Lecture 13
Chapter # 5
MARKOV CHAINS

STOCHASTIC PROCESSES

Processes that evolve over time in a probabilistic manner.
These processes include the fluctuations & stock market & exchange rate. These processes are also used to represents the signals such as speech audio & video, are also used in the representation of medical data such as the patient's EEG, BP or temperature etc.

Mathematically, a stochastic process is defined to be an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T.
It will always index through a variable t which is usually a time variable.

- Often T is taken to be the set of non-negative integers, and X_t represents a measurable characteristic of interest at time t.
- e.g., X_t might represent the inventory level of a particular product at the end of week t.

Lets suppose i want to keep the track of the stock of medicines by the end of week 2.
X_2 might represent the availaibility level of stock of medicines by the end of week 2.

- Stochastic processes are of interest for describing the behavior of a system operating over some period of time.

These proccesses actually used to describe that how systems actually respond over certain period of time or it actually behave

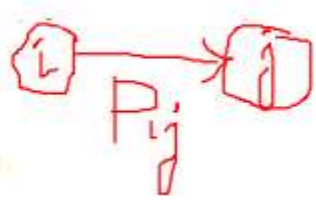
in a certain period of time.
so we will consider the states of the system; at any one point in time the system will be in a certain state & after some point in time it will go to another state. In this way you can actually describe the behaviour of the system.

- State: any one of M + 1 mutually exclusive categories or states possible. For notational convenience, states are labeled 0, 1, ..., M.

STRUCTURE OF STOCHASTIC PROCESSES

• Let $\{X_t , t = 0, 1, 2, . . . , \}$ be a stochastic process that takes on a finite or countable number of possible values.
• This set of possible values of the process will be denoted by the set of non negative integers $\{0, 1, 2, . . .\}$
• If $X_t = i$, then the process is said to be in state i at time t.
$X_4 = 3$; the stochastic process is at state 3 at time=4

• We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j.
let suppose the process is in state i (it can simply be represented by state transition diagram) & after certain point in time it will change its state & go to state j.
whenever it will do so then there will be a fixed transition probability which will be represented by P_{ij}



$P\{X_{t+1} = j|X_t = i, X_{t-1} = i_{t-1}, . . . ,X_1 = i_1 ,X_0 = i_0\} = P_{ij}$
the next state probability $(X_{t+1}=j)$

for all states $i_0, i_1, \ldots, i_{t-1}, i, j$ and all $t > 0$

STOCHASTIC PROCESSES - Example

- The weather in the town of Centerville can change rather quickly from day to day.
- However, the chances of being dry (no rain) tomorrow are somewhat larger if it is dry today than if it rains today.
- In particular, the probability of

the present state values are that it could either dry or it could either rain today.
we want to actually predict the tommorows value provided that the today weather conditions with probabilities.

  - being dry tomorrow is 0.8 if it is dry today,
  - but is only 0.6 if it rains today.

- These probabilities do not change if information about the weather before today is also taken into account.

- The evolution of the weather from day to day in Centerville is a stochastic process.
- Starting on some initial day (labeled as day 0), the weather is observed on each day t, for t = 0, 1, 2, ...

- The state of the system on day t can be either

State 0 = Day t is dry

or

State 1 = Day t has rain.

- Thus, for t = 0, 1,2, ..., the random variable X_t takes on the values,

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

MARKOV CHAINS

Markov chain: A stochastic process $\{X_t\}$ ($t = 0, 1, \ldots$) with Markovian property.
Markovian property says that the conditional probability of any future "event," given any past "event" and the present state $X_t = i$, is independent of the past event and depends only upon the present state.
i.e. the future only depends on the present value & not on the past value.
The conditional probabilities $P\{X_{t+1} = j|X_t = i\}$ for a Markov chain are called one-step transition probabilities.
this future state value is equal to j given that the present state is equal to i.

- If, for each i and j,

$P\{X_{t+1} = j|X_t = i\} = P\{X_1 = j|X_o = i\}$ for all $t = 0, 1, 2, \ldots,$
state of X_1 will only be j if the state of X_o is i

- then the (one-step) transition probabilities are said to be stationary. i.e by means of markov chain we can actually predict so many future values if i know about the present state. Lets suppose if i want to know about the inventory level of laptops at some point in time like i know its present value then if the transitional probabilities remains stationary or constant that i can predict its value after certain time period like after 1st week or 2nd week

- implies that the transition probabilities do not change over time.

• The existence of stationary (one-step) transition probabilities also implies that,
for each i, j, and n (n = 0, 1, 2, ...),

$$P\{X_{t+n} = j|X_t = i\} = P\{X_n = j|X_o = i\}$$
$$\text{for all } t = 0, 1, \ldots.$$

▪ These conditional probabilities are called ***n-step transition probabilities.***
▪ To simplify notation with stationary transition probabilities, let

- $p_{ij} = P\{X_{t+1} = j|X_t = i\}$; 1-step transition
- $p_{ij}^{(n)} = P\{X_{t+n} = j|X_t = i\}$; n-step transition

▪ Thus, the n-step transition probability $p_{ij}^{(n)}$ is just the conditional probability that the system will be in state j after exactly n steps (time units), given that it starts in
state i at any time t.
▪ When n = 1, note that $p_{ij}^{(1)} = p_{ij}$.

▪ Because the $p^{ij(n)}$ are conditional probabilities, they must be nonnegative, and since the process must make a transition into some state, they must satisfy the
properties:

$$p_{ij}^{(n)} \geq 0, \qquad \text{for all } i \text{ and } j;\ n = 0, 1, 2, \ldots,$$

and

$$\sum_{j=0}^{M} p_{ij}^{(n)} = 1 \qquad \text{for all } i;\ n = 0, 1, 2, \ldots.$$

& the sum of all the transitional probabilities starting from 0^{th} state till n^{th} state, the total sum should be 1.

- A convenient way of showing all the n-step transition probabilities is the n-step transition matrix.

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

Probability P_{01} actually represents the transitional probability of going from state 0 to till 1.
If I am actually standing at state 0 & not changing my state than probability is represented by P_{00}

Structure of Transition Matrix

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

Structure of Transition Matrix

• Note that the transition probability in a particular row and column is for the transition from the row state to the column state. i.e. these are the row states & these are the column states & whenever we are jumping from the row state to column state we can write our probability here.

▪ When n = 1, we drop the superscript n and simply refer to this as the transition matrix.

▪ The Markov chains to be considered have following properties:-

1. A finite number of states.
2. Stationary transition probabilities. only when the initial transitional probabilities are constant then we can predict the future.

▪ We also will assume that we know the initial probabilities $P\{X_o = i\}$ for all i.

<u>Example 1 (Forecasting the weather)</u>

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two-state Markov chain whose transition probabilities are given by:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{Vmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{Vmatrix} \end{array}$$

$P_{00} = \alpha$

$P_{01} = 1-\alpha$

P_{00} = it rains today & it'll raining tommorow
P_{01} = its rains today but it'll not raining tommorow
& the total probability of this particular row will be 1. i.e. we are doing it correct.
P_{10} = it does'nt rains today but it'll raining tommorow
P_{11} = its does'nt rains today & it'll not raining tommorow
& the total probability of this particular row will be 1. i.e. we are doing it correct.
i.e. by Constructing the transitional probability matrix.

<u>Example 2 (Gary's Mood)</u>

On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he
will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.
Letting X_n denote Gary's mood on the nth day,
then $\{X_n, n \geq 0\}$ is a three-state Markov chain (state 0 = C, state 1 = S, state 2 = G) with transition probability matrix:
provide transition probability matrix

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccc} 0 & \quad 1 & \quad 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \left\| \begin{array}{ccc} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{array} \right\| \end{array} \begin{array}{c} = 1 \\ = 1 \\ = 1 \end{array}$$

sum of all probability will be 1

make state transition diagram as an exercise?