Elements of Statistical Learning, Solutions

Marios

Exercises for Section 2

1. Suppose each of K-classes has an associated target t_k, which is a vector of all zeros, except a one in the kth position. Show that classifying to the largest element of $\hat{y}$ amounts to choosing the closest target, $\min_k \|t_k - \hat{y}\|$, if the elements of $\hat{y}$ sum to one.

   Solution. Let $k^* = \arg\max_k \hat{y}_k$ and suppose that there is $k' \leq k^*$ such that $\|t_{k'} - \hat{y}\| < \|t_{k^*} - \hat{y}\|$.

   - ℓ_1 norm. It holds that $\|t_k - \hat{y}\|_1 = \sum_i |t_{k,i} - \hat{y}_i| = \sum_{i \neq k} |\hat{y}_i| + |1 - \hat{y}_k|$. Hence, we get

   $$\sum_{i \neq k'} |\hat{y}_i| + |1 - \hat{y}_{k'}| < \sum_{i \neq k^*} |\hat{y}_i| + |1 - \hat{y}_{k^*}| \Rightarrow |\hat{y}_{k^*}| - |1 - \hat{y}_{k^*}| < |\hat{y}_{k'}| - |1 - \hat{y}_{k'}|. \tag{1}$$

   But the function $f(y) = |y| - |1-y|$ is increasing in $[0, 1]$ hence Equation (1) implies that $\hat{y}_{k^*} < \hat{y}_{k'}$, reaching a contradiction.

   - ℓ_2 norm. Similarly, we get that $\hat{y}_{k^*}(1 - \hat{y}_{k^*}) < \hat{y}_{k'}(1 - \hat{y}_{k'})$ and since the function $f(y) = y(1-y)$ is increasing in $[0, 1]$, we get that $\hat{y}_{k^*} < \hat{y}_{k'}$, reaching a contradiction.

2. Show how to compute the Bayes decision boundary for the simulation example in Figure 2.5.

   Solution. If we know the exact probability distribution $\Pr[G, X]$, $X \in \mathbb{R}^p$, $G \in \mathcal{G} = \{B, O\}$, then we can probably also derive $f(X) = \Pr[B|X] = \Pr[B, X]/\Pr[X]$, namely the probability that X maps to blue in reality. This assume that we also know $\Pr[X]$ which is not necessary. Of course, $\Pr[O|X] = 1 - \Pr[B|X]$. So now, all we have to do is to check for each $x \in \mathbb{R}^p$, whether $f(x) > 1/2$. For the case where $x \in \mathbb{R}$, this is trivial. We simply solve the equation $f(x) = 1/2$. This also hold in general. So the points (in $\mathbb{R}$), the line (in $\mathbb{R}^2$), and the $(p-1)$-dimensional hyperplane (in $\mathbb{R}^p$), is the solution to the equation $f(x) = \Pr[B|X] = 1/2$. See Figure 2 for another example.

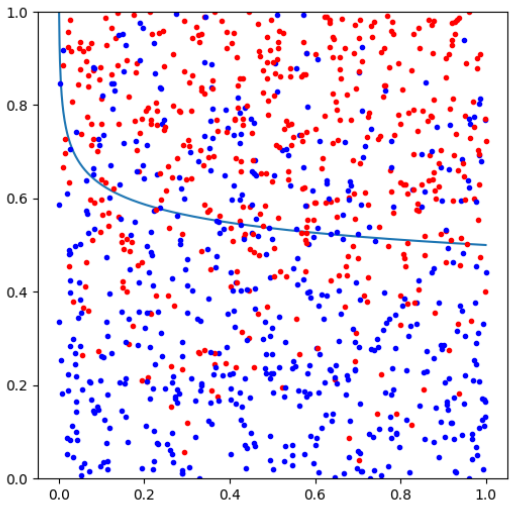

Figure 1: In this example we have computed the Bayes decision boundary when $X \sim U(0,1)^2$ and $\Pr[Y = \text{red}|X] = X_1^{1/10} X_2$. Therefore, the line is the solution to the equation $X_1^{1/10} X_2 = 1/2$.

3. Derive equation 2.24. Consider N data points uniformly sampled in a p-dimensional unit ball centered at the origin. Show that the median distance from the origin to the closest data point is given by the expression

$$d(p, N) = \left(1 - \frac{1}{2}^{1/N}\right)^{1/p}.$$

Solution. We start with the cumulative distribution function (CDF) of the distance of a random point from the origin. The volume of a p-dimensional ball of radius d is $V_p(d) = c_p d^p$, where c_p is a value that does not depend on d. Therefore,

$$F_D(d) = \Pr[D \leq d] = \frac{V_p(d)}{V_p(1)} = d^p. \qquad \text{(First trick to remember)}$$

Now it is useful to compute the CDF of the distance of the closest point $C = \min_{i\in[N]} D_i$. We have that

$$\begin{aligned} F_C(d) &= \Pr[C \leq d] \\ &= 1 - \Pr[C \geq d] \\ &= 1 - \Pr\left[\min_{i\in[N]} D_i \geq d\right] \\ &= 1 - \Pr[\forall i \in [N], D_i \geq d] \\ &= 1 - \prod_{i\in[N]} \Pr[D_i \geq d] \\ &= 1 - \Pr[D \geq d]^N \\ &= 1 - (1 - \Pr[D \leq d])^N \\ &= 1 - (1 - d^p)^N. \end{aligned} \tag{2}$$

By definition, the median m is defined as $F_C(m) = 1/2$. Hence, we get that $(1 - m^p)^N = 1/2$ and solving for m, we get

$$m = \left(1 - \frac{1}{2}^{1/N}\right)^{1/p}.$$

4. Consider inputs drawn from a spherical multinormal distribution $X \sim N(0, \mathbf{I}_p)$. The squared distance from any sample point to the origin has a χ^2_p distribution with mean p. Consider a prediction point x_0 drawn from this distribution, and let $a = x_0/\|x_0\|$ be an associated unit vector. Let $z_i = a^T x_i$ be the projection of each of the training points on this direction.

   Show that the z_i are distributed according to $N(0, 1)$ with expected squared distance from the origin 1, while the target point has expected squared distance p from the origin.

   Solution. We use the fact that for any $a \in \mathbb{R}^p$, if $x \sim N(0, \mathbf{I}_p)$, then $a^T x \sim N\left(\sum_j a_j\mu_j, \sum_j a_j^2\sigma_j^2\right)$, where $\mu_j = E(x_j)$ and $\sigma_j = V(x_j)$ and $j \in [p]$. Since $\sigma_j = 1$ and $\mu_j = 0$, we get that $a^T x \sim N\left(0, \sum_j a_j^2\right)$. Given that $\|a\|$ is a unit vector, we get that $a^T x \sim N(0, 1)$. Hence $|z| = |a^T x| \sim \chi^2_1$ and $E[|z|] = 1$.

5. Suppose that we know that the true relationship between Y and X is linear,

$$Y = X^T\beta + \varepsilon, \tag{2.26}$$

   where $\varepsilon \sim N(0, \sigma^2)$ and we fit the model by least squares to the training data. For an arbitrary test point x_0, we have $\hat{y}_0 = x_0^T\hat{\beta}$, which can be written as $\hat{y}_0 = x_0^T\beta + \sum_{i-1}^N \ell_i(x_0)\varepsilon_i$, where $\ell_i(x_0)$ is the ith element of $\mathbf{X}(\mathbf{X}^T\mathbf{X})^{-1}x_0$. Show that

$$\text{EPE}(x_0) = \sigma^2 + E_{\mathcal{T}} x_0^T(\mathbf{X}^T\mathbf{X})^{-1}x_0\sigma^2 + 0^2, \tag{2.27}$$

where you can use the fact that for any $\mathbf{X}$,

$$\mathrm{Cov}[\hat{\beta}] = (\mathbf{X}^T\mathbf{X})^{-1}\sigma^2. \tag{3.8}$$

Additionally, suppose N is large and $\mathcal{T}$ were selected at random. Assuming $E(X) = 0$, then $\mathbf{X}^T\mathbf{X} \to N\,\mathrm{Cov}(X)$. Show that

$$\begin{aligned} E_{x_0}\,\mathrm{EPE}(x_0) &\approx E_{x_0} x_0^T\,\mathrm{Cov}[X]^{-1} x_0\sigma^2/N + \sigma^2 \\ &= \mathrm{trace}[\mathrm{Cov}[X]^{-1}\,\mathrm{Cov}[x_0]]\sigma^2/N + \sigma^2 \\ &= \sigma^2(p/N) + \sigma^2. \end{aligned} \tag{2.28}$$

Make use of the cyclic property of the trace operator ($\mathrm{trace}(AB) = \mathrm{trace}(BA)$), and its linearity (which allows us to interchange the order of trace and expectation).

Solution. In the first question, the test point x_0 is arbitrary and not sampled from the distribution. Thus the randomness is only over:

- the samples $\mathcal{T}$,
- the error ε.

In the second part, we also sample x_0 and hence we consider the expectation of $\mathrm{EPE}(x_0)$.

(a) We start by showing that the expected prediction error equals the sum of the variance of the system, the variance of the model and the squared bias of the model:

$$\begin{aligned} \mathrm{EPE}(x_0) &= E_{\mathcal{T},\varepsilon}[(y_0 - \hat{y}_0)^2] \\ &= E_{\mathcal{T},\varepsilon}[y_0^2 - 2y_0\hat{y}_0 + \hat{y}_0^2] \\ &= E_{\mathcal{T},\varepsilon}[y_0^2] - 2E_{\mathcal{T},\varepsilon}[y_0\hat{y}_0] + E_{\mathcal{T},\varepsilon}[\hat{y}_0^2] \\ &= E_{\mathcal{T},\varepsilon}[y_0^2] - 2x_0^T\beta E_{\mathcal{T},\varepsilon}[\hat{y}_0] + E_{\mathcal{T},\varepsilon}[\hat{y}_0^2] \\ &= E_{\mathcal{T},\varepsilon}[y_0^2]\boxed{-E_{\mathcal{T},\varepsilon}[y_0]^2 + E_{\mathcal{T},\varepsilon}[y_0]^2} - 2x_0^T\beta E_{\mathcal{T},\varepsilon}[\hat{y}_0] + E_{\mathcal{T},\varepsilon}[\hat{y}_0^2] - \boxed{E_{\mathcal{T},\varepsilon}[\hat{y}_0]^2 + E_{\mathcal{T},\varepsilon}[\hat{y}_0]^2} \\ &= \mathrm{Var}[y_0] + \mathrm{Var}[\hat{y}_0] + \left(E_{\mathcal{T},\varepsilon}[\hat{y}_0] - x_0^T\beta\right)^2, \end{aligned}$$

where the third line follows from the fact that

$$E_{\mathcal{T},\varepsilon}[y_0\hat{y}_0] = E_{\mathcal{T},\varepsilon}[(x_0^T\beta + \varepsilon)\hat{y}_0] = x_0^T\beta E_{\mathcal{T},\varepsilon}[\hat{y}_0] + E_{\mathcal{T},\varepsilon}[\varepsilon\hat{y}_0]$$

and $E_{\mathcal{T},\varepsilon}[\varepsilon\hat{y}_0] = E_{\mathcal{T},\varepsilon}[\varepsilon]E_{\mathcal{T},\varepsilon}[\hat{y}_0] = 0$, since ε is independent from $\hat{y}_0$. Now, we have that $\mathrm{Var}[y_0] = \sigma^2$. Moreover,

$$\begin{aligned} E_{\mathcal{T},\varepsilon}[\hat{y}_0] &= E_{\mathcal{T},\varepsilon}\left[x_0^T\beta + \sum_{i-1}^{N}\ell_i(x_0)\varepsilon_i\right] \\ &= x_0^T\beta + \sum_{i-1}^{N} E_{\mathcal{T},\varepsilon}[\ell_i(x_0)\varepsilon_i] \\ &= x_0^T\beta + \sum_{i-1}^{N} E_{\mathcal{T},\varepsilon}[\ell_i(x_0)]E_{\mathcal{T},\varepsilon}[\varepsilon_i] \\ &= x_0^T\beta, \end{aligned} \tag{3}$$

since ε_i is independent of x_0 and $\mathbf{X}$. Hence, $\mathrm{Bias}(\hat{y}_0) = \left(E_{\mathcal{T},\varepsilon}[\hat{y}_0] - x_0^T\beta\right) = 0$. Last, we want to calculate the variance of our prediction $\mathrm{Var}[\hat{y}_0]$. We have

$$\begin{aligned} \mathrm{Var}[\hat{y}_0] &= \mathrm{Var}[x_0^T\hat{\beta}] \\ &= x_0^T\,\mathrm{Cov}[\hat{\beta}]x_0 \\ &= x_0^T(\mathbf{X}^T\mathbf{X})^{-1}\sigma^2 x_0, \end{aligned} \tag{4}$$

where the last line comes from eq. (3.8).

(b) Now we have that

$$\begin{aligned} E_{x_0}\,\mathrm{EPE}(x_0) &\approx E_{x_0}[x_0^T\,\mathrm{Cov}[X]^{-1}x_0]\sigma^2/N+\sigma^2 \\ &= E_{x_0}[\mathrm{trace}[x_0^T\,\mathrm{Cov}[X]^{-1}x_0]]\sigma^2/N+\sigma^2 \\ &= E_{x_0}[\mathrm{trace}[\mathrm{Cov}[X]^{-1}x_0x_0^T]]\sigma^2/N+\sigma^2 \\ &= \mathrm{trace}[E_{x_0}[\mathrm{Cov}[X]^{-1}x_0x_0^T]]\sigma^2/N+\sigma^2 \\ &= \mathrm{trace}[\mathrm{Cov}[X]^{-1}E_{x_0}[x_0x_0^T]]\sigma^2/N+\sigma^2 \\ &= \mathrm{trace}[\mathrm{Cov}[X]^{-1}\,\mathrm{Cov}[x_0]]\sigma^2/N+\sigma^2 \\ &= \mathrm{trace}[\mathbf{I}_p]\sigma^2/N+\sigma^2 \\ &= p\sigma^2/N+\sigma^2 \end{aligned} \tag{5}$$

We see that the function $f(p) = E_{x_0}\,\mathrm{EPE}(x_0) = \sigma^2(p/N)+\sigma^2$ increases linearly with p, with slope $f'(p) = \sigma^2/N$. Hence as long as we have sufficiently many samples N this increase becomes negligible. In other words, even if we have a lot of dimensions, the expected EPE remains constant. Of course, the reason is that we imposed heavy restrictions on the class of models being fitted.

6. Consider a regression problem with inputs x_i and outputs y_i, and a parameterized model $f_\theta(x)$ to be fit by least squares. Show that if there are observations with *tied* or *identical* values of x, then the fit can be obtained from a reduced weighted least squares problem.

Solution. The weighted least squares problem is defined as the problem of minimizing the value

$$\mathrm{WRSS}_{(x_i,y_i,w_i)}(\beta) = \sum_{i=1}^{N} w_i(y_i - \hat{f}_\beta(x_i))^2.$$

It generasizes RSS since by setting $w_i = 1$ we get the RSS. The idea behind WRSS is that some pairs (x_i, y_i) may have errors and some may be more accurate. By giving them weights, we reward the more accurate ones and we penalize the less accurate.

A reduced least squares problem is one that uses fewer observations than available; $N' < N$.

Suppose that we have N observations (x_i, y_i) with some of them sharing the same x_i. Suppose we have N' distinct x_is and for each distinct x_i, we have N_i observations, so $N = \sum_{i=1}^{N'} N_i$. We wish to compute the value $\arg\min_\beta \mathrm{RSS}(\beta)$. We will show that

$$\arg\min_\beta \mathrm{RSS}_{(x_i,y_i)}(\beta) = \arg\min_\beta \mathrm{WRSS}_{(x_i,\overline{y}_i,N_i)}(\beta).$$

We have

$$\begin{aligned}
\arg\min_{\beta} \mathrm{RSS}_{(x_i,y_i)}(\beta) &= \arg\min_{\beta} \sum_{i=1}^{N} (y_i - \hat{f}_\beta(x_i))^2 \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} \sum_{j=1}^{N_i} (y_i - \hat{f}_\beta(x_i))^2 \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} \sum_{j=1}^{N_i} (y_i^2 - 2y_i \hat{f}_\beta(x_i) + \hat{f}_\beta^2(x_i)) \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} \sum_{j=1}^{N_i} (-2y_i \hat{f}_\beta(x_i) + \hat{f}_\beta^2(x_i)) \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} (-2N_i \overline{y}_i \hat{f}_\beta(x_i) + N_i \hat{f}_\beta^2(x_i)) \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} N_i (-2\overline{y}_i \hat{f}_\beta(x_i) + \hat{f}_\beta^2(x_i)) \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} N_i (\overline{y}_i^2 - 2\overline{y}_i \hat{f}_\beta(x_i) + \hat{f}_\beta^2(x_i)) \\
&= \arg\min_{\beta} \sum_{i=1}^{N'} N_i (\overline{y}_i - \hat{f}_\beta(x_i))^2 \\
&= \arg\min_{\beta} \mathrm{WRSS}_{(x_i,\overline{y}_i,N_i)}(\beta),
\end{aligned} \tag{6}$$

where the fourth and the seventh lines come from the fact that for any function f and $c \in \mathbb{R}$, it holds that $\arg\min_\beta (f(\beta) + c) = \arg\min_\beta f(\beta)$.

7. Suppose we have a sample of N pairs x_i, y_i drawn i.i.d from the distribution characterized as follows:

   $x_i \sim h(x)$, the design density
   $y_i = f(x_i) + \varepsilon_i$, f is the regression function
   $\varepsilon_i \sim (0, \sigma^2)$ (mean zero, variance σ^2)

   We construct an estimator for f *linear* in the y_i,

   $$\hat{f}(x_0) = \sum_{i=1}^{N} \ell_i(x_0; \mathcal{X}) y_i,$$

   where the weights $\ell_i(x_0; \mathcal{X})$ do not depend on the y_i, but do depend on the entire training sequence of x_i, denoted here by $\mathcal{X}$.

   (a) Show that the linear regression and k-nearest-neighbor regression are members of this class of estimators. Describe explicitly the weights $\ell_i(x_0; \mathcal{X})$ in each of these cases.

   (b) Decompose the conditional mean-squared error

   $$E_{\mathcal{Y}|\mathcal{X}}(f(x_0) - \hat{f}(x_0))^2$$

   into a conditional squared bias and a conditional variance component. Like $\mathcal{X}$, $\mathcal{Y}$ represents the entire training sequence of y_i.

   (c) Decompose the mean-squared error $E_{\mathcal{X},\mathcal{Y}}(f(x_0) - \hat{f}(x_0))^2$ into a squared bias and a variance component.

(d) Establish a relationship between the squared biases and variances in the above two cases.

Solution. (a) For linear regression, we have that $\hat{f}(x_0) = x^T(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}y = \sum_{i=1}^N \ell_i(x_0;\mathcal{X})y_i$, where $\ell_i(x_0;\mathcal{X})$ is the ith element of the vector $x^T(\mathcal{X}^T\mathcal{X})^{-1}\mathcal{X}$. For the k-nearest neighbor, we have that $\ell_i(x_0;\mathcal{X}) = \frac{1}{k}I(x_i \in N_k(x_0,\mathcal{X}))$, where $N_k(x_0,\mathcal{X})$ is the set of the k closest points to x_0.

(b) We have that

$$\begin{aligned} E_{\mathcal{Y}/\mathcal{X}}[(f(x_0) - \hat{f}(x_0))^2] &= E_{\mathcal{Y}/\mathcal{X}}[f^2(x_0) - 2f(x_0)\hat{f}(x_0) + \hat{f}^2(x_0)] \\ &= \mathrm{Var}_{\mathcal{Y}/\mathcal{X}}(f(x_0)) + \mathrm{Var}_{\mathcal{Y}/\mathcal{X}}(\hat{f}(x_0)) + E_{\mathcal{Y}/\mathcal{X}}[(f(x_0) - \hat{f}(x_0))]^2 \\ &= \mathrm{Var}_{\mathcal{Y}/\mathcal{X}}(f(x_0)) + \mathrm{Var}_{\mathcal{Y}/\mathcal{X}}(\hat{f}(x_0)) + \mathrm{Bias}_{\mathcal{Y}/\mathcal{X}}[\hat{f}(x_0)]^2 \\ &= \mathrm{Var}_{\mathcal{Y}/\mathcal{X}}(\hat{f}(x_0)) + \mathrm{Bias}_{\mathcal{Y}/\mathcal{X}}[\hat{f}(x_0)]^2 \end{aligned} \tag{7}$$

since $f(x_0)$ does not have any randomness.

(c) Similarly,

$$E_{\mathcal{Y},\mathcal{X}}[(f(x_0) - \hat{f}(x_0))^2] = \mathrm{Var}_{\mathcal{Y},\mathcal{X}}(\hat{f}(x_0)) + \mathrm{Bias}_{\mathcal{Y},\mathcal{X}}[\hat{f}(x_0)]^2. \tag{8}$$

(d) We have that

$$\begin{aligned} \mathrm{Var}_{\mathcal{Y},\mathcal{X}}(\hat{f}(x_0)) &= E_{\mathcal{X}\sim h}[\mathrm{Var}_{\mathcal{Y},\mathcal{X}}(\hat{f}(x_0))] \\ \mathrm{Bias}_{\mathcal{Y},\mathcal{X}}(\hat{f}(x_0)) &= E_{\mathcal{X}\sim h}[\mathrm{Bias}_{\mathcal{Y},\mathcal{X}}(\hat{f}(x_0))] \end{aligned}$$

8. Compare the classification performance of linear regression and k-nearest neighbor classification on the `zipcode` data. In particular, consider only the 2's and 3's and $k = 1, 3, 5, 7, 15$. Show both the training and test error for each choice. The `zipcode` data are available from the book website `https://hastie.su.domains/ElemStatLearn/`.

Solution. We use the `sklearn` library of Python.

```
Linear Regression
Training Error
2.48%
Testing Error
15.17%
k-nearest neighbors classifier
k = 1
Training Error
0.00%
Testing Error
2.47%
k = 3
Training Error
0.50%
Testing Error
3.02%
k = 5
Training Error
0.58%
Testing Error
3.02%
k = 7
Training Error
0.65%
Testing Error
3.30%
k = 15
Training Error
0.94%
Testing Error
3.85%
```

9. Consider a linear regression model with p parameters, fit by least squares to a set of training data $(x_1, y_1), \ldots, (x_N, y_N)$ drawn at random from a population. Let $\hat{\beta}$ be the least squares estimate. Suppose we have some test data $(\tilde{x}_1, \tilde{y}_1), \ldots, (\tilde{x}_M, \tilde{y}_M)$ drawn at random from the same population as the training data. If $R_{\mathsf{tr}}(\beta) = \frac{1}{N}\sum_{i=1}^{N}(y_i - \beta^T x_i)^2$ and $R_{\mathsf{te}}(\beta) = \frac{1}{M}\sum_{i=1}^{M}(\tilde{y}_i - \beta^T \tilde{x}_i)^2$, prove that

$$E\left[R_{\mathsf{tr}}(\hat{\beta})\right] \leq E\left[R_{\mathsf{te}}(\hat{\beta})\right],$$

where the expectations are over all that is random in each expression (meaning the population).

Solution. We will prove a more general result. Let S be the training set and T be the testing set. Moreover, let $f(S, \beta) = E_i[f(S_i, \beta)]$ be the function we want to minimize (in our case it will be the RSS or its normalized version, the mean squared error MSE). Observe that for any β,

$$\begin{aligned} E_S[f(S,\beta)] &= E_S E_i[f(S_i,\beta)] \\ &= E_i E_S[f(S_i,\beta)] \\ &= E_i E_S[f(S_1,\beta)] \\ &= E_S[f(S_1,\beta)], \end{aligned} \tag{9}$$

since all S_i are i.i.d.. Let $\beta_S = \arg\min_\beta f(S,\beta)$ and observe that for any β, it holds that $f(S,\beta_S) \leq f(S,\beta)$. Let T' be the set T, truncated or augmented by sampling more data to match the size of S. We have

$$\begin{aligned} E_{S,T}[f(T,\beta_S)] &= E_S E_{T/S}[f(T,\beta_S)] \\ &\geq E_S E_{T/S}[f(T,\beta_{T'})] \\ &= E_S E_T[f(T,\beta_{T'})] \\ &= E_T[f(T,\beta_{T'})] \\ &= E_T[f(T_1,\beta_{T'})] \\ &= E_{T'}[f(T_1,\beta_{T'})] \\ &= E_S[f(S_1,\beta_S)] \\ &= E_S[f(S,\beta_S)] \end{aligned} \tag{10}$$

where the second line comes from the above inequality and the 5th and the 8th lines come from Equation (9).

For our case, S is the training set and T is the testing set. Moreover, we have that

$$\arg\min_\beta(\mathrm{RSS}(\beta)) = \arg\min_\beta(\frac{1}{N}\mathrm{RSS}(\beta)) = \arg\min_\beta(\mathrm{MSE}(\beta)).$$

Hence it is enough to consider the function $f(S,\beta) = \mathrm{MSE}(\beta)$, $\mathsf{tr} = S$, $\mathsf{te} = T$ and $\hat{\beta} = \beta_S$.

We see that this inequality is illustrated in Exercise 8.