Cinc Cèntims de Models

Models Matemàtics i Sistemes Dinàmics

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$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 1 & \text{if } n = 1 \\ f(n-1) + f(n-2) & \text{if } n \geq 2 \end{cases}$$

Amb 51 pàgines, aquest cop *C*inc *C*èntims és un dir...

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I

Discrete dynamical systems

A Dynamical System consists of:

1. A phase space: set of possible states;
2. A time set: $\mathbb{N}$ or $\mathbb{Z}$ (discrete) or $\mathbb{R}$ (continuous).
3. An *evolution law*: a rule which determines the present state in terms of the previous ones.

Example 1.1 (A discrete DS)**.** Take the population of tropical butterflies and its annual reporudction cycle. No interaction between generation.

1. *Time set:* $\mathbb{N}$ ($n = 0, 1, 2, \ldots$ years).
2. *Phase space*: $\mathbb{R}$ ($x_n = \{$hundreds of butterflies at year $n\}$).
3. *Evolution law*: $x_{n+1} = 2x_n$.

It was a model suggested by Malthus in 1798 for human population with no control. It's easy to see that the explicit solution is $x_n = 2^n x_0$ (that is, an exponential grow).

Example 1.2 (A continuous DS)**.** Thomas Austin in 1860 introduced 24 rabbits in an Australian region, with disastrous consequences. In 10 years, they had spread all around and, in 20 years, they had devastated all the land. A prize was offered for anybody who could find a solution.

1. *Time set:* $\mathbb{R}$ (rabbits reproduce continuously).
2. *Phase space*: $\mathbb{R}$ ($x(t) = \{$millions of rabbits at time $t\}$).
3. *Evolution law*, assuming no predators and no food restriction: let Δt the increment of time. Then, $x(t + \Delta t) - x(t) \sim x(t)\Delta t$ and take the limit when $\Delta t \to 0$:
$$\frac{dx(t)}{dt} = kx(t).$$
Taking $x(0) = x_0$ the initial condition, we can observe that $x(t) = x_0 e^{kt}$ is an explicit solution, because:
$$x'(t) = x_0 k e^{kt} = k x_0 e^{kt} = kx(t).$$

It was a model suggested by Malthus in 1798 for human population with no control. It's easy to see that the explicit solution is $x_n = 2^n x_0$ (that is, an exponential grow).

Remark 1.3. A MODEL is an idealization of reality. The goal is to capture the important properties of the phenomenon under study, with enough precision to be able to make predictions for the future. Making a model is an art: If it is too accurate, very difficult to study, if it is too simple, results do not match reality.

Once the model is made, we can study it in several ways.

1. *Quantitatively*: Trying to find explicit solutions. (Ex: $x_n = a^n x_0$). This is rarely possible!

2. *Qualitatively*: Describing orbits without finding an explicit formula (for example, $\forall x_0 > 0$, $a > 1$, $\lim_n x_n = +\infty$). More often possible, but not always.

Example 1.4 (Newton's method). In many cases, the equation $f(x) = 0$ cannot be solved explicitly. Newton's method can be seen as a dynamical system, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, which finds approximations to the roots of f (x_0 close to the root).

Example 1.5 (Square root approximation). Compute $(x, y) \longmapsto (\frac{x+y}{2}, \frac{2A}{x+y})$, this is actually a system of one variable: $(x, y) \longmapsto (\frac{x^2+A}{2x}, \frac{A}{x})$.

2 One-dimensional maps

A map $f : \mathbb{R} \longrightarrow \mathbb{R}$, or f:$I \longrightarrow I$ with $I \subset \mathbb{R}$, determines the evolution law of a discrete dynamical system. We write $x_{n+1} = f(x_n)$.

2.1 Orbits

Definition 2.1 (Orbit). Given an initial condition x_0, its orbit is the sequence:

$$O(x_0) = \{x_0, x_1 = f(x_0), \ldots, x_n = f(x_{n-1}) = f^2(x_{n-2}) = \cdots = f^n(x_0)\}.$$

Definition 2.2 (Periodic orbits). A periodic point of period $p \geq 1$ is an x_0 such that $f^p(x_0) = x_0$, but $f^k(x_0) \neq x_0$, $0 < k < p$. All points in the periodic orbit $\{x_0, x_1, \ldots, x_{p-1}\}$ satisfy $f^p(x_i) = x_i$, $0 \leq i < p$.

With graphical analysis and other tools, in some cases (few) the behaviour of all orbits in the system can be understood. The different orbits define a partition of the phase space.

2.2 Phase portrait and sequences

Definition 2.3 (Phase portrait). The phase portrait is partition of the phase space into orbits.

Definition 2.4 (Limit). A sequence $(z_n)_n$ has a limit $p \in \mathbb{R}^N$ (or converges to p) if $\forall \varepsilon > 0$ exists $N_0 > 0$ such that $\forall n \geq N_0$, $\|z_n - p\| < \varepsilon$. We write $\lim_n x_n = p$ or $z_n \xrightarrow{n\to\infty} p$.

Definition 2.5 (Cauchy sequence). A sequence $(z_n)_n$ is called a Cauchy sequence if $\forall \varepsilon > 0$ exists $N_0 > 0$ such that for all $n, m \geq N_0$, $\|z_n - z_m\| < \varepsilon$.

Because $\mathbb{R}^N$ is a complete space (metric space that satisfies that a sequence is a Cauchy sequence if an only if it converges):

Theorem 2.6. *In $\mathbb{R}^N$, a sequence is a Cauchy sequence if an only if it converges.*

Proposition 2.7 (Continuity criterion for sequences). *$g : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is continuous at $a \in \mathbb{R}^n$ if and only if for all $(z_n)_n$ sequences that satisfy $z_n \xrightarrow{n\to\infty} a$, $g(z_n) \xrightarrow{n\to\infty} g(a)$. In this case, we write $\lim_n g(z_n) = g(\lim_n z_n)$, which means if g is continuous, g commutes with the operation of taking limit.*

Property 2.8. *Let $(x_n)_n$ be a sequence in $\mathbb{R}$. If $(x_n)_n$ is decreasing and bounded below, then $(x_n)_n$ converges. If $(x_n)_n$ is increasing and bounded above, then $(x_n)_n$ converges.*

2.3 Asymptotic behaviour and stability of orbits

Proposition 2.9. *If f is continuous and an orbit $(x_n)_n$ satisfies $x_n \xrightarrow{n\to\infty} p$, then p is a fixed point.*

Proof. Observe that $x_{n+1} \xrightarrow{n\to\infty} p$. From $x_{n+1} = f(x_n)$ and taking limits as $n \to \infty$, we obtain that $\lim_n f(x_n) = f(\lim_n x_n) = f(p)$; using that $\lim_n x_n = p$ too, then $p = f(p)$, as we wanted to prove. ∎

Some fixed points "attract" nearby orbits, others "repel" them, and others do neither.

Definition 2.10 (Attractor). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, p fixed point. We say that p is an attracting fixed point if points close enough to p have orbits tending to p, i.e:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ st. } \forall x \mid |x - p| < \delta \implies |f^n(x) - f(p)| = |f^n(x) - p| < \varepsilon \equiv f^n(x) \xrightarrow{n\to\infty} p.$$

Definition 2.11 (Repellor). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ continuous, p fixed point. We say that p is a repelling fixed point if the orbits of nearby points get away from p, i.e:

$$\exists \varepsilon > 0, \forall \delta > 0 \text{ st. } \forall x \mid |x - p| < \delta \implies |f^n(x) - p| \geq \varepsilon \equiv f^n(x) \not\to p,\ n \to \infty.$$

Sometimes, this is equivalent to p is attracting for f^{-1}.

Now let's study local stability (i.e. in a neighbourhood of a fixed point). Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ differentiable and $x_0 = f(x_0)$ a fixed point. Near x_0, the function is close to the tangent line and therefore we are led to the study of linear maps.

Example 2.12. Let $f(x) = ax$, so $f^n(x) = a^n x$. The fixed points are $x = ax \iff (a-1)x = 0$: so $x = 0$ is the only fixed point except when $a = 1$, when all $x \in \mathbb{R}$ is fixed.

- For $a \in (0, 1)$, we approach $x = 0$ with rate a, *monotonically*: take $x_0 > 0$ and $x_0 > ax_0 > a^2x_0 > \cdots$ (similarly, when $x_0 < 0$, $x_0 < ax_0 < a^2x_0 < \cdots$). For $a \in (-1, 0)$, $x = 0$ is also attracting, but it does so oscillating (because a^n sign changes depending on the parity of n). All in all, when $a \in (-1, 0) \cup (0, 1)$, $x = 0$ is an attracting fixed point.
- For $a > 1$ we are getting away with rate a, monotonically. For $a < -1$, $x = 0$ is also a repelling fixed point, but now oscillating for the same reason as before.

- When $a = -1$, all points are 2-periodic (periodic with period two) except $x = 0$.

Theorem 2.13 (Stability of fixed points)**.** *Let $I \subset \mathbb{R}$, $f : I \longrightarrow \mathbb{R}$ differentiable, $x_0 = f(x_0) \in I$ fixed point.*

1. *If $|f'(x_0)| < 1$ then x_0 is attracting.*
2. *If $|f'(x_0)| > 1$ then x_0 is repelling.*

Proof.

1. Let λ be such that $|f'(x_0)| < \lambda < 1$. We know $\lim_{x \to x_0} \frac{|f(x)-f(x_0)|}{|x-x_0|} = |f'(x_0)|$. Hence, $\exists \varepsilon > 0$ such that if $x \in (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}$, then $\frac{|f(x)-f(x_0)|}{|x-x_0|} < \lambda$. In other words, $|f(x) - f(x_0)| \leq \lambda |x - x_0|$ for all $x \in (x_0 - \varepsilon, x_0 + \varepsilon)$. But $f(x) - f(x_0) = f(x) - x_0$ and $\lambda|x - x_0| < |x - x_0| < \varepsilon$ and $f(x) \in (x_0 - \varepsilon, x_0 + \varepsilon)$. Then, $|f^2(x) - x_0| \leq \lambda |f(x) - x_0| \leq \lambda^2 |x - x_0|$. We can iterate this procedure:
$$|f^n(x) - x_0| \leq \lambda |f^{n-1}(x) - x_0| \leq \cdots \leq \lambda^n |x - x_0| \xrightarrow{n \to \infty} 0.$$
2. Assuming that f is C^1, there exists a local inverse of f. Computing $(f^{-1})'(x_0)$:
$$|f^{-1}|'(x_0) = \frac{1}{|f'(x_0)|} < 1 \implies x_0 \text{ is attracting for } f^{-1}.$$
We have used that $|f'(x_0)| > 1$, so clearly $\frac{1}{|f'(x_0)|} < 1$. ■

Definition 2.14 (Superattracting)**.** We say that a fixed point x_0 is superattracting when $f'(x_0) = 0$.

In many cases, attracting points *attract* points other than those in $(x_0 - \varepsilon, x_0 + \varepsilon)$.

Definition 2.15 (Basin of attraction)**.** The basin of attraction of a fixed point x_0 is the set $A(x_0) = \{x \in \mathbb{R}^2 \mid f^n(x) \xrightarrow{n \to \infty} x_0\}$.

Definition 2.16 (Neutral fixed points)**.** Let $I \subset \mathbb{R}$, $f : I \longrightarrow I$ differentiable. If x_0 is a fixed point of f such that $|f'(x_0)| = 1$ then x_0 is called a *neutral* fixed point.

Remark 2.17. The linear part of the Taylor polynomial does not tell us anything, and the remainder of Taylor approximation (Higher Order Taylor, HOT) can tilt f to either side. If $f \in C^2$, then: $f(x) - x_0 = (x - x_0) + o(|x - x_0|^2)$

Definition 2.18 (Periodic attracting/repelling point)**.** A point of period k of f is attracting if it is attracting as a fixed point of f^k, and is repelling if it is repelling as a fixed point of f^k.

$$\{\text{periodic points of period } k\} \subset \{\text{fixed points of } f^k\}.$$

Proposition 2.19. *If $\{x_0, \dots, x_{k-1}\}$ is a periodic orbit of period k, then $(f^k)'(x_i) = (f^k)'(x_j)$ for all $i, j \in \{0, 1, \dots, k-1\}$, and:*

$$(f^k)'(x_i) = \prod_{\ell=0}^{k-1} f'(x_\ell) = f'(x_0) \cdots f'(x_{k-1}).$$

Definition 2.20 (Multiplier of the periodic orbit). We define the multiplier of the periodic orbit as $\rho = \rho(O(x_i)) = f'(x_0) \cdots f'(x_{k-1})$. Orbits have similar behaviour as fixed points:

1. If $|\rho| < 1$, the periodic orbit is attracting.
2. If $|\rho| > 1$, the periodic orbit is repelling.
3. If $|\rho| = 1$, the periodic orbit is neutral.

Definition 2.21 (Basin of attraction (periodic orbits)). If $\{x_0, \dots, x_{k-1}\}$ is a periodic orbit of period k for f, we define its *basin of attraction* as $A(O(x_0)) := \{x \in \mathbb{R} \mid f^{nk}(x) \xrightarrow{n\to\infty} x_i, 0 \le i < k\}$[1]. $A(O(x_0))$ contains at least k disjoint intervals.

Remark 2.22 (Iteration of homeomorphisms in $\mathbb{R}$). If f is an homeomorphism in $\mathbb{R}$, then f is increasing or drcreasing (f is bijective, in particular injective so either $f' > 0$ or $f' < 0$). If f is increasing then there are no periodic orbits of period greater than 1[2]. If f is decreasing, f^2 is increasing.

In each interval, the orbits converge to one of the endpoints. If p_n is the rightmost fixed point, what happens in $(p_n, +\infty)$?

Exercise 2.23. *Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly decreasing homeomorphism. Prove that for each $x \in \mathbb{R}$, the orbit $O(x)$, either:*

1. *converges to a unique fixed point,*
2. *converges to a periodic orbit of period 2 or*
3. *tends to infinity.*

2.4 Bifurcations

Let $f_\lambda : I \longrightarrow \mathbb{R}$, $I \subset \mathbb{R}$, $\lambda \in J \subset \mathbb{R}$ a uniparametric family of dynamical systems depending $\mathbb{C}^r$ (r big enough) on $x \in I$ and $\lambda \in J$. When moving the parameters, we have seen examples for which there is a qualitative change in the phase portrait. The value for which this change occurs is called a bifurcation parameter.

Theorem 2.24 (Saddle-node bifurcation). *Let $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be C^r, $r \ge 2$ and $(x_0, y_0) \in \mathcal{U}$. Assume:*

[1] Similarly, $\forall \varepsilon > 0, \exists \delta > 0 \mid 0 < |x - x_i| < \delta \implies |f^{nk}(x) - f^{nk}(x_i)| = |f^{nk}(x) - x_i| < \varepsilon$. Because $f^k(x_i) = x_i$ for some $0 \le i < k$, and $k \mid nk$, then $f^{nk}(x_i) = x_i$ for every $n \in \mathbb{N}$.

[2] Given $(x_n)_n$ a sequence, and f increasing, $x_{n+1} = f(x_n) > f(x_{n-1}) = x_n$. It means that $x_i \neq x_j$ for every i, j.

1. $f(x_0, \lambda_0) = x_0$,
2. $f'(x_0, \lambda_0) = 1$,
3. $A = \frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \neq 0$,
4. $B = \frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) \neq 0$.

Then, there exists $\lambda^* : (x_0 - \eta, x_0 + \eta) \longrightarrow \mathbb{R}$, $\lambda^*(x) = \lambda$ *such that:*

1. $\lambda^*(x_0) = \lambda_0$,
2. $f(x, \lambda^*(x)) = x$,
3. $(\lambda^*)'(x_0) = 0$,
4. $(\lambda^*)''(x_0) = -\frac{B}{A} \neq 0$.

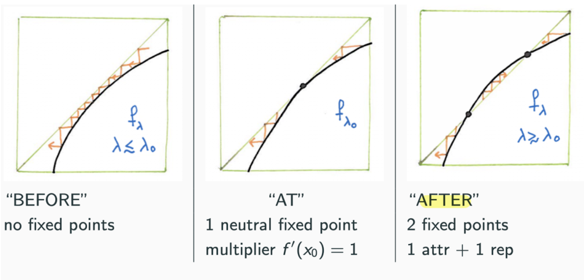


Figure 1: Sketch of the situation.

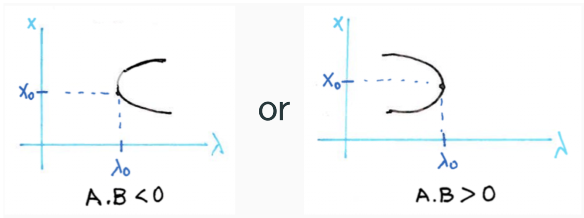


Figure 2: Curve λ^*.

Theorem 2.25 (Pitchfork bifurcation). *Let* $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be* C^3. *If there exists* $(x_0, y_0) \in \mathcal{U}$ *such that:*

1. $f(x_0, \lambda_0) = x_0$,
2. $f'(x_0, \lambda_0) = 1$,
3. $\frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) = 0$,
4. $\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) = 0$,
5. $A = \frac{\partial^2 f}{\partial \lambda \partial x}(x_0, \lambda_0) \neq 0$,
6. $B = \frac{\partial^3 f}{\partial x^3}(x_0, \lambda_0) \neq 0$.

Then, there exist two curbes of fixed points, $x^* : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \longrightarrow \mathbb{R}$ *and* $\lambda^* : (x_0 - \eta, x_0 + \eta) \longrightarrow \mathbb{R}$, $\lambda^*(x) = \lambda$ *such that:*

1. $\lambda^*(x_0) = \lambda_0$,
2. $x^*(\lambda_0) = x_0$,
3. $f(x, \lambda^*(x)) = x$,
4. $f(x^*(\lambda), \lambda) = x^*(\lambda)$,
5. $(\lambda^*)'(x_0) = 0$,
6. $(x^*)'(\lambda_0) = \frac{-\frac{\partial^2 f}{\partial \lambda^2}(x_0, \lambda_0)}{2A}$,
7. $(\lambda^*)''(x_0) = -\frac{B}{3A} \neq 0$.

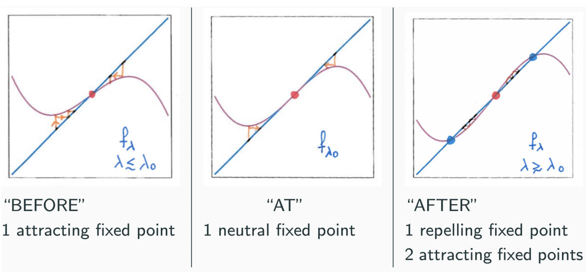


Figure 3: Sketch of the situation.

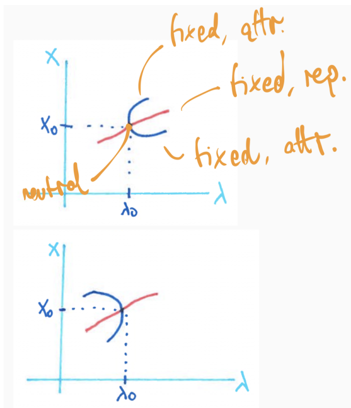


Figure 4: Curves x^* and λ^*.

Theorem 2.26 (Transcritical bifurcation). *Let $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be C^2. If there exists $(x_0, y_0) \in \mathcal{U}$ such that:*

1. $f(x_0, \lambda_0) = x_0$,
2. $f'(x_0, \lambda_0) = 1$,
3. $\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) = 0$,
4. $A_{11} = \frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) \neq 0$,
5. $A_{11} \cdot A_{22} - A_{12}^2 < 0$,

where:

$$A_{11} \cdot A_{22} - A_{12}^2 = \frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) \cdot \frac{\partial^2 f}{\partial \lambda^2}(x_0, \lambda_0) - \left(\frac{\partial^2 f}{\partial x \partial \lambda}(x_0, \lambda_0)\right)^2.$$

Then, there exist two curbes of fixed points, $x_1^*, x_2^* : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \longrightarrow \mathbb{R}$ such that:

1. $x_1^*(\lambda_0) = x_0$,
2. $x_2^*(\lambda_0) = x_0$,
3. $f(x_1^*(\lambda), \lambda) = x_1^*(\lambda)$,
4. $f(x_2^*(\lambda), \lambda) = x_2^*(\lambda)$,
5. $(x_1^*)'(\lambda_0) = -\frac{A_{12} + \sqrt{A_{12}^2 - A_{11}A_{22}}}{A_{11}}$,
6. $(x_2^*)'(\lambda_0) = -\frac{A_{12} + \sqrt{A_{12}^2 - A_{11}A_{22}}}{A_{11}}$.

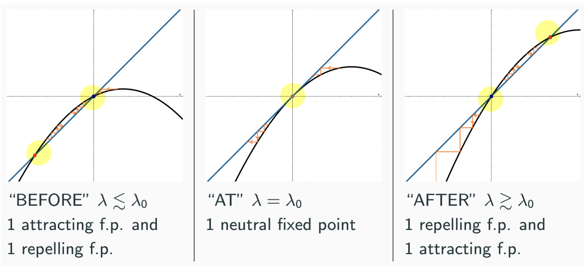


Figure 5: Sketch of the situation.

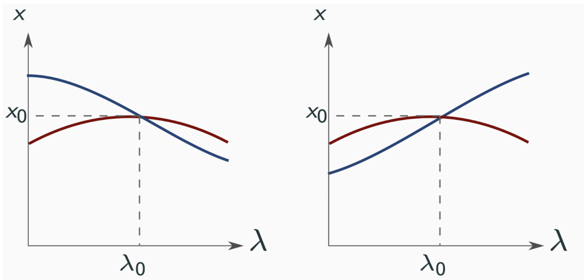


Figure 6: Curves x_1^* and x_2^*.

Theorem 2.27 (Period-doubling bifurcation). *Let $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be C^r, $r \geq 3$. Assume that:*

1. $f(x_0, \lambda_0) = x_0$,
2. $f'(x_0, \lambda_0) = 1$,

and:

$$A = \frac{\partial^2 f}{\partial x \partial \lambda}(x_0, \lambda_0) + \frac{1}{2}\frac{\partial f}{\partial \lambda}(x_0, \lambda_0) \cdot \frac{\partial^2 f}{\partial x^2}(x_0, \lambda_0) \neq 0,$$
$$B = \frac{1}{6}\frac{\partial^3 f}{\partial x^3}(x_0, \lambda_0) + \left(\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(x_0, y_0)\right)^2 \neq 0.$$

Then, there exist two curbes, one of fixed points, $x = x^*(\lambda) : (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \longrightarrow \mathbb{R}$ and one of periodic points of period two $\lambda^* : (x_0 - \eta, x_0 + \eta) \longrightarrow \mathbb{R}$ such that:

1. $x^*(\lambda_0) = x_0$,
2. $\lambda^*(x_0) = \lambda_0$,
3. $f(x^*(\lambda), \lambda) = x^*(\lambda)$,
4. $f^2(x, \lambda^*(x)) = x$,
5. $(\lambda^*)'(x_0) = 0$,
6. $(\lambda^*)''(x_0) = -\frac{2B}{A} \neq 0$.

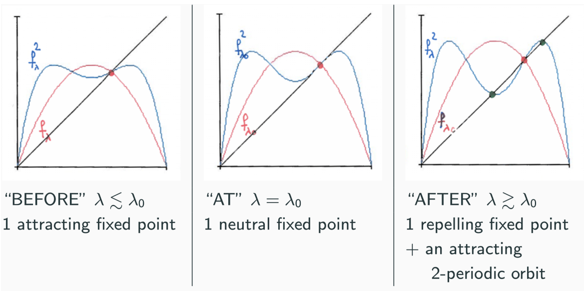


Figure 7: Sketch of the situation.

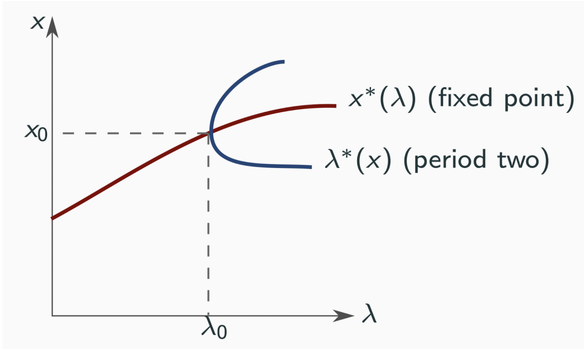


Figure 8: Curves x^* and λ^*.

3 N-dimensional maps

Up to now, we studied models in which the state was represented by a number (population, capital...). Next we study systems where the states are represented by N numbers (phase portrait $\subset \mathbb{R}^N$). We always assume that functions are (at least) continuous.

Definition 3.1 (Orbits, in $\mathbb{R}^N$). The orbits are sequences of points in $\mathbb{R}^N$: $O(v_0) = \{v_0, v_1, \dots\}, v_i \in \mathbb{R}^N$.

The phase portrait is the partition of the phase space ($\subset \mathbb{R}^N$) in orbits.

Definition 3.2 (Invariant curves). In the phase portraits we can see invariant curves: $\gamma = \gamma(s) \subset \mathbb{R}^N$ such that $\forall x \in \gamma, O(x) \subset \gamma$. Finding invariant curves (in general, manifolds) is always desirable. They form dynamical subsystems of lower dimension which are easier to study.

Definition 3.3 (Stable/attracting fixed point). A fixed point p is stable if nearby orbits cannot get far away, but they do not necessarily need to tend to p:

$$\forall \varepsilon > 0, \exists \delta > 0 \,\middle|\, \|x - p\| < \delta \implies \|f^n - p\| < \varepsilon, \forall n \geq 0.$$

We can also say $\|x - p\| < \delta \equiv x \in B_\delta(p)$ and $\|f^n - p\| < \varepsilon \equiv f^n(x) \in B_\varepsilon(p)$. A fixed point p is asymptotically stable or attracting if nearby orbits cannot get far away and in fact tend to p, too. Moreover, it is stable and $\exists \varepsilon > 0$ such that $\forall x \in B_\varepsilon(p)$ it holds that $f^n(x) \xrightarrow{n\to\infty} p$.

A fixed point is unstable if it's not stable (nearby orbits do get away).

Definition 3.4 (Repelling fixed point). A fixed point p is repelling if it's attrating for f^{-1}. f^{-1} can be a local inverse and the orbits have to abandon a neighbourhood of p at some point, but they could return afterwards.

Definition 3.5 (k-periodic orbit). $\{x_0, x_1, \dots, x_{k-1}\}$ is a k-periodic orbit if $x_j = f(x_{j-1}), j = 1, \dots, k-1$ and $x_0 = f(x_{k-1})$.

Lemma 3.6. *Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map. Assume that $\{x_0, x_1, \dots, x_{k-1}\}$ is a k-periodic orbit. Then x_j is stable (resp. attracting) fixed point for f^k if, and only if x_i is stable (resp. attracting).*

Proof. We only shall prove the $k = 2$ case. Let's assume that x_0 is stable for f^2, which means that $f(x_0) = x_1$ and $f(x_1) = x_0$. Take $B_{\varepsilon_1}(x_1)$, we have to find δ_1 such that $x \in B_{\delta_1}(x_1) \implies f^{2n}(x) \in B_{\varepsilon_1}(x_1)$, for all n (i.e. prove that x_1 is stable for f^2).

- If f is continuous, then $\exists \varepsilon_0 > 0 \mid f(B_{\varepsilon_0}(x_0)) \subset B_{\varepsilon_1}(x_1)$.
- If x_0 is stable, then $\exists \delta_0 > 0$ such that $x \in B_{\delta_0}(x_0) \implies f^{2n}(x) \in B_{\varepsilon_0}(x_0)$, for all n.
- Again, if f is continuous, then $\exists \delta_1 > 0$ such that $f(B_{\delta_1}(x_1)) \subset B_{\delta_0}(x_0)$.

Now write $f^{2n} = f \circ f^{2n-2} \circ f$. If $x \in B_{\delta_1}(x_1)$, then $f^{2n}(x) = f(f^{2n-2}(f(x)))$ and x_1 is stable for f^2. That is because $f(x) \in B_{\delta_0}(x_0)$, $f^{2n-2}(f(x)) \in B_{\varepsilon_0}(x_0)$ and then $f(f^{2n-2}(f(x))) = f^{2n}(x) \in B_{\varepsilon_1}(x_1)$, for all n.

Now for the attracting part: assume that x_0 is attracting. Then x_0 is stable and therefore x_1 is stable. We need to find $\varepsilon_1 > 0$ such that $x \in B_{\varepsilon_1}(x_1) \implies f^{2n}(x) \to x_1$.

- If x_0 attracting, then $\exists \varepsilon_0 > 0 \mid x \in B_{\varepsilon_0}(x_0) \implies f^{2n}(x) \to x_0$.
- If f is continuous, then $\exists \varepsilon_1 > 0 \mid f(B_{\varepsilon_1}(x_1)) \subset B_{\varepsilon_0}(x_0)$.

If $x \in B_{\varepsilon_1}(x_1)$, then $f(x) \in B_{\varepsilon_0}(x_0) \implies f^{2n-2}(f(x)) \xrightarrow{n\to\infty} x_0$. Applying that f is continuous, $f(f^{2n-2}(f(x))) \xrightarrow{n\to\infty} f(x_0) = x_1$ and x_1 is attracting. ■

Definition 3.7 (Stable/attracting orbit). Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map. A k-periodic orbit O is *stable* (resp. attracting) if there exists a point in O such that is stable (resp. attracting) as a fixed point of f^k.

For functions $f : I \longrightarrow I, I \subset \mathbb{R}$, the stability of a fixed point x_0 is given by the multiplier $\rho = f'(x_0)$, because $\tilde{y} = \rho \cdot \tilde{x}$ is the tangent line and best linear approximation near x_0 (change of variable is $\tilde{x} = x - x_0$). If $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ and p is a fixed point, the linear approximation is provided by the differential:

$$Df(p) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_N}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial x_1}(p) & \cdots & \frac{\partial f_N}{\partial x_N}(p) \end{pmatrix}$$

The equivalent to the tangent line is now the graph of $\mathcal{L}(x) = p + Df(p) \cdot (x - p)$. *We need to make a parenthesis in order to understand linear systems.*

Remark 3.8 (Linear systems). Consider $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N, v \longmapsto A \cdot v$ where A is an $N \times N$ matrix and v is a point (or vector) in $\mathbb{R}^N$.

- Linear property: $A(\alpha v + \beta w) = \alpha Av + \beta Aw$, for all $\alpha, \beta \in \mathbb{R}, v, w \in \mathbb{R}^N$.
- Dynamical system: $v_{n+1} = Av_n \in \mathbb{R}^N \implies v_n = A^n v_0$.

An important property of linear systems is the following: $p = (0, \dots, 0) \in \mathbb{R}^N$ is always a fixed point.

Remark 3.9 (Eigenvalues and eigenvectors)**.**

1. **Eigenvalue**: $\lambda \in \mathbb{R}$ (or $\mathbb{C}$) is an eigenvalue of the $N \times N$ matrix A if there exists a vector $v \neq 0$, $v \in \mathbb{R}^N$ (or $\mathbb{C}^N$) such that $Av = \lambda v$.
2. **Eigenvector**: Such a vector $v \in \mathbb{R}^N$ (or $\mathbb{C}^N$) is called an eigenvector of eigenvalue λ.
3. If $v \in \mathbb{R}^N$ is an eigenvector of eigenvalue $\lambda \in \mathbb{R}$, then all its multiples also are:

$$A(\alpha v) = \alpha A v = \alpha \lambda v = \lambda(\alpha v), \ \ \alpha \in \mathbb{R}.$$

Definition 3.10 (Invariant line)**.** Therefore, when $\lambda \in \mathbb{R}$ there is at least one line L of eigenvectors. The line L is invariant for f: $f(u) = \lambda u, \forall u \in L$ and $u_n = \lambda^n u_0$.

3.1 Linear systems in the plane

We consider three fundamental examples.

Example 3.11. The first fundamental example has for matrix:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \ \lambda, \mu \in \mathbb{R},$$

and can be dissected in the following cases:

1. $0 < |\lambda|, |\mu| < 1$,
2. $|\lambda|, |\mu| > 1$,
3. $0 < |\lambda| < 1 < |\mu|$,
4. $|\lambda| = 1$ and/or $|\mu| = 1$,
5. $\lambda = 0$ and/or $\mu = 0$.

Remark 3.12. The eigenvalues are λ, μ and eigenvectors $v_\lambda = (1, 0)$ and $v_\mu = (0, 1)$ i.e. the axis are invariant. The orbit of a given point (x_0, y_0) is:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \lambda^n & 0 \\ 0 & \mu^n \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \lambda^n x_0 \\ \mu^n y_0 \end{pmatrix}.$$

Exercise 3.13 (Invariant curves)**.** *Fix $c \in \mathbb{R}$. Prove that the curves:*

$$|y| = c \cdot |x|^{\frac{\log|\mu|}{\log|\lambda|}}, \ \lambda, \mu \in \mathbb{R} \setminus \{0\}, \ |\lambda| \neq 1, \ x = 0.$$

are invariant. If $|\lambda| = 1$ then the curve $C_c = \{(x, y) \in \mathbb{R}^2 \mid |x| = c\}$ is invariant. This means that if $|y_0| = C|x_0|^{\frac{\log|\mu|}{\log|\lambda|}}$, then the entire orbit $(\lambda^n x_0, \mu^n y_0)$ belongs to the curve with the same constant C. For each value of C we have 4 branches of the curve, one for each quadrant. If $\lambda, \mu > 0$, then each branch is invariant.

1. For $0 < |\lambda|, |\mu| < 1$:

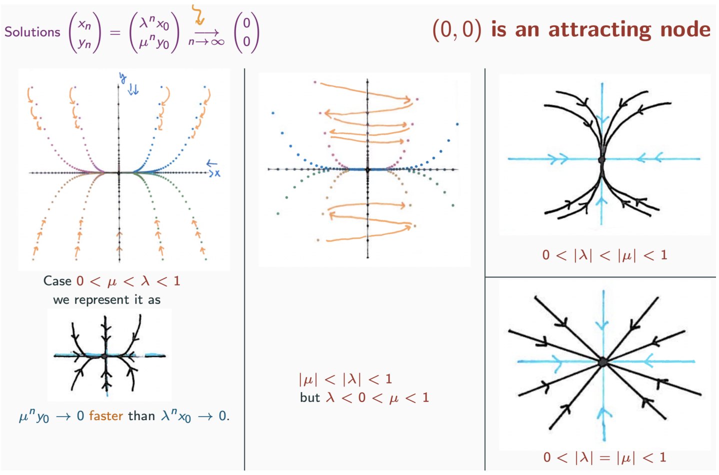


Figure 9: First case, $0 < |\lambda|, |\mu| < 1$.

Remember that $v_\lambda = (1,0)$ and $v_\mu = (0,1)$. The orbit depends on which λ or μ is greater or smaller. Take the case $0 < \mu < \lambda < 1$: if μ is smaller, μ^n dominates over λ^n (i.e. $\mu^n y_0 \longrightarrow 0$ faster than $\lambda^n y_0 \longrightarrow 0$). If $|\mu| < |\lambda| < 1$ but $\lambda < 0 < \mu < 1$ the drawing is similar, but the branches of $|y| = c \cdot |x|^{\frac{\log|\mu|}{\log|\lambda|}}$ aren't invariant *among them* (as we can see in the drawing, convergence is achieved oscillating between branches). And so on for the rest.

2. For $|\lambda|, |\mu| > 1$:

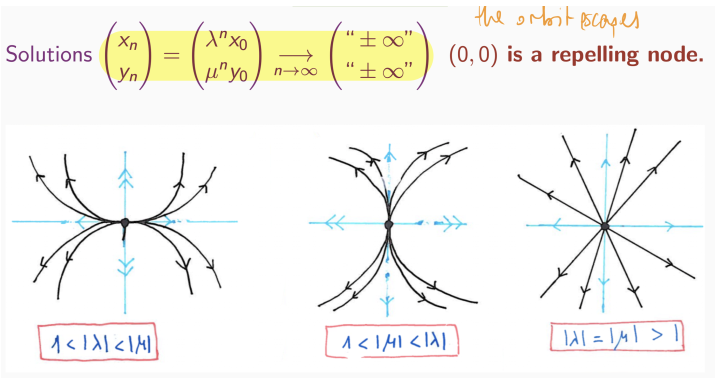


Figure 10: Second case, $|\lambda|, |\mu| > 1$.

Faré aquesta observació en català per comoditat i perquè és menys formal. Fixemnos que l'eix (és a dir, el VEP) que atrau o repel amb més força és on la corba agafa forma de conca. Això realment és gràcies al quocient $\frac{\log|\mu|}{\log|\lambda|}$, si $|\mu| > |\lambda|$, aleshores $\frac{\log|\mu|}{\log|\lambda|} > 1$ i $|y| = c \cdot |x|^{\frac{\log|\mu|}{\log|\lambda|}}$ es comporta quadràticament; si $|\mu| < |\lambda|$, aleshores $\frac{\log|\mu|}{\log|\lambda|} < 1$ i $|y| = c \cdot |x|^{\frac{\log|\mu|}{\log|\lambda|}}$ es comporta com una arrel.

3. For $0 < |\lambda| < 1 < |\mu|$:

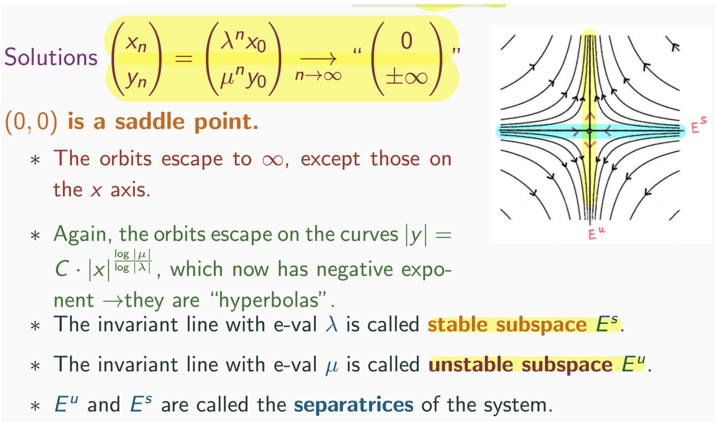


Figure 11: Third case, $0 < |\lambda| < 1 < |\mu|$.

There is nothing else to add other than what is already written in this slide. Perhaps one may notice that the exponent is negative because $\log|\mu| > \log 1 = 0$ (log is strictly increasing) and $\log|\lambda| < \log 1 = 0$. Hence, the invariant curves are of the form $|y| = \frac{C}{|x|^r}$, for some positive r.

4. Regarding one of the *non-hyperbolic* case, when the eigenvalue has module 1 (i.e. $|\lambda| = 1$ or $|\mu| = 1$, we shall study the second case).

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda^n x_0 \\ y_0 \end{pmatrix}.$$

And so:

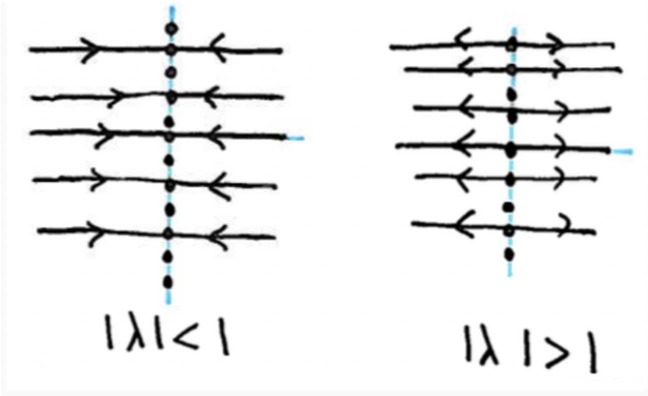


Figure 12: Fourth case, $|\lambda| = 1$.

5. There still remains one case, very similar to the last one. Take, for example, $|\mu| = 0$ ($|\lambda| = 0$ is done equivalently). Then,

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix} \implies A^n = \begin{pmatrix} \lambda^n & 0 \\ 0 & 0 \end{pmatrix} \implies \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} \lambda^n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \lambda^n x_0 \\ 0 \end{pmatrix}.$$

Example 3.14. The second fundamental example has for matrix:

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ \lambda \in \mathbb{R},$$

and can be dissected in the following cases:

1. $0 < |\lambda| < 1$,
2. $|\lambda| > 1$,
3. $|\lambda| = 1$,
4. $\lambda = 0$.

Exercise 3.15. *Show that:*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \ \lambda \neq 0 \implies A^n = \lambda^{n-1} \begin{pmatrix} \lambda & n \\ 0 & \lambda \end{pmatrix}$$

So there is one single e-val λ *and one e-vector* $(1, 0)$. *Furthermore, check that:*

1. *If* $|\lambda| < 1$ *then* $\lim_{n\to\infty} \frac{y_n}{x_n} = 0$,
2. *If* $|\lambda| > 1$ *then* $\lim_{n\to-\infty} \frac{y_n}{x_n} = 0$.

Using last exercise, the orbits of this example must be:

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \lambda^{n-1} \begin{pmatrix} \lambda x_0 + n y_0 \\ \lambda y_0 \end{pmatrix}$$

And the orbits are contained in invariant curves tangent to the x axis.

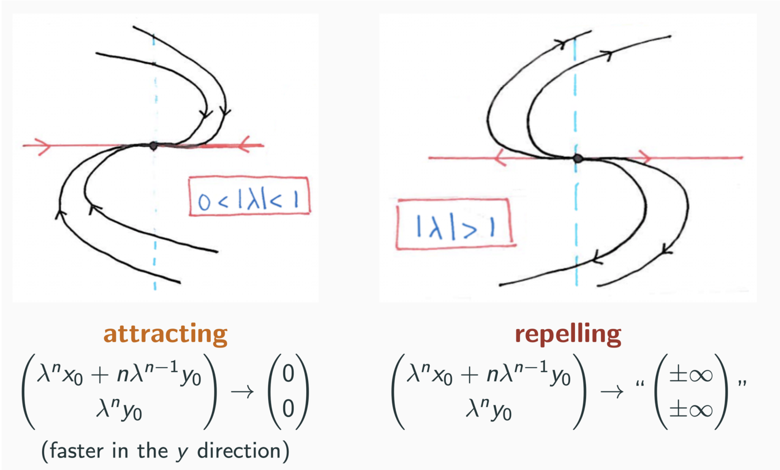


Figure 13: First and second case, $|\lambda| \neq 1$.

There is the third case, when the eigenvalue has module 1 (i.e. $|\lambda| = 1$).

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0 + n\,y_0 \\ y_0 \end{pmatrix}.$$

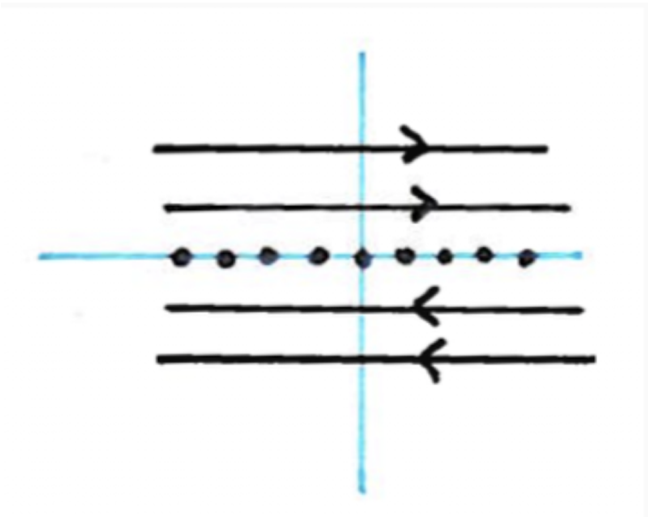


Figure 14: Third case, $|\lambda| = 1$.

And regarding the last one, it is trivial. It can be checked that $(x_n, y_n) = (y_0, 0)$, for any $n \geq 1$; not only it does not depend on n, the orbit is a single point!

Example 3.16. The third fundamental example has for matrix:

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \quad \alpha, \beta \in \mathbb{R}, \ \lambda = \alpha + i\beta;$$

and can be dissected in the following cases:

1. $0 < |\lambda| < 1$,
2. $|\lambda| > 1$,
3. $|\lambda| = 1\ (\lambda = e^{i\theta}, \theta \in \mathbb{R}, \theta \neq 0, \pi)$.

Let $\mathbb{R}^2 \simeq \mathbb{C}$ and $z = x + iy = re^{i\varphi} = r(\cos\varphi + i\sin\varphi)$. We have that:

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \alpha x - \beta y \\ \beta x + \alpha y \end{pmatrix} \implies \text{e-vals are } \alpha \pm i\beta \begin{cases} \lambda = \alpha + i\beta \\ \bar{\lambda} = \alpha - i\beta \end{cases}$$
$$\implies \lambda z = (\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y).$$

Multiplying by A (in $\mathbb{R}^2$) is the same as multiplying by the complex number $\lambda = \alpha + i\beta = \rho e^{i\theta}, \theta \in (-\pi, \pi)$. Remember:

$$\lambda z = \rho e^{i\theta} r e^{i\varphi} = \underbrace{\rho r}_{\text{homotecy by } \rho} \times \underbrace{e^{i(\theta+\varphi)}}_{\text{rotation by } \theta}.$$

Hence, the orbits are $z_n = \lambda^n z_0 = \rho^n e^{i(n\theta)} z_0$.

Exercise 3.17. *The spirals $r(\gamma) = C \cdot e^{\frac{\log|\lambda|}{\theta}\gamma}$ are invariant.*

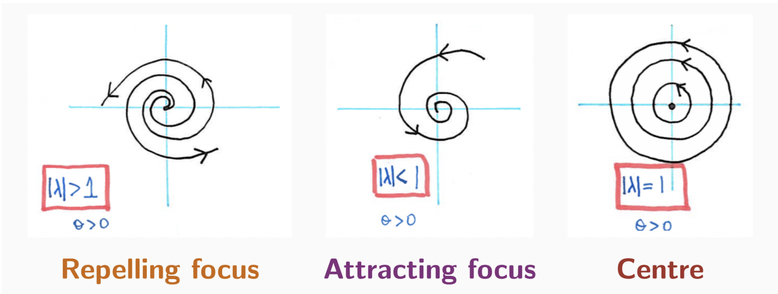


Figure 15: Third fundamental example.

Remark 3.18. There are still more cases where some eigenvalue has module 1. For example:

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

For the example 3.16 too:

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \lambda = e^{i\theta} \implies \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} x_0\cos n\theta - y_0 \sin n\theta \\ x_0 \cos n\theta + y_0 \sin n\theta \end{pmatrix}$$

Example 3.19 (General case). In the basis $\{v_\lambda, v_\mu\}$ write $v = (x_0, y_0) = a_1 v_\lambda + a_2 v_\mu$, then:

$$A^n v = A^n(a_1 v_\lambda + a_2 v_\mu) = a_1 A^n v_\lambda + a_2 A^n v_\mu = a_1 \lambda^n v_\lambda + a_2 \mu^n v_\mu.$$

Given any 2×2 matrix A, there exists a change of basis (matrix C, $\det C \neq 0$) such that $A = CJC^{-1}$, where J takes the form of matrix A from either example 3.11, 3.14 or 3.16. A and J have the same eigenvalues, fixed points, orbits, asymptotic behaviour of corresponding orbits... This tells us that all linear systems in $\mathbb{R}^2$ are linearly conjugate to one of the fundamental examples, **they have the same phase portrait, modulo a change of basis**.

SUMMARY: STABILITY OF THE ORIGIN $f(v) = Av$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, eigenvalues λ and μ

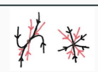
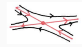
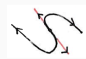
HYPERBOLIC	$\lvert\lambda\rvert, \lvert\mu\rvert < 1$ $(0,0)$ **attracting**	$\lambda, \mu \in \mathbb{R}$ **node**	2 linearly independent eigenvectors $\to$ attracting node $J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	
			1 unique eigenvector $\to$ (degenerate) attracting node $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	
		$\lambda, \mu \in \mathbb{C}$ **focus**	$(\Rightarrow \bar{\lambda} = \mu) \to$ attracting focus, $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	
	$\lvert\lambda\rvert < 1 < \lvert\mu\rvert$ $(0,0)$ **saddle**			
	$\lvert\lambda\rvert, \lvert\mu\rvert > 1$ $(0,0)$ **repelling**	$\lambda, \mu \in \mathbb{R}$ **node**	2 linearly independent eigenvectors $\to$ repelling node $J = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$	
			1 unique eigenvector $\to$ (degenerate) repelling node $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$	
		$\lambda, \mu \in \mathbb{C}$ **focus**	$(\Rightarrow \bar{\lambda} = \mu) \to$ repelling focus, $J = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$	
NON-HYP.	$\lvert\lambda\rvert = 1$ or $\lvert\mu\rvert = 1$	**degenerate cases**		

Figure 16: Summary of all orbits in the plane.

Remark 3.20. In higher dimension, there is a greater variety of cases. We will not give a detailed classification, but instead, we will study the stability of the origin and the stable and unstable subspaces. As before, exists a change of basis matrix C such that $A = CJC^{-1}$ and J is in Jordan form. Each Jordan block J_i of $\dim(m_i \times m_i)$ determines an invariant subspace.

Theorem 3.21 (Stability of the origin). *Let $f : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, $v \longmapsto Av$, where A is an $N \times N$ matrix with eigenvalues $\{\lambda_1, \dots, \lambda_N\}$.*

1. *If $|\lambda_i| < 1$, $1 \le i \le N$, then the origin is a global attractor.*
2. *If $|\lambda_i| > 1$, $1 \le i \le N$, then the origin is a repellor.*
3. *If $|\lambda_1| \le \dots \le |\lambda_p| < 1 < |\lambda_{p+1}| \le \dots \le |\lambda_N|$ for $1 \le p < N$, then the origin is a generalized saddle point.*

Moreover, $\mathbb{R}^N = E^u \oplus E^s$, where E^s is a stable subspace of $\dim p$, and E^u is the unstable subspace, of $\dim N - p$. If $v_0 \in E^s$, then $f^n(v_0) \in E^s$, for all n, and $f^n(v_0) \xrightarrow{n\to\infty} 0$. If $v_0 \in E^u$, then $f^n(v_0) \in E^u$, for all n, and $f^n(v_0) \xrightarrow{n\to-\infty} 0$.

Definition 3.22 (Spectrum). If A is an $N \times N$ matrix, the spectrum of A is the set of eigenvalues of A, $\sigma(A) = \{\lambda_1, \dots, \lambda_N\}$. The spectral radius of A is $\rho(A) = \max\{|\lambda_i| \mid \lambda_i \in \sigma(A)\}$, i.e. the maximal absolut value of the eigenvalues.

Theorem 3.23. *Let A be an $N \times N$ matrix. Then, $\forall \delta > 0$ there exists a norm $\|\cdot\|_*$ in $\mathbb{R}^N$ such that $\|A\|_* < \rho(A) + \delta$.*

Proof. We shall prove that if $|\lambda_i| < 1, 1 \leq i \leq N$, then the origin is a global attractor. Start with an $N \times N$ matrix A with all its eigenvalues having absolute values < 1. Then $\rho(A) < 1$ and let $\|\cdot\|_*$ be the norm provided by the previous theorem, with δ such that $\rho(A) + \delta < 1$. We have $\|A\|_* < \rho(A) + \delta$. The orbit of every point $v_0 \in \mathbb{R}^N$ is given by $v_n = A^n v_0$. We want to show that $\|v_n\| \xrightarrow{n\to\infty} 0$. Compute:

$$\|v_n\|_* = \|A^n v_0\|_* \leq \|A^n\|_* \|v_0\|_* \leq \|A\|_*^n \|v_0\|_* \leq (\rho(A) + \delta)^n \|v_0\|_* \xrightarrow{n\to\infty} .$$

We have used the properties of a norm for the first and second inequality, and $\|A\|_* < \rho(A) + \delta$ in the last one. Therefore, $\|v_n\|_* \xrightarrow{n\to\infty} 0$. But all norms in $\mathbb{R}^N$ are equivalent and:

$$\|v_n\| \leq c\|v_n\|_* \xrightarrow{n\to\infty} 0 \implies \|v_n\| \xrightarrow{n\to\infty} 0.$$

■

In general, for nonlinear systems, we cannot obtain explicit formulas for their orbits. Assume $f : \mathcal{U} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^n$ ofclass $\mathbb{C}^r$; in general, f has lots of fixed points and periodic orbits and if p is a fixed point of f, $Df(p) = A$ is an $N \times N$ matrix. We will see that the **linear system** $v \longmapsto Av$ at the origin (if not degenerate) gives us **information about f in a neighbourhood of p**. This will also be true for periodic orbits.

3.2 Local attractors and repellors

Theorem 3.24 (Local stability criterion for fixed points). *Let $\mathcal{U}$ be an open set $\mathbb{R}^N$, $p \in \mathcal{U}$, $f : \mathcal{U} \longrightarrow \mathbb{R}^N$ such that:*

1. *$f(p) = p$,*
2. *f is differentiable at p,*
3. *$\rho(Df(p)) < 1$ (all absolute values of eigenvalues lower than 1).*

Then, p is asymptotically stable (attractor). If all eigenvalues are greater than 1, the fixed point p is repelling.

Proof. We may assume $p = 0$ (otherise, change coordinates to $w = v - p$. Let $A = Df(0)$. Consider the norm $\|\cdot\|$ in $\mathbb{R}^N$ such that $a := \|A\| < 1$ (given by a previous theorem: for all $\delta > 0$ there exists $\|\cdot\|$ such that $\|A\| < \rho(A) + \delta$). In a neighbourhood of the origin, f is written as:

$$f(v) = Av + \mu(v), \quad \lim_{v\to 0} \frac{\mu(v)}{\|v\|} = 0.$$

This means that $\forall \eta > 0, \exists r > 0 \mid \|v\| < r \implies \|\mu(v)\| \leq \eta\|v\|$. Choose η such that $a + \eta < 1$ and let r be the corresponding radius. We will show that $\forall n \geq 0$, and $\forall v \in B(0, r)$, $f^n(v) \in B(0, r)$ and

$\|f^n(v)\| \le (a+\eta)^n\|v\|$. We proceed by induction: for $n=0$ it is clearly true, and if it is true for n then:

$$\begin{aligned}\|f^{n+1}(v)\| &= \|f(f^n(v))\| \le \|Af^n(v)\| + \|\mu(f^n(v))\| \\ &\le \|A\|\cdot\|f^n(v)\| + \eta\|f^n(v)\| \le (a+\eta)\|f^n(v)\| \le (a+\eta)^{n+1}\|v\|.\end{aligned}$$

Now for the stability, given $\varepsilon > 0$ let $\delta = \min(\varepsilon, r)$. If $\|v\| < \delta$ then $\|f^n(v)\| \le (a+\eta)^n\|v\| \le \|v\| < \delta \le \varepsilon, \forall n \ge 0$. Now, for the asymptotical stability, if $\|v\| < \delta$, $\|f^n(v)\| \le (a+\eta)^n\|v\| \xrightarrow{n\to\infty} 0$. Lastly, regarding that last statement. If $A = Df(p)$ has all eigenvalues with modulus greater than 1, then A^{-1} exists and has the inverse eigenvalues (thus, with modulus < 1). Now, because $Df^{-1}(p) = (Df(p))^{-1}$, we obtain that p is locally stable for f^{-1}. ∎

Remark 3.25.

1. If we do not assume $p=0$ we would prove:

$$\|f^n(v) - p\| \le (a+\eta)^n\|v-p\|.$$

   The proof works if $Df(p)$ is not invertible as well (i.e. det $Df(p) = 0$).
2. If we have a periodic orbit of period k, $\{p_0, \dots, p_{k-1}\}$, then each p_i is a fixed point of f^k. The stability of the periodic orbit is given by the eigenvalues of the matrix:

$$A = Df^k(p_i) = Df(p_{i-1})\cdots Df(p_k)\cdots Df(p_{i-1}), \text{ in that order.}$$

We have seen, for linear maps, that the saddle points have two associated invariant subspaces (lines, planes, etc): E^s the stable subspace, for which $f^n(v) \xrightarrow{n\to\infty} 0$; and E^u the unstable subspace, for which $f^n(v) \xrightarrow{n\to-\infty} 0$. When the map is nonlinear, locally **the situation is similar** for each saddle point p, but instead of invariant subspaces we have invariant **manifolds** (curves, surfaces, etc).

Definition 3.26 (Stable/unstable manifold). $W^s(p) = \{v \in \mathbb{R}^N \mid f^n(v) \xrightarrow{n\to\infty} p\}$ is the stable manifold of p and $W^u(p) = \{v \in \mathbb{R}^N \mid f^n(v) \xrightarrow{n\to-\infty} p\}$ is the unstable manifold of p.

Theorem 3.27 (Stable manifold theorem, $n=2$). *Let $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, C^r, $r \ge 1$. Let $p \in \mathbb{R}^2$ such that $f(p) = p$ and p is a saddle point. Then, there exists $\mathbb{C}^r$ curve $\gamma_1 = \gamma^s : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$ such that:*

1. $\gamma_1(0) = p$
2. $\gamma_1'(t) \neq 0$,
3. $\gamma_1'(0) = v$[3],
4. *γ_1 is invariant for f.*
5. $f^n(\gamma_1(t)) \xrightarrow{n\to\infty} p$,
6. $\gamma_1 = W^s_{loc}(p) = \{q \in B \mid f^n(q) \in B, \forall n \ge 0\}$, *where B is a neighbourhood of p.*

[3] The eigenvector of $Df(p)$ with eigenvalue λ, $|\lambda| < 1$.

Theorem 3.28 (Unstable manifold theorem, $n = 2$). *Let $f : \mathcal{U} \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, C^r, $r \geq 1$. Let $p \in \mathbb{R}^2$ such that $f(p) = p$ and p is a saddle point. Then, there exists $\mathbb{C}^r$ curve $\gamma_2 = \gamma^s : (-\varepsilon, \varepsilon) \longrightarrow \mathbb{R}^2$ such that:*

1. $\gamma_2(0) = p$
2. $\gamma_2'(t) \neq 0$,
3. $\gamma_2'(0) = v$[4],
4. γ_2 *is invariant for* f.
5. $f^n(\gamma_2(t)) \xrightarrow{n \to -\infty} p$.

Note that here we either assume that f is invertible, or $f^{-n} = f^{-1} \circ \cdots \circ f^{-1}$ where f^{-1} is the branch of the inverse map that fixes p (such locally exists since the differential at p is non-degenerate.

Iterating the curves γ^s and γ^u for f^{-1} and f, we would obtain $W^s(p)$ and $W^u(p)$. If f is globally invertible, then both sets are manifolds. Globally, such manifolds can be very complicated. Their intersections are a source of chaos.

Remark 3.29. The stable and unstable manifolds theorem is also valid in $\mathbb{R}^N$, $N \geq 2$. In this case, if $f : \mathcal{U} \subset \mathbb{R}^N \longrightarrow \mathbb{R}^N$ has a saddle point p with eigenvalues $|\lambda_1| \leq \cdots \leq |\lambda_k| < 1 < |\lambda_{k+1}| \leq \cdots \leq |\lambda_N|$ then p has two associated manifolds:

1. $W^s(p)$, which is stable and $\dim W^s = k$, tangent to $E^s(p)$.
2. $W^u(p)$, which is unstable and $\dim W^u = N - k$, tangent to $E^u(p)$.

3.3 Banach

In general, it is difficult to make a complete global study. However, if we have a contractive map in a domain, then there is always a fixed point which is a global attractor.

Definition 3.30 (Contractive map). Let $E \subset \mathbb{R}^N$ be a closed set and a contractive map is $f : E \longrightarrow E$ such that exists a $\lambda < 1$ satisfying $\|f(x) - f(y)\| \leq \lambda \|x - y\|, \forall x, y \in E$.

Theorem 3.31 (Banach Fixed Point Theorem). *Let $E \subset \mathbb{R}^N$ be a closed set and f a contractive map. Then, f has a unique fixed point $p \in E$ and p is globally asymptotically stable (global attractor). Moreover,*

$$\|f^n(x) - p\| \leq \lambda^n \|x - p\| \text{ and } \|f^n(x) - p\| \leq \frac{\lambda^n}{1 - \lambda} \|x - f(x)\|.$$

Proof. Let $z_0 \in E$ and set $z_n := f^n(z_0)$, $n \geq 0$. Since f is contractive, we have that $\forall x_0, y_0 \in E$, $\|x_n - y_n\| \leq \lambda^n \|x_0 - y_0\|$. In particular, taking $y_0 := x_1 = f(x_0)$, we get $\|x_n - x_{n+1}\| \leq \lambda^n \|x_0 - x_1\|$. We will get see that

[4] The eigenvector of $Df(p)$ with eigenvalue μ, $|\mu| < 1$.

it is a Cauchy sequence, that is, $\forall \varepsilon > 0, \exists N \in \mathbb{N} \mid m > n > N \implies \|x_n - x_m\| < \varepsilon$:

$$\begin{aligned}\|x_n - x_m\| &= \|x_n - x_{n+1} + x_{n+1} - x_{n+2} + \cdots + x_{m-1} - x_m\| \\ &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - x_{n+2}\| + \cdots + \|x_{m-1} - x_m\| \\ &\leq \lambda^n \|x_0 - x_1\| + \lambda^{n+1}\|x_0 - x_1\| + \cdots + \lambda^{m-1}\|x_0 - x_1\| \\ &\leq \sum_{j=n}^{+\infty} \lambda^k \|x_0 - x_1\| = \frac{\lambda^n}{1-\lambda}\|x_0 - x_1\| \xrightarrow{n\to\infty} 0.\end{aligned}$$

Cauchy sequences in $\mathbb{R}^N$ do converge. Thus, $(x_n)_n$ has a limit. Let p be this limit, which lies in E because E is closed. Since contractive maps are continuous[5] we have:

$$f(p) = f\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = p,$$

i.e. p is a fixed point.

Remark 3.32. Notice that E cannot have 2 different fixed points. If $p \neq q$ were fixed, then:

$$\|p - q\| = \|f^n(p) - f^n(q)\| \leq \lambda^n \|p - q\| \xrightarrow{n\to\infty} 0,$$

but that would imply that $p = q$. As a consequence, the limit must be the same for all sequences $(x_n)_n$ indepently of which $x_0 \in E$ we take.

Finally, p is global attracting, because:

$$\|f^n(x) - p\| = \|f^n(x) - f^n(p)\| \leq \lambda^n \|x - p\| \xrightarrow{n\to\infty} 0. \qquad \blacksquare$$

4 Dynamics

4.1 Complex Dynamics

Let $P : \mathbb{C} \longrightarrow \mathbb{C}$ be a complex polynomial of degree at least 2. Our goal is to study the asymptotic behaviour of $\{P^n(z_0)\}$, $z_0 \in \mathbb{C}$.

Remark 4.1.

1. Polynomials are particular holomorphic functions: they are analytical functions in $\mathbb{C}$. In particular,

$$P'(z_0) = \lim_{z\to z_0} \frac{P(z) - P(z_0)}{z - z_0} \in \mathbb{C}.$$

[5] Let $E \subset \mathbb{R}^N$ a closed set, and $f : E \longrightarrow E$ a function be such that $d(f(x), f(y)) \leq c d(x, y)$, for all $x, y \in E$, where $0 \leq c < 1$ is given. Fix $\varepsilon > 0$ and choose $a \in E$. The case where $c = 0$ is trivial. Assume $c > 0$ and let $\delta = \frac{\varepsilon}{c}$. For all $x \in E$ with $d(x, a) < \delta$, we have $d(f(x), f(a)) \leq c d(x, a) < \varepsilon$, i.e. f is continuous.

2. If $z_0 \in \mathbb{C}$ is such that $P'(z_0) = 0$ we say that z_0 is a critical point and its image $P(z_0)$ a critical value. Those points will play a key role for the dynamics. Otherwise, we say that z_0 is a regular point, and then P is as a local diffeomorphism (inverse function theorem) *in some neighbourhood* of z_0.
3. Complex polynomials are open maps: if $\mathcal{U}$ is open, then $P(\mathcal{U})$ is also open.

Theorem 4.2 (Stability of fixed points)**.** *Let $P : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial, and $z_0 \in \mathbb{C}$ such that $P(z_0) = z_0$. Let $\lambda = P'(z_0)$ be the multiplier of z_0.*

1. *If $|\lambda| < 1$, then z_0 is attracting.*
2. *If $|\lambda| > 1$, then z_0 is repelling.*

Moreover,

1. *As in the case of maps in $\mathbb{R}^2$, if $|\lambda| = 1$, then there are lots of possibilities for the dynamics.*
2. *If $z_0, z_1, \dots, z_{k-1}$ is a periodic orbit, its multiplier is given by:*

$$(P^k)'(z_i) = P'(z_0) \cdot P'(z_1) \cdots P'(z_{k-1}).$$

Remark 4.3.

1. P is never globally invertible ($\deg(P) \geq 2$). This means that the stable and unstable manifolds are only well-defined locally. That is, in a neighbourhood of the fixed point or periodic orbit.
2. The *local* stable manifold of an attracting fixed point (or unstable or a repelling fixed point) has always dimension 2. This is because the eigenvalues are always complex (or double real). There are no saddle points!

Definition 4.4 (Basin of attraction)**.** Given an attracting fixed point z_0 for P, we define its basin of attraction as $A = A(z_0) = \{z \in \mathbb{C} \mid P^n(z_0) \xrightarrow{z \to \infty} z_0\}$. If $z_0, \dots, z_{p-1}$ is a periodic orbit of period p, its basin of attraction is given by:

$$A = A(\{z_0, \dots, z_{p-1}\}) = \{z \in \mathbb{C} \mid P^{pn}(z) \longrightarrow z_i,\ i = 0, 1, \dots, p-1\}.$$

In general, basins of attraction can have infinitely-many connected components.

Definition 4.5 (Supersensitive for F)**.** We say that $z_0 \in \mathbb{C}$ is supersensitive for F if $\forall \mathcal{U}$ neighbourhood of z_0, $\bigcup_{n \geq 0} F^n(\mathcal{U})$ is equal to $\mathbb{C}$ except, at most, one point.

Example 4.6 (Particular example for the quadratic family $Q_c(z) = z^2 + c$)**.** If we define $z_{n+1} = Q_0(z_n)$, $z_0 = re^{i\theta} \in \mathbb{C}$, then $z_1 = r^2 e^{2i\theta}$, $z_2 = r^4 e^{4i\theta}$, and so on. The general term is $z_n = r^{2^n} e^{i(2^n \theta)}$, which means that $z_n \xrightarrow{n \to \infty} \infty$ if $r > 1$, $z_n \xrightarrow{n \to \infty} 0$ if $r < 1$ and $|z_n| = 1$ for all n if $r = 1$. ∞ is an attracting fixed point and $z = 0$ is an attracting fixed point, too ($Q_0'(0) = 0$).

Property 4.7. *The function $\theta \longmapsto 2\theta \pmod{2\pi}$ on $\mathbb{S}^1$ satisfies:*

1. *Periodic points are repelling and dense.*
2. *It depends sensibly on initial conditions.*

Proof.

1. Points of period p are $2^p\theta = \theta + 2k\pi, k \in \mathbb{Z}$, and $\theta = \frac{2k\pi}{2^p-1}, k \in \mathbb{Z}$. Fixing the period p, the points are at distance $\frac{2\pi}{2^p-1}$. Equidistant on $\mathbb{S}^1$: separation goes to zero, as $p \to \infty$. All are repelling since $|Q_0'(z)| = 2|z| = 2$, for all $z \in \mathbb{S}^1$.
2. (Idea). $\theta \longmapsto 2\theta$ is expansive: it doubles the lengths of the segments at each iteration, and ends up covering the whole $\mathbb{S}^1$.

■

4.1.1 Invariant sets

Let $P(z) = a_d z^d + a_{d-1}z^{d-1} + \cdots + a_0$, $a_i, z \in \mathbb{C}$, $a_d \neq 0$ be a polynomial in $\mathbb{C}$ of degree $d \geq 2$. We first show that ∞ is an attracting fixed point as for z^d.

Proposition 4.8. *There exists $R > 0$ such that if $|z| > R$, then $P^n(z) \xrightarrow{n\to\infty} \infty$.*

Proof. We have that:

$$|P(z)| = |z| \cdot \left|a_d z^{d-1} + \cdots + a_2 z + a_1 + \frac{a_0}{z}\right| > 2|z|, \quad \left|a_d z^{d-1} + \cdots + a_2 z + a_1 + \frac{a_0}{z}\right| \xrightarrow{z\to\infty} \infty,$$

if $|z| > R$. Therefore, by induction, if $|z| > R$, $|P^n(z)| > 2^n|z|$. Hence, if $|z| > R$, $|P^n(z)| \xrightarrow{n\to\infty} \infty$. ■

Definition 4.9. Let P be a polynomial of degree $d \geq 2$.

1. The basin of infinity is the open set $A_p(\infty) = \{z \mid P^n(z) \xrightarrow{n\to\infty} \infty\}$.
2. The filled-in Julia set is the closed set $K_p = \mathbb{C} \setminus A_p(\infty) = \{z \mid P^n(z) \not\to \infty\}$.
3. The Julia set J_p is the closed set $J_p = \partial A_p(\infty) = \partial K_p$.

Proposition 4.10. *K_p and $A_p(\infty)$ are totally invariant i.e. they are invariant for P and any branch of P^{-1}. Thus, J_p is totally invariant, too.*

Therefore, these sets form a partition of the plane into dynamical subsystems (orbits from A, K and J do not mix).

Notation 4.11. If we consider $Q_c(z) = z^2 + c$, then we write $A_c(\infty) := A_{Q_c}(\infty)$, $K_c := K_{Q_c}$ and $J_c := J_{Q_c}$.

4.1.2 Conjugacies

Let X, Y be subsets of $\mathbb{C}$ (or of $\mathbb{R}^n$) and $F_1 : X \longrightarrow X$, $F_2 : Y \longrightarrow Y$ functions (at least continuous).

Definition 4.12 (Topologically conjugate). We say that F_1 and F_2 are topologically conjugate if there exists a homeomorphism $h : X \longrightarrow Y$ such that $h \circ F_1 = F_2 \circ h$:

$$\begin{array}{ccc} X & \xrightarrow{F_1} & X \\ \Big\downarrow h & & \Big\downarrow h \\ Y & \xrightarrow{F_2} & Y \end{array}$$

If we can choose h to be C^r (resp. linear), we say that F_1 and F_2 are C^r-conjugate (resp. linearly conjugate).

Proposition 4.13. *A conjugacy $h : X \longrightarrow Y$ sends orbits of F_1 to orbits of F_2. In particular:*

1. *h sends f9xed points of F_1 to f9xed points of F_2.*
2. *h sends period orbits of period p of F_1 to period orbits of period p of F_2.*
3. *h sends attractors of F_1 to attractors of F_2.*
4. *h sends repellors of F_1 to repellors of F_2.*
5. *Phase portraits correspond each other by h: $h(K_{F_1}) = K_{F_2}$ and $h(J_{F_1}) = J_{F_2}$.*

Proof.

1. Let x_0 a fixed point of F, i.e. $F_1(x_0) = x_0$.

$$h(x_0) = h(F_1(x_0)) = F_2(h(x_0)) \implies h(x_0) \text{ is a fixed point of } F_2.$$

2. We want to prove that if x_0 is a periodic point of period $k \geq 1$ for F_1, then $h(x_0)$ is a periodic point of period k for F_2. Define $F_2(y) := F_2(h(x)) = h(F_1(x))$, so $F_2(x) = h(F_1(h^{-1}(x)))$ and if we define $h(x) := y$, along with the fact that h is an homeomorphism, $x = h^{-1}(y)$. Hence,

$$h(F_1(h^{-1}(y))) = h(F_1(x)) = F_2(y) \implies F_2(y) = h(F_1(h^{-1}(y))) \implies F_2 = h \circ F_1 \circ h^{-1}, \ \forall y \in Y.$$

By induction, $F_2^n = h \circ F_1^n \circ h^{-1}, \forall n \geq 1$. If we apply the first part to F_1^k, we are done.

3. Let x_0 be an attracting fixed point of F_1. Then, $\exists \varepsilon > 0$ such that $y \in \mathbb{C}$ and $|x_0 - y| < \varepsilon \implies F_1^n(y) \xrightarrow{n\to\infty} x_0$. By the first part, $h(x_0)$ is a fixed point of F_2. We need to show that there exists $\varepsilon' > 0$ such that $w \in \mathbb{C}, |w - h(x_0)| < \varepsilon' \implies F_2(w) \xrightarrow{n\to\infty} h(x_0)$. Choose $\varepsilon' > 0$ such that $h^{-1}(\{z \mid |h(x_0) - z| < \varepsilon'\}) \subset h^{-1}(\{z \mid |x_0 - z| < \varepsilon\})$. Then $h^{-1}(w)$ will be within ε from x_0. Hence, $F_1^n(h^{-1}(w)) \xrightarrow{n\to\infty} x_0$. Applying h (we can because h is C), $F_2^n(w) = h(F_1^n(h^{-1}(w))) \xrightarrow{n\to\infty} h(x_0)$, as we wanted to prove.

4. Similar to last part.
5. Consequence of all previous parts. ■

Conjugacies are used to classify dynamical systems. Two conjugate dynamical systems are qualitatively *equal*.

Definition 4.14 (Semiconjugate). If there exists $h : X \longrightarrow Y$ such that $hF_1 = F_2 h$, but h is not injective (instead, it has degree $k > 1$), we say that F_1, F_2 are *semiconjugate*.

Exercise 4.15. *Semiconjugacies also send orbits to orbits, but the period of periodic orbits may be different.*

Example 4.16 ($Q_{-2}(z) = z^2 - 2$). The function Q_{-2} in $\mathbb{C} \setminus [-2, 2]$ is conjugate to $Q_0(z) = z^2$ in $\mathbb{C} \setminus \overline{\mathbb{D}}$. That is, the following diagram commutes:

$$\begin{tikzcd}
\mathbb{C} \setminus \overline{\mathbb{D}} \arrow[r, "z^2"] \arrow[d, "h"] & \mathbb{C} \setminus \overline{\mathbb{D}} \arrow[d, "h"] \\
\mathbb{C} \setminus [-2, 2] \arrow[r, "Q_{-2}"] & \mathbb{C} \setminus [-2, 2]
\end{tikzcd}$$

As a consequence, $A_{-2}(\infty) = \mathbb{C} \setminus [-2, 2]$ and $K_{-2} = J_{-2} = [-2, 2]$. Let $h(z) = z + \frac{1}{z}$, considered in $\mathbb{C} \setminus \overline{\mathbb{D}} = \{\|z\| > 1\}$.

1. If h was, in fact, a conjugacy, then $h(P(z)) = Q_{-2}(h) \iff h(z^2) = h(z)^2 - 2$. Let's check it:

$$h(z^2) = z^2 + \frac{1}{z^2} \text{ and } h(z)^2 - 2 = \left(z + \frac{1}{z}\right)^2 - 2 = z^2 + \frac{1}{z^2} + 2z\frac{1}{z} = z^2 + \frac{1}{z^2}.$$

2. For it to be injective, $h(z) = h(w) \implies z = w$.

$$z + \frac{1}{z} = w + \frac{1}{w} \iff z - w = \frac{z - w}{zw} \implies \begin{cases} z = w \\ z \cdot w = 1 \end{cases}$$

   but if that were true, $|z| = \frac{1}{|w|}$ and if $|z| > 1$ then $|w| < 1$ and $w \notin \mathbb{C} \setminus \mathbb{D}$, contradiction.

3. For it to be surjective, $z + \frac{1}{z} = w \in \mathbb{C} \setminus [-2, 2]$:

$$z^2 - wz + 1 = 0 \iff z_\pm = \frac{1}{2}(w \pm \sqrt{w^2 - 4}) \implies z_+ \cdot z_- = 1.$$

   Either one of the solutions is in $\mathbb{C} \setminus \mathbb{D}$ or both lie on $\mathbb{S}^1$.

4. h is continuous at $\mathbb{C} \setminus \{0\}$, so it is certainly continuous at $\mathbb{C} \setminus \mathbb{D}$. By the next exercise, h^{-1} is continuous, too.

Exercise 4.17. *If $h : X \longrightarrow Y$ is continuous, bijective and open, then h^{-1} is continuous.*

Theorem 4.18. *Let $P : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial of degree $d \geq 2$.*

1. *Periodic points are dense in J_P.*
2. *All points in J_P are supersensitive for P.*

That is, dynamics are chaotic in the Julia set.

Proposition 4.19 (Escape criterion). *Let $c \in \mathbb{C}$. If $|z| \geq |c|$ and $|z| > 2$, there exists $\varepsilon > 0$ such that $|Q_c^n(z)| \geq (1+\varepsilon)^n |z|$, and therefore $Q_c^n(z) \xrightarrow{n\to\infty} \infty$. In other words, if $R = \max\{|c|, 2\}$, then $\mathbb{C} \setminus \overline{D(0, R)} \subset A_c(\infty)$.*

Proof. Take $|Q_c(z)| = |z^2 + c|$. Using $|z|^2 \leq |z^2 + c| + |c|$, and $|z| \geq |c|$,

$$|Q_c(z)| = |z^2 + c| \geq |z|^2 - |c| \geq |z|^2 - |z| = |z|(|z| - 1).$$

If $|z| > 2$, then $|z| - 1 > 1$, in particular, $|z| - 1 = 1 + \varepsilon$, $\varepsilon > 0$. Therefore,

$$|Q_c(z)| \geq (1 + \varepsilon)|z| \implies |Q_c^n(z)| \geq (1 + \varepsilon)^n |z| \implies Q_c^n(z) \xrightarrow{n \to \infty} \infty. \qquad \blacksquare$$

Corollary 4.20.

1. *If $|c| > 2$, then $0 \in A_c(\infty)$. That is, $0 \longmapsto c \longmapsto c^2 + c \longmapsto \cdots$*
2. *If for some k we have $|Q_c^k(z)| \geq |c|$ and $|Q_c^k(z)| > 2$. Then, $z \in A_c(\infty)$.*

Connectedness. We will see that the orbit of the critical point $z = 0$ (derivative zero) plays an important role on connectedness.

Proposition 4.21 (Fundamental dichotomy). *Let $Q_c(z) = z^2 + c$. Then either:*

1. *$0 \in K_c$ and K_c is connected, or*
2. *$0 \in A_c(\infty)$ and K_c has infinitely many connected components (**Cantor set**).*

Notice that connectedness only depends on whether the orbit of $z = 0$ tends to ∞ or not. We observe that K_c cannot have more than one piece. It has either one piece or infinite.

4.1.3 Mandelbrot set

We have seen that Julia sets can only be connected ($0 \in K_c$) or Cantor sets ($0 \notin A_c(\infty)$). We may define (in the parameter plane, i.e. the plane of all c).

Definition 4.22 (Mandelbrot set). The Mandelbrot set M consists of the values $c \in \mathbb{C}$ for which K_c is connected. Equivalently, $M = \{c \in \mathbb{C} \mid |Q_c^n(0) \not\to \infty, n \to \infty\}$.

Because of the escape criterion, we know that if $|c| > 2$, then $c \in A_c(\infty)$ and therefore $M \subset \overline{D(0, 2)}$. The Mandelbrot set is a *catalog* of Julia sets. For each value of c we have a different Julia set. Let us study M in more detail. First, its interior, and then its boundary.

Proposition 4.23. *Assume $Q_{c_0}^p(z_0) = z_0$, $|(Q_{c_0}^p)'(z_0)| < 1$. Then, z_0 belongs to an attracting periodic orbit. Then:*

1. *The basin of attraction is in K_{c_0}. In consequence, K_{c_0} is not a Cantor set and $c_0 \in M$.*
2. *There exists $\varepsilon > 0$ such that if $|c - c_0| < \varepsilon$, Q_c has an attracting periodic orbit (of the same period). Then, $c \in M$ if $|c - c_0| < \varepsilon$, which means $c_0 \in \operatorname{int}(M)$.*

Proof. We will prove only the second part. It is a consequence of the Implicit Function Theorem. Assuming $Q_{c_0}^p(z_0) = z_0$, define $|(Q_{c_0}^p)'(z_0)| = |\lambda_0| < 1$. Let $F(c,z) = Q_c^p(z) - z$ ($\mathbb{C}$-differentiable). Then:

$$F(c_0, z_0) = 0, \quad \frac{\partial F}{\partial z}(c,z)|_{(c_0,z_0)} = (Q_{c_0}^p)'(z_0) - 1 = \lambda_0 - 1 \neq 0.$$

By the Inverse Function Theorem (complex version), there exists a continuous function $z(c)$ defined for $|c - c_0| < \varepsilon'$ such that $F(c, z(c)) = 0$ and so $z(c)$ periodic point of period p. Since $(Q_c^p)'(z)$ and $z(c)$ are continuous, it is also attracting (reducing ε' if necessary). ■

Interior of the Mandelbrot set. Let's define C_i, the i-th hyperbolic component (more generally, region) of the interior of M. i is the period.

1. Period 1. We find the region C_1 of values c for which Q_c has an attracting fixed point. So $z^2 + c = z$ and $|Q_c'(z)| = |2z| < 1$. It is clear that it contains $c = 0$. The boundary will be given by the cardioid:

$$\begin{cases} z^2 + c = z \\ |2z| = 1 \end{cases} \implies \begin{cases} c = z - z^2 \\ |z| = \frac{1}{2} \end{cases} \implies \begin{cases} c = \frac{1}{2}e^{i\theta} - \frac{1}{4}e^{2i\theta}, \\ z = \frac{1}{2}e^{i\theta}, \ \theta \in [0, 2\pi). \end{cases}$$

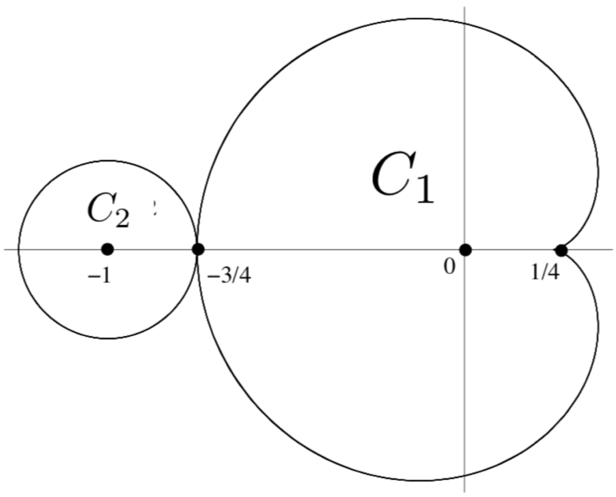
2. Period 2. Values of c for which Q_c has an attracting periodic orbit of period 2.

$$(z^2 + c)^2 + c = z \implies z_\pm = \frac{-1 \pm \sqrt{-3 - 4c}}{2}.$$

The multiplier (remember, $\lambda = P'(z_0)$)) is given by $2z_+ \cdot 2z_- = 4(c+1)$ and the orbit is attracting because:

$$4|c+1| < 1 \implies |c - (-1)| < \frac{1}{4},$$

so C_2 is the disk centered at -1 and of radius $\frac{1}{4}$.



Theorem 4.24. *Let P be a complex polynomial. Then, every attracting basin must have at least one critical point*[6].

[6] Remember that a critical point $z_0 \in \mathbb{C}$ is one such that $P'(z_0) = 0$.

Corollary 4.25. *For each c, Q_c has at most one attracting periodic orbit.*

For the rest of the periods, the sets are not explicitly computable. Numerically we see that for periods ≥ 3 there is more than one component.

Bifurcations happen at the contact points between components. We will study two examples: $c = \frac{1}{4}, -\frac{3}{4}$. Notice these are real numbers.

1. For $c = \frac{1}{4}$, $z^2 + \frac{1}{4} = z$, then $z_0 = \frac{1}{2}$. And so $Q'_{\frac{1}{4}}(\frac{1}{2}) = 1$. On the real line, saddle-node bifurcation.
2. For $c = -\frac{3}{4}$, $z^2 - \frac{3}{4} = z$, then $z_0 = -\frac{1}{2}$. And so $Q'_{-\frac{3}{4}}(-\frac{1}{2}) = -1$. On the real line, period doubling. The periodic points of period two are:
$$z_{\pm} = \frac{-1 \pm \sqrt{-3 - 4c}}{2}.$$

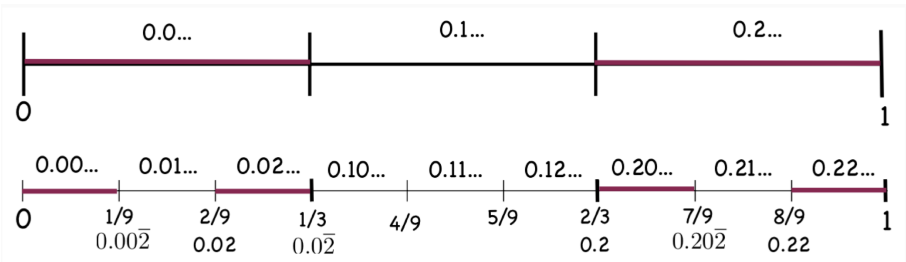
4.2 Fractals

We shall construct fractals geometrically starting from a generator and an initiator.

Definition 4.26 (Ternary Cantor set). The Cantor ternary set C is created iteatively by deleting the open middle third from a set of line segments. One starts by deleting the open middle third $(\frac{1}{3}, \frac{2}{3})$ from $[0, 1]$. Next, the open middle third of each of these remaining segments is deleted, and so on.

1. At least all segment endpoints. For every $n \geq 1$, $x = \frac{k}{3^n}$, for 2^{n+1} values of k.
2. The points in the Ternary Cantor set are those without 1s in their base 3 expansion.

Proof. In base 10 expansion we have $0.s_1s_2s_3\ldots = \frac{s_1}{10} + \frac{s_2}{100} + \cdots$. Multiplying by 10 (i.e. displacing the decimal point one place), $s_1.s_2s_3\ldots$ The same may be done in base 3: we have $(0.s_1s_2s_3\ldots)_3 = \frac{s_1}{3} + \frac{s_2}{9} + \cdots$. Multiplying by 3 (i.e. displacing the decimal point one place), $(s_1.s_2s_3\ldots)_3$. And now we can observe that:



So points in the Ternary Cantor set are those without 1s in their base 3 expansion. ■

3. The interval endpoints are only those that end in either $\overline{0}$ or $\overline{2}$. Therefore, the set has many other points which are not interval endpoints.
4. C is uncountable and in bijection with the real numbers.

Definition 4.27 (Cantor set). If C is a set in $\mathbb{R}^2$ checking these next three properties, it is called Cantor set.

1. C is closed.
2. C is perfect (i.e. no isolated points). Each point in C can be approximated by endpoints.
3. C is totally disconnected (its Lebesgue measure is 0).

Figure 17: Ternary Cantor set.

Definition 4.28 (Koch curve/snowflake). Begin with a straight line. Divide it into three equal segments and replace the middle segment by the two sides of an equilateral triangle of the same length as the segment being removed. Now repeat, taking each of the four resulting segments, dividing them into three equal parts and replacing each of the middle segments by two sides of an equilateral triangle. Iterate.

1. $\text{area(flake)} < \text{area(circle)} = \pi \cdot 2^2 = 4\pi$.
2. The perimeter is $P_n = (\frac{4}{3})^n \xrightarrow{n\to\infty} +\infty$. So it has infinite perimeter.

The Koch curve is a fractal.

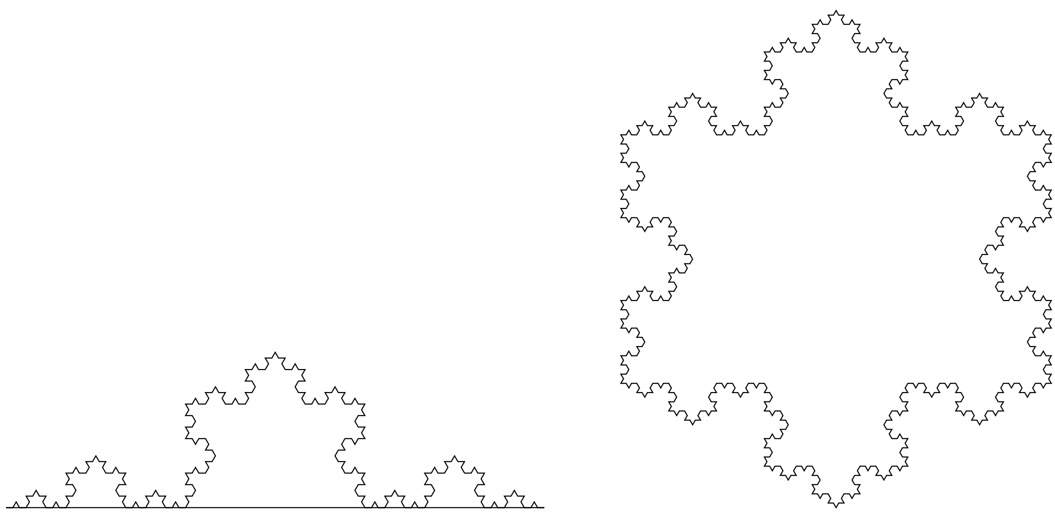


Figure 18: Koch curve/snowflake

Definition 4.29 (Sierpinski triangle). We notice the initiator has three copies of the generator.

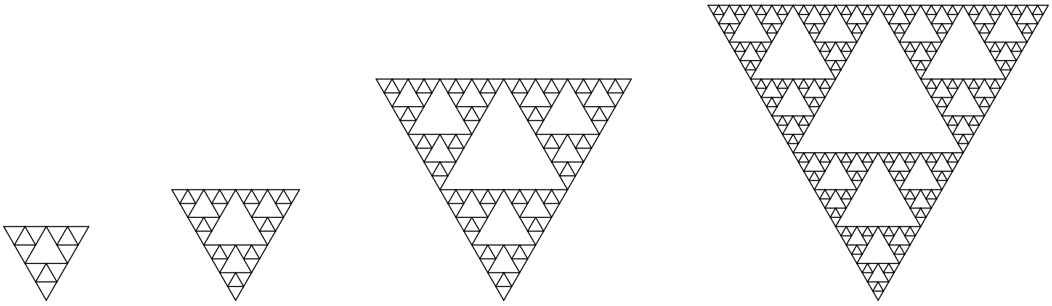


Figure 19: Sierpinksi triangle.

The limit object (the points that remain) after iterating infinite times is known as the Sierpinski triangle. We build the initiator copies from the generator, using planar affinities (scaling, rotating, translating):

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} rx \\ sy \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \cos\theta & -\sin\varphi \\ \sin\theta & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}.$$

We will take $r = s = 0.5$, $\theta = \varphi = 60$, $(e, f) = (1, 0.5)$. A set of rules like this one (Sierpinski triangle) is called an IFS (Iterated Function System).

affinities	r	θ	e	f
A_1	0.5	0	0	0
A_2	0.5	0	0.5	0
A_3	0.5	0	0	0.5

Any limit must be invariant for the rules (cf. 4.30) that generate it. This means that if we apply the rules we obtain the fractal itself again.

Definition 4.30 (IFS). An IFS (Iterated Function System) is a set of contractions of the plane $T_1, \ldots, T_n$, $T_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\|T_i(p) - T_i(q)\| < \lambda \|p - q\|$, for all $p, q \in \mathbb{R}^2$.

Proposition 4.31. *An IFS acts on the set $K(\mathbb{R}^2)$ of compact sets of $\mathbb{R}^2$ (closed and bounded). For $P_0 \in K(\mathbb{R}^2)$ we generate a sequence $P_1 = T_1(P_0) \cup \cdots \cup T_n(P_0)$, $P_2 = T_1(P_1) \cup \cdots \cup T_n(P_1), \ldots$, such that $P_{k+1} = T_1(P_k) \cup \cdots \cup T_n(P_k)$. If this sequence converges to a set $P = T_1(P) \cup \cdots \cup T_n(P)$, this must be invariant for the described operation.*

Theorem 4.32. *Let $T_i : \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, \ldots, N$ be contractions. Let $T : K(\mathbb{R}^2) \to K(\mathbb{R}^2)$ be the collage function acting on a set $K(\mathbb{R}^2)$ of compact sets of $\mathbb{R}^2$ by $T(C) = T_1(C) \cup \cdots \cup T_n(C)$, where $T_i(C) = \{T_i(x, y); (x, y) \in C\}$. Then, there exists a unique compact set A in $\mathbb{R}^2$ satisfying $T(A) = A$. Moreover, for any compact set B,*

$$\lim_{k\to\infty} T^k(B) = A$$

with the Hausdorff metric in $K(\mathbb{R}^2)$.

Idea of proof.

1. We must define a distance between two compact sets in the plane, this is the Hausdorff metric in $K(\mathbb{R}^2)$.
2. With this metric, we need to show the collage function is a contraction in $K(\mathbb{R}^2)$. This means that $d_H(T(A), T(B)) \leq L d_H(A, B)$ for some $L < 1$.
3. Next, we apply the Fixed Point Theorem, which, as we said, works in more general spaces than the Euclidean space. Particularly, it works for $K(\mathbb{R}^2)$ equipped with the Hausdorff metric.
4. The fixed point theorem gives the existence of a compact set A, fixed by the map T, i.e, $T(A) = A$. Additionally, this fixed «point» is a global attractor and therefore, for all B we have that $\lim_{k\to\infty} T^k(B) = A$. ■

Definition 4.33 (Self-similar). In a fractal, if we enlarge any small part of them, we see a copy of the whole set. All fractal sets we have seen are self-similar. Every number of self-similar pieces has an associated contraction factor.

$$3^n \text{ self-similar pieces} \leftrightarrow \left(\frac{1}{2}\right)^n \text{ contraction factor.}$$

Definition 4.34 (Topological dimension). Let S be a subset of $\mathbb{R}^N$. We say that:

1. S has dimension zero, if each of its points has arbitrarily small neighbourhoods whose boundaries do not intersect the set. Actually, every totally disconnected set has dimension zero.
2. S has dimension k, if each point of S has arbitrarily small neighbourhoods whose boundaries intersect S in a set of dimension $k-1$, and k is the smallest natural number for which this happens.

All of fractals we have seen have topological dimension 1.

$$\dim_S = \frac{\log(\#\text{pieces})}{\log(\frac{1}{\text{contraction factor}})}$$

For fractals coming from Iterated Function Systems, number of self-similar pieces equals number of contractions. Contraction factor equals r.

$$\dim_S = \frac{\log(\#\text{contractions})}{\log(\frac{1}{r})}$$

This formula only works when the contraction factor r is the same for every contraction.

Definition 4.35 (Moran's formula). If we are applying N different affinities with contraction factors $r_1, \ldots, r_N$ (not necessarily equal), then the self-similar dimension of the fractal we obtain after iterating the IFS is the only real number $d > 0$ such that:

$$\sum_{k=1}^{N} r_k^d = r_1^d + \cdots + r_N^d = 1.$$

There is only one value of d for which $\sum_{k=1}^{N} r_k^d = 1$.

A good approximation of Hausdorff dimension is known as the Box-counting dimension or Minkowski dimension. It is denoted by $\dim_M$. For self-similar sets, $\dim_H = \dim_M = \dim_S$.

Definition 4.36 (Minkowskii's dimension). Let $K \subset \mathbb{R}^2$ be a bounded set. Then:

$$\dim_M = \lim_{r \to 0} \frac{\log N(r)}{\log(\frac{1}{r})},$$

where $N(r)$ is the number of squares with side r covering the set.

Definition 4.37 (Fractal). A plane set is called fractal if its Hausdorff dimension is strictly larger than its topological dimension.

5 Differential Equations

5.1 Introduction

Definition 5.1 (Corba parametritzada). Una *corba parametritzada* a $\mathbb{R}^n$ ($n \geq 2$) és una aplicació contínua:

$$\begin{array}{rccl} \gamma : & [a,b] & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & \gamma(t) = (\gamma_1(t), \dots, \gamma_n(t)). \end{array}$$

Es diu *paràmetre* de la corba a la variable t. Al conjunt imatge $\gamma^* = \{\gamma(t) \mid t \in [a,b]\}$ se'n diu *recorregut*, o *trajectòria*, de la corba. Si és contínua (resp. diferenciable) en direm que és contínua (resp. diferenciable).

Definition 5.2 (Corba de classe C^1). Diem que una corba $\gamma : [a,b] \longrightarrow \mathbb{R}^n$ és de classe C^1 (s'entén $C^1([a,b])$) si cada component $\gamma_j(t)$ amb $j = 1 \div n$ és de classe C^1 (derivable i amb derivada contínua, per a tot $t \in [a,b]$).

Definition 5.3 (Vector i recta tangents). Sigui $\gamma : I \longrightarrow \mathbb{R}^n$ una corba diferenciable, $t \in I$ tal que $\gamma'(t) \neq 0$.

1. El vector tangent a la corba γ en el punt t (o en $\gamma(t)$) és el vector $\gamma'(t)$:

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h} = (\gamma_1'(t), \dots, \gamma_n'(t))$$

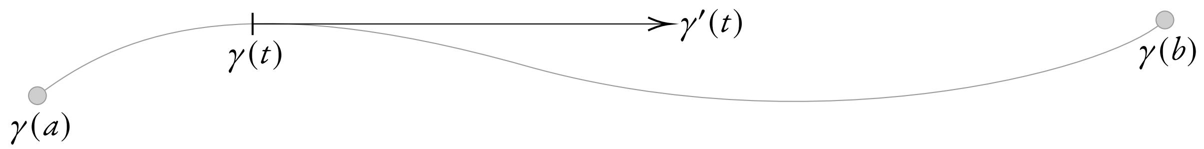


Figure 20: Vector tangent a la corba.

2. La recta tangent a γ en t (o en $\gamma(t)$) és la recta $\gamma(t) + \langle \gamma'(t) \rangle$.

Definition 5.4 (Solution). Consider:

$$f: \begin{array}{ccc} \mathcal{U} \subset (\mathbb{R} \times \mathbb{R}^n) & \longrightarrow & \mathbb{R}^n \\ (t, x) & \longmapsto & f(t, x) \end{array} \qquad I \subset \mathbb{R}$$

The function $x : I \longrightarrow \mathbb{R}^n$ is a solution of $x' = f(t, x)$ if $x'(t) = f(t, x(t))$ for all $t \in I$.

1. $x(t)$ is called *trajectory* or *solution*.
2. The trace of $x(t)$ in $\mathbb{R}^n$ is called the *orbit*.

If $x(t)$ is a solution, $f(t^*, x^*)$ is the tangent vector of $x(t)$ at time t^* (i.e. at $x(t^*) = x^*$)[7]. Constant solutions $x(t) \equiv c$ are called *equilibrium points*. They are solutions of $f(t, x) = 0$, for all t.

Example 5.5.

1. $x' = 0$. General solution: $x(t) = c, c \in \mathbb{R}$.
2. $x' = x$. General solution: $x(t) = ce^t, c \in \mathbb{R}$.
3. $x' = x^2$. General solution: $x(t) = -\frac{1}{t-c}, c \in \mathbb{R}$.
4. $x' = t + 1$. General solution: $x(t) = t^2 + t + c, c \in \mathbb{R}$.
5. $x' = g(t)$. General solution: $x(t) = G(t) + c, c \in \mathbb{R}$.
6. $x' = xt$. General solution: $x(t) = ce^{\frac{t^2}{2}}, c \in \mathbb{R}$.

We obtained infinitely many solutions for every differential equation. The family of solutions is called the **general solution** of the differential equation and it depends on one or more parameters.

Definition 5.6 (Cauchy problem). If we require the solution to pass through a given point (i.e. impose an initial condition $x(t_0) = x_0$), then we obtain a particular solution. A differential equation together with an initial condition is called an initial value problem or a Cauchy problem.

$$\begin{cases} x' = f(t, x) \\ x(t_0) = x_0 \end{cases} \quad \text{where} \quad f: \begin{array}{ccc} \Omega & \longrightarrow & \mathbb{R}^n \\ (t, x) & \longmapsto & f(t, x) \end{array} \tag{5.1}$$

and $(t_0, x_0) \in \Omega \subset \mathbb{R} \times \mathbb{R}^n$.

Theorem 5.7 (Existence and uniqueness theorem). *Under some regularity, assumptions on f (e.g. C^r, $r \geq 1$), the initial value problem from (5.1) has a solution:*

$$x: \begin{array}{ccc} I \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ t & \longmapsto & x(t) \end{array}$$

of class C^r such that $x(t)$ is maximal and unique. This means that if $\tilde{x} : J \subset \mathbb{R} \longrightarrow \mathbb{R}^n$ is another solution, then $J \subset I$ and $\tilde{x}(t) = x(t)$, for all $t \in J$.

Remark 5.8. What this tells us is that any other solution going through (t_0, x_0) needs to be a piece of this one. Thus, different trajectories do not intersect. When the equation is autonomous ($x' = f(x)$) orbits do not intersect either.

[7] If $n = 1$, we draw $x(t)$ in a $t - x$ graph, and then $f(t, x)$ is the slope of the tangent line at $(t, x(t))$.

5.2 Differential Equations in dimension 1

5.2.1 Separation of variables

Proposition 5.9. *Let g, h continuous, $h(x) \neq 0$, for all x. Suppose $x' = g(t) \cdot h(x)$ and $x(t)$ a solution. Then, $x(t) = u = H^{-1}(G(t) + c)$.*

Proof. Suppose $x(t)$ is a solution. We can write $\frac{x'(t)}{h(x(t))} = g(t)$. We integrate over t:

$$\int \frac{x'(t)}{h(x(t))}\, dt = \int g(t)\, dt\,.$$

We change variables to $u = x(t) \implies du = x'(t)\, dt$ obtaining:

$$\int \frac{1}{h(u)}\, du = \int g(t)\, dt\,.$$

Rewrite as $H(u) = G(t) + c$. There exists H because $\frac{1}{h}$ continuous, and so $H' = \frac{1}{h}$ and $G' = g$. Then, $x(t) = u = H^{-1}(G(t) + c)$. And H^{-1} exists because $\frac{1}{h} \neq 0$ and $H' \neq 0$, so H is monotone. We check that $x(t)$ is a solution, using the Inverse Function Theorem:

$$x'(t) = \frac{d}{dt}(H^{-1}(G(t) + c)) = \frac{1}{H'(H^{-1}(G(t) + c))} \cdot G'(t) = \frac{1}{\frac{1}{h(H^{-1}(G(t)+c))}} \cdot g(t).$$ ∎

Remark 5.10. In practice, we shall «separate» $\frac{dx}{dt}$ and integrate directly:

$$\int \frac{dx}{h(x)} = \int g(t)\, dt\,.$$

We are renaming $u = x$.

Example 5.11. Let $x' = \frac{t}{x^2}$. Formally,

$$\int x'(t) x(t)^2\, dt = \int t\, dt \quad \begin{Bmatrix} u = x(t) \\ du = x'(t)\, dt \end{Bmatrix} \implies \int u^2\, du = \int t\, dt \implies u^3 = \frac{t^2}{2} + c.$$

And so $u = x(t) = \left(\frac{3t^2}{2} + k\right)^{\frac{1}{3}}$. We can also prove it via 5.10:

$$\frac{dx}{dt} = \frac{t}{x^2} \iff \int x^2\, dx = \int t\, dt \iff \frac{x^3}{3} = \frac{t^2}{2} + c.$$

Example 5.12. Let $x' = x$. Observe that $x(t) \equiv 0$ is a solution. All others $x(t) > 0$ or $x(t) < 0$ for all t, because **solutions do not intersect**.

1. If $x(t) > 0$ for all t:

$$\int \frac{dx}{x} = \int dt \implies \ln(x) = t + c \implies x(t) = e^{t+c} = e^c e^t \implies x(t) = ke^t,\ k > 0.$$

2. Analogously, if $x(t) < 0$ for all t:

$$\int \frac{dx}{x} = \int dt \implies \ln(-x) = t + c \implies x(t) = -e^{t+c} = -e^c e^t \implies x(t) = ke^t,\ k < 0.$$

Joining the three cases, $x(t) = ke^t$, for $k \in \mathbb{R}$.

Example 5.13. Let $x' = a(t)x$, which is a linear equation of order 1 and homogeneous (no independent term). Since $x(t) \equiv 0$ is a solution, all others satisfy $x(t) \neq 0$, for all t. If $x \neq 0$, then $\int \frac{dx}{x} = \int a(t)\,dt$.

$$\int \frac{1}{x}\,dx = \ln|x| = \begin{cases} \ln(x), & \text{si } x > 0; \\ \ln(-x), & \text{si } x < 0; \end{cases} \qquad \int a(t)\,dt = A(t) + c,$$

where $A'(t) = a(t)$ is any primitive. And so $|x(t)| = e^{A(t)}e^c = \tilde{c}e^{A(t)}$, $\tilde{c} > 0$, implies that:

$$x(t) = k \cdot e^{A(t)},\ k \in \mathbb{R},\ \text{which covers the three cases.}$$

5.2.2 Linear equations

We can dissect the following cases:

1. Special case, linear with constant coefficients $x' = ax + b$:

$$\int \frac{dx}{ax+b} = \int dt \implies x(t) = ke^{at} - \frac{b}{a},\ k \in \mathbb{R}.$$

2. Special case, linear homogeneous $x' = a(t)x$, of 5.13.
3. General case, which we will explain now $x' = a(t)x + b(t)$.

Proposition 5.14. *Suppose $x' = a(t) \cdot x + b(t)$, $a(t), b(t)$ functions of t and $x(t)$ a solution. Then,*

$$x(t) = e^{A(t)}\left(c + \int e^{-A(t)}b(t)\,dt\right),\ c \in \mathbb{R}.$$

Proof. Let $\mu(t) = e^{-A(t)}$, where $A'(t) = a(t)$.

$$\begin{aligned} \mu(t) \cdot x' = \mu(t)a(t) \cdot x + \mu(t)b(t) &\iff x'e^{-A(t)} - x \cdot a(t)e^{-A(t)} = e^{-A(t)}b(t) \\ &\iff \frac{d}{dt}\left(xe^{-A(t)}\right) = e^{-A(t)}b(t). \end{aligned}$$

If we integrate both sides, the general solution is, as we wanted to prove:

$$xe^{-A(t)} = \int e^{-A(t)}b(t)\,dt + c \implies x(t) = e^{A(t)}\left(c + \int e^{-A(t)}b(t)\,dt\right),\ c \in \mathbb{R}.$$ ■

Example 5.15. Let $x' = -\frac{2}{t}x + 4t$. By the formula $x' = a(t)x + b(t)$, $\mu(t) = \exp\{\int a(t)\,dt\}$:

$$\mu(t) = \exp\left(\int \frac{2}{t}\,dt\right) = \exp(2\ln(|t|)) = t^2.$$

Then,

$$t^2x' + 2tx = \frac{d}{dt}(t^2x) = 4t^3 \implies t^2x = t^4 + c \implies x(t) = t^2 + \frac{c}{t^2}.$$

5.2.3 Change of variables

Start with $x' = f(t,x)$ and consider $t = t(s)$ where s is the new independent variable. Then,

$$\frac{d}{ds}x(t(s)) = \frac{dx}{dt}\cdot\frac{dt}{ds} = f(t,x)\cdot\frac{dt}{ds} \implies x'(s) = g(s,x(s)) \implies x' = g(s,x).$$

Formally,

$$\begin{array}{rccc} \varphi: & J & \longrightarrow & I \\ & s & \longmapsto & \varphi(s) \end{array} \qquad \varphi\in C^1,\ \varphi'(s)\neq 0,\ \forall s\in J.$$

Define $y(s) = x(\varphi(s))$. Then:

$$y'(s) = x'(\varphi(s))\cdot\varphi'(s) = f(\varphi(s), x(\varphi(s)))\cdot\varphi'(s) = f(\varphi(s), y(s))\varphi'(s) =: g(s, y(s)).$$

If $y(s)$ is a solution, then $x(t) = y(\varphi^{-1}(t))$ satisfies:

$$x'(t) = y'(\varphi^{-1}(t))\cdot(\varphi^{-1})'(t) = g(\varphi^{-1}(t), y(\varphi^{-1}(t)))\cdot\frac{1}{\varphi'(\varphi^{-1}(t))} = f(t, x(t)).$$

Example 5.16. Let $x' = (x^2+9)t^2$. Change $t(s) = \sqrt[3]{3s}$:

$$\frac{dx}{ds} = \frac{dx}{dt}\cdot\frac{dt}{ds} = (x^2+9)(t(s))^2\frac{1}{3}(3s)^{-\frac{2}{3}}\cdot 3 = (x^2+9)\cdot(3s)^{\frac{2}{3}}\cdot(3s)^{-\frac{2}{3}} = x^2+9.$$

The new equation is $x' = x^2+9$, the resulting equation does not depend on t (autonomous). Autonomous equations are simpler to understand qualitatively, as we will see later on.

Exercise 5.17. *In general, if $g\neq 0$ and $x' = h(x)\cdot g(t)$ the change $t(s) = G^{-1}(s)$ with $G' = g$ transforms the equation to $x' = h(x)$ which is autonomous.*

Example 5.18. Let $x' = tx + t^3x^2$. Let the change $y = \frac{1}{x}$, and $x = \frac{1}{y}$:

$$y' = -\frac{1}{x^2}(tx + t^3x^2) = -\frac{t}{x} - t^3 \implies y' = -ty - t^3.$$

Taking $\mu(t) = e^{\frac{t^2}{2}}$ $(a(t) = -t)$,

$$e^{\frac{t^2}{2}}y' + te^{\frac{t^2}{2}}y = -t^3e^{\frac{t^2}{2}} \implies (e^{\frac{t^2}{2}}y)' = -t^3e^{\frac{t^2}{2}},\ y(t) = e^{-\frac{t^2}{2}}((-t^2+2)e^{\frac{t^2}{2}} + c) = 2 - t^2 + ce^{-\frac{t^2}{2}}.$$

Undoing the change, $x(t) = \frac{1}{y(t)}$, $c\in\mathbb{R}$ or $x(t)\equiv 0$. **Warning**. Do not forget the solution $x(t)\equiv 0$, it is easy to forget it when doing the change $y = \frac{1}{x}$.

A more general change. Start with $x' = f(t,x)$, change $y = \phi(t,x)$ (suppose $\frac{\partial\phi}{\partial x}\neq 0$). By the IFT we can solve for x, i.e. $\exists\psi$ such that $x = \psi(t,y)$. Then,

$$y'(t) = \frac{\partial\phi}{\partial x}(t,x(t))\cdot x'(t) + \frac{\partial\phi}{\partial t}(t,x(t)) = \frac{\partial\phi}{\partial x}(t,\psi(t,y(t)))f(t,\psi(t,y(t))) + \frac{\partial\phi}{\partial t}(t,\psi(t,y(t))),$$

and we define it as $g(t,y(t))$.

Example 5.19. Let $x' = \frac{x+t}{t}$, change $y = \frac{x}{t}$ so that $x(t) = t\,y(t)$. Therefore,

$$y' = x'\frac{1}{t} + x\left(-\frac{1}{t^2}\right) = \frac{x+t}{t^2} - \frac{x}{t^2} = \frac{1}{t} \implies y' = \frac{1}{t}.$$

Integrating, $y(t) = \ln|t| + C$. Reverse changing, $x(t) = t(\ln|t| + C)$.

5.2.4 Exact equations

Definition 5.20 (Exact equation). Let P, Q polynomials $\mathbb{C}^1(D)$, $D \subset \mathbb{R}^2$ such that:

$$P(x,y) + Q(x,y)y' = 0 \iff P_y = Q_x \iff P(x,y)\,dx + Q(x,y)\,dy = 0.$$

The previous equation is exact if there exists a **potential function**, that is, a C^1-smooth function $U(x,y) : D \longrightarrow \mathbb{R}$, $D \subset \mathbb{R}^2$, such that:

$$U_x = P, U_y = Q \implies U_{xy} = P_y, U_{yx} = Q_x.$$

If U is smooth, then $U_{xy} = U_{yx}$ and $P_y = Q_x$.

Proof. We write:

$$U_x(x,y) + U_y(x,y)\cdot y' = 0 \iff U_x(x,y(x)) + U_y(x,y(x))y'(x) = 0$$

i.e. $\frac{d}{dx}U(x,y(x)) = 0 \implies U(x,y(x)) = C$ (implicit solution). If, moreover, $U_y \neq 0$, then by the Implicit Function Theorem we can solve for y: there exists $y(x)$ an explicit solution such that $U(x,y(x)) = C$.

$$y'(x) = -\frac{U_x(x,y(x))}{U_y(x,y(x))} \implies P + Q\frac{-U_x}{U_y} = 0. \qquad \blacksquare$$

1. *How do we know if U exists?* If U exists and is C^2, then $U_{xy} = U_{yx} \implies P_y = Q_x$, a necessary condition. If P, Q are differentiable and the domain is convex, this is also a sufficient condition.
2. *If we know that U exists, how do we find it?* We know that:

$$\left.\begin{aligned} U_x &= P \\ U_y &= Q \end{aligned}\right\} \implies \left.\begin{aligned} U(x,y) &= \textstyle\int P(x,y)\,dx + c(y) \\ U_y(x,y) &= \tfrac{d}{dy}\textstyle\int P(x,y)\,dx + c'(y) \end{aligned}\right\}.$$

   From the second one we obtain a differential equation for $c'(y)$ and then we find $c(y)$ and substitute in the first one.

Example 5.21. Propose $2x + y^2 + 2xyy = P(x,y) + Q(x,y)y' = 0$. Because $P_y = \frac{\partial P}{\partial y} = 2y$ and $Q_x = \frac{\partial Q}{\partial x} = 2y$, the equation is exact. Regarding the potential function, we want to find $U(x,y)$ such that $U_x = P$ and $U_y = Q$:

$$U_x = P \implies U(x,y) = \int (2x + y^2)\,dx + c(y) = x^2 + y^2x + c(y);$$

$$U_y = Q \implies 2yx + c'(y) = 2xy \implies c'(y) = 0 \implies c(y) = c' \in \mathbb{R}.$$

And so $U(x,y) = x^2 + y^2x + c'$ gives the implicit solution $x^2 + y^2x = K$ (group the constants). The explicit solution $y = \pm\sqrt{\frac{K-x^2}{x}}$ is not defined everywhere.

5.2.5 Integrating factor

Even if the equation is not exact, sometimes it can be transformed into an exact one, multiplying all of it by an integrating factor, $\mu(x, y)$.

Definition 5.22 (Integrating factor). We say that the equation $P(x, y)+Q(x, y)y' = 0$ admits an integrating factor $\mu(x, y)$ if $\mu P + \mu Q y'\mu(P + Qy') = 0$ is exact, i.e. if $(\mu P)_y = (\mu Q)_x$. If $\mu(x, y) \neq 0$ for all (x, y), the solutions of both equations coincide.

In general it is not easy to find integrating factors, but if we assume that μ depends only on y and x, then it is easier. If $\mu(x, y) = \mu(x)$, then $(\mu P)_y = \mu P_y$ and $(\mu Q)_x = \mu_x Q + \mu Q_x$:

$$(\mu P)_y = (\mu Q)_x \implies \mu P_y = \mu_x Q + \mu Q_x \implies \mu_x = \mu\left(\frac{P_y - Q_x}{Q}\right).$$

$\frac{P_y - Q_x}{Q}$ depending only on x is a necessary condition and $\mu'(x) = \mu(x) \cdot \frac{P_y - Q_x}{Q}$ allows us to find μ.

Exercise 5.23. *The same holds for $\mu(x, y) = \mu(y)$, that is, $\frac{Q_x - P_y}{P}$ depending only on y is a necessary condition and $\mu'(y) = \mu(y) \cdot \frac{Q_x - P_y}{P}$ allows us to find μ.*

Example 5.24. Let $(3xy + y^2) + (x^2 + xy)y' = P(x, y) + Q(x, y)y' = 0$. Is it exact? Evidently, it is not:

$$P_y = 3x + 2y,\ Q_x = 2x + y \implies P_y \neq Q_x.$$

Sometimes it can be transformed into an exact one, multiplying all of it by an integrating factor, $\mu(x, y)$. So let's try and find this integrating actor:

$$\frac{P_y - Q_x}{Q} = \frac{-2x - y + (3x + 2y)}{x^2 + xy} = \frac{x + y}{x(x + y)} = \frac{1}{x} \implies \mu(x, y) = \mu(x).$$

Now we can find $\mu'(x) = \mu(x) \cdot \frac{P_y - Q_x}{Q} = \mu(x) \cdot \frac{1}{x}$:

$$\int \frac{d\mu}{\mu} = \int \frac{dx}{x} \implies \ln|\mu| = \ln|x| + c \implies |\mu(x)| = |x| \implies \mu(x) = x.$$

We can take $c = 0$ because finding one integrating factor that works is enough. We are not looking for uniqueness but existence. Multiplying by $\mu(x)$:

$$2x^2y + y^2x + (x^3 + yx^2)y' = 0 \implies \tilde{P}_y = 3x^2 + 2xy,\ \tilde{Q}_x = 3x^2 + 2xy.$$

5.2.6 Autonomous equations

Definition 5.25 (Autonomous equation). An autonomous equation is $x' = f(x)$ such that $f \in C^1$ at least and:

$$\begin{array}{rccc} f: & J \subset \mathbb{R} & \longrightarrow & \mathbb{R} \\ & x & \longmapsto & f(x) \end{array}$$

Proposition 5.26. *For all $x_0 \in J$, the Cauchy problem $\{x' = f(x) \mid x(0) = x_0\}$ (x_0 is given) has a unique solution which we denote by $x(t)$ or by $\varphi_{x_0}(t)$ or $\varphi(t, x_0)$. This solution must satisfy:*

$$\varphi'_{x_0}(t) = f(\varphi_{x_0}(t)) \text{ and } \varphi_{x_0}(0) = x_0.$$

Solutions do not intersect. *One can also conisder initial conditions of the form $x(t_0) = x_0$, $t_0 \neq 0$.*

1. *Explicit solving:* Autonomous equations can be solved by separation of variables. But not always can $\int \frac{1}{f(x)}\,dx$ can be expressed in terms of elementary functions.
2. *Qualitative study:* Analysis of the asymptotic behaviour of solutions. It is simple and often is all we need. We draw the results on the phase portrait (set of orbits without temporal parametrization).

Definition 5.27 (Equilibrium solutions). These are the constant solutions $x(t) \equiv p$, $p \in \mathbb{R}$. $x(t) \equiv p$ is an equilibrium solution if, and only if, $f(p) = 0$.

Proof. $x(t) \equiv p$ for some $p \in \mathbb{R}$ means $f(x(t)) = 0 \iff f(p) = 0$. ∎

Equilibrium solutions can be:

1. **Attracting**: if they *attract* solutions with nearby initial conditions.

   Definition 5.28 (Attracting). $x(t) \equiv p$ is attracting if $\exists \varepsilon > 0$ such that $\forall x_0$ with $|x_0 - p| < \varepsilon$ we have $\varphi_{x_0}(t) \xrightarrow{t\to+\infty} p$.

2. **Repelling**: if they *repel* solutions with nearby initial conditions.

   Definition 5.29. $x(t) \equiv p$ is repelling if $\exists \varepsilon > 0$ such that $\forall x_0$ with $|x_0 - p| < \varepsilon$ we have $\varphi_{x_0}(t) \xrightarrow{t\to-\infty} p$.

3. **Neutral**: if they are neither attracting nor repelling.

Property 5.30.

1. *If $x(t)$ is a solution such that $x(t) \xrightarrow{t\to\infty} p$, then $\tilde{x}(t) \equiv p$ is an equilibrium solution (i.e. $f(p) \equiv 0$).*
2. *Every non-constant solution $\varphi_{x_0}(t)$ is strictly monotonous.*

$$\begin{cases} \varphi_{x_0}(t) & \text{increases} \iff f(x_0) > 0 \\ \varphi_{x_0}(t) & \text{decreases} \iff f(x_0) < 0 \end{cases} \quad (f(x_0) = \varphi'_{x_0}(0).$$

Proof.

1. Suppose $x(t)$ is a solution such that $x(t) \xrightarrow{t\to\infty} p$. Then:

$$f(x(t)) \xrightarrow{t\to\infty} f(p) \implies x'(t) \xrightarrow{t\to\infty} f(p).$$

Suppose $f(p) \equiv k > 0$ and $t \to +\infty$. Then, there exists $t_0 > 0$ such that for all $t > t_0$ we have $x'(t) > \frac{k}{2}$[8]. Via Mean Value Theorem,

$$x(t) - x(t_0) = x'(\tilde{t}) = (t - t_0) \geq \frac{k}{2}(t - t_0), \quad x(t) - x(t_0) \to p - x(t_0),\ t - t_0 \to +\infty.$$

Which means that $p - x(t_0) \geq +\infty$, but we saw that $p - x(t_0) \xrightarrow{t\to\infty} 0$, contradiction. Cases $k < 0$ and $t \to -\infty$ are treated similarly.

2. Suppose there exists $\tilde{t}$ such that $\varphi'_{x_0}(\tilde{t}) = 0$ and define $p := \varphi_{x_0}(\tilde{t})$. Then,

$$f(p) = f(\varphi_{x_0}(\tilde{t})) = \varphi'_{x_0}(\tilde{t}) = 0 \implies x(t) \equiv p \text{ is a solution.}$$

But $\varphi_{x_0}(\tilde{t}) = p$, and by uniqueness of the solution $\varphi_{x_0}(t) \equiv x(t) \equiv p$, which is a contradiction. An alternative proof would be that if $x(t)$ is a solution, then $\tilde{x}(t) = x(t + c)$ is also a solution, for all $c \in \mathbb{R}$. In other words, if $x(t)$ is a solution all its horizontal translates are also solutions:

$$\tilde{x}'(t) = x'(t + c) = f(x(t + c)) = f(\tilde{x}(t)).$$

But solutions cannot intersect, so $x(t)$ cannot have maxima or minima. ■

Example 5.31. If we have the expression $x' = (1 - \frac{x}{20})^3(\frac{x}{5} - 1)x^7$, the equilibria is met at $x = 20$, $x = 5$, $x = 0$. To study the monotonicity of solutions we only need to look at the sign of $f(x) = (1 - \frac{x}{20})^3(\frac{x}{5} - 1)x^7$.

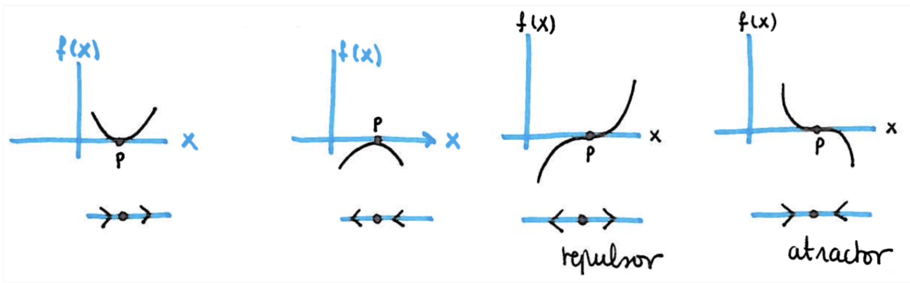
Theorem 5.32 (Local stability criterium). *Let $f(p) = 0$ and p an isolated zero of f, $x' = f(x)$.*

1. *If $f'(p) < 0$, then the solution $x(t) \equiv p$ is attracting.*
2. *If $f'(p) > 0$, then the solution $x(t) \equiv p$ is repelling.*
3. *If $f'(p) = 0$, then the solution $x(t) \equiv p$ is indifferent, and its stability depends on the higher order derivatives (if they exist).*

Proof.

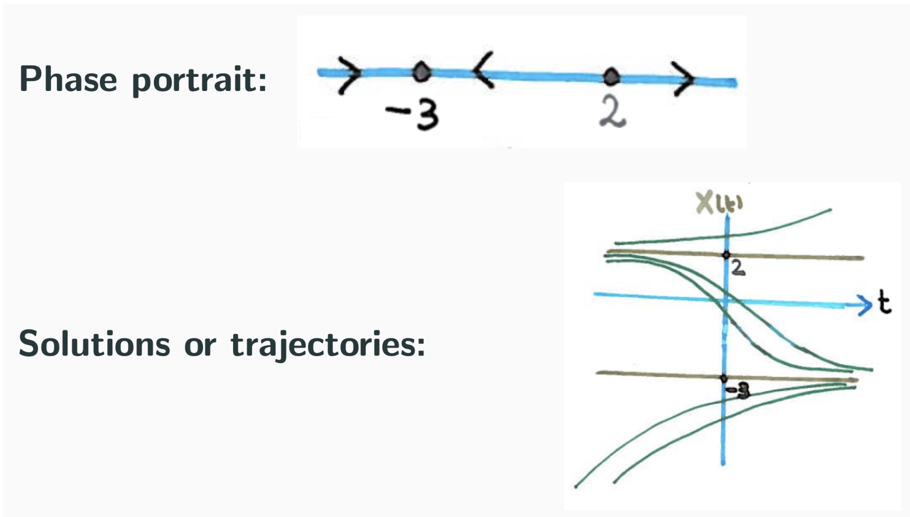
1. If $f'(p) < 0$ and $f(p) = 0$, then f is decreasing at p. Furthermore:
   - $x \lesssim p \implies f(x) > 0$, and $\varphi'_x(0) > 0$. In consequence, $\varphi_x(t)$ is increasing.
   - $x \gtrsim p \implies f(x) < 0$, and $\varphi'_x(0) < 0$. In consequence, $\varphi_x(t)$ is decreasing.
2. Exercise.
3. The proof is in this next photo:

[8] When $t \to \infty$, $\forall \varepsilon > 0$ exists t_0 such that if $t > t_0$, $|f(p) - x'(t)| < \varepsilon$. Using $f(p) = k$, $|k - x'(t)| < \varepsilon$, and taking $\varepsilon = \frac{k}{2}$ we have $-\varepsilon < k - x'(t) < \varepsilon$, so $x'(t) > k - \varepsilon = \frac{k}{2}$.



Indifferent equilibrium points can be attracting, repelling or neutral. ■

Example 5.33. Let $x' = x^2 + x - 6 = (x+3)(x-2)$. The equilibria is met at $f(x) = (x-3)(x-2) = 0$ and so $x(t) \equiv -3$, $x(t) \equiv 2$. Regarding the stability, $f'(x) = 2x + 1$, and at $p = -3, 2$: $f'(-3) = -5 < 0$ (i.e. $x(t) \equiv -3$ is attracting) and $f'(2) = 5 > 0$ (i.e. $x(t) \equiv 2$ is repelling). Phase and solutions can be drawn.



5.3 Differential Equations in dimension N

Let us consider systems of n autonomous differential equations in $\mathbb{R}^n$:

$$\begin{cases} x_1' = f_1(x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(x_1, \dots, x_n) \end{cases} \qquad \text{where} \quad \underset{i=1,\dots,n}{f_i} : \Omega \subset \mathbb{R}^n \longrightarrow \mathbb{R} \text{ are } C^r,\ r \geq 1.$$

Abbreviated: $X' = f(X)$, X solution is a parametrized curve in $\mathbb{R}^n$ satisfying the previous equation, where:

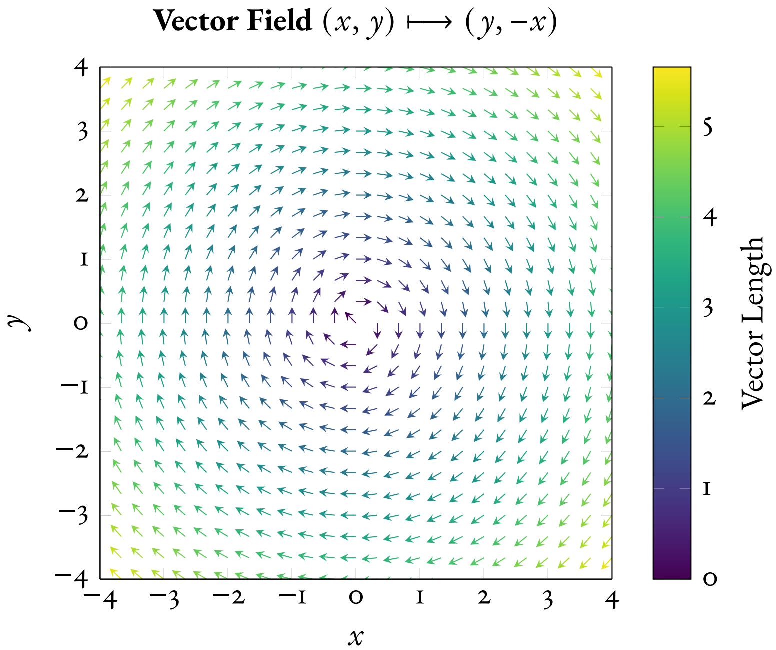
$$\begin{array}{rccl} X: & I \subset \mathbb{R} & \longrightarrow & \mathbb{R}^n \\ & t & \longmapsto & (x_1(t), \dots, x_n(t)) = X(t) \end{array} \qquad \begin{array}{rccl} f: & \Omega & \longrightarrow & \mathbb{R}^n \\ & X & \longmapsto & (f_1(X), \dots, f_n(X)) = f(X) \end{array}$$

Example 5.34. Let $x' = x + y$ and $y = y'$. Then, $(x(t), y(t)) = (te^t, e^t)$ is a solution because:

$$\begin{cases} x'(t) = e^t + te^t \text{ while } x(t) + y(t) = te^t + e^t, \\ y'(t) = e^t \text{ while } y(t) = e^t. \end{cases}$$

Definition 5.35 (Tanger vector). $X'(t) = (x_1'(t), \dots, x_n'(t))$ is the tangent vector to the curve $X(t) = (x_1(t), \dots, x_n(t))$ at time t. Therefore, since $X'(t) = f(X(t))$, we obtain that the tangent vector is given by $f(X(t))$.

Definition 5.36 (Vector field). For each $X \in \mathbb{R}^n$, we assign a vector $f(X)$ which must be tangent to the orbit passing through X.



Remark 5.37. Using existence and uniqueness theorem and the fact that we are dealing with a differential autonomous equation, for each $X_0 \in \Omega$ (domain) there exists a unique orbit passing through the point X_0 (corresponding to ∞ solutions). Since, as we said, the differential equation is autonomous, vectors in the vector field only depend on the point $X = (x_1, \dots, x_n)$ and NOT on t.

Proposition 5.38. *If f is C^1, orbits do not cross.*

Proof. If they crossed transversally, we would have two different tangent vectors at the same point X, but this is impossible because $f(X)$ is one single vector. ■

Exercise 5.39. *Prove the general case using Existence and Uniqueness, and the fact that if $X(t)$ is a solution, then $X(t + c)$ is also a solution.*

Definition 5.40. The phase portrait is the partition of the phase space into different orbits. The equilibrium point or solution is the constant solution $X(t) \equiv p$.

Definition 5.41 (Stable equilibrium point). An equilibrium $p \in \Omega$ is stable if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $|x_0 - p| < \delta$, then $\varphi_{x_0}(t)$ is defined for all $t \geq 0$ and satisfies $|\varphi_{x_0}(t) - p| < \varepsilon$, for all $t \geq 0$.

Definition 5.42 (Asymptotically stable equilibrium point). An equilibrium $p \in \Omega$ is asymptotically stable or attracting if is stable and there exists $\nu > 0$ such that if $|x_0 - p| < \nu$, then $\varphi_{x_0}(t) \xrightarrow{t\to\infty} p$.

Definition 5.43 (Repelling equilibrium point). The equilibrium point $p \in \Omega$ is repelling if it is attracting for the vector field obtained by the time change $s = -t$.

Remark 5.44. A good understading of linear systems will be useful to study the (local) stability of equilibrium points (as in discrete dynamical systems). $X' = AX$, where A is an $n \times n$ matrix and $X : \mathbb{R} \longrightarrow \mathbb{R}^n$. Phase space $\Omega = \mathbb{R}^n$. Solutions are defined for all $t \in \mathbb{R}$. Observe that $X(t) \equiv 0$ is always an equilibrium solution.

Suppose $v = (v_1, \ldots, v_n)$ is an e-vector of A with e-value $\lambda \in \mathbb{R}$. For any $\alpha \in \mathbb{R}$ at the point $X = \alpha v$ the vector field is:

$$AX = A\alpha v = \alpha A v = \alpha\lambda v = \lambda(\alpha v) = \lambda X.$$

Hence, the line of e-vectors $\{\alpha v \mid \alpha \in \mathbb{R}\}$ is invariant by the vector field. On the line $\{\alpha v \mid \alpha \in \mathbb{R}\}$ we can write the solution as $X(t) = \alpha(t) \cdot v$ and then, the equation $X' = \lambda X$ becomes:

$$\alpha'(t) \cdot v = \lambda\alpha(t)v \implies \alpha' = \lambda\alpha \implies \alpha(t) = ce^{\lambda t}.$$

Hence $(x_1(t), \ldots, x_n(t)) = ce^{\lambda t}(v_1, \ldots, v_n)$ (or, conversely, $X(t) = ce^{\lambda t}v$) is a solution. If $\lambda > 0$, $X(t) \xrightarrow{t\to\infty} \infty$, and if $\lambda < 0$, $X(t) \xrightarrow{t\to\infty} 0$.

Example 5.45. The first fundamental example has for matrix:

$$A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R} \setminus \{0\},$$

and can be dissected in the following cases:

1. $0 < \lambda < \mu$,
2. $0 < \mu < \lambda$,
3. $0 < \lambda = \mu$,
4. $\mu < \lambda < 0$,
5. $\lambda < \mu < 0$,
6. $\lambda = \mu < 0$,
7. $\lambda < 0 < \mu$,
8. $\mu < 0 < \lambda$.

The eigenvalues are λ, μ and eigenvectors $v_\lambda = (1, 0)$ and $v_\mu = (0, 1)$ i.e. the axis are invariant. Furthermore,

$$\begin{cases} x' = \lambda x \\ y' = \mu y \end{cases} \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 e^{\lambda t} \\ c_2 e^{\mu t} \end{pmatrix}.$$

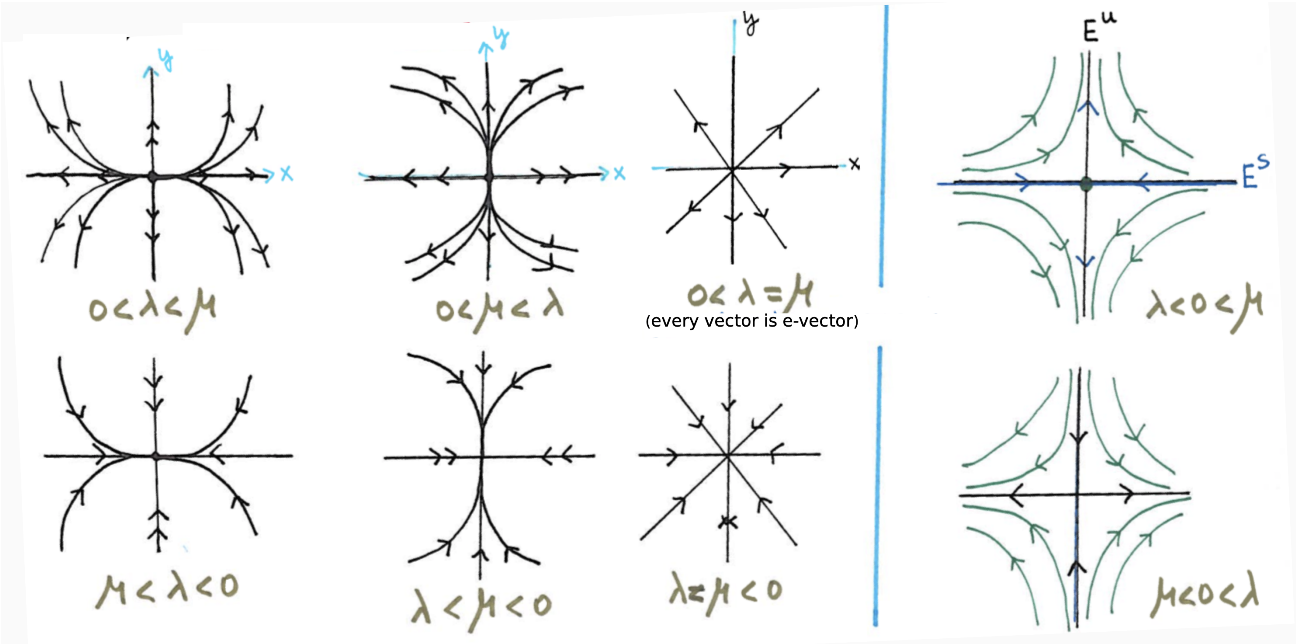


Figure 21: First fundamental example cases.

Example 5.46. The second fundamental is described by:

$$\begin{cases} x' = \lambda x + y \\ y' = \lambda y \end{cases} \qquad y(t) \equiv c_2 e^{\lambda t} \implies x' = \lambda x + c_2 e^{\lambda t} \text{ (linear).}$$

The solution is of the form:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (c_2 t + c_1) e^{\lambda t} \\ c_2 e^{\lambda t} \end{pmatrix}.$$

And has for matrix:

$$A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}, \ \lambda \in \mathbb{R} \setminus \{0\},$$

and can be dissected in the cases $\lambda < 0$ and $\lambda > 0$:

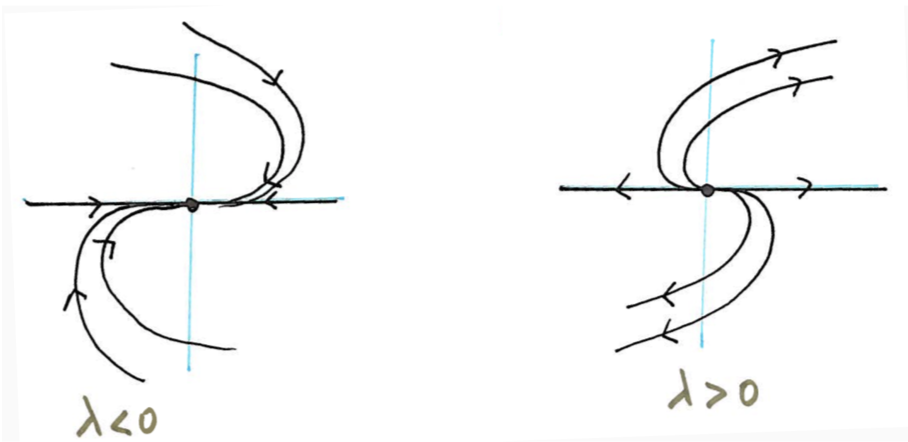


Figure 22: Second fundamental example cases.

Example 5.47. The third fundamental is described by:

$$\begin{cases} x' = \alpha x - \beta y \\ y' = \beta x + \alpha y \end{cases} \qquad y(t) \equiv c_2 e^{\lambda t} \implies \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\alpha t} \begin{pmatrix} \cos \beta t & -\sin \beta t \\ \sin \beta t & \cos \beta t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

And has for matrix:

$$A = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}, \ \alpha \in \mathbb{R} \setminus \{0\}, \ \beta \in \mathbb{R}.$$

And can be dissected in the cases $\alpha, \beta > 0$, $\alpha, \beta < 0$, $\alpha < 0, \beta > 0$ and $\alpha > 0, \beta < 0$:

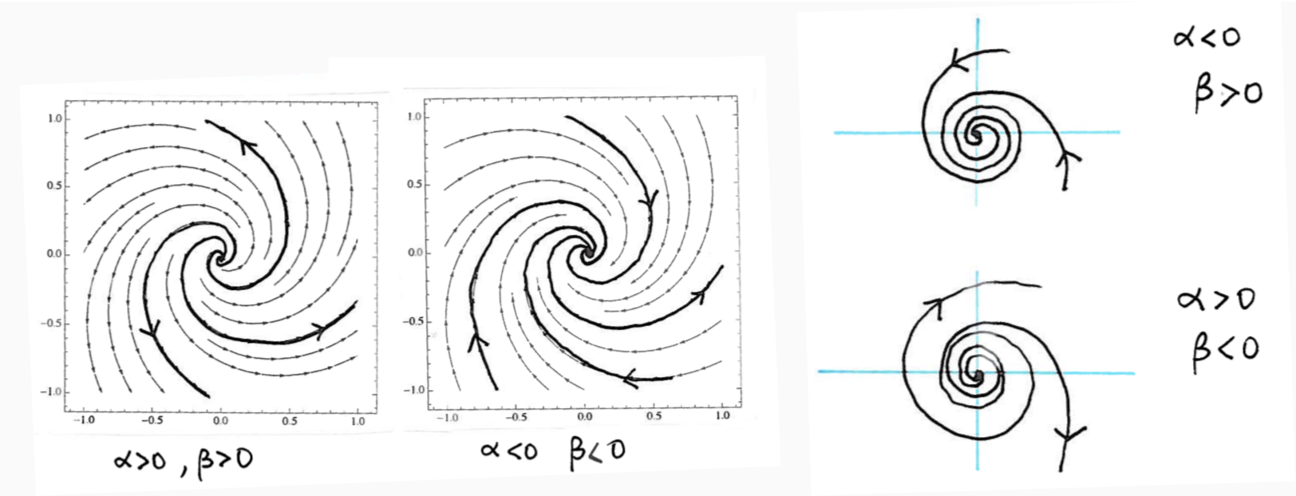


Figure 23: Second fundamental example cases.

E-vals are $\lambda_\pm = \alpha \pm \beta i$ and e-vectors are $(1, \mp i)$.

Proof.

1. We look for a complex solution using one complex e-value and its e-vector:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = e^{(\alpha - \beta i)t} \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{\alpha t}(\cos(-\beta t) + i \sin(-\beta t)) \begin{pmatrix} 1 \\ i \end{pmatrix} = e^{\alpha t} \begin{pmatrix} \cos \beta t - i \sin \beta t \\ \sin \beta t + i \cos \beta t \end{pmatrix}.$$

2. We can write:

$$X(t) = e^{\alpha t} \begin{pmatrix} \cos \beta t \\ \sin \beta t \end{pmatrix} + i e^{\alpha t} \begin{pmatrix} -\sin \beta t \\ \cos \beta t \end{pmatrix}.$$

If $X(t) = X_1(t) + iX_2(t)$ is a complex solution, then $X_1(t)$ and $X_2(t)$ are real solutions. That is because if $X'(t) = AX(t)$, then $X_1'(t) + iX_2'(t) = A(X_1(t) + iX_2(t))$ and hence $X_1'(t) = AX_1(t)$ and $X_2'(t) = AX_2(t)$[9]. We need to prove the following proposition in order to conclude the demonstration.

Proposition 5.48 (Linearity principle, general for linear systems in $\mathbb{R}^2$). *If $X_1(t)$ and $X_2(t)$ are solutions, and the vectors $X_1(0)$ and $X_2(0)$ are linearly independent, then the general solution is $X(t) = C_1X_1(t) + C_2X_2(t)$.*

[9] Application: $X_1(t) = e^{\alpha t} \begin{pmatrix} \cos \beta t \\ \sin \beta t \end{pmatrix}$ and $X_2(t) = e^{\alpha t} \begin{pmatrix} -\sin \beta t \\ \cos \beta t \end{pmatrix}$ are solutions

Proof. The idea is that $\forall v, \exists C_1, C_2$ such that $v = C_1X_1 + C_2X_2$. $X(t)$ is a solution because:

$$X'(t) = C_1X_1'(t) + C_2X_2'(t) = C_1AX_1(t) + C_2AX_2(t) = A(C_1X_1(t) + C_2X_2(t)) = AX(t).$$

We need to see that taking all possible values of C_1 and C_2 we obtain all possible solutions. Since $X_1(0)$ and $X_2(0)$ are linearly independent, they form a base of $\mathbb{R}^2$. Hence, every vector of $\mathbb{R}^2$ can be written as $C_1X_1(0)+C_2X_2(0)$ for some $C_1, C_2 \in \mathbb{R}$. Thus, we can get all the possible initial conditions $X(0)$. ∎

Once proven[10], we are done. ∎

Example 5.49 (Non-hyperbolic examples). First matrix refers to the second fundamental example, and the second one to the third.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{cases} x' = y \\ y' = 0 \end{cases} \quad \Longrightarrow \quad y(t) = c_1 \implies x' = c_1 \implies x(t) = c_1t + c_2.$$

Keep in mind that for the special case $c_1 = 0$, $x(t) = c_2$, $y(t) = 0$, which is not a curve, but the point $(c_2, 0)$. And:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \begin{cases} x' = -y \\ y' = x \end{cases} \quad \Longrightarrow \quad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Exercise 5.50. *Prove that orbits are contained in circles, that is, $x(t)^2 + y(t)^2 = c_1^2 + c_2^2$.*

Proposition 5.51 (General case). *Let $X' = AX$, A an $n \times n$ matrix.*

1. *For $n = 2$, A can be written as $A = CJC^{-1}$, where C is a linear transformation matrix and J is one of the fundamental examples.*
2. *With the change $Y = C^{-1}X$, we obtain:*

$$Y' = C^{-1}X' = C^{-1}CJC^{-1}X = JC^{-1}X = JY \iff Y' = JY.$$

3. *At the end, we must undo the change using $X = CY$.*

Alternative: *if we can find solutions $X_1(t), \dots, X_n(t)$ such that $X_1(0), \dots, X_n(0)$ are linear independent, then the general solution is $X(t) = C_1X_1(t) + \cdots + C_nX_n(t)$.*

Example 5.52. Let

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 3 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

E-vals are $\lambda = 2, \mu = -4$, which means that $(0, 0)$ is a saddle point. E-vecs are $(1, 1)$ and $(-1, 1)$. General solution can be computed as follows:

$$C = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \; J = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix} \implies \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = C^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} c_1e^{2t} \\ c_2e^{-4t} \end{pmatrix}.$$

[10] Application: $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1e^{\alpha t} \begin{pmatrix} \cos \beta t \\ \sin \beta t \end{pmatrix} + C_2e^{\alpha t} \begin{pmatrix} -\sin \beta t \\ \cos \beta t \end{pmatrix}$ is the general solution, because $X_1(0) = (1, 0)$ and $X_2(0) = (0, 1)$ are linearly independent.

And so:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C \begin{pmatrix} \tilde{x}(t) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 e^{2t} \\ c_2 e^{-4t} \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} - c_2 e^{-4t} \\ c_1 e^{2t} + c_2 e^{-4t} \end{pmatrix}.$$

We shall study stability of the origin and invariant subspaces. As mentioned, there exists a tranformation matrix C such that $A = CJC^{-1}$ where

$$J = \begin{pmatrix} J_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & J_p \end{pmatrix}$$

is a Jordan matrix. Every block J_i, $m \times m$, is associated to an invariant subspace of dimension m.

Theorem 5.53 (Stability of the origin). *Let $X' = AX$, A an $n \times n$ matrix, $X : \mathbb{R} \longrightarrow \mathbb{R}^n$. Let $\{\lambda_1, \ldots, \lambda_n\}$ be the e-values of the matrix A.*

1. *If $\mathrm{Re}(\lambda_i) < 0$ for all $1 \le i \le n$, then the origin is a global attractor.*
2. *If $\mathrm{Re}(\lambda_i) > 0$ for all $1 \le i \le n$, then the origin is a global repellor.*
3. *If $\mathrm{Re}(\lambda_i) < 0$ for $1 \le i \le p$ and $\mathrm{Re}(\lambda_i) > 0$ for $p+1 \le i \le n$, where $1 \le p \le n-1$, then the origin is a saddle point.*

Moreover, $\mathbb{R}^n = E^s \oplus E^u$, where E^s is the stable subspace of dimension p and E^u is the unstable subspace of dimension $n - p$.

- *If $x_0 \in E^s$, then $\varphi_{x_0}(t) \in E^s$, for all t, and $\varphi_{x_0}(t) \xrightarrow{t \to \infty} 0$.*
- *If $x_0 \in E^u$, then $\varphi_{x_0}(t) \in E^u$, for all t, and $\varphi_{x_0}(t) \xrightarrow{t \to -\infty} 0$.*

Proof. Analogous to the one we did for discrete dynamical systems. ■

Example 5.54. Let

$$\begin{cases} x' = -2x + 9y - 2z \\ y' = 2x + 5y - 2z \\ z' = -3x + 13y - z \end{cases} \qquad A = \begin{pmatrix} -2 & 9 & -2 \\ 2 & 5 & -2 \\ -3 & 13 & -1 \end{pmatrix}.$$

The e-values are $\lambda_0 = -4$, $\lambda_1 = 3 + 2i$, $\lambda_2 = 3 - 2i$. Since $\mathrm{Re}(\lambda_0) < 0$ and $\mathrm{Re}(\lambda_1) = \mathrm{Re}(\lambda_2) > 0$, 0 must be a saddle point.

1. The stable subspace has dimension 1. Its e-vec is $v_0 = (1, 0, 1)$, which gives the line $E^s = \langle (1, 0, 1) \rangle = \{(x, y, z) \mid x = s, y = 0, z = s\}$.
2. The unstable subspace has dimension 2. Its e-vecs are $v_1 = (2 + i, 2 + i, 1)$ and $v_2 = (2 - i, 2 - i, 5)$, which gives the plane $\langle \mathrm{Re}(v_1), \mathrm{Im}(v_1) \rangle = \langle (2, 2, 5), (1, 1, 0) \rangle$.

Definition 5.55 (Linear differential equations of order n). They are linear equations of order n and they can be transformed into a system of equations of order 1 and dimension n.

$$x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = 0 \iff X' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix} X$$

Given the equation, we consider the characteristic polynomial $\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$. Assume that:

1. $\lambda_1, \ldots, \lambda_k$ are real roots with multiplicity $m_1, \ldots, m_k$
2. $\alpha_{k+1} \pm i\beta_{k+1}, \ldots, \alpha_r \pm i\beta_r$ are complex roots with multiplicity $m_{k+1}, \ldots, m_r$.

Then, the general solution is of the form:

$$X(t) = \sum_{i=1}^{k} q_i(t)e^{\lambda_i t} + \sum_{i=k+1}^{r} (s_i(t)e^{\alpha_i t}\cos(\beta_i t) + p_i(t)e^{\alpha_i t}\sin(\beta_i t)),$$

where $q_i(t), p_i(t), s_i(t)$ are polynomials of degree $m_i - 1$ with arbitrary coefficients.

Definition 5.56 (Topologically conjugate). Consider $f : \mathcal{U} \longrightarrow \mathbb{R}^n$ and $g : \mathcal{V} \longrightarrow \mathbb{R}^n$, $\mathcal{U}, \mathcal{V} \subset \mathbb{R}^n$, and the equations $X' = f(X)$ and $Y' = g(Y)$. Let $\varphi_{X_0}(t)$ and $\psi_{Y_0}(t)$ be the respective solutions. We say the equations (or the vector fields f, g) are topologically conjugate if there exists a homeomorphism $h : \mathcal{U} \longrightarrow \mathcal{V}$ such that $h(\varphi_{X_0}(t)) = \psi_{h(X_0)}(t)$, for all $X_0 \in \mathcal{U}$ and t for which $\varphi_{X_0}(t)$ is defined.

Definition 5.57 (Locally conjugate). If the conjugacy h is defined only in a neighbourhood $\mathcal{U}$ of an equilibrium point p, and $h(\varphi_{X_0}(t)) = \psi_{h(X_0)}(t)$ is satisfied for all $X_0 \in \mathcal{U}$, we say that $X' = f(X)$ and $Y' = g(Y)$ are locally conjugate in a neighbourhood of p.

Proposition 5.58. *If h is a conjugacy between $X' = f(X)$ and $Y' = g(Y)$, then:*

1. *h sends orbits to orbits. In particular, it sends equilibrium points of f to equilibrium points of g, periodic orbits to periodic orbits, etc.*
2. *h sends attractors of f to attractors of g and repellors of f to repellors of g.*

Definition 5.59 (Hyperbolic point). Let $X' = f(X)$ with $f : \mathcal{U} \longrightarrow \mathbb{R}^n$, at least C^1. $p \in \mathbb{R}^n$ equilibrium point (i.e. $f(p) = 0$). The equilbrium point p is called hyperbollic if all e-values of $Df(p)$ have a non-zero real part.

Theorem 5.60 (Local stability criterion). *Let $p \in \mathbb{R}^n$ be a hyperbolic point and $\lambda_1, \ldots, \lambda_n$ the e-values of $Df(p)$. Then, the system $X' = f(X)$ is topologically conjugate to $Y' = Df(p)Y$ in a neighbourhood of p. Consequently,*

1. *If $\mathrm{Re}(\lambda_i) < 0$ for all i, then p is (locally) attracting.*
2. *If $\mathrm{Re}(\lambda_i) > 0$ for all i, then p is (locally) repelling.*

Remark 5.61.

1. As it was the case for discrete dynamical systems, the linear system $Y' = AY$ where $A = Df(p)$ gives us *local* information about $X' = f(X)$ (in a neighbourhood of p, a hyperbolic equilibrium point). The orbits can connect different local neighbourhoods, depending on the particular system.

2. For linear systems, we saw that saddle points carry with them invariant *subspaces* (lines, planes...). Remember that if E^s is the stable subspace, $\varphi_{X_0}(t) \xrightarrow{t\to\infty} 0$ if $X_0 \in E^s$ and if E^s is the unstable subspace, $\varphi_{X_0}(-t) \xrightarrow{t\to\infty} 0$ if $X_0 \in E^u$.
3. For non-linear systems there are no invariant subspaces, but **invariant manifolds** (curves, surfaces, etc.) associated to the saddle point p, and tangent to the invariant subspaces of the associated linear system at p, $Y' = DF(p)Y$.

Definition 5.62 (Stable/unstable manifold). If $p \in \mathbb{R}^n$ is a saddle equilibrium point of $X' = f(X)$, with solutions $\varphi_X(t)$, we define:

1. Stable manifold of p: $W^s(p) = \{X \in \mathbb{R}^n \mid \varphi_X(t) \xrightarrow{t\to\infty} p\}$.
2. Unstable manifold of p: $W^u(p) = \{X \in \mathbb{R}^n \mid \varphi_X(t) \xrightarrow{t\to-\infty} p\}$.

Definition 5.63 (Periodic solution/orbit). We say that the non-constant solution $X(t)$ is periodic (or that $X(t)$ forms a periodic orbit) if there exists $T > 0$ such that $X(t+T) = X(t)$ for all $t \in \mathbb{R}$. The minimum $T > 0$ with this property is called period of $X(t)$.

Example 5.64. For the linear system $x' = -y$ and $y' = x$, every orbit (except $(0,0)$) is periodic of period $T = 2\pi$.

$$X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 \cos t - c_2 \sin t \\ c_1 \sin t + c_2 \cos t \end{pmatrix}, \ X(t) = X(t+2\pi), \ \forall t \in \mathbb{R}.$$

Definition 5.65 (Isolated). A periodic orbit $\gamma \subset \mathbb{R}^n$ is isolated if there exists a neighbourhood $\mathcal{U}$ of γ such that for all $x_0 \in \mathcal{U} \setminus \gamma$, the solution $\varphi_{x_0}(t)$ is not periodic.

Example 5.66 (Change of coordinates). Let's prove how we can change to one system to another with polar coordinates:

$$\begin{cases} x' = x - y - x(x^2+y^2) \\ y' = x + y - y(x^2+y^2) \end{cases} \implies \begin{cases} r' = r(1-r^2) \\ \theta' = 1 \end{cases}$$

The change to polar coordinates is given by $x = r\cos\theta$ and $y = r\sin\theta$, so:

$$\begin{cases} x^2+y^2 = r^2 \\ \tan\theta = \frac{y}{x} \end{cases} \implies \begin{cases} 2rr' = 2xx' + 2yy' \\ \frac{\theta'}{\cos^2\theta} = \frac{y'x - x'y}{x^2} = \frac{y'x-x'y}{r^2\cos^2\theta} \end{cases} \implies \begin{cases} r' = \frac{1}{r}(xx'+yy') \\ \theta' = \frac{1}{r^2}(xy' - yx') \end{cases}$$

Changing $x' = x - y - x(x^2+y^2)$ and $y' = x + y - y(x^2+y^2)$, we obtain $r' = r(1-r^2)$ and $\theta' = 1$.

- When $r = 1$ (or 0), $r' = 0$ (the circle of radius 1) is invariant.
- $\theta' = 1$ implies that every orbit turns counterclockwise with constant angular velocity. Hence, there are no equilibrium points except $r = 0$.
- $r' > 0$ if $0 < r < 1$ and $r' < 0$ if $r > 1$.

$\{x^2+y^2 = 1\}$ is an isolated periodic orbit, also known as *limit cycle*.

Proposition 5.67. *Let $X(t)$ be a solution of $X' = f(X)$ with $f \in C^1$ and*

1. *$X(t_1) = X(t_2)$ for some $t_1 < t_2$, $t_1, t_2 \in \mathbb{R}$;*
2. *$X(t) \neq X(t_1)$ for all $t \in (t_1, t_2)$.*

Then, $X(t)$ is a periodic solution of period $T = t_1 - t_2$.

Proof. Define the function $\tilde{X}(t) = X(t+T)$. Check it is a solution:

$$\tilde{X}'(t) = X'(t+T) = f(X(t+T)) = f(\tilde{X}(t)).$$

But $\tilde{X}(t_1) = X(t_1 + T) = X(t_1 + t_2 - t_1) = X(t_2)$, and using the hypothesis $X(t_2) = X(t_1)$, and so $\tilde{X}(t_1) = X(t_1)$. Hence, we have two solutions which match for $t = t_1$. By the uniqueness of the solution:

$$\tilde{X}(t) = X(t),\ \forall t \implies X(t+T) = X(t),\ \forall t.$$

Which means that X is a periodic solution. ■

Definition 5.68 (First integral). Given a vector field $X' = f(X)$, a first integral of the system is a C^1 function $H : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that $H(X(t))$ is constant on every solution.

$$H \text{ first integral} \iff \frac{d}{dt}H(X(t)) = 0 \text{ for all solution } X(t),\ \forall t.$$

If H is a first integral of $X' = f(X)$, then every orbit $\varphi_{X_0}(t)$ must live on one level curve $H(x, y) = c$ for some c.

Example 5.69. The function $H(x, y) = x^2 + y^2$ is a first integral of the sytem $x' = -y$, $y' = x$. Because

$$\frac{d}{dt}(H(X(t))) = 2xx' + 2yy' = 2x(-y) + 2yx = 0.$$

It might be a good idea to check the Lotka-Volterra system example, as it gives a thorough resolution of a non-linear differential system.

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