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# A MODEL ON THE ENERGY TRANSPORT INSIDE A STAR SIMILAR TO THE SUN.

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## ABSTRACT

The transportation of energy inside a theoretical star based on the initial parameters of the sun and a fluid mechanical model of convection. This model and paper aims to explain the transport of energy in the stellar interior by the means of radiation and convection, then a set of optimal parameters and a reasonable model for the stellar interior shall be attempted to be found. A set of sanity checks shall be conducted to verify the mathematical internal consistency to make sure that the model is consistent with itself.

*Subject headings:*

## 1. INTRODUCTION

Using a fluid mechanical basic model for convection inside the stellar interior, the transportation is then modelled as a continuous function of several differential equations that describe the properties of the star. This then makes it possible to approximate the star as a series of parameter evolutions. A numerical solution to the differential equations is essential to make the model work as these equations are coupled and have no analytical solution thus an iterative method is necessary. These differential equations are then solved with a regular euler method and iterated this way, a higher order solver was not necessary as the equations are fairly stable.

A set of best fit parameters will be estimated and used in the star, this is reasonable as several of the initial conditions is not necessarily constant and are only approximations, as the evolution of these parameters is not necessarily known and only approximated through measurement.

Should be noted that for the entire simulation the star is assumed to be an ideal gas and that all elements are fully ionized. As the temperatutre at the outer regions is not as energetic as the core, elements such as helium may not be entirely ionized at this stage, but it is still like the assumption of ideal gas not too unreasonable, but should be noted that the gas inside a star is not ideal, but for the purposes of this paper it is enough.

## 2. METHOD

To begin with how this model was created we start by stating and explaining a few equations that will be used later for the simulations and dictates how the energy in the stellar interior is transported. First we begin with the following:

$$\nabla_{ad} = (\frac{\partial \ln(T)}{\partial \ln(P)})_s = 2/5 \tag{1}$$

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As to why this equation is equal to a constant is due to the assumption of ideal gas, thus it becomes obvious that it has to be constant, this again is more obvious when one considers the alternate form of the adiabatic gradient:

$$\nabla_{ad} = \frac{P\delta}{T\rho c_P} \tag{2}$$

Thus for this to be obvious we need to know $\delta$ and $c_P$, thus due to the assumption of ideal gas and no energy exchange between the rising convective gas bubbles it becomes 1. Then for $c_P$ we have the following;

$$c_P = \frac{5}{2}\frac{k_B}{\mu m_u} \tag{3}$$

Thus we have an expression for the adiabatic gradient, although this is the easiest gradient to find, it also dictates the convection criteria, which will become apparent at later stages.

Thus for the other gradients we need to consider the following equations;

$$F_{con} = \rho c_P T\sqrt{g\delta} H_P^{-3/2} (\frac{l_m}{2})^{\frac{1}{2}} (\nabla^* - \nabla_p)^{3/2} \tag{4}$$

Then we need then following:

$$F_{rad} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* \tag{5}$$

This then completes our set for equations that we need to find an expression for $(\nabla^* - \nabla_P)$ Then we use the previous two equations in the following equation:

$$F_{con} + F_{rad} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{stable} \tag{6}$$

We then have the following situation by combining 4 and 5 into equation 6 we get the following situation:

$$\frac{16\sigma T^4}{3\kappa\rho H_P}\nabla^* + \rho c_P T\sqrt{g\delta} H_P^{-3/2} (\frac{l_m}{2})^{\frac{1}{2}} (\nabla^* - \nabla_p)^{3/2} = \frac{16\sigma T^4}{3\kappa\rho H_P}\nabla_{stab} \tag{7}$$

Thus we may simplify this equation to the following:

$$(\nabla^* - \nabla_p)^{3/2} = \frac{64\sigma T^3 H_P^{1/2}}{3\kappa\rho^2\sqrt{g\delta} l_m^2}(\nabla_{stable} - \nabla^*) \tag{8}$$

This equation will become useful at a later point in eliminating the left hand side and make a more usable expression that will not depend necessarily on the gradients but a substituted variable.

Now we will begin by inserting the following equation into the equation that describes the gradient and their relation.

$$(\nabla_p - \nabla_{ad}) = \frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}(\nabla^* - \nabla_p) \tag{9}$$

We will insert this into the following equation, we will then insert an expression for v, then we will solve this as a polynomial by inserting $\xi$ for $(\nabla^* - \nabla_p)^(1/2)$.
Thus:

$$\frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}(\nabla^* - \nabla_p) = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{10}$$

$$v = \sqrt{\frac{g\delta l_m^2}{4H_P}(\nabla^* - \nabla_p)} \tag{11}$$

Then by inserting we get the following:

$$\frac{32\sigma T^3}{3\kappa\rho^2 c_P v}\frac{S}{Qd}\sqrt{\frac{4H_P}{g\delta l_m^2}}(\nabla^* - \nabla_p)^{1/2} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{12}$$

Then by simplifying by setting U to:

$$U = \frac{64\sigma T^3}{3\kappa\rho^2 c_P}\sqrt{\frac{H_P}{g\delta}} \tag{13}$$

Then we have the following:

$$U l_m^{-1}(\nabla^* - \nabla_p)^{1/2} = (\nabla^* - \nabla_{ad}) - (\nabla^* - \nabla_p) \tag{14}$$

Thus we want a second degree polynomial for this so we simplify this to:

$$\xi^2 + U(\frac{S}{Qdl_m})\xi - (\nabla^* - \nabla_{ad}) = 0 \tag{15}$$

So we move forward to eliminate the $\nabla^*$ in all forms except in the form of $\xi$ thus we move on to the following:

$$\xi^2 + (U\frac{S}{Qdl_m})\xi - -(\nabla^* - \nabla_{ad}) \tag{16}$$

Then due to equation 8 We have the following equation:

$$\xi^3 = \frac{U}{l_m^2}(\nabla_{stable} - \nabla^*) \tag{17}$$

Thus once more by insertion we have the following:

$$\xi^3 = \frac{U}{l_m^2}(\nabla_{stable} - \xi^2 + U(\frac{S}{Qdl_m})\xi - \nabla_{ad}) \tag{18}$$

Thus by simplification and algebraic juggling we are left with the following:

$$\xi^3 + \frac{U}{l_m^2}\xi^2 + 4\frac{U^2}{l_m^4}\xi - U(\nabla_{stable} - \nabla_{ad}) \tag{19}$$

It should be noted that this polynomial only has one viable solution and that is due to the property of the third degree polynomial as determined by the algebraic fundamental theorem. This polynomial will only have one real root and thus only one viable solution as imaginary solutions for the stellar interior is undesirable.

So for explaining these gradients , $\nabla_{ad}$ is the gradient that determines the limit for convection in the gas, as this limit is well defined and constant for ideal gas the gradient $\nabla_{stable}$ has to be greater than this adiabatic gradient for convection to take place as the adiabatic gradient for an ideal gas turns out to be constant, actually 2/5 thus we have a constant limit for convection. These then turn out to be our governing equations, since the gradients determine the convection we are then left with radiative flux as can be found in equation 5 Thus based on these and the changes in the other parameters of the star a model of the radiative and convective flux may be found and simulated.

Then we have a few things left such as the mean molecular weight ($\mu$), finding this is fairly easy and we use the following:

$$\mu = \frac{1}{\sum(X \cdot (z+1) \cdot_z^A Y)} \tag{20}$$

This then gives us the basis for calculating the mean molecular weight, now as to what this means. We take the mass fraction of each element and multiply that by the total amount of particles generated per particle, here the nucleus is one particle and since all elements are fully ionized all electrons associated with the element in the unionized state is included. Then we multiply this with the mass of the particles generated. Now doing this for all elements shall yield a $\mu \approx 0.618$ Here the other elements that are not $^4He$ or $^1H$ may be omitted due to such small mass fractions and one may still yield the same answer. Although the $\mu$ used in the simulations included all other mass fractions.

## 3. SIMULATION

The simulation strategy in this case was to firstly define all the necessary methods to calculate the stellar variables, this being pressure, density, energy production, pressure scale height, gravitational acceleration, a method to extract and interpolate a set of opacity values must be implemented amongst other minor parts of the simulations. To solve the differential equations the euler method is employed this is not the most accurate method but it is good enough for this case, for the standard solar parameters the loop then runs 3005 times. The chosen variable to iterate over is the mass as this is the variable that gives the most stable differential equations. Then to make sure further that the equations are numerically stable a variable step function is employed given that for a constant step size the solutions may run away or spiral out of control this is beneficial as it allows the step size to be minimized and not run out of control.

In the solution of the differential equations all the steps are calculated iteratively and the loop runs for as long as the mass is larger than the mass step, if the next mass step is going to produce a negative mass the loop is broken and mass is set to zero. The last step is then neglected, but this is not too much of an issue as the temperature and the other variables will not change that much in the last mass shell if at all. Thus the loop will break itself when a reasonable mass condition is reached. A secondary condition is set as a limit on the number of solved values, this may be problematic but as long as the condition is not too strict it should not affect the solution significantly.

To make sure the gradients behave correctly we follow the conditions on the relative sizes of the gradients described bye the inequality defined in the method. The gradients are then calculated in the same loop as the differential equations this is then as the gradients dictate how the temperature has to be calculated, as the gradients dictate how the energy is transported, whether this is by radiation or by convection. Thus if the gradients satisfy the conditions for convection described in the method the temperature is calculated accordingly. The radiation flux and the convective flux is also calculated for the current steps of the loop, this then gives us the conditions for making a cross section plot of the star, where convective transport is present, as there is a set condition for convective transport deciding whether or not it is allowed thus implying that it is 0 at points the radiative flux does not behave like this, but when convection is present it becomes the lesser transport method. Thus this then dictates the cross section and the convective zones. Conduction is not modelled in the simulations.

The sanity checks for this paper was the verification of the interpolation method by interpolating between data points then known values were compared with the interpolated ones and a set tolerance was used to verify their validity. Another test was implemented to verify the internal mathematical consistency of the model and a series of gradient values and corresponding variables was calculated and then compared to some that would be known for the exact values.

## 4. RESULTS

The model generated the following for intial parameters equal to that of the sun, thus we should be left with something that resembles known parameters for the sun. The model produced the following results:

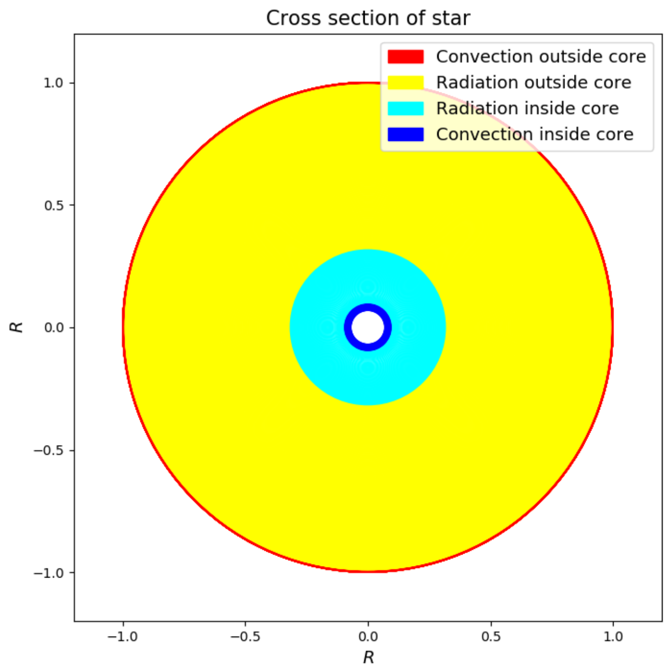



Fig. 1.— A cross section of the star marking the radiative and convective zones in the interior, this is not an optimal model, but verifies the internal mathematical validity of the model.

Then for a look at the evolution of the other stellar parameters we first consider the pressure and density.

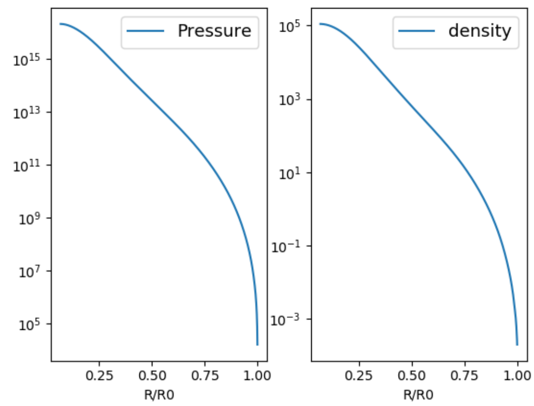



Fig. 2.— The evolution of the density and pressure in SI units plotted logarithmically as a function of the radius of the star, as can be seen the radius does not reach 0, but as can be seen the density and pressure begins to plateau, this makes phsyical sense as the density and pressure must converge towards a value as there is a limit to how dense and how high the pressure of the star with given parameters can get.

Further the resulting convective flux and the plot of the gradients, as should be noted due to the implications of the gradients dictating the onset and decline of convection as described in the method and simulation sections.

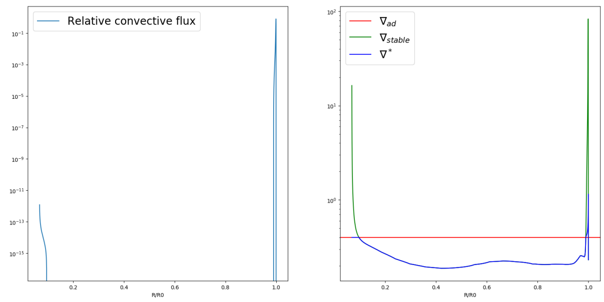



FIG. 3.— The convective flux and gradients plotted as a function of the radius, as can be seen the gradient and the convective flux follow each other perfectly and the gradient dictates perfectly when there is convection or not.

Further the resulting mass and luminosity as a function of radius:

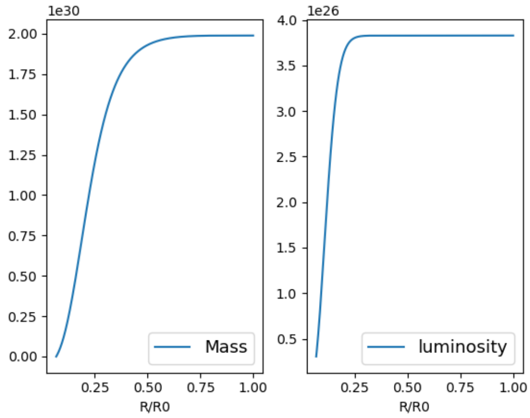



FIG. 4.— The luminosity and the mass evolution as a function of radius. Here the mass is the remainder of mass after the total mass has been reduced by a step size, thus as the stepsize grows the mass decreases faster and faster, as we see in figure 2 the density increases substantially in the core as does the decrease of mass, this makes sense as these two depend integrally on each other. The luminosity is practically constant before $R/R_0 \approx 0.3$ this also makes sense as the temperature in this region is way too low for fusion to take place on a significant scale.

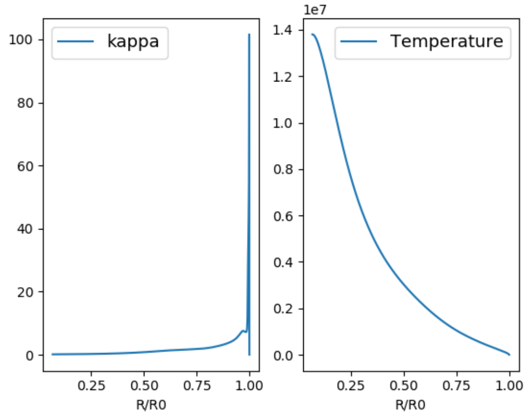



FIG. 5.— The opacity (kappa) and the temperature plotted as a functions of the radius. As can be seen the temperature increases rapidly towards the core. The opacity has a massive spike in the middle of the convective zone corresponding to a drop in radiative flux. Both variables are in SI units.

The relative total energy production is also interesting as it could be telling which branch of the chains that is the dominant, for the energy production.

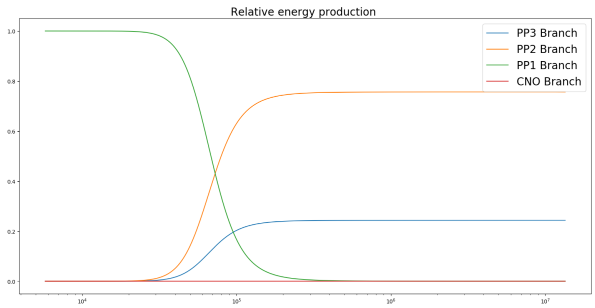



FIG. 6.— Relative energy production of the differing branches for the temperatures produced for the stellar interior for the model. As can be seen the PP branches dominate for the entire temperature range, the CNO never rises to prominence as should be expected from the temperature range.

### 4.1. *Changes in initial paramaters and model effects*

Now an interesting result is the change of the other variables at the change of on initial condition with the others staying the same, for this purpose the cross section plot will be ignored but the gradient and convective flux plots are very much relevant and shall be considered.

So for what happens when one changes the initial temperature, by increasing it the temperature starts off higher but will converge towards the same area, the density and pressure plots remain unchanged, the opacity has a rather high spike in the beginning and a secondary minor spike in place of the spike of the solar initial conditions. This then spikes the convective flux in these areas but not enough for it to be a significant part of the energy transport of the star except in the core and at the edge of the star. This was achieved by a doubling of $T_0$ A further increase will just exacerbate these effects.

The increase of density by a factor 5 from the solar values led to a wider convective zone at the surface and a higher spike in opacity as should be expected was found by the model. The pressure evolved similarly but had a higher initial and end value as was to be expected, the density evolution remained mostly unchanged. Mass evolution and luminosity remained mostly unchanged, the mass declined slightly faster as a function of radii, but general behaviour remained unchanged.

The change of initial mass led to a faster decline of mass paradoxically, this also led to radius not converging correctly and leaving out a rather significant part of the star. It also minimized the convective zone at the surface and left the core unchanged but then again this was affected by the star not having a radius converge towards 0. Density and pressure behaved similarly to the changes described when increasing the density.

Increasing the initial luminosity by a substantial amount leads to what seems like numerical problems, the mass at the end takes a dive, as does the gradient along the core, this is rather peculiar, but as seen from the pressure change it increases significantly and is higher than from any other change. This significant pressure may suppress any convection and explain the gradient in the core. Since this gradient takes such a significant dive, it may explain the numerical problem of the mass as the dynamic step size then becomes unreasonably large and then kicks in the condition of ending the simulation described in the simulation section.

By changing the radius the model produces a higher core temperature and a dive in the $\nabla_{stable}$ gradient just like increasing the luminosity as described above. It also dramatically will increase the density and pressure in the center of the core, and suppress any convection in the core. The opacity has a spike in the outer parts of the star not unlike the solar parameters

### 4.2. *Best fit parameters and model*

So for the bewst fit, these parameters were found by random experimentation, although they were done with a "hunch" based on the changes as described above. So for the found parameters we aimed for a convergence towards the following changes. A convection zone consisting of 15% of the stellar radii, furthermore a core consisting of L ¡ 0.995 $L_0$ and all other parameters going to 0 or within 5% of the initial parameters.

Thus after considering the paragraph above and the changes mentioned previously we are left with the following parameters:

TABLE 1
BEST FIT PARAMETERS

| Parameter | Changed to: |
|---|---|
| R | $0.94 \cdot R_\odot$ |
| M | $1.05 \cdot M_\odot$ |
| L | $1.65 \cdot L_\odot$ |
| $\rho$ | $85 \cdot 1.42 \cdot 10^{-7} \cdot \bar{\rho_\odot}$ |
| T | $2 \cdot T_\odot$ |

Table 1: The best fit parameters expressed in solar units, should be noted the temperature is the surface temperature of the sun as initial condition.

So for the parameter evolutions for these initial conditions we have the following results:

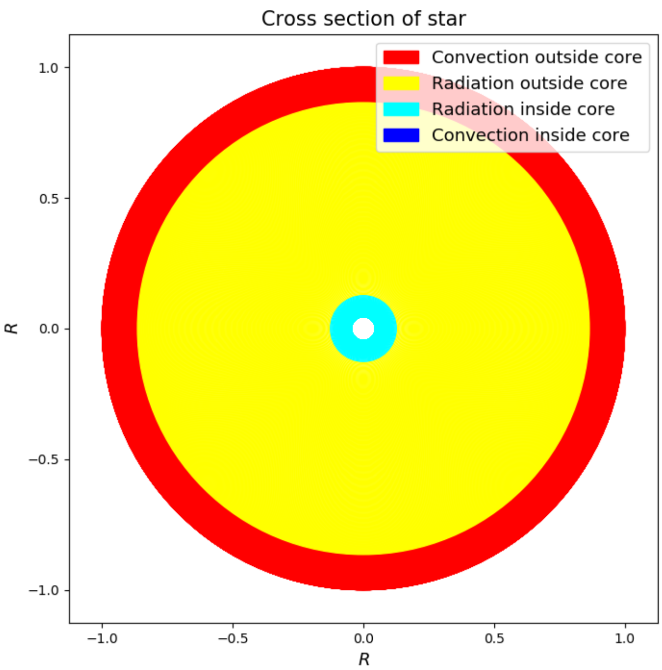



FIG. 7.— A cross section of the star marking the radiative and convective zones in the interior, as should be noted the difference from this to the original plot is that the convective zone in the core is gone and the core itself has shrunk to about 10% of the stellar radii.

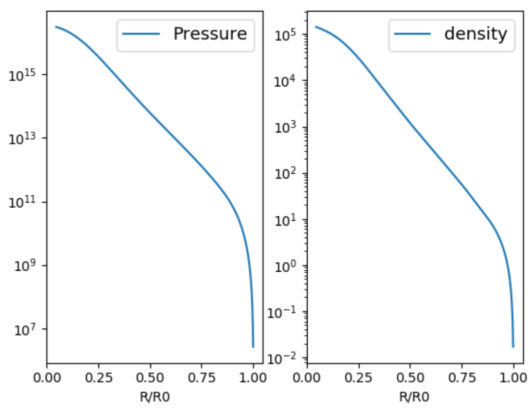



FIG. 8.— So this is the evolution of the density and pressure with the best fit initial parameters. Should be noted that the evolution is very similar to the evolution previously seen.

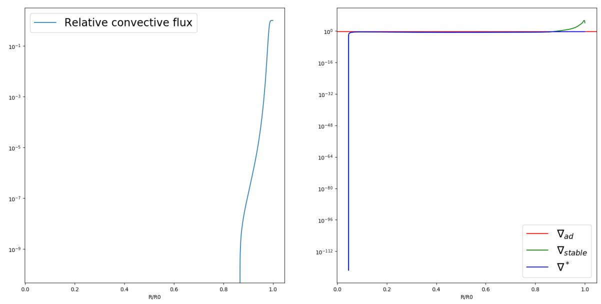



FIG. 9.— The gradient and convective flux with best fit parameters, as seen from the cross section the convection inside the core is gone as the gradient takes a massive dive and does not allow for convection.

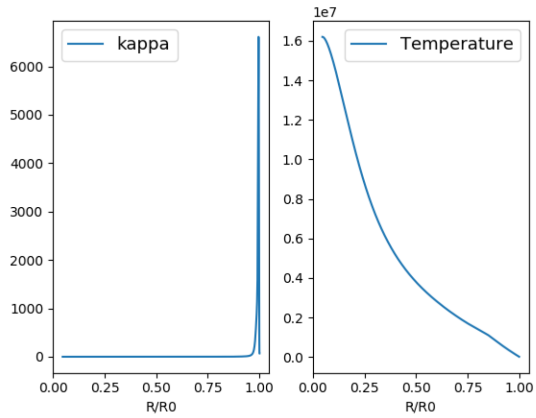



FIG. 10.— The opacity and temperature evolution as a function of radius, should be noted that the the temperature evolution has not changed significantly and has the approximate same end point. The opacity also corresponds to the convective flux value and is not too different in shape than the one for solar values, although significantly higher.

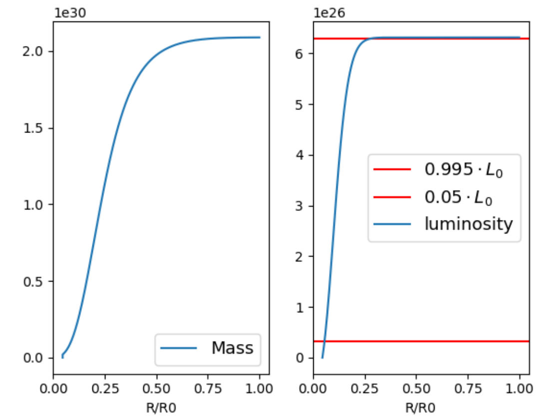



FIG. 11.— The plot of the mass and luminosity as functions of the radius. The mass is very similar to the one for the original initial conditions, but the luminosity changes somewhat and is constant for a longer period of time and then declines more rapidly than for the solar initial conditions. This corresponds with the diminished core seen in the cross section.

This summarizes the results from the change of variable, as should be noted for these values the convective zone is only 14% not the 15 desired, this is unfortunate, although as the old saying goes this is considered close enough for government work.

## 5. DISCUSSION AND CONCLUSIONS

To adress the final best fit parameters these values are fairly reasonable for the most part as they do not differ to much from the actual solar values, this strengthens the validity of the model, but as should be noted the density is a significant factor larger than the change in all other parameters. This again is slightly unrealistic as this density is approaching something resembling the density in the stellar interior. Thus this is not probable, but given as the sun is a complex big ball of gas this is not that much of an issue as this model is meant for interior energy transport. This massive discrepancy is then not the greatest problem in the world.

As should be noted this is a 1 dimensional model of the energy transport, as this is only treated as a continuous gradient, this is fairly unrealistic as an actual star is a 3 dimensional system that evolves over time, for example in some types of stars the opacity varies over time and will not be constant. The energy porduction of the core may not be entirely continuous as well thus leaving this as an interesting step in an effort to make a more realistic model of the stellar interior. The natural next step is to move towards a complete model is to move to a 2-dimensional model, although this model may prove to be a good basis to base a more complex model on. But still its linearity as a model remains its main weakness.

But the model is still a good enough depiction of how the star transports energy, as is predicted by the model the gas at the surface is transported there by convection and this is supported by observations as convection can clearly be observed by solar telescopes at the solar surface. Thus we must conclude that for this purpose it has something going for it, as should be noted about the core, there has to exist convection at the core otherwise the radiative flux would not be sufficient to transport the energy at the core to the outer parts of the star. The main difference between this and an actual cross section of the sun is that the outer convective zone in the sun is much larger than generated in 1. The real sun has the same values as in this project, but still has a larger convective zone. The main difference then is the size of this zone.

In conclusion the model is a decent representation of the energy transportation given its initial condition of being a linear model and thus omitting changes in time and the other 2 dimensions. Although it is a fair assumption that the star evolves similarly towards the core from all directions it still is insufficient as a model as several observed stellar phenomena depend on direction and time it would be insufficient for all other purposes than an approximation of transport form.

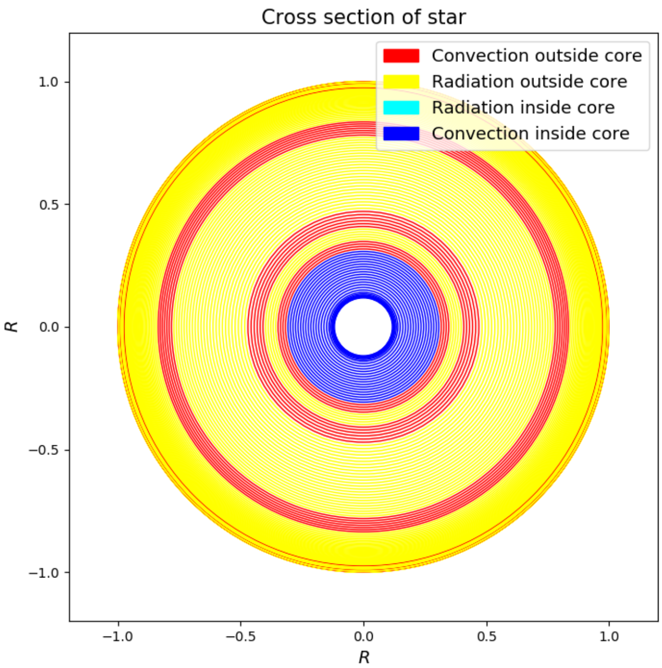
## 6. APPENDIX

### *LEARNING OUTCOME*

The main learning outcome for me from this project has been how the gradients govern the internal processes of the star. Specifically how they relate to each other. I also learned that i really should start doing object oriented programming more to simplify the programming bit. I also learned about the variable step size which is something i probably will use alot more in other courses as well. I have also gotten to refresh the coupled ODE solution methods.

### *ART*

During this project I encountered true art as shown below, I have included this as it should be enjoyed by more people:

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