Proof. Trying to find other fixed points than $(0,0)$ we resolve $Av = v$, for $v = (x, y) \in \mathbb{R}^2$, and this linear system returns as a solution $x + y = 0$, $a\,y = 0$. So if $a = 0$ there are infinitely many fixed points because y can take any real value and $(-y, y)$, $y \in \mathbb{R}$ (or $\langle(-1,1)\rangle$), are solutions of $Ax = x$. Else, $a \neq 0$ and the only fixed point is $(0,0)$.

1. This procedure is much ordinary:

$$\det(A_a - x\,\mathrm{id}_{2\times 2}) = \begin{vmatrix} -x & -1 \\ \frac{a+1}{2} & a + \frac{3}{2} - x \end{vmatrix} = \frac{2x^2 - (2a+3)x + a + 1}{2}$$
$$= \frac{(2x-1)(x-(a+1))}{2} = \frac{2x-1}{2} \cdot \frac{x-(a+1)}{2} = 0.$$

The eigenvalues are $\lambda = \frac{1}{2}$ and $\mu(a) = a + 1 \neq 0$. The eigenvectors can be computed easily: $v_\lambda = (-2, 1)$ and $v_\mu = (-\frac{1}{a+1}, 1)$. Note that when $\mu \to +\infty$, $v_\mu \to (0,1)$ ($\langle v_\mu \rangle$ would become the vertical axis). Therefore, the line L of eigenvectors we are looking for is invariant for L_a, and the only one that doesn't depend on a is $r : (0,0) + \langle(-2,1)\rangle$. The dynamics on this straight line does not depend on a because the eigenvalue associated with the invariant straight line is $\frac{1}{2}$, which remains constant for all values of a. Now, to determine the dynamics on r, we apply the vector that generates r onto the map given by A_a, and confirm what we already know:

$$\begin{pmatrix} 0 & -1 \\ \frac{a+1}{2} & \frac{2a+3}{2} \end{pmatrix} \begin{pmatrix} -2x \\ x \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2x \\ x \end{pmatrix}$$

Every point in r maps via L_a to another point in r, hence r is invariant.

Remark A.7. The two eigenvectors are $v_\lambda = (-2, 1)$ and $v_\mu = (-\frac{1}{a+1}, 1)$. Note that $\langle(-1, a+1)\rangle$ might not be the same as $\langle v_\mu \rangle$, when $a = -1$. Anyway, even though they seem linearly independent they necessarily aren't, it all depends on the values that a takes.

$$v_\lambda = k \cdot v_\mu \iff (-2, 1) = \left(-\frac{k}{a+1}, k\right) \iff \begin{cases} 2 = \frac{k}{a+1} \\ k = 1 \end{cases} \iff a = -\frac{1}{2}.$$

So we must acknowledge this value of a for further evaluation when needed.

2. From now on, keep in mind that $|\lambda| < 1$, but $|\mu|$ can take any positive value, depending on a. Now we are constraining a to $|a| < 1$, so $\mu(a) = 1 + a \in (0, 2)$. For any value of a, both $\lambda, \mu \in \mathbb{R}$. Nevertheless, we are forced to segregate in these three cases:

2.1. $a \in (-1, 0)$. Then, $|\lambda|, |\mu| < 1$, but more precisely, we should distinguish when $|\mu| \lesseqgtr |\lambda|$. Remembering A.7, we need to make one more distinction: when $a = -\frac{1}{2}$, $\{v_\lambda, v_\mu\}$ are linearly dependent (i.e. $\lambda = \mu$), which means that $(0,0)$ is a degenerate attracting node. If $0 < |\lambda| = |\mu| < 1$, the invariant lines and the dynamics can be drawn like:

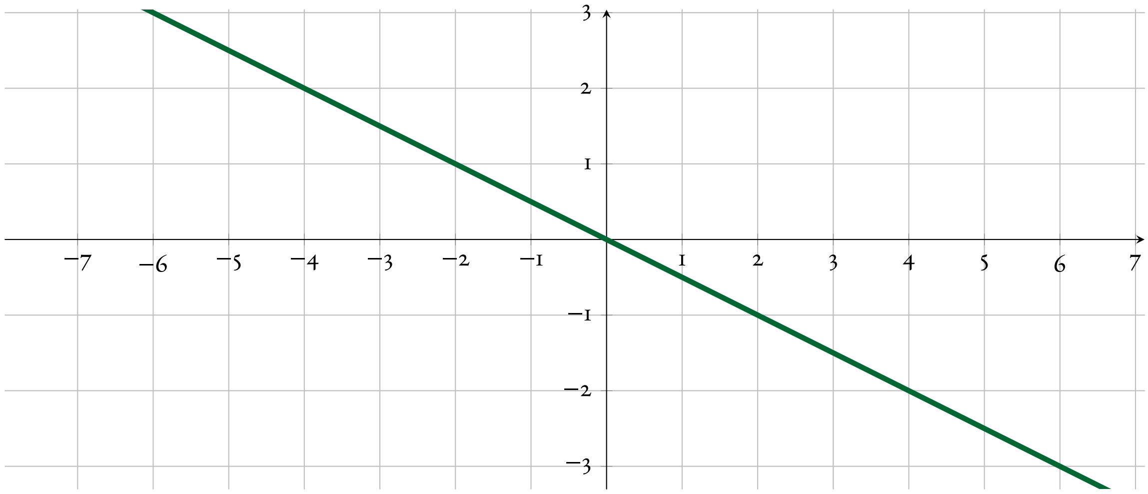

Figure 8: First case, for a value of $a = -\frac{1}{2}$. Invariant line is $\langle v_\lambda \rangle = \langle v_\mu \rangle$ (green).

When $a \neq -\frac{1}{2}$, then $(0,0)$ is a non-degenerate attracting node. In more detail, if $a < -\frac{1}{2}$, $0 < \mu = |\mu| < |\lambda| = \lambda < 1$:

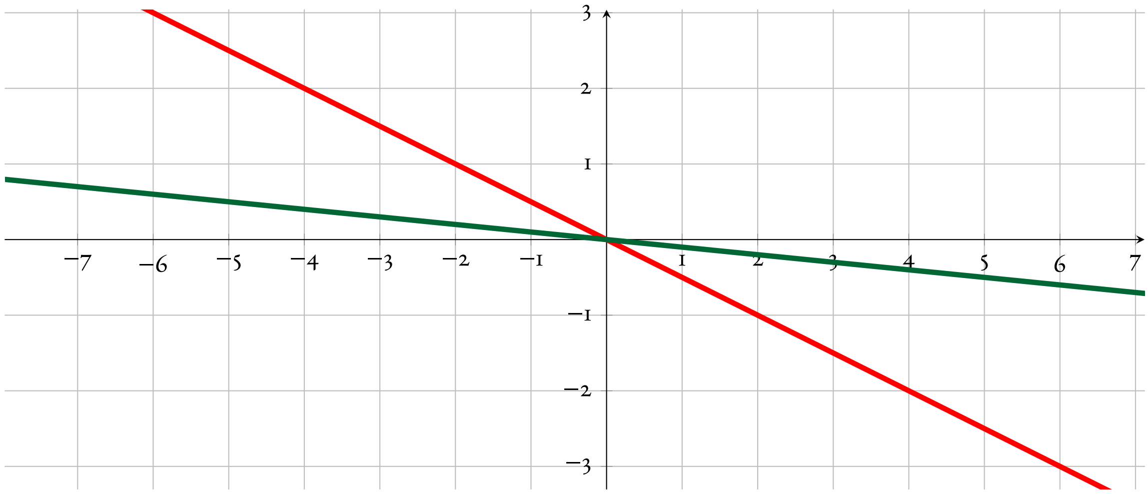

Figure 9: Second case, for a value of $a = -\frac{9}{10}$. Invariant lines are $\langle v_\lambda \rangle$ (green) and $\langle v_\mu \rangle$ (red).

Lastly, if $a > -\frac{1}{2}$, $0 < \lambda = |\lambda| < |\mu| = \mu < 1$:

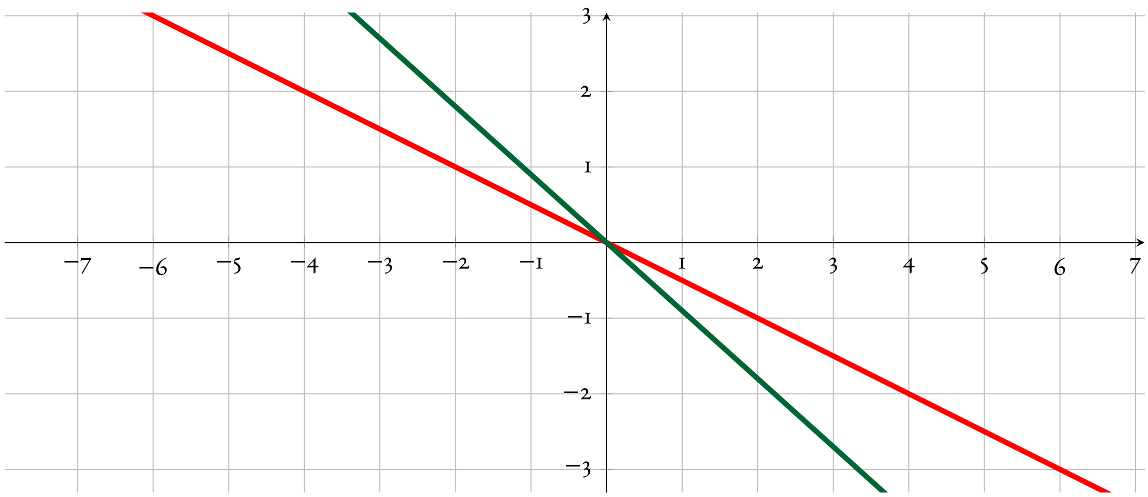

Figure 10: Third case, for a value of $a = -\frac{1}{10}$. Invariant lines are $\langle v_\lambda \rangle$ (green) and $\langle v_\mu \rangle$ (red).

2.2. $a = 0$. Then, $|\lambda| < 1$, $|\mu| = 1$ and we have encountered with a non-hyperbolic degenerate case. Moreover, $\langle v_\mu \rangle = \langle (-1, 1) \rangle$ is a line of fixed points (every point $(x, y) \in \mathbb{R}^2$ in $\langle v_\mu \rangle$ meets $A_0(x, y) = (x, y)$).

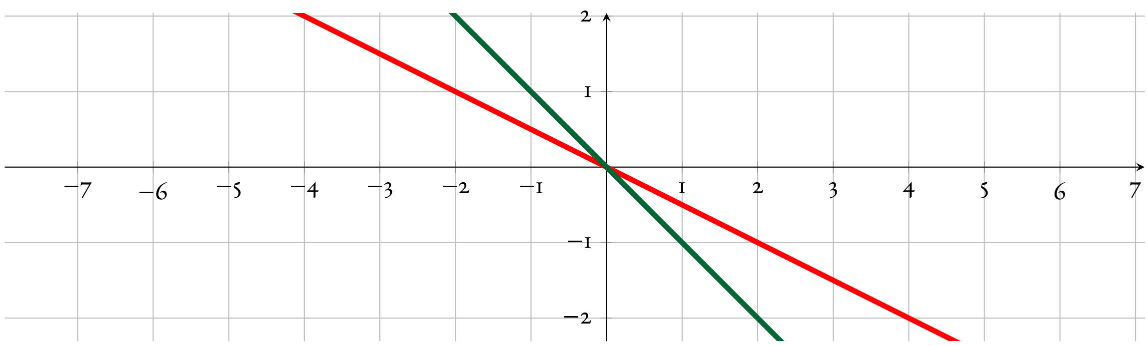

Figure 11: Fourth case, for a value of $a = 0$. Invariant lines are $\langle v_\lambda \rangle$ (green) and $\langle v_\mu \rangle$ (red).

2.3. $a \in (0,1)$. Then, $|\lambda| < 1 < |\mu|$ and $(0,0)$ is a hyperbolic saddle node.

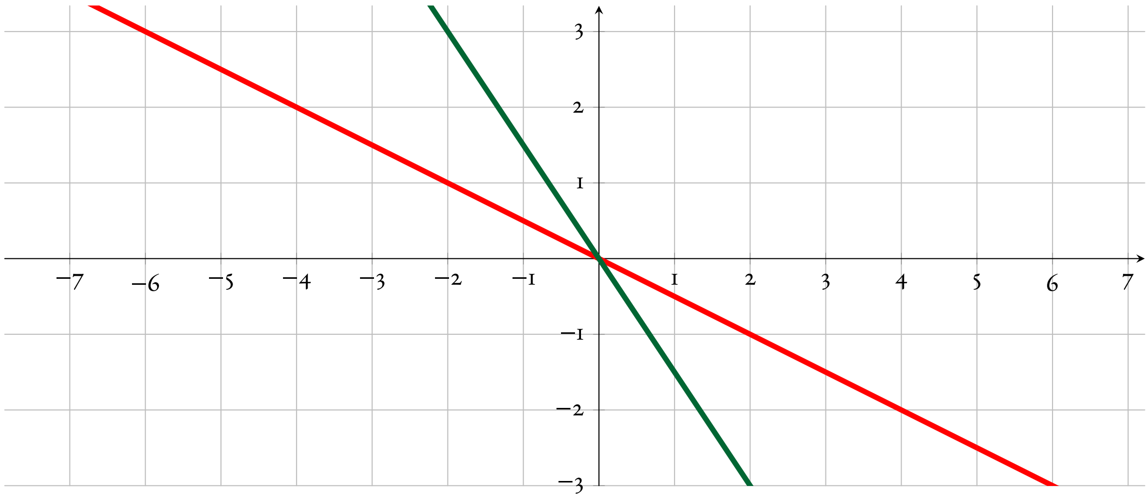

Figure 12: Fourth case, $a = 0$. Stable subspace is $E^s = \langle v_\lambda \rangle$ (green) and unstable is $E^u = \langle v_\mu \rangle$ (red).

3. If $a = -1$, then the eigenvalue μ is equal to 0. Having a 0 eigenvalue tells us that A_a is singular or non-invertible. In fact, v_μ is not defined, so $\langle v_\mu \rangle$ isn't either and there is only one invariant line. The invariant curves are $|y| = C|x|^{\frac{\log|\mu|}{\log|\lambda|}}$ ($c \in \mathbb{R}$) *in a certain basis*, but the exponent is a constant that does not depend on the basis, and in this case cannot be computed because $\frac{\log|\mu|}{\log|\lambda|}$ is not defined, as $\mu = 0$. We could also argue that for every point $(x_0, y_0) \in \mathbb{R}^2$, $A_{-1}(x_0, y_0) = (-y_0, \frac{1}{2}y_0) \in \langle v_\lambda \rangle$.

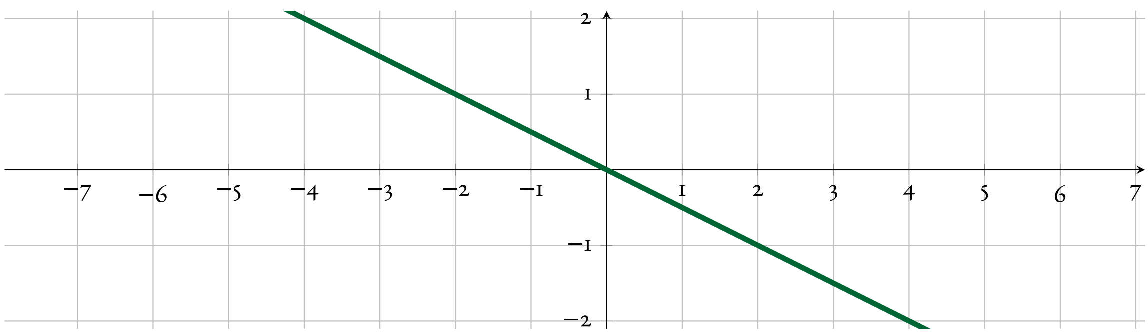

Figure 13: Invariant line is $\langle v_\lambda \rangle$ (green).

4. If the first and third quadrant cannot be forward invariant, then we have to prove that there's at least one point in the first quadrant (resp. third) which orbit does not *stay there*. For example, if we take (x_0, y_0), $x_0, y_0 > 0$[4], we claim that there exists an $n \in \mathbb{N}$ such that $A_a^n(x_0, y_0) = (x_1, y_1)$ and either $x_1 < 0$ or $y_1 < 0$. Take $n = 1$:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{a+1}{2} & \frac{2a+3}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -y_0 \\ \frac{ax_0+x_0+2ay_0+3y_0}{2} \end{pmatrix}, \quad x_1 = -y_0 < 0. \tag{A.5}$$

So at least the x coordinate will be outside the scope of the first quadrant. Hence, first quadrant cannot be invariant. Same argument works for third quadrant: if (x_0, y_0), $x_0, y_0 < 0$:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ \frac{a+1}{2} & \frac{2a+3}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} -y_0 \\ \frac{ax_0+x_0+2ay_0+3y_0}{2} \end{pmatrix}, \quad x_1 = -y_0 > 0. \tag{A.6}$$

5. For the second and third quadrant it isn't that trivial. Note that we aren't trying to refuse, but to prove: the *whole* semi-plane has to be invariant, so the point (x_0, y_0) we consider has to be arbitrary. Hence, we take $(x_0, y_0) \in \mathbb{R}^2$ an arbitrary point

[4] We discard the points along both axis because if the axis were invariant, they would be linear invariant, but there are, at most, only two invariant lines, $\langle v_\lambda \rangle$ and $\langle v_\mu \rangle$.

that meets the conditions $x_0 < 0$, $y_0 > 0$ (i.e. is located in the second quadrant). Following (A.5) and (A.6):

$$\left.\begin{array}{l} x_1 = -y_0 < 0 \checkmark \\ y_1 = \frac{ax_0+x_0+2ay_0+3y_0}{2} > 0 \end{array}\right\} \implies \frac{ax_0 + x_0 + 2ay_0 + 3y_0}{2} > 0 \iff \frac{(a+1)x_0 + (2a+3)y_0}{2} > 0.$$

So if both $\frac{a+1}{2} < 0$ and $\frac{2a+3}{2} > 0$, and because $x_0 < 0$, $y_0 > 0$ the inequality $y_1 > 0$ will hold, and $x_1 < 0$, $y_1 > 0$ as we wanted. For that reason, $a \in (-\frac{3}{2}, -1)$. Note that we have given a *sufficient* condition: it may be the case that other values of a outside the scope of this interval also fulfill the conditions. Analogously for the third quadrant, take (x_0, y_0) such that $x_0 > 0$ and $y_0 < 0$:

$$\left.\begin{array}{l} x_1 = -y_0 > 0 \checkmark \\ y_1 = \frac{ax_0+x_0+2ay_0+3y_0}{2} < 0 \end{array}\right\} \implies \frac{ax_0 + x_0 + 2ay_0 + 3y_0}{2} < 0 \iff \frac{(a+1)x_0 + (2a+3)y_0}{2} < 0.$$

And if we want both $\frac{a+1}{2} < 0$ and $\frac{2a+3}{2} > 0$, and because $x_0 > 0$, $y_0 < 0$ the inequality $y_1 < 0$ will hold, and $x_1 > 0$, $y_1 < 0$ as we wanted: $a \in (-\frac{3}{2}, -1)$ as before. Regarding the phase portrait, if $a \in (-\frac{3}{2}, -1)$, then $\mu \in (-\frac{1}{2}, 0)$ and $0 < |\mu| < |\lambda| < 1$ but $\mu < 0 < \lambda < 1$.

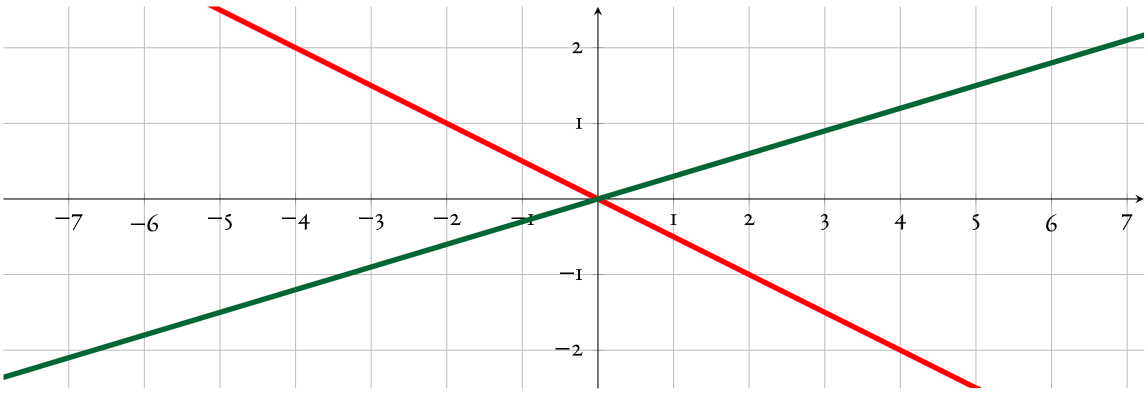

Figure 14: $a = -1.3$. $\langle v_\lambda \rangle$ (green) and $\langle v_\mu \rangle$ (red).

6. If $a = 5$, then $|\lambda| < 1 < |\mu| = 6$ and $(0, 0)$ becomes an hyperbolic saddle node (by the slides, we know what will J be). Note that $\omega_{n+1} = L_5(\omega_n)$, $\omega_i \in \mathbb{R}^2$, for all $i \in \mathbb{N}$ and that means that A_5 takes the form:

$$A_5 = \begin{pmatrix} 0 & -1 \\ 3 & \frac{13}{2} \end{pmatrix}, \quad v_\lambda = \langle(-2, 1)\rangle, \quad v_\mu = \left\langle\left(-\frac{1}{6}, 1\right)\right\rangle = \langle(-1, 6)\rangle.$$

As it was already stated in the slides, $\omega_{n+1} = A_5^{n+1}\omega_0$, $\omega \in \mathbb{R}^2$, but A_5^{n+1} might be uncomfortable to iterate. All linear systems in $\mathbb{R}^2$ are linearly conjugate to one of the fundamental examples with same phase portrait, modulo a change of basis.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = CJ^{n+1}C^{-1} = \left(C \cdot \begin{pmatrix} \lambda^{n+1} & 0 \\ 0 & \mu^{n+1} \end{pmatrix} \cdot C^{-1}\right) \cdot \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \frac{1}{11 \cdot 2^n} \begin{pmatrix} -6 \cdot 2^{2n} \cdot 3^n + 6 & -12 \cdot 2^{2n} \cdot 3^n + 1 \\ 36 \cdot 2^{2n} \cdot 3^n - 3 & \frac{144 \cdot 2^{2n} \cdot 3^n - 1}{2} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{A.7}$$

Previously, we have had to compute C, which is the change of basis matrix, from the canonical $\{(1, 0), (0, 1)\}$ to the one set by the eigenvectors, $\{(-2, 1), (-1, 6)\}$. Because A_5 is diagonalitzable:

$$\left.\begin{array}{l} (1, 0) = x_1(-2, 1) + y_1(-1, 6) \\ (0, 1) = x_2(-2, 1) + y_1(-1, 6) \end{array}\right\} \implies C = \begin{pmatrix} -2 & -1 \\ -1 & 6 \end{pmatrix} \implies C^{-1} = \frac{1}{11}\begin{pmatrix} -6 & -1 \\ 1 & 2 \end{pmatrix}$$

$$\implies CJ^{n+1}C^{-1} = \begin{pmatrix} -2 & -1 \\ -1 & 6 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2^{n+1}} & 0 \\ 0 & 6^{n+1} \end{pmatrix} \cdot \frac{1}{11}\begin{pmatrix} -6 & -1 \\ 1 & 2 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} \frac{12}{2^{n+1}} - 6^{n+1} & \frac{2}{2^{n+1}} - 2 \cdot 6^{n+1} \\ \frac{-6}{2^{n+1}} + 6 \cdot 6^{n+1} & -\frac{1}{2^{n+1}} + 12 \cdot 6^{n+1} \end{pmatrix}$$

The latter expression can be arranged in order to look as in (A.7). The explicit solutions for $\omega_n = (x_n, y_n)$ taking $\omega_0 = (x_0, y_0)$ are:

$$x_n = \frac{1}{11 \cdot 2^{2n}}\left(-6 \cdot 2^{2n} \cdot 3^n \cdot x_0 + 6x_0 - 12 \cdot 2^{2n} \cdot 3^n \cdot y_0 + y_0\right),$$

$$y_n = \frac{1}{11 \cdot 2^{n+1}}\left(72 \cdot 2^{2n} \cdot 3^n \cdot x_0 - 6x_0 + 144 \cdot 2^{2n} \cdot 3^n \cdot y_0 + y_0\right).$$

Once and for all, we are done. ■