② ⓐ $A = (a_{ij}) \quad 1 \le i,j \le n$

$a_{ij} = 0$ quan $i > j+2$. O sigui $A = \begin{pmatrix} \times & \times & \times & \times & \times & \times & \cdots \\ \times & \times & \times & \times & \times & \times & \\ \times & \times & \times & \times & \times & \times & \vdots \\ 0 & \times & \times & \times & \times & \times & \\ 0 & 0 & \times & \times & \times & \times & \\ 0 & 0 & 0 & \times & \times & \times & \\ \vdots & & & & \ddots & & \ddots \end{pmatrix}$

Es calcula $A = LU$ mitjançant elim. gauss. <u>sense pivotatge</u>:

els 0 en les posicions $i > j+2$ donen multiplicadors 0. O sigui "es conserven" en L

En cada etapa $k = 1 \div n-2$ (excepte l'última) només es calculen 2 multiplicadors i es modifiquen 2 files: $i = k+1, k+2$

En l'etapa $k = n-1$ (última) només es calcula 1 multiplicador es modifica l'última fila (de fet, 1 element)

O sigui: $\begin{cases} U \text{ serà "normal": triang. superior} \\ L \text{ serà triang. inferior amb uns a la diagonal, amb 0's quan } i > j+2 \end{cases}$

(o sigui, <u>només 2 subdiagonals</u> no nul·les)

<u>Fórmules</u>

	(/)	(*)
$\forall k = 1 \div n-2$		
$\quad \forall i = k+1, k+2$		
$\quad\quad a_{ik} \leftarrow a_{ik}/a_{kk}$	(1)	
$\quad\quad \forall j = k+1 \div n$		
$\quad\quad\quad a_{ij} \leftarrow a_{ij} - a_{ik}a_{kj}$		(1)
$k = n-1$		
$\quad i = k+1 = n$		
$\quad\quad a_{ik} \leftarrow a_{ik}/a_{kk}$	(1)	
$\quad\quad j = k+1 = n$		
$\quad\quad\quad a_{ij} \leftarrow a_{ij} - a_{ik}a_{kj}$		(1)

<u>Operacions</u>

$$(/) : \left(\sum_{k=1}^{n-2} 2 \cdot 1 \right) + 1 = 2(n-2) + 1 = \boxed{2n-3}$$

$$(*) : \left(\sum_{k=1}^{n-2} 2 \cdot \sum_{j=k+1}^{n} 1 \right) + 1 = \left(2 \sum_{k=1}^{n} (n-k) \right) + 1 = 2 \underbrace{\left[(n-1) + (n-2) + \dots + 2 \right]}_{\frac{(n+1)(n-2)}{2}} + 1 = \boxed{n^2 - n - 1}$$

(b) L té det 1 $\Rightarrow \exists L^{-1}$

L triang. infer. amb uns a la diag. $\Rightarrow$ L^{-1} també. Siguin x_{ij} els elements essencials

Imposem $L L^{-1} = Id$ en el $\boxed{\text{cas } n=6}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a_{21} & 1 & 0 & 0 & 0 & 0 \\ a_{31} & a_{32} & 1 & 0 & 0 & 0 \\ 0 & a_{42} & a_{43} & 1 & 0 & 0 \\ 0 & 0 & a_{53} & a_{54} & 1 & 0 \\ 0 & 0 & 0 & a_{64} & a_{65} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ x_{21} & 1 & 0 & 0 & 0 & 0 \\ x_{31} & x_{32} & 1 & 0 & 0 & 0 \\ x_{41} & x_{42} & x_{43} & 1 & 0 & 0 \\ x_{51} & x_{52} & x_{53} & x_{54} & 1 & 0 \\ x_{61} & x_{62} & x_{63} & x_{64} & x_{65} & 1 \end{pmatrix} = Id_6$$

Anem imposant igualtats, element a element, avançant per files (per exemple) i anem aïllant elements desconeguts en funció d'elements coneguts.

Fila 1: No aporta res.

Fila 2: $a_{21} + x_{21} = 0 \Rightarrow \boxed{x_{21} = -a_{21}}$ (I)

Fila 3:
$$\left.\begin{array}{l} a_{31} + a_{32}x_{21} + x_{31} = 0 \\ a_{32} + x_{32} = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{ll} x_{31} = -a_{31} - a_{32}x_{21} & \text{(II)} \\ x_{32} = -a_{32} & \text{(I)} \end{array}}$$

Fila 4:
$$\left.\begin{array}{l} a_{42}x_{21} + a_{43}x_{31} + x_{41} = 0 \\ a_{42} + a_{43}x_{32} + x_{42} = 0 \\ a_{43} + x_{43} = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{ll} x_{41} = -a_{42}x_{21} - a_{43}x_{31} & \text{(III)} \\ x_{42} = -a_{42} - a_{43}x_{32} & \text{(II)} \\ x_{43} = -a_{43} & \text{(I)} \end{array}}$$

Fila 5:
$$\left.\begin{array}{l} a_{53}x_{31} + a_{54}x_{41} + x_{51} = 0 \\ a_{53}x_{32} + a_{54}x_{42} + x_{52} = 0 \\ a_{53} + a_{54}x_{43} + x_{53} = 0 \\ a_{54} + x_{54} = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{ll} x_{51} = -a_{53}x_{31} - a_{54}x_{41} & \text{(III)} \\ x_{52} = -a_{53}x_{32} - a_{54}x_{42} & \\ x_{53} = -a_{53} - a_{54}x_{43} & \text{(II)} \\ x_{54} = -a_{54} & \text{(I)} \end{array}}$$

Fila 6:
$$\left.\begin{array}{l} a_{64}x_{41} + a_{65}x_{51} + x_{61} = 0 \\ a_{64}x_{42} + a_{65}x_{52} + x_{62} = 0 \\ a_{64}x_{43} + a_{65}x_{53} + x_{63} = 0 \\ a_{64} + a_{65}x_{54} + x_{64} = 0 \\ a_{65} + x_{65} = 0 \end{array}\right\} \Rightarrow \boxed{\begin{array}{ll} x_{61} = -a_{64}x_{41} - a_{65}x_{51} & \\ x_{62} = -a_{64}x_{42} - a_{65}x_{52} & \text{(III)} \\ x_{63} = -a_{64}x_{43} - a_{65}x_{53} & \\ x_{64} = -a_{64} - a_{65}x_{54} & \text{(II)} \\ x_{65} = -a_{65} & \text{(I)} \end{array}}$$

Hi ha 3 tipus de fórmules: I, II i III

(I): 1a subdiagonal $(i = j+1)$

(II): 2a " $(i = j+2)$

(III): resta de subdiagonals $(i > j+2)$

$\boxed{\text{Cas } n \text{ general}}$

$$\boxed{\begin{array}{l} \forall i = 2 \div n \quad x_{i,i-1} = -a_{i,i-1} \\ \forall i = 3 \div n \quad x_{i,i-2} = -a_{i,i-2} - a_{i,i-1} \cdot x_{i-1,i-2} \\ \forall i = 4 \div n \\ \quad \forall j = 1 \div i-3 \quad x_{i,j} = -a_{i,i-2}\,x_{i-2,j} - a_{i,i-1}\,x_{i-1,j} \end{array}} \quad \begin{array}{l} (*) \\ \leftarrow 0 \\ \leftarrow 1 \\ \\ \leftarrow 2 \end{array}$$

$\boxed{\text{Quantitat de } (*)}$

$$\sum_{i=3}^{n} 1 + \sum_{i=4}^{n} \sum_{j=1}^{i-3} 2 = (n-2) + 2\sum_{i=4}^{n}(i-3) = (n-2) + 2\underbrace{(1+2+\dots+(n-3))}_{\frac{(n-2)(n-3)}{2}} = (n-2)[1+(n-3)] = \boxed{(n-2)^2}$$

ⓒ Es poden usar les fórmules finals de ⓑ, o:
Es pot fer directament, repetint ⓑ en aquest cas particular : $a_{i,i-1} = a_{i,i-2} = 1$, o

Es pot fer directament, com a ⓑ però "avançant per columnes".

Fem això últim, per exemple.

<u>1a columna</u>

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ & & \cdots & & & \end{pmatrix} \begin{pmatrix} 1 \\ x_{21} \\ x_{31} \\ x_{41} \\ x_{51} \\ x_{61} \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix} \longrightarrow \text{s'obté} \quad \begin{matrix} x_{21} = -1 \\ x_{31} = 0 \\ x_{41} = 1 \\ x_{51} = -1 \\ x_{61} = 0 \\ x_{71} = 1 \\ \vdots \end{matrix}$$

<u>2a columna</u>

$$\left(\begin{array}{c|c} & \\ \hline & \text{idem} \end{array}\right) \begin{pmatrix} 0 \\ 1 \\ x_{32} \\ x_{42} \\ x_{52} \\ x_{62} \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

eliminant 1a columna i 1a fila, queda "el mateix sistema" del cas d'abans $\Rightarrow \begin{matrix} x_{32} = -1 \\ x_{42} = 0 \\ x_{52} = 1 \\ \text{etc.} \end{matrix}$

anàlogament per a la resta de columnes.

Així doncs $L^{-1} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ x_{ij} & & 1 \end{pmatrix}$ amb $\begin{cases} x_{i,i-1} = -1 \\ x_{i,i-2} = 0 \\ x_{i,i-3} = 1 \\ \text{etc (es repeteix això)} \end{cases}$

O sigui:

$$L^{-1} = \begin{pmatrix} 1 & & & & & \\ -1 & \ddots & & & & \\ 0 & \ddots & \ddots & & & \\ 1 & \ddots & \ddots & \ddots & & \\ -1 & \ddots & \ddots & \ddots & \ddots & \\ 0 \;\cdots & -1 & 1 & 0 & -1 & 1 \end{pmatrix}$$

Llavors: $\|L\|_\infty = 3$

$\|L^{-1}\|_\infty$ = quantitat de ± 1 de la fila n $\Rightarrow$ Alguns casos són

n	$\|L^{-1}\|_\infty$
4	3
5	4
6	4
7	5
8	6
9	6
	etc

En general: $\forall k \geq 2$ $\begin{cases} \text{si } n = 3k \text{ o } n = 3k-1 & \Rightarrow \|L^{-1}\|_\infty = 2k \\ \text{si } n = 3k-2 & \Rightarrow \|L^{-1}\|_\infty = 2k-1 \end{cases}$

Llavors

$$\boxed{\forall k \geq 2. \begin{cases} \text{si } n = 3k \text{ o } n = 3k-1 & \Rightarrow \kappa_\infty(L) = 6k \\ \text{si } n = 3k-2 & \Rightarrow \kappa_\infty(L) = 6k-3 \end{cases}}$$