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CHAPTER 3

APPLICATIONS OF DERIVATIVES

REVIEW AND PREVIEW TO CHAPTER 3

Problem Solving

There are no hard and fast rules that will ensure success in solving problems. However, it is possible to outline some general steps in the problem-solving process and to give some principles that may be useful in the solution of certain problems. These steps and principles are just common sense made explicit. They have been adapted from George Polya's book *How to Solve It*.

1. **UNDERSTAND THE PROBLEM.** The first step is to read the problem and make sure that it is clearly understood. Ask yourself the following questions:

What is the unknown?
What are the given quantities?
What are the given conditions?

For many problems it is useful to

draw a diagram

and identify the given and required quantities on the diagram.

Usually it is necessary to

introduce suitable notation.

In choosing symbols for the unknown quantities we often use letters such as a, b, c, ..., m, n, ..., x, y, but in some cases it helps to use initials as suggestive symbols, for instance, V for volume, t for time.

2. **THINK OF A PLAN.** Find a connection between the given information and the unknown that will enable you to calculate the unknown. If you do not see a connection immediately, the following ideas may be helpful in devising a plan.

(a) ***Try to recognize something familiar.*** Relate the given situation to previous knowledge. Look at the unknown and try to recall a more familiar problem having a similar unknown.

(b) ***Try to recognize patterns.*** Some problems are solved by recognizing that some kind of pattern is occurring. The pattern could be geometric, or numeric, or algebraic. If you can see regularity or repetition in a problem, then you might be able to guess what the continuing pattern is and then prove it.

(c) ***Use analogy.*** Try to think of an analogous problem, that is, a similar problem, a related problem, but one that is easier than the original problem. If you can solve the similar, simpler problem, then it might give you the clues you need to solve the original, more difficult problem. For instance, if a problem involves very large numbers, you could first try a similar problem with smaller numbers. Or if the problem is in three-dimensional geometry, you could look for a similar problem in two-dimensional geometry. Or if the problem you start with is a general one, you could first try a special case.

(d) ***Introduce something extra.*** It may sometimes be necessary to introduce something new, an auxiliary aid, to help make the connection between the given and the unknown. For instance, in geometry the auxiliary aid could be a new line drawn in a diagram. In algebra it could be a new unknown that is related to the original unknown.

(e) ***Take cases.*** You may sometimes have to split a problem into several cases and give a different argument for each of the cases. We used this strategy in dealing with absolute value and other functions in Section 1.3 and in connection with geometric series in Example 2 in Section 1.7.

(f) ***Work backwards.*** Sometimes it is useful to imagine that your problem is solved and work backwards, step by step, till you arrive at the given data. Then you may be able to reverse your steps and thereby construct a solution to the original problem.

(g) ***Use indirect reasoning.*** Sometimes it is appropriate to attack a problem indirectly. For instance, in a counting argument it might be best to count the total number of objects and subtract the number of objects that do *not* have the required property. Another example of indirect reasoning is *proof by contradiction* in which we assume that the desired conclusion is false and eventually arrive at a contradiction.

3. **CARRY OUT THE PLAN.** In Step 2 a plan was devised. In carrying out that plan we have to check each stage of the plan and write the details that prove that each stage is correct.

4. **LOOK BACK.** Having completed our solution, it is wise to look back over it, partly to see if there are errors in the solution and partly to see if there is an easier way to solve the problem. Another reason for looking back is that it will familiarize us with the method of solution and this may be useful for solving a future problem. Descartes said, "Every problem that I solved became a rule which I then used to solve other problems."

Example Express the hypotenuse h of a right triangle in terms of its area A and its perimeter P.

Solution Let us first sort out the information by identifying the unknown quantity and the data.

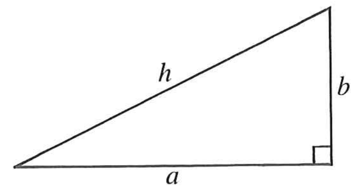
UNDERSTAND THE PROBLEM

Unknown: h

Given quantities: A, P

It helps to draw a diagram and we do so.

DRAW A DIAGRAM



CONNECT THE GIVEN WITH THE UNKNOWN

In order to connect the given quantities to the unknown, we introduce two extra variables a and b, which are the lengths of the other two sides of the triangle. This enables us to express the given condition, which is that the triangle is right-angled, by the Pythagorean Theorem:

$$h^2 = a^2 + b^2$$

The other connections among the variables come by writing expressions for the area and perimeter:

$$A = \tfrac{1}{2}ab \qquad P = a + b + h$$

Since A and P are given, notice that we now have three equations in the three unknowns a, b, and h:

$$h^2 = a^2 + b^2 \qquad ①$$

$$A = \tfrac{1}{2}ab \qquad ②$$

$$P = a + b + h \qquad ③$$

RELATE TO THE FAMILIAR

Although we have the correct number of equations, they are not easy to solve in a straightforward fashion. But if we use the problem-solving strategy of trying to recognize something familiar, then we can solve these equations by an easier method. Look at the right sides of Equations 1, 2, and 3. Do these expressions remind you of anything familiar? Notice that they contain the ingredients of a familiar formula:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Using this idea, we express $(a + b)^2$ in two ways. From Equations 1 and 2 we have

$$(a + b)^2 = (a^2 + b^2) + 2ab = h^2 + 4A$$

From Equation 3 we have

$$(a + b)^2 = (P - h)^2 = P^2 - 2Ph + h^2$$

Thus,
$$h^2 + 4A = P^2 - 2Ph + h^2$$
$$2Ph = P^2 - 4A$$
$$h = \frac{P^2 - 4A}{2P}$$

This is the required expression.

EXERCISE 1

1. Solve the equation $||3x + 1| - x| = 2$.
2. Use your calculator to evaluate
$$\frac{\sqrt{2} + \sqrt{6}}{\sqrt{2 + \sqrt{3}}}$$
The answer looks very simple. Show that the calculated value is correct.
3. A man drives from home to work at a speed of 80 km/h. The return trip from work to home is covered at the more leisurely pace of 50 km/h. What is the average speed for the round trip?
4. In a right triangle, the hypotenuse has length 5 cm and another side has length 3 cm. What is the length of the altitude that is perpendicular to the hypotenuse?
5. A car with tires having radius 33 cm was driven on a trip and the odometer indicated that the distance travelled was 640 km. Two weeks later, with snow tires installed, the odometer indicated that the distance for the return trip over the same route was 625 km. Find the radius of the snow tires.
6. Bob and Jim, next-door neighbours, use hoses from both houses to fill Bob's swimming pool. They know it takes eighteen hours using both hoses. They also know that Bob's hose, used alone, can fill the pool in six hours less than Jim's hose. How much time is required by each hose alone?

INTRODUCTION

Now that we know how to calculate derivatives, we use them in this chapter to compute velocity, acceleration, and other rates of change in the natural and social sciences. Another application of derivatives occurs when we use them in Newton's method for finding approximate solutions of equations.

3.1 VELOCITY

We have already defined and computed velocities in Sections 1.5 and 2.1, but now we can compute them more easily with the aid of the differentiation formulas that were developed in Chapter 2.

Suppose that an object moves along a straight line. (Think of a ball being thrown vertically upward or a car being driven along a road or a stone being dropped from a cliff.) The position function is $s = f(t)$, where s is the displacement (directed distance) of the object from the origin at time t. Recall that the **(instantaneous) velocity** of the object at time t is defined as the limit of average velocities over shorter and shorter time intervals:

$$v = f'(t) = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Thus, the velocity is the derivative of the position function and in Leibniz notation we write

$$v = \frac{ds}{dt}$$

Example 1 If a stone is dropped from a cliff that is 122.5 m high, then its height in metres after t seconds is $h = 122.5 - 4.9t^2$ (until it hits the ground).

(a) Find its velocity after 1 s and 2 s.

(b) When will the stone hit the ground?

(c) With what velocity will it hit the ground?

Solution (a) The position function is $h = 122.5 - 4.9t^2$, so the velocity at time t is

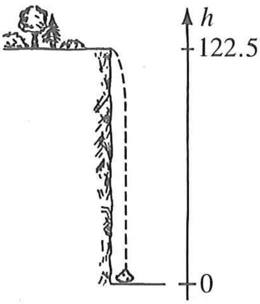
$$\frac{dh}{dt} = -9.8t$$

Thus, the velocity after 1 s is

$$\left.\frac{dh}{dt}\right]_{t=1} = -9.8(1) = -9.8 \text{ m/s}$$

and after 2 s it is

$$\left.\frac{dh}{dt}\right]_{t=2} = -9.8(2) = -19.6 \text{ m/s}$$



(b) The stone will hit the ground when the height is 0, that is,

$$h(t) = 122.5 - 4.9t^2 = 0$$

$$t^2 = \frac{122.5}{4.9} = 25$$

Since $t > 0$, we have $t = 5$. So the stone hits the ground after 5 s.

(c) The stone hits the ground with velocity

$$h'(5) = -9.8(5) = -49 \text{ m/s}$$

Example 2 The position of a particle moving on a line is given by the equation

$$s = f(t) = 2t^3 - 21t^2 + 60t, \ t \geqslant 0$$

where t is measured in seconds and s in metres.

(a) What is the velocity after 3 s and after 6 s?
(b) When is the particle at rest?
(c) When is the particle moving in the positive direction?
(d) Find the total distance travelled by the particle during the first 6 s.

Solution (a) The velocity after t seconds is

$$v = f'(t) = 6t^2 - 42t + 60$$

so the velocity after 3 s is

$$f'(3) = 6(3)^2 - 42(3) + 60 = -12 \text{ m/s}$$

and after 6 s it is

$$f'(6) = 6(6)^2 - 42(6) + 60 = 24 \text{ m/s}$$

(b) The particle is at rest when $v(t) = 0$, that is,

$$\begin{aligned} 6t^2 - 42t + 60 &= 0 \\ t^2 - 7t + 10 &= 0 \\ (t - 2)(t - 5) &= 0 \\ t = 2 \quad \text{or} \quad t &= 5 \end{aligned}$$

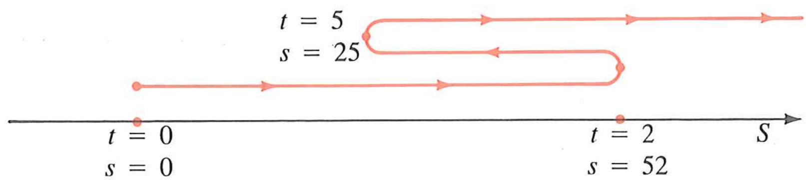
Thus, the particle is at rest when $t = 2$ s and when $t = 5$ s.

(c) The particle moves in the positive direction when $v(t) > 0$, that is,

$$t^2 - 7t + 10 = (t - 2)(t - 5) > 0$$

This inequality is true when both factors are positive ($t > 5$) or when both factors are negative ($t < 2$). Thus the particle moves in the positive direction in the time intervals $0 \leqslant t < 2$ and $t > 5$. It moves in the negative direction when $2 < t < 5$.

The motion of the particle is illustrated schematically in the following figure.



(d) The distance travelled in the first 2 s is

$$|f(2) - f(0)| = |52 - 0| = 52 \text{ m}$$

From $t = 2$ to $t = 5$ the distance travelled is

$$|f(5) - f(2)| = |25 - 52| = 27 \text{ m}$$

From $t = 5$ to $t = 6$ the distance travelled is

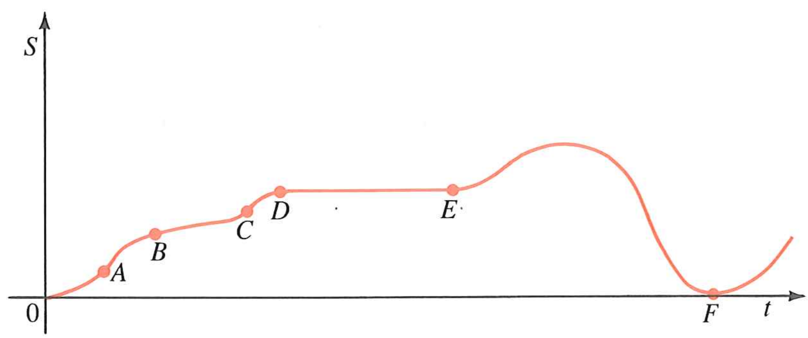
$$|f(6) - f(5)| = |36 - 25| = 11 \text{ m}$$

The total distance is

$$52 + 27 + 11 = 90 \text{ m}$$

EXERCISE 3.1

A **1.** The graph shows the position function of a car.



(a) What was the initial velocity of the car?
(b) Was the car going faster at B or at C?
(c) Was the car slowing down or speeding up at A, B, and C?
(d) What happened between D and E?
(e) What happened at F?

B **2.** The position functions give s (in metres) as a function of t (in seconds). Find the velocity as a function of time and the velocities after 2 s and 4 s.

(a) $s = 5 + 12t$ (b) $s = 8t^2 - 24t + 5$

(c) $s = t^3 - 6t^2$ (d) $s = \dfrac{5t}{1 + t}$

3. If a stone is thrown downward with a speed of 15 m/s from a cliff that is 80 m high, its height in metres after t seconds is $h = 80 - 15t - 4.9t^2$. Find the velocity after 1 s and after 2 s.

4. If a ball is thrown directly upward with an initial velocity of 24.5 m/s, then its height after t seconds, in metres, is

$$h = 24.5t - 4.9t^2$$

(a) Find the velocity after 1 s, 2 s, 3 s, and 4 s.
(b) When does the ball reach its maximum height?
(c) What is its maximum height?
(d) When does it hit the ground?
(e) With what velocity does it hit the ground?

5. The distance travelled by a car is given by $s = 160t^2 + 20t$, where t is measured in hours and s in kilometres. When did the velocity reach 100 km/h?

6. The position function of a particle is $s = t^3 - 3t^2 - 5t$, $t \geqslant 0$, where t is measured in seconds and s in metres. When does the particle reach a velocity of 4 m/s?

7. The position of a particle is given by

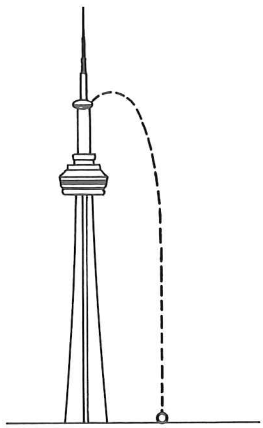
$$s = t^2 - 4t + 4, \; t \geqslant 0$$

where s is measured in metres and t in seconds.
(a) Find the velocity after 1 s and 3 s.
(b) When is the particle at rest?
(c) When is the particle moving in the positive direction?
(d) Draw a diagram to illustrate the motion of the particle.

8. The motion of a particle is described by the position function

$$s = t^3 - 15t^2 + 63t, \; t \geqslant 0$$

where t is measured in metres and s in seconds.
(a) When is the particle at rest?
(b) When is the particle moving in the positive direction?
(c) Draw a diagram to illustrate the motion of the particle.
(d) Find the total distance travelled in the first 10 s.



9. If a ball is thrown upward with a velocity of 10 m/s from the upper observation deck of the CN Tower, 450 m above the ground, then the distance, in metres, of the ball above ground level after t seconds is

$$s = 450 + 10t - 5t^2$$

(a) When does the ball reach its maximum height?
(b) Use the quadratic formula to find how long it takes for the ball to reach the ground.
(c) Find the approximate velocity with which the ball strikes the ground.

3.2 ACCELERATION

If an object moves along a straight line, its **acceleration** is the rate of change of velocity with respect to time. Therefore, the acceleration $a(t)$ at time t is the derivative of the velocity function:

$$a(t) = v'(t) = \frac{dv}{dt}$$

Since the velocity is the derivative of the position function $s = f(t)$, it follows that the acceleration is the second derivative of the position function:

$$a(t) = v'(t) = f''(t)$$

or, in Leibniz notation,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

If s is measured in metres and t in seconds, then the units for acceleration are metres/second^2, or m/s^2.

Example 1 The position function of a particle is given by $s = t^3 + 2t^2 + 2t$, where s is measured in metres and t in seconds.
(a) Find the velocity and acceleration as a function of time.
(b) Find the acceleration at 3 s.

Solution (a) The velocity is

$$v = \frac{ds}{dt} = 3t^2 + 4t + 2$$

and the acceleration is

$$a = \frac{dv}{dt} = 6t + 4$$

(b) After 3 s the acceleration is

$$a = 6(3) + 4 = 22 \text{ m/s}^2$$

Example 2 If a ball is thrown directly upward with an initial velocity of 24.5 m/s, then its distance above the ground in metres after t seconds is

$$s = 24.5t - 4.9t^2$$

(until it hits the ground). Find the acceleration of the ball.

Solution

$$s = 24.5t - 4.9t^2$$

$$\frac{ds}{dt} = 24.5 - 9.8t$$

$$a = \frac{d^2s}{dt^2} = -9.8$$

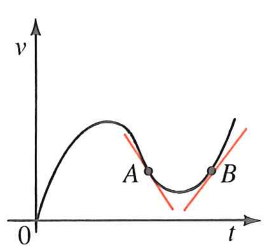
The acceleration is -9.8 m/s^2.

Notice that the acceleration in Example 2 is constant, and is called the *acceleration due to gravity*. The fact that it is negative means that the ball slows down as it rises and speeds up as it falls.

In general, a negative acceleration

$$a = \frac{dv}{dt} < 0$$

indicates that the velocity is decreasing (as at point A in the figure). This follows from the fact that the acceleration is the slope of the tangent to the graph of the velocity function. Likewise, a positive acceleration

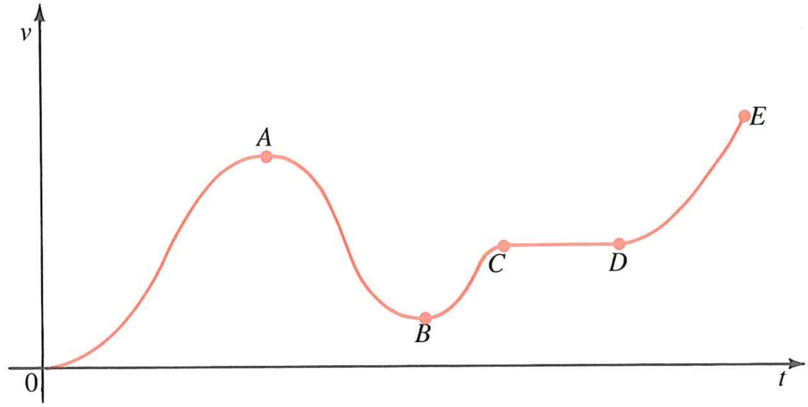


$$a = \frac{dv}{dt} > 0$$

means that the velocity is increasing (as at B).

EXERCISE 3.2

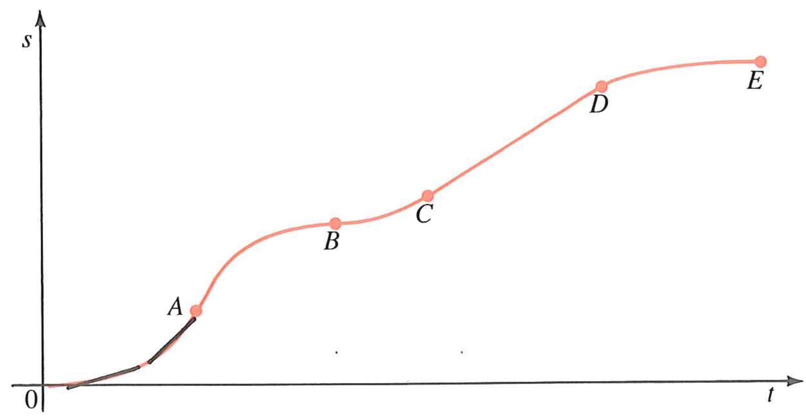
A **1.**



The graph of a velocity function is shown. State whether the acceleration is positive, zero, or negative

(a) from O to A,
(b) from A to B,
(c) from B to C,
(d) from C to D,
(e) from D to E.

2.



The graph of a position function is shown.

(a) For the part of the graph from O to A, use slopes of tangents to decide whether the velocity is increasing or decreasing. Is the acceleration positive or negative?

(b) State whether the acceleration is positive, zero, or negative

(i) from A to B
(ii) from B to C
(iii) from C to D
(iv) from D to E

B **3.** The position functions give the displacement s as a function of the time t. Find the velocity and acceleration as functions of t.
(a) $s = 12 + 30t$ (b) $s = 16t^2 + 5t - 10$
(c) $s = t^3 + 5t^2 + t + 1$ (d) $s = \sqrt{t^2 + t}$

4. The position functions give s (in metres) as a function of t (in seconds). Find the acceleration at 4 s.
(a) $s = 100 - 15t - 4.9t^2$ (b) $s = t^3 - t^2$
(c) $s = t^3 - 2t^2 + 3t - 5$ (d) $s = \dfrac{5t}{1 + t}$

5. A position function is given by $s = s_0 + v_0t + \frac{1}{2}gt^2$, where s_0, v_0, and g are constants. Find
(a) the initial position
(b) the initial velocity
(c) the acceleration

6. The position function of a particle is $s = t^3 - 12t$, $t \geqslant 0$, where s is measured in metres and t is measured in seconds. Find the acceleration at the instant when the velocity is 0.

7. A particle moves according to the equation of motion $s = t^3 - 9t^2 + 18t$, where s is measured in metres and t is measured in seconds.
(a) When is the acceleration 0?
(b) Find the displacement and velocity at that time.

8. The position function of a particle is $s = t^4 - 12t^3 + 30t^2 + 5t$, $t \geqslant 0$. When is the acceleration positive and when is it negative?

9. A car is travelling at 72 km/h and the brakes are fully applied, producing a constant deceleration of 12 m/s^2.
(a) Verify that the velocity function $v(t) = -12t + 20$, where t is measured in seconds, gives this deceleration and initial velocity.
(b) How long does it take for the car to come to a complete stop?

3.3 RATES OF CHANGE IN THE NATURAL SCIENCES

Recall from Section 2.1 that a derivative can be interpreted as a rate of change. In this section, we use derivatives to find rates of change in physics, biology, and chemistry.

First we recall from Section 1.5 the basic ideas behind rates of change. If y is a quantity that depends on another quantity x, we can write y as a function of x: $y = f(x)$. If x changes from x_1 to x_2, then the change in x is

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)$$

The **(instantaneous) rate of change** of y with respect to x at x_1 is the limit of the average rate of change as x_2 approaches x_1:

$$\text{rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_1)$$

Example 1 A spherical balloon is being inflated. Find the rate of change of the volume with respect to the radius when the radius is 10 cm.

Solution We solved this problem as Example 4 in Section 1.5, but now we can use our differentiation formulas.

If the radius of the balloon, in centimetres, is r, then the volume V, in cubic centimetres, is given by

$$V(r) = \tfrac{4}{3}\pi r^3$$

Therefore $\quad V'(r) = \tfrac{4}{3}\pi(3r^2) = 4\pi r^2$

and so the rate of change of V with respect to r when $r = 10$ cm is

$$V'(10) = 4\pi(10)^2 = 400\pi \text{ cm}^3/\text{cm}$$

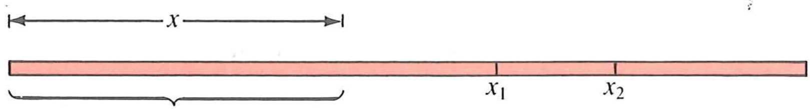
Applications to Physics

We have already considered velocity (the rate of change of displacement with respect to time) in Section 3.1 and acceleration (the rate of change of velocity with respect to time) in Section 3.2. Other occurrences in physics include current (the rate of flow of charge), power (the rate at which work is done), temperature gradient (the rate of change of temperature with respect to position), and rate of heat flow. In what follows, we discuss in detail the linear density of a wire.

If a rod or piece of wire is homogeneous, then its *linear density* is uniform and is defined as mass per unit length:

$$\rho = \frac{m}{L}$$

If the mass m is measured in kilograms and the length L in metres, then the linear density ρ is measured in kilograms per metre. If the rod is not homogeneous, let $m = f(x)$ be its mass measured from its left end to a point x as shown in the figure.



The mass of the part of the rod that lies between $x = x_1$ and $x = x_2$ is $\Delta m = f(x_2) - f(x_1)$, so the average density of that part of the rod is

$$\text{average density} = \frac{\Delta m}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

If we now let $\Delta x \to 0$ (that is, $x_2 \to x_1$), we are computing the average density over a smaller and smaller interval. The **linear density** ρ at x_1 is the limit of these average densities as $\Delta x \to 0$; that is, the linear density is the rate of change of mass with respect to length. Symbolically, we can write

$$\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x} = \frac{dm}{dx}$$

Thus, the linear density of the rod is the derivative of mass with respect to length.

Example 2 The mass of the left-hand x metres of a rod is $f(x) = x^2$ kilograms.
(a) Find the average density of the part of the rod given by $2 \leqslant x \leqslant 2.3$.
(b) Find the linear density at $x = 2$.

Solution (a) The average density for $2 \leqslant x \leqslant 2.3$ is

$$\frac{\Delta m}{\Delta x} = \frac{f(2.3) - f(2)}{2.3 - 2} = \frac{(2.3)^2 - 2^2}{0.3} = 4.3 \text{ kg/m}$$

(b) The linear density at $x = 2$ is

$$\rho = \left.\frac{dm}{dx}\right]_{x=2} = 2x]_{x=2} = 4 \text{ kg/m}$$

Applications to Biology

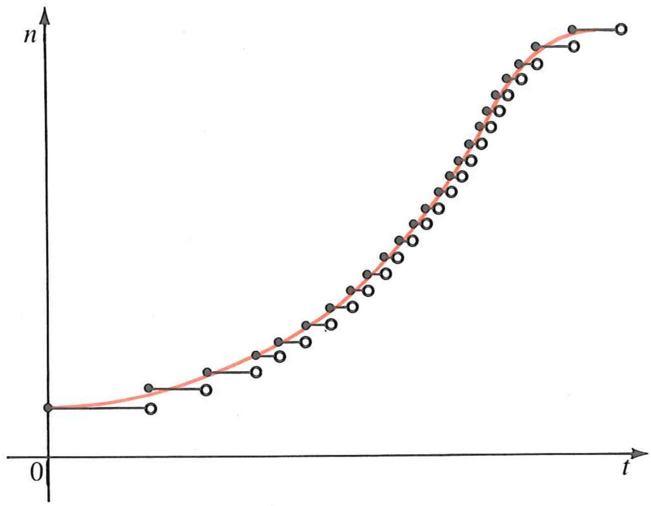
If $n = f(t)$ is the number of individuals in a bacteria or animal population at time t, then the change in the population size between the times $t = t_1$ and $t = t_2$ is $\Delta n = f(t_2) - f(t_1)$. Over the time period $t_1 \leqslant t \leqslant t_2$ we have

$$\text{average rate of growth} = \frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

The **(instantaneous) rate of growth** is the rate of change of the population size with respect to time:

$$\text{growth rate} = \lim_{\Delta t \to 0} \frac{\Delta n}{\Delta t} = \frac{dn}{dt}$$

Thus, the rate of growth is the derivative of the population function. Strictly speaking, this is not quite accurate because the actual graph of a population function $n = f(t)$ would be a step function that is discontinuous whenever a birth or death occurs and therefore not differentiable. However, for a large population, we can replace the graph by a smooth approximating curve as in the following figure.



In Chapter 8, we will use the exponential function to construct models for population growth, and at that time we will be able to compute growth rates for exponentially increasing populations. The model for a population function in the next example is more appropriate for a slowly growing bacteria colony.

Example 3 The population of a bacteria culture after t hours is given by $n = 500 + 200t + 12t^2$. Find the rate of growth after 5 h.

Solution The rate of growth is

$$\frac{dn}{dt} = 200 + 24t$$

After 5 h it is

$$\left.\frac{dn}{dt}\right]_{t=5} = 200 + 24(5) = 320 \text{ bacteria/h}$$

Another occurrence of rates of change in biology is given in Exercise 3.3 as Question 9.

Applications to Chemistry

The *concentration* of a substance A is the number of moles (6.022×10^{23} molecules) per litre and is denoted by $[A]$. During a chemical reaction the concentration will vary and so $[A]$ is a function of time. During a time interval $t_1 \leq t \leq t_2$, the average rate of reaction of a reactant A is

The minus sign is used to make the rate of reaction positive.

$$\frac{\Delta[A]}{\Delta t} = -\frac{[A](t_2) - [A](t_1)}{t_2 - t_1}$$

and the **(instantaneous) rate of reaction** is the rate of change of concentration with respect to time:

$$\text{rate of reaction} = -\lim_{\Delta t \to 0} \frac{\Delta[A]}{\Delta t} = -\frac{d[A]}{dt}$$

Since the rate of reaction is the derivative of the concentration function, chemists often determine the rate of reaction by measuring the slope of a tangent (see Question 8 in Exercise 3.3).

Another application of rates of change in chemistry is described in Question 7.

EXERCISE 3.3

1. Find the rate of change of the volume of a cube with respect to its edge length x when $x = 4$.
2. Find the rate of change of the area of a circle with respect to its radius r when $r = 5$ cm.
3. If a tank holds 1000 L of water, which takes an hour to drain from the bottom of the tank, then the volume V of water remaining in the tank after t minutes is

$$V = 1000\left(1 - \frac{t}{60}\right)^2 \qquad 0 \leq t \leq 60$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) after 10 min.

4. The mass of the part of a wire that lies between its left end and a point x metres to the right is $\sqrt{x}$ kilograms.
   (a) Find an approximate value for the average density of the part of the wire from $x = 1$ m to $x = 1.1$ m.
   (b) Find the linear density when $x = 1$ m.

5. The mass of the left x centimetres of a string is $x + \frac{1}{2}x^2$ grams. Find the linear density when $x = 6$ cm.

6. The population of a bacteria colony after t hours is given by $n = 1000 + 180t + 25t^2 + 3t^3$. Find the growth rate after 3 h.

7. The volume V of a substance kept at constant temperature will depend on the pressure P. The **isothermal compressibility** β is defined by

$$\beta = -\frac{1}{V}\frac{dV}{dP}$$

and measures how fast, per unit volume, the volume of the substance decreases as the pressure increases at constant temperature.

The volume V (in cubic metres) of a sample of air at 25°C was related to the pressure P (in kilopascals) by the equation

$$V = \frac{5.3}{P}$$

Find the compressibility when the pressure is 40 kPa.

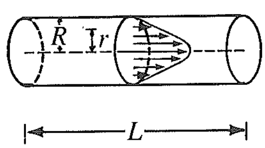
8. The concentrations of dinitrogen pentoxide, N_2O_5, in the reaction

$$2N_2O_5 \rightarrow 4NO_2 + O_2$$

were measured at one-minute intervals as in the table below.

time (min)	0	1	2	3	4
$[N_2O_5]$	0.160	0.113	0.080	0.056	0.040

Draw the graph of $[N_2O_5]$ as function of time and use it to estimate the rate of reaction after two minutes.



9. When blood flows through a blood vessel, such as a vein or artery, we can assume that the blood vessel has the shape of a cylindrical tube with radius R and length L. Because of friction at the walls of the tube, the velocity v of the blood is greatest along the central axis of the tube and decreases as the distance r from the axis increases until v becomes 0 at the wall (see the figure).

The relationship between v and r is given by the *Law of Laminar Flow* discovered by the French physician Poiseuille in 1840. This states that

$$v = \frac{P}{4\eta L}(R^2 - r^2)$$

where η is the viscosity of the blood and P is the pressure difference between the ends of the tube. If P and L are constant, then v is a function of r. In a typical human artery, the values are $\eta = 0.027$, $R = 0.008$ cm, $L = 2$ cm, and $P = 4000$ dynes/cm^2. Find the rate of change of v with respect to r (which is called the **velocity gradient**) when $r = 0.005$ cm.

3.4 RATES OF CHANGE IN THE SOCIAL SCIENCES

Although calculus has been applied to the natural sciences for centuries, it has only been recently that the social sciences, such as psychology, sociology, urban geography, and economics, have been making use of calculus.

Psychologists interested in learning theory study the so-called learning curve, which graphs the performance level $P(t)$ of someone learning a skill as a function of the training time t. Of particular interest is the rate at which performance improves as time passes, that is, the derivative $P'(t)$.

Sociologists use calculus to analyze the spread of rumours (or innovations or fads or fashions). If $f(t)$ is the fraction of the population that knows a rumour by time t, then the derivative $f'(t)$ represents the rate of spread of the rumour.

These applications to psychology and sociology will be explored in Chapter 8. In this section, we examine rates of change in business and economics.

If it costs a company $C(x)$ to produce x units of a certain commodity, then the function C is called a **cost function**. If the number of items produced is increased from x_1 to x_2, the additional cost is $\Delta C = C(x_2) - C(x_1)$ and the average rate of change of the cost is

$$\frac{\Delta C}{\Delta x} = \frac{C(x_2) - C(x_1)}{x_2 - x_1} = \frac{C(x_1 + \Delta x) - C(x_1)}{\Delta x}$$

The limit of this quantity as $\Delta x \to 0$, that is, the instantaneous rate of change of cost with respect to the number of items produced, is called the **marginal cost** by economists:

$$\text{marginal cost} = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

Since x can usually take on only integer values, it may not make literal sense to let Δx approach 0, but we can always replace $C(x)$ by a smooth approximating function as we did for growth functions in Section 3.3.

Thus, the marginal cost is the derivative of the cost function. To see how to interpret the rate of change in this situation, we recall the definition of a derivative at $x = n$:

$$C'(n) = \lim_{h \to 0} \frac{C(n + h) - C(n)}{h}$$

Taking $h = 1$ and n large, we see that

$$C'(n) \doteq C(n + 1) - C(n)$$

Therefore, the marginal cost of producing n units is approximately equal to the cost of producing one more unit, the $(n + 1)$st unit.

It is often appropriate to represent a cost function by a polynomial

$$C(x) = a + bx + cx^2 + dx^3$$

where a represents the fixed cost (rent, heat, maintenance) and the other terms represent the cost of raw materials, labour, and so on. (The cost of raw materials may be proportional to x, but labour costs might depend partly on higher powers of x because of overtime costs and inefficiencies involved in large-scale operations.)

Example 1 Quinton Mills is a large producer of flour. Management estimates that the cost (in dollars) of producing x 5-kg bags of flour is

$$C(x) = 140\ 000 + 0.43x + 0.000\ 001x^2$$

(a) Find the marginal cost at a production level of $x = 1000$ bags.
(b) Find the actual cost of producing the 1001st bag.

Solution (a) The marginal cost function is

$$C'(x) = 0.43 + 0.000\ 002x$$

The marginal cost when $x = 1000$ is

$$C'(1000) = 0.43 + (0.000\ 002)(1000) = \$0.432/\text{bag}$$

(b) The cost of producing the 1001st bag is

$$\begin{aligned} C(1001) - C(1000) &= [140\ 000 + (0.43)(1001) + (0.000\ 001)(1001)^2] \\ &\quad - [140\ 000 + (0.43)(1000) + (0.000\ 001)(1000)^2] \\ &= \$0.432\ 001 \end{aligned}$$

Of course a businessman is interested not only in costs but also in revenue and profit. Let $p(x)$ be the price per unit that a company can charge if it sells x units. Then p is called the **demand function** (or **price function**). If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = xp(x)$$

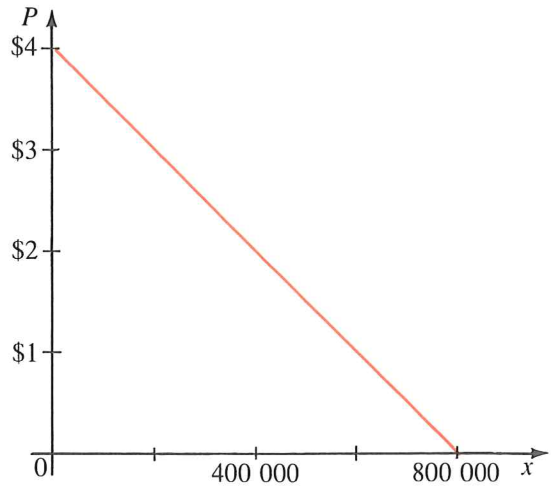
and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

Example 2 Howard's Hamburgers has taken a market survey and has found that the yearly demand for their hamburgers is given by

$$p = \frac{800\ 000 - x}{200\ 000} \qquad (p \text{ in dollars})$$

(a) Graph the demand function.
(b) What is the demand for hamburgers corresponding to prices of \$0.00, \$0.50, \$1.00, \$1.50, \$2.00, \$2.50, \$3.00, \$3.50, \$4.00?
(c) Find the marginal revenue when $x = 300\,000$.

Solution (a) Notice from the graph that, unsurprisingly, more hamburgers are sold as the price decreases.



(b) The table shows the demand at the given prices.

p	0	\$0.50	\$1.00	\$1.50	\$2.00	\$2.50	\$3.00	\$3.50	\$4.00
x	800 000	700 000	600 000	500 000	400 000	300 000	200 000	100 000	0

(c) The revenue function is

$$R(x) = xp(x) = x\left(\frac{800\,000 - x}{200\,000}\right) = \frac{1}{200\,000}(800\,000x - x^2)$$

So the marginal revenue function is

$$R'(x) = \frac{1}{200\,000}(800\,000 - 2x)$$

When $x = 300\,000$ the marginal revenue is

$$R'(300\,000) = \frac{1}{200\,000}(800\,000 - 600\,000) = \$1/\text{hamburger}$$

In Example 2(c), the marginal revenue of \$1 per hamburger is the rate at which revenue is increasing with respect to increase in sales. It represents the approximate additional income to the company per additional item sold.

In general, if x units of a commodity are sold, the total profit is obtained by subtracting the cost from the revenue:

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit** function is P', the derivative of the profit function.

Example 3 Howard's Hamburgers estimates that the cost, in dollars, of making x hamburgers is

$$C(x) = 125\ 000 + 0.42x$$

Using the demand function from Example 2, find the profit and the marginal profit when (a) $x = 300\ 000$, (b) $x = 400\ 000$.

Solution From Example 2, the revenue function is

$$R(x) = \frac{1}{200\ 000}(800\ 000x - x^2)$$

and so the profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \frac{1}{200\ 000}(800\ 000x - x^2) - (125\ 000 + 0.42x) \\ &= -\frac{x^2}{200\ 000} + 3.58x - 125\ 000 \end{aligned}$$

The marginal profit function is

$$P'(x) = -\frac{x}{100\ 000} + 3.58$$

(a) When $x = 300\ 000$, the profit is

$$\begin{aligned} P(300\ 000) &= -\frac{(300\ 000)^2}{200\ 000} + (3.58)(300\ 000) - 125\ 000 \\ &= \$499\ 000 \end{aligned}$$

and the marginal profit is

$$P'(300\ 000) = -\frac{300\ 000}{100\ 000} + 3.58 = \$0.58/\text{hamburger}$$

(b) When $x = 400\ 000$, the profit is

$$\begin{aligned} P(400\ 000) &= -\frac{(400\ 000)^2}{200\ 000} + (3.58)(400\ 000) - 125\ 000 \\ &= \$507\ 000 \end{aligned}$$

and the marginal profit is

$$P'(400\ 000) = -\frac{400\ 000}{100\ 000} + 3.58 = -\$0.42/\text{hamburger}$$

In Example 3(a), the marginal profit of \$0.58/hamburger represents the approximate additional income per additional hamburger sold when 300 000 hamburgers have been sold. The negative marginal profit in part (b) shows that, when 400 000 hamburgers have been sold, additional sales will increase revenue but decrease profits. In Section 4.5 we will see how to choose x so as to maximize profits.

EXERCISE 3.4

1. A company determines that the cost, in dollars, of producing x items is

$$C(x) = 55\ 000 + 23x + 0.012x^2$$

(a) Find the marginal cost function.
(b) Find the marginal cost at a production level of 100 items.
(c) Find the cost of producing the 101st item.

2. The cost in dollars for the production of x units of a commodity is

$$C(x) = 1500 + \frac{x}{10} + \frac{x^2}{1000}$$

(a) Find the marginal cost function.
(b) Find the marginal cost at a production level of 800 units.
(c) Find the cost of producing the 801st unit.

3. A manufacturer determines that the revenue derived from selling x units of one of their products is $R(x) = 8000x - 0.02x^3$.

(a) Find the marginal revenue function.
(b) Find the marginal revenue when 300 units are sold.
(c) Compare this to the actual gain in revenue when the 301st unit is sold.

4. The Manchester Pen Company estimates that the cost of manufacturing x pens is

$$C(x) = 23\ 000 + 0.24x + 0.0001x^2$$

and the revenue is

$$R(x) = 0.98x - 0.0002x^2$$

(a) Find the profit function.
(b) Find the marginal profit function.
(c) Find the marginal profit when 1000 pens are sold.
(d) Compare this to the actual increase when the 1001st pen is sold.

5. Sue's Submarines has determined that the monthly demand for their submarines is given by

$$p = \frac{30\,000 - x}{10\,000}$$

and the cost of making x submarines is

$$C(x) = 6000 + 0.8x$$

(a) Graph the demand function.
(b) Fill in the following table to illustrate the demand at the given prices.

p	0	\$0.50	\$1.00	\$1.50	\$2.00	\$2.50	\$3.00
x							

(c) Find the revenue function.
(d) Find the marginal revenue function.
(e) Find the marginal revenue when $x = 1000$.
(f) Find the profit function.
(g) Find the marginal profit function.
(h) Find the marginal profit when $x = 10\,000$.

6. A company estimates that its production costs, in dollars, for x items is

$$C(x) = 82\,000 + 23x + 0.001x^2$$

and the demand function for this product is given by

$$p = 100 - 0.01x$$

(a) Find the marginal cost function.
(b) Find the marginal revenue function.
(c) Find the marginal profit function.
(d) Find the marginal profit at a production level of 50 items.

3.5 RELATED RATES

In a related rates problem, we are given the rate of change of one quantity and we are asked to find the rate of change of a related quantity. To do this, we find an equation that relates the two quantities and use the Chain Rule to differentiate both sides of the equation with respect to time.

Example 1 A spherical snowball is melting in such a way that its volume is decreasing at a rate of 1 cm³/min. At what rate is the radius decreasing when the radius is 5 cm?

Solution Let V be the volume of the snowball and r its radius. Then V and r are related by the equation

$$V = \tfrac{4}{3}\pi r^3 \tag{1}$$

We are given the rate of change of V:

$$\frac{dV}{dt} = -1 \text{ cm}^3/\text{min}$$

The minus sign is used because the volume is decreasing.

We are asked to find $\frac{dr}{dt}$ when $r = 5$. Using the Chain Rule to differentiate Equation 1 with respect to time, we have

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

so

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

Now we put $r = 5$ and $\frac{dV}{dt} = -1$ in this equation and we get

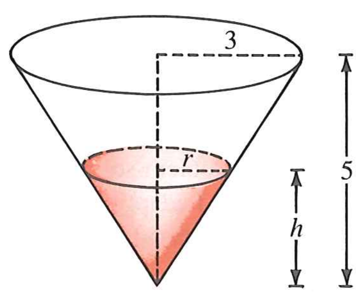
$$\frac{dr}{dt} = \frac{1}{4\pi(5)^2}(-1) = -\frac{1}{100\pi}$$

The radius of the snowball is decreasing at a rate of

$\frac{1}{100\pi} \doteq 0.003$ cm/min.

Example 2 A water tank is built in the shape of a circular cone with height 5 m and diameter 6 m at the top. Water is being pumped into the tank at a rate of 1.6 m³/min. Find the rate at which the water level is rising when the water is 2 m deep.

Solution First we sketch the cone.



Let V be the volume of the water and let r and h be the radius of the surface and the height at time t, where t is measured in minutes.

We are given the rate of increase of V, that is,

$$\frac{dV}{dt} = 1.6 \text{ m}^3/\text{min}$$

and we are asked to find $\frac{dh}{dt}$ when $h = 2$ m.

The quantities V and h are related by the equation

$$V = \tfrac{1}{3}\pi r^2 h$$

but we have to express V as a function of h alone. To eliminate r we look for a relationship between r and h. We use the similar triangles in the figure to write

$$\frac{r}{h} = \frac{3}{5}$$

Thus $r = \frac{3}{5}h$ and we have

$$V = \frac{1}{3}\pi\left(\frac{3}{5}h\right)^2 h = \frac{3\pi}{25}h^3$$

Differentiating both sides with respect to t, we have

$$\frac{dV}{dt} = \frac{3\pi}{25}(3h^2)\frac{dh}{dt} = \frac{9\pi}{25}h^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{9\pi}\frac{1}{h^2}\frac{dV}{dt}$$

When $h = 2$ and $\frac{dV}{dt} = 1.6$, we have

$$\frac{dh}{dt} = \frac{25}{9\pi}\frac{1}{2^2}(1.6) = \frac{10}{9\pi}$$

The water level is rising at a rate of $\frac{10}{9\pi} \doteq 0.4$ m/min.

In solving related rates problems it is useful to recall some of the problem-solving principles from the Review and Preview to this chapter and adapt them to the present situation:

1. Read the problem carefully until you understand it.
2. Draw a diagram if possible.

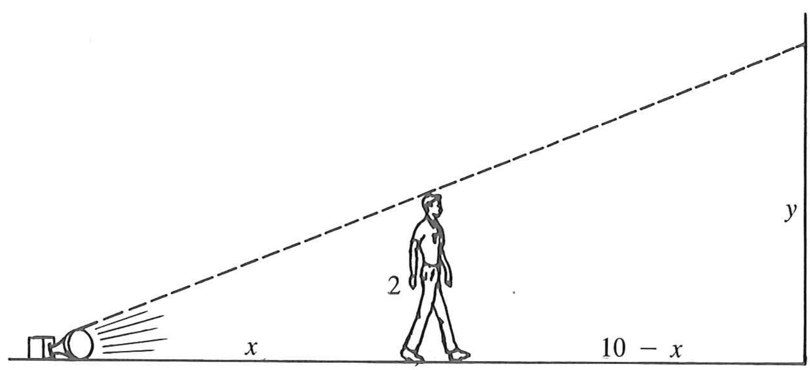
3. Introduce notation. Assign symbols to all quantities that are functions of time.
4. Express the given information and the required rate in terms of derivatives.
5. Write an equation that relates the various quantities of the problem. If necessary, use the geometry of the situation to eliminate one of the variables by substitution (as in Example 2).
6. Use the Chain Rule to differentiate both sides of the equation with respect to t.
7. Substitute the given information into the resulting equation and solve for the unknown rate.

A common error is to substitute the given numerical information (for quantities that vary with time) at too early a stage. This should only be done *after* the differentiation. (Step 7 follows Step 6.) For instance, in Example 2 we dealt with general values of h until we finally substituted $h = 2$ at the last stage. (If we had put $h = 2$ earlier, we would have got $\frac{dV}{dt} = 0$, which is clearly wrong.)

The remaining two examples further illustrate this strategy.

Example 3 A spotlight on the ground shines on a wall 10 m away. A man 2 m tall walks from the spotlight toward the wall at a speed of 1.2 m/s. How fast is his shadow on the wall decreasing when he is 3 m from the wall?

Solution



As in the figure, let x be the distance from the light to the man and let y be the height of his shadow, in metres.

We are given that $\frac{dx}{dt} = 1.2$ m/s, and we wish to find $\frac{dy}{dt}$ when $10 - x = 3$ m, that is, $x = 7$ m.

To relate y to x we use similar triangles:

$$\frac{y}{10} = \frac{2}{x}$$

Thus, $$y = \frac{20}{x}$$

and so, $$\frac{dy}{dt} = -\frac{20}{x^2}\frac{dx}{dt}$$

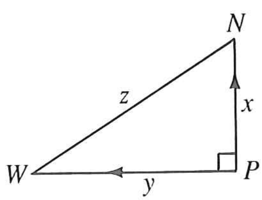
When $x = 7$ and $\frac{dx}{dt} = 1.2$, we have

$$\frac{dy}{dt} = -\frac{20}{7^2}(1.2) = -\frac{24}{49}$$

The shadow is decreasing at a rate of $\frac{24}{49}$ m/s.

Example 4 A man starts walking north at a speed of 1.5 m/s and a woman starts at the same point P at the same time walking west at a speed of 2 m/s. At what rate is the distance between the man and the woman increasing one minute later?

Solution At any given time t after they start, let x be the distance travelled by the man, y be the distance travelled by the woman, and z be the distance between them. We are given that



$$\frac{dx}{dt} = 1.5 \text{ m/s} \quad \text{and} \quad \frac{dy}{dt} = 2 \text{ m/s}$$

and we are required to find $\frac{dz}{dt}$ when $t = 60$.

The equation that relates x, y, and z is given by the Pythagorean Theorem:

$$z^2 = x^2 + y^2 \tag{1}$$

Differentiating with respect to t, we get

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{1}{z}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right) \tag{2}$$

When $t = 60$, we have $x = 90$ m and $y = 120$ m, so Equation 1 gives

$$z = \sqrt{90^2 + 120^2} = 150 \text{ m}$$

Putting these values in Equation 2, we have

$$\frac{dz}{dt} = \frac{1}{150}[90(1.5) + 120(2)] = 2.5$$

The distance between the man and the woman is increasing at a rate of 2.5 m/s.

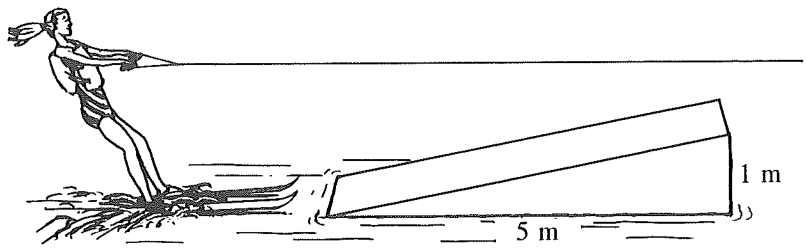
EXERCISE 3.5

1. If $xy^2 = 12$ and $\frac{dy}{dt} = 6$, find $\frac{dx}{dt}$ when $y = 2$.
2. If $x^3 + y^3 = 9$ and $\frac{dx}{dt} = 4$, find $\frac{dy}{dt}$ when $x = 2$.
3. How fast is the area of a square increasing when the side is 3 m in length and growing at a rate of 0.8 m/min?
4. How fast is the edge length of a cube increasing when the volume of the cube is increasing at a rate of 144 cm^3/s and the edge length is 4 cm?
5. A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 25 cm/s. Find the rate at which the area within the circle is increasing after 4 s.
6. A spherical balloon is being inflated so that the volume is increasing at a rate of 8 m^3/min. How fast is the radius of the balloon increasing when the diameter is 2 m?
7. A snowball melts so that its surface area decreases at a rate of 0.5 cm^2/min. Find the rate at which the radius decreases when the radius is 4 cm.
8. The side of an equilateral triangle decreases at the rate of 2 cm/s. At what rate is the area decreasing when the area is 100 cm^2?
9. The area of a triangle is increasing at a rate of 4 cm^2/min and its base is increasing at a rate of 1 cm/min. At what rate is the altitude of the triangle increasing when the altitude is 20 cm and the area is 80 cm^2?
10. A man 2 m tall walks away from a lamppost whose light is 5 m above the ground. If he walks at a speed of 1.5 m/s, at what rate is his shadow growing when he is 10 m from the lamppost?
11. A ladder 4 m long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a speed of 30 cm/s, how quickly is the top of the ladder sliding down the wall when the bottom of the ladder is 2 m from the wall?

12. Joe is driving west at 60 km/h and Dave is driving south at 70 km/h. Both cars are approaching the intersection of the two roads. At what rate is the distance between the cars decreasing when Joe's car is 0.4 km and Dave's is 0.3 km from the intersection?

13. At 1:00 p.m. ship A was 80 km south of ship B. Ship A is sailing north at 30 km/h and ship B is sailing east at 40 km/h. How fast is the distance between them changing at 3:00 p.m.?

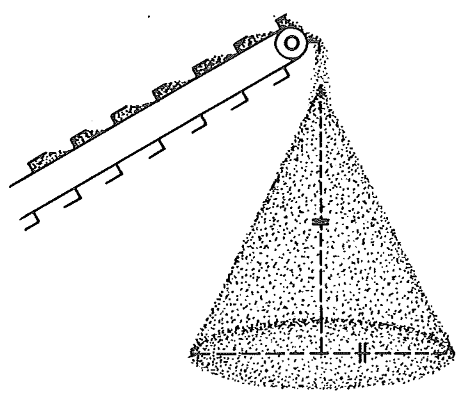
14. A waterskier skis over the ramp shown in the figure at a speed of 12 m/s. How fast is she rising as she leaves the ramp?



15. A plane flies horizontally with a speed of 600 km/h at an altitude of 10 km and passes directly over the town of Quinton. Find the rate at which the distance from the plane to Quinton is increasing when it is 20 km away from Quinton.

16. A water trough is 10 m long and a cross-section has the shape of an isosceles triangle that is 1 m across at the top and is 50 cm high. The trough is being filled with water at a rate of 0.4 m^3/min. How fast will the water level rise when the water is 40 cm deep?

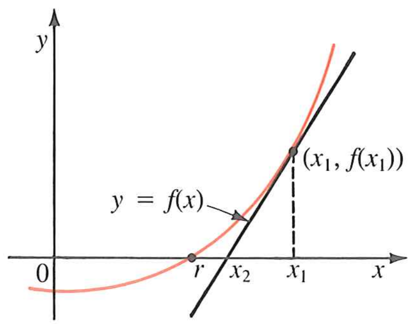
17. Sand is being dumped from a conveyor belt at a rate of 1.2 m^3/min and forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile growing when the pile is 3 m high?



3.6 NEWTON'S METHOD

A quadratic equation $ax^2 + bx + c = 0$ can be solved either by factoring or by using the quadratic formula. For a cubic equation $ax^3 + bx^2 + cx + d = 0$ there is also a formula, but it is so complicated that it is seldom used. Likewise, the formula for the solutions of a fourth-degree equation is extremely difficult and there is no formula at all for equations of degrees higher than four. Using **Newton's method**, however, we can find *approximations* to the solutions of such equations.

Suppose we want to solve an equation of the form $f(x) = 0$, where f is a differentiable function. Let r be a **solution**, or **root**, of the equation; that is, $f(r) = 0$. Our aim is to find a good approximation to r. The idea behind Newton's method is seen in the figure, where r is shown as the x-intercept of the graph of f.



We start with a first approximation x_1 to r, obtained by guessing, or by numerical experimentation, or by roughly sketching the graph of f. We draw the tangent line to the graph of f at the point $(x_1, f(x_1))$. Let x_2 be the x-intercept of this tangent line. It appears from the figure that if x_1 is close to r, then x_2 is even closer to r and so we use it as the second approximation to r.

To express x_2 in terms of x_1, we first write the equation of the tangent line in slope-point form:

$$y - f(x_1) = f'(x_1)(x - x_1)$$

Since the x-intercept is x_2, we put $y = 0$ and $x = x_2$ in this equation:

$$0 - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, we can solve for x_2:

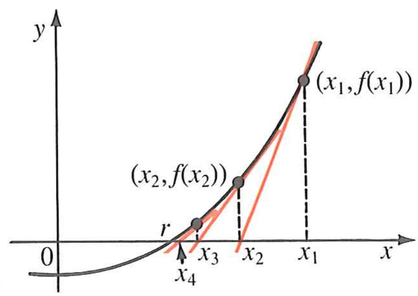
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad ①$$

If we repeat this procedure with x_1 replaced by x_2, we get a third approximation x_3 given by

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

In fact, if we repeat the procedure indefinitely, we get a sequence of approximations $x_1, x_2, x_3, x_4, \ldots$ as shown in the following diagram. If, as we hope, these numbers become closer and closer to the desired root r, then, in the notation of Section 1.6, we can write

$$\lim_{n\to\infty} x_n = r$$



Newton's Method

If x_1 is a first approximation to a root of the equation $f(x) = 0$, then successive approximations are given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots \tag{2}$$

if $f'(x_n) \neq 0$.

Example 1 Starting with $x_1 = 1$, find the third approximation x_3 to the root of the equation $x^3 + x - 1 = 0$.

Solution Applying Newton's method with

$$f(x) = x^3 + x - 1$$

and

$$f'(x) = 3x^2 + 1$$

we have

$$\begin{aligned} x_2 &= x_1 - \frac{f(x_1)}{f'(x_1)} \\ &= x_1 - \frac{x_1^3 + x_1 - 1}{3x_1^2 + 1} \\ &= 1 - \frac{1^3 + 1 - 1}{3(1)^2 + 1} \\ &= \tfrac{3}{4} \end{aligned}$$

Then, using this value to get the next approximation, we have

$$\begin{aligned} x_3 &= x_2 - \frac{f(x_2)}{f'(x_2)} \\ &= \frac{3}{4} - \frac{f\left(\frac{3}{4}\right)}{f'\left(\frac{3}{4}\right)} \\ &= \frac{3}{4} - \frac{\frac{27}{64} - \frac{1}{4}}{\frac{27}{16} + 1} \\ &= \frac{59}{86} \end{aligned}$$

The third approximation is $\frac{59}{86} \doteq 0.6860$.

If we wish to find a root correct to six decimal places, say, we use Formula 2 for $n = 1, 2, 3, \ldots$ and we stop when successive approximations x_n and x_{n+1} agree to six decimal places.

The procedure in going from stage n to stage $n + 1$ is the same for all values of n, and we call Formula 2 a **recursion formula**. As a result, it is especially convenient to use a computer or a programmable calculator when using Newton's method.

Example 2 Use Newton's method to find $\sqrt[4]{13}$ correct to four decimal places.

Solution Notice that finding $\sqrt[4]{13}$ is equivalent to finding the positive root of the equation

$$x^4 - 13 = 0$$

Therefore we take $f(x) = x^4 - 13$ in Newton's method, so $f'(x) = 4x^3$ and Equation 2 becomes

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 13}{4x_n^3} \tag{3}$$

If we take $x_1 = 2$ as our initial approximation and we put $n = 1$ in Equation 3, we get

$$x_2 = 2 - \frac{2^4 - 13}{4(2)^3} = 1.906\,250$$

With this value of x_2, Equation 3 gives

$$x_3 \doteq 1.898\,872$$

Repeating this procedure using a calculator, we obtain

$$x_4 \doteq 1.898\ 829$$
$$x_5 \doteq 1.898\ 829$$

These values agree to six decimal places, so we conclude that

$$\sqrt[4]{13} \doteq 1.898\ 829$$

correct to six decimal places.

If we had used a different positive initial value for x_1 in Example 2, we would have arrived at the same approximation for $\sqrt[4]{13}$, though more steps might be required. For instance, if the initial guess is $x_1 = 5$, then we get

$$\begin{aligned} x_2 &\doteq 3.776\ 000 \\ x_3 &\doteq 2.892\ 365 \\ x_4 &\doteq 2.303\ 589 \\ x_5 &\doteq 1.993\ 561 \\ x_6 &\doteq 1.905\ 370 \\ x_7 &\doteq 1.898\ 863 \\ x_8 &\doteq 1.898\ 829 \\ x_9 &\doteq 1.898\ 829 \end{aligned}$$

Example 3 Find the coordinates of the point of intersection of the curves $y = x^5$ and $y = x^2 + 1$ correct to six decimal places.

Solution First notice that the x-coordinate of the point of intersection satisfies

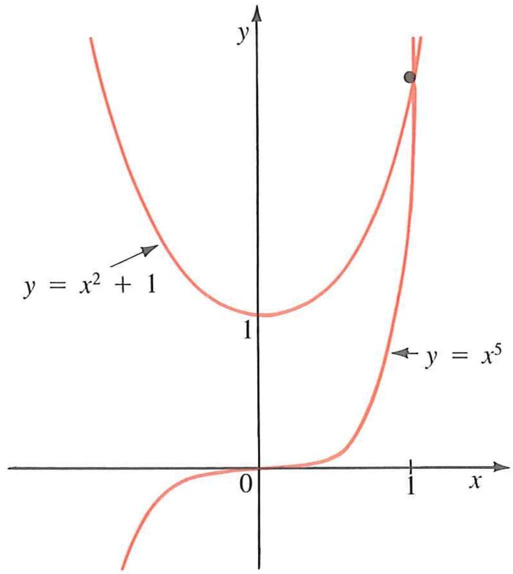
$$x^5 = x^2 + 1$$

so it is a root of the equation

$$x^5 - x^2 - 1 = 0$$

and we can employ Newton's method with $f(x) = x^5 - x^2 - 1$ and $f'(x) = 5x^4 - 2x$.

To find a first approximation x_1 we sketch the curves $y = x^5$ and $y = x^2 + 1$. It appears that the curves intersect when x is slightly larger than 1, so we take $x_1 = 1$.



Newton's method gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^5 - x_n^2 - 1}{5x_n^4 - 2x_n}$$

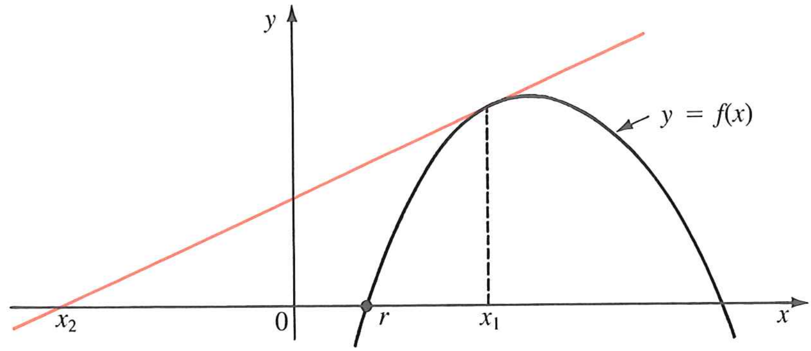
So we have, successively,

$$\begin{aligned} x_2 &\doteq 1.333\,333 \\ x_3 &\doteq 1.223\,997 \\ x_4 &\doteq 1.195\,608 \\ x_5 &\doteq 1.193\,865 \\ x_6 &\doteq 1.193\,859 \\ x_7 &\doteq 1.193\,859 \end{aligned}$$

The y-coordinate of the point of intersection can be approximated using either of the equations $y = x^5$ or $y = x^2 + 1$. Correct to six decimal places, the point of intersection is

$$(1.193\,859,\ 2.425\,300)$$

Finally, we note that care should be taken to ensure a reasonable first approximation x_1. If x_1 is not chosen close enough to r, it could happen that x_2 is a worse approximation than x_1. (See the diagram below and Question 3(a) in Exercise 3.6.)



EXERCISE 3.6

1. Start with $x_1 = 0$ and use Newton's method to find the second approximation x_2 to the root of the equation $x^3 + 2x + 1 = 0$.
2. Find the second and third approximations to the root of the equation $x^3 + x^2 + 1 = 0$ using Newton's method and taking $x_1 = -1$.
3. (a) Use Newton's method with $x_1 = 2$ to find the root of the equation $x^3 - x - 2 = 0$ correct to six decimal places.
   (b) Solve the equation in part (a) using $x_1 = 1$ as the initial approximation.
   (c) Solve the equation in part (a) using $x_1 = 0.57$. Sketch the graph of $f(x) = x^3 - x - 2$ to show why x_2 is such a poor approximation.
4. Use Newton's method to approximate the root of the equation in the given interval correct to six decimal places.
   (a) $x^4 - x^2 + x - 5 = 0,\ 1 < x < 2$
   (b) $x^3 - x^2 + 2x = 9,\ 2 < x < 3$
   (c) $x^6 = \sqrt{x + 7},\ 1 < x < 2$
5. Use Newton's method to find all roots of the equation correct to six decimal places.
   (a) $x^3 - 5x + 1 = 0$ (b) $x^5 = 4x^2 - 1$
6. (a) Apply Newton's method to derive the following square-root algorithm (used by the ancient Babylonians to compute $\sqrt{a}$):
   $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$$
   (b) Use part (a) to compute $\sqrt{17.2}$ correct to six decimal places.
7. Use Newton's method to approximate the following numbers correct to six decimal places.
   (a) $\sqrt[5]{28}$ (b) $\sqrt[8]{1.23}$
8. Sketch the following pairs of curves and find the coordinates of their point of intersection correct to six decimal places.
   (a) $y = x^3,\ y = x + 1$ (b) $y = x^2 + 1,\ xy = 1$

COMPUTER APPLICATION

Newton's method for solving equations is well suited to computer implementation. We present a pseudocode (the logic of the algorithm) for Newton's method, together with a computer program in BASIC. This program relates to Example 3 in Section 3.6.

Pseudocode

```
define the function
define the derivative
establish a first root
loop through calculations of successive
  approximations
    calculate next approximation using
      Newton's formula
    check for desired accuracy
    reset the variables in preparation
      for the next pass through the loop
continue looping until desired accuracy
  reached
print out the root of the equation
```

A BASIC version

```
DEF function(x)=X^5-X^2-1
DEF derivative(x)=5*X^4-2*X
PRINT ``enter a first approximation for
  the root...'';
INPUT xn

DO
  LET xnext=xn-funtion(xn)/
    derivative(xn)
  LET difference=abs(xn-xnext)
  LET xn=xnext
LOOP until difference<0.00000001
PRINT ``correct to 8 decimal places, the
  root is ...'';
PRINT using ``#########'' : xnext

END
```

The Output

```
enter a first approximation for the root
  ...? 1
correct to 8 decimal places, the root is
  ... 1.19385911
```

3.7 REVIEW EXERCISE

1. The position function of a particle is given by $s = 2t^3 + 4t^2 - t$, where s is measured in metres and t in seconds.
   (a) Find the velocity and acceleration as functions of t.
   (b) Find the velocity and acceleration after 4 s.
2. The motion of a particle is described by the position function
   $$s = t^3 - 12t^2 + 45t + 3,\ t \geqslant 0$$
   where t is measured in seconds and s in metres.
   (a) When is the particle at rest?
   (b) When is the velocity positive and when is it negative?
   (c) When is the acceleration positive and when is it negative?
   (d) Find the velocity when the acceleration is 0.
   (e) Draw a diagram to illustrate the motion of the particle.
   (f) Find the total distance travelled in the first 8 s.
3. If a ball is thrown upward on the moon with a velocity of 65 m/s, its height in metres after t seconds is
   $$h = 65t - 0.83t^2$$
   (a) Find the velocity of the ball after 1 s.
   (b) Find the acceleration of the ball after 1 s.
   (c) When will the ball hit the moon?
   (d) With what velocity will it hit the moon?
4. Find the rate of change of the area of a square with respect to the length L of a side when $L = 5$.
5. The mass of a length of wire from its left end to a point x metres to the right is $\left(2 + x + \frac{1}{2}x^2\right)$ kilograms.
   (a) Find the average density of the part of the wire from $x = 2$ m to $x = 2.1$ m.
   (b) Find the linear density when $x = 2$ m.
6. A company estimates that the cost, in dollars, of manufacturing x units of their product is $C(x) = 19\,000 + 16.2x + 0.06x^2$.
   (a) Find the marginal cost function.
   (b) Find the marginal cost at a production level of 200 units.
   (c) Find the cost of manufacturing the 201st item.
7. Pasquale's Pizza makes only one size of pizza and has determined that the cost of making x pizzas is
   $$C(x) = 12\,500 + 1.08x$$
   The monthly demand for their pizzas is given by
   $$p = \frac{20\,000 - x}{1000}$$
   (a) Find the marginal cost function.
   (b) Find the revenue function.
   (c) Find the marginal revenue function.
   (d) Find the profit function.

(e) Find the marginal profit function.
(f) Find the marginal profit when $x = 8000$.

8. A rectangle is expanding so that its length is always twice its width. The perimeter of the rectangle is increasing at a rate of 6 cm/min. Find the rate of increase of the area of the rectangle when the perimeter is 40 cm.

9. Boyle's Law states that when a sample of gas is compressed at a constant temperature, the pressure P and volume V satisfy the equation $PV = C$, where C is a constant. At a certain instant, the volume is 480 cm^3, the pressure is 160 kPa, and the pressure is increasing at a rate of 15 kPa/min. At what rate is the volume decreasing at this instant?

10. At 9 a.m. ship A is 50 km east of ship B. Ship A is sailing north at 40 km/h and ship B is sailing south at 30 km/h. How fast is the distance between them changing at noon?

11. Use Newton's method with initial approximation $x_1 = 1$ to find the second approximation x_2 to the root of the equation $x^4 + x - 1 = 0$ that lies between 0 and 1.

12. Use Newton's method to find all the roots of the equation $x^3 - x^2 + 1 = 0$ correct to six decimal places.

13. (a) Sketch the curves $y = x^6$ and $y = 3 - 2x$ using the same axes.
(b) Find the coordinates of the points of intersection of these curves correct to six decimal places.

3.8 CHAPTER 3 TEST

1. The position function of a particle is given by
$$s = t^3 - 6t^2 + 9t + 1,\ t \geqslant 0$$
where t is measured in seconds and s in metres.
(a) Find the velocity after 4 s.
(b) Find the acceleration after 4 s.
(c) When is the particle at rest?
(d) When is the particle moving in the positive direction?
(e) Find the velocity when the acceleration is 0.
(f) Find the total distance travelled in the first 4 s.

2. A manufacturer of CD players estimates that the cost of making x machines is
$$C(x) = 87\ 000 + 122x$$
and the demand function is given by
$$p = \frac{600\ 000 - x}{1000}$$
(a) Find the marginal cost function.
(b) Find the revenue function.
(c) Find the marginal revenue function.
(d) Find the profit function.
(e) Find the marginal profit function.

3. A paper cup has the shape of a cone with height 8 cm and radius 3 cm at the top. Water is poured into the cup at a rate of 2 cm^3/s. How fast is the water level rising when the water level is 6 cm deep?

4. Find the root of the equation $x^5 = x + 2$ correct to six decimal places.

CUMULATIVE REVIEW FOR CHAPTERS 1 TO 3

1. Find each limit.

(a) $\lim_{x\to 2} \dfrac{2x^2+1}{3x^2-4}$
(b) $\lim_{x\to 2} \dfrac{x^2-3x+2}{x^2-4}$

(c) $\lim_{x\to -1} \dfrac{x^2+2x+1}{x^3+1}$
(d) $\lim_{h\to 0} \dfrac{(2+h)^3-8}{h}$

(e) $\lim_{h\to 0} \dfrac{\dfrac{1}{(3+h)^2}-\dfrac{1}{9}}{h}$
(f) $\lim_{x\to -2} \dfrac{x^2-x-6}{2x^2+5x+2}$

(g) $\lim_{x\to 2^+} \sqrt{x^2-x-2}$
(h) $\lim_{x\to -5^-} \dfrac{2x+10}{|x+5|}$

2. Let

$$f(x) = \begin{cases} -x-2 & \text{if } x < -2 \\ 1-x^2 & \text{if } -2 \leqslant x \leqslant 2 \\ x-5 & \text{if } x > 2 \end{cases}$$

(a) Find the following limits, if they exist.

(i) $\lim_{x\to -2^-} f(x)$ (ii) $\lim_{x\to -2^+} f(x)$ (iii) $\lim_{x\to -2} f(x)$

(iv) $\lim_{x\to 2^-} f(x)$ (v) $\lim_{x\to 2^+} f(x)$ (vi) $\lim_{x\to 2} f(x)$

(b) Sketch the graph of f.
(c) Where is f discontinuous?
(d) Where is f not differentiable?

3. Find the following limits.

(a) $\lim_{n\to\infty} \dfrac{1-2n^3}{n+n^3}$
(b) $\lim_{n\to\infty} 3^{-n}$

4. Find the sum of the series or state that it is divergent.

(a) $2 - 3 + \dfrac{9}{4} - \dfrac{27}{8} + \cdots$
(b) $3 + 2 + \dfrac{4}{3} + \dfrac{8}{9} + \dfrac{16}{27} + \cdots$

5. (a) If $f(x) = 6 - 5x + 3x^2$, find $f'(x)$ directly from the definition of a derivative.

(b) Find the equation of the tangent line to the parabola $y = 6 - 5x + 3x^2$ at the point where $x = 1$.

6. Find the derivative of the function $g(x) = \sqrt{3-x}$ directly from the definition.

7. Differentiate.

(a) $f(x) = 12x^5 - \frac{1}{2}x^4 - 4x$
(b) $f(x) = \dfrac{6}{x^2}$

(c) $g(x) = \sqrt[3]{x}\left(2x + \dfrac{1}{x}\right)$
(d) $g(x) = \dfrac{x^2}{2x-3}$

(e) $f(t) = \sqrt{2t - t^3}$
(f) $f(y) = \left(\dfrac{2-y}{1+2y}\right)^4$

(g) $y = (3x + 5)(x^3 - 1)^3$ (h) $y = \dfrac{1}{\sqrt[5]{x^5 + 1}}$

(i) $y = \dfrac{\sqrt{x} - x}{x^2}$ (j) $y = \sqrt{\dfrac{x}{1 + x^2}}$

8. If $f(x) = \dfrac{1}{\sqrt{x^2 - 1}}$, find $f'(x)$ and state the domains of both f and f'.

9. Find $\dfrac{dy}{dx}$ if $x^4 + 2x^2y^3 + y^2 = 21$.

10. Find y' and y''.

(a) $y = \dfrac{x + 1}{x + 2}$ (b) $x^2 - y^3 = 7$

11. Find the equation of the tangent line to the curve at the given point.

(a) $y = \dfrac{2}{1 + x^2}$, $(1, 1)$ (b) $x^2 - y^2 = 3$, $(-2, -1)$

12. Find the equation of both tangent lines to the curve $y = x^3 - x$ that are parallel to the line $22x - 2y + 1 = 0$.

13. Suppose that $f(2) = -3$, $f'(2) = 10$, $f'(4) = 6$, $g(2) = 4$, and $g'(2) = 1$. Evaluate

(a) $(fg)'(2)$ (b) $\left(\dfrac{f}{g}\right)'(2)$ (c) $(f \circ g)'(2)$

14. If f is a differentiable function, find an expression for $F'(x)$ in terms of $f'(x)$.

(a) $F(x) = f(x^5)$ (b) $F(x) = [f(x)]^5$

(c) $F(x) = x^5f(x)$ (d) $F(x) = \sqrt{\dfrac{f(x)}{x}}$

15. If a ball is thrown downward from a 120 m high cliff with an initial speed of 18 m/s, then its height after t seconds, before it hits the ground, is

$$h = 120 - 18t - 4.9t^2$$

(a) Find the average velocity of the stone for the following time periods.

(i) $2 \leqslant t \leqslant 3$ (ii) $2 \leqslant t \leqslant 2.1$ (iii) $2 \leqslant t \leqslant 2.01$

(b) Find the velocity after 2 s.

(c) Find the acceleration after 2 s.

16. The motion of a particle is described by the position function

$$s = t^3 - 6t^2 + 9t + 5, \ t \geqslant 0$$

where s is measured in metres and t in seconds.

(a) Find the velocity after 2 s and 4 s.

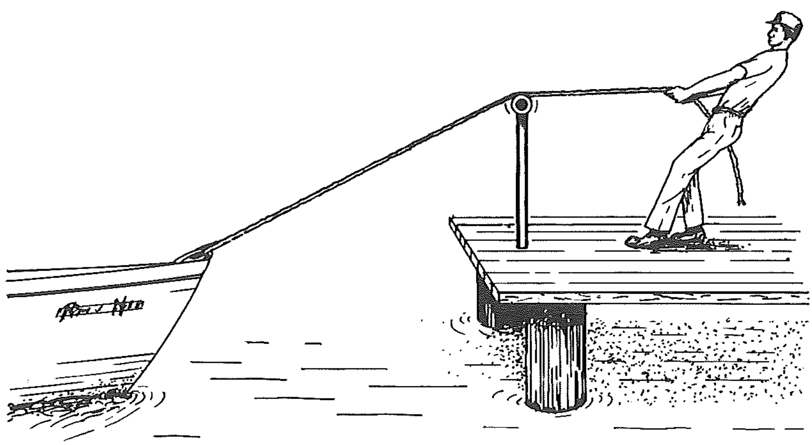
(b) Find the acceleration after 2 s and 4 s.

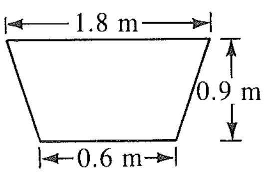
(c) When is the particle at rest?

(d) When is the velocity positive and when is it negative?
(e) When is the acceleration positive and when is it negative?
(f) Find the total distance travelled in the first 5 s.

17. A spherical balloon is being inflated.
(a) Find the rate of change of the volume with respect to the radius when the radius is 0.5 m.
(b) If the volume of the balloon is increasing at a rate of 10 m^3/min, how fast is the radius increasing when the radius is 3 m?

18. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at rate of 0.8 m/s, how fast does the boat approach the dock when it is 10 m from the dock?





19. A water trough is 6 m long and has a cross-section in the shape of an isosceles trapezoid with dimensions as shown in the diagram. Water is being pumped into the trough at a rate of 0.5 m^3/min. How fast is the water level rising when it is 0.5 m deep?

20. Use Newton's method to find the root of the equation $x^3 = 2x + 5$ correct to six decimal places.

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ANSWERS

CHAPTER 3 APPLICATIONS OF DERIVATIVES

REVIEW AND PREVIEW TO CHAPTER 3

EXERCISE 1

1. $\frac{1}{2}, -\frac{3}{4}$ **2.** 2 **3.** $\frac{800}{13} \doteq 61.5$ km/h

4. 2.4 cm **5.** $\frac{4224}{125} \doteq 33.8$ cm

6. $15 + 3\sqrt{37} \doteq 33.25$ h with Bob's hose, $21 + 3\sqrt{37} \doteq 39.25$ h with Jim's hose

EXERCISE 3.1

1. (a) 0 (b) C (c) The car was speeding up at A and C, slowing down at B. (d) The car is stopped. (e) The car returned to the point at which it started.

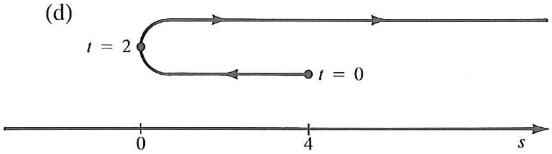
2. (a) $v(t) = 12$, 12 m/s, 12 m/s
(b) $v(t) = 16t - 24$, 8 m/s, 40 m/s
(c) $v(t) = 3t^2 - 12t$, -12 m/s, 0 m/s
(d) $v(t) = \dfrac{5}{(1 + t)^2}$, $\frac{5}{9}$ m/s, $\frac{1}{5}$ m/s

3. -24.8 m/s, -34.6 m/s

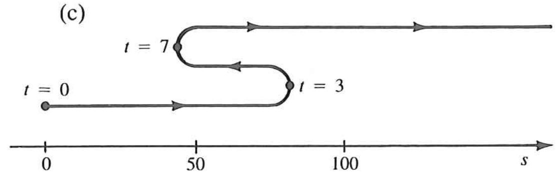
4. (a) 14.7 m/s, 4.9 m/s, -4.9 m/s, -14.7 m/s
(b) 2.5 s (c) 30.6 m (d) at 5 s
(e) -24.5 m/s

5. at 15 min **6.** at 3 s

7. (a) -2 m/s, 2 m/s (b) at 2 s (c) after 2 s
(d)



8. (a) at 3 s and 7 s (b) when $0 \leqslant t < 3$ or $t > 7$
(c)



(d) 194 m

9. (a) at 1 s (b) at $1 + \sqrt{91} \doteq 10.5$ s
(c) -95.4 m/s

EXERCISE 3.2

1. (a) positive (b) negative (c) positive
(d) zero (e) positive

2. (a) velocity increasing, acceleration positive
(b) (i) negative (ii) positive (iii) zero
(iv) negative

3. (a) $v = 30$, $a = 0$ (b) $v = 32t + 5$, $a = 32$
(c) $v = 3t^2 + 10t + 1$, $a = 6t + 10$
(d) $v = \dfrac{2t + 1}{2\sqrt{t^2 + t}}$, $a = \dfrac{-1}{4\sqrt{(t^2 + t)^3}}$

4. (a) -9.8 m/s^2 (b) 22 m/s^2 (c) 20 m/s^2
(d) -0.08 m/s^2

5. (a) s_0 (b) v_0 (c) g **6.** 12 m/s^2

7. (a) at 3 s (b) 0 m, -9 m/s

8. positive when $0 \leqslant t < 1$ or $t > 5$, negative when $1 < t < 5$ **9.** (b) $\frac{5}{3}$ s

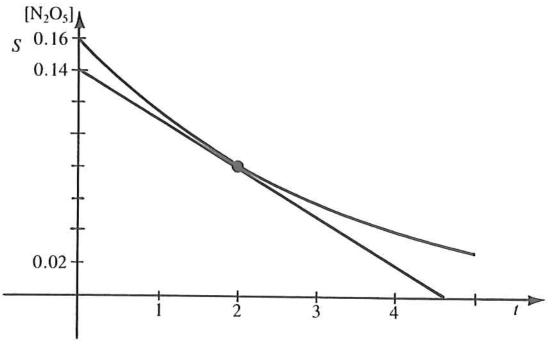
EXERCISE 3.3

1. 48 **2.** 10π cm²/cm **3.** $\frac{250}{9}$ L/min

4. (a) 0.488 kg/m (b) 0.5 kg/m **5.** 7 g/cm

6. 411 bacteria/h **7.** $\frac{1}{40}$ m³/kPa/m³

8. -0.03 moles/L/min



9. -185.2 cm/s/cm

EXERCISE 3.4

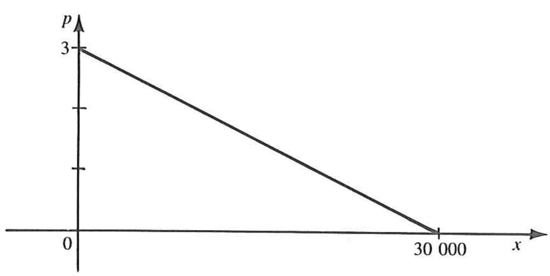
1. (a) $C'(x) = 23 + 0.024\,x$ (b) \$25.40/item
(c) \$25.41

2. (a) $C'(x) = \frac{1}{10} + \dfrac{x}{500}$ (b) \$1.70/unit
(c) \$1.70

3. (a) $R'(x) = 8000 - 0.06x^2$ (b) \$2600/unit
(c) \$2581.92

4. (a) $P(x) = 0.74x - 0.0003x^2 - 23\ 000$
(b) $P'(x) = 0.74 - 0.0006x$ (c) \$0.14/pen
(d) \$0.1397

5. (a)



p	0	\$0.50	\$1.00	\$1.50	\$2.00	\$2.50	\$3.00
x	30 000	25 000	20 000	15 000	10 000	5000	0

(c) $R(x) = \frac{1}{10\ 000}(30\ 000x - x^2)$

(d) $R'(x) = \frac{1}{10\ 000}(30\ 000 - 2x)$ (e) \$2.80

(f) $P(x) = 2.2x - \dfrac{x^2}{10\ 000} - 6000$

(g) $P'(x) = 2.2 - \dfrac{x}{5000}$ (h) \$0.20

6. (a) $C'(x) = 23 + 0.002x$
(b) $R'(x) = 100 - 0.02x$
(c) $P'(x) = 77 - 0.022x$ (d) \$75.90

EXERCISE 3.5

1. -18 **2.** -16 **3.** 4.8 m²/min

4. 3 cm/s **5.** $5000\pi \doteq 15\ 700$ cm²/s

6. $\dfrac{2}{\pi} \doteq 0.64$ m/min **7.** $\dfrac{1}{64\pi} \doteq 0.005$ cm/min

8. $\dfrac{20}{\sqrt{3}} \doteq 11.5$ cm²/s **9.** -1.5 cm/min

10. 1 m/s **11.** $\frac{\sqrt{3}}{10} \doteq 0.17$ m/s

12. 90 km/h **13.** $\dfrac{130}{\sqrt{17}} \doteq 31.5$ km/h

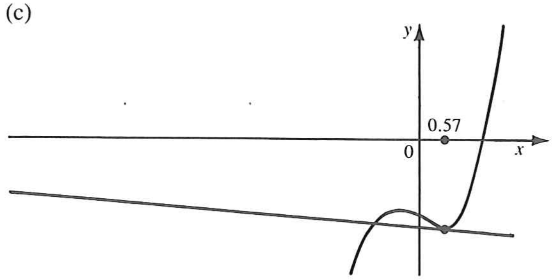
14. $\dfrac{6\sqrt{26}}{13} \doteq 2.35$ m/s **15.** $240\sqrt{5} \doteq 537$ km/h

16. 5 cm/min **17.** $\dfrac{8}{15\pi} \doteq 0.17$ m/min

EXERCISE 3.6

1. $-\frac{1}{2}$ **2.** $-2, -\frac{13}{8}$ **3.** (a) 1.521 380

(c)



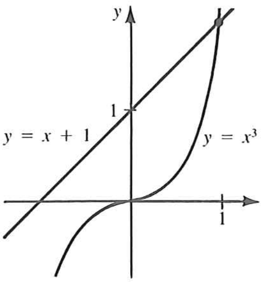
4. (a) 1.556 250 (b) 2.095 366 (c) 1.191 554

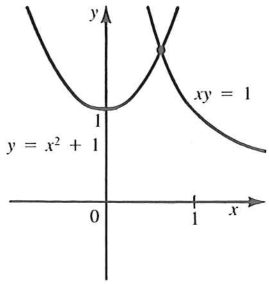
5. (a) $-2.330\ 059$, 0.201 640, 2.128 419
(b) $-0.492\ 689$, 0.508 422, 1.528 643

6. (b) 4.147 288

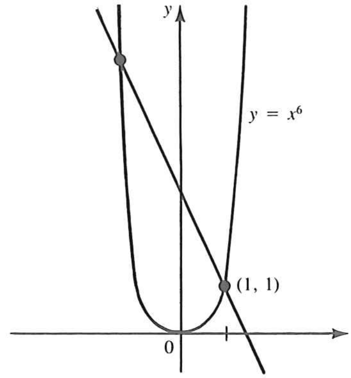
7. (a) 1.947 294 (b) 1.026 214

8. (a) (1.324 718, 2.324 718)
(b) (0.682 328, 1.465 571)





13. (a)



(b) (1, 1), ($-1.335\ 387$, $-5.670\ 774$)

3.8 CHAPTER 3 TEST

1. (a) 9 m/s (b) 12 m/s^2 (c) at 1 s, 3 s
(d) when $0 \leqslant t < 1$ or $t > 3$ (e) -3 m/s
(f) 12 m

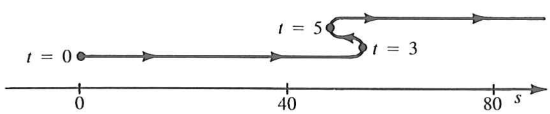
2. (a) $C'(x) = 122$ (b) $R(x) = 600x - 0.001x^2$
(c) $R'(x) = 600 - 0.002x$
(d) $P(x) = -0.001x^2 + 478x - 87\ 000$
(e) $P'(x) = -0.002x + 478$

3. $\dfrac{32}{81\pi} \doteq 0.13$ cm/s **4.** 1.267 168

3.7 REVIEW EXERCISE

1. (a) $v(t) = 6t^2 + 8t - 1$, $a(t) = 12t + 8$
(b) 127 m/s, 56 m/s^2

2. (a) at 3 s and 5 s (b) positive when $t > 5$ or $0 \leqslant t < 3$, negative when $3 < t < 5$
(c) positive when $t > 4$, negative when $0 \leqslant t < 4$
(d) -3 m/s
(e)



(f) 112 m

3. (a) 63.34 m/s (b) -1.66 m/s^2 (c) 78.3 s
(d) -65 m/s

4. 10 **5.** (a) 3.05 kg/m (b) 3 kg/m

6. (a) $C'(x) = 16.2 + 0.12x$ (b) \$40.20/unit
(c) \$40.26/unit

7. (a) $C'(x) = 1.08$ (b) $R(x) = 20x - 0.001x^2$
(c) $R'(x) = 20 - 0.002x$
(d) $P(x) = -0.001x^2 + 18.92x - 12\ 500$
(e) $P'(x) = -0.002x + 18.92$ (f) \$2.92/pizza

8. $\frac{80}{3}$ cm^2/min **9.** 45 cm^3/min

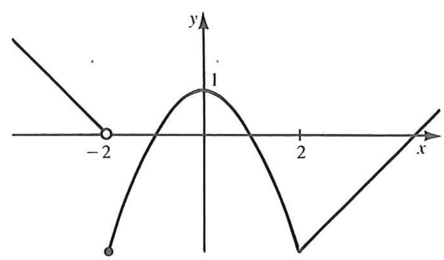
10. $\dfrac{1470}{\sqrt{466}} \doteq 68$ km/h **11.** $\frac{4}{5}$

12. $-0.754\ 878$

CUMULATIVE REVIEW FOR CHAPTERS 1 TO 3

1. (a) $\frac{9}{8}$ (b) $\frac{1}{4}$ (c) 0 (d) 12 (e) $-\frac{2}{27}$ (f) $\frac{5}{3}$
(g) 0 (h) -2

2. (a) (i) 0 (ii) -3 (iii) does not exist (iv) -3
(v) -3 (vi) -3
(b)



(c) -2 (d) -2, 2

3. (a) -2 (b) 0 **4.** (a) divergent (b) 9

5. (a) $6x - 5$ (b) $x - y + 3 = 0$

6. $g'(x) = \dfrac{-1}{2\sqrt{3 - x}}$

7. (a) $f'(x) = 60x^4 - 2x^3 - 4$

(b) $f'(x) = -\dfrac{12}{x^3}$ (c) $g'(x) = \frac{8}{3}x^{\frac{1}{3}} - \frac{2}{3}x^{-\frac{5}{3}}$

(d) $g'(x) = \dfrac{2x(x-3)}{(2x-3)^2}$ (e) $f'(t) = \dfrac{2-3t^2}{2\sqrt{2t-t^3}}$

(f) $f'(y) = -\dfrac{20(2-y)^3}{(1+2y)^5}$

(g) $y' = 3(x^3-1)^2(10x^3+15x^2-1)$

(h) $y' = \dfrac{-x^4}{\sqrt[5]{(x^5+1)^6}}$ (i) $y' = \dfrac{2\sqrt{x}-3}{2\sqrt{x^5}}$

(j) $y' = \dfrac{1-x^2}{2\sqrt{x}\sqrt{(1+x^2)^3}}$

8. $f'(x) = \dfrac{-x}{\sqrt{(x^2-1)^3}}$, domain of f = domain of $f' = \{x | x < -1 \text{ or } x > 1\}$

9. $\dfrac{dy}{dx} = -\dfrac{2x^3+2xy^3}{3x^2y^2+y}$

10. (a) $y' = \dfrac{1}{(x+2)^2}$, $y'' = \dfrac{-2}{(x+2)^3}$

(b) $y' = \dfrac{2x}{3y^2}$, $y'' = \dfrac{6y^3-8x^2}{9y^5}$

11. (a) $x + y - 2 = 0$ (b) $2x - y + 3 = 0$

12. $11x - y + 16 = 0$, $11x - y - 16 = 0$

13. (a) 37 (b) $\frac{43}{16}$ (c) 6

14. (a) $5x^4f'(x^5)$ (b) $5[f(x)]^4f'(x)$

(c) $5x^4f(x) + x^5f'(x)$ (d) $\frac{1}{2}\sqrt{\dfrac{x}{f(x)}}\left(\dfrac{xf'(x) - f(x)}{x^2}\right)$

15. (a) (i) -42.5 m/s (ii) -38.09 m/s (iii) -37.65 m/s (b) -37.6 m/s (c) -9.8 m/s^2

16. (a) -3 m/s, 9 m/s (b) 12 m/s^2

(c) at 1 s, 3 s

(d) positive when $t > 3$ or $0 \leqslant t < 1$, negative when $1 < t < 3$

(e) positive when $t > 2$, negative when $0 \leqslant t < 2$ (f) 28 m

17. (a) π m^3/m (b) $\dfrac{5}{18\pi} \doteq 0.088$ m/min

18. $\dfrac{2\sqrt{101}}{25} \doteq 0.8$ m/s

19. $\frac{5}{76}$ m/min $\doteq$ 6.6 cm/min **20.** 2.094 551