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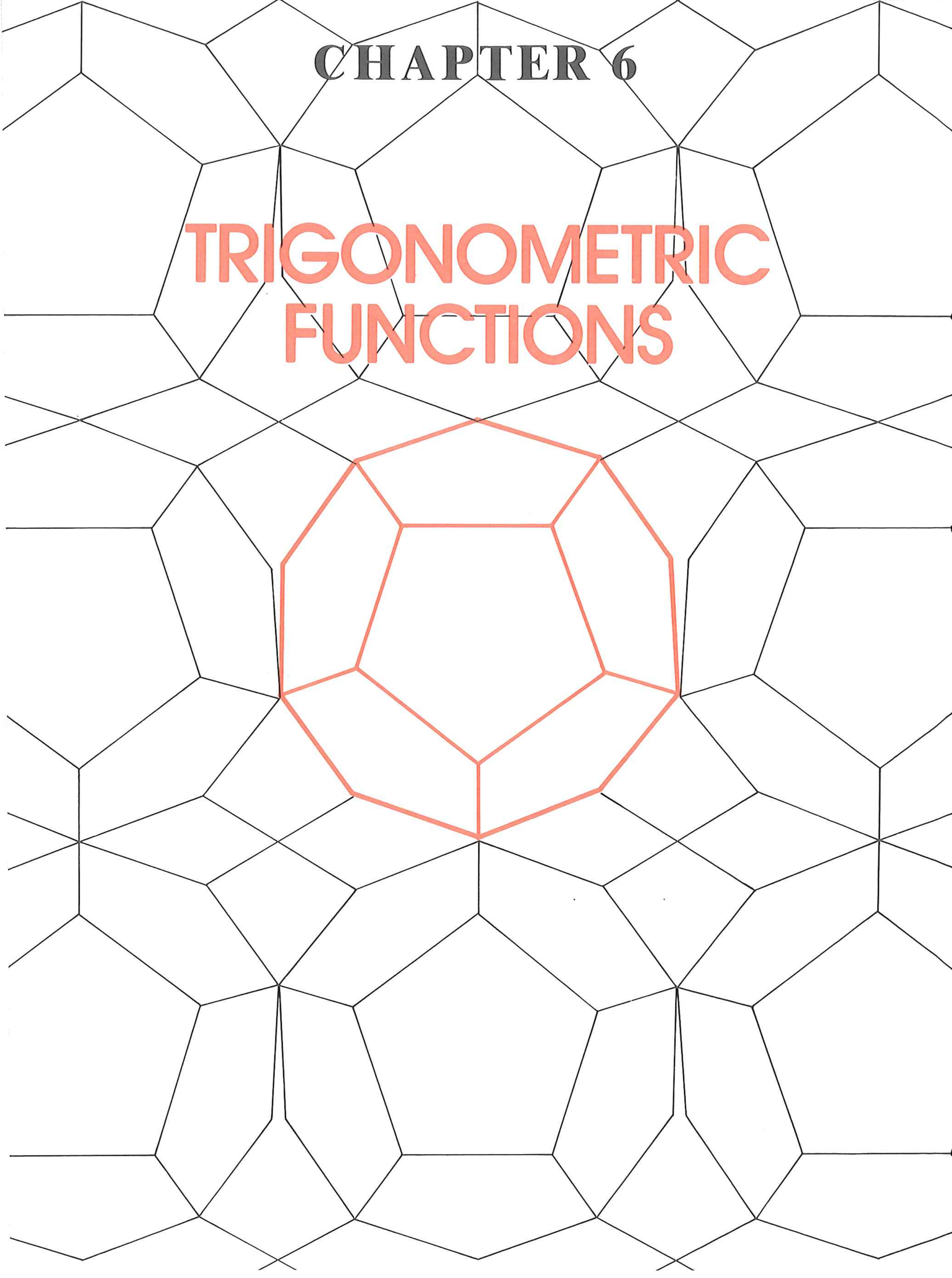
CHAPTER 6

TRIGONOMETRIC FUNCTIONS



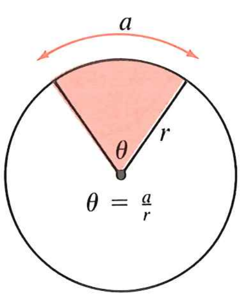
REVIEW AND PREVIEW TO CHAPTER 6

Angles and Arcs

The measure of an angle is an amount of rotation, where one complete revolution is divided into 360 equal parts, each of which is called a **degree**.

$$\boxed{1 \text{ rev} = 360^\circ}$$

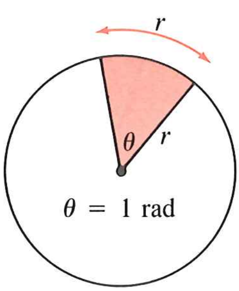
This unit of measure is seldom used in calculus. In order to simplify the formulas for the derivatives of the trigonometric functions, we use the unit of measurement called a **radian**.



> The measure of an angle in radians is the ratio of the arc length that subtends the angle at the centre of a circle to the radius of the circle.

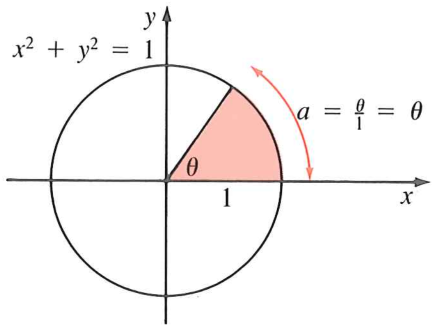
In the diagram the arc with length a subtends angle θ at the centre of the circle with radius r. The radian measure of angle θ is

$$\boxed{\theta = \frac{a}{r}}$$

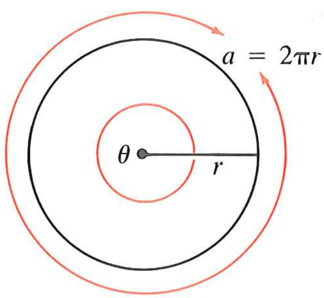


Since a and r represent lengths measured in the same units, the ratio $\frac{a}{r}$ has no unit. Thus the radian measure of angle θ is a pure number. In particular, if a is equal to r, the radian measure of θ is one.

If we let $r = 1$, we can interpret radian measure as the length of an arc on the unit circle. Now θ, the angle subtended at the centre of the unit circle by arc a, is just equal to a.



The radian measure of the angle θ subtended by an arc equal in length to the circumference of the circle is



$\theta = \frac{2\pi r}{r} = 2\pi$

$$\theta = \frac{2\pi r}{r} = 2\pi$$

Since θ is 1 rev it has degree measure 360.

Therefore $\quad 2\pi \text{ rad} = 360°$

which simplifies to

$$\boxed{\pi \text{ rad} = 180°}$$

This enables us to convert any angle expressed in degrees to radians and vice versa.

Example 1 Convert 2 radians to degrees.

Solution A simple method of conversion is presented.

$$\text{Since} \quad \pi \text{ rad} = 180°$$

$$1 \text{ rad} = \frac{180°}{\pi}$$

$$\text{Therefore} \quad 2 \text{ rad} = \frac{2 \times 180°}{\pi} \doteq 115°$$

Example 2 Convert 60° to radian measure.

Solution Since

$$180° = \pi \text{ rad}$$

$$1° = \frac{\pi}{180} \text{ rad}$$

so

$$60(1°) = 60\left(\frac{\pi}{180}\right) \text{ rad}$$

Finally

$$60° = \frac{\pi}{3} \text{ rad} \doteq 1.047 \text{ rad}$$

EXERCISE 1

1. Calculate the degree measure of the angles having radian measures

(a) $\frac{\pi}{6}$ (b) $\frac{-3\pi}{2}$ (c) $\frac{5\pi}{4}$ (d) 3π

(e) 4 (f) $-\frac{3}{4}$ (g) -12

2. Calculate the radian measure of the angles whose measures are

(a) 45° (b) 315° (c) −210° (d) 570°

(e) 2° (f) −28° (g) 601°

3. The radius of a circle is 10 cm. Calculate:

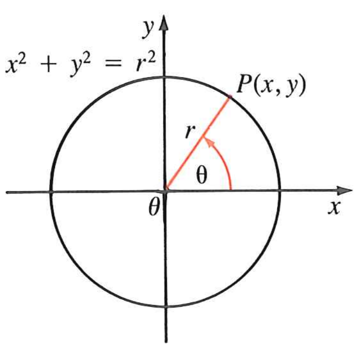
(a) the arc length that determines an angle of 2.5 rad.

(b) the angle in radians that determines an arc length of 12 cm.

4. The length of an arc of a circle is 32 cm and the angle it determines is 72°. Find the radius of the circle.

Definitions of the Trigonometric Functions

A point $P(x, y)$ in the Cartesian plane determines an angle θ as shown in the diagram. This angle θ, with one arm along the positive x-axis, is said to be in **standard position**. If we let the distance from P to the origin be r, six ratios are determined. Their names, abbreviations, and definitions are summarized in the following chart.



Trigonometric Ratios

sine:	$\sin \theta = \frac{y}{r}$	cosecant:	$\csc \theta = \frac{r}{y}$
cosine:	$\cos \theta = \frac{x}{r}$	secant:	$\sec \theta = \frac{r}{x}$
tangent:	$\tan \theta = \frac{y}{x}$	cotangent:	$\cot \theta = \frac{x}{y}$

Eight relationships involving the trigonometric ratios are listed below. The proof of each is requested in Exercise 2.

Fundamental Relationships	
$\csc\theta = \dfrac{1}{\sin\theta}$ $\quad \sec\theta = \dfrac{1}{\cos\theta}$ $\quad \cot\theta = \dfrac{1}{\tan\theta}$	
$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	$\sin^2\theta + \cos^2\theta = 1$ $\sec^2\theta = 1 + \tan^2\theta$ $\csc^2\theta = 1 + \cot^2\theta$

EXERCISE 2

1. $P(3, 4)$ determines an angle θ in standard position. Determine the six trigonometric ratios of angle θ.
2. $P(-2, -1)$ determines an angle θ in standard position. Determine the sine, cosine, and tangent of angle θ.
3. $P(5, -12)$ determines an angle θ in standard position. Determine the cosecant, secant, and cotangent of angle θ.
4. If $\sin\theta = \frac{1}{3}$, $0 \leqslant \theta \leqslant \frac{\pi}{2}$, find $\cos\theta$ and $\tan\theta$.
5. If $\cos\theta = -\frac{1}{2}$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$, find $\csc\theta$ and $\cot\theta$.
6. If $\tan\theta = -\frac{5}{3}$, $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, find $\cos\theta$ and $\csc\theta$.
7. If $\cot\theta = \frac{5}{12}$, $\pi \leqslant \theta \leqslant \frac{3\pi}{2}$, find $\sec\theta$ and $\sin\theta$.
8. If $\tan\theta = -\frac{4}{3}$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$, show that $\sin^2\theta + \cos^2\theta = 1$.
9. If $\csc\theta = 2$, $\frac{\pi}{2} \leqslant \theta \leqslant \pi$, show that $\frac{\sin\theta}{\cos\theta} = \tan\theta$.
10. If $\sin\theta = -\frac{1}{2}$, $\frac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sec^2\theta = 1 + \tan^2\theta$.
11. Use the basic definitions and the relationship $x^2 + y^2 = r^2$ to prove each of the following relationships:

(a) $\csc\theta = \dfrac{1}{\sin\theta}$ (b) $\sec\theta = \dfrac{1}{\cos\theta}$

(c) $\cot\theta = \dfrac{1}{\tan\theta}$ (d) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(e) $\cot \theta = \dfrac{\cos \theta}{\sin \theta}$ (f) $\sin^2 \theta + \cos^2 \theta = 1$

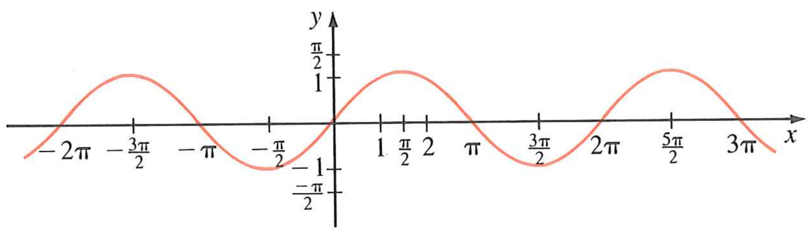
12. Use $\sin^2 \theta + \cos^2 \theta = 1$ to prove:
(a) $\sec^2 \theta = 1 + \tan^2 \theta$ (b) $\csc^2 \theta = 1 + \cot^2 \theta$

Graphs of the Trigonometric Functions

When we talk about the function f defined for all real numbers x by $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose *radian* measure is x. A similar convention holds for the other trigonometric functions.

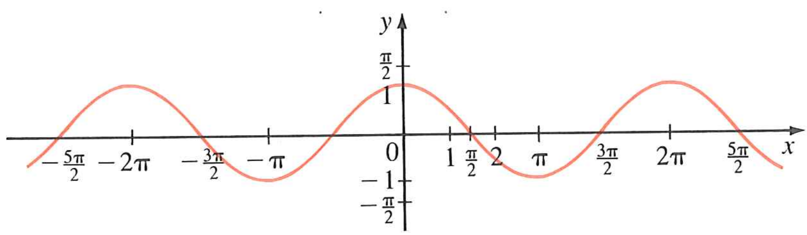
The graphs of the basic trigonometric functions and knowledge of their domain, range, and periodicity are useful. In Section 6.5, we use them to help us solve trigonometric equations. In Chapter 7, the graphs of the sine and cosine functions are instrumental in the development of the derivative of $f(x) = \sin x$.

$y = \sin x$



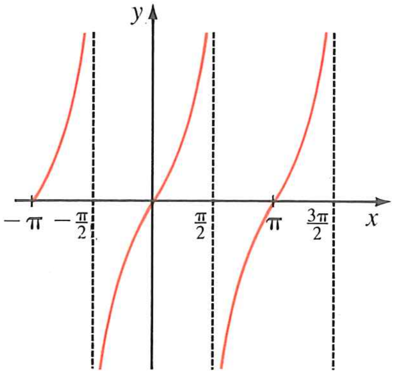
Domain $= R$
Range $= \{y \mid -1 \leq y \leq 1\}$
Period $= 2\pi$

$y = \cos x$



Domain $= R$
Range $= \{y \mid -1 \leq y \leq 1\}$
Period $= 2\pi$

$y = \tan x$



Domain $= \{x \mid x \neq (2n - 1)\frac{\pi}{2}, n \in I\}$

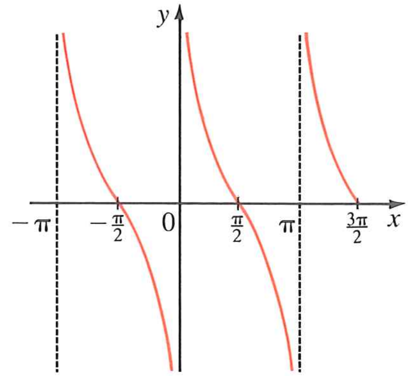
Range $= R$

Period $= \pi$

Vertical asymptotes:

$x = (2n - 1)\frac{\pi}{2}, n \in I$

$y = \cot x$



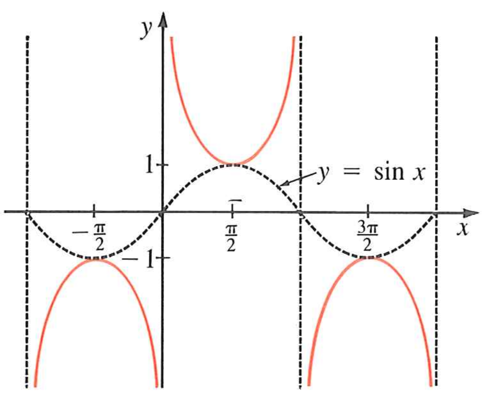
Domain $= \{x \mid x \neq n\pi, n \in I\}$

Range $= R$

Period $= \pi$

Vertical asymptotes: $x = n\pi, n \in I$

$y = \csc x$



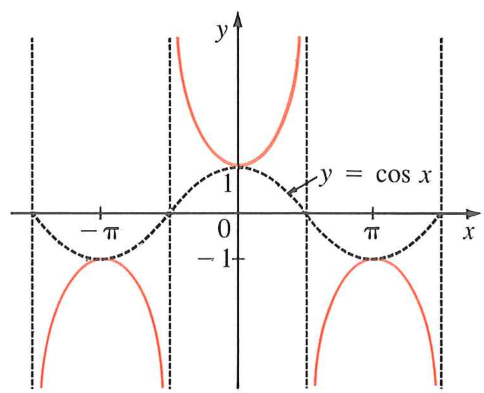
Domain $= \{x \mid x \neq n\pi, n \in I\}$

Range $= \{y \mid y \leqslant -1 \text{ or } y \geqslant 1\}$

Period $= 2\pi$

Vertical asymptotes: $x = n\pi, n \in I$

$y = \sec x$



Domain $= \{x \mid x \neq (2n - 1)\frac{\pi}{2}, n \in I\}$

Range $= \{y \mid y \leqslant -1 \text{ or } y \geqslant 1\}$

Period $= 2\pi$

Vertical asymptotes: $x = (2n - 1)\frac{\pi}{2}, n \in I$

Transformations, such as translations and reflections, can reduce the amount of work in graphing functions.

Vertical and Horizontal Shifts

If $c > 0$ the graph of

$$\left.\begin{array}{l} y = f(x) + c \\ y = f(x) - c \\ \\ y = f(x + c) \\ y = f(x - c) \end{array}\right\} \text{is the graph of } y = f(x) \text{ shifted } c \text{ units} \left\{\begin{array}{l} \text{upward.} \\ \text{downward.} \\ \\ \text{to the left.} \\ \text{to the right.} \end{array}\right.$$

Stretching and Reflecting

The graph of $y = af(x)$ is obtained from the graph of $y = f(x)$ by

stretching in the y-direction	if $a > 1$.
shrinking in the y-direction	if $0 < a < 1$.
reflection in the x-axis	if $a = -1$.
shrinking and reflecting in the x-axis	if $-1 < a < 0$.
stretching and reflecting in the x-axis	if $a < -1$.

The graph of $y = f(ax)$ is obtained from the graph of $y = f(x)$ by

shrinking in the x-direction	if $a > 1$.
stretching in the x-direction	if $0 < a < 1$.

EXERCISE 3

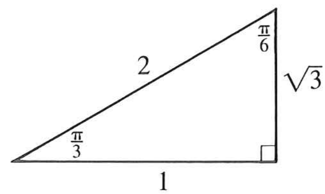
1. Sketch $y = \sin x$, $y = 2 \sin x$, and $y = \sin 2x$, $0 \leq x \leq 2\pi$, on the same set of axes.

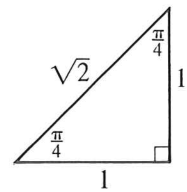
2. Sketch $y = \cos x$, $y = \cos\left(x + \dfrac{\pi}{4}\right)$, and $y = \cos\left(x - \dfrac{\pi}{4}\right)$, $-2x \leq x \leq 2\pi$, on the same set of axes.

3. Sketch $y = \tan \frac{1}{2}x$ for $-2\pi \leq x \leq 2\pi$. Sketch $y = -\tan \frac{1}{2}x$ on the same set of axes over the same interval.

Trigonometric Ratios of Special Angles

The values of the trigonometric functions at $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$ can be found from the special right triangles illustrated. In problems dealing with these angles or multiples of these angles, knowing the values can give us exact answers.





EXERCISE 4

1. Evaluate each of the following using the special triangles.

(a) $\sin \frac{\pi}{3} - \cos \frac{\pi}{6}$ (b) $\sec \frac{\pi}{6} + 2 \cot \frac{\pi}{4}$

(c) $\sin^2 30° + \cos^2 45°$ (d) $4 \sin \frac{\pi}{6} + \sec^2 \frac{\pi}{4}$

(e) $\sqrt{3} \cos 30° - \csc 45° + 3 \sin^2 45°$

(f) $\tan \frac{\pi}{3} \cos \frac{\pi}{4} \csc \frac{\pi}{3} - \sec \frac{\pi}{6} \tan \frac{\pi}{6}$

PROBLEMS PLUS

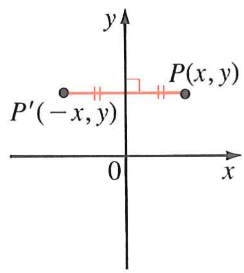
Find the number of solutions of the equation $\sin x = \frac{x}{100}$.

INTRODUCTION

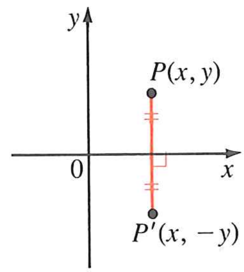
In this chapter, we take a short break from our study of calculus to extend our knowledge of the trigonometric functions. Relationships that will enable us to apply the calculus to the trigonometric functions are developed. Attention is devoted to the proofs of trigonometric identities and the solution of trigonometric equations, two skills that are frequently used when the techniques of calculus are applied to problems involving the trigonometric functions.

6.1 FUNCTIONS OF RELATED VALUES

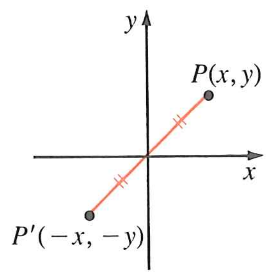
In this section, we develop formulas that express the trigonometric function of any angle in terms of the trigonometric function of an acute angle. We use reflections in the coordinate axes or the origin to develop the formulas. The following diagrams illustrate the coordinate of P', the image of $P(a, b)$ after a reflection in the y-axis, the x-axis, and the origin.



Reflection in y-axis

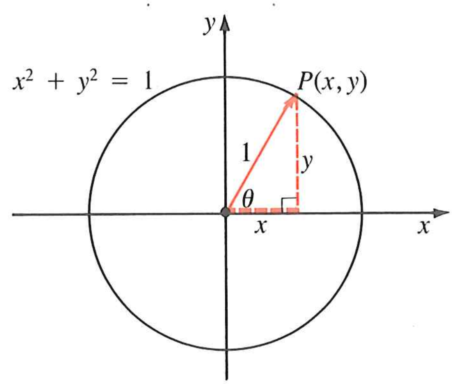


Reflection in x-axis



Reflection in origin

Now we express a point on the unit circle in terms of the angle it determines at the origin. $P(x, y)$ determines angle θ in the diagram.



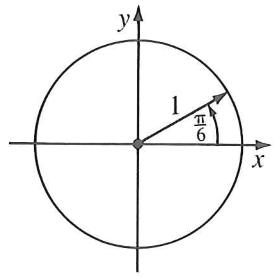
Since $\cos\theta = \frac{x}{1} = x$ and $\sin\theta = \frac{y}{1} = y$, $P(x, y)$ can be expressed as $P(\cos\theta, \sin\theta)$.

Example 1 Find the coordinates of the point on the unit circle determined by

(a) angle $\frac{\pi}{6}$ in standard position.

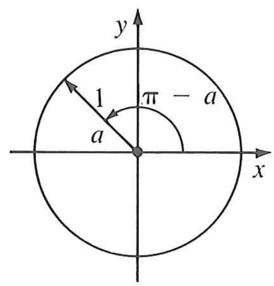
(b) angle $\pi - a$ in standard position.

Solution (a)



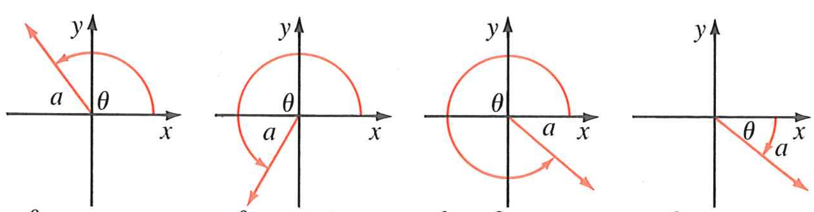
The required point is $\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(b)



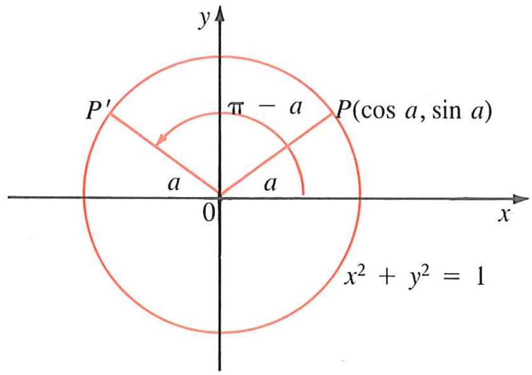
The required coordinates are $(\cos(\pi - a), \sin(\pi - a))$.

If an angle is in standard position, the acute angle between its terminal arm and the x-axis is called the **related acute angle**. In each diagram, θ has related acute angle a and can be expressed in terms of a.



$\theta = \pi - a$ $\qquad \theta = \pi + a$ $\qquad \theta = 2\pi - a$ $\qquad \theta = -a$

We are now ready to develop the related angle formulas. In the diagram, point P on the unit circle determines acute angle a. P' is the image of P after a reflection in the y-axis. By symmetry, the acute angle between OP' and the x-axis is equal to a. Therefore P' determines the second quadrant angle $\pi - a$ having related acute angle a.



Since P' is the image of P, its coordinates are $(-\cos a, \sin a)$. Since P' determines angle $\pi - a$, its coordinates are $(\cos(\pi - a), \sin(\pi - a))$.

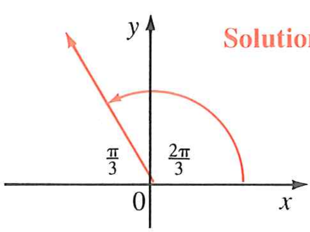
Therefore $\sin(\pi - a) = \sin a$

and $\cos(\pi - a) = -\cos a$

and
$$\tan(\pi - a) = \frac{\sin(\pi - a)}{\cos(\pi - a)} = \frac{\sin a}{-\cos a} = -\tan a$$

$$\boxed{\begin{aligned}\sin(\pi - a) &= \sin a\\ \cos(\pi - a) &= -\cos a\\ \tan(\pi - a) &= -\tan a\end{aligned}}$$

Example 2 Find the exact value of $\sin \frac{2\pi}{3}$.

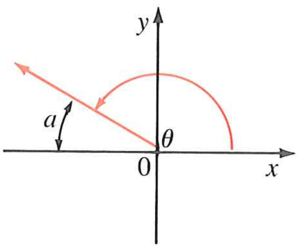


Solution Express $\frac{2\pi}{3}$ in terms of its related acute angle $\frac{\pi}{3}$.

$$\sin \frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Example 3 If $\tan\theta = -\frac{1}{\sqrt{3}}, \frac{\pi}{2} \leqslant \theta \leqslant \pi$, find θ.

Solution In the diagram, a is the related acute angle of second quadrant angle θ.



Now $$\theta = \pi - a$$

Therefore $$\tan(\pi - a) = -\frac{1}{\sqrt{3}}$$

Using our new formula we get

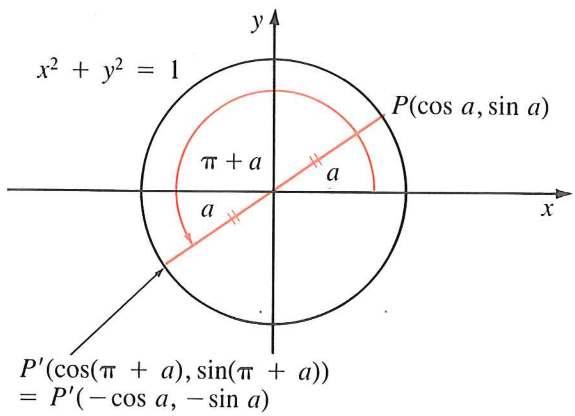
$$-\tan a = -\frac{1}{\sqrt{3}}$$

$$\tan a = \frac{1}{\sqrt{3}}$$

$$a = \frac{\pi}{6}$$

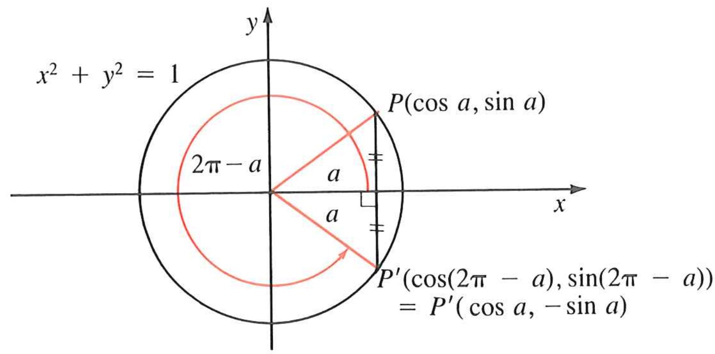
and so $$\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

When $P(\cos a, \sin a)$ is reflected in the origin its image P' determines angle $\pi + a$ in the third quadrant with a as its related acute angle.



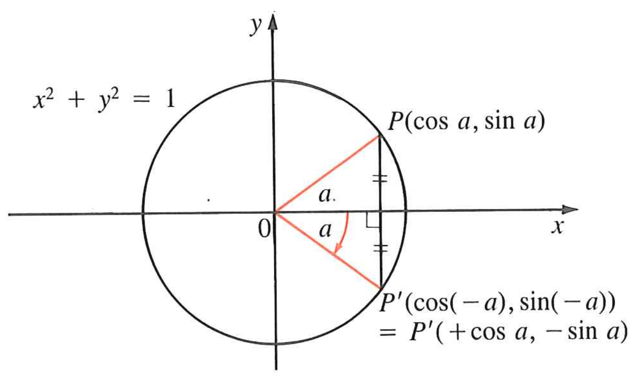
$$\boxed{\begin{aligned} \sin(\pi + a) &= -\sin a \\ \cos(\pi + a) &= -\cos a \\ \tan(\pi + a) &= \tan a \end{aligned}}$$

When P is reflected in the x-axis, the image determines angle $2\pi - a$ in the fourth quadrant. The related acute angle is a.



$$\boxed{\begin{aligned} \sin(2\pi - a) &= -\sin a \\ \cos(2\pi - a) &= \cos a \\ \tan(2\pi - a) &= -\tan a \end{aligned}}$$

A point on the unit circle is determined by more than one angle. When $P(\cos a, \sin a)$ is reflected in the x-axis, the image P' is not uniquely determined by the angle $2\pi - a$. In fact, all of the positive and negative angles that are coterminal with $2\pi - a$ also determine the position of P'. Let us consider the particular coterminal angle $-a$.



$$\boxed{\begin{aligned} \sin(-a) &= -\sin a \\ \cos(-a) &= \cos a \\ \tan(-a) &= -\tan a \end{aligned}}$$

It is important to note that, even though we developed these formulas with a as an acute angle, they are true for all values of a. The following example illustrates this fact.

Example 4 Prove that $\cos\left(\pi + \frac{2\pi}{3}\right) = -\cos\frac{2\pi}{3}$.

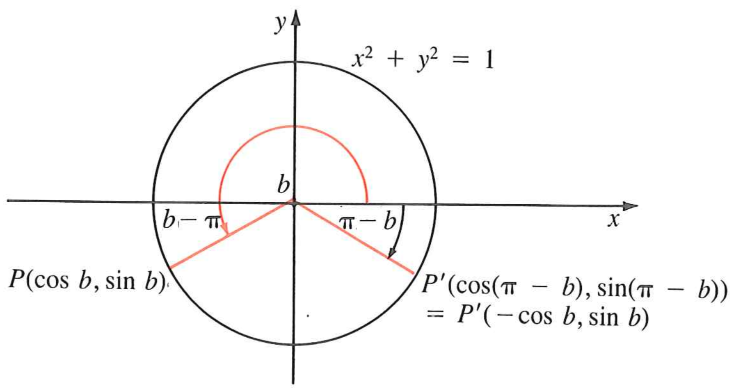
Solution We manipulate the argument of each function to create one of the forms introduced in this section. We work with each side separately to achieve the same result.

$$\begin{aligned}\cos\left(\pi + \frac{2\pi}{3}\right) &= \cos\frac{5\pi}{3} \\ &= \cos\left(2\pi - \frac{\pi}{3}\right) \\ &= \cos\frac{\pi}{3}\end{aligned} \qquad \begin{aligned}-\cos\frac{2\pi}{3} &= -\cos\left(\pi - \frac{\pi}{3}\right) \\ &= -\left(-\cos\frac{\pi}{3}\right) \\ &= \cos\frac{\pi}{3}\end{aligned}$$

Therefore $$\cos\left(\pi + \frac{2\pi}{3}\right) = -\cos\frac{2\pi}{3}$$

Example 5 Prove that $\cos(\pi - b) = -\cos b$ if b is in the interval $\left(\pi, \frac{3\pi}{2}\right)$.

Solution In the diagram, P is a point on the unit circle and P' is the image of P after a reflection in the y-axis.

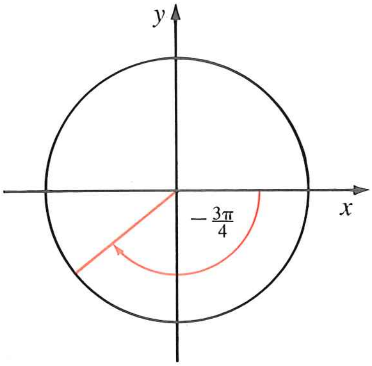


Since P' is the image of P, it has coordinates $(-\cos b, \sin b)$. But P' determines the angle $\pi - b$ and has coordinates $(\cos(\pi - b), \sin(\pi - b))$.

Therefore $$\cos(\pi - b) = -\cos b$$

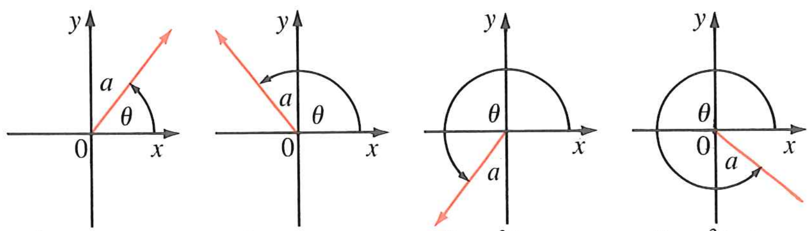
Example 6 Find the exact value of $\sin\left(-\frac{3\pi}{4}\right)$.

Solution Since $\sin(-a) = -\sin a$ for all values of a, we proceed as follows:



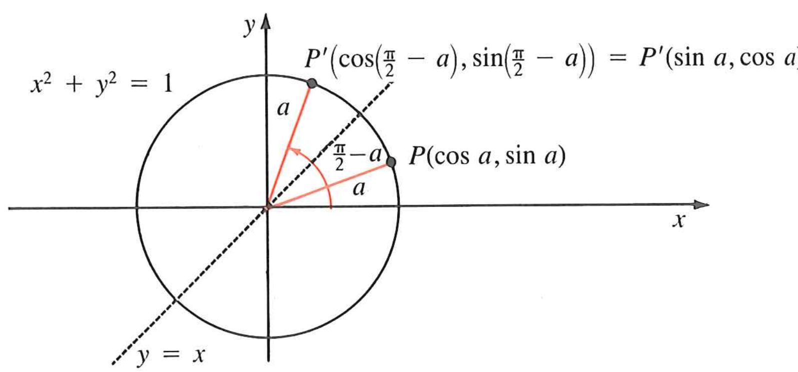
$$\begin{aligned}\sin\left(-\frac{3\pi}{4}\right) &= -\sin\frac{3\pi}{4}\\ &= -\sin\left(\pi-\frac{\pi}{4}\right)\\ &= -\sin\frac{\pi}{4}\\ &= -\frac{1}{\sqrt{2}}\end{aligned}$$

The complement of the related acute angle is the *co-related acute angle*. It is the acute angle between the terminal arm of an angle in standard position and the y-axis. In each diagram, θ has co-related acute angle a and can be expressed in terms of a.



$\theta = \frac{\pi}{2} - a$ $\qquad$ $\theta = \frac{\pi}{2} + a$ $\qquad$ $\theta = \frac{3\pi}{2} - a$ $\qquad$ $\theta = \frac{3\pi}{2} + a$

Any angle can also be expressed as a function of its co-related acute angle. In the diagram, $P(\cos a, \sin a)$ determines the acute angle a. P' is the image of P after a reflection in the line $y = x$. Thus P' has coordinates $(\sin a, \cos a)$.



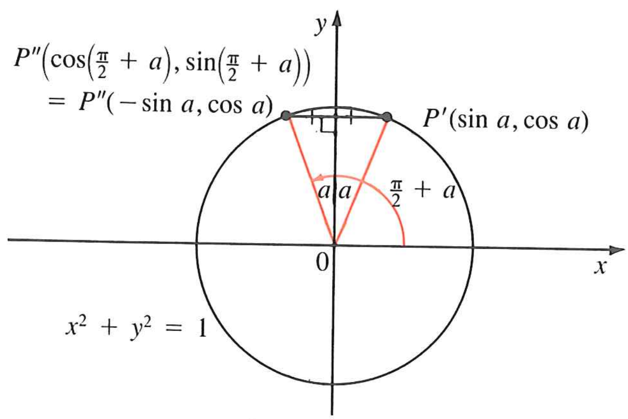
But P' determines the angle $\frac{\pi}{2} - a$ and has coordinates $\left(\cos\left(\frac{\pi}{2}-a\right), \sin\left(\frac{\pi}{2}-a\right)\right)$.

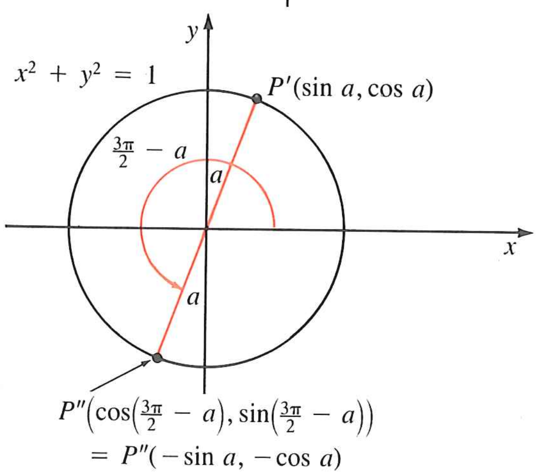
Therefore $\sin\left(\frac{\pi}{2} - a\right) = \cos a$

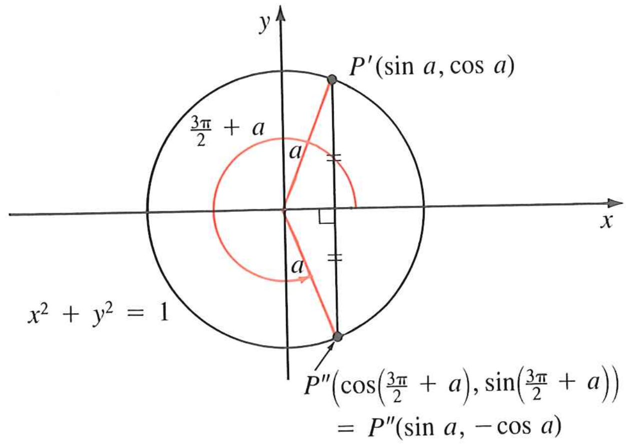
and $\cos\left(\frac{\pi}{2} - a\right) = \sin a$

Now
$$\tan\left(\frac{\pi}{2} - a\right) = \frac{\sin\left(\frac{\pi}{2} - a\right)}{\cos\left(\frac{\pi}{2} - a\right)} = \frac{\cos a}{\sin a} = \cot a$$

We now reflect $P'(\sin a, \cos a)$ in the y-axis, the origin, and the x-axis to generate the rest of the co-related angle formulas.







The results are summarized in the following charts.

$$\sin\left(\frac{\pi}{2} - a\right) = \cos a$$
$$\cos\left(\frac{\pi}{2} - a\right) = \sin a$$
$$\tan\left(\frac{\pi}{2} - a\right) = \cot a$$

$$\sin\left(\frac{\pi}{2} + a\right) = \cos a$$
$$\cos\left(\frac{\pi}{2} + a\right) = -\sin a$$
$$\tan\left(\frac{\pi}{2} + a\right) = -\cot a$$

$$\sin\left(\frac{3\pi}{2} - a\right) = -\cos a$$
$$\cos\left(\frac{3\pi}{2} - a\right) = -\sin a$$
$$\tan\left(\frac{3\pi}{2} - a\right) = \cot a$$

$$\sin\left(\frac{3\pi}{2} + a\right) = -\cos a$$
$$\cos\left(\frac{3\pi}{2} + a\right) = \sin a$$
$$\tan\left(\frac{3\pi}{2} + a\right) = -\cot a$$

Example 7 Express $\cos\left(-\frac{4\pi}{3}\right)$ as a function of its co-related acute angle and evaluate.

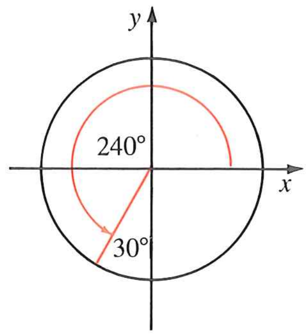
Solution Use the identity $\cos(-a) = \cos a$ to produce a positive argument.

$$\begin{aligned}
\cos\left(-\frac{4\pi}{3}\right) &= \cos\frac{4\pi}{3} \\
&= \cos\left(\frac{3\pi}{2} - \frac{\pi}{6}\right) \\
&= -\sin\frac{\pi}{6} \\
&= -\tfrac{1}{2}
\end{aligned}$$

Example 8 Express tan 240° as a function of its co-related acute angle and evaluate.

Solution

$$\begin{aligned}\tan 240^\circ &= \tan(270^\circ - 30^\circ)\\ &= \cot 30^\circ\\ &= \sqrt{3}\end{aligned}$$



As was the case with the related angle formulas, the co-related angle formulas are true for all values of a. Example 9 illustrates this fact.

Example 9 Prove that $\cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3}$.

Solution We work with each side separately to achieve an identical result.

$$\begin{aligned}\cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) &= \cos\left(-\frac{\pi}{6}\right) \\ &= \cos\frac{\pi}{6}\end{aligned} \qquad \begin{aligned}\sin\frac{2\pi}{3} &= \sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right)\\ &= \cos\frac{\pi}{6}\end{aligned}$$

Therefore

$$\cos\left(\frac{\pi}{2} - \frac{2\pi}{3}\right) = \sin\frac{2\pi}{3}$$

EXERCISE 6.1

B **1.** Express each of the following as a function of its related acute angle and evaluate.

(a) $\sin\left(-\frac{7\pi}{6}\right)$ (b) $\cos\frac{15\pi}{4}$

(c) $\tan\left(-\frac{8\pi}{3}\right)$ (d) $\tan\frac{33\pi}{4}$

(e) $\sin 240^\circ$ (f) $\cos(-135^\circ)$

(g) $\tan 330^\circ$ (h) $\sin 495^\circ$

2. Express each of the following as a function of its co-related acute angle and evaluate.

(a) $\cos \frac{11}{6}\pi$ (b) $\sin\left(-\frac{7\pi}{6}\right)$

(c) $\sin 120°$ (d) $\tan\left(-\frac{5\pi}{3}\right)$

(e) $\tan 510°$ (f) $\cos(-315°)$

3. Simplify.

(a) $\cos x + \cos(\pi - x) - \cos(\pi + x) - \cos(-x)$

(b) $\tan x + \tan(\pi - x) + \cot\left(\frac{\pi}{2} - x\right) - \tan(2\pi - x)$

(c) $\sin(\pi + x) + \cos\left(\frac{\pi}{2} - x\right) + \tan\left(\frac{\pi}{2} + x\right) + \tan\left(\frac{3\pi}{2} - x\right)$

(d) $\sin\left(\frac{\pi}{2} + x\right) - \cos\left(\frac{3\pi}{2} - x\right) + \sin\left(\frac{3\pi}{2} - x\right)$

(e) $\sin\left(\frac{\pi}{2} - x\right) + \sin(\pi - x) + \sin\left(\frac{3\pi}{2} - x\right) + \sin(2\pi - x)$

4. Find the cosecant, secant, and cotangent of each of the following. Express your answers in terms of cosecant, secant, or cotangent of x.

(a) $\pi - x$ (b) $\frac{\pi}{2} + x$ (c) $\pi + x$ (d) $\frac{3\pi}{2} + x$

5. Simplify.

(a) $\sin(x - \pi)$ (b) $\cos\left(x - \frac{\pi}{2}\right)$ (c) $\tan(-x - \pi)$

6. Evaluate.

(a) $\sec\left(\pi + \frac{\pi}{3}\right)$ (b) $\csc\left(\frac{3\pi}{2} - \frac{\pi}{6}\right)$

(c) $\cot\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$ (d) $\sec\left(\frac{3\pi}{4}\right)$

(e) $\csc\left(\frac{3\pi}{2} - \frac{\pi}{4}\right)$ (f) $\cot\left(-\pi + \frac{\pi}{4}\right)$

7. Simplify.

(a) $\dfrac{\cos(\pi + x)\cos\left(\frac{\pi}{2} + x\right)}{\cos(\pi - x)} - \dfrac{\sin\left(\frac{3\pi}{2} - x\right)}{\sec(\pi + x)}$

(b) $\dfrac{\sin\left(x - \frac{\pi}{2}\right)}{\cos(\pi - x)} + \dfrac{\tan\left(x - \frac{3\pi}{2}\right)}{-\tan(\pi + x)}$

8. If $b + c = \pi$, prove that $2(1 - \sin b \sin c) = \cos^2 b + \cos^2 c$.

9. If A, B, and C are angles in a triangle prove that $\sin B = \sin(A + C)$.

10. Use a reflection to prove that $\cos\left(\frac{\pi}{2} - b\right) = \sin b$ if b is in the interval $\left(\frac{\pi}{2}, \pi\right)$.

11. Use a reflection to prove that $\sin(2\pi - b) = -\sin b$ if b is in the interval $\left(\frac{3\pi}{2}, 2\pi\right)$.

6.2 ADDITION AND SUBTRACTION FORMULAS

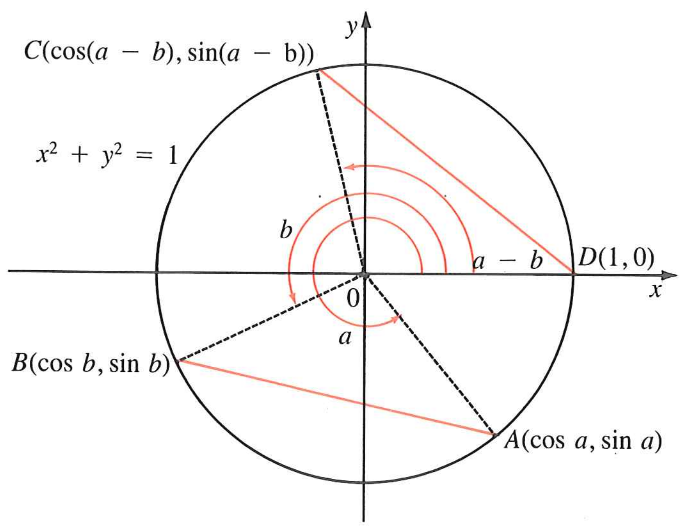
In this section we show that it is possible to express the sine and the cosine of $a + b$ and of $a - b$ in terms of sines and cosines of a and of b. Similarly, the tangent of $a + b$ and of $a - b$ can be expressed in terms of tangents of a and of b.

Subtraction Formula for Cosine

$$\cos(a - b) = \cos a \cos b + \sin a \sin b$$

Proof

Points A and B on the unit circle determine angles a and b respectively. We can assume that $a > b$. D is the point $(1, 0)$ and C is selected so that $\angle DOC = a - b$.



Now	$\angle AOB = \angle COD = a - b$
and	$OA = OB = OC = OD$
Therefore	$\triangle AOB$ is congruent to $\triangle COD$
and so	$AB = CD$

$$\sqrt{(\cos a - \cos b)^2 + (\sin a - \sin b)^2} = \sqrt{[\cos(a - b) - 1]^2 + [\sin(a - b)]^2}$$

$$\cos^2 a - 2 \cos a \cos b + \cos^2 b + \sin^2 a - 2 \sin a \sin b + \sin^2 b = \cos^2(a - b) - 2 \cos(a - b) + 1 + \sin^2(a - b)$$

We apply the identity $\sin^2 \theta + \cos^2 \theta = 1$ for $\theta = a$, $\theta = b$, and $\theta = a - b$ to obtain

$$\begin{aligned} 2 - 2(\cos a \cos b + \sin a \sin b) &= 2 - 2 \cos(a - b) \\ 2(\cos a \cos b + \sin a \sin b) &= 2 \cos(a - b) \\ \cos(a - b) &= \cos a \cos b + \sin a \sin b \end{aligned}$$

It is no longer necessary to create a diagram to develop the rest of the formulas of this section. We apply the previously established formula and results derived in Section 6.1 to establish each successive formula.

Addition Formula for Cosine

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

Proof

Using the Subtraction Formula for Cosine, we have

$$\begin{aligned} \cos(a + b) &= \cos[a - (-b)] \\ &= \cos a \cos(-b) + \sin a \sin(-b) \\ &= \cos a \cos b + \sin a(-\sin b) \\ &= \cos a \cos b - \sin a \sin b \end{aligned}$$

Addition Formula for Sine

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

Proof

Using a co-related angle identity and the Subtraction Formula for Cosine we get

$$\begin{aligned}\sin(a+b) &= \cos\left[\frac{\pi}{2} - (a+b)\right] \\ &= \cos\left[\left(\frac{\pi}{2} - a\right) - b\right] \\ &= \cos\left(\frac{\pi}{2} - a\right)\cos b + \sin\left(\frac{\pi}{2} - a\right)\sin b \\ &= \sin a \cos b + \cos a \sin b\end{aligned}$$

The proof of the following theorem is requested in Exercise 6.2.

Subtraction Formula for Sine

$$\sin(a - b) = \sin a \cos b - \cos a \sin b$$

The identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is used to develop the Addition Formula for Tangent.

Addition Formula for Tangent

$$\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

In the proof that follows and in all subsequent proofs it is understood that we are dealing with values of the variables for which the trigonometric expressions are defined.

Proof

$$\begin{aligned}\tan(a+b) &= \frac{\sin(a+b)}{\cos(a+b)} \\ &= \frac{\sin a \cos b + \cos a \sin b}{\cos a \cos b - \sin a \sin b} \\ &= \frac{\dfrac{\sin a \cos b}{\cos a \cos b} + \dfrac{\cos a \sin b}{\cos a \cos b}}{\dfrac{\cos a \cos b}{\cos a \cos b} - \dfrac{\sin a \sin b}{\cos a \cos b}} \\ &= \frac{\dfrac{\sin a}{\cos a} + \dfrac{\sin b}{\cos b}}{1 - \dfrac{\sin a}{\cos a}\dfrac{\sin b}{\cos b}} \\ &= \frac{\tan a + \tan b}{1 - \tan a \tan b}\end{aligned}$$

The next formula follows from the Addition Formula for Tangent. Its proof is left as an exercise.

Subtraction Formula for Tangent

$$\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$$

Example 1 Find the exact value of $\sin \dfrac{\pi}{12}$.

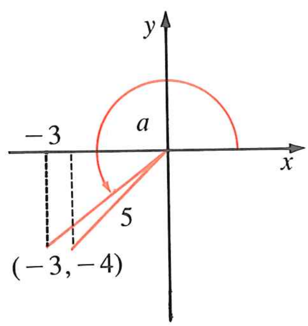
Solution

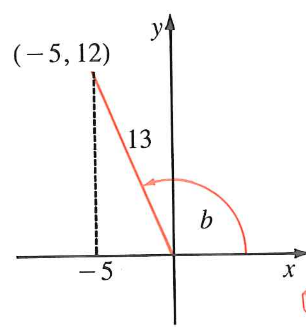
$$\begin{aligned}
\sin \frac{\pi}{12} &= \sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right) \\
&= \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6} \\
&= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2} \\
&= \frac{\sqrt{3} - 1}{2\sqrt{2}}
\end{aligned}$$

Example 2 If $\sin a = -\dfrac{4}{5}$, $\pi \leqslant a \leqslant \dfrac{3\pi}{2}$ and $\cos b = -\dfrac{5}{13}$, $\dfrac{\pi}{2} \leqslant b \leqslant \pi$, evaluate $\tan(a + b)$.

Solution

$$\begin{aligned}
\tan(a + b) &= \frac{\tan a + \tan b}{1 - \tan a \tan b} \\
&= \frac{\frac{4}{3} + \left(-\frac{12}{5}\right)}{1 - \left(\frac{4}{3}\right)\left(-\frac{12}{5}\right)} \\
&= \frac{20 - 36}{15 + 48} \\
&= -\frac{16}{63}
\end{aligned}$$





Example 3 Prove that $\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$.

Solution We know that $\sin(a+b) = \sin a \cos b + \cos a \sin b$

and $\sin(a-b) = \sin a \cos b - \cos a \sin b$

Adding, we get $\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$

Let
$$a + b = x$$
$$a - b = y$$

Therefore $2a = x + y$ and $2b = x - y$

$$a = \frac{x+y}{2} \quad \text{and} \quad b = \frac{x-y}{2}.$$

and $\sin(a+b) + \sin(a-b) = 2 \sin a \cos b$

becomes
$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$

In Example 3 we have changed the form of a trigonometric expression from a sum into a product. It is but one of a group of formulas called the *transformation formulas*. You will encounter more of them in Exercise 6.2.

The Addition and Subtraction Formulas provide an alternative method for simplifying expressions involving related angles.

Example 4 Prove that $\cos(\pi + x) = -\cos x$.

Solution
$$\begin{aligned}\cos(\pi + x) &= \cos \pi \cos x - \sin \pi \sin x\\ &= (-1)\cos x - (0) \sin x\\ &= -\cos x\end{aligned}$$

Example 5 Prove that $\cos\left(\frac{\pi}{2} - x\right) = \sin x$.

Solution
$$\begin{aligned}\cos\left(\frac{\pi}{2} - x\right) &= \cos \frac{\pi}{2} \cos x + \sin \frac{\pi}{2} \sin x\\ &= (0) \cos x + (1) \sin x\\ &= \sin x\end{aligned}$$

EXERCISE 6.2

A **1.** Express as a single trigonometric function.

(a) $\cos 2a \cos a - \sin 2a \sin a$ (b) $\cos x \cos 4x + \sin x \sin 4x$

(c) $\sin 5 \cos 2 - \cos 5 \sin 2$

(d) $\sin 2m \cos m + \cos 2m \sin m$

(e) $\dfrac{\tan 2a + \tan 3a}{1 - \tan 2a \tan 3a}$ (f) $\dfrac{\tan 7 - \tan 9}{1 + \tan 7 \tan 9}$

(g) $\cos^2 x - \sin^2 x$ (h) $\sin a \cos a + \cos a \sin a$

(i) $\dfrac{\tan x + \tan x}{1 - \tan^2 x}$ (j) $\cos^2 2 + \sin^2 2$

2. Evaluate using formulas developed in this section.

B (a) $\sin \frac{11\pi}{12}$ (b) $\cos \frac{13\pi}{12}$

(c) $\tan\left(-\frac{7}{12}\pi\right)$ (d) $\tan\left(-\frac{5}{12}\pi\right)$

(e) $\sin 75°$ (f) $\cos(-15°)$

3. Find the value of each of the following.

(a) $\sin\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$ (b) $\cos\left(-\frac{\pi}{6} - \frac{\pi}{4}\right)$ (c) $\tan\left(-\frac{3\pi}{4} + \frac{2\pi}{3}\right)$

4. If x and y are in the interval $\left(0, \frac{\pi}{2}\right)$ and $\sin x = \frac{3}{5}$ and $\cos y = \frac{12}{13}$, evaluate each of the following.

(a) $\sin(x - y)$ (b) $\cos(x + y)$ (c) $\tan(x + y)$

5. If x is in the interval $\left(\frac{\pi}{2}, \pi\right)$ and y is in the interval $\left(\pi, \frac{3\pi}{2}\right)$ and $\cos x = -\frac{5}{13}$ and $\tan y = \frac{4}{3}$, evaluate each of the following.

(a) $\sin(x + y)$ (b) $\cos(x - y)$ (c) $\tan(x - y)$

6. Find the exact value of each of the following.

(a) $\sin 50° \cos 20° - \cos 50° \sin 20°$

(b) $\cos \frac{\pi}{7} \cos \frac{4\pi}{21} - \sin \frac{\pi}{7} \sin \frac{4\pi}{21}$

(c) $\dfrac{\tan 7° + \tan 8°}{1 - \tan 7° \tan 8°}$

(d) $\sin \frac{5\pi}{36} \cos \frac{5\pi}{18} + \cos \frac{5\pi}{36} \sin \frac{5\pi}{18}$

7. Use the Addition Formula for Sine to prove the Subtraction Formula for Sine, namely, $\sin(a - b) = \sin a \cos b - \cos a \sin b$.

8. Use the identity $\tan \theta = \dfrac{\sin \theta}{\cos \theta}$ to prove the Subtraction Formula for Tangent, namely $\tan(a - b) = \dfrac{\tan a - \tan b}{1 + \tan a \tan b}$.

9. Use the Addition Formula for Tangent to prove the Subtraction Formula for Tangent.

10. Prove each of the following.

(a) $\sin(\pi + x) = -\sin x$ (b) $\tan(2\pi - x) = -\tan x$

(c) $\cos\left(\frac{3\pi}{2} + x\right) = \sin x$ (d) $\sin\left(\frac{3\pi}{2} - x\right) = -\cos x$

(e) $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$ (f) $\tan\left(\frac{\pi}{2} + x\right) = -\cot x$

(g) $\sin(x - \pi) = -\sin x$ (h) $-\tan(-x - \pi) = \tan x$

11. Using the method developed in Example 3 of this section, prove each of the following Transformation Formulas.

(a) $\sin x - \sin y = 2 \cos \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

(b) $\cos x + \cos y = 2 \cos \dfrac{x+y}{2} \cos \dfrac{x-y}{2}$

(c) $\cos x - \cos y = -2 \sin \dfrac{x+y}{2} \sin \dfrac{x-y}{2}$

12. Express each of the following as a product and simplify.

(a) $\sin 60° + \sin 20°$ (b) $\cos 70° - \cos 110°$

(c) $\cos 40° + \cos 80°$ (d) $\sin 6x - \sin 2x$

(e) $\sin 130° - \sin 40°$ (f) $\cos 4x - \cos 2x$

13. Simplify.

(a) $\dfrac{\sin(x - 30°) + \cos(60° - x)}{\sin x}$ (b) $\dfrac{\tan\left(\frac{\pi}{4} - x\right) - \tan\left(\frac{\pi}{4} + x\right)}{\tan x}$

(c) $\dfrac{\cos 4x + \cos 3x}{\sin 4x - \sin 3x}$

14. If $\sin x = -\frac{1}{3}$, $\pi < x < \dfrac{3\pi}{2}$ and $\cos y = \frac{2}{5}$, $\dfrac{3\pi}{2} < y < 2\pi$, find the value of $\sec(x - y)$.

15. Express $\csc(x + y)$ in terms of secants and cosecants of x and y.

16. Develop a formula for $\sin(x + y + z)$. Start by rewriting it in form $\sin[(x + y) + z]$.

17. If x, y, and z are second quadrant angles and $\cos x = -\frac{1}{3}$, $\sin y = \frac{1}{4}$, and $\sin z = \frac{1}{5}$, find the value of $\cos(x + y - z)$.

C **18.** If $\dfrac{\sin x}{\sin y} = \dfrac{1}{2}$ and $\dfrac{\cos x}{\cos y} = 3$ prove:

(a) $\sin(x + y) = \frac{7}{3} \sin x \cos x$

(b) $\cos(x + y) = \dfrac{7 \cos^2 x - 6}{3}$

19. If $2 \sin(x - y) = \sin(x + y)$, prove that $\tan x = 3 \tan y$.

20. If $\tan\left(\dfrac{\pi}{4} + x\right) = 3 \tan\left(\dfrac{\pi}{4} - x\right)$, find the value of $\tan x$.

21. Simplify.

(a) $\cos\left(\dfrac{3\pi}{4} + x\right) + \sin\left(\dfrac{3\pi}{4} - x\right)$

(b) $\cos\left(\dfrac{\pi}{12} - x\right) \sec \dfrac{\pi}{12} - \sin\left(\dfrac{\pi}{12} - x\right) \csc \dfrac{\pi}{12}$

(c) $\dfrac{\sin(x - y)}{\cos x \cos y} + \dfrac{\sin(z - x)}{\cos z \cos x} + \dfrac{\sin(y - z)}{\cos y \cos z}$

6.3 DOUBLE ANGLE FORMULAS

In this section we use the addition formulas for sine, cosine, and tangent to generate some frequently used trigonometric relationships.
If we start with

$$\sin(a + b) = \sin a \cos b + \cos a \sin b$$

then, setting $a = b = x$ gives

$$\begin{aligned} \sin(x + x) &= \sin x \cos x + \cos x \sin x \\ \sin 2x &= 2 \sin x \cos x \end{aligned}$$

Double Angle Formula for Sine

$$\sin 2x = 2 \sin x \cos x$$

Example 1 If $\sin x = \frac{4}{5}, \frac{\pi}{2} < x < \pi$, find the value of $\sin 2x$.

Solution Now $\qquad \sin 2x = 2 \sin x \cos x$

and, since $\frac{\pi}{2} < x < \pi$,

$$\cos x = -\tfrac{3}{5}$$

Therefore $\qquad \sin 2x = 2\left(\tfrac{4}{5}\right)\left(-\tfrac{3}{5}\right) = -\tfrac{24}{25}$

A similar development produces the Double Angle Formula for Cosine.
If we start with

$$\cos(a + b) = \cos a \cos b - \sin a \sin b$$

then, letting $a = b = x$ gives

$$\begin{aligned} \cos(x + x) &= \cos x \cos x - \sin x \sin x \\ \cos 2x &= \cos^2 x - \sin^2 x \end{aligned}$$

The identity $\sin^2 \theta + \cos^2 \theta = 1$ is applied to the original result to produce two alternative forms of the formula.

$$\begin{aligned} \cos 2x &= \cos^2 x - (1 - \cos^2 x) \\ &= 2 \cos^2 x - 1 \end{aligned} \qquad \begin{aligned} \cos 2x &= (1 - \sin^2 x) - \sin^2 x \\ &= 1 - 2 \sin^2 x \end{aligned}$$

Double Angle Formulas for Cosine

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\cos 2x = 2 \cos^2 x - 1$$

$$\cos 2x = 1 - 2 \sin^2 x$$

The three results are equivalent, but as you gain experience working with these formulas, you will learn that one form may be superior to the others in a particular problem.

Example 2 If $\cos a = \frac{2}{3}$, find the value of $\cos 4a$.

Solution We must express $\cos 4a$ in terms of trigonometric functions of a. This is accomplished by applying the Double Angle Formula for Cosine twice.

$$\begin{aligned}\cos 4a &= \cos[2(2a)]\\ &= 2\cos^2 2a - 1\\ &= 2(2\cos^2 a - 1)^2 - 1\\ &= 2\left[2\left(\tfrac{2}{3}\right)^2 - 1\right]^2 - 1\\ &= 2\left(-\tfrac{1}{9}\right)^2 - 1\\ &= -\tfrac{79}{81}\end{aligned}$$

The application of the Double Angle Formula for Cosine in the next example should be examined carefully.

Example 3 Evaluate $\sin \frac{\pi}{8}$.

Solution Since $$\cos\frac{\pi}{4} = \cos 2\left(\frac{\pi}{8}\right)$$

the Double Angle Formula gives $$\cos\frac{\pi}{4} = 1 - 2\sin^2\frac{\pi}{8}$$

Solve for $\sin\frac{\pi}{8}$: $$\sin^2\frac{\pi}{8} = \frac{1 - \cos\frac{\pi}{4}}{2}$$

Since $\sin\frac{\pi}{8}$ is positive, $$\begin{aligned}\sin\frac{\pi}{8} &= \sqrt{\frac{1 - \frac{1}{\sqrt{2}}}{2}}\\ &= \sqrt{\frac{\sqrt{2} - 1}{2\sqrt{2}}}\end{aligned}$$

Now we develop the Double Angle Formula for Tangent. If we start with

$$\tan(a + b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$$

then, setting $a = b = x$ gives

$$\tan(x + x) = \frac{\tan x + \tan x}{1 - \tan x \tan x}$$

Therefore

Double Angle Formula for Tangent

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Example 4 If $\tan x = \frac{4}{3}$, $\pi < x < \dfrac{3\pi}{2}$, find the value of $\tan \dfrac{x}{2}$.

Solution

$$\tan x = \frac{4}{3}$$

$$\tan\left[2\left(\frac{x}{2}\right)\right] = \frac{4}{3}$$

$$\frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{4}{3}$$

$$6 \tan \frac{x}{2} = 4 - 4 \tan^2 \frac{x}{2}$$

$$2 \tan^2 \frac{x}{2} + 3 \tan \frac{x}{2} - 2 = 0$$

$$\left(2 \tan \frac{x}{2} - 1\right)\left(\tan \frac{x}{2} + 2\right) = 0$$

Therefore

$$\tan \frac{x}{2} = \frac{1}{2} \quad \text{or} \quad \tan \frac{x}{2} = -2$$

Since $\pi < x < 2\pi$, we have $\dfrac{\pi}{2} < \dfrac{x}{2} < \pi$, so

$$\tan \frac{x}{2} = -2$$

EXERCISE 6.3

B **1.** Use a Double Angle Formula to rewrite each expression.

(a) $\cos 2(2x)$ (b) $\sin 3x$ (c) $\tan 6x$

(d) $\sin \frac{1}{2}x$ (e) $\cos \frac{2}{3}x$ (f) $\tan(-7x)$

2. Express as a single sine or cosine function.

(a) $2 \sin 3\theta \cos 3\theta$ (b) $6 \sin \theta \cos \theta$

(c) $\frac{1}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}$ (d) $\cos^2 \frac{3\theta}{2} - \sin^2 \frac{3\theta}{2}$

(e) $1 - 2 \sin^2 \frac{\theta}{4}$ (f) $2 \cos^2 \left(\frac{7}{2}\theta\right) - 1$

(g) $8 \sin^2 2\theta - 4$ (h) $1 - 2 \sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)$

3. If $\cos \theta = -\frac{4}{5}, \frac{\pi}{2} \leq \theta \leq \pi$, find the value of $\sin 2\theta$ and $\cos 2\theta$. Determine the quadrant of angle 2θ.

4. If $\sin \theta = \frac{12}{13}, 0 \leq \theta \leq \frac{\pi}{2}$, evaluate $\sin 2\theta$ and $\cos 2\theta$. Determine the quadrant of angle 2θ.

5. If $\sin \theta = \frac{2}{3}, 0 \leq \theta \leq \frac{\pi}{2}$, find the value of $\sin 4\theta$.

6. If $\cos \theta = \frac{2}{5}, \frac{3\pi}{2} \leq \theta \leq 2\pi$, find the values of $\csc 2\theta$ and $\sec 2\theta$.

7. If $\tan a = \frac{1}{2}, 0 \leq a \leq \frac{\pi}{2}$, find the value of $\tan 2a$.

8. If $\tan a = 2, -2\pi \leq a \leq -\frac{3\pi}{2}$, evaluate $\tan 4a$.

9. Develop formulas for

(a) $\sin 3\theta$ in terms of $\sin \theta$. (b) $\cos 3\theta$ in terms of $\cos \theta$.

(c) $\tan 3\theta$ in terms of $\tan \theta$. (d) $\cos 4\theta$ in terms of $\cos \theta$.

10. Find the exact values.

(a) $\sin 67\frac{1}{2}°$ (b) $\cos 112\frac{1}{2}°$ (c) $\tan 22.5°$

(d) $\sin\left(-\frac{\pi}{8}\right)$ (e) $\cos \frac{\pi}{16}$ (f) $\tan 33.75°$

11. (a) Express $\sin \frac{\theta}{2}$ in terms of $\cos \theta$.

(b) Express $\cos \frac{\theta}{2}$ in terms of $\cos \theta$.

(c) Express $\tan \frac{\theta}{2}$ in terms of $\cos \theta$.

12. If $\cos x = -\frac{12}{13}$ and x is in the interval $\left(\pi, \frac{3\pi}{2}\right)$, find the following.

(a) $\sin \frac{x}{2}$ (b) $\cos \frac{x}{2}$ (c) $\tan \frac{x}{2}$

13. Express $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan \theta$.

14. Express $\dfrac{\sin 2x}{1 + \cos 2x}$ in terms of $\tan x$.

15. Express $\dfrac{1 - \cos 2x}{\sin 2x}$ in terms of $\tan x$.

16. Express $\dfrac{1 - \tan^2 x}{1 + \tan^2 x}$ as a trigonometric function of $2x$.

C **17.** If $\cos \theta + \sin \theta = \frac{2}{3}$, find the value of $\sin 2\theta$.

18. If $\cos \theta + \sin \theta = \dfrac{1 + \sqrt{3}}{2}$ and $\cos \theta - \sin \theta = \dfrac{1 - \sqrt{3}}{2}$ find the value of $\sin 2\theta$.

19. If $2a + b = \frac{\pi}{2}$, prove that $\cos a = \pm\sqrt{\dfrac{1 + \sin b}{2}}$.

20. If $\tan a = \frac{1}{5}$ and $\tan b = \frac{1}{239}$, find the value of $\tan(4a - b)$.

21. If $\sec 4\theta - \sec 2\theta = 2$, find the value of $\cos^2 \theta$, $0 < \theta < \frac{\pi}{2}$.

22. Simplify $\sin^2\left(\frac{\pi}{8} + \frac{\theta}{2}\right) - \sin^2\left(\frac{\pi}{8} - \frac{\theta}{2}\right)$.

6.4 TRIGONOMETRIC IDENTITIES

Statements of equality in mathematics generally fall into two categories. A *conditional equation* is valid for certain values of the variable or variables involved. Each of the following fall into this category.

$$3x - 2 = 7 \quad ①$$
$$x^2 - 7x + 12 = 0 \quad ②$$
$$2x - 3y = 6 \quad ③$$

Equation 1 is only true for $x = 3$. Equation 2 is true for $x = 3$ or $x = 4$, but no other values of x. Equation 3 is true for an infinite number of values of x and y, but values chosen at random will generally fail to satisfy it.

The second type of equation is called an *identity*. It holds for all values of the variable or variables for which it is defined. Each of the following is an identity.

$$7x - 4x = 3x \qquad (4)$$

$$x^2 - 7xy + 12y^2 = (x - 3y)(x - 4y) \qquad (5)$$

$$\frac{x^2 - 9}{x + 3} = x - 3 \qquad (6)$$

Equations 4 and 5 are obviously true for all values of their respective variables. Equation 6 is undefined at $x = -3$, but is true for all other values of x.

We have encountered most of the fundamental trigonometric identities in earlier courses. Many more have been established in the first three sections of this chapter. By means of these established identities, trigonometric expressions can be transformed or simplified. They can also be used to prove other identities.

To prove an identity we may:

(1) transform the left side to the exact form of the right side; or

(2) transform the right side to the exact form of the left side; or

(3) transform both sides to an identical form.

To accomplish this we may:

(1) perform substitutions based on established identities;

(2) employ algebraic manipulation, (factoring and creating common denominators are two of the most popular); and

(3) use our ingenuity.

Here is the complete list of identities and formulas required to attempt Exercise 6.4.

Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$\csc x = \dfrac{1}{\sin x}$	$\tan x = \dfrac{\sin x}{\cos x}$	$\sin^2 x + \cos^2 x = 1$
$\sec x = \dfrac{1}{\cos x}$	$\cot x = \dfrac{\cos x}{\sin x}$	$\sec^2 x = 1 + \tan^2 x$
$\cot x = \dfrac{1}{\tan x}$		$\csc^2 x = 1 + \cot^2 x$

Addition and Subtraction Formulas	Double Angle Formulas
$\sin(x+y) = \sin x \cos y + \cos x \sin y$	$\sin 2x = 2 \sin x \cos x$
$\sin(x-y) = \sin x \cos y - \cos x \sin y$	
$\cos(x+y) = \cos x \cos y - \sin x \sin y$	$\cos 2x = \cos^2 x - \sin^2 x$
$\cos(x-y) = \cos x \cos y + \sin x \sin y$	$= 2\cos^2 x - 1$
$\tan(x+y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$= 1 - 2\sin^2 x$
$\tan(x-y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$	$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

Related Angle Identities

$\sin(\pi - x) = \sin x$	$\sin(2\pi - x) = -\sin x$
$\cos(\pi - x) = -\cos x$	$\cos(2\pi - x) = \cos x$
$\tan(\pi - x) = -\tan x$	$\tan(2\pi - x) = -\tan x$
$\sin(\pi + x) = -\sin x$	$\sin(-x) = -\sin x$
$\cos(\pi + x) = -\cos x$	$\cos(-x) = \cos x$
$\tan(\pi + x) = \tan x$	$\tan(-x) = -\tan x$

Corelated Angle Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$	$\sin\left(\dfrac{3\pi}{2} - x\right) = -\cos x$
$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$	$\cos\left(\dfrac{3\pi}{2} - x\right) = -\sin x$
$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$	$\tan\left(\dfrac{3\pi}{2} - x\right) = \cot x$
$\sin\left(\dfrac{\pi}{2} + x\right) = \cos x$	$\sin\left(\dfrac{3\pi}{2} + x\right) = -\cos x$
$\cos\left(\dfrac{\pi}{2} + x\right) = -\sin x$	$\cos\left(\dfrac{3\pi}{2} + x\right) = \sin x$
$\tan\left(\dfrac{\pi}{2} + x\right) = -\cot x$	$\tan\left(\dfrac{3\pi}{2} + x\right) = -\cot x$

Once you know the formulas, the key to becoming adept at proving identities is perseverance. If you relish the challenge of a good puzzle, this will prove to be a particularly enjoyable section.

A list of the most common strategies used in performing substitutions is provided.

(1) Move from the complex to the simple.
(2) Express all functions in terms of sine and cosine.
(3) Look for squares and the use of the Pythagorean identities.
(4) Express all functions with the same argument.

Example 1 Prove $1 + \cos x = \dfrac{\sin^2 x}{1 - \cos x}$.

Solution We start with the more complex right side and focus on $\sin^2 x$.

$$\begin{aligned}\frac{\sin^2 x}{1 - \cos x} &= \frac{1 - \cos^2 x}{1 - \cos x}\\ &= \frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x}\\ &= 1 + \cos x\end{aligned}$$

Example 2 Prove $\cos(x + y)\cos(x - y) = \cos^2 x + \cos^2 y - 1$.

Solution The left side contains two familiar factors, so let's start with it. As we move through the proof, we are constantly aware of the terms on the right side that we need to produce on the left side.

$$\begin{aligned}\cos(x + y)\cos(x - y) &= (\cos x \cos y - \sin x \sin y)(\cos x \cos y + \sin x \sin y)\\ &= \cos^2 x \cos^2 y - \sin^2 x \sin^2 y\end{aligned}$$

Noting that $\sin^2 x$ and $\sin^2 y$ do not appear on the right side, we replace them and get

$$\begin{aligned}\cos(x + y)\cos(x - y) &= \cos^2 x \cos^2 y - (1 - \cos^2 x)(1 - \cos^2 y)\\ &= \cos^2 x \cos^2 y - (1 - \cos^2 x - \cos^2 y + \cos^2 x \cos^2 y)\\ &= \cos^2 x \cos^2 y - 1 + \cos^2 x + \cos^2 y - \cos^2 x \cos^2 y\\ &= \cos^2 x + \cos^2 y - 1\end{aligned}$$

Example 3 Prove $\dfrac{\sin 2x}{1 - \cos 2x} = 2 \csc 2x - \tan x$.

Solution We express csc and tan in terms of sin and cos. Next we note the presence of two different arguments, $2x$ and x. We work with each side separately to achieve a common result.

$$\begin{aligned}\frac{\sin 2x}{1 - \cos 2x} &= \frac{2 \sin x \cos x}{1 - (1 - 2\sin^2 x)}\\ &= \frac{2 \sin x \cos x}{2 \sin^2 x}\\ &= \frac{\cos x}{\sin x}\end{aligned}$$

$$\begin{aligned}2 \csc 2x - \tan x &= \frac{2}{\sin 2x} - \tan x\\ &= \frac{2}{2 \sin x \cos x} - \frac{\sin x}{\cos x}\\ &= \frac{2 - 2\sin^2 x}{2 \sin x \cos x}\\ &= \frac{1 - \sin^2 x}{\sin x \cos x}\\ &= \frac{\cos^2 x}{\sin x \cos x}\\ &= \frac{\cos x}{\sin x}\end{aligned}$$

Therefore $\dfrac{\sin 2x}{1 - \cos 2x} = 2 \csc 2x - \tan x$

EXERCISE 6.4

B *The following identities involve the reciprocal, quotient, and Pythagorean relationships. Prove each one.*

1. $\sin x \tan x = \sec x - \cos x$
2. $\cos^4 x - \sin^4 x = 1 - 2\sin^2 x$
3. $\csc^2 x + \sec^2 x = \csc^2 x \sec^2 x$
4. $\cos^2 x \cos^2 y + \sin^2 x \sin^2 y + \sin^2 x \cos^2 y + \sin^2 y \cos^2 x = 1$
5. $\sec^2 x - \sec^2 y = \tan^2 x - \tan^2 y$
6. $\dfrac{\tan x + \tan y}{\cot x + \cot y} = (\tan x)(\tan y)$
7. $(\sec x - \cos x)(\csc x - \sin x) = \dfrac{\tan x}{1 + \tan^2 x}$
8. $\cos^6 x + \sin^6 x = 1 - 3\sin^2 x + 3\sin^4 x$
9. $\sec^6 x - \tan^6 x = 1 + 3\tan^2 x \sec^2 x$

The following involve the addition and subtraction formulas.

10. $1 + \cot x \tan y = \dfrac{\sin(x + y)}{\sin x \cos y}$
11. $\cos(x + y)\cos y + \sin(x + y)\sin y = \cos x$
12. $\sin x - \tan y \cos x = \dfrac{\sin(x - y)}{\cos y}$
13. $\cos\left(\dfrac{3\pi}{4} + x\right) + \sin\left(\dfrac{3\pi}{4} - x\right) = 0$
14. $\dfrac{\tan\left(\dfrac{\pi}{4} + x\right) - \tan\left(\dfrac{\pi}{4} - x\right)}{\tan\left(\dfrac{\pi}{4} + x\right) + \tan\left(\dfrac{\pi}{4} - x\right)} = 2\sin x \cos x$
15. $\sin(x + y)\sin(x - y) = \cos^2 y - \cos^2 x$
16. $\tan(x + y)\tan(x - y) = \dfrac{\sin^2 x - \sin^2 y}{\cos^2 x - \sin^2 y}$
17. $\dfrac{\tan(x - y) + \tan y}{1 - \tan(x - y)\tan y} = \tan x$
18. $\sin 5x = \sin x\,(\cos^2 2x - \sin^2 2x) + 2\cos x \cos 2x \sin 2x$

The following involve related and co-related angles.

19. $\sin\left(\dfrac{\pi}{2} - x\right)\cot\left(\dfrac{\pi}{2} + x\right) = -\sin x$
20. $\cos(-x) + \cos(\pi - x) = \cos(\pi + x) + \cos x$

21. $\dfrac{\sin(\pi - x)}{\tan(\pi + x)} \dfrac{\cot\left(\dfrac{\pi}{2} - x\right)}{\tan\left(\dfrac{\pi}{2} + x\right)} \dfrac{\cos(2\pi - x)}{\sin(-x)} = \sin x$

22. $\dfrac{\sin(-x)}{\sin(\pi + x)} - \dfrac{\tan\left(\dfrac{\pi}{2} + x\right)}{\cot x} + \dfrac{\cos x}{\sin\left(\dfrac{\pi}{2} + x\right)} = 3$

23. $\dfrac{\csc(\pi - x)}{\sec(\pi + x)} \dfrac{\cos(-x)}{\cos\left(\dfrac{\pi}{2} + x\right)} = \cot^2 x$

24. $\dfrac{\cos\left(\dfrac{\pi}{2} + x\right)\sec(-x)\tan(\pi - x)}{\sec(2\pi + x)\sin(\pi + x)\cot\left(\dfrac{\pi}{2} - x\right)} = -1$

25. $\dfrac{\sin(\pi - x)\cos(\pi + x)\tan(2\pi - x)}{\sec\left(\dfrac{\pi}{2} + x\right)\csc\left(\dfrac{3\pi}{2} - x\right)\cot\left(\dfrac{3\pi}{2} + x\right)} = \sin^4 x - \sin^2 x$

The following involve the double angle formulas.

26. $\dfrac{\sin 2x}{1 + \cos 2x} = \tan x$

27. $\dfrac{1 + \cos x}{\sin x} = \cot \dfrac{x}{2}$

28. $2 \csc 2x = \sec x \csc x$

29. $2 \cot 2x = \cot x - \tan x$

30. $\dfrac{\cos 2x}{1 + \sin 2x} = \tan\left(\dfrac{\pi}{4} - x\right)$

31. $\dfrac{\cos x - \sin x}{\cos x + \sin x} = \sec 2x - \tan 2x$

32. $\dfrac{1 - \cos 2x + \sin 2x}{1 + \cos 2x + \sin 2x} = \tan x$

33. $\cos^6 x - \sin^6 x = \cos 2x\left(1 - \frac{1}{4}\sin^2 2x\right)$

34. $4(\cos^6 x + \sin^6 x) = 1 + 3\cos^2 2x$

35. $\sec x - \tan x = \tan\left(\dfrac{\pi}{4} - \dfrac{x}{2}\right)$

36. $\dfrac{\sin 2x}{1 + \cos 2x} \dfrac{\cos x}{1 + \cos x} = \tan \dfrac{x}{2}$

The following involve a variety of formulas and identities.

37. $\sin^2 x + \cos^4 x = \cos^2 x + \sin^4 x$

38. $\tan x - \cot x = (\tan x - 1)(\cot x + 1)$

39. $\cos x = \sin x \tan^2 x \cot^3 x$

40. $(\sin x + \cos x)(\tan x + \cot x) = \sec x + \csc x$

41. $\sin^4 x + \cos^4 x = \sin^2 x(\csc^2 x - 2\cos^2 x)$

42. $\sin^3 x + \cos^3 x = (1 - \sin x \cos x)(\sin x + \cos x)$

43. $\cos\left(\frac{\pi}{12} - x\right)\sec\frac{\pi}{12} - \sin\left(\frac{\pi}{12} - x\right)\csc\frac{\pi}{12} = 4\sin x$

44. $\tan(x - y) + \tan(y - z) = \dfrac{\sec^2 y\,(\tan x - \tan z)}{(1 + \tan x \tan y)(1 + \tan y \tan z)}$

45. $\sin 8x = 8 \sin x \cos x \cos 2x \cos 4x$

46. $\sin x = 1 - 2\sin^2\left(\frac{\pi}{4} - \frac{x}{2}\right)$

47. $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

48. $\dfrac{\sin(x - y)}{\sin x \sin y} + \dfrac{\sin(y - z)}{\sin y \sin z} + \dfrac{\sin(z - x)}{\sin z \sin x} = 0$

49. $\tan x + \tan(\pi - x) + \cot\left(\frac{\pi}{2} + x\right) = \tan(2\pi - x)$

50. $\sin\left(\frac{\pi}{2} + x\right)\cos(\pi - x)\cot\left(\frac{3\pi}{2} + x\right)$
$= \sin\left(\frac{\pi}{2} - x\right)\sin\left(\frac{3\pi}{2} - x\right)\cot\left(\frac{\pi}{2} + x\right)$

51. $\tan\left(\frac{\pi}{2} - x\right) - \cot\left(\frac{3\pi}{2} - x\right) + \tan(2\pi - x) - \cot(\pi - x)$
$= \dfrac{4 - 2\sec^2 x}{\tan x}$

52. $\tan(x + y + z) = \dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}$

53. $\csc^2\left(\frac{\pi}{2} - x\right) = 1 + \sin^2 x \csc^2\left(\frac{\pi}{2} - x\right)$

54. $\tan\left(\frac{\pi}{4} + x\right) + \tan\left(\frac{\pi}{4} - x\right) = 2\sec 2x$

55. $\dfrac{1 - \sin 2x}{\cos 2x} = \dfrac{\cos 2x}{1 + \sin 2x}$

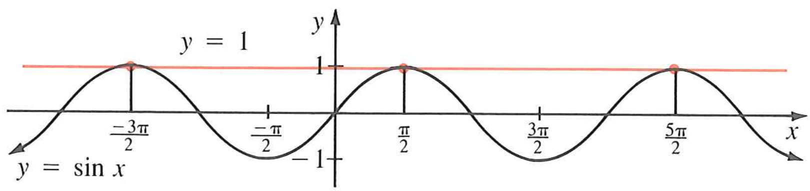
56. $\dfrac{\sin 4x}{1 - \cos 4x} \times \dfrac{1 - \cos 2x}{\cos 2x} = \tan x$

6.5 SOLVING TRIGONOMETRIC EQUATIONS

Because of the periodic nature of the trigonometric functions, if no domain is specified, an infinite number of solutions to a trigonometric equation will exist, providing the equation has a solution. Knowledge of the range of the basic trigonometric functions and an ability to sketch their graphs may be very useful in this section. Refer to the Review and Preview at the beginning of this chapter if your memory needs jogging.

Example 1 If $\sin x = 1$, solve for x.

Solution Since no restrictions have been placed on x, an infinite number of solutions exist. We draw the graph of $y = \sin x$ and the horizontal line $y = 1$.

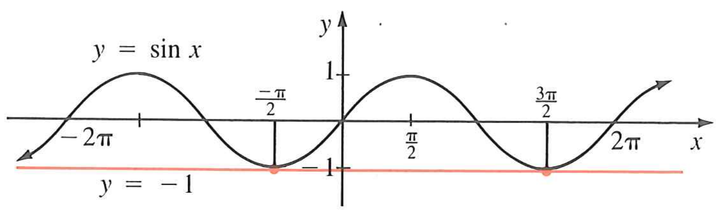


They intersect at $x = \ldots, -\frac{3}{2}\pi, \frac{1}{2}\pi, \frac{5}{2}\pi, \ldots$

or generally $x = \frac{\pi}{2} + 2k\pi,\ k \in I$

Example 2 If $\sin x = -1$, solve for x in the interval $[-2\pi, 2\pi]$.

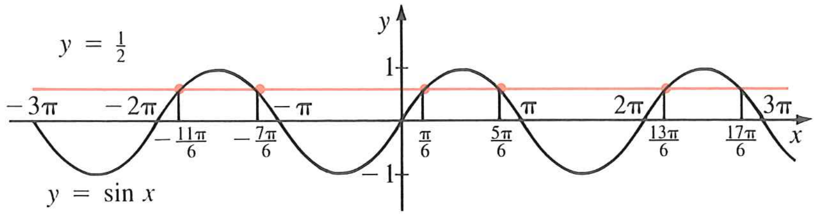
Solution The given interval limits us to two solutions.



$$x = -\tfrac{1}{2}\pi \text{ or } x = \tfrac{3}{2}\pi$$

Example 3 If $\sin 3x = \frac{1}{2}$, solve for x in the interval $[-\pi, \pi]$.

Solution The form of the restriction is changed to fit the argument $3x$. Since $-\pi \leqslant x \leqslant \pi$, we have $-3\pi \leqslant 3x \leqslant 3\pi$.



In our domain $\quad 3x = -\frac{11}{6}\pi, -\frac{7}{6}\pi, \frac{1}{6}\pi, \frac{5}{6}\pi, \frac{13}{6}\pi, \text{ or } \frac{17}{6}\pi$

Therefore $\quad x = -\frac{11}{18}\pi, -\frac{7}{18}\pi, \frac{1}{18}\pi, \frac{5}{18}\pi, \frac{13}{18}\pi, \text{ or } \frac{17}{18}\pi$

Although a variety of methods may be used to solve trigonometric equations, the following strategies are most frequently applied.

(1) *Make use of the established formulas and identities to express the equation in terms of a single trigonometric function*
(2) *Apply the double angle formulas to match the arguments*
(3) *Use algebraic methods to solve for the trigonometric function*
(4) *Finally, within the specified domain, solve for the variable*

Example 4 If $\sec 2x + \dfrac{1}{\cos x} = 0$, $0 \leqslant x \leqslant \pi$, solve for x.

Solution Since $\sec 2x = \dfrac{1}{\cos 2x}$, we can express the equation in terms of the same function.

$$\frac{1}{\cos 2x} + \frac{1}{\cos x} = 0$$
$$\frac{\cos x + \cos 2x}{\cos 2x \cos x} = 0$$

Therefore $\quad \cos x + \cos 2x = 0 \quad \text{if } \cos 2x \cos x \neq 0$

We match the arguments using the formula $\cos 2x = 2\cos^2 x - 1$.

$$\cos x + (2\cos^2 x - 1) = 0$$
$$2\cos^2 x + \cos x - 1 = 0$$
$$(2\cos x - 1)(\cos x + 1) = 0$$

Solving for $\cos x$, $\quad \cos x = \frac{1}{2}$ or $\cos x = -1$

Solving for x, $x \in [0, \pi]$, $\quad x = \dfrac{\pi}{3} \quad$ or $\quad x = \pi$

We verify that $\cos 2x \cos x \neq 0$. If $x = \dfrac{\pi}{3}$,

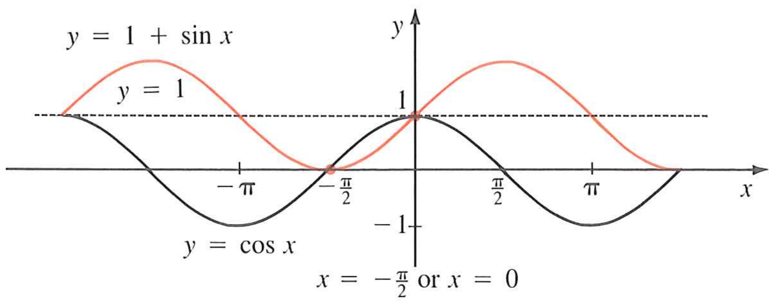
$$\cos \frac{2\pi}{3} \cos \frac{\pi}{3} = \frac{-1}{2} \times \frac{1}{2} = \frac{-1}{4}$$

If $x = \pi$, $\quad \cos 2\pi \cos \pi = 1(-1) = -1$

We examine three completely different solutions for the next example.

Example 5 If $1 + \sin x = \cos x$, solve for x in the interval $[-\pi, \pi]$.

Solution 1 We graph $y = 1 + \sin x$ and $y = \cos x$ on the same set of axes and read the value of x at the points of intersection.



Solution 2 In order to express the equation in terms of a single trigonometric function, we square both sides to create a $\cos^2 x$ and replace it with $1 - \sin^2 x$.

$$\begin{aligned} 1 + 2\sin x + \sin^2 x &= \cos^2 x \\ &= 1 - \sin^2 x \end{aligned}$$

Therefore $\qquad 2\sin^2 x + 2\sin x = 0$

$\qquad\qquad\qquad 2\sin x(\sin x + 1) = 0$

Now $\qquad \sin x = 0 \quad$ or $\quad \sin x = -1$

and for $x \in [-\pi, \pi]$ $\qquad x = -\pi, -\dfrac{\pi}{2}, 0$, or π

Since we squared both sides, we may have introduced roots that do not satisfy the original equation. We must verify each value of x.

If $x = -\pi$, $1 + \sin x = 1 + 0 = 1$ and $\cos x = -1$. Therefore $x = -\pi$ is inadmissible.

If $x = -\frac{\pi}{2}$, $1 + \sin x = 1 - 1 = 0$ and $\cos x = 0$. Therefore $x = -\frac{\pi}{2}$ is a root.

Similarly it can be proved that $x = 0$ is a root, but $x = \pi$ is inadmissible.

Solution 3 The equation is not expressed in terms of a single trigonometric function. A Double Angle Formula enables us to eliminate the 1. Note that when the argument is changed the restriction on the variable must change accordingly.

$$1 + \sin x = \cos x$$

$$1 + 2 \sin \frac{x}{2} \cos \frac{x}{2} = 1 - 2 \sin^2 \frac{x}{2}$$

$$2 \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2} = 0$$

$$2 \sin \frac{x}{2}\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) = 0$$

Therefore $\sin \frac{x}{2} = 0$ or $\sin \frac{x}{2} = -\cos \frac{x}{2}$

$\sin \frac{x}{2} = 0$ or $\tan \frac{x}{2} = -1$

Since $-\frac{\pi}{2} < \frac{x}{2} < \frac{\pi}{2}$, $\frac{x}{2} = 0$ or $\frac{x}{2} = -\frac{\pi}{4}$

and $x = 0$ or $x = -\frac{\pi}{2}$

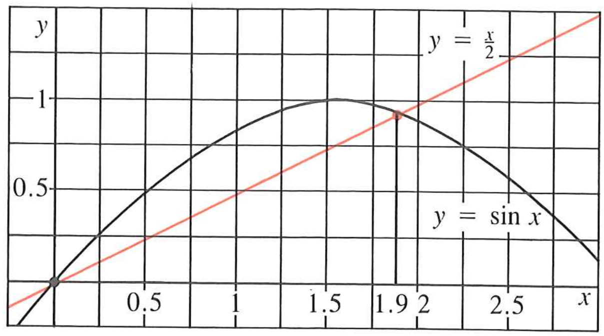
If an equation contains algebraic as well as trigonometric functions it cannot be expressed in terms of a single function. In many cases the best we can do is approximate the root.

Example 6 Solve $\sin x = \frac{x}{2}$, $0 < x < 2.5$.

Solution The equation $\sin x = \frac{x}{2}$ can be replaced by the equivalent system

$$\begin{cases} y = \sin x \\ y = \frac{x}{2} \end{cases}$$

An approximation can be obtained by graphing each on the same set of axes and estimating the value of x at the points of intersection.



One solution occurs when $x = 0$. The other solution appears to be approximately 1.9. When $x = 1.9$, $\sin x \doteq 0.946\ 300$ and $\frac{x}{2} = .95$. Their difference is approximately .0037.

In Exercise 7.2 we will use Newton's method to solve this type of equation and our accuracy will be significantly improved. Our initial guess will be arrived at by approximating a solution using a graph as in Example 6.

EXERCISE 6.5

A **1.** Solve for x in the interval $[0, 2\pi]$.

(a) $\sin x = \frac{\sqrt{3}}{2}$ (b) $\cos x = \frac{1}{2}$ (c) $\tan x = -1$

(d) $\sec x = -2$ (e) $\sin x = -\frac{1}{2}$ (f) $\cos^2 x = \frac{1}{4}$

B **2.** Solve for x in the interval $[-\pi, 0]$.

(a) $\cos x = -\frac{1}{\sqrt{2}}$ (b) $\tan^2 x = \tan x$

(c) $\sin^2 x - \sin x = 2$ (d) $\sin^2 x = \frac{3}{4}$

(e) $4\cos^2 x - 3 = 0$ (f) $(2\csc x - 1)^2 = 9$

3. Solve for x in the given interval.

(a) $\sin x - \sin x \tan x = 0$, $[0, \pi]$
(b) $\sin x \tan 3x = 0$, $[-\pi, 0]$
(c) $6\sin^2 x - 5\cos x - 2 = 0$, $[0, 2\pi]$
(d) $\sqrt{2}\sin x + \tan x = 0$, $[-\pi, \pi]$
(e) $\cos^2 x - 3\sin^2 x = 1$, $[-2\pi, 2\pi]$
(f) $2\tan x = \sec x$, $[-2\pi, 0]$

4. Solve for x.
   (a) $\cos 2x = \cos^2 x, \ -\pi \leqslant x \leqslant \pi$
   (b) $\sin 2x = \cos x, \ -\pi \leqslant 2x \leqslant \pi$
   (c) $\cos^2 x - 2 \sin x \cos x - \sin^2 x = 0, \ 0 \leqslant 2x \leqslant \pi$
   (d) $\tan 2x = 8 \cos^2 x - \cot x, \ 0 \leqslant x \leqslant \dfrac{\pi}{2}$
   (e) $\tan x + \sec 2x = 1, \ -\dfrac{\pi}{2} \leqslant x \leqslant \dfrac{\pi}{2}$
   (f) $2(\sin^4 x + \cos^4 x) = 1, \ -\pi \leqslant x \leqslant \pi$
5. If $2 \tan x \cos^2 x + \sin x \tan x - 2 \tan x = 0, \ |x| < 2\pi$, solve for x.
6. If $\tan 2x = -\frac{24}{7}$, find the values of $\sin x$ and $\cos x$.
7. Find the values of x in the interval $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$ that satisfy the equations. Approximating may be required.
   (a) $\sin x = x$ (b) $\cos x = x$ (c) $\tan x = -x$
   (d) $\cos x = -\dfrac{x}{3}$ (e) $\sin x = x \sin x$ (f) $\tan x = 2x$

C 8. If $x \sin A + y \cos A = p$ and $x \cos A - y \sin A = q$, find $x^2 + y^2$.
9. If $\sin A + \cos A = p$ and $\tan A + \cot A = q$, prove that $q(p^2 - 1) = 2$.

6.6 REVIEW EXERCISE

1. If x is in the interval $\left[0, \dfrac{\pi}{2}\right]$ and y is in the interval $\left[\dfrac{\pi}{2}, \pi\right]$ and $\tan x = \frac{4}{3}$ and $\csc y = \frac{13}{5}$, evaluate.
   (a) $\sin(x + y)$ (b) $\tan(x - y)$ (c) $\cos 2(x + y)$
2. Evaluate.
   (a) $\sin \dfrac{13\pi}{12}$ (b) $\cos\left(\dfrac{-11\pi}{12}\right)$ (c) $-\tan\left(-\dfrac{5\pi}{12}\right)$
   (d) $\sin 15°$ (e) $\cos(-75°)$ (f) $-\tan 105°$
3. If $\tan x = -\frac{3}{4}, \ \dfrac{\pi}{2} \leqslant x \leqslant \pi$, evaluate.
   (a) $\sin 2x$ (b) $\cos 2x$ (c) $\tan 2x$
4. If $\sin \dfrac{x}{2} = \dfrac{2}{3}, \ 0 \leqslant x \leqslant \dfrac{\pi}{2}$, evaluate.
   (a) $\cos x$ (b) $\tan x$ (c) $\sin \dfrac{x}{4}$

5. Find the value of

(a) $\sin 112\frac{1}{2}^\circ$ (b) $\cos \frac{\pi}{8}$ (c) $\tan \frac{3\pi}{16}$

6. Express each of the following as a function of its related acute angle and evaluate.

(a) $\sin 120^\circ$ (b) $\cos \frac{11\pi}{6}$ (c) $\tan\left(-\frac{7}{3}\pi\right)$

7. Express each of the following as a function of its co-related acute angle and evaluate.

(a) $\sin\left(-\frac{7\pi}{6}\right)$ (b) $\cos 495^\circ$ (c) $\tan \frac{39\pi}{4}$

8. Prove the following identities.

(a) $\tan x = \csc 2x - \cot 2x$ (b) $\dfrac{1 - \sin 2x}{\cos 2x} = \dfrac{1 - \tan x}{1 + \tan x}$

(c) $\cos x - \tan y \sin x = \sec y \cos(x + y)$

(d) $\sin(\pi + x) + \cos\left(\frac{\pi}{2} - x\right) + \tan\left(\frac{\pi}{2} + x\right) = -\cot x$

(e) $\dfrac{\sin 4x - \sin 2x}{\sin 2x} = \dfrac{\cos 3x}{\cos x}$

(f) $\cos x + \cos 2x + \cos 3x = \cos 2x\,(1 + 2\cos x)$

(g) $\sin(x + y) + \sin(x - y) = 2 \sin x \cos y$

9. Solve.

(a) $2 \sin x \cos x = 0,\ 0 \leqslant x \leqslant \pi$

(b) $\sin^2 x + \sin x = 0,\ -\pi \leqslant x \leqslant \pi$

(c) $\cos^2 x - \cos x = 0,\ 0 \leqslant x \leqslant 2\pi$

(d) $\sin^2 x - 2 \sin x + 1 = 0,\ -2\pi \leqslant x \leqslant 2\pi$

(e) $\cos^2 2x + 2 \cos 2x + 1 = 0,\ -\pi \leqslant x \leqslant \pi$

(f) $\sec^2 2x - 1 = 0,\ -2\pi \leqslant x \leqslant 2\pi$

(g) $\tan 4x - \tan 2x = 0,\ 0 < x < \pi$

(h) $\sqrt{3} \cos x + \sin x = 0,\ -2\pi \leqslant x \leqslant 0$

10. If $\tan x = 1$ and $\sin y = \frac{24}{25}$ with x and y in the interval $\left[0, \frac{\pi}{2}\right]$, evaluate.

(a) $\csc(x + y)$ (b) $\sec(x - y)$

11. Express $\cos 12a$ in terms of $\cos 3a$ and in terms of $\sin 3a$.

12. Use reflections to prove

(a) $\sin(b - \pi) = -\sin b,\ \pi < b < \frac{3\pi}{2}$.

(b) $\cos\left(b - \frac{3}{2}\pi\right) = -\sin b,\ \frac{\pi}{2} < b < \pi$.

13. Express $y = \sqrt{3}\sin x + \cos x$ as a sine function in the form $y = a\sin(x + b)$, $a > 0$. Use the result to graph the original function. (*Hint:* Use the Addition Formula for Sine and compare the coefficients of $\sin x$ and $\cos x$ in both equations.)

14. Express $y = \sqrt{3}\sin x - \cos x$ as a cosine function in the form $y = a\cos(x + b)$, $a > 0$.

15. If a, b, c, and d are the four smallest positive angles, in ascending order of magnitude, which have their sines equal to k, $k > 0$, prove that

$$4\sin a + 3\sin\frac{b}{2} + 2\cos\frac{c}{2} + \sin\frac{d}{2} = 4k.$$

16. If $\theta = 18°$, prove that $\sin 2\theta = \cos 3\theta$. Find the exact value of (a) $\sin 18°$ and (b) $\cos 18°$.

17. If $x = 3\sin\theta - \sin 3\theta$ and $y = \cos 3\theta + 3\cos\theta$, prove that $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4^{\frac{2}{3}}$.

18. If $\sin(x + y) = a\sin(x - y)$ and $\cos(x + y) = b\cos(x - y)$ where neither a nor b is ± 1, find an expression for $\cos 2x$ in terms of a and b.

6.7 CHAPTER 6 TEST

1. Find the exact value of each of the following.

(a) $\cos\left(-\frac{\pi}{12}\right)$ (b) $\sin \frac{3}{8}\pi$

(c) $\frac{\tan 67° - \tan 22°}{1 + \tan 67° \tan 22°}$ (d) $\left(\sin \frac{\pi}{8} + \cos \frac{\pi}{8}\right)^2$

(e) $\sin \frac{13}{36}\pi \cos \frac{5}{36}\pi + \cos \frac{13}{36}\pi \sin \frac{5}{36}\pi$

2. If $\sin x = \frac{12}{13}$, x in the interval $\left[0, \frac{\pi}{2}\right]$ and $\cos y = \frac{4}{5}$, y in the interval $\left[-\frac{\pi}{2}, 0\right]$, find the value of $\sin[2(x - y)]$.

3. Find the value of $\tan 2x$, $\frac{\pi}{2} < x < \pi$, given $\sec x = -\frac{5}{4}$.

4. If $\sqrt{2} \cos x - 1 = \frac{1 + \sqrt{3}}{2}$ and $\sqrt{2} \cos x + 1 = \frac{1 - \sqrt{3}}{2}$, find the value of $\cos 4x$.

5. Prove.

(a) $\frac{1 + \sin x + \cos x}{1 + \sin x - \cos x} = \cot \frac{x}{2}$

(b) $-\sin^2 x - \sin^2 y + 1 = \cos(x + y)\cos(x - y)$

(c) $\frac{\sin(x - \pi)}{\cos(\pi + x)} - \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin(-\pi - x)} = \frac{\sin x - \cos x}{\cos x}$

6. Solve for x in the given interval.

(a) $\tan^2\left(2x - \frac{\pi}{12}\right) = 3$, $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) $\cos 2x - \cos^2 x - 2 \sin x + 3 = 0$, $[0, 2\pi]$

7. Develop a formula for $\tan(a + b - c)$ in terms of $\tan a$, $\tan b$, and $\tan c$.

FOUNDERS OF CALCULUS

In the seventeenth and eighteenth centuries the development of calculus and its applications was due in a large part to members of a single family, the Bernoullis. They, along with Euler, made Basel, Switzerland, famous as the birthplace of great mathematicians. In the course of a century, no less than eight members of this remarkable family distinguished themselves in mathematics.

- Nicolaus Bernoulli, the Father (1623–1708)
  - **Jacob I (1654–1705)**
  - Nicolaus I (1662–1716)
    - Nicolaus II (1687–1759)
  - **Johann I (1667–1748)**
    - Nicolaus III (1695–1726)
    - **Daniel (1700–1782)**
    - Johann II (1710–1790)
      - Johann III (1746–1807)
      - Jacob II (1759–1789)

The most famous were the two brothers Jacob and Johann, staunch supporters and co-workers of Leibniz, and Johann's son Daniel, a close friend of Euler.

Johann I was more prolific than his brother Jacob, who had taught him mathematics. The two were often embroiled in controversy, especially in light of the fact that Johann had attempted to steal some of Jacob's ideas. Johann's competitive nature is evidenced by the fact that he threw his own son, Daniel, out of the house for having won a prize from the French Academy of Sciences that Johann himself coveted. He championed the cause of Leibniz over Newton, whom he detested, in their priority dispute over the invention of calculus.

It was a practice at that time for mathematicians to pose problems for each other. In 1696, Bernoulli posed the problem of the *brachistochrone*, the problem of finding the *curve of quickest descent* for "the shrewdest mathematicians of all the world," and fixed a six month limit for its solution. Leibniz solved the problem the day he received it and correctly predicted that only five solutions would be forthcoming, the solvers being Jacob and Johann Bernoulli, l'Hospital, Newton and himself.

Correspondence between scholars was the norm in these times, but Bernoulli was prodigious in his letter writing. He wrote more than 2500 letters to an estimated 110 scholars. He and Jacob picked up calculus where Newton and Leibniz left off. His contributions to integral calculus and the solution of differential equations were particularly significant. Johann developed *exponential calculus* when it was pointed out that Leibniz' published differential methods did not apply to the exponential function $y = a^x$.

ANSWERS

CHAPTER 6 TRIGONOMETRIC FUNCTIONS

REVIEW AND PREVIEW TO CHAPTER 6

EXERCISE 1

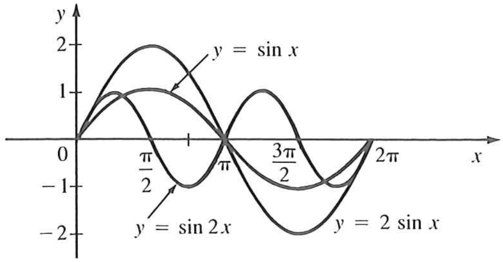
1. (a) 30° (b) −270° (c) 225° (d) 540° (e) 229° (f) −43° (g) −688°
2. (a) 0.79 (b) 5.50 (c) 3.67 (d) 9.95 (e) 0.03 (f) −0.49 (g) 10.5
3. (a) 25 (b) 1.2
4. 25.46

EXERCISE 2

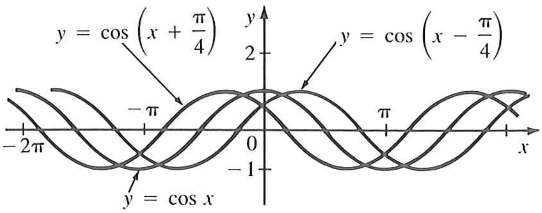
1. $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$, $\tan\theta = \frac{4}{3}$, $\csc\theta = \frac{5}{4}$, $\sec\theta = \frac{5}{3}$, $\cot\theta = \frac{3}{4}$
2. $\sin\theta = \dfrac{-1}{\sqrt{5}}$, $\cos\theta = \dfrac{-2}{\sqrt{5}}$, $\tan\theta = \frac{1}{2}$
3. $\csc\theta = -\frac{13}{12}$, $\sec\theta = \frac{13}{5}$, $\cot\theta = -\frac{5}{12}$
4. $\cos\theta = \dfrac{2\sqrt{2}}{3}$, $\tan\theta = \dfrac{1}{2\sqrt{2}}$
5. $\csc\theta = \dfrac{2}{\sqrt{3}}$, $\cot\theta = -\dfrac{1}{\sqrt{3}}$
6. $\cos\theta = \dfrac{3}{\sqrt{34}}$, $\csc\theta = -\dfrac{\sqrt{34}}{5}$
7. $\sec\theta = -\frac{13}{5}$, $\sin\theta = -\frac{12}{13}$

EXERCISE 3

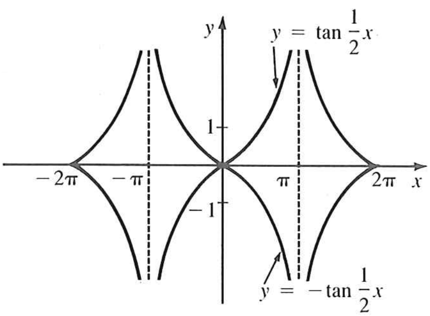
1.



2.



3.



EXERCISE 4

1. (a) 0 (b) $\dfrac{2 + 2\sqrt{3}}{\sqrt{3}}$ (c) $\frac{3}{4}$ d) 4 (e) $3 - \sqrt{2}$ (f) $\sqrt{2} - \frac{2}{3}$

EXERCISE 6.1

1. (a) $\frac{1}{2}$ (b) $\dfrac{1}{\sqrt{2}}$ (c) $\sqrt{3}$ (d) 1 (e) $-\dfrac{\sqrt{3}}{2}$ (f) $-\dfrac{1}{\sqrt{2}}$ (g) $-\dfrac{1}{\sqrt{3}}$ (h) $\dfrac{1}{\sqrt{2}}$

2. (a) $\dfrac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $\sqrt{3}$ (e) $-\dfrac{1}{\sqrt{3}}$ (f) $\dfrac{1}{\sqrt{2}}$

3. (a) 0 (b) $2 \tan x$ (c) 0 (d) $\sin x$ (e) 0

4. (a) $\csc x$, $-\sec x$, $-\cot x$ (b) $\sec x$, $-\csc x$, $\dfrac{1}{-\cot x}$ (c) $-\csc x$, $-\sec x$, $\cot x$ (d) $-\sec x$, $\csc x$, $-\dfrac{1}{\cot x}$

5. (a) $-\sin x$ (b) $\sin x$ (c) $-\tan x$

6. (a) -2 (b) $-\dfrac{2}{\sqrt{3}}$ (c) $-\sqrt{3}$ (d) $-\sqrt{2}$ (e) $-\sqrt{2}$ (f) 1

7. (a) $-\sin x - \cos^2 x$ (b) $\csc^2 x$

EXERCISE 6.2

1. (a) $\cos 3a$ (b) $\cos 3x$ (c) $\sin 3$ (d) $\sin 3m$ (e) $\tan 5a$ (f) $-\tan 2$ (g) $\cos 2x$ (h) $\sin 2a$ (i) $\tan 2x$ (j) 1

2. (a) $\dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ (b) $\dfrac{-1 - \sqrt{3}}{2\sqrt{2}}$ (c) $-\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$ (d) $\dfrac{\sqrt{3} + 1}{1 - \sqrt{3}}$ (e) $\dfrac{\sqrt{3} + 1}{2\sqrt{2}}$ (f) $\dfrac{\sqrt{3} + 1}{2\sqrt{2}}$

3. (a) $\dfrac{1 - \sqrt{3}}{2\sqrt{2}}$ (b) $\dfrac{\sqrt{3} - 1}{2\sqrt{2}}$ (c) $\dfrac{1 - \sqrt{3}}{1 + \sqrt{3}}$

4. (a) $\frac{16}{65}$ (b) $\frac{33}{65}$ (c) $\frac{56}{33}$

5. (a) $-\frac{16}{65}$ (b) $-\frac{33}{65}$ (c) $\frac{56}{33}$

6. (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\dfrac{\sqrt{3} - 1}{\sqrt{3} + 1}$ (d) $\dfrac{1 + \sqrt{3}}{2\sqrt{2}}$

12. (a) $2 \sin 40° \cos 20°$ (b) $2 \sin 20°$ (c) $\cos 20°$ (d) $2 \cos 4x \sin 2x$ (e) $\dfrac{2}{\sqrt{2}} \cos 85°$ (f) $-2 \sin 3x \sin x$

13. (a) $\sqrt{3}$ (b) $\dfrac{-4}{1 - \tan^2 x}$ (c) $\cot \dfrac{x}{2}$

14. $\dfrac{15}{\sqrt{21} - 4\sqrt{2}}$

15. $\dfrac{\csc x \csc y \sec x \sec y}{\sec x \csc y + \csc x \sec y}$

16. $\sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$

17. $\dfrac{-6\sqrt{10} + 8\sqrt{3} - 2\sqrt{30} - 1}{60}$

20. $2 \pm \sqrt{3}$

21. (a) 0 (b) $4 \sin x$ (c) 0

EXERCISE 6.3

1. (a) $\cos^2 2x - \sin^2 2x$ or $1 - 2 \sin^2 2x$ or $2 \cos^2 2x - 1$ (b) $2 \sin \frac{3}{2}x \cos \frac{3}{2}x$ (c) $\dfrac{2 \tan 3x}{1 - \tan^2 3x}$ (d) $2 \sin \frac{1}{4}x \cos \frac{1}{4}x$

(e) $\cos^2 \frac{1}{3}x - \sin^2 \frac{1}{3}x$ (f) $\dfrac{-2\tan\frac{7}{2}x}{1-\tan^2\frac{7}{2}x}$

2. (a) $\sin 6\theta$ (b) $3\sin 2\theta$ (c) $\frac{1}{4}\sin\theta$ (d) $\cos 3\theta$

(e) $\cos\dfrac{\theta}{2}$ (f) $\cos 7\theta$ (g) $-4\cos 4\theta$ (h) $\sin x$

3. $\sin 2\theta = -\frac{24}{25}$, $\cos 2\theta = \frac{7}{25}$, 4th quadrant

4. $\sin 2\theta = \frac{120}{169}$, $\cos 2\theta = -\frac{119}{169}$, 2nd quadrant

5. $\dfrac{8\sqrt{5}}{81}$

6. $\csc 2\theta = \dfrac{25}{-4\sqrt{21}}$, $\sec 2\theta = \dfrac{25}{-17}$

7. $\frac{4}{3}$ **8.** $\frac{24}{7}$

9. (a) $3\sin\theta - 4\sin^3\theta$ (b) $4\cos^3\theta - 3\cos\theta$

(c) $\dfrac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$

(d) $8\cos^4\theta - 8\cos^2\theta + 1$

10. (a) $\sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$ (b) $-\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$

(c) $-1+\sqrt{2}$ (d) $-\sqrt{\dfrac{\sqrt{2}-1}{2\sqrt{2}}}$

(e) $\sqrt{\dfrac{\dfrac{1+\sqrt{2}}{2\sqrt{2}}+1}{2}}$ (f) $\dfrac{-1+\sqrt{4+2\sqrt{2}}}{1+\sqrt{2}}$

11. (a) $\pm\sqrt{\dfrac{1-\cos\theta}{2}}$ (b) $\pm\sqrt{\dfrac{1+\cos\theta}{2}}$

(c) $\pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

12. (a) $\dfrac{5}{\sqrt{26}}$ (b) $-\dfrac{1}{\sqrt{26}}$ (c) -5

13. (a) $\dfrac{2\tan\theta}{1+\tan^2\theta}$ (b) $\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$ **14.** $\tan x$

15. $\tan x$ **16.** $\cos 2x$ **17.** $-\frac{5}{9}$ **18.** $\dfrac{\sqrt{3}}{2}$

20. 1 **21.** $\dfrac{5\pm\sqrt{5}}{8}$ **22.** $\dfrac{1}{\sqrt{2}}\sin\theta$

EXERCISE 6.5

1. (a) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}$ (b) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$ (c) $\frac{3}{4}\pi, \frac{7}{4}\pi$ (d) $\dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

(e) $\dfrac{7\pi}{6}, \dfrac{11\pi}{6}$ (f) $\dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$

2. (a) $-\dfrac{3\pi}{4}$ (b) $-\dfrac{3\pi}{4}, -\pi, 0$ (c) $-\dfrac{\pi}{2}$

(d) $-\dfrac{2\pi}{3}, -\dfrac{\pi}{3}$ (e) $-\dfrac{5\pi}{6}, -\dfrac{\pi}{6}$ (f) $-\dfrac{\pi}{2}$

3. (a) $0, \dfrac{\pi}{4}, \pi$ (b) $-\pi, -\dfrac{2\pi}{3}, -\dfrac{\pi}{3}, 0$

(c) $\dfrac{\pi}{3}, \dfrac{5\pi}{3}$ (d) $-\pi, -\dfrac{3\pi}{4}, 0, \dfrac{3\pi}{4}, \pi$

(e) $-2\pi, -\pi, 0, \pi, 2\pi$ (f) $-\dfrac{11\pi}{6}, -\dfrac{7\pi}{6}$

4. (a) $-\pi, 0, \pi$ (b) $-\dfrac{\pi}{2}, \dfrac{\pi}{6}, \dfrac{\pi}{2}$ (c) $\dfrac{\pi}{8}$ (d) $\dfrac{\pi}{24}, \dfrac{5\pi}{24}$ (e) $-\dfrac{\pi}{8}, 0, \dfrac{3\pi}{8}$ (f) $-\dfrac{3\pi}{4}, -\dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{3\pi}{4}$

5. $-2\pi, -\dfrac{11\pi}{6}, -\dfrac{7\pi}{6}, -\pi, 0, \dfrac{\pi}{6}, \dfrac{5\pi}{6}, \pi, 2\pi$

6. $\sin x = \pm\frac{4}{5}$, $\cos x = \pm\frac{3}{5}$ or $\sin x = \pm\frac{3}{5}$, $\cos x = \pm\frac{4}{5}$

7. (a) 0 (b) 0.7391 (c) 0 (d) ± 1.17 (e) 0, 1 (f) ± 1.165 **8.** $p^2 + q^2$

6.6 REVIEW EXERCISE

1. (a) $-\frac{33}{65}$ (b) $\frac{63}{16}$ (c) $\frac{2047}{4225}$

2. (a) $\dfrac{-\sqrt{3}+1}{2\sqrt{2}}$ (b) $\dfrac{-\sqrt{3}-1}{2\sqrt{2}}$ (c) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

(d) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (e) $\dfrac{\sqrt{3}-1}{2\sqrt{2}}$ (f) $\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$

3. (a) $-\frac{24}{25}$ (b) $\frac{7}{25}$ (c) $-\frac{24}{7}$

4. (a) $\frac{1}{9}$ (b) $4\sqrt{5}$ (c) $\sqrt{\dfrac{3-\sqrt{5}}{6}}$

5. (a) $\sqrt{\dfrac{\sqrt{2}+1}{2\sqrt{2}}}$ (b) $\sqrt{\dfrac{1+\sqrt{2}}{2\sqrt{2}}}$

(c) $\dfrac{-1+\sqrt{4+2\sqrt{2}}}{1+\sqrt{2}}$

6. (a) $\dfrac{\sqrt{3}}{2}$ (b) $\dfrac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$

7. (a) $\frac{1}{2}$ (b) $-\dfrac{1}{\sqrt{2}}$ (c) -1

9. (a) $0, \dfrac{\pi}{2}, \pi$ (b) $-\pi, -\dfrac{\pi}{2}, 0, \pi$

(c) $0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, 2\pi$ (d) $-\dfrac{3\pi}{2}, \dfrac{\pi}{2}$ (e) $\pm\dfrac{\pi}{2}$

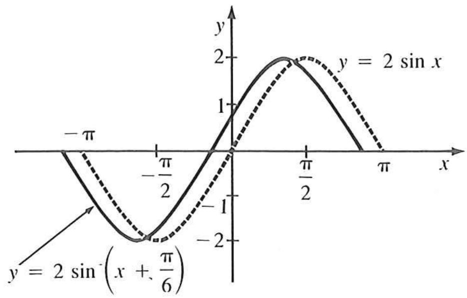
(f) $0, \pm\dfrac{\pi}{2}, \pm\pi, \pm\dfrac{3\pi}{2}, \pm 2\pi$

(g) $0, \dfrac{\pi}{2}, \pi$ (h) $-\dfrac{4\pi}{3}, -\dfrac{\pi}{3}$

10. (a) $\dfrac{25\sqrt{2}}{31}$ (b) $\dfrac{25\sqrt{2}}{31}$

11. $1 - 8\sin^2 3a + 8\sin^4 3a$ and $8\cos^4 3a - 8\cos^2 3a + 1$

13. $2\sin\left(x + \frac{\pi}{6}\right)$



14. $2\cos\left(x + \frac{4\pi}{3}\right)$

16. (a) $\frac{-1 + \sqrt{5}}{4}$ (b) $\sqrt{\frac{5 + \sqrt{5}}{8}}$

18. $\frac{ab - 1}{a - b}$

6.7 CHAPTER 6 TEST

1. (a) $\frac{\sqrt{3} + 1}{2\sqrt{2}}$ (b) $\sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}}$ (c) 1 (d) $\frac{\sqrt{2} + 1}{\sqrt{2}}$ (e) 1 **2.** $-\frac{2016}{4225}$ **3.** $-\frac{24}{7}$

4. $-\frac{1}{2}$ **6.** (a) $-\frac{7\pi}{24}, -\frac{\pi}{8}, \frac{5\pi}{24}, \frac{3\pi}{8}$ (b) $\frac{\pi}{2}$

7. $\frac{\tan a + \tan b - \tan c + \tan a \tan b \tan c}{1 - \tan a \tan b + \tan a \tan c + \tan b \tan c}$