant \cos \theta \sin \theta \leqslant \theta$$

$$\frac{\theta \cos^2 \theta}{\theta \cos \theta} \leqslant \frac{\cos \theta \sin \theta}{\theta \cos \theta} \leqslant \frac{\theta}{\theta \cos \theta}, \quad \theta \cos \theta > 0$$

$$\cos \theta \leqslant \frac{\sin \theta}{\theta} \leqslant \frac{1}{\cos \theta}$$

and $\lim_{\theta \to 0^+} \cos\theta \leqslant \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \leqslant \lim_{\theta \to 0^+} \frac{1}{\cos\theta}$

Applying the squeeze or sandwich theorem.

Therefore $1 \leqslant \lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} \leqslant 1$

and $\lim_{\theta \to 0^+} \frac{\sin\theta}{\theta} = 1$

Since $f(\theta) = \frac{\sin\theta}{\theta}$ is an even function, $\lim_{\theta \to 0^-} \frac{\sin\theta}{\theta} = 1.$

Therefore $\boxed{\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1}$ ①

Most of the limit evaluations in this section will require us to produce this particular limit, since straight substitution will often produce $\frac{0}{0}$.

Example 1 Evaluate $\lim_{x \to 0} \frac{\sin x}{2x}$.

Solution We make use of the fact that $\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x)$. (See Property 3 of limits in Section 1.2.) The technique is simple and is used in a variety of questions.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin x}{2x} &= \lim_{x \to 0} \frac{1}{2}\left(\frac{\sin x}{x}\right) \\
&= \frac{1}{2}\left(\lim_{x \to 0} \frac{\sin x}{x}\right) \\
&= \frac{1}{2}(1) \quad \text{(From ①)} \\
&= \frac{1}{2}
\end{aligned}$$

Example 2 Evaluate $\lim_{x \to 0} \frac{\sin 2x}{x}$.

Solution In order to use ① the denominator must be identical to the argument of the sine function. Multiply the numerator and denominator by 2.

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 2x}{x} &= \lim_{x \to 0} 2\left(\frac{\sin 2x}{2x}\right) \\
&= 2\left(\lim_{2x \to 0} \frac{\sin 2x}{2x}\right) \quad (\text{as } x \to 0,\ 2x \to 0) \\
&= 2(1) \\
&= 2
\end{aligned}$$

Example 3 Evaluate $\lim_{x\to 0} \frac{\sin 7x}{\sin 4x}$.

Solution

$$\lim_{x\to 0} \frac{\sin 7x}{\sin 4x} = \lim_{x\to 0} \frac{7x\left(\frac{\sin 7x}{7x}\right)}{4x\left(\frac{\sin 4x}{4x}\right)}$$

$$= \frac{7}{4}\left(\frac{\lim_{7x\to 0} \frac{\sin 7x}{7x}}{\lim_{4x\to 0} \frac{\sin 4x}{4x}}\right)$$

$$= \left(\frac{7}{4}\right)\frac{(1)}{(1)}$$

$$= \frac{7}{4}$$

Calculator Approximation

x	$\frac{\sin 7x}{\sin 4x}$
0.1	1.654 308
0.01	1.749 038
0.001	1.749 990

Example 4 Evaluate $\lim_{x\to 0} \frac{\cos x - 1}{x}$.

Solution 1 Multiply the numerator and denominator by $\cos x + 1$ and apply the identity $\sin^2 x + \cos^2 x = 1$ to get the expression in terms of $\sin x$.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}$$

$$= \lim_{x\to 0} \frac{-\sin^2 x}{x(\cos x + 1)}$$

$$= -\lim_{x\to 0} \frac{\sin x}{x} \lim_{x\to 0} \frac{\sin x}{1 + \cos x}$$

$$= -(1)\left(\frac{0}{1 + 1}\right)$$

$$= 0$$

Solution 2 We can also use the formula $\cos 2x = 1 - 2\sin^2 x$ to change from the cosine to the sine function.

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = \lim_{x\to 0} \frac{1 - 2\sin^2 \frac{x}{2} - 1}{x}$$

$$= \lim_{x\to 0} \frac{-2\sin^2 \frac{x}{2}}{2\left(\frac{x}{2}\right)}$$

$$= -\lim_{\frac{x}{2}\to 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \lim_{x\to 0} \sin \frac{x}{2}$$
$$= -(1)(0)$$
$$= 0$$

The limit in Example 4 is an important result. It will be used to develop the derivative of $y = \sin x$ in Section 7.2.

Example 5 Evaluate $\lim_{x\to\pi} \frac{\sin x}{\pi - x}$.

Solution In order to use ① we need a variable to approach 0.

Now as $x \to \pi$, $x - \pi \to 0$

Therefore $\pi - x \to 0$

Since $\sin x = \sin(\pi - x)$

we get $\lim_{x\to\pi} \frac{\sin x}{\pi - x} = \lim_{\pi - x\to 0} \frac{\sin(\pi - x)}{\pi - x} = 1$

EXERCISE 7.1

B Use a calculator to estimate the value of each of the following limits.

1. $\lim_{x\to 0} \frac{\sin 3x}{x}$ **2.** $\lim_{x\to 0} \frac{\sin 2x}{\sin 3x}$ **3.** $\lim_{x\to 0} \frac{\sin^3 2x}{\sin^3 3x}$

4. $\lim_{x\to 0} \frac{1 - \cos^2 x}{x^2}$ **5.** $\lim_{x\to 0} \frac{1 - \cos x}{\tan x}$ **6.** $\lim_{x\to 0} \frac{\sin(\cos x)}{\sec x}$

Evaluate each of the following limits.

7. $\lim_{x\to 0} \frac{\sin 3x}{x}$ **8.** $\lim_{x\to 0} \frac{\sin ax}{\sin bx}$ **9.** $\lim_{x\to 0} \frac{\sin^3 2x}{\sin^3 3x}$

10. $\lim_{x\to 0} \frac{1 - \cos x}{x}$ **11.** $\lim_{x\to 0} (x^2 + \cos x)$ **12.** $\lim_{x\to \frac{\pi}{3}} (\sin x - \cos x)$

13. $\lim_{x\to \frac{\pi}{4}} \frac{\sin x}{3x}$ **14.** $\lim_{x\to -3\pi} x^3 \sin^4 x$ **15.** $\lim_{x\to 0} \frac{\sin 5x}{5}$

16. $\lim_{x\to \frac{\pi}{4}} \frac{\tan x}{4x}$ **17.** $\lim_{x\to 0} \frac{\tan 3x}{3 \tan 2x}$ **18.** $\lim_{x\to 0} \frac{\sin^2 3x}{x^2}$

19. $\lim\limits_{x \to \frac{\pi}{6}} \sqrt{\sin x}$ **20.** $\lim\limits_{x \to 0} \dfrac{\sin 6x}{\cos 4x}$ **21.** $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{\sin x}$

22. $\lim\limits_{x \to 0} \dfrac{\tan x}{4x}$ **23.** $\lim\limits_{x \to 0} \dfrac{x^3}{\tan^3 2x}$ **24.** $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{2x^2}$

25. $\lim\limits_{x \to 0} \dfrac{x}{\sin \frac{x}{2}}$ **26.** $\lim\limits_{x \to 0} \dfrac{2 \tan x}{x \sec x}$ **27.** $\lim\limits_{x \to 0} \dfrac{1 - \cos^2 x}{x^2}$

28. $\lim\limits_{x \to 0} \dfrac{1 - \cos 2x}{x^2}$ **29.** $\lim\limits_{x \to \frac{\pi}{2}} \dfrac{\cot x}{\frac{\pi}{2} - x}$ **30.** $\lim\limits_{x \to \pi} \dfrac{\sin x}{x - \pi}$

31. $\lim\limits_{x \to 0} \dfrac{\sin^2 x \cos x}{1 - \cos x}$ **32.** $\lim\limits_{x \to 0} \dfrac{\sin x}{\tan x}$

33. $\lim\limits_{x \to 0} \dfrac{2 \sin x - \sin 2x}{x \cos x}$ **34.** $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x \cos x}$

35. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{\tan x}$ **36.** $\lim\limits_{x \to 0} \dfrac{\csc x - \cot x}{\sin x}$

37. $\lim\limits_{x \to 0} \dfrac{\sin 2x}{2x^2 + x}$ **38.** $\lim\limits_{x \to 0} \dfrac{\sin(\cos x)}{\sec x}$

39. (a) Use a calculator to approximate the value of $\dfrac{\tan x - x}{x^3}$ for $x = 0.1, 0.01, 0.001$, and 0.0001.

(b) Estimate the value of $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3}$ using the results from part (a).

(c) Use a calculator to approximate the value of $\dfrac{\tan x - x}{x^3}$ for $x = 0.00001, 0.000001$, and 0.0000001 and examine your answer to part (b). Can you explain what went wrong?

C **40.** Does the $\lim\limits_{x \to 0} \dfrac{\sin x}{|x|}$ exist? If so, what is it? If not, why not?

41. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin x}{x + \sin x}$.

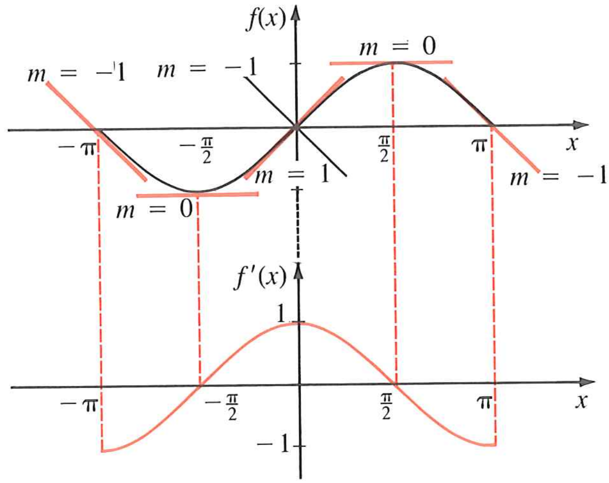
42. Evaluate $\lim\limits_{x \to 1^-} \dfrac{\sin(x - 1)}{|x - 1|}$.

43. Evaluate $\lim\limits_{h \to 0} \dfrac{\sin(a + h) - \sin a}{h}$.

44. Evaluate $\lim\limits_{h \to 0} \dfrac{\cos(a + h) - \cos a}{h}$.

7.2 DERIVATIVES OF THE SINE AND COSINE FUNCTIONS

If we apply the interpretation of $f'(x)$ as the slope of the tangent line at $(x, f(x))$ to the function $f(x) = \sin x$, it appears that $f'(x) = \cos x$. That is, the derivative of the sine function is the cosine function.



This conjecture is confirmed when we apply the definition of the derivative to calculate $f'(x)$ when $f(x) = \sin x$.

$$\boxed{\frac{d}{dx}\sin x = \cos x}$$

Proof

$$\begin{aligned}
f(x) &= \sin x \\
f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x(\cos h - 1) + \cos x\sin h}{h} \\
&= \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} \\
&= \sin x(0) + \cos x(1) \qquad \text{(From Example 4, Section 7.1)} \\
&= \cos x
\end{aligned}$$

The derivative of the cosine function can now be found. We use a co-related angle identity to obtain an equivalent sine function. The derivative is calculated and a co-related angle identity is used to create the desired form.

$$\boxed{\frac{d}{dx}\cos x = -\sin x}$$

Proof

$$f(x) = \cos x$$
$$= \sin\left(\frac{\pi}{2} - x\right)$$

Therefore, using the Chain Rule,

$$f'(x) = \cos\left(\frac{\pi}{2} - x\right)\frac{d}{dx}\left(\frac{\pi}{2} - x\right)$$
$$= (\sin x)(-1)$$
$$= -\sin x$$

Example 1 Differentiate.

(a) $y = \sin 3x$ (b) $y = \sin(x + 2)$ (c) $y = \sin(kx + d)$

Solution We must always be aware of the use of the Chain Rule when we differentiate the trigonometric functions.

(a) $y = \sin 3x$

$$\frac{dy}{dx} = \cos 3x \frac{d}{dx} 3x$$
$$= 3\cos 3x$$

(b) $y = \sin(x + 2)$

$$\frac{dy}{dx} = \cos(x + 2)\frac{d}{dx}(x + 2)$$
$$= \cos(x + 2)$$

(c) $y = \sin(kx + d)$

$$\frac{dy}{dx} = \cos(kx + d)\frac{d}{dx}(kx + d)$$
$$= k\cos(kx + d)$$

Example 2 Differentiate (a) $y = \sin(x^3)$, (b) $y = \sin^3 x$ and (c) $y = \sin^3(x^2 - 1)$.

Solution Again we must be aware of the Chain Rule.

(a) $y = \sin(x^3)$

$$\frac{dy}{dx} = \cos(x^3)\frac{d}{dx}x^3$$
$$= 3x^2\cos x^3$$

(b) $y = \sin^3 x$

$$y = (\sin x)^3$$
$$\frac{dy}{dx} = 3(\sin x)^2\frac{d}{dx}\sin x$$
$$= 3\sin^2 x\cos x$$

(c) We use the Chain Rule twice.

$$y = \sin^3(x^2 - 1)$$
$$y = [\sin(x^2 - 1)]^3$$
$$\frac{dy}{dx} = 3[\sin(x^2 - 1)]^2 \frac{d}{dx} \sin(x^2 - 1)$$
$$= 3 \sin^2(x^2 - 1)\cos(x^2 - 1) \frac{d}{dx}(x^2 - 1)$$
$$= 3 \sin^2(x^2 - 1)[\cos(x^2 - 1)](2x)$$
$$= 6x \sin^2(x^2 - 1)\cos(x^2 - 1)$$

Example 3 Differentiate $y = x^2 \cos x$.

Solution We use the Product Rule.

$$y = x^2 \cos x$$
$$\frac{dy}{dx} = x^2 \frac{d}{dx} \cos x + \cos x \frac{d}{dx} x^2$$
$$= x^2(-\sin x) + (\cos x)(2x)$$
$$= -x^2 \sin x + 2x \cos x$$

Example 4 If $\sin x + \sin y = 1$ find the derivative of y with respect to x.

Solution Differentiate implicitly.

$$\sin x + \sin y = 1$$
$$\cos x + \cos y \frac{dy}{dx} = 0$$
$$\cos y \frac{dy}{dx} = -\cos x$$
$$\frac{dy}{dx} = -\frac{\cos x}{\cos y}$$

Example 5 Find the equation of the tangent line to $y = \dfrac{\sin x}{\cos 2x}$ at the point where $x = \dfrac{\pi}{6}$.

Solution To find the equation of a straight line, we need its slope and a point on the line.

When $x = \dfrac{\pi}{6}$, $\quad y = \dfrac{\sin \frac{\pi}{6}}{\cos \frac{\pi}{3}} = \dfrac{\frac{1}{2}}{\frac{1}{2}} = 1$. The required point is $\left(\dfrac{\pi}{6}, 1\right)$.

Now $$\frac{dy}{dx} = \frac{\cos 2x \dfrac{d}{dx}\sin x - \sin x \dfrac{d}{dx}\cos 2x}{(\cos 2x)^2}$$

$$= \frac{\cos 2x \cos x - \sin x(-\sin 2x)\dfrac{d}{dx}2x}{\cos^2 2x}$$

$$= \frac{\cos x \cos 2x + 2 \sin x \sin 2x}{\cos^2 2x}$$

Thus the slope of the tangent line at $x = \dfrac{\pi}{6}$ is

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = \frac{\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2}\right)^2} = 3\sqrt{3}$$

The equation of the required tangent line is

$$y - 1 = 3\sqrt{3}\left(x - \frac{\pi}{6}\right)$$

$$y - 1 = 3\sqrt{3}x - \frac{\sqrt{3}\,\pi}{2}$$

$$6\sqrt{3}x - 2y = \pi\sqrt{3} - 2$$

Example 6 Use the procedure developed in Chapter 5 to sketch $f(x) = x - \cos x$, $-2\pi \leqslant x \leqslant 2\pi$.

Solution A. ***Domain.*** The restricted domain $-2\pi \leqslant x \leqslant 2\pi$ is given.

B. ***Intercepts.*** The y-intercept is $f(0) = 0 - \cos 0 = -1$. The x-intercept occurs when $\cos x = x$. It cannot be given exactly, but a graph suggests that it is approximately 0.7. (Newton's Method gives the x-intercept $\doteq 0.739$. See Question 8 in Exercise 7.2.)

C. ***Symmetry.*** Since $$f(-x) = (-x) - \cos(-x) = -x - \cos x \neq f(x) \quad \text{or} \quad -f(x)$$

The function is neither even nor odd. Therefore the curve is not symmetric about the y-axis or the origin.

D. ***Asymptotes.*** None.

E. ***Intervals of increase or decrease.*** The first derivative is

$$f'(x) = 1 + \sin x$$

Since $\quad -1 \leqslant \sin x \leqslant 1$

we have $\quad f'(x) \geqslant 0$

and the function is always increasing.

F. ***Local Maximum and Minimum Values.*** Since the curve is always increasing we examine the end points of the domain.
When $x = -2\pi$, the minimum value of y is

$$-2\pi - \cos(-2\pi) = -2\pi - 1$$

When $x = 2\pi$, the maximum value of y is

$$2\pi - \cos 2\pi = 2\pi - 1$$

G. ***Concavity and Points of Inflection.*** Find the points at which the second derivative is equal to zero and set up a chart to determine the concavity.

$$f''(x) = \cos x$$

Now $\quad \cos x = 0, \ -2\pi \leqslant x \leqslant 2\pi$

when $\quad x = -\frac{3}{2}\pi, \ -\frac{1}{2}\pi, \ \frac{1}{2}\pi$, or $\frac{3}{2}\pi$

The chart summarizes our results.

Interval	$f''(x)$	$f(x)$
$-2\pi < x < -\frac{3}{2}\pi$	$+$	concave up on $\left(-2\pi, -\frac{3}{2}\pi\right)$
$-\frac{3}{2}\pi < x < -\frac{1}{2}\pi$	$-$	concave down on $\left(-\frac{3}{2}\pi, -\frac{1}{2}\pi\right)$
$-\frac{1}{2}\pi < x < \frac{1}{2}\pi$	$+$	concave up on $\left(-\frac{1}{2}\pi, \frac{1}{2}\pi\right)$
$\frac{1}{2}\pi < x < \frac{3}{2}\pi$	$-$	concave down on $\left(\frac{1}{2}\pi, \frac{3}{2}\pi\right)$
$\frac{3}{2}\pi < x < 2\pi$	$+$	concave up on $\left(\frac{3}{2}\pi, 2\pi\right)$

Values of x at which the concavity changes determine points of inflection.

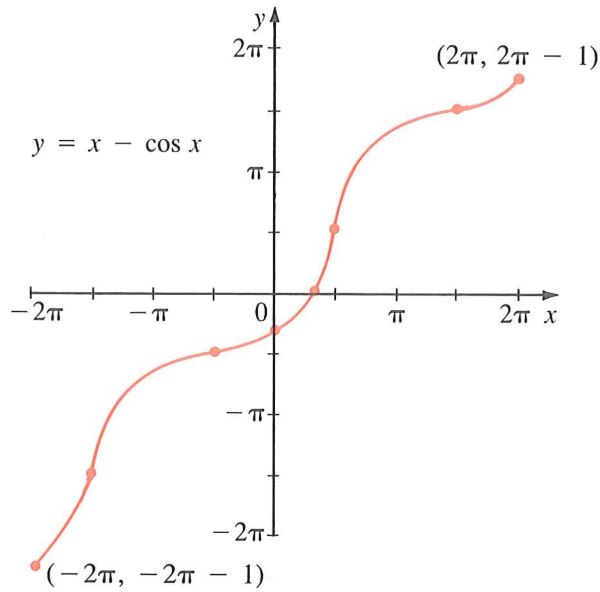
$f\left(-\frac{3}{2}\pi\right) = -\frac{3}{2}\pi$ gives point of inflection $\left(-\frac{3}{2}\pi, -\frac{3}{2}\pi\right)$

$f\left(-\frac{1}{2}\pi\right) = -\frac{1}{2}\pi$ gives point of inflection $\left(-\frac{1}{2}\pi, -\frac{1}{2}\pi\right)$

$f\left(\frac{1}{2}\pi\right) = \frac{1}{2}\pi$ gives point of inflection $\left(\frac{1}{2}\pi, \frac{1}{2}\pi\right)$

$f\left(\frac{3}{2}\pi\right) = \frac{3}{2}\pi$ gives point of inflection $\left(\frac{3}{2}\pi, \frac{3}{2}\pi\right)$

G. ***Sketch of the graph.***



EXERCISE 7.2

B **1.** Find the derivative of y with respect to x in each of the following.

(a) $y = \cos(-4x)$ (b) $y = \sin(3x + 2\pi)$

(c) $y = 4\sin(-2x^2 - 3)$ (d) $y = -\frac{1}{2}\cos(4 + 2x)$

(e) $y = \sin x^2$ (f) $y = -\cos x^2$

(g) $y = \sin^{-2}(x^3)$ (h) $y = \cos(x^2 - 2)^2$

(i) $y = 3\sin^4(2 - x)^{-1}$ (j) $y = x\cos x$

(k) $y = \dfrac{x}{\sin x}$ (l) $y = \dfrac{\sin x}{1 + \cos x}$

(m) $y = (1 + \cos^2 x)^6$ (n) $y = \sin\dfrac{1}{x}$

(o) $y = \sin(\cos x)$ (p) $y = \cos^3(\sin x)$

(q) $y = x\cos\dfrac{1}{x}$ (r) $y = \dfrac{\sin^2 x}{\cos x}$

(s) $y = \dfrac{1 + \sin x}{1 - \sin 2x}$ (t) $y = \sin^3 x + \cos^3 x$

(u) $y = \cos^2\left(\dfrac{1 - \sqrt{x}}{1 + \sqrt{x}}\right)$

2. Find $\dfrac{dy}{dx}$ in each of the following.

(a) $\sin y = \cos 2x$ (b) $x \cos y = \sin(x + y)$
(c) $\sin y + y = \cos x + x$ (d) $\sin(\cos x) = \cos(\sin y)$
(e) $\sin x \cos y + \cos x \sin y = 1$
(f) $\sin x + \cos 2x = 2xy$

3. Find an equation of the tangent line to the given curve at the given point.

(a) $y = 2 \sin x$ at $\left(\dfrac{\pi}{6}, 1\right)$ (b) $y = \dfrac{\sin x}{\cos x}$ at $\left(\dfrac{\pi}{4}, 1\right)$

(c) $y = \dfrac{1}{\cos x} - 2 \cos x$ at $\left(\dfrac{\pi}{3}, 1\right)$

(d) $y = \dfrac{\cos^2 x}{\sin^2 x}$ at $\left(\dfrac{\pi}{4}, 1\right)$

(e) $y = \sin x + \cos 2x$ at $\left(\dfrac{\pi}{6}, 1\right)$

(f) $y = \cos(\cos x)$ at $x = \dfrac{\pi}{2}$

4. Find the critical numbers, the intervals of increase and decrease, and any maximum or minimum values.
(a) $y = \sin^2 x, \ -\pi \leqslant x \leqslant \pi$
(b) $y = \cos x - \sin x, \ -\pi \leqslant x \leqslant \pi$

5. Determine the concavity and find the points of inflection.
(a) $y = 2 \cos x + \sin 2x, \ 0 \leqslant x \leqslant 2\pi$
(b) $y = 4 \sin^2 x - 1, \ -\pi \leqslant x \leqslant \pi$

6. Use the procedure of Example 5 to sketch the graph of each of the following.
(a) $y = x + \sin x, \ 0 \leqslant x \leqslant 2\pi$ (b) $y = x \cos x, \ 0 \leqslant x \leqslant \pi$

7. If $f(x) = \sin x \cos 3x$, evaluate $f''\left(\dfrac{\pi}{3}\right)$.

8. Use Newton's method to find all roots of the given equation correct to 6 decimal places.

(a) $\cos x - x = 0$ (b) $2 \sin x = 2 - x$ (c) $\sin x = \dfrac{x}{2}$

C **9.** Use the results of this section to find the derivative of $y = \tan x$ and $y = \csc x$.

10. If $\sin y + \cos x = 1$ find $\dfrac{d^2y}{dx^2}$.

11. Find $\dfrac{dy}{dx}$ in each of the following.

(a) $y = \dfrac{1}{\sin(x - \sin x)}$ (b) $y = \sqrt{\sin\sqrt{x}}$

(c) $y = \sqrt[3]{x \cos x}$

(d) $y = \cos^3(\cos x) + \sin^2(\cos x)$

(e) $y = \sqrt{\cos(\sin^2 x)}$

12. Find an equation for the tangent line to the curve $x \sin 2y = y \cos 2x$ at the point $\left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right)$

13. Find the derivative of y with respect to x if $x + \tan(xy) = \sin y + \cos x$

PROBLEMS PLUS

If $f(x) = x \sin x$, find $f^{(100)}(0)$.

7.3 DERIVATIVES OF OTHER TRIGONOMETRIC FUNCTIONS

The trigonometric identities allow us to express the remaining trigonometric functions in terms of sine or cosine or both. We can then generate the derivatives of the remaining trigonometric functions using the Quotient and Chain Rules in conjunction with our differentiation formulas for sine and cosine.

$$\boxed{\frac{d}{dx} \tan x = \sec^2 x}$$

Proof

A basic identity transforms

$$y = \tan x$$

into $$y = \frac{\sin x}{\cos x}$$

Applying the Quotient Rule, we get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\cos x \frac{d}{dx} \sin x - \sin x \frac{d}{dx} \cos x}{\cos^2 x} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} \\
&= \sec^2 x
\end{aligned}$$

$$\boxed{\frac{d}{dx} \csc x = -\csc x \cot x}$$

Proof

A basic identity transforms

$$y = \csc x$$

into $\quad y = \dfrac{1}{\sin x} = (\sin x)^{-1}$

Therefore

$$\begin{aligned}
\frac{dy}{dx} &= -(\sin x)^{-2} \frac{d}{dx} \sin x \\
&= -\frac{1}{\sin^2 x} \cos x \\
&= \frac{-1}{\sin x} \frac{\cos x}{\sin x} \\
&= -\csc x \cot x
\end{aligned}$$

The development of the formulas to differentiate the secant and cotangent functions is requested in Exercise 7.3. We collect the derivatives of the trigonometric functions in the table that follows.

Derivatives of the Trigonometric Functions

$$\begin{aligned}
\frac{d}{dx} \sin x &= \cos x & \qquad \frac{d}{dx} \csc x &= -\csc x \cot x \\
\frac{d}{dx} \cos x &= -\sin x & \qquad \frac{d}{dx} \sec x &= \sec x \tan x \\
\frac{d}{dx} \tan x &= \sec^2 x & \qquad \frac{d}{dx} \cot x &= -\csc^2 x
\end{aligned}$$

To help in the memorization of these formulas, note that there is a minus sign in the derivative of each of the "co" functions, that is, the cosine, cosecant, and cotangent functions.

Example 1 Differentiate $f(x) = \dfrac{1}{1 + \tan x}$

Solution

$$\begin{aligned} f(x) &= \frac{1}{1 + \tan x} \\ &= (1 + \tan x)^{-1} \\ f'(x) &= -(1 + \tan x)^{-2}\frac{d}{dx}(1 + \tan x) \\ &= -(1 + \tan x)^{-2}(\sec^2 x) \\ &= \frac{-\sec^2 x}{(1 + \tan x)^2} \end{aligned}$$

It is possible to use algebraic manipulation and trigonometric identities to change the form of the answer in Example 1 to

$$y = \frac{-1}{1 + \sin 2x}$$

If we need to find the second derivative it is easier to work with this form.

Example 2 Differentiate $y = 2\csc^3(3x^2)$

Solution We rewrite in the form $y = 2[\csc(3x^2)]^3$. Note the repeated use of the Chain Rule.

$$\begin{aligned} \frac{dy}{dx} &= 6[\csc(3x^2)]^2\frac{d}{dx}\csc(3x^2) \\ &= 6\csc^2(3x^2)[-\csc(3x^2)]\cot(3x^2)\frac{d}{dx}(3x^2) \\ &= -6\csc^3(3x^2)[\cot(3x^2)](6x) \\ &= -36\, x\csc^3(3x^2)\cot(3x^2) \end{aligned}$$

Example 3 If $\tan y = x^2$ find the derivative of y with respect to x.

Solution We differentiate implicitly.

$$\begin{aligned} \tan y &= x^2 \\ \sec^2 y\frac{dy}{dx} &= 2x \\ \frac{dy}{dx} &= \frac{2x}{\sec^2 y} = 2x\cos^2 y \end{aligned}$$

Example 4 Find the slope of the tangent line to $y = \tan(\csc x)$ when $\sin x = \dfrac{1}{\pi}$, x in the interval $\left(0, \dfrac{\pi}{2}\right)$.

Solution Since the derivative is the slope of the tangent, first find the derivative.

$$\begin{aligned}\frac{dy}{dx} &= \sec^2(\csc x)\frac{d}{dx}\csc x \\ &= [\sec^2(\csc x)](-\csc x \cot x)\end{aligned}$$

When $\sin x = \dfrac{1}{\pi}$, we have $\csc x = \pi$ and $\cot x = \sqrt{\pi^2 - 1}$.

The slope of the tangent is

$$\begin{aligned}\left.\frac{dy}{dx}\right|_{\sin x = \frac{1}{\pi}} &= (\sec^2 \pi)(-\pi)(\sqrt{\pi^2 - 1}) \\ &= -\pi\sqrt{\pi^2 - 1}\end{aligned}$$

Example 5 Prove that $y = \sec x + \tan x$ is concave up on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Solution First we find the second derivative.

$$\begin{aligned}\frac{dy}{dx} &= \sec x \tan x + \sec^2 x \\ \frac{d^2y}{dx^2} &= \sec x \sec^2 x + \tan x \sec x \tan x + 2 \sec x \sec x \tan x \\ &= \sec x(\sec^2 x + 2 \sec x \tan x + \tan^2 x) \\ &= \sec x(\sec x + \tan x)^2\end{aligned}$$

Now $\sec x > 0$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Thus, the second derivative is always positive and the curve is always concave up on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

Example 6 Find the vertical asymptotes of $y = \sec x + \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

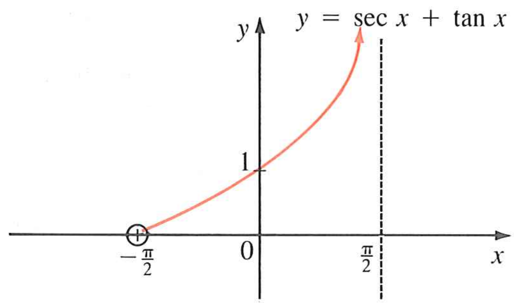
Solution Since $\cos\left(-\dfrac{\pi}{2}\right) = \cos\dfrac{\pi}{2} = 0$, possible vertical asymptotes are $x = -\dfrac{\pi}{2}$ and $x = \dfrac{\pi}{2}$.

$$\begin{aligned}
\lim_{x \to -\frac{\pi}{2}^+} (\sec x + \tan x) &= \lim_{x \to -\frac{\pi}{2}^+} \frac{1 + \sin x}{\cos x} \\
&= \lim_{x \to -\frac{\pi}{2}^+} \frac{1 - \sin^2 x}{\cos x(1 - \sin x)} \\
&= \lim_{x \to -\frac{\pi}{2}^+} \frac{\cos x}{1 - \sin x} \\
&= \frac{0}{1 - (-1)} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\lim_{x \to \frac{\pi}{2}^-} (\sec x + \tan x) &= \lim_{x \to \frac{\pi}{2}^-} \frac{1 + \sin x}{\cos x} \\
&= \infty
\end{aligned}$$

Therefore $x = \dfrac{\pi}{2}$ is a vertical asymptote.

The function considered in Examples 5 and 6 is graphed in the following diagram.



EXERCISE 7.3

B **1.** Find the derivative of each of the following.

(a) $y = 3 \tan 2x$

(b) $y = \frac{1}{3} \cot 9x$

(c) $y = 12 \sec \frac{1}{4}x$

(d) $y = -\frac{1}{4} \csc(-8x)$

(e) $y = \tan x^2$

(f) $y = \tan^2 x$

(g) $y = \sec \sqrt[3]{x}$

(h) $y = x^2 \csc x$

(i) $y = \cot^3(1 - 2x)^2$

(j) $y = \sec^2 x - \tan^2 x$

(k) $y = \dfrac{1}{\sqrt{(\sec 2x - 1)^3}}$

(l) $y = \dfrac{x^2 \tan x}{\sec x}$

(m) $y = 2x(\sqrt{x} - \cot x)$

(n) $y = \sin(\tan x)$

(o) $y = \tan^2(\cos x)$

(p) $y = [\tan(x^2 - x)^{-2}]^{-3}$

2. Find $\dfrac{dy}{dx}$.

(a) $\tan x + \sec y - y = 0$ (b) $\tan 2x = \cos 3y$

(c) $\cot(x + y) + \cot x + \cot y = 0$

(d) $y^2 - \csc(xy) = 0$

(e) $x^2 + \sec\left(\dfrac{x}{y}\right) = 0$

(f) $y^2 = \sin(\tan y) + x^2$

3. Find the equations of the tangent lines.

(a) $y = \cot^2 x$ when $x = \dfrac{\pi}{4}$ (b) $y = \sin x \tan \dfrac{x}{2}$ when $x = \dfrac{\pi}{3}$

(c) $y = \csc 2x$ when $x = -\dfrac{\pi}{8}$

(d) $y = \sec x + \csc x$ when $x = \dfrac{3\pi}{4}$

4. Prove that $y = \sec x + \tan x$ is always increasing on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

5. Find the vertical asymptotes.

(a) $y = \csc x - \cot x$, $0 < x < \pi$

(b) $y = \sin x - \tan x$, $-\dfrac{\pi}{2} < x < \dfrac{3\pi}{2}$

6. Find the critical numbers, intervals of increase and decrease, and maximum and minimum values of $y = \csc x - \cot x$ on $(0, \pi)$.

7. Determine the concavity of $y = \sin x - \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.

8. Use the procedure described in Chapter 5 to sketch $y = x \tan x$ on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$.

9. Prove the following.

(a) $\dfrac{d}{dx} \sec x = \sec x \tan x$ (b) $\dfrac{d}{dx} \cot x = -\csc^2 x$

C 10. If $f(x) = \cot 2x$, $0 \leq x \leq 2\pi$ find all values of x for which $f(x) = f''(x)$.

11. If $x^2 + \tan^2 y = \sec^2 y - y$ find the values of x for which $\dfrac{dy}{dx} = \dfrac{dx}{dy}$.

12. If $f(x) = \sqrt{\sec^3(\sqrt[4]{x})}$ find $f'(x)$.

13. If $x = \cos 3t$ and $y = \sin^2 3t$ find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

7.4 APPLICATIONS

First we examine problems of optimization involving the trigonometric functions. In many problems trigonometric functions are inherent. Such is the case in Example 1.

Example 1 The position of a particle as it moves horizontally is described by the equation $s = 2 \sin t - \cos t$, $0 \leq t \leq 2\pi$ where s is the displacement in metres and t is the time in seconds. Find the maximum and minimum displacements.

Solution Recall that local maximum or minimum displacement occurs when the velocity is zero and that velocity is the rate of change of displacement with respect to time.

$$v = \frac{ds}{dt} = 2 \cos t - (-\sin t) = 2 \cos t + \sin t$$

For critical numbers

$$\begin{aligned} 2 \cos t + \sin t &= 0 \\ \sin t &= -2 \cos t \\ \tan t &= -2 \end{aligned}$$

It is not necessary to solve for t. Only the values of $\sin t$ and $\cos t$ are needed.

for $\frac{\pi}{2} < t < \pi$

$$\sin t = \frac{2}{\sqrt{5}} \text{ and } \cos t = \frac{-1}{\sqrt{5}}$$

$$\begin{aligned} s &= 2\left(\frac{2}{\sqrt{5}}\right) - \left(-\frac{1}{\sqrt{5}}\right) \\ &= \frac{5}{\sqrt{5}} \\ &= \sqrt{5} \end{aligned}$$

for $\frac{3\pi}{2} < t < 2\pi$

$$\sin t = \frac{-2}{\sqrt{5}} \text{ and } \cos t = \frac{1}{\sqrt{5}}$$

$$\begin{aligned} s &= 2\left(\frac{-2}{\sqrt{5}}\right) - \left(\frac{1}{\sqrt{5}}\right) \\ &= -\frac{5}{\sqrt{5}} \\ &= -\sqrt{5} \end{aligned}$$

Finally, we test the end points of the domain.

When $t = 0$, $s = 2 \sin 0 - \cos 0 = -1$

When $t = 2\pi$, $s = 2 \sin 2\pi - \cos \pi = 1$

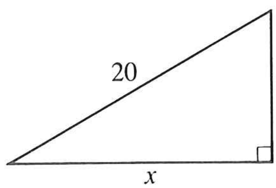
Therefore, the absolute maximum displacement is $\sqrt{5}$ m and the absolute minimum displacement is $-\sqrt{5}$ m.

Although the statement of a problem may not contain reference to trigonometric functions, it is often advantageous to introduce an angle and apply a trigonometric solution. Two different solutions are given for Example 2. Compare them and decide which one you prefer.

Example 2 Find the maximum perimeter of a right triangle with hypotenuse 20 cm.

Solution 1 Let the base be x cm, $0 < x < 20$. Therefore the height is

$$\sqrt{400 - x^2}$$



Let the perimeter be P cm. Therefore

$$P = x + (400 - x^2)^{\frac{1}{2}} + 20$$

$$\frac{dP}{dx} = 1 + \tfrac{1}{2}(400 - x^2)^{-\frac{1}{2}}(-2x) = 1 - x(400 - x^2)^{-\frac{1}{2}}$$

For critical numbers $\quad 1 - \dfrac{x}{\sqrt{400 - x^2}} = 0$

Squaring both sides $\quad 1 = \dfrac{x^2}{400 - x^2}$

Therefore $\quad 400 - x^2 = x^2$

$$400 = 2x^2$$

Since x is positive, $\quad x = 10\sqrt{2}$

We test the second derivative:

$$\frac{d^2P}{dx^2} = 0 - \left[x\left(-\tfrac{1}{2}\right)(400 - x^2)^{-\frac{3}{2}}(-2x) + (400 - x^2)^{-\frac{1}{2}}(1)\right]$$

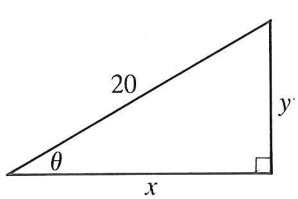
$$= \frac{-x^2}{\sqrt{(400 - x^2)^3}} + \frac{-1}{\sqrt{400 - x^2}}$$

< 0 for all values of x.

Therefore the maximum perimeter is

$$10\sqrt{2} + \sqrt{400 - 200} + 20 = 20 + 20\sqrt{2} \text{ cm.}$$

Solution 2 Let the base angle be θ, $0 < \theta < \dfrac{\pi}{2}$, as in the diagram. Let the perimeter be P cm.



Now $\quad \dfrac{x}{20} = \cos\theta$ and $\dfrac{y}{20} = \sin\theta$

So $\quad x = 20\cos\theta$ and $y = 20\sin\theta$

and $\quad P = 20 + 20\cos\theta + 20\sin\theta$

Therefore $\quad \dfrac{dP}{d\theta} = -20\sin\theta + 20\cos\theta$

For critical numbers $\quad -20 \sin \theta + 20 \cos \theta = 0$

so
$$\sin \theta = \cos \theta$$
$$\tan \theta = 1$$
$$\theta = \frac{\pi}{4}$$

Test the second derivative:

$$\frac{d^2P}{d\theta^2} = -20 \cos \theta - 20 \sin \theta$$
$$< 0 \text{ for all values of } \theta$$

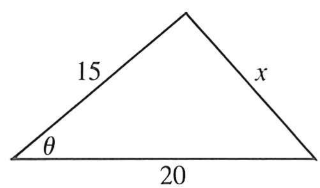
Therefore the maximum perimeter is

$$20 + \frac{20}{\sqrt{2}} + \frac{20}{\sqrt{2}} = 20 + \frac{40}{\sqrt{2}} = 20 + 20\sqrt{2} \text{ cm.}$$

The following examples are related rates problems involving trigonometric functions.

Example 3 Two sides of a triangle have lengths 15 m and 20 m. The angle between them is increasing at $\frac{\pi}{90}$ rad/s. How fast is the length of the third side changing when the angle between the sides is $\frac{\pi}{3}$?

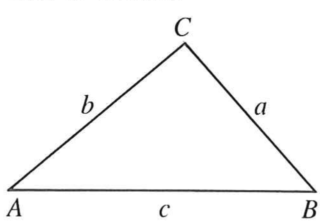
Solution Let the angle between the given sides be θ radians and the variable side be x m. Note that both are functions of time t measured in seconds.



Applying the Law of Cosines,

$$x^2 = 15^2 + 20^2 - 2(15)(20) \cos \theta$$
$$x^2 = 625 - 600 \cos \theta$$

Law of Cosines



$a^2 = b^2 + c^2 - 2bc \cos A$

We differentiate both sides with respect to t:

$$2x \frac{dx}{dt} = -600\left(-\sin \theta \frac{d\theta}{dt}\right)$$
$$\frac{dx}{dt} = \frac{300 \sin \theta}{x} \frac{d\theta}{dt}$$

Now $\frac{d\theta}{dt}$ is the rate at which the angle is increasing, namely $\frac{\pi}{90}$ rad/s.

Therefore $$\frac{dx}{dt} = \frac{300 \sin \theta}{x}\frac{\pi}{90} = \frac{10\pi \sin \theta}{3x}$$

When $\theta = \frac{\pi}{3}$ $$x^2 = 625 - 600 \cos \frac{\pi}{3} = 625 - 600\left(\tfrac{1}{2}\right) = 325$$

and $$x = \sqrt{325} = 5\sqrt{13}$$

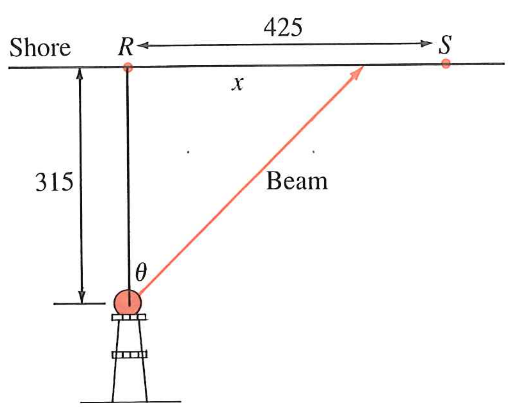
Therefore $$\frac{dx}{dt} = \frac{10\pi \sin \frac{\pi}{3}}{15\sqrt{13}} = \frac{10\pi \frac{\sqrt{3}}{2}}{15\sqrt{13}} = \frac{5\sqrt{3}\pi}{15\sqrt{13}} = \frac{\pi}{\sqrt{39}} \doteq 0.50$$

The third side is increasing at the rate of approximately 0.50 m/s.

Remember that you must develop a formula before you use the specific value that is given to solve the problem. For example, in Example 3 the value $\frac{\pi}{3}$ is substituted at the very end of the problem.

Example 4 A beacon, located a perpendicular distance of 315 m from point R on a straight shoreline, revolves at 1 rev/min. How fast does its beam sweep along the shoreline at point S on the shoreline 425 m from R?

Solution Let the distance that the beam has swept from point R be x m. Let the angle between the perpendicular to point R and the beam be θ. The elapsed time t in seconds is measured from the moment the beam hits point R.



Now $$\frac{x}{315} = \tan \theta$$

Thus $$x = 315 \tan \theta$$

We differentiate implicitly with respect to t.

Therefore $\dfrac{dx}{dt} = 315 \sec^2 \theta \dfrac{d\theta}{dt}$

and so $\dfrac{dx}{dt} = 315 \sec^2 \theta\,(2\pi) = 630\pi \sec^2 \theta$

When $x = 425$ the distance from the beacon to point S is $\sqrt{315^2 + 425^2} \doteq 529$.

Now $\sec \theta \doteq \dfrac{529}{315}$

and $\dfrac{dx}{dt} \doteq 630\pi\left(\dfrac{529}{315}\right)^2 \doteq 5580$

Thus the beam is sweeping along the shore at approximately 5580 m/min.

EXERCISE 7.4

B 1. Find the local maxima and/or minima of each of the following functions.
(a) $f(x) = x - 2 \sin x,\ 0 \leqslant x \leqslant 2\pi$
(b) $f(x) = x + \cos x,\ 0 \leqslant x \leqslant 2\pi$
(c) $f(x) = \sin^4 x + \cos^4 x,\ 0 \leqslant x \leqslant 2\pi$
(d) $f(x) = x \sin x + \cos x,\ -\pi \leqslant x \leqslant \pi$

2. The position of a particle as it moves horizontally is described by the given equations. If s is the displacement in metres and t is the time in seconds find the absolute maximum and absolute minimum displacements.
(a) $s = 2 \sin t + \sin 2t,\ -\pi \leqslant t \leqslant \pi$
(b) $f(t) = \sin^2 t - 2 \cos^2 t,\ -\pi \leqslant t \leqslant \pi$

3. Triangle ABC is inscribed in a semicircle with diameter $BC = 10$ cm. Find the value of angle B that produces the triangle of maximum area.

4. Points A and B lie on a circle, centre O, radius 5 cm. Find the value of angle AOB that produces a maximum area for triangle AOB.

5. Triangle ABC has $AB = AC$. It is inscribed in a circle centre O, radius 10 cm. Find the value of angle BAC that produces a maximum area for triangle ABC.

6. Rectangle $ABCD$ has A and D on the equal sides of an isosceles triangle and B and C on its base. If $AB = 2$ cm and $BC = 6$ cm, find the value of the base angle that produces the triangle of minimum area.

7. A wall of height 8 m stands parallel to and 27 m from a tall building. A ladder with its foot on the ground is to pass over the wall and lean on the building. What angle will the shortest such ladder make with the ground?

8. The angle of elevation of the sun is decreasing at $\frac{1}{4}$ rad/h. How fast is the shadow cast by a building of height 50 m lengthening, when the angle of elevation of the sun is $\frac{\pi}{4}$?

9. A kite 40 m above the ground moves horizontally at the rate of 3 m/s. At what rate is the angle between the string and the horizontal decreasing when 80 m of string has been let out?

10. A revolving beacon is situated 925 m from a straight shore. It turns at 2 rev/min. How fast does the beam sweep along the shore at its nearest point? How fast does it sweep along the shore at a point 1275 m from the nearest point?

11. Two sides of a triangle are six and eight metres in length. If the angle between them decreases at the rate of 0.035 rad/s, find the rate at which the area is decreasing when the angle between the sides of fixed length is $\frac{\pi}{6}$.

12. A ladder 10 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 2 m/s, how fast is the angle between the top of the ladder and the wall changing when the angle is $\frac{\pi}{4}$?

13. The base of an isosceles triangle is 20 cm and the altitude is increasing at the rate of 1 cm/min. At what rate is the base angle increasing when the area is 100 cm^2?

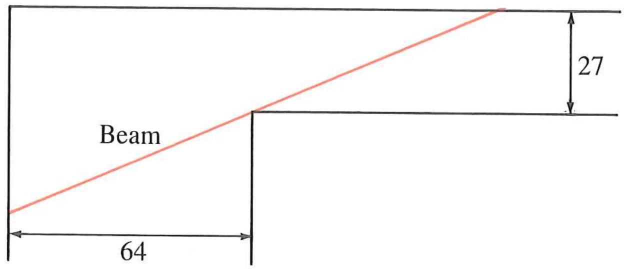
14. A vehicle moves along a straight path with a speed of 4 m/s. A searchlight is located on the ground 20 m from the path and is kept focused on the vehicle. At what rate (in rad/s) is the searchlight rotating when the vehicle is 15 m from the point on the path closest to the searchlight?

C 15. Triangle ABC has $AB = AC$. It is circumscribed about a circle with a radius of 5 cm. What value of angle BAC produces the triangle of minimum area?

16. Rectangle $ABCD$ has $AB = 3$ m and $BC = 4$ m. Find the maximum area of the rectangle that can be circumscribed about rectangle $ABCD$.

17. Prove that the maximum area of the quadrilateral that can be constructed with sides 2, 3, 4, and 5 m occurs when the opposite angles are supplementary.

18. Two corridors whose widths are 64 units and 27 units respectively meet at a right angle. Calculate the length of the longest rigid beam that can pass from one corridor to the other when it is slid along the floor. Assume that the width of the beam is negligible.

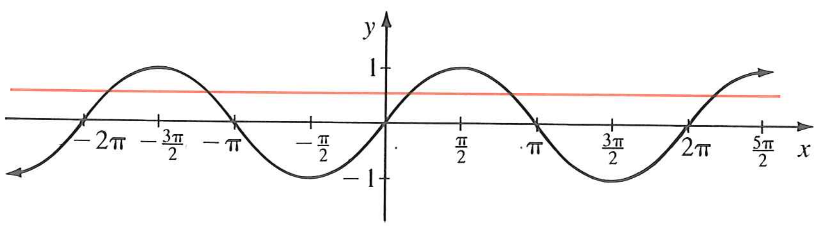


PROBLEMS PLUS

(a) Sketch the graph of the function $f(x) = |\cos x|$.
(b) At what values of x is f not differentiable?
(c) Give a formula for f' and sketch its graph.

**7.5 INVERSE TRIGONOMETRIC FUNCTIONS

At the outset we deal with the apparent contradiction in the title of this section. Examine the graph of $y = \sin x$ in the diagram.

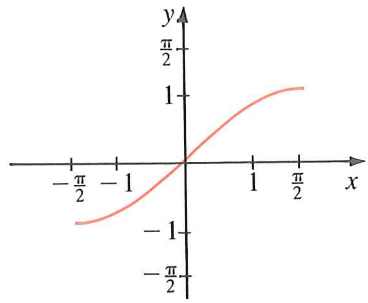


Different calculators use different notations. The most common for inverse sine are
(1) $\sin^{-1}$
(2) arcsin
(3) inv sin
(4) asn

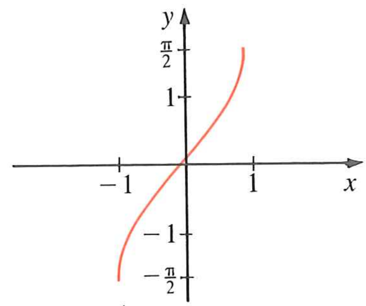
The Horizontal Line Test indicates that it is not 1–1. Therefore the inverse of $y = \sin x$ is not a function. We create the **inverse sine function** or the **arcsine function** by restricting the domain of the sine function so that the resulting function is 1–1. The interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is the domain that we choose. The notation we use for the inverse sine function is $\sin^{-1}$ or arcsin.

Example 1 Sketch the graph of $y = \sin x$, $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$, and its inverse. State the domain and range of each.

Solution



$y = \sin x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$,
$-1 \leq y \leq 1$



$y = \sin^{-1}x$, $-1 \leq x \leq 1$,
$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Therefore, if $-1 \leqslant x \leqslant 1$, $\sin^{-1} x$ is the number y between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$ having the value of its sine equal to x.

$$\boxed{\sin^{-1} x = y \iff \sin y = x,\ -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}}$$

Example 2 Find the exact value of

(a) $\sin^{-1} \dfrac{\sqrt{3}}{2}$ (b) $\sin^{-1}\left(-\frac{1}{2}\right)$.

Solution (a) Let $y = \sin^{-1} \dfrac{\sqrt{3}}{2}$

$$\sin y = \frac{\sqrt{3}}{2},\ -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}.$$

$$y = \frac{\pi}{3}$$

(b) Let $y = \sin^{-1}\left(-\frac{1}{2}\right)$

$$\sin y = \left(-\tfrac{1}{2}\right),\ -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$

$$y = -\frac{\pi}{6}$$

Example 3 Find the exact value of

(a) $\sin^{-1}\left(\sin \dfrac{\pi}{4}\right)$ (b) $\sin\left(\sin^{-1}\left(-\frac{1}{2}\right)\right)$ (c) $\sin^{-1}\left(\sin \dfrac{7\pi}{6}\right)$

Solution (a) Let $$y = \sin^{-1}\left(\sin \frac{\pi}{4}\right)$$

Therefore
$$y = \sin^{-1} \frac{1}{\sqrt{2}}$$
$$\sin y = \frac{1}{\sqrt{2}}, \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$
$$y = \frac{\pi}{4}$$

The $\sin^{-1}$ undoes the sin and the result is $\frac{\pi}{4}$.

(b) Let
$$y = \sin^{-1}\left(-\tfrac{1}{2}\right)$$
Therefore
$$\sin y = \left(-\tfrac{1}{2}\right), \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$
$$y = -\frac{\pi}{6}$$
and
$$\sin\left(-\frac{\pi}{6}\right) = -\tfrac{1}{2}$$

The sin undoes the $\sin^{-1}$ and the result is $-\frac{1}{2}$.

(c) Let
$$y = \sin^{-1}\left(\sin \frac{7\pi}{6}\right)$$
Then
$$y = \sin^{-1}\left(-\tfrac{1}{2}\right)$$
$$\sin y = -\tfrac{1}{2}, \quad -\frac{\pi}{2} \leqslant y \leqslant \frac{\pi}{2}$$
$$y = -\frac{\pi}{6}$$

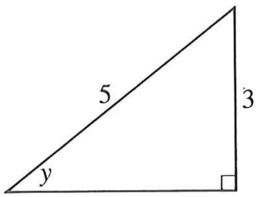
In this case, the $\sin^{-1}$ does not undo the sin because $\frac{7\pi}{6}$ does not fall within the domain we chose to create the inverse sine function.

Examples 3(a) and 3(b) are particular cases of the general equations that are summarized below.

$$\boxed{\begin{aligned} \sin^{-1}(\sin x) &= x \quad \text{for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \\ \sin(\sin^{-1} x) &= x \quad \text{for } -1 \leqslant x \leqslant 1 \end{aligned}}$$

Example 4 Find the exact value of $\tan\left(\sin^{-1}\frac{3}{5}\right)$.

Solution We have done this type of question before. See Exercise 2, Question 4, in the Review and Preview to Chapter 6. We want the tangent of the angle whose sine has value $\frac{3}{5}$.



Let $\qquad y = \sin^{-1}\frac{3}{5}$

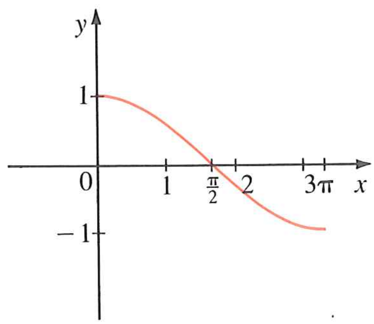
Then $\quad \sin y = \frac{3}{5}, 0 \leqslant y \leqslant \frac{\pi}{2}$

and from our basic definitions

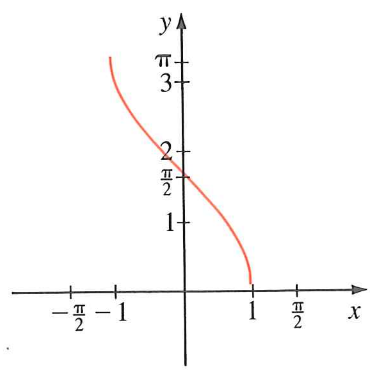
$$\tan y = \frac{3}{4}$$

Thus $\quad \tan\left(\sin^{-1}\frac{3}{5}\right) = \frac{3}{4}$

We create the **inverse cosine function** or the **arccosine function** by restricting the domain of the cosine function so that the resulting function is $1-1$. The interval $[0, \pi]$ is the domain we choose. The notation for the inverse cosine function is $\cos^{-1}$ or arccos.



$y = \cos x, 0 \leqslant x \leqslant \pi,$
$-1 \leqslant y \leqslant 1$



$y = \cos^{-1} x, -1 \leqslant x \leqslant 1,$
$0 \leqslant y \leqslant \pi$

$$\boxed{\cos^{-1} x = y \iff \cos y = x, 0 \leqslant y \leqslant \pi}$$

Example 5 Find the exact value of

(a) $\cos^{-1}\left(\cos \dfrac{5\pi}{6}\right)$ (b) $\cos\left(\cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)\right)$

(c) $\cos^{-1}\left(\cos \dfrac{7\pi}{6}\right)$

Solution (a) Let $y = \cos^{-1}\left(\cos \dfrac{5\pi}{6}\right)$

Therefore $y = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

$$\cos y = -\frac{\sqrt{3}}{2},\ 0 \leqslant y \leqslant \pi$$

$$y = \frac{5\pi}{6}$$

The $\cos^{-1}$ undoes the cos.

(b) Let $y = \cos^{-1}\left(-\dfrac{1}{\sqrt{2}}\right)$

Therefore $\cos y = -\dfrac{1}{\sqrt{2}},\ 0 \leqslant y \leqslant \pi$

$$y = \frac{3\pi}{4}$$

and $\cos\left(\dfrac{3\pi}{4}\right) = -\dfrac{1}{\sqrt{2}}$

The cos undoes the $\cos^{-1}$.

(c) Let $y = \cos^{-1}\left(\cos \dfrac{7\pi}{6}\right)$

Then $y = \cos^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

Therefore $\cos y = -\dfrac{\sqrt{3}}{2},\ 0 \leqslant y \leqslant \pi$

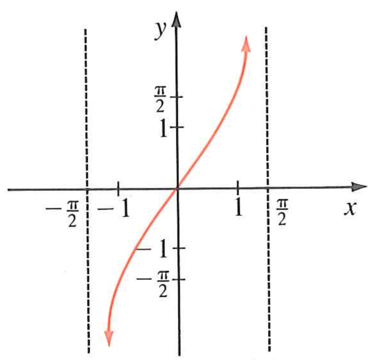
and $y = \frac{5}{6}\pi$

In this case, the $\cos^{-1}$ does not undo the cos because $\dfrac{7\pi}{6}$ falls outside the domain of the inverse cosine function.

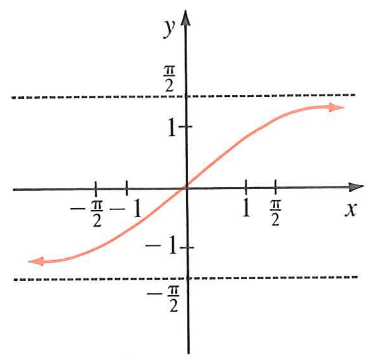
Examples 5(a) and 5(b) are particular cases of the general equations summarized below.

$$\boxed{\begin{array}{ll} \cos^{-1}(\cos x) = x & \text{for } 0 \leqslant x \leqslant \pi \\ \cos(\cos^{-1} x) = x & \text{for } -1 \leqslant x \leqslant 1 \end{array}}$$

We create the **inverse tangent function** by restricting the domain of the tangent function so that the resulting function is 1–1. The interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$ is the domain we choose. The notation for the inverse tangent function is $\tan^{-1}$ or arctan.



$y = \tan x,\ -\frac{\pi}{2} < x < \frac{\pi}{2}$



$y = \tan^{-1}x,\ -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\boxed{\tan^{-1} x = y \iff \tan y = x,\ -\frac{\pi}{2} < y < \frac{\pi}{2}}$$

Notice that the inverse tangent function has two horizontal asymptotes. This fact is expressed by the following limits.

$$\boxed{\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}} \qquad \boxed{\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}}$$

The vertical asymptotes of the restricted tangent function have become the horizontal asymptotes of the inverse tangent function.

Example 6 Find the exact value of $\sin\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$.

Solution Let $$y = \tan^{-1}\left(-\tfrac{4}{3}\right)$$

Therefore $$\tan y = -\tfrac{4}{3}, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

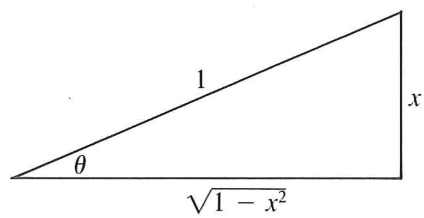
Now
$$\begin{aligned}\sin\left[2\tan^{-1}\left(-\tfrac{4}{3}\right)\right] &= \sin 2y \\ &= 2\sin y\cos y \\ &= 2\left(-\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right) \\ &= -\tfrac{24}{25}\end{aligned}$$

Example 7 Find an equivalent algebraic expression for $\tan(\sin^{-1} x)$.

Solution Let $$\theta = \sin^{-1} x$$

Therefore $$\sin\theta = x, \quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$$

If $\theta > 0$, we can draw a right angle triangle with angle θ as in the diagram and use the Pythagorean Theorem to determine the length of the third side.



Now
$$\begin{aligned}\tan(\sin^{-1} x) &= \tan\theta \\ &= \frac{x}{\sqrt{1-x^2}}\end{aligned}$$

Although the solution illustrates the case for $\theta > 0$, it is true for $\theta < 0$ as well.

Example 8 Evaluate $\lim\limits_{x\to\infty} \tan^{-1}(1-x)$

Solution Let $t = 1 - x$

As $x \to \infty$, $1 - x \to -\infty$, so $t \to -\infty$

$$\begin{aligned}\lim_{x\to\infty} \tan^{-1}(1-x) &= \lim_{t\to-\infty} \tan^{-1} t \\ &= -\frac{\pi}{2}\end{aligned}$$

EXERCISE 7.5

B **1.** Evaluate.

(a) $\sin^{-1} \frac{1}{2}$ (b) $\cos^{-1}\left(-\frac{1}{2}\right)$ (c) $\tan^{-1}(-1)$

(d) $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$ (e) $\cos^{-1} \frac{1}{\sqrt{2}}$ (f) $\tan^{-1} \frac{\sqrt{3}}{3}$

2. Evaluate.

(a) $\sin\left(\tan^{-1} \frac{12}{13}\right)$ (b) $\cos\left(\sin^{-1} \frac{4}{5}\right)$ (c) $\tan\left(\cos^{-1}\left(-\frac{1}{3}\right)\right)$

(d) $\sin\left(\cos^{-1} \frac{7}{8}\right)$ (e) $\cos\left(\tan^{-1} \frac{7}{5}\right)$ (f) $\tan\left(\sin^{-1}\left(-\frac{2}{\sqrt{5}}\right)\right)$

3. Evaluate.

(a) $\sin\left(\sin^{-1} \frac{\sqrt{3}}{2}\right)$ (b) $\sin^{-1}\left(\sin \frac{3\pi}{7}\right)$ (c) $\sin^{-1}\left(\sin \frac{3\pi}{4}\right)$

(d) $\cos\left(\cos^{-1} \frac{\sqrt{2}}{2}\right)$ (e) $\cos^{-1}\left(\cos \frac{7\pi}{8}\right)$ (f) $\cos^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right)$

(g) $\tan(\tan^{-1} \sqrt{3})$ (h) $\tan^{-1}\left(\tan \frac{5\pi}{6}\right)$ (i) $\tan^{-1}\left(\tan\left(-\frac{\pi}{6}\right)\right)$

4. Express as an algebraic function in terms of x.

(a) $\cos(\sin^{-1} x)$ (b) $\sin(\cos^{-1} x)$ (c) $\tan(\sin^{-1} x)$

(d) $\tan(\cos^{-1} x)$ (e) $\sin(\tan^{-1} x)$ (f) $\cos(\tan^{-1} x)$

(g) $\cos(2 \sin^{-1} x)$

5. Evaluate.

(a) $\sin\left(2 \sin^{-1} \frac{3}{5}\right)$ (b) $\cos\left(2 \sin^{-1} \frac{5}{13}\right)$

(c) $\sin\left(\sin^{-1} \frac{1}{3} + \sin^{-1} \frac{2}{3}\right)$ (d) $\cos\left(\sin^{-1} \frac{3}{4} + \cos^{-1} \frac{1}{4}\right)$

6. Find the domain.

(a) $y = \sin^{-1}(1 - x)$ (b) $y = \sin^{-1} x^2$

(c) $y = \sin^{-1}(1 - x^2)$ (d) $y = \cos^{-1}(-x^2)$

(e) $y = \cos^{-1}(x^2 - 4)$ (f) $y = \cos^{-1} \sqrt{x - 1}$

7. Evaluate.

(a) $\lim_{x \to 2^-} \tan^{-1}\left(\frac{3}{x - 2}\right)$ (b) $\lim_{x \to \infty} \tan^{-1}(x^2)$

(c) $\lim_{x \to \infty} \tan^{-1}(x - x^2)$ (d) $\lim_{x \to 3^+} \tan^{-1}\left(\frac{x}{3 - x}\right)$

C **8.** Prove each of the following.

(a) $\tan^{-1} \frac{3}{5} + \sin^{-1} \frac{3}{5} = \tan^{-1} \frac{27}{11}$

(b) $\tan^{-1} \frac{1}{7} + \tan^{-1} \frac{1}{8} + \tan^{-1} \frac{1}{18} = \cot^{-1} 3$

(c) $\sin(2 \sin^{-1} x) = 2x\sqrt{1 - x^2}$

(d) $\tan^{-1} m + \tan^{-1} n = \cos^{-1} \frac{1 - mn}{\sqrt{(1 + m^2)(1 + n^2)}}$

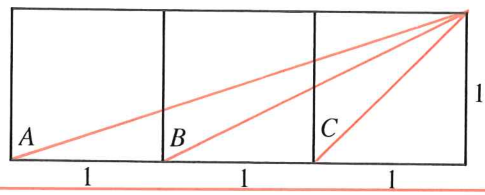
9. Prove that $\tan^{-1} a - \tan^{-1} c = \tan^{-1} \dfrac{a-b}{1+ab} + \tan^{-1} \dfrac{b-c}{1+bc}$.

10. If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, prove that $x + y + z = xyz$.

11. Sketch the graphs of the following functions.
(a) $f(x) = \sin(\sin^{-1} x)$
(b) $f(x) = \sin^{-1}(\sin x)$

PROBLEMS PLUS

Find $\angle A + \angle B + \angle C$ in the diagram.



7.6 DERIVATIVES OF THE INVERSE TRIGONOMETRIC FUNCTIONS

In Section 7.5 we created the inverse sine function $y = \sin^{-1} x$, $-1 \leqslant x \leqslant 1$, $-\dfrac{\pi}{2} \leqslant y \leqslant \dfrac{\pi}{2}$ by restricting the domain of the function $y = \sin x$ to $-\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$. We differentiate implicitly to find the derivative of $y = \sin^{-1} x$.

$$\boxed{\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.}$$

Proof

Recall that $y = \sin^{-1} x$ means that y is the angle having x as the value of its sine and can be written in form $x = \sin y$.

Now $$\frac{d}{dx}(x) = \frac{d}{dx}(\sin y)$$

$$1 = \cos y \frac{dy}{dx}$$

If $y \neq \pm\dfrac{\pi}{2}$, $$\frac{dy}{dx} = \frac{1}{\cos y}$$

We arrive at the desired form by noting that $\cos y$ is positive since $\frac{-\pi}{2} < y < \frac{\pi}{2}$ and isolating $\cos y$ in the identity $\sin^2 y + \cos^2 y = 1$ to produce $\cos y = +\sqrt{1 - \sin^2 y}$.

Thus $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \sin^2 y}}$$

Since $\sin y = x$, $$\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$$

The derivatives of $y = \cos^{-1} x$ and $y = \tan^{-1} x$ are developed in a similar manner.

$$\boxed{\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}}$$

Proof

$$y = \cos^{-1} x$$
$$x = \cos y,\ 0 \leq y \leq \pi$$
$$\frac{d}{dx}(x) = \frac{d}{dx}(\cos y)$$
$$1 = -\sin y \frac{dy}{dx}$$

If $y \neq 0$ or π, $$\frac{dy}{dx} = -\frac{1}{\sin y}$$

We arrive at the desired form by noting that $\sin y$ is positive since $0 < y < \pi$ and isolating $\sin y$ in the identity $\sin^2 y + \cos^2 y = 1$.

Therefore $$\frac{dy}{dx} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}}$$

Proof

$$y = \tan^{-1} x$$

$$x = \tan y, \quad -\frac{\pi}{2} < y < \frac{\pi}{2}$$

$$\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$$

$$1 = \sec^2 y \frac{dy}{dx}$$

Since $\sec^2 y = 1 + \tan^2 y$, we have

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

The derivatives of the inverse trigonometric functions that we have developed are summarized in the table.

Derivatives of the Inverse Trigonometric Functions

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1 + x^2}$$

Example 1 Differentiate $y = \sin^{-1}(1 - x^2)$.

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - (1 - x^2)^2}} \frac{d}{dx}(1 - x^2)$$

$$= \frac{-2x}{\sqrt{2x^2 - x^4}}$$

$$= \frac{-2x}{|x|\sqrt{2 - x^2}}$$

Since $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$, we could write the answer as

$$\frac{dy}{dx} = \begin{cases} \dfrac{2}{\sqrt{2 - x^2}} & \text{if } x < 0 \\ \dfrac{-2}{\sqrt{2 - x^2}} & \text{if } x > 0 \end{cases}$$

Example 2 Differentiate $y = x \tan^{-1} \sqrt{x}$.

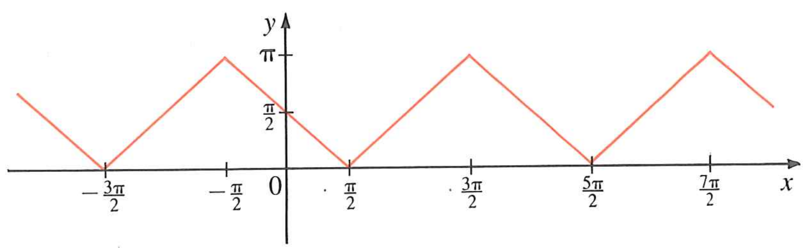
Solution We use the Product Rule.

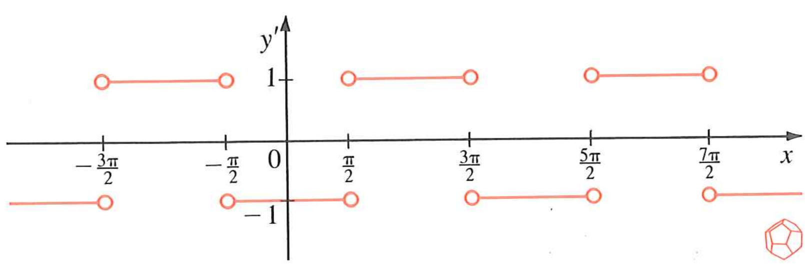
$$\begin{aligned}
\frac{dy}{dx} &= x \frac{d}{dx}(\tan^{-1} \sqrt{x}) + \tan^{-1} \sqrt{x} \frac{d}{dx}(x) \\
&= x \frac{1}{1 + (\sqrt{x})^2} \frac{d}{dx}(x^{\frac{1}{2}}) + \tan^{-1} \sqrt{x}(1) \\
&= \frac{x}{1 + x}\left(\tfrac{1}{2}x^{-\frac{1}{2}}\right) + \tan^{-1} \sqrt{x} \\
&= \frac{x^{\frac{1}{2}}}{2(1 + x)} + \tan^{-1} \sqrt{x} \\
&= \frac{\sqrt{x}}{2(1 + x)} + \tan^{-1} \sqrt{x}
\end{aligned}$$

Example 3 Differentiate $y = \cos^{-1}(\sin x)$ and use the result to sketch the graph.

Solution

$$\begin{aligned}
\frac{dy}{dx} &= \frac{-1}{\sqrt{1 - \sin^2 x}} \frac{d}{dx} \sin x \\
&= \frac{-1}{\sqrt{\cos^2 x}} \cos x \qquad \text{(using the identity } \sin^2 x + \cos^2 x = 1) \\
&= \frac{-\cos x}{|\cos x|} \\
&= \begin{cases} 1 \text{ if } \cos x < 0 \\ -1 \text{ if } \cos x > 0 \end{cases}
\end{aligned}$$





Example 4 If $y = \tan^{-1}\left(\frac{x}{y}\right)$ find $\frac{dy}{dx}$.

Solution

$$\frac{dy}{dx} = \frac{1}{1 + \left(\frac{x}{y}\right)^2} \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{y\frac{d}{dx}(x) - x\frac{d}{dx}(y)}{y^2}$$

$$\frac{dy}{dx} = \frac{y^2}{x^2 + y^2} \cdot \frac{y(1) - x\frac{dy}{dx}}{y^2}$$

$$(x^2 + y^2)\frac{dy}{dx} = y - x\frac{dy}{dx}$$

$$(x^2 + y^2 + x)\frac{dy}{dx} = y$$

$$\frac{dy}{dx} = \frac{y}{x^2 + y^2 + x}$$

EXERCISE 7.6

B **1.** Find $\frac{dy}{dx}$ in each of the following.

(a) $y = \sin^{-1}(x + 1)$
(b) $y = \cos^{-1}(x^2)$
(c) $y = \tan^{-1}(3x)$
(d) $y = (\sin^{-1} x)^2$
(e) $y = \cos^{-1}\left(\frac{x^3}{2}\right)$
(f) $y = (1 + x^2)\tan^{-1} x$
(g) $y = \cos^{-1}\sqrt{2x - 1}$
(h) $y = \tan^{-1}(\sin x)$
(i) $y = \sin^{-1}\left(\frac{\cos x}{1 + \sin x}\right)$
(j) $y = \frac{\sin^{-1} x}{\cos^{-1} x}$
(k) $y = (\tan^{-1} x)^{-1}$
(l) $y = (\cos^{-1} x^2)^{-2}$
(m) $y = \tan^{-1} x + \tan^{-1}\frac{1}{x}$
(n) $y = \frac{\sqrt{1 - x^2}}{x} + \sin^{-1} x$
(o) $y = \frac{x}{\sqrt{1 - x^2}} - \sin^{-1} x$
(p) $y = \sin^{-1} x + \cos^{-1}\sqrt{1 - x^2}$
(q) $y = \sin(\sin^{-1} x^2)$
(r) $y = \sin^{-1}(\tan^{-1} x)$
(s) $y = x^2 \cos^{-1}\left(\frac{2}{x}\right)$

2. Find the slope of the tangent line to $f(x) = x \tan^{-1} x$ at the point where $x = 1$.

3. Find the equation of the tangent line to

   $f(x) = x \sin^{-1}\left(\frac{x}{4}\right) + \sqrt{16 - x^2}$ at the point where $x = 2$.

4. If $f(x) = (3 \tan^{-1} x)^4$, find $f'(\sqrt{3})$.

5. If $y^2 \sin x = \tan^{-1}x - y$ find y'.

6. If $f(x) = (x - 3)\sqrt{6x - x^2} + 9 \sin^{-1}\left(\frac{x - 3}{3}\right)$ find $f'(3)$.

7. Differentiate and use your result to sketch the graph of the given function.

   (a) $y = \sin^{-1}(\cos x)$ (b) $y = \cos^{-1}(\cos x)$
   (c) $y = \sin^{-1}(\sin x)$ (d) $y = \sin^{-1}(\cos 2x)$

7.7 REVIEW EXERCISE

1. Evaluate each of the following limits.

   (a) $\lim\limits_{x\to 0} \dfrac{\sin \frac{1}{2}x}{x}$ (b) $\lim\limits_{x\to 0} \dfrac{\cos\left(\frac{\pi}{2} - x\right)}{x}$ (c) $\lim\limits_{x\to 0} \dfrac{\sin 3x}{5x}$

   (d) $\lim\limits_{x\to 0} \dfrac{\sin^2 x}{x \cos x}$ (e) $\lim\limits_{x\to 0} x \csc x$ (f) $\lim\limits_{x\to 0} \dfrac{\sin x \cos x}{x}$

   (g) $\lim\limits_{x\to 0} \dfrac{1 - \cos 2x}{x}$ (h) $\lim\limits_{x\to 0} \dfrac{2 \tan^2 x}{x^2}$ (i) $\lim\limits_{x\to \frac{\pi}{2}} \dfrac{\frac{\pi}{2} - x}{\cos x}$

2. Differentiate y with respect to x.

   (a) $y = \tan^4 3x$ (b) $y = \dfrac{\sin x}{1 - 2 \cos x}$ (c) $y = \sec x^2$

   (d) $y = \dfrac{\cot^2 2x}{1 + x^2}$ (e) $y = \csc(x^3 + 1)$ (f) $y = 2 \sec \sqrt{x}$

   (g) $y = \sqrt[3]{x} \tan x$ (h) $y = \cos^2(\tan x)$

   (i) $y = \dfrac{1}{\sin(x - \sin x)}$

3. Find $\dfrac{dy}{dx}$ by implicit differentiation.

   (a) $y = \cos(x - y)$ (b) $\sin(x + y) + \sin(x - y) = 1$
   (c) $y = \tan(x + y)$ (d) $\cos(x + y) = y \sin x$
   (e) $\cot xy + xy = 0$ (f) $\csc(x - y) + \sec(x + y) = x$

4. At what points on the curve $y = \sin x + \cos x$, $0 \leq x \leq 2\pi$, is the tangent line horizontal?

5. Find the equation of the tangent line to $y = \tan x$ when $x = \frac{\pi}{3}$.

6. Find the slope of the tangent line to $x \tan y = y - 1$ when $y = \frac{\pi}{4}$.

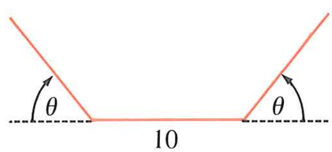
7. The length of the hypotenuse of a right triangle is 10 cm. One of the acute angles is decreasing at the rate of 5°/s. How fast is the area decreasing when this angle is 30°?

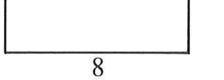
8. Triangle ABC has sides AB and AC increasing at rates of 2 cm/min and 3 cm/min respectively. The angle A between these sides is increasing at 1°/min. How fast is the area changing when $AB = 40$ cm, $AC = 75$ cm and $A = 30°$?

9. A radar antenna is located on a ship that is 4 km from a straight shore. It is rotating at 32 rev/min. How fast does the radar beam sweep along the shore when the angle between the beam and the shortest distance to the shore is $\frac{\pi}{4}$?

10. A triangle has adjacent sides of 4 cm and 6 cm. Prove that the triangle has maximum area when the angle enclosed by these sides is 90°.

11. A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one third of the sheet on each side through an angle θ. How should θ be chosen so that the gutter will carry the maximum amount of water?





12. The upper left-hand corner of a piece of paper 8 cm wide by 12 cm long is folded over to the right hand edge. How would you fold it if you wish to minimize the length of the fold?

13. A rigid beam 25 m long is leaning against a vertical wall. If the bottom of the beam is pulled horizontally away from the wall at 3 m/s, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 15 m from the wall?

14. Use the procedure established in Chapter 5 to sketch the following.
(a) $y = 2\cos x + \cos^2 x,\ 0 \leqslant x \leqslant 2\pi$
(b) $y = x + \sin 2x,\ -\pi \leqslant x \leqslant \pi$

15. Find the point of intersection of the curve $y = \cos x$ with the line $y = -x$ to five decimal places.

16. Find the value of each of the following.

(a) $\cos^{-1} \dfrac{\sqrt{3}}{2}$ (b) $\sin^{-1} \dfrac{1}{\sqrt{2}}$ (c) $\tan^{-1}(-\sqrt{3})$

(d) $\sin\left(\cos^{-1} \dfrac{1}{2}\right)$ (e) $\cos\left(\tan^{-1} \dfrac{3}{4}\right)$ (f) $\tan\left(\sin^{-1} \dfrac{1}{\sqrt{5}}\right)$

(g) $\tan^{-1}\left(\cot \dfrac{\pi}{4}\right)$ (h) $\cos^{-1}\left(\sin \dfrac{\pi}{3}\right)$ (i) $\sin^{-1}\left(\dfrac{1}{2} \tan \dfrac{\pi}{4}\right)$

(j) $\sin^{-1}\left(\sin \dfrac{\pi}{3}\right)$ (k) $\sin^{-1}\left(\sin \dfrac{2\pi}{3}\right)$ (l) $\cos^{-1}\left(\cos \dfrac{4\pi}{3}\right)$

17. Find the derivative of y with respect to x.

(a) $y = \sin^{-1} x^2$ (b) $y = x^3 \sin^{-1}\left(\dfrac{x}{3}\right)$

(c) $y = \cos^{-1} \sqrt{x}$ (d) $y = x - \tan^{-1} x$

(e) $y = \sin^{-1} \dfrac{x-1}{x+1}$ (f) $y = \tan^{-1}(\sin^2 x)$

(g) $y = \cos^{-1}\left(\dfrac{1}{x}\right)$ (h) $y = \sin^{-1}(\sin^{-1} x)$

(i) $y = \tan^{-1}(\tan^{-1} x)$

18. Evaluate

(a) $\lim\limits_{x \to 0^-} \tan^{-1}(x^{-1})$ (b) $\lim\limits_{x \to 0^+} \tan^{-1}[(\sin x)^{-1}]$

19. Find $\dfrac{dy}{dx}$ in the following.

(a) $x \sin y + x^3 = \tan^{-1} y$ (b) $\sin^{-1}(xy) = \cos^{-1}(x + y)$

7.8 CHAPTER 7 TEST

1. Evaluate each of the following limits.

(a) $\lim_{x\to 0} \dfrac{2\tan^2 x}{x^2}$ (b) $\lim_{x\to 0} \dfrac{1-\cos x}{x \sin x}$

2. Find $\dfrac{dy}{dx}$ in each of the following.

(a) $y = \sin^2(x^3 - 2)^{-4}$ (b) $y = \tan(\cos^3 x)$

(c) $y = \dfrac{\csc x}{1 + \cot x}$ (d) $y = \dfrac{\cot^2 2x}{1 + x^2}$

(e) $y = \tan^4 x - \sec^4 x$ (f) $y = (\cos x \sin 2x)^{-2}$

3. If $x \cos y + y \cos x = xy$, find $\dfrac{dy}{dx}$.

4. Find the exact value of $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$.

5. Find the derivative of each of the following.

(a) $y = \sin^{-1}\dfrac{x}{2}$ (b) $y = \tan^{-1}\dfrac{x-1}{x+1}$

(c) $y = \cos^{-1}(\sin x)$

6. For $y = 2\sin x + \cos 2x$, $-\pi \leqslant x \leqslant \pi$,

(a) find the critical numbers;

(b) find the intervals of increase and decrease;

(c) find the absolute maximum and minimum values.

7. An airplane flies west at 150 m/s at an altitude of 1000 m. A searchlight on the ground in the same vertical plane must be kept on the airplane. What is the rate of revolution of the searchlight when the airplane is 500 m due east of the searchlight?

8. The cross section of a trough is an inverted isosceles triangle. Prove that the trough has maximum capacity when the vertex angle is $\frac{\pi}{2}$ rad.

9. If $f(x) = \sin x + \cos x$, find the values of n for which $f^{(n)}(x) = f(x)$.

10. If $y = \cos^{-1}(\cos^{-1} x)$ prove that $\dfrac{dy}{dx} = \dfrac{1}{|\sin y|\sqrt{1 - x^2}}$.

CUMULATIVE REVIEW FOR CHAPTERS 4 TO 7

1. Find each limit.

(a) $\lim\limits_{x\to -3^+} \dfrac{x}{x+3}$ (b) $\lim\limits_{x\to\infty} \dfrac{x}{x+3}$

(c) $\lim\limits_{x\to 1} \dfrac{x+2}{(x-1)^4}$ (d) $\lim\limits_{x\to -\infty} (x^2 + x^3)$

(e) $\lim\limits_{x\to\infty} \dfrac{2x^2+1}{x^2+3x+2}$ (f) $\lim\limits_{x\to -2^-} \dfrac{2x^2+1}{x^2+3x+2}$

(g) $\lim\limits_{x\to 0^-} \cot x$ (h) $\lim\limits_{x\to 0} \dfrac{\sin 5x}{8x}$

(i) $\lim\limits_{x\to 0} \dfrac{\tan 6x}{2x}$ (j) $\lim\limits_{x\to 0} \dfrac{\cos 2x - 1}{2x^2}$

2. If $\sin x = \frac{3}{5}$, x in the interval $\left(0, \dfrac{\pi}{2}\right)$, and $\sin(x+y) = \frac{56}{65}$, y in the interval $\left(0, \dfrac{\pi}{2}\right)$, find the value of $\sin y$ if $\sin y + \cos y = \frac{17}{13}$.

3. Find the exact value of each of the following.

(a) $\sin \dfrac{11\pi}{6} - \cos \dfrac{4\pi}{3}$ (b) $\dfrac{\tan \frac{7}{18}\pi + \tan \frac{1}{9}\pi}{1 - \tan \frac{7}{18}\pi \tan \frac{1}{9}\pi}$

(c) $\tan \dfrac{\pi}{12}$ (d) $\cos^2 \dfrac{\pi}{16} - \sin^2 \dfrac{\pi}{16}$

(e) $\sin \frac{5}{8}\pi \cos \frac{5}{8}\pi$ (f) $\left(\sin \dfrac{\pi}{8} - \cos \dfrac{\pi}{8}\right)^2$

4. Prove.

(a) $\sin(a+b)\sin(a-b) + \cos(a+b)\cos(a-b) = \cos 2b$

(b) $\sin 3a \csc a - \cos 3a \sec a = 2$

(c) $\tan 4a = \dfrac{4\tan a\,(1-\tan^2 a)}{1 - 6\tan^2 a + \tan^4 a}$

(d) $\dfrac{\sin(\pi - a)\cos\left(\dfrac{3\pi}{2} + a\right)}{(\tan \dfrac{\pi}{2} + a)\sin(-a)} = \tan a \sin a$

5. Solve for x.

(a) $\sin x = \cos(2x - \pi)$, $-2\pi \le x \le 2\pi$

(b) $2\cos^2 x \sin^2 x - \cos x \sin x = 0$, $0 \le x \le 2\pi$

(c) $\sin \dfrac{x}{2} + \cos \dfrac{x}{2} = \sqrt{2}$, $0 \le x \le 2\pi$

(d) $\sin^2\left(x + \dfrac{\pi}{6}\right) - \cos^2\left(x + \dfrac{\pi}{6}\right) = \dfrac{\sqrt{2}}{2}$, $0 \le x \le \pi$

6. Evaluate.

(a) $\tan^{-1} \dfrac{1}{\sqrt{3}}$ (b) $\tan \left[\cos^{-1}\left(-\frac{1}{2}\right)\right]$ (c) $\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$

7. Find the derivative of y with respect to x.

(a) $y = \dfrac{x}{\sin x + \cos x}$ (b) $y = \sqrt{\cos^2 x - \sin^2 x}$

(c) $y = \dfrac{\sin ax}{bx}$ (d) $y = \cos\left(\dfrac{2}{\sin x}\right)$

(e) $y = \dfrac{\tan 2x}{1 - \cot 2x}$ (f) $y = x^{-2} \csc x$

(g) $\sin(x - y) = y \cos x$ (h) $\tan^2 x = \sec^2 y$

(i) $y = \tan(\cos(\sin x))$ (j) $y = x^2 \sin^{-1} \dfrac{x}{2}$

(k) $y = \cos^{-1}\left(\dfrac{x}{x - 1}\right)$ (l) $y = \tan^{-1} \sqrt{x}$

8. Find the local maximum and minimum values of f.

(a) $f(x) = 4x^3 - 9x^2 + 6x - 1$

(b) $f(x) = \dfrac{x^3}{x^2 - 1}$

(c) $f(x) = \sin x - \cos x, \dfrac{-\pi}{2} \leq x \leq \dfrac{\pi}{2}$

(d) $f(x) = 2 \sec x + \tan x, -\pi \leq x \leq 2\pi$

9. Find the absolute maximum and minimum values of each function.

(a) $f(x) = x^3 - 6x^2 + 9x + 2, \dfrac{1}{2} \leq x \leq \dfrac{9}{2}$

(b) $f(x) = \sin^2 x + 2 \sin x, 0 \leq x \leq \pi$

10. On what interval is the curve $y = \sin^2 x + 2 \cos x, 0 \leq x \leq 2\pi$, concave upward?

11. Discuss each curve under the headings A–H given in Section 5.5.

(a) $y = 2x^2 - x^4$ (b) $y = x\sqrt{x + 1}$ (c) $y = \dfrac{x^2}{x^2 - 1}$

(d) $y = \cos 2x - x$ in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

12. Sketch the graph of a function f that satisfies all the following conditions.

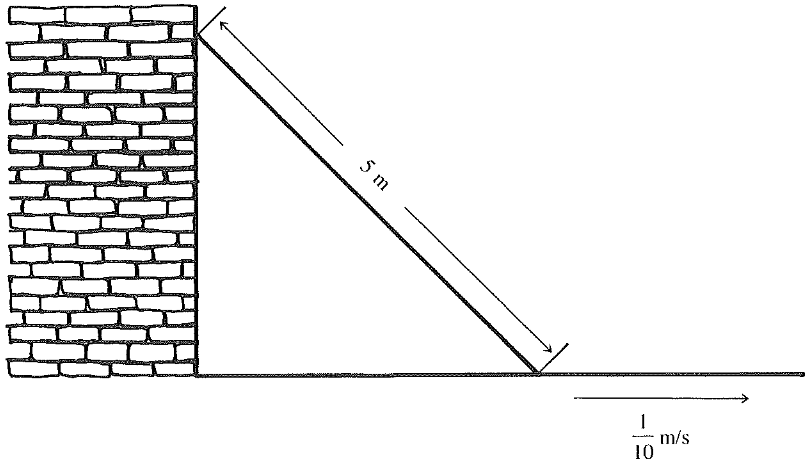
(a) $f(0) = 1, f(1) = 0$

(b) $f'(x) > 0$ for $|x| > 1, f'(x) < 0$ for $|x| < 1$

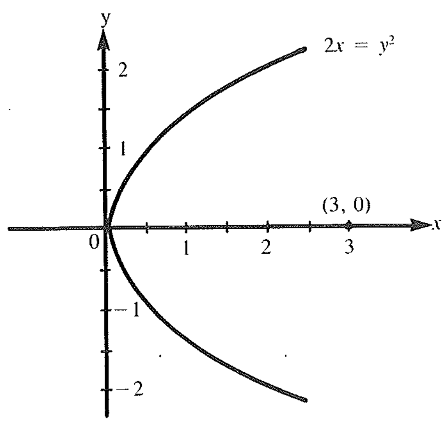
(c) $f''(x) > 0$ for $x > 0$ and $x < -2, f''(x) < 0$ for $-2 < x < 0$

(d) $\lim_{x \to -\infty} f(x) = 0$

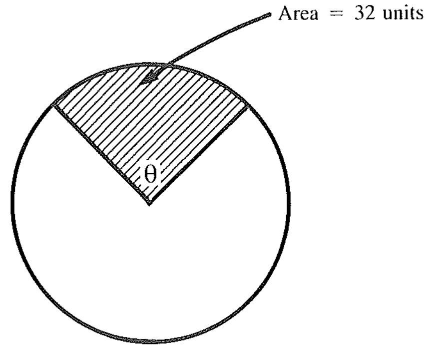
13. A ladder 5 m long rests against a vertical wall. The bottom of the ladder slides away from the wall at the rate of $\frac{1}{10}$ m/s. How fast is the angle between the ladder and the wall increasing when the bottom of the ladder is 3 m from the wall?



14. Find the points on the parabola $2x = y^2$ that are closest to the point $(3, 0)$.



15. A sector of a circle is to have an area of 32. What value of the sector angle θ will give a sector with minimum perimeter?



16. A manufacturer of microwave ovens will, on the average, sell 800 units a month at \$400 per unit. It has been determined that the company can sell an additional 100 ovens for each reduction of \$20 in price.

(a) Find the demand function, assuming that it is linear.

(b) What price will maximize revenue?

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ANSWERS

CHAPTER 7 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

REVIEW AND PREVIEW TO CHAPTER 7

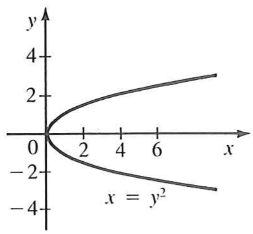
EXERCISE 1

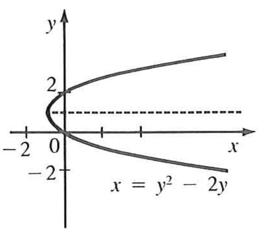
1. (a) 25 cm (b) 1.2 (c) 100 cm^2 (d) $\dfrac{100\pi}{3}$ cm^2

2. $a = \dfrac{\pi}{3}$ cm and $\theta = \dfrac{\pi}{18}$

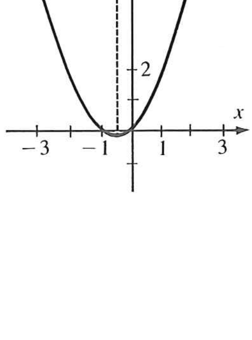
EXERCISE 2

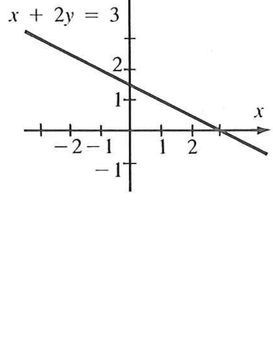
1. (a) $x = y^2$ is not a function
(b) $x = y^2 - 2y$ is not a function



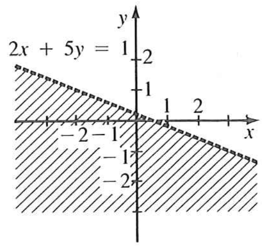


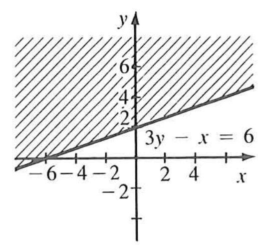
(c) $y = x^2 + x$ is a function
(d) $x + 2y = 3$ is a function



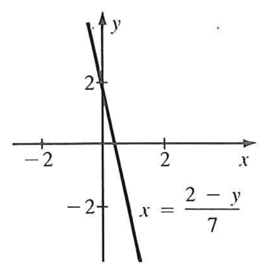


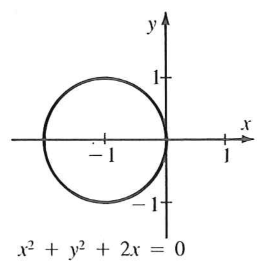
(e) $2x + 5y < 1$ is not a function
(f) $3y - x \geqslant 6$ is not a function





(g) $x = \dfrac{2 - y}{7}$ is a function
(h) $x^2 + y^2 + 2x = 0$ is not a function

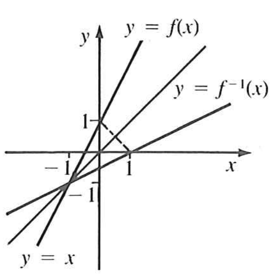


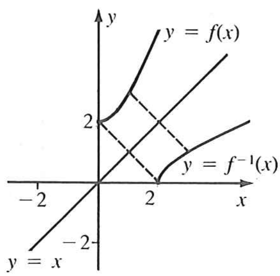


2. (a) 1 − 1 (b) not 1 − 1 (c) 1 − 1
(d) 1 − 1 (e) not 1 − 1 (f) not 1 − 1
(g) 1 − 1 (h) not 1 − 1 (i) 1 − 1
(j) 1 − 1

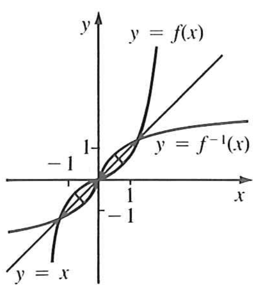
3. (a) $y = 2x + 7$ (b) $y = 36 - 5x$
(c) $y = \sqrt[3]{\dfrac{x+6}{5}}$ (d) $y = x^2, x \geqslant 0$
(e) $y = x^2 + 3, x \geqslant 0$ (f) $y = \dfrac{1}{x-1}$
(g) $y = \dfrac{1}{x} - 1$
(h) $y = \dfrac{1-x}{1+x}$ (i) $y = \dfrac{2x+1}{4-3x}$
(j) $y = \dfrac{\pi}{x+3}$ (k) $y = x^{\frac{1}{4}}$
(l) $y = \sqrt{\dfrac{x}{3}} + 1$ (m) $y = \sqrt{x^2 - 9}$
(n) $y = -\sqrt{25 - x^2}, x \geqslant 0$

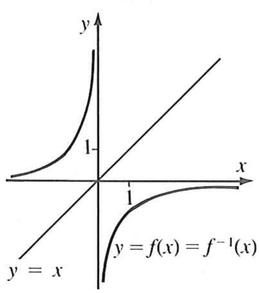
4. (a) $f^{-1}(x) = \dfrac{x-1}{2}$ (b) $f^{-1}(x) = \sqrt{x-2}$





(c) $f^{-1}(x) = x^{\frac{1}{3}}$ (d) $f^{-1}(x) = -\dfrac{1}{x}$





EXERCISE 7.1

1. 2.999 999 955 **2.** 0.666 666 672
3. 0.296 296 304 **4.** 0.998 **5.** 0.000 049 9
6. 0.841 470 978 **7.** 3 **8.** $\dfrac{a}{b}$ **9.** $\frac{8}{27}$
10. 0 **11.** 1 **12.** $\dfrac{\sqrt{3}-1}{2}$ **13.** $\dfrac{2\sqrt{2}}{3\pi}$
14. 0 **15.** 0 **16.** $\dfrac{1}{\pi}$ **17.** $\frac{1}{2}$ **18.** 9
19. $\sqrt{\frac{1}{2}}$ **20.** 0 **21.** 0 **22.** $\frac{1}{4}$ **23.** $\frac{1}{8}$
24. $\frac{1}{4}$ **25.** 2 **26.** 2 **27.** 1 **28.** 2
29. 1 **30.** -1 **31.** 2 **32.** 1 **33.** 0
34. 0 **35.** 0 **36.** $\frac{1}{2}$ **37.** 2 **38.** sin 1

39. (a) 0.334 672 085, 0.333 346 6, 0.333 33, 0.333 (b) Appears to be approaching 0.3
(c) 0.3, -10, -1000. The correct value of this limit is $\frac{1}{3}$. Eventually all calculators will give incorrect values. Different calculators will give different incorrect values. Because of loss of significant digits in the process of rounding off, when two numbers that are very close together are subtracted, this type of error is likely to occur.

40. Limit does not exist. Left-hand limit $= -1$ and right-hand limit $= 1$.

41. $\frac{1}{2}$ **42.** -1 **43.** $\cos a$ **44.** $-\sin a$

EXERCISE 7.2

1. (a) $-4 \sin 4x$ (b) $3 \cos(3x + 2\pi)$
(c) $-16x \cos(-2x^2 - 3)$ (d) $\sin(4 + 2x)$
(e) $2x \cos x^2$ (f) $2x \sin x^2$ (g) $\dfrac{-6x^2 \cos x^3}{\sin^3 x^3}$
(h) $-4x(x^2 - 2) \sin(x^2 - 2)^2$
(i) $\dfrac{12 \cos (2-x)^{-1} \sin^3(2-x)^{-1}}{(2-x)^2}$
(j) $\cos x - x \sin x$ (k) $\dfrac{\sin x - x \cos x}{\sin^2 x}$
(l) $\dfrac{1}{1 + \cos x}$ (m) $-6 \sin 2x(1 + \cos^2 x)^5$
(n) $-\dfrac{1}{x^2} \cos \dfrac{1}{x}$ (o) $-\sin x \cos(\cos x)$
(p) $-3 \cos x \sin(\sin x) \cos^2(\sin x)$
(q) $\dfrac{1}{x} \sin \dfrac{1}{x} + \cos \dfrac{1}{x}$
(r) $\dfrac{\sin x(2 \cos^2 x + \sin^2 x)}{\cos^2 x}$
(s) $\dfrac{\cos x - \cos x \sin 2x + 2 \cos 2x + 2 \sin x \cos 2x}{(1 - \sin 2x)^2}$
(t) $3 \sin x \cos x(\sin x - \cos x)$
(u) $\sin 2\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)\left(\dfrac{1}{\sqrt{x}(1+\sqrt{x})^2}\right)$

2. (a) $\dfrac{-2 \sin 2x}{\cos y}$ (b) $\dfrac{\cos y - \cos(x+y)}{x \sin y + \cos(x+y)}$

(c) $\dfrac{1 - \sin x}{1 + \cos y}$ (d) $\dfrac{\sin x \cos(\cos x)}{\cos y \sin(\sin y}$ (e) -1

(f) $\dfrac{\cos x - 2 \sin 2x - 2y}{2x}$

3. (a) $6\sqrt{3}x - 6y - \pi\sqrt{3} + 6 = 0$
(b) $4x - 2y + 2 - \pi = 0$
(c) $3\sqrt{3}x - y + 1 - \pi\sqrt{3} = 0$
(d) $4x + y - 1 - \pi = 0$
(e) $6\sqrt{3}x + 12y - 12 - \pi\sqrt{3} = 0$
(f) $y = 1$

4. (a) critical numbers $\pm\pi, \pm\dfrac{\pi}{2}, 0$;

increasing on $\left(-\pi, -\dfrac{\pi}{2}\right)$ and $\left(0, \dfrac{\pi}{2}\right)$,

decreasing on $\left(-\dfrac{\pi}{2}, 0\right)$ and $\left(\dfrac{\pi}{2}, \pi\right)$;

local maximums $f\left(-\dfrac{\pi}{2}\right) = 1, f\left(\dfrac{\pi}{2}\right) = 1$;

local minimum $f(0) = 0$.

(b) critical numbers $-\dfrac{\pi}{4}, \dfrac{3\pi}{4}$;

increasing on $\left(-\pi, -\dfrac{\pi}{4}\right)$ or $\left(\dfrac{3\pi}{4}, \pi\right)$;

decreasing on $\left(-\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$;

local maximum $f\left(-\dfrac{\pi}{4}\right) = \dfrac{2}{\sqrt{2}}$;

local minimum $f\left(\dfrac{3\pi}{4}\right) = -\dfrac{2}{\sqrt{2}}$

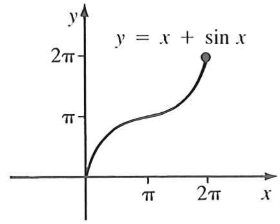
5. (a) concave down on $\left(0, \dfrac{\pi}{2}\right)$ or $\left(3.394, \dfrac{3\pi}{2}\right)$ or $(6.030, 2\pi)$; concave up on $\left(\dfrac{\pi}{2}, 3.394\right)$ or $\left(\dfrac{3\pi}{2}, 6.030\right)$; points of inflection $\left(\dfrac{\pi}{2}, 0\right)$, $(3.394, -1.453)$, $\left(\dfrac{3\pi}{2}, 0\right)$, and $(6.030, 1.451)$

(b) concave down on $\left(-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}\right)$ or $\left(\dfrac{\pi}{4}, \dfrac{3\pi}{4}\right)$;

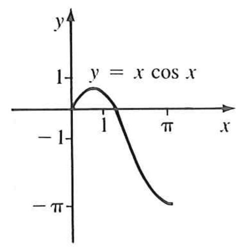
concave up on $\left(-\pi, -\dfrac{3\pi}{4}\right)$ or $\left(-\dfrac{\pi}{4}, \dfrac{\pi}{4}\right)$ or $\left(\dfrac{3\pi}{4}, \pi\right)$;

points of inflection $\left(-\dfrac{3\pi}{4}, 1\right)$, $\left(-\dfrac{\pi}{4}, 1\right)$, $\left(\dfrac{\pi}{4}, 1\right)$ and $\left(\dfrac{3\pi}{4}, 1\right)$.

6. (a) A. $[0, 2\pi]$ B. y-intercept $= 0$, x-intercept $= 0$ C. none D. none E. always increasing F. minimum $f(0) = 0$, maximum $f(2\pi) = 2\pi$ G. CD on $(0, \pi)$ and CU on $(\pi, 2\pi)$; IP (π, π)



(b) A. $(0, \pi)$ B. y-intercept 0, x-intercepts 0, $\dfrac{\pi}{2}$ C. none D. none E. increasing on $(0, 0.86)$, decreasing on $(0.86, \pi)$ F. local maximum $f(0.86) = 0.56$, minimums $f(0) = 0$ and $f(\pi) = -\pi$ G. CD on $(0, 2.29)$ and CU on $(2.29, \pi)$; IP $(2.29, -1.51)$



7. $5\sqrt{3}$

8. (a) 0.739 085 (b) 0.704 576 (c) 1.895 494

9. $\dfrac{d}{dx} \tan x = \sec^2 x$, $\dfrac{d}{dx} \csc x = -\csc x \cot x$

10. $\dfrac{\cos^2 y \cos x + \sin^2 x \sin y}{\cos^3 y}$

11. (a) $(\cos x - 1) \cot(x - \sin x) \csc(x - \sin x)$
(b) $\dfrac{\cos \sqrt{x}}{4\sqrt{x \sin \sqrt{x}}}$ (c) $\dfrac{\cos x - x \sin x}{3\sqrt[3]{x^2 \cos^2 x}}$
(d) $\sin x[\sin(\cos x)][\cos(\cos x)][3 \cos (\cos x) - 2]$
(e) $\dfrac{-\sin x \cos x[\sin(\sin^2 x)]}{\sqrt{\cos(\sin^2 x)}}$

12. $2x - y = 0$ **13.** $\dfrac{\sin x + y \sec^2(xy) + 1}{\cos y - x \sec^2(xy)}$

EXERCISE 7.3

1. (a) $6 \sec^2 2x$ (b) $-3 \csc^2 9x$

(c) $3 \sec \frac{x}{4} \tan \frac{x}{4}$ (d) $-2 \csc 8x \cot 8x$
(e) $2x \sec^2 x^2$ (f) $2 \tan x \sec^2 x$
(g) $\frac{\sec \sqrt[3]{x} \tan \sqrt[3]{x}}{3\sqrt[3]{x^2}}$ (h) $x \csc x(2 - x \cot x)$
(i) $12(1 - 2x) \csc^2(1 - 2x)^2 \cot^2(1 - 2x)^2$
(j) 0 (k) $\frac{-3 \sec 2x \tan 2x}{\sqrt{(\sec 2x - 1)^5}}$ (l) $x(x \cos x + 2 \sin x)$ (m) $3\sqrt{x} + 2x \csc^2 x - 2 \cot x$
(n) $\sec^2 x[\cos(\tan x)]$ (o) $-2 \sin x[\tan(\cos x)][\sec^2(\cos x)]$
(p) $\frac{6(2x - 1)\sec^2(x^2 - x)^{-2}}{(x^2 - x)^3 \tan^4(x^2 - x)^{-2}}$

2. (a) $\frac{\sec^2 x}{1 - \sec y \tan y}$ (b) $-\frac{2 \sec^2 2x}{\sin 3y}$
(c) $-\frac{\csc^2 x + \csc^2(x + y)}{\csc^2(x + y) + \csc^2 y}$
(d) $-\frac{y \csc(xy) \cot(xy)}{2y + x \csc(xy)\cot(xy)}$
(e) $\frac{2xy^2 + y \sec\left(\frac{x}{y}\right) \tan\left(\frac{x}{y}\right)}{x \sec\left(\frac{x}{y}\right) \tan\left(\frac{x}{y}\right)}$
(f) $\frac{2x}{2y - \sec^2 y[\cos(\tan y)]}$

3. (a) $4x + y - 1 - \pi = 0$
(b) $3\sqrt{3}x - 6y - \sqrt{3}\pi + 3 = 0$
(c) $4y + 8\sqrt{2}x + 4\sqrt{2} + \pi\sqrt{2} = 0$
(d) $4\sqrt{2}x - 2y - 3\sqrt{2}\pi = 0$

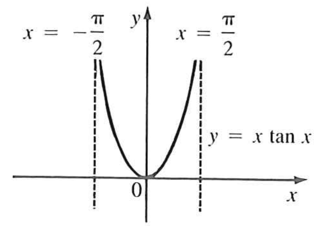
4. No critical numbers exist and $y' > 0$

5. (a) $x = \pi$
(b) $x = -\frac{\pi}{2}$, $x = \frac{\pi}{2}$, and $x = \frac{3\pi}{2}$

6. No critical numbers; always increasing

7. CU on $\left(-\frac{\pi}{2}, 0\right)$ or $\left(\frac{\pi}{2}, \pi\right)$;
CD on $\left(0, \frac{\pi}{2}\right)$ or $\left(\pi, \frac{3\pi}{2}\right)$

8. A. $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ B. y-intercept is 0, x-intercept is 0 C. y-axis D. $x = \pm\frac{\pi}{2}$
E. increasing on $\left(0, \frac{\pi}{2}\right)$, decreasing on $\left(-\frac{\pi}{2}, 0\right)$
F. minimum $f(0) = 0$ G. CU on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
H.



10. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ 11. $\pm\frac{1}{2}$

12. $\frac{3 \sec^{\frac{3}{2}}(\sqrt[4]{x}) \tan(\sqrt[4]{x})}{8(\sqrt[4]{x^3})}$

13. $\frac{dy}{dx} = -2x$, $\frac{d^2y}{dx^2} = -2$

EXERCISE 7.4

1. (a) local minimum $f\left(\frac{\pi}{3}\right) = -0.685$; local maximum $f\left(\frac{5\pi}{3}\right) \doteq 6.968$ (b) none (c) local minima $f\left(\frac{\pi}{4}\right), f\left(\frac{3\pi}{4}\right), f\left(\frac{5\pi}{4}\right), f\left(\frac{7\pi}{4}\right) = \frac{1}{2}$; local maxima $f(0), f\left(\frac{\pi}{2}\right), f(\pi), f\left(\frac{3\pi}{2}\right), f(2\pi) = 1$
(d) local minimum $f(0) = 1$; local maxima $f\left(-\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}\right) \doteq 1.571$

2. (a) abs min $= -\frac{3\sqrt{3}}{2}$; abs max $= \frac{3\sqrt{3}}{2}$
(b) abs min $= -2$; abs max $= 1$

3. $\frac{\pi}{4}$ 4. $\frac{\pi}{2}$ 5. $\frac{\pi}{3}$ 6. 0.588 00

7. 0.588 00 8. 25 m/h 9. 0.02 m/s

10. 33 708 m/min and 11 624 m/min

11. 0.727 m²/min 12. $\frac{\sqrt{2}}{5}$ rad/s

13. 0.05 rad/s 14. 0.128 rad/s 15. $\frac{\pi}{3}$

16. 24.5 m² 18. 125 m

EXERCISE 7.5

1. (a) $\frac{\pi}{6}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{4}$ (d) $-\frac{\pi}{4}$ (e) $\frac{\pi}{4}$
(f) $\frac{\pi}{6}$

2. (a) $\dfrac{12}{\sqrt{313}}$ (b) $\frac{3}{5}$ (c) $-2\sqrt{2}$ (d) $\dfrac{\sqrt{15}}{8}$
(e) $\dfrac{5}{\sqrt{74}}$ (f) -2

3. (a) $\dfrac{\sqrt{3}}{2}$ (b) $\dfrac{3\pi}{7}$ (c) $\dfrac{\pi}{4}$ (d) $\dfrac{\sqrt{2}}{2}$ (e) $\dfrac{7\pi}{8}$
(f) $\dfrac{\pi}{4}$ (g) $\sqrt{3}$ (h) $-\dfrac{\pi}{6}$ (i) $-\dfrac{\pi}{6}$

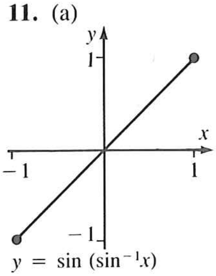
4. (a) $\sqrt{1-x^2}$ (b) $\sqrt{1-x^2}$ (c) $\dfrac{x}{\sqrt{1-x^2}}$
(d) $\dfrac{\sqrt{1-x^2}}{x}$ (e) $\dfrac{x}{\sqrt{1+x^2}}$ (f) $\dfrac{1}{\sqrt{1+x^2}}$
(g) $1-2x^2$

5. (a) $\frac{24}{25}$ (b) $\frac{119}{169}$ (c) $\dfrac{\sqrt{5}+4\sqrt{2}}{9}$
(d) $\dfrac{\sqrt{7}-3\sqrt{15}}{16}$

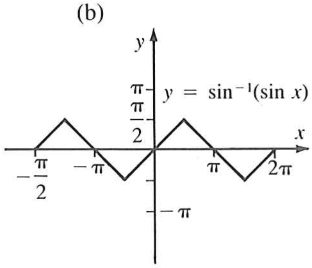
6. (a) $[0, 2]$ (b) $[-1, 1]$ (c) $[-\sqrt{2}, \sqrt{2}]$
(d) $[-1, 1]$ (e) $[-\sqrt{5}, -\sqrt{3}] \cup [\sqrt{3}, \sqrt{5}]$
(f) $[1, 2]$

7. (a) $-\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{2}$ (c) $-\dfrac{\pi}{2}$ (d) $-\dfrac{\pi}{2}$

11. (a)



(b)



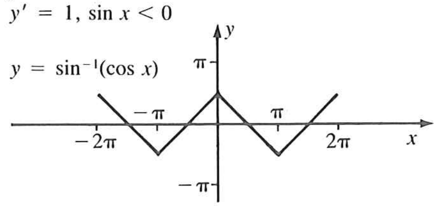
EXERCISE 7.6

1. (a) $\dfrac{1}{\sqrt{-x^2-2x}}$ (b) $\dfrac{-2x}{\sqrt{1-x^4}}$ (c) $\dfrac{3}{1+9x^2}$
(d) $\dfrac{2\sin^{-1}x}{\sqrt{1-x^2}}$ (e) $\dfrac{-3x^2}{\sqrt{4-x^6}}$
(f) $1+2x\tan^{-1}x$
(g) $\dfrac{-1}{\sqrt{(2-2x)(2x-1)}}$ (h) $\dfrac{\cos x}{2-\cos^2 x}$
(i) $\dfrac{-1}{\sqrt{2\sin x(\sin x+1)}}$ (j) $\dfrac{\cos^{-1}x-\sin^{-1}x}{(\cos^{-1}x)^2\sqrt{1-x^2}}$
(k) $\dfrac{-1}{(1+x^2)(\tan^{-1}x)^2}$ (l) $\dfrac{4x}{(\cos^{-1}x^2)^3\sqrt{1-x^4}}$
(m) 0 (n) $\dfrac{x^2-1}{x^2\sqrt{1-x^2}}$ (o) $\dfrac{x^2}{\sqrt{(1-x^2)^3}}$
(p) $\dfrac{|x|+x}{|x|\sqrt{1-x^2}}$ (q) $\dfrac{2x\cos(\sin^{-1}x^2)}{\sqrt{1-x^4}}$
(r) $\dfrac{1}{(1+x^2)\sqrt{1-(\tan^{-1}x)^2}}$
(s) $\dfrac{2|x|}{\sqrt{x^2-4}}+2x\cos^{-1}\left(\dfrac{2}{x}\right)$ **2.** $\dfrac{2+\pi}{4}$

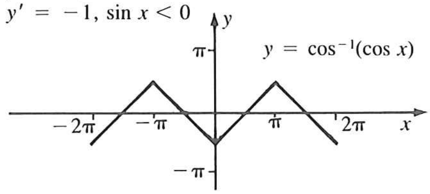
3. $\pi x - 6y + 12\sqrt{3} = 0$ **4.** $3\pi^3$

5. $\dfrac{1-y^2\cos x - x^2y^2\cos x}{(1+x^2)(1+2y\sin x)}$ **6.** 12

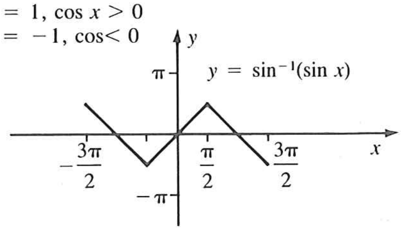
7. (a) $y' = -1, \sin x > 0$
$y' = 1, \sin x < 0$



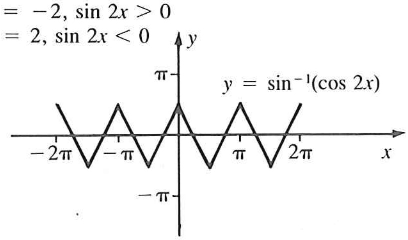
(b) $y' = 1, \sin x > 0$
$y' = -1, \sin x < 0$



(c) $y' = 1, \cos x > 0$
$y' = -1, \cos < 0$



(d) $y' = -2, \sin 2x > 0$
$y' = 2, \sin 2x < 0$



7.7 REVIEW EXERCISE

1. (a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{5}$ (d) 0 (e) 1 (f) 1 (g) 0
(h) 2 (i) 1

2. (a) $12\tan^3 3x\sec^2 3x$ (b) $\dfrac{\cos x - 2}{(1-2\cos x)^2}$
(c) $2x\sec x^2\tan x^2$
(d) $\dfrac{-2\cot 2x[2(1+x^2)\csc^2 2x + x\cot 2x]}{(1+x^2)^2}$
(e) $-3x^2\csc(x^3+1)\cot(x^3+1)$

(f) $\dfrac{\sec\sqrt{x}\tan\sqrt{x}}{\sqrt{x}}$ (g) $\dfrac{\tan x + x\sec^2 x}{3\sqrt[3]{x^2\tan^2 x}}$

(h) $-\sec^2 x \sin(2\tan x)$

(i) $\dfrac{(\cos x - 1)[\cos(x - \sin x)]}{\sin^2(x - \sin x)}$

3. (a) $\dfrac{-\sin(x-y)}{1-\sin(x-y)}$

(b) $\dfrac{\cos(x-y)+\cos(x+y)}{\cos(x-y)-\cos(x+y)}$ (c) $-\dfrac{\sec^2(x+y)}{y^2}$

(d) $-\dfrac{y\cos x + \sin(x+y)}{\sin x + \sin(x+y)}$ (e) $-\dfrac{y}{x}$

(f) $\dfrac{1+\csc(x-y)\cot(x-y)-\sec(x+y)\tan(x+y)}{\csc(x-y)\cot(x-y)+\sec(x+y)\tan(x+y)}$

4. $\left(\dfrac{\pi}{4}, \dfrac{2}{\sqrt{2}}\right)$ and $\left(\dfrac{5\pi}{4}, -\dfrac{2}{\sqrt{2}}\right)$

5. $12x - 3y + 3\sqrt{3} - 4\pi = 0$

6. $\dfrac{2}{6-\pi}$ **7.** $\frac{25}{36}\pi$ cm/s^2

8. increasing 90 cm^2/min **9.** 512π km/min

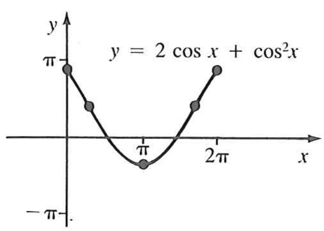
11. 60° **12.** fold $6\sqrt{2}$ cm of the 12 cm side or 6 cm of the 8 cm side **13.** $-\frac{3}{20}$ rad/min

14. (a) A. $[0, 2\pi]$ B. y-intercept 3, x-intercepts $\dfrac{\pi}{2}$, $\dfrac{3\pi}{2}$ C. none D. none E. decreasing on $(0, \pi)$; increasing on $(\pi, 2\pi)$ F. local minimum $f(\pi) = -1$, maxima $f(0) = 3$ and $f(2\pi) = 3$

G. CD on $\left(0, \dfrac{\pi}{3}\right)$, $\left(\dfrac{5\pi}{3}, 2\pi\right)$, CU on $\left(\dfrac{\pi}{3}, \dfrac{5\pi}{3}\right)$,

Points of inflection $\left(\dfrac{\pi}{3}, 1.25\right)$, $\left(\dfrac{5\pi}{3}, 1.25\right)$

H.



(b) A. $[-\pi, \pi]$ B. y-intercept 0, x-intercept 0 C. origin D. none E. increasing on $\left(-\pi, -\dfrac{2\pi}{3}\right)$, $\left(-\dfrac{\pi}{3}, \dfrac{\pi}{3}\right)$, $\left(\dfrac{2\pi}{3}, \pi\right)$;

decreasing on $\left(-\dfrac{2\pi}{3}, -\dfrac{\pi}{3}\right)$, $\left(\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right)$

F. local maxima: $f\left(-\dfrac{2\pi}{3}\right) \doteq -1.2$, $f\left(\dfrac{\pi}{3}\right) \doteq 1.9$,

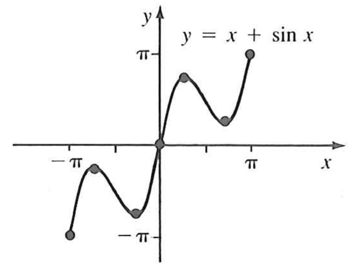
local minima: $f\left(-\dfrac{\pi}{3}\right) \doteq -1.9$, $f\left(\dfrac{2\pi}{3}\right) \doteq 1.2$

G. CD on $\left(-\pi, -\dfrac{\pi}{2}\right)$, $\left(0, \dfrac{\pi}{2}\right)$;

CU on $\left(-\dfrac{\pi}{2}, 0\right)$, $\left(\dfrac{\pi}{2}, \pi\right)$.

Points of inflection $\left(-\dfrac{\pi}{2}, -\dfrac{\pi}{2}\right)$, $(0, 0)$, $\left(\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

H.



15. $(-0.739\,08, 0.739\,08)$

16. (a) $\dfrac{\pi}{6}$ (b) $\dfrac{\pi}{4}$ (c) $-\dfrac{\pi}{3}$ (d) $\dfrac{\sqrt{3}}{2}$ (e) $\frac{4}{5}$ (f) $\dfrac{1}{2}$

(g) $\dfrac{\pi}{4}$ (h) $\dfrac{\pi}{6}$, $\dfrac{\pi}{6}$ (j) $\dfrac{\pi}{3}$ (k) $\dfrac{\pi}{3}$ (l) $\dfrac{2\pi}{3}$

17. (a) $\dfrac{2x}{\sqrt{1-x^4}}$ (b) $\dfrac{x^3}{\sqrt{9-x^2}} + 3x^2\sin^{-1}\left(\dfrac{x}{3}\right)$

(c) $\dfrac{-1}{2\sqrt{x(1-x)}}$ (d) $\dfrac{x^2}{1+x^2}$ (e) $\dfrac{|x+1|}{(x+1)^2\sqrt{x}}$

(f) $\dfrac{\sin 2x}{1+\sin^4 x}$ (g) $\dfrac{|x|}{x^2\sqrt{x^2-1}}$

(h) $\dfrac{1}{\sqrt{(1-x^2)[1-(\sin^{-1}x)^2]}}$

(i) $\dfrac{1}{(1+x^2)[1+(\tan^{-1}x)^2]}$

18. (a) $-\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{2}$

19. (a) $\dfrac{(1+y^2)(3x^2+\sin y)}{x\cos y(1+y^2)+1}$

(b) $\dfrac{-y\sqrt{1-(x+y)^2}-\sqrt{1-x^2y^2}}{x\sqrt{1-(x+y)^2}+\sqrt{1-x^2y^2}}$

7.8 CHAPTER 7 TEST

1. (a) 2 (b) $\frac{1}{2}$

2. (a) $\dfrac{-24x^2\sin(x^3-2)^{-4}\cos(x^3-2)^{-4}}{(x^3-2)^5}$

(b) $-3\sin x\cos^2 x\sec^2(\cos^3 x)$

(c) $\dfrac{-\csc x(\cot x - 1)}{(1+\cot x)^2}$

(d) $\dfrac{-2\cot 2x[2(1+x^2)\csc^2 2x + x\cot 2x]}{(1+x^2)^2}$

(e) $-4\sec^2 x\tan x$ (f) $\dfrac{4(3\sin^2 x - 1)}{\cos^2 x\sin^3 2x}$

3. $\dfrac{y - \cos y + y\sin x}{-x\sin y + \cos x - x}$ **4.** $-\frac{24}{25}$

5. (a) $\dfrac{1}{\sqrt{4-x^2}}$ (b) $\dfrac{1}{x^2+1}$ (c) ± 1

6. (a) $-\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$ (b) increasing on $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{6}\right), \left(\dfrac{\pi}{2}, \dfrac{5\pi}{6}\right)$,

decreasing on $\left(-\pi, -\dfrac{\pi}{2}\right), \left(\dfrac{\pi}{6}, \dfrac{\pi}{2}\right), \left(\dfrac{5\pi}{6}, \pi\right)$

(c) abs min $= -3$; abs max $= 1.5$

7. $\frac{3}{25}$ rad/s counter-clockwise **9.** $n = 4i,\ i \in N$

CUMULATIVE REVIEW FOR CHAPTERS 4 TO 7

1. (a) $-\infty$ (b) 1 (c) ∞ (d) $-\infty$ (e) 2 (f) ∞ (g) $-\infty$ (h) $\frac{5}{8}$ (i) 3 (j) -1 **2.** $\frac{5}{13}$

3. (a) 0 (b) undefined (c) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

(d) $\sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$ (e) $-\dfrac{1}{2\sqrt{2}}$ (f) $\dfrac{\sqrt{2}-1}{\sqrt{2}}$

5. (a) $-\dfrac{3\pi}{2}, -\dfrac{5\pi}{6}, -\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}$

(b) $0, \dfrac{\pi}{4}, \dfrac{\pi}{2}, \pi, \dfrac{5\pi}{4}, \dfrac{3\pi}{2}, 2\pi$ (c) $\dfrac{\pi}{2}$

(d) $\dfrac{5\pi}{24}, \dfrac{11\pi}{24}$

6. (a) $\dfrac{\pi}{6}$ (b) $-\sqrt{3}$ (c) $-\dfrac{\pi}{6}$

7. (a) $\dfrac{(1-x)\cos x + (1+x)\sin x}{(\sin x + \cos x)^2}$ (b) $\dfrac{-\sin 2x}{\sqrt{\cos 2x}}$

(c) $\dfrac{ax\cos ax - \sin ax}{bx^2}$

(d) $2\csc x\cot x\sin(2\csc x)$

(e) $\dfrac{2\sec 2x(\sec 2x - 2\csc 2x)}{(1-\cot 2x)^2}$

(f) $-\dfrac{\csc x(x\cot x + 2)}{x^3}$

(g) $\dfrac{\cos(x-y) + y\sin x}{\cos(x-y) + \cos x}$

(h) $\dfrac{\tan x\sec^2 x}{\tan y\sec^2 y}$

(i) $-\cos x\sin(\sin x)\sec^2[\cos(\sin x)]$

(j) $\dfrac{x^2}{\sqrt{4-x^2}} + 2x\sin^{-1}\dfrac{x}{2}$

(k) $\dfrac{|x-1|}{(x-1)^2\sqrt{1-2x}}$ (l) $\dfrac{1}{2\sqrt{x}(1+x)}$

8. (a) local maximum $f\left(\frac{1}{2}\right) = \frac{1}{4}$, local minimum $f(1) = 0$

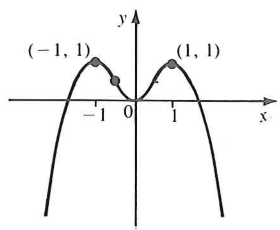
(b) local maximum, $f(-\sqrt{3}) = \dfrac{-3\sqrt{3}}{2}$, local minimum $f(\sqrt{3}) = \dfrac{3\sqrt{3}}{2}$ (c) local minimum $f\left(-\dfrac{\pi}{4}\right) = -\sqrt{2}$ (d) local maxima $f\left(-\dfrac{5\pi}{6}\right) = f\left(\dfrac{7\pi}{6}\right) = -\sqrt{3}$, local minima $f\left(-\frac{\pi}{6}\right) = f\left(\dfrac{11\pi}{6}\right) = \sqrt{3}$

9. (a) $f\left(\dfrac{9}{2}\right) = \frac{97}{8}$, $f(3) = 2$ (b) $f\left(\dfrac{\pi}{2}\right) = 3$, $f(0) = f(\pi) = 0$

10. $\left(\dfrac{2\pi}{3}, \dfrac{4\pi}{3}\right)$

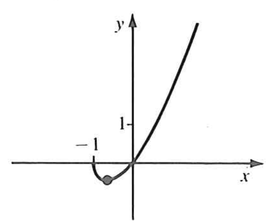
11. (a) A. R B. y-intercept 0, x-intercepts 0, $\pm\sqrt{2}$ C. about the y-axis D. none E. increasing on $(-\infty, -1)$, $(0, 1)$, decreasing on $(-1, 0)$, $(1, \infty)$ F. local maximum $f(-1) = f(1) = 1$, local minimum $f(0) = 0$ G. CU on $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$, CD on $\left(-\infty, -\dfrac{1}{\sqrt{3}}\right)$, $\left(\dfrac{1}{\sqrt{3}}, \infty\right)$; IP $\left(\pm\dfrac{1}{\sqrt{3}}, \frac{5}{9}\right)$

H.



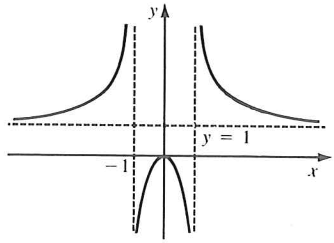
(b) A. $[-1, \infty)$ B. y-intercept 0, x-intercepts -1, 0 C. none D. none E. increasing on $\left(-\frac{2}{3}, \infty\right)$, decreasing on $\left(-1, -\frac{2}{3}\right)$ F. local minimum $f\left(-\frac{2}{3}\right) = -\dfrac{2\sqrt{3}}{9}$ G. CU on $(-1, \infty)$

H.



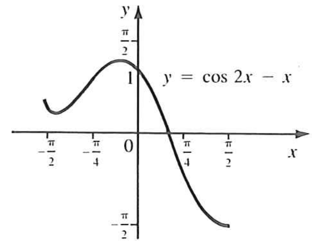
(c) A. $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$
B. intercepts 0 C. about y-axis D. VA: $x = \pm 1$, HA: $y = 1$ E.. increasing on $(-\infty, -1)$, $(-1, 0)$, decreasing on $(0, 1)$, $(1, \infty)$ F. local maximum $f(0) = 0$
G. CU on $(-\infty, -1)$, $(1, \infty)$, CD on $(-1, 1)$

H.

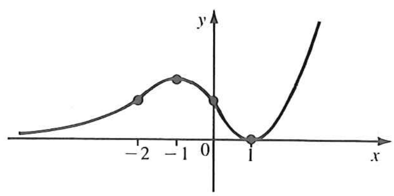


(d) A. $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ B. y-intercept $= 1$, x-intercept $\doteq 0.51$ C. none D. none E. increasing on $\left(-\frac{5\pi}{12}, -\frac{\pi}{12}\right)$, decreasing on $\left(-\frac{\pi}{2}, -\frac{5\pi}{12}\right)$, $\left(-\frac{\pi}{12}, \frac{\pi}{2}\right)$ F. local minimum $f\left(-\frac{5\pi}{12}\right) \doteq 0.44$, local maximum $f\left(-\frac{\pi}{12}\right) \doteq 1.13$ G. CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$, $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, CD on $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$; IP $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, -\frac{\pi}{4}\right)$

H.



12.



13. $\frac{1}{40}$ rad/sec **14.** $(2, \pm 2)$ **15.** 2 rad

16. (a) $p(x) = 560 - \frac{1}{5}x$ (b) $280