(4) $f(x) \equiv x^3+x+c=0 \quad , c \approx 0.$

(a) $\left.\begin{array}{l}\lim\limits_{x\to+\infty} f(x)=+\infty \\ \lim\limits_{x\to-\infty} f(x)=-\infty\end{array}\right\} \overset{f \text{ contínua}}{\Rightarrow} \quad f(x)$ té arrels reals

$f'(x)=3x^2+1 \geq 1 > 0 \quad \forall x\in\mathbb{R} \Rightarrow f(x)$ estrict. monòtona creixent

$\Rightarrow \boxed{\exists!\text{ arrel real } \alpha=\alpha(c)}$

Si $c>0$: $\left.\begin{array}{l} f(0)=c>0 \\ f(-c)=-c^3-c+c=-c^3<0 \end{array}\right\} \Rightarrow \boxed{\alpha\in(-c,0)}$

Si $c=0 \Rightarrow \alpha=0$

Si $c<0$, el canvi $\left\{\begin{array}{l} c\to -c \\ x\to -x\end{array}\right\}$ deixa l'equació igual $-(x^3+x+c)=0$

Per tant $\alpha(-c)=-\alpha(c)\in(0,-c)$

(b) $c\approx 0$ (suposem $c>0$); $x_0=0$

M1: $x_{k+1}=g(x_k), \quad g(x)=-x^3-c$

M2: $x_{k+1}=h(x_k), \quad h(x)=-(x+c)^{1/3}$

Són "consistents"?

$x=g(x) \Rightarrow f(x)=0$. ok!

$x=h(x) \Rightarrow f(x)=0$. ok!

Convergència local

$g'(x)=-3x^2 \Rightarrow |g'(\alpha)|=3\alpha^2\leq 3c^2<1$ si $c\approx 0 \Rightarrow \boxed{\text{Sí}}$

$h'(x)=-\frac{1}{3}(x+c)^{-2/3} \Rightarrow |h'(\alpha)|=\frac{1}{3(\alpha+c)^{2/3}} \underset{\alpha\in(-c,0)}{\geq} \frac{1}{3c^{2/3}}>1$ si $c\approx 0 \Rightarrow \boxed{\text{No}}$

$\Rightarrow \boxed{\text{Millor usar } g(x)}$

(c) Estimació d'iterats per $g(x)$? Sigui $I=(-c,0)$. És fàcil veure: $\boxed{\text{si } c>0 \text{ petit} \Rightarrow g(I)\subset I}$

Sigui $e_k\equiv x_k-\alpha \quad \forall k\geq 0$

$e_k=x_k-\alpha=g(x_{k-1})-g(\alpha)=g'(\xi)(x_{k-1}-\alpha)=g'(\xi)\,e_{k-1}$; $\xi\in\langle x_{k-1},\alpha\rangle\subset I$

Com que $|g'(\xi)|=|-3\xi^2|\leq 3c^2 \Rightarrow \boxed{|e_k|\leq(3c^2)|e_{k-1}|}$

$\left.\begin{array}{l}\text{Recurrentment } |e_k|\leq(3c^2)^k\cdot|e_0| \\ \text{Com que } |e_0|<c\end{array}\right\} \Rightarrow \boxed{|e_k|\leq c\cdot(3c^2)^k \quad \forall k\geq 0}$

Així doncs, només cal imposar $c\cdot(3c^2)^k\leq 10^{-100}$

i s'aïlla k: $\boxed{k\geq\frac{-100-\log_{10}c}{\log_{10}(3c^2)}}$. Exemple: $c=0.1 \Rightarrow n\geq 65.01$

(d) Estimació d'iterats per a NR

Teoria: Si $f(\alpha)=0$, $f'(\alpha)\neq 0$, $f''(\alpha)\neq 0$ i NR convergeix $\Rightarrow \begin{cases} \text{ordre } 2 \\ \text{coef. asimpt. } \left|\frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}\right| \end{cases}$

O sigui, si $e_n = |x_n-\alpha|$ llavors $\boxed{e_n \approx \frac{1}{2}\left|\frac{f''(\alpha)}{f'(\alpha)}\right| e_{n-1}^2}$

En aquest problema $\left.\begin{array}{l} |f'(\alpha)| = 3\alpha^2+1 \geq 1 \\ |f''(\alpha)| = |6\alpha| < 6c \end{array}\right\} \Rightarrow e_n \lesssim \frac{1}{2}\,\frac{6c}{1}\,e_{n-1}^2$ o $\boxed{e_n \lesssim (3c)\cdot e_{n-1}^2}$

Busquem una fita (aproximada) explícita de e_n en funció de n (i no e_{n-1})

$$\left.\begin{array}{l}
e_0 < c \\
e_1 \lesssim (3c)\cdot c^2 = 3^1\cdot c^3 \\
e_2 \lesssim (3c)(3^1\cdot c^3)^2 = 3^3\cdot c^7 \\
e_3 \lesssim (3c)(3^3\cdot c^7)^2 = 3^7\cdot c^{15} = \frac{(\sqrt{3}c)^{15}}{\sqrt{3}} \\
e_4 \lesssim (3c)(3^7\cdot c^{15})^2 = 3^{15}\cdot c^{31} = \frac{(\sqrt{3}c)^{31}}{\sqrt{3}} \\
\text{etc.}
\end{array}\right\} \text{En general: } \boxed{e_k \lesssim \frac{(\sqrt{3}c)^{2^{k+1}-1}}{\sqrt{3}} \quad \forall k \geq 0}$$

Així doncs, només cal imposar $(\sqrt{3}c)^{2^{k+1}-1} \leq (\sqrt{3})\cdot 10^{-100}$

i aïllar k: $(2^{k+1}-1)\cdot \log_{10}(\sqrt{3}c) \leq \log_{10}(\sqrt{3}) - 100$

(si $\sqrt{3}c<1$) $\boxed{2^{k+1} \geq 1 + \frac{\log_{10}(\sqrt{3})-100}{\log_{10}(\sqrt{3}c)}}$

Exemple: $\boxed{c=0.1}$ $\Rightarrow$ és suficient $2^{k+1} \geq 132.017$

$\Rightarrow k+1 = 8 \Leftrightarrow \boxed{k=7}$