Lec # 8

Commonly used Discrete Probability Distributions

5) Poisson Distribution

• This models occurrences of an event, These events are generally regarded as successes (or failures, depending upon the context) in a given time duration.

• The Poisson pmf is given by:

$$p_X(k) = \begin{cases} \dfrac{\alpha^k e^{-\alpha}}{k!}, & k = 0, 1, 2, \cdots \qquad \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

• The parameter α is related to time duration t as follows:

$$\alpha = \lambda t$$

where, λ is interpreted as the rate of occurring successes. or failures Its exact explanation depends on the context & application. For example it may be the arrival rate of customer in a queing system, it could be the failure rate of components in a reliability application or it may represent message transmission rate of communication channel.

• When n is large and p is small, the Poisson pmf (distribution) can also be used as a convenient approximation to the binomial pmf (distribution).

$$\binom{n}{k} p^k q^{n-k} \cong \frac{\alpha^k e^{-\alpha}}{k!}, \qquad \alpha = np$$

• As a rule of thumb, we use it when $n \geq 20$ and $p \leq 0.05$

Example

Queries to a database server arrive at a rate of 12/hour. Calculate the probability that:

a) Exactly six queries will arrive in next 30 min?

b) Three or more queries will arrive in next 15 min?

c) Two, three or four queries will arrive in next 5 minutes?

Solution:

First identify the probability distribution to be applied over here given: arrival rate of some queries to a server

so whenever we are tallking about arrival rate or λ & we have been given some time duration in our question then we can safely say that poisson distribution will be applied.

first calculating α,

$\alpha = \lambda t$, here $\lambda = 12$/hour

$\alpha = 12t$

we have to make units same. in part a the value of t is 30 min & we will work in unit of hour so 30 min = 0.5 hr

Part a)

t = 30 mins/60 = 0.5 hour

Therefore $\alpha = 6$. exactly 6 queries will arrive in 0.5 hrs

$$P[X=6] = \frac{e^{-\alpha}\alpha^k}{k!} = \frac{e^{-6}6^6}{6!} = 0.1606$$

Part b)

$\alpha = 12x1/4 = 3$ either 3 or more queries will arrive in next 15 min ; 1/4 of hr & value of α is 3.

$\alpha = 12t = 12(15min/60) = 3$ hr

$P[X\geq3] = 1 - P[X < 3]$

$$= 1 - \sum_{k=0}^{2} \frac{e^{-\alpha}\alpha^k}{k!}$$
$$= 1 - [e^{-\alpha} \{1 + \frac{3}{1!} + \frac{3^2}{2!} \}]$$
$$= 0.5768$$

Part c)
α = 12x5/60 = 1 hr
P[X=2] + P[X=3] + P[X=4] = 0.2606

Commonly used Discrete Probability Distributions (Cont'd)

6) Hypergeometric Distribution
• Let X be a random variable with Hypergeometric pmf (distribution) giving number of defectives in a random sample drawn without replacement from a lot having certain number of defective components.
recall the difference between sampling with replacement & sampling without replacement.
Lets take an example to understand it;
suppose we have a ball of 100 unique numbers from 0 to 99 & we want to select a random sample of numbers from the ball. So after we pick a number we can put the number aside or we can put it back aswell. So if we put a number back it may be selected one more time & if we putted aside it means it can only be selected only one time.
So when a population element can be selected more than one time it is called that we are sampling with replacement.
& when a population element can be selected only one time then we are sampling without replacement.
• The probability of k defectives in a random sample of m components drawn without replacement from a lot of n

components having d defectives can be calculated as:

$$h(k;m,n,d) = \frac{\binom{d}{k}\binom{n-d}{m-k}}{\binom{n}{m}}, \qquad k = 0, 1, 2, \cdots, min\{d, m\}$$

7) Uniform Distribution

• Consider a random variable X that can acquire n different values $\{x_1, x_2, ..., x_n\}$.

• The variable X is said to be uniformly distributed if:
if the probability of every element is going to be 1/n

$$p_X(x_i) = \begin{cases} \frac{1}{n} & i = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

• plays an important role in the theory of random numbers and its applications to discrete event simulation.
• In the average-case analysis of programs, it is often assumed that the input data are uniformly distributed over the input space.
The most common example; lets suppose a deck of cards which has a uniform distribution it's because an individual has a unequal chance of drawing a spade, a heart, a clu or a diamond. Another example of uniform distribution is when an unbiased coin is tossed, the likelihood of getting a tial or head is the same. A very good example of discrete uniform distribution would be the possible outcome of rolling a 6 sided die; the possible values would be either 1 2 3 4 5 6 & in this case each of 6 numbers will have equal chance of appearing therefore in this case the probability is going to be 1/6 when the die is thrown.

8) Constant Distribution

• A random variable X is said to have constant distribution if:

if its holds the given formula that is the probabilty is going to be 1 if the value of x is real number c & its going to be 0 otherwise.

$$p_X(x) = \begin{cases} 1, & x = c \\ 0, & \text{otherwise} \end{cases}$$

• This means that every sample point maps to just one real number c.

9) Indicator Distribution

• Suppose that an event A partitions the sample space S into two mutually exclusive subsets A and A'. The indicator of event A is a random variable I_A defined by:

$$I_A(s) = \begin{cases} 1, & \text{if } s \in A \\ 0, & \text{if } s \in \bar{A} \end{cases}$$

The indicator function of an event is a random variable that takes value 1 when the event happens & value 0 when the vent does'nt happen.

• Then event A occurs if and only if $I_A = 1$.

The pmf of I_A is given by:

$$p_{I_A}(0) = P(\bar{A}) = 1 - P(A)$$

$$p_{I_A}(1) = P(A)$$

The indicator functions are often used in the probability theory to simplify notation & to prove theorem.

One example to understand indicator distribution could be that we are rolling a die & the 6 possibilities are being divided into 2 mutually exclusive events the first event is that the odd number will face up when we roll the die & the 2^{nd} event is that the even number will face up when we roll the die.

10) Multinomial Distribution

• Multinomial pmf is the generalized binomial pmf.
• There are more than two possible outcomes on each trial.

For example; the switch case constuct in a high level language actually represents the multinomial distribution.

Another example could be a process receiving service of one of the many I/O devices after lets suppose availing CPU time slice in a time sharing environment.

So in this particular example we can consider that we have caught multiple queues & initially the process was in the ready queue waiting for the processor & once it gets the chance of execution it first executes some compute intensive tasks & then let suppose then it request some I/O device it will get block by OS for making the I/O request then it will then move to some another queue of blocked processes here it will then wait for some time for the availaibility of I/O here there can be multiple queues for I/O as well & lets suppose some of the process are waiting for getting a page from the virtual memory, in other case some may want to read some data from disk or lets suppose in another queue some processes are waiting for input from another I/O device or I/O channel.

So again we can have multiple choices or multiple possibilities when we are talking about multiple ready queue even it actually represents the multinomial distribution.

• Let us define a random vector $X = (X_1, X_2, \ldots , X_r)$ such that Xi gives the number of trials resulting in the i^{th} outcome. Let pi be the probability of i^{th} outcome. The joint pmf of X is given by:

$$p_X(n_1, n_2, \ldots, n_r) = P[X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r]$$
$$= \frac{n!}{n_1! n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$$

<u>Task</u>

Q.1) Data packets transmitted by a modem over a phone line form a Poisson process at the rate of 10 packets/sec.

- What is the probability that exactly four packets will be transmitted per second?
- What is the probability that more than three packets will be transmitted per second?
- What is the probability that at least four packets will be transmitted in 2 seconds?

First identify the probability distribution to be used then apply the appropriate formula & solve it

Q.2) The probability of error in the transmission of a bit over a communication channel is 10-3. What is the probability of more than three errors in transmitting a block of 1000 bits?
solve it !

<u>Answers</u>

Q1)
- 0.0189
- 0.9896
- 0.9999

Q2)
- 0.01898

Exponential Distribution (Continuous Distribution)

It is one of the widely used continous distribution &

- It is used to model the time elapsed between events.

In most queueing situation the arrival of customer occur in a totally random fasion. so,

- If the number of arrivals at a service facility during a specified period follows Poisson distribution, then,
    - automatically, the distribution of the time interval between successive arrivals must follow the negative exponential (or, simply, exponential) distribution.

here i would like to highlight the various application of this distribution so it is used to model time in various fields like the interarrival of customers entering a system. It is also used to model the lifetime of hardware components & waiting in service time in a queueing system.

- Specifically, if λ is the rate at which Poisson events occur, then the distribution of time between successive arrivals, t, $f(t) = \lambda e^{-\lambda t}$, $t > 0$

- The mean and variance of the exponential distribution are:

$E\{t\} = \frac{1}{\lambda}$	$var\{t\} = \frac{1}{\lambda}$

- The mean E{t} is consistent with the definition of λ.
- If λ is the rate at which events occur, then 1/λ is the average time interval between successive events.

The fact that the exponential distribution is completely random can be illustrated by one example; Lets suppose that the time right now is 8:20am & the last arrival has occured at 8:05am, So

that the probability that the next arrival will occur by 8:30am is the function of the interval between 8:20 to 8:30 only & is totally independant of the length of time that has elapsed since the occurrence of last event & that was happened at 8:05am.
So this result is often reffered to as the forgetfulness or lack of memory of the exponential distribution.
Remember; that exponential distribution is the only continous distribution that exhibits this forgetfulness property.

Rare Events

• When two events are extremely unlikely to occur simultaneously or within a very short period of time, they are called rare or Poissonian events, as they are modeled using Poisson distribution.
Examples
• Job arrivals to a system, telephone calls, e-mail messages, network breakdowns, virus attacks, software errors are examples of rare events.

Inter-arrival Times of Rare Events are Exponential

• Let N be a Poisson random variable denoting number of customers arriving to a system in the interval (0, t]. Hence,
the interval is taking such that the time starts with 0 & it can go beyond this particular time t as well.

$$P_N(k) = \frac{\alpha^k e^{-\alpha}}{k!}$$

• Let T be a random variable denoting interarrival time of customers.

Then,

$$P[T > t] = P[N = 0] = \frac{\alpha^0 e^{-\alpha}}{0!} = e^{-\alpha}$$

• where the parameter $\alpha = \lambda t$

$$P[T > t] = e^{-\lambda t}$$

• which shows that interarrival times are exponentially distributed when arrivals occur according to Poisson distribution.

Memoryless Property

Exponential distribution exhibits a unique property known as the Memoryless property.

• Let T be a random variable denoting service time of a server.

• Suppose that a job currently with the server has already consume service time t.

• We are interested in the probability that the job will stay with the server for additional time s; i.e. we wish to calculate the conditional probability $P[T > t + s \mid T > t]$

that is the time it has already spent with the server plus the additional time given that the random variable T is greater than the time that has already been spent in the server. So here we will use our knowledge of conditional probability.

$$P[T > t + s \mid T > t] = \frac{P[T > t + s]}{P[T > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = \underline{e^{-\lambda s}}$$

If we closely follow than we can easily & safely say that this

result is independant of t which means that the probabilty of spending an additional service time does'nt depend on the time that has already been spent on the server.
So this relation is independant of t & whatever additional time the jobs wants to spent with the server does'nt depend opon the time that is already been spent with the server.
So the past has no bearing on future & hence the name is called Memoryless property.
The Memoryless property for exponentially distributed lifetime would mean that the probability of a component serviving for some additional time would be independant of how long it has been operating. This means that the component shows no sign of aging*.