Exercise A.15. *Construct the binary Cantor set by removing from* $[0, 1]$ *the central interval of length* $\frac{1}{2}$. *Find its Lebesgue measure.*

Proof. $C_0 = [0, 1]$, in the first iteration $C_1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, the second is $C_2 = [0, \frac{1}{16}] \cup [\frac{3}{16}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{13}{16}] \cup [\frac{15}{16}, 1]$ and so on. C_n has 2^n intervals and the length of each subinterval is $\frac{1}{2^{2^n}}$. Cantor set is defined as $C = \bigcap_{n=1}^{\infty} C_n$.

- Using that $m([0, 1]) = 1$. Note the size of the set can be thought of as the propition of the interval $[0, 1]$ that is removed. Having said that, we remove $\frac{1}{2}$ of each interval at each step, which means that at step $n - 1$, a length of $\frac{1}{2^n}$ is removed 2^{n-1} times:

$$m(C^c) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2} = +\infty \ldots$$

  Not even close. Let's try again.

- The intersection of any collection amb closed sets is closed. Therefore, C must be closed. Each closed set is measurable, so that each C_n and C itself is measurable. Each C_n is the disjoint union of 2^n intervals of length $\frac{1}{2^{2^n}}$, so that by finite additivity (cf. properties of Lebesgue measure), $m(C_n) = \frac{1}{2^n}$. By monotonicity of measure, since $m(C) \leq m(C_n) = \frac{1}{2^n}$ for all n, so $m(C) = 0$. ■

Remark A.16. I have not done it using bases (as in the slides), as I suspect is what the correctors intended. Frankly, I do not see why this approach would be incorrect, but it is true that the binary Cantor set has not been detailed. $C_0 = [0, 1]$, in the first iteration $C_1 = [0, \frac{1}{4}] \cup [\frac{3}{4}, 1]$, the second is $C_2 = [0, \frac{1}{16}] \cup [\frac{3}{16}, \frac{1}{4}] \cup [\frac{3}{4}, \frac{13}{16}] \cup [\frac{15}{16}, 1]$ and so on. In base 4,

$$(0.s_1 s_2 s_3 \ldots)_4 = \frac{s_1}{4} + \frac{s_2}{4^2} + \frac{s_3}{4^3} + \cdots, \quad s_i \in \{0, 1, 2, 3\}.$$

But $s_i \notin \{1, 2\}$ because then $x = \sum_i s_i$ would not lie in the Cantor set[6], so $s_i \in \{0, 3\}$.

Exercise A.17. *In a similar way that we construct the snowflake, do the construction by substituting the central interval of length* $\frac{1}{2^n}$, $n \geq 1$, *by the corresponding equilateral triangle. Compute the length of the perimeter and the exact area.*

Proof. Start with $[0, 1]$ as a segment. Let's compute the first two iterations of the resulting perimeter Koch curve:

$$\frac{1}{4} + \left(\frac{1}{2} + \frac{1}{2}\right) + \frac{1}{4} = \frac{3}{2}$$

$$2 \cdot \left(\frac{1}{16} + 2 \cdot \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + 2 \cdot \frac{1}{4} + \frac{1}{8}\right) = \frac{9}{4} = 2 \cdot \left(\frac{3}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{2}\right) = \left(\frac{3}{2}\right)^2.$$

$$2 \cdot \left(\frac{3}{2} \cdot \frac{1}{16} + 2 \cdot \frac{3}{2} \cdot \frac{1}{8} + \frac{3}{2} \cdot \frac{1}{16} + \frac{3}{2} \cdot \frac{1}{8} + 2 \cdot \frac{3}{2} \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{8}\right) = \frac{27}{8} = \left(\frac{3}{2}\right)^3.$$

[6] For example, if $s_2 = 1$ then depending on $s_1 \in \{0, 3\}$ we have $\frac{1}{16} < x < \frac{2}{16}$ or $\frac{13}{16} < x < \frac{14}{16}$, meaning $x \notin C$. With $s_1 \in \{1, 2\}$, x wouldn't lie in C either.

Proceeding by induction, the perimeter is $P_n = \left(\frac{3}{2}\right)^n$ and $P_\infty = \lim_n \left(\frac{3}{2}\right)^n = +\infty$, so the perimeter is infinite. Regarding the area, we have to sum to the total area every triangle that *arises* at every iteration. The area of a general equilateral triangle with side length a is $\frac{\sqrt{3}}{4}a^2$. Unfortunately, we will have to do some iterations in order to identify a pattern:

1. In 17, the only triangle has side $\frac{1}{2}$ and its area is $\frac{\sqrt{3}}{4^2}$.
2. In 18, we are adding 4 triangles, two of side $\frac{1}{4}$ and two of $\frac{1}{8}$, which areas are $\frac{\sqrt{3}}{4^3}$ and $\frac{\sqrt{3}}{4^4}$, respectively. New area adds up to:
$$\frac{\sqrt{3}}{4}\left(2\cdot\frac{1}{4^2}+2\cdot\frac{1}{4^3}\right)=\frac{\sqrt{3}}{4}\cdot\frac{5}{2^5}.$$
3. In 19, we are adding 16 triangles. 4 have a side of $\frac{1}{8}$, 8 of $\frac{1}{16}$ and 4 more of $\frac{1}{32}$, with areas of $\frac{\sqrt{3}}{4^4}$, $\frac{\sqrt{3}}{4^5}$ and $\frac{\sqrt{3}}{4^6}$, respectively. New area adds up to:
$$\frac{\sqrt{3}}{4}\left(4\cdot\frac{1}{4^3}+8\cdot\frac{1}{4^4}+4\cdot\frac{1}{4^5}\right)=\frac{\sqrt{3}}{4}\cdot\frac{5^2}{2^8}.$$
4. In general, it can be proven (out of the scope of this laboratory) that in every n-th iteration we will be adding 4^n triangles, $n \geq 0$, and the added area will be:
$$\frac{\sqrt{3}}{4}\cdot\frac{5^n}{2^{3n+2}}=\frac{\sqrt{3}}{4^2}\cdot\left(\frac{5}{8}\right)^n=\frac{\sqrt{3}}{4^2}\cdot\frac{10^n}{4^{2n}} \implies A=\sum_{n=0}^{\infty}\frac{\sqrt{3}}{4^2}\cdot\frac{10^n}{4^{2n}}=\frac{\sqrt{3}}{4^2}\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n.$$

Note that $\sum_{n=0}^{\infty}\left(\frac{5}{8}\right)^n$ is the infinite geometric series of rate $\frac{5}{8}<1$, so $A=\frac{\sqrt{3}}{4^2}\cdot\frac{1}{1-\frac{3}{8}}=\frac{\sqrt{3}}{10}$. ∎

Exercise A.18. *Compute the fractal dimension of the geometric fractals of the previous questions.*

Proof. The first is a binary Cantor set and the latter a Koch curve. Both have topological dimension 1 (will not trouble ourselves to demonstrate it because is written in the slides).

1. Regarding the binary Cantor set, we can use that the contraction factor r does not change and take advantage of the formula seen in Theory:
$$\dim_S=\frac{\log(\#\text{contractions})}{\log(\frac{1}{\text{contraction factor}})}=\frac{\log(2)}{\log(\frac{1}{4})}=\frac{1}{2}.$$
The number of contractions is 2 and $r=\frac{1}{4}$, per iteration.
2. We are applying four different affinities, with respective contraction factors: $r_1=r_4=\frac{1}{4}$ and $r_2=r_3=\frac{1}{2}$ (i incremental from left to right in 17). The self-similar dimension of the fractal we obtain after iterating is the only real number d such that $r_1^d+\cdots+r_4^d=1$.
$$2\left(\frac{1}{2^d}\right)+2\left(\frac{1}{2^{2d}}\right)=1\xrightarrow{t=\frac{1}{2^d}}t+t^2=\frac{1}{2}\iff t+t^2-\frac{1}{2}=0\iff t=\frac{-1\pm\sqrt{3}}{2}$$
$$\iff 2^{-d}=\frac{-1\pm\sqrt{3}}{2}\xrightarrow{\text{discard negative solution}}d=-\frac{\ln\left(\frac{-1+\sqrt{3}}{2}\right)}{\ln 2}\approx 1.45.$$
So $d\approx 1.45$, as we wanted to prove. ∎

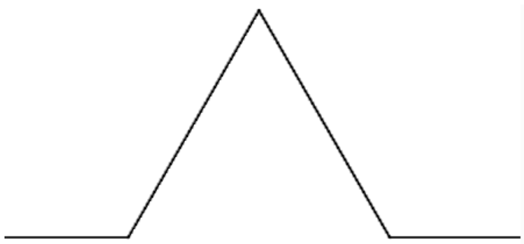

Figure 17: First iteration.

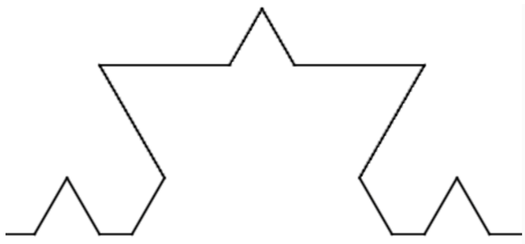

Figure 18: Second iteration.

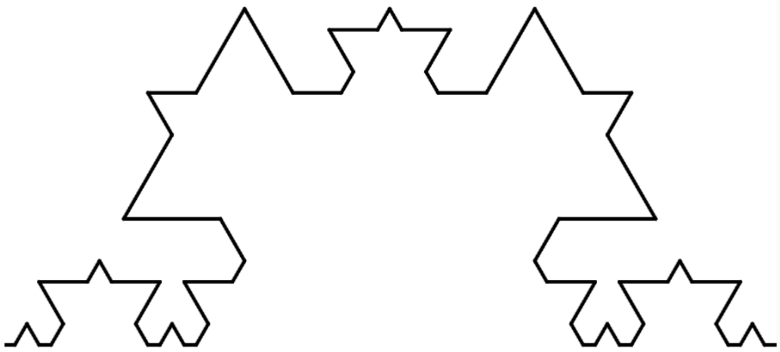

Figure 19: Third iteration.

Exercise A.19. *Compute the fractal dimension of the subset of* $[0, 1]$ *of the real numbers such that their representation in base 10 only contains multiples of* $k = \{1, 2, \ldots, 9\}$.

Proof. The dimension depends on the number of pieces, and the number of pieces depends on k. We will be applying that in this case $\dim_S = \frac{\log(\#\text{pieces})}{\log(\frac{1}{\text{contraction}})}$. The contraction factor is $\frac{1}{10}$ for all the cases.

- If $k = 2$, its multiples are $\{0, 2, 4, 6, 8\}$ and $\#\text{pieces} = \#\{0, 2, 4, 6, 8\} = 5$, so $\dim_S = \frac{\log 5}{\log 10} \approx 0.70$.
- If $k = 3$, its multiples are $\{0, 3, 6, 9\}$ and $\#\text{pieces} = \#\{0, 3, 6, 9\} = 4$, so $\dim_S = \frac{\log 4}{\log 10} \approx 0.60$.
- If $k = 4$, its multiples are $\{0, 4, 8\}$ and $\#\text{pieces} = \#\{0, 4, 8\} = 3$, so $\dim_S = \frac{\log 3}{\log 10} \approx 0.48$.
- If $k = 5$, its multiples are $\{0, 5\}$ and $\#\text{pieces} = \#\{0, 5\} = 2$, so $\dim_S = \frac{\log 2}{\log 10} \approx 0.30$.
- If $k = 6$, its multiples are $\{0, 6\}$ and $\#\text{pieces} = \#\{0, 6\} = 2$, so $\dim_S = \frac{\log 2}{\log 10} \approx 0.30$.
- If $k = 7$, its multiples are $\{0, 7\}$ and $\#\text{pieces} = \#\{0, 7\} = 2$, so $\dim_S = \frac{\log 2}{\log 10} \approx 0.30$.
- If $k = 8$, its multiples are $\{0, 8\}$ and $\#\text{pieces} = \#\{0, 8\} = 2$, so $\dim_S = \frac{\log 2}{\log 10} \approx 0.30$.
- If $k = 9$, its multiples are $\{0, 9\}$ and $\#\text{pieces} = \#\{0, 9\} = 2$, so $\dim_S = \frac{\log 2}{\log 10} \approx 0.30$.
- Finally, if $k = 1$ it is trivial, because every number is multiple of 1, so #pieces is equal to 10 and $\dim S = \frac{\log 10}{\log 10} = 1$.

■