Multiobjective Minimum Spanning Trees Using Dynamic Programming

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Output
The MO-MST problem is to find $\mathcal{T}^*(V)$.

Lemma

Consider a subset $V' \subseteq V$ containing k nodes. Its efficient spanning trees are given by

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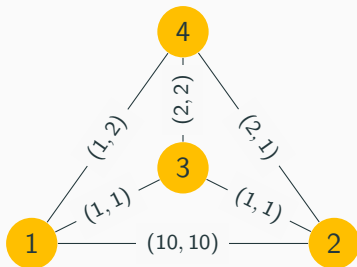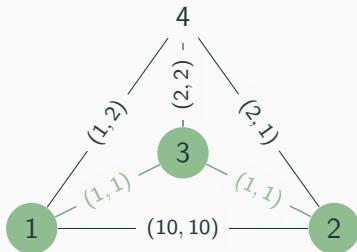*Consider a subset $V' \subseteq V$ containing k nodes. Its efficient spanning trees are given by*

$$\mathcal{T}^*(V') := \min_{\preceq_D} \left\{ t \circ \{u, v\} \mid t \in \mathcal{T}^*(V''), \ \forall V'' \subset V', \ |V''| = k - 1 \right.$$

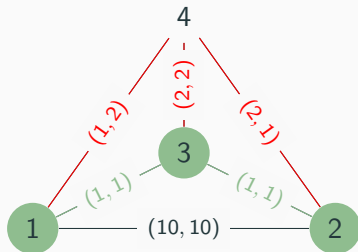$$\left. and \ u \in V'', \ v \in V' \backslash V'' \right\}.$$

Bellman Condition for MO-MST

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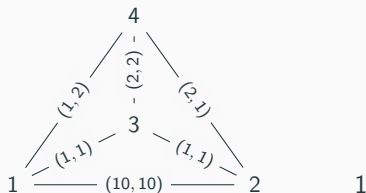
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MO-MST → MO Shortest Path

(a) Input (G, c).

(b) Search Graph D_G with costs $\bar{c}$ induced by c.

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The implicit search graph D_G of a MO-MST instance (G, c) is a directed, acyclic multigraph with $\mathcal{O}(2^{n-1})$ nodes, one for every subset $\{v_1\} \cup V'$, $V' \subseteq \{v_2, \ldots, v_n\}$.

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Theorem

Efficient $\{v_1\}$-V-paths in D_G uniquely correspond to efficient spanning trees of G.
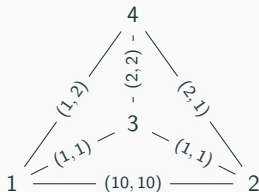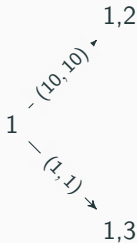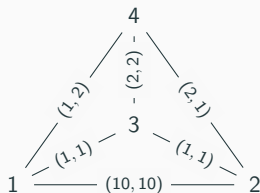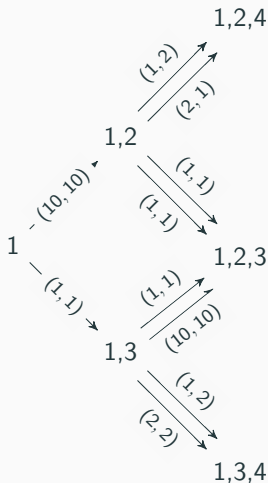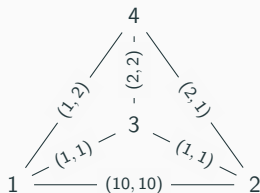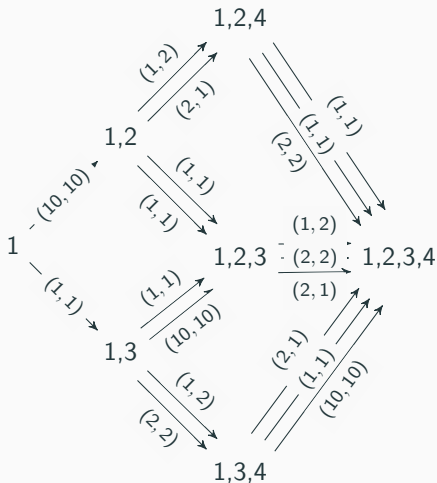
MO-MST instance (G, c) ⇔ MOSP instance $(D_G, \bar{c}, \{1\})$

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Proof.

Follows from the Bellman Condition for MO-MST. $\qquad\qquad\qquad\square$

MO-MST → MO Shortest Path

Tailored MOSP algorithm for MO-MST instances

- Preprocessing: eliminate *red* edges and contract *blue* edge components.

Designing an efficient MOSP algorithm for MO-MST

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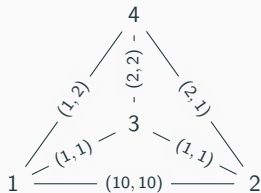
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Designing an efficient MOSP algorithm for MO-MST

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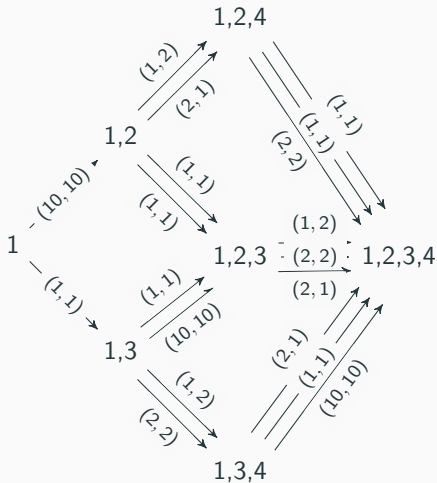- Dimensionality reduction (e.g. Pulido et al., 2014)

MO-MST → MO Shortest Path

Preprocessing Edges

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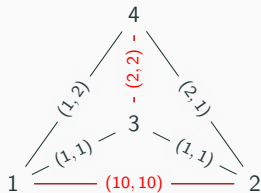
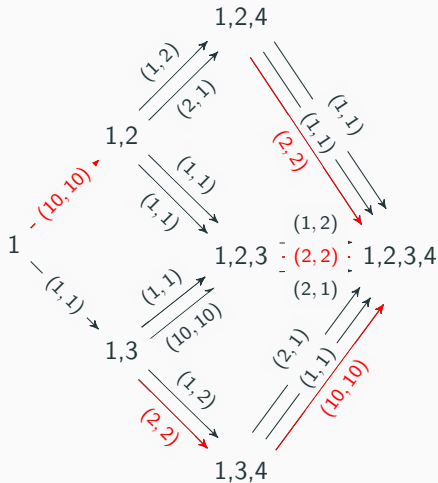(a) Input (G, c).

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Cut and Cycle Optimality Conditions

Red/Blue coloring inspired by...

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Preprocessing

Red and Blue edges can be determined running two DFS searches on G. Cycle and Cut Optimality conditions are checked.

MO-MST → MO Shortest Path

Remove dominated parallel arcs

Delete dominated parallel arcs – Chen et al., 2007

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Use lex. sorting of
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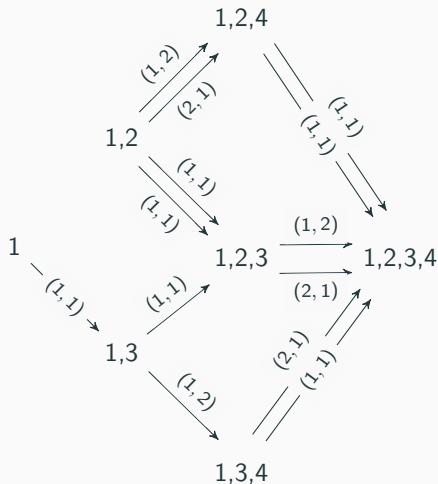
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What MOSP algorithm to choose?

Multiobjective Dijkstra Algorithm (MDA). M. et al., 2021

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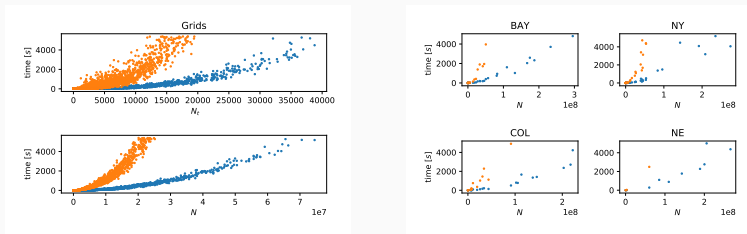

Figure 4: MDA vs. (Martins, 1984) algorithm on 3-d MOSP instances.

Experiments

Setup

Choice Justification

Fernandes et al., 2019. Empirical study of exact algorithms for the multi-objective spanning tree.

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	BN algo. original	Our BN algo.	New algo.
Coloring	No	Yes	Yes
Arc ordering	Head/Tail Ids	Head/Tail Ids	Lex.
Lower bound pruning	No	Yes	Yes
Dim. reduction	No	Yes	Yes
MOSP algo.	(Martins, 1984)	(Martins, 1984)	MDA

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Table 1: Original BN vs. Our BN on Santos et al., 2018 instances with 14 nodes.

Instance dimension	BN algo. original	Our BN algo.
2	1.12s	0.09s
3	86.33s	44.34s
4	3656.91s	1823.48s

Experiments

Biobjective $(d = 2)$ **MST**

Solutions on complete graphs with 2 anticorrelated objectives

n	TIME NEW [s]	TIME BN [s]	SPEEDUP	$\mathcal{T}^*(V)$ cardinality
10	0.05	0.05	1	143
12	0.57	0.58	1	218
15	19.97	20.14	1	359
17	163.06	163.47	1	453
20	677.97	712.44	1.05	374
22	1877.95	1978.14	1.05	289
25	4192.325	-	-	398

Two phase approaches for BO-MST problems

Graph	n	m'	blue	LS	Sort	Shaving	BnB	Total
Clique	50	156.5	1.4	0.62	0.05	0.05	0.75	1.54
	100	339.8	2.8	4.34	0.56	0.23	3.17	8.56
	150	534.3	3.0	12.71	1.89	0.40	3.87	19.68
	200	727.0	3.5	28.22	4.10	0.45	4.10	38.64
	250	905.3	4.8	52.26	7.06	0.46	2.28	65.74
	300	1075.9	5.2	87.94	9.47	0.43	7.61	105.82
	350	1181.8	8.7	137.92	13.27	0.58	4.87	154.01
	400	1650.4	7.4	208.94	-	-	1.56	225.68
	450	1839.3	10.1	308.38	-	-	0.44	331.68
	500	1980.9	13.6	438.75	-	-	1.88	473.34

Figure 5: Extremely fast Biobjective algorithm by Sourd and Spanjaard.

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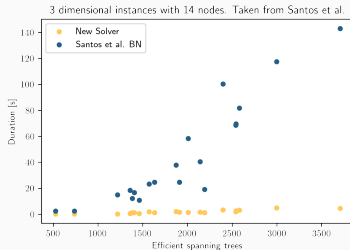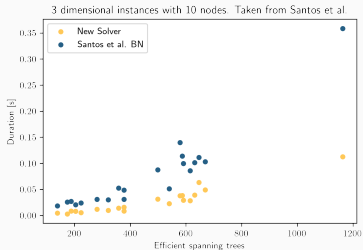- Sourd and Spanjaard, 2008
- Amorosi and Puerto, 2022

Experiments

Random graphs from Santos et al., 2018

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Table 2: 3-d instances with anticorrelated edge costs.

NODES	TIME NEW [s]	TIME BN [s]	SPEEDUP
10	0.03	0.08	3.44
11	0.06	0.22	3.56
12	0.18	1.63	7.45
13	0.49	7.69	14.35
14	1.76	44.34	25.33

Random graphs from Santos et al., 2018

Table 3: 4-d instances with anticorrelated edge costs.

NODES	TIME NEW [s]	TIME BN [s]	SPEEDUP
10	1.35	1.48	1.22
11	4.90	7.39	1.50
12	35.00	82.87	2.35
13	102.79	311.90	2.86
14	523.48	1823.48	3.96

Complete graphs
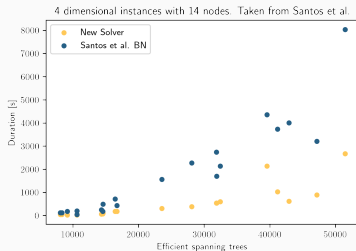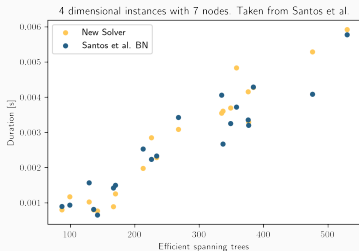
Table 4: 4-d instances with anticorrelated edge costs.

NODES	TIME NEW [s]	TIME BN [s]	SPEEDUP
7	0.01	0.02	2
10	8.36	150.75	18.03
12	272.45	4044.89	14.85

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