(1) $f(x,y) = \dfrac{x-y^2}{y+x^2}$; $x \approx 10$; $y \approx 5$

(a) Propag. de l'error : $|\Delta f| \approx \left|\frac{\partial f}{\partial x}\right| |\Delta x| + \left|\frac{\partial f}{\partial y}\right| \cdot |\Delta y|$

Costa un esforç similar reduir $|\Delta x|$ o $|\Delta y|$.

Per tal que $|\Delta f|$ sigui petit, $\begin{cases} \text{si } \left|\frac{\partial f}{\partial x}\right| > \left|\frac{\partial f}{\partial y}\right| \text{ llavors és millor reduir } |\Delta x| \\ \text{" } < \text{ " " " " } |\Delta y| \end{cases}$

$$\frac{\partial f}{\partial x} = \frac{1\cdot(y+x^2) - (x-y^2)\cdot 2x}{(y+x^2)^2} \quad ; \quad \frac{\partial f}{\partial y} = \frac{(-2y)(y+x^2) - (x-y^2)\cdot 1}{(y+x^2)^2}$$

Per a comparar $\left|\frac{\partial f}{\partial x}\right|$ i $\left|\frac{\partial f}{\partial y}\right|$, només cal comparar els numeradors

$\text{num}\left|\frac{\partial f}{\partial x}\right| = |y - x^2 + 2xy^2| \approx 405$

$\text{num}\left|\frac{\partial f}{\partial y}\right| = |-y^2 - 2yx^2 - x| \approx |-1035| = 1035$

(s'ha avaluat en $x=10, y=5$)

Per tant, quan $x \approx 10, y \approx 5$, serà $\left|\frac{\partial f}{\partial y}\right| > \left|\frac{\partial f}{\partial x}\right|$. $\Rightarrow$ **És millor dedicar l'esforç a millorar la precisió en y.**

(b) $$fl\left(f(x,y)\right) = \frac{[x - y^2(1+\delta_1)](1+\delta_2)}{[y+x^2(1+\delta_3)](1+\delta_4)}(1+\delta_5) =$$ $|\delta_i| \le u << 1 \quad \forall i = 1 \div 5$

$$= \frac{x-y^2-y^2\delta_1}{y+x^2+x^2\delta_3}\left[1+\delta_2+\delta_5-\delta_4+O(u^2)\right] = \frac{(x-y^2)\left(1-\frac{y^2}{x-y^2}\delta_1\right)}{(y+x^2)\left(1+\frac{x^2}{y+x^2}\delta_3\right)}\left(1+\delta_2+\delta_5-\delta_4+O(u^2)\right) =$$

$$= \frac{x-y^2}{y+x^2}\left[1 - \frac{y^2}{x-y^2}\delta_1 - \frac{x^2}{y+x^2}\delta_3 + \delta_2 + \delta_5 - \delta_4 + O(u^2)\right]$$

$\Rightarrow$ A 1r ordre en u, és $\quad e_r(fl(x,y)) = \dfrac{-y^2}{x-y^2}\delta_1 - \dfrac{x^2}{y+x^2}\delta_3 + \delta_2 + \delta_4 - \delta_5$

Usant $x=10, y=5$ i $|\delta_i| \le u$ s'obté

$$|e_r| \le \left(\underbrace{\frac{5^2}{|10-5^2|}}_{5/3} + \underbrace{\frac{10^2}{5+10^2}}_{20/21} + 1+1+1\right)u = \boxed{\frac{118}{21}u} \approx 5.619u$$

(c) $x = 10.17$, $y = 5.115$ i es treballa arrodonint a 4 dígits

$$\frac{10.17 - 5.115^2}{5.115 + 10.17^2} = \frac{10.17 - 26.163225}{5.115 + 103.4289} \approx \frac{10.17 - 26.16}{5.115 + 103.4} = \frac{-15.99}{108.515} \approx \frac{-15.99}{108.5} = -0.147373\ldots \approx \boxed{-0.1474}$$

② ⓐ Es vol $A_n = LU$ d'una A_n banda $(1,2)$. Se suposa $\exists$.

Conservació caràcter banda $\Rightarrow$ L serà banda $(1,0)$ i U serà banda $(0,2)$

Es treballa amb vectors diagonals. Imposem:

$$\underbrace{\begin{pmatrix} a_1 & c_1 & d_1 & & & 0 \\ b_1 & a_2 & c_2 & d_2 & & \\ & b_2 & a_3 & c_3 & d_3 & \\ & & b_3 & a_4 & c_4 & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix}}_{} = \underbrace{\begin{pmatrix} 1 & & & & & 0 \\ \beta_1 & 1 & & & & \\ & \beta_2 & 1 & & & \\ & & \beta_3 & 1 & & \\ 0 & & & \beta_4 & 1 & \\ & & & & \ddots & \ddots \end{pmatrix} \begin{pmatrix} \alpha_1 & \gamma_1 & \delta_1 & & & 0 \\ & \alpha_2 & \gamma_2 & \delta_2 & & \\ & & \alpha_3 & \gamma_3 & \delta_3 & \\ 0 & & & \alpha_4 & \gamma_4 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}}$$

$$\begin{pmatrix} \underline{\alpha_1} & \underline{\gamma_1} & \underline{\delta_1} & & & 0 \\ \underline{\beta_1}\alpha_1 & \beta_1\gamma_1 + \underline{\alpha_2} & \beta_1\delta_2 + \underline{\gamma_2} & \underline{\delta_2} & & \\ & \underline{\beta_2}\alpha_2 & \beta_2\gamma_2 + \underline{\alpha_3} & \beta_2\delta_2 + \underline{\gamma_3} & \underline{\delta_3} & \\ & & \underline{\beta_3}\alpha_3 & \beta_3\gamma_3 + \underline{\alpha_4} & \beta_3\delta_3 + \underline{\gamma_4} & \ddots \\ 0 & & & \beta_4\alpha_4 & \beta_4\gamma_4 + \alpha_5 & \ddots \\ & & & & \ddots & \ddots \end{pmatrix}$$

Igualant element a element, es van calculant els elements subratllats. S'obté:

	Operacions (/)	Operacions (*)
$\delta_i = d_i \quad \forall i = 1 \div n-2$		
$\alpha_1 = a_1$		
$\gamma_1 = c_1$		
$\forall i = 1 \div n-2$: $\beta_i = b_i/\alpha_i$	1	
$\alpha_{i+1} = a_{i+1} - \beta_i\gamma_i$		1
$\gamma_{i+1} = c_{i+1} - \beta_i\delta_i$		1
$\beta_{n-1} = b_{n-1}/\alpha_{n-1}$	1	
$\alpha_n = a_n - \beta_{n-1}\gamma_{n-1}$		1

Total (/)

$$\left(\sum_{i=1}^{n-2} 1\right) + 1 = \boxed{n-1}$$

Total (*)

$$\left(\sum_{i=1}^{n-2} 2\right) + 1 = \boxed{2n-3}$$

ⓑ Per a deduir la fórmula lineal recurrent, és suficient usar $n=4$

$$D_4 = \begin{vmatrix} 3 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 3 & -1 \\ 0 & 0 & -1 & 3 \end{vmatrix} = 3\cdot D_3 - (-1)\left[(-1)\cdot D_2 - (-1)(-1)\cdot D_1\right] = 3D_3 - D_2 - D_1$$

En general, serà $\boxed{D_n = 3D_{n-1} - D_{n-2} - D_{n-3} \quad \forall n \geq 4}$

Calculem directament els 3 primers i usem després la fórmula recurrentment

$$\left.\begin{aligned} D_1 &= 3 \\ D_2 &= 9 - 1 = 8 \\ D_3 &= 27 - 1 - 3 - 3 = 20 \end{aligned}\right\} \quad \begin{aligned} D_4 &= 3\cdot 20 - 8 - 3 = 49 \\ D_5 &= 3\cdot 49 - 20 - 8 = 119 \\ D_6 &= 3\cdot 119 - 49 - 20 = 288 \\ D_7 &= 3\cdot 288 - 119 - 49 = 696 \\ D_8 &= 3\cdot 696 - 288 - 119 = \boxed{1681} \end{aligned}$$

(2) (c) $n=4$

Fem els 3 passos d'eliminació gaussiana (sense pivotatge)

En cadascun, imposem que el pivot sigui no nul.

Mirem de no fer càlculs innecessaris.

$$\left(\begin{array}{c|ccc} \boxed{c} & -1 & -1 & 0 \\ \hline -1 & c & -1 & -1 \\ 0 & -1 & c & -1 \\ 0 & 0 & -1 & c \end{array}\right) \xrightarrow{\text{Pas 1}} \left(\begin{array}{c|ccc} & & & \\ \hline -1/c & c-(-\frac{1}{c})(-1) & -1-(-\frac{1}{c})(-1) & -1 \\ 0 & -1 & c & -1 \\ 0 & 0 & -1 & c \end{array}\right) = \left(\begin{array}{c|cc} & & \\ \hline c-\frac{1}{c} & -1-\frac{1}{c} & -1 \\ \hline -1 & c & -1 \\ 0 & -1 & c \end{array}\right) \rightarrow$$

Cal $\boxed{c \neq 0}$

$$\xrightarrow{\text{Pas 2}} \left(\begin{array}{c|cc} & & \\ \hline \frac{-c}{c^2-1} & c-\left(\frac{-c}{c^2-1}\right)\left(-1-\frac{1}{c}\right) & (*) \\ & -1 & c \end{array}\right) \xrightarrow{\text{Pas 3}}$$

Cal $\boxed{c-\frac{1}{c} \neq 0}$

(no cal fer-lo explícitament, només saber quan es pot fer, o no)

$\Leftrightarrow \frac{c^2-1}{c} \neq 0 \Leftrightarrow c \neq \pm 1$

No cal calcular (*)

El pivot del 3r pas (i últim) és $c-\left(\frac{c}{c^2-1}\right)\left(\frac{c+1}{c}\right) = c-\frac{1}{c-1} \Rightarrow$

$\Rightarrow$ Cal $\boxed{c-\frac{1}{c-1} \neq 0}$

$\Leftrightarrow c^2-c-1 \neq 0 \Leftrightarrow c \neq \frac{1 \pm \sqrt{1+4}}{2} = \frac{1\pm\sqrt{5}}{2}$

$\Rightarrow$ Els valors de c per als quals <u>no</u> es pot fer E.G. sense pivotatge són

$$\boxed{c=0,\ c=1,\ c=-1,\ c=\underset{\approx 1.618}{\frac{1+\sqrt{5}}{2}},\ c=\underset{\approx -0.618}{\frac{1-\sqrt{5}}{2}}}$$

(3) Diferències dividides generalitzades:

(a)

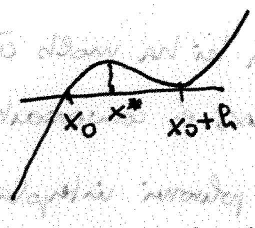
x_0	f_0		
		$\frac{f_1-f_0}{h}$	
x_1	f_1		$\frac{g_1}{h}-\frac{f_1-f_0}{h^2}$
		g_1	
x_1	f_1		

$$\Rightarrow p(x) = f_0 + \frac{f_1-f_0}{h}(x-x_0) + \left[\frac{g_1}{h} - \frac{f_1-f_0}{h^2}\right](x-x_0)(x-x_0-h)$$

$$f(x)-p(x) = \frac{f^{(3)}(\xi(x))}{3!}\,\omega(x) \;; \quad \omega(x) = (x-x_0)(x-x_0-h)^2 = (x-x_0)^3 - 2h(x-x_0)^2 + h^2(x-x_0)$$

$$0 = \omega'(x) = 3(x-x_0)^2 - 4h(x-x_0) + h^2 \iff$$

$$\iff x-x_0 = \frac{4h \pm \sqrt{16h^2-12h^2}}{6} = \frac{4h\pm 2h}{6} = \left\langle \begin{array}{l} h \\ \frac{h}{3} \end{array}\right.$$



L'extrem (màxim) és a $x^*-x_0 = \frac{h}{3}$. Llavors $\omega(x^*) = \left(\frac{h}{3}\right)^3 - 2h\left(\frac{h}{3}\right)^2 + h^2\left(\frac{h}{3}\right) = h^3\underbrace{\left(\frac{1}{27} - \frac{2}{9} + \frac{1}{3}\right)}_{4/27}$

$$\omega(x^*) = \frac{h}{3}\left(-\frac{2h}{3}\right)^2 = \frac{4}{27}h^3$$

$$\Rightarrow \forall x \in [x_0, x_0+h] \quad |f(x)-p(x)| \le \frac{M_3}{6}\,\frac{4}{27}h^3 = \boxed{\frac{2M_3}{81}h^3}$$

(b)

$$I \approx \int_{x_0}^{x_0+h} p(x)\,dx = \left[f_0(x-x_0) + \frac{f_1-f_0}{h}\,\frac{(x-x_0)^2}{2} + \left(\frac{g_1}{h} - \frac{f_1-f_0}{h^2}\right)\left[\frac{(x-x_0)^3}{3} - h\,\frac{(x-x_0)^2}{2}\right]\right]_{x_0}^{x_0+h} =$$

$$= f_0 h + \frac{f_1-f_0}{h}\,\frac{h^2}{2} + \left(\frac{g_1}{h} - \frac{f_1-f_0}{h^2}\right)\left(\frac{h^3}{3} - h\,\frac{h^2}{2}\right) =$$

$$= h\underbrace{\left[f_0 + \frac{f_1-f_0}{2} - (f_1-f_0)\left(\frac{1}{3}-\frac{1}{2}\right)\right]}_{f_0\underbrace{\left[1-\frac{1}{2}-\frac{1}{6}\right]}_{1/3} + f_1\underbrace{\left[\frac{1}{2}+\frac{1}{6}\right]}_{2/3}} + h^2 g_1\underbrace{\left(\frac{1}{3}-\frac{1}{2}\right)}_{-1/6} = \boxed{\frac{h}{3}(f_0+2f_1) - \frac{h^2}{6}g_1}$$

$$\text{Error} = \int_{x_0}^{x_0+h}[f(x)-p(x)]\,dx = \int_{x_0}^{x_0+h}\frac{f^{(3)}(\xi(x))}{6}\,\underbrace{\omega(x)}_{\text{no canvia de signe}}\,dx \underset{\text{T.V.M.}}{=} \frac{f^{(3)}(\eta)}{6}\int_{x_0}^{x_0+h}\omega(x)\,dx = \boxed{\frac{1}{72}f^{(3)}(\eta)\cdot h^4}$$

$$\int_{x_0}^{x_0+h}\omega(x)\,dx = \left[\frac{(x-x_0)^4}{4} - 2h\,\frac{(x-x_0)^3}{3} + h^2\,\frac{(x-x_0)^2}{2}\right]_{x_0}^{x_0+h} = h^4\left[\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\right] = h^4\cdot\frac{1}{12}$$

(c)

x	$f(x)$	$f'(x)$
0	0.2955	
0.1	0.3894	0.9211
0.2	0.4794	0.8776

$$\int_0^{0.1} f(x)\,dx \approx \frac{0.1}{3}[0.2955 + 2(0.3894)] - \frac{0.01}{6}(0.9211) = 0.0342748\widehat{3}$$

$$\int_{0.1}^{0.2} f(x)\,dx \approx \frac{0.1}{3}[0.3894 + 2(0.4794)] - \frac{0.01}{6}(0.8776) = 0.043477\widehat{3}$$

$$\Rightarrow \boxed{I \approx 0.077752\widehat{1}\widehat{6}}$$

→

Les dades són de la funció $f(x)=\sin(x+0.3)$

$$\Rightarrow \left.\begin{array}{l} \int_0^{0.1} f(x)dx = \cos(0.3)-\cos(0.4) = \quad 0.03427549512 \\ \int_{0.1}^{0.2} f(x)dx = \cos(0.4)-\cos(0.5) = \quad 0.04347843211 \end{array}\right\} \text{suma } 0.07775392723$$

Nota. Quan hi ha molts càlculs, es poden cometre errors.

Per això és convenient fer algunes comprovacions.

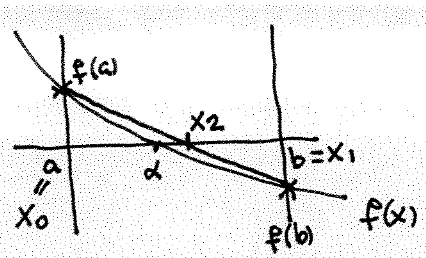
a) El polinomi interpolador $p(x)$ s'ha de comprovar : $\begin{cases} p(x_0)=f_0 \\ p(x_1)=f_1 \\ p'(x_1)=g_1 \end{cases}$

b) Prenent $\overset{f(x)}{p(x)} \in \mathcal{P}_2$, l'aproximació ha de ser una igualtat

Prenent $\overset{f(x)=}{p(x)} \in \mathcal{P}_3$ llavors $f^{(3)}$ és ctant i es comprova la fórmula de l'error.

4

ⓐ Idea gràfica



i) Recta per $(x_0,f_0),(x_1,f_1)$: $y-f_1=\dfrac{f_0-f_1}{x_0-x_1}(x-x_1)$ (*)

Fent $y=0$ i aïllant x trobem x_2: $\boxed{x_2 = x_1 - f_1\dfrac{(x_0-x_1)}{(f_0-f_1)}}$

¿ $\alpha < x_2 < x_1$?

$\left.\begin{array}{l} x_0<x_1 \\ f_0=f(a)>0>f(b)=f(x_1)=f_1 \\ f_1<0 \end{array}\right\} \Rightarrow f_1\dfrac{(x_0-x_1)}{(f_0-f_1)}>0 \Rightarrow \boxed{x_2<x_1}$

Intercanviant $x_0 \leftrightarrow x_1$, també és $x_2 = x_0 - f_0\dfrac{(x_1-x_0)}{f_1-f_0}$ i, per tant, $x_0<x_2$.

$f'(x)<0 \quad \forall x\in(a,b) \Rightarrow f(x)$ estrict. mon. decreixent en (a,b)

Per tant, $\alpha < x_2 \iff f(x_2)<0$. Vegem això darrer.

Si $y=p(x)$ és el pol. interpolador (*), llavors $f(x)-p(x) = \dfrac{f''(\xi(x))}{2}(x-x_0)(x-x_1)$

Posant $x=x_2 \Rightarrow p(x_2)=0$ i $f(x_2) = \underbrace{\dfrac{f''(\xi)}{2}}_{\vee\atop 0}\underbrace{(x_2-x_0)}_{\vee\atop 0}\underbrace{(x_2-x_1)}_{\wedge\atop 0} < 0$

ii) El que s'ha fet a $[x_0,x_1]$ es pot repetir a $[x_0,x_2]$, després a $[x_0,x_3]$, etc.

S'obté: ($x_0=a$ fixat, canviem x_1 per x_k)

$$\boxed{x_{k+1} = x_k - f(x_k)\dfrac{a-x_k}{f(a)-f(x_k)} \qquad \forall k\geq 0} \qquad (**)$$

I la successió $(x_k)_{k\geq 0}$ és monòtona decreixent i fitada inferiorment per α.

$\Rightarrow$ $\boxed{\text{té límit}}$. Sigui $\underset{[\alpha,b)\subsetneq(a,b)}{\beta}$. Verificarà $\beta=\beta-f(\beta)\dfrac{a-\beta}{f(a)-f(\beta)} \Rightarrow f(\beta)=0 \Rightarrow \boxed{\beta=\alpha}$

$\uparrow$ f estrict. monòtona.

ⓑ Ordre 1 ?

Sigui $e_k = x_k-\alpha \quad \forall k\geq 0 \Rightarrow (e_k)\xrightarrow[k\to\infty]{} 0$. Operant a (**) (restant α)

$$\Rightarrow e_{k+1} = e_k - \Big(\underset{0}{\underset{\|}{f(\alpha)}}+f'(\alpha)e_k+O_2\Big)\dfrac{(a-\alpha)-e_k}{f(a)-[\underset{0}{\underset{\|}{f(\alpha)}}+f'(\alpha)e_k+O_2]} =$$

$$= \dfrac{[f(a)e_k - f'(\alpha)e_k^2+O_3] - f'(\alpha)(a-\alpha)e_k + O_2}{f(a)-f'(\alpha)e_k+O_2}$$

$$\Rightarrow \dfrac{e_{k+1}}{e_k^1} \longrightarrow \dfrac{f(a)-f'(\alpha)(a-\alpha)}{f(a)} = \boxed{1-f'(\alpha)\dfrac{a-\alpha}{f(a)}} \quad \text{coef. asimptòtic}$$

$$= 1+f'(\alpha)\dfrac{\alpha-a}{f(a)} \equiv L$$

<u>Observació</u> (exercici). $|L|<1$