CS-417
COMPUTER SYSTEMS MODELING

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(LECTURE # 20)

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Recap of Lecture # 19

Kendall’s Notation

Little’s Law and Utilization Law

Operational and Stochastic Analysis

Chapter # 6 (Cont’d)

FUNDAMENTALS OF QUEUING MODELS

Transient State & Steady State

- When a queuing system has recently begun operation,
  - the state of the system (number of customers in the system) will be greatly affected by the initial state and by the time that has since elapsed.
- The system is said to be in a **transient condition.**
- However, after sufficient time has elapsed, the state of the system becomes essentially independent of the initial state and the elapsed time (except under unusual circumstances).
- The system has now essentially reached a **steady-state condition,** where the probability distribution of the state of the system remains the same (the *steady-state* or *stationary* distribution) over time.
- Queuing theory has tended to focus largely on the steady-state condition.

Steady State Notations

The following notation assumes that the system is in a *steady-state condition:*

P_n = probability of exactly n customers in queueing system.

L = expected number of customers in queueing system $= \sum_{n=0}^{\infty} nP_n$.

L_q = expected queue length (excludes customers being served) $= \sum_{n=s} (n - s)P_n$.

$\mathcal{W}$ = waiting time in system (includes service time) for each individual customer.

$W = E(\mathcal{W})$.

$\mathcal{W}_q$ = waiting time in queue (excludes service time) for each individual customer.

$W_q = E(\mathcal{W}_q)$.

M / M / 1 QUEUE ANALYSIS

THE BIRTH-AND-DEATH PROCESS

- Most elementary queuing models assume that the inputs (arriving customers) and outputs (leaving customers) of the queuing system occur according to the *birth-and-death process.*

- The *state* of the system at time t ($t \geq 0$), denoted by $N(t)$,
  - number of customers in the queuing system at time t.

- The birth-and-death process describes *probabilistically* how $N(t)$ changes as t increases.

- Broadly speaking, it says that *individual* births and deaths occur *randomly,* where their mean occurrence rates depend only upon the current state of the system.

THE BIRTH-AND-DEATH PROCESS (Cont'd)

- The next transition in the state of the process is either
  - $n \rightarrow n + 1$ (a single birth)

    or
  - $n \rightarrow n - 1$ (a single death),
  - depending on whether the former or latter random variable is smaller.

- Because of these assumptions, the birth-and-death process is a special type of *continuous time Markov chain.*

- Queuing models that can be represented by a continuous time Markov chain are far more tractable analytically than any other.

THE BIRTH-AND-DEATH PROCESS (Cont'd)

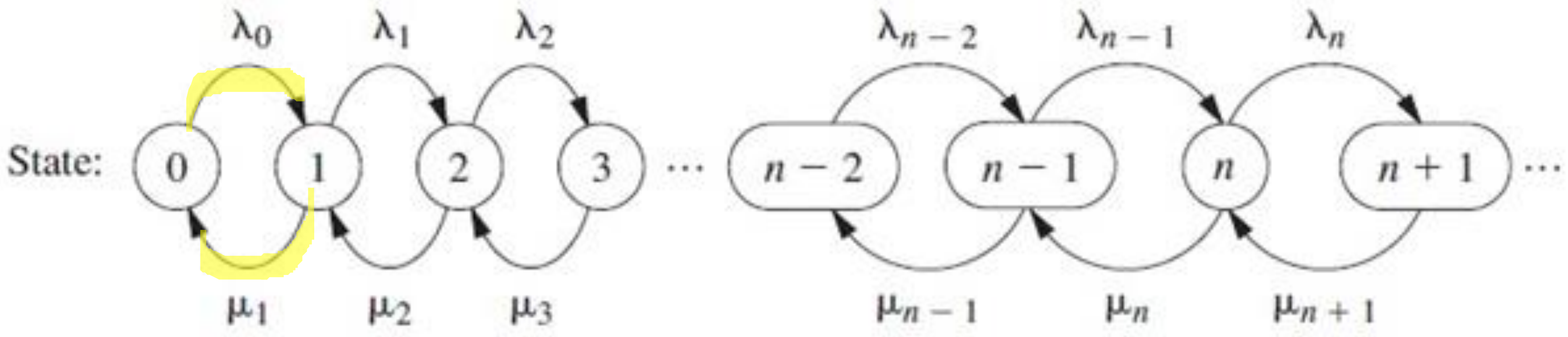

Fig 1: Rate diagram for the birth-and-death process

- The arrows in this diagram show the only possible *transitions* in the state of the system.
- The entry for each arrow gives the mean rate for that transition.
- These results yield the following key principle:

Rate In = Rate Out Principle

THE BIRTH-AND-DEATH PROCESS (Cont'd)

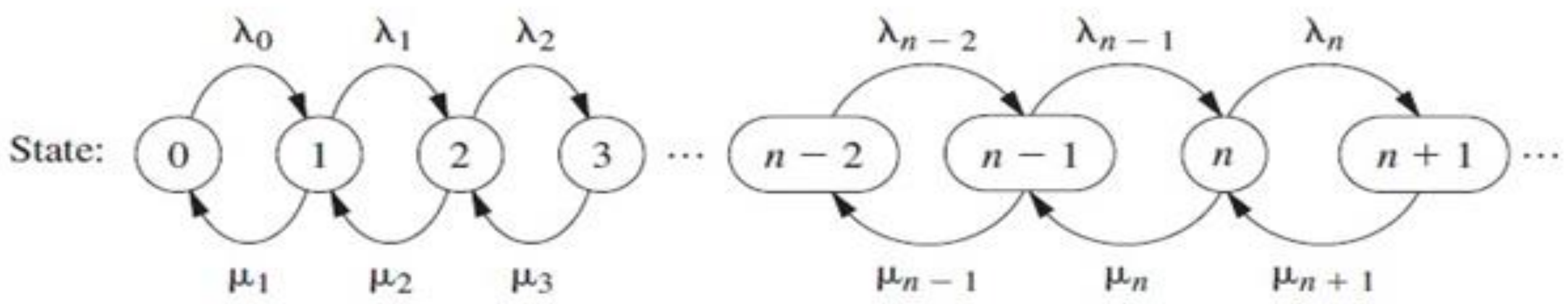

Fig 1: Rate diagram for the birth-and-death process

- For any state of the system n ($n = 0, 1, 2, \ldots$),
  - mean entering rate = mean leaving rate.
- The equation expressing this principle is called the **balance equation** for state n.
- After constructing the balance equations for all the states in terms of the *unknown* P_n probabilities,
  - we can solve this system of equations (plus an equation stating that the probabilities must sum to 1) to find these probabilities.

THE BIRTH-AND-DEATH PROCESS (Cont'd)

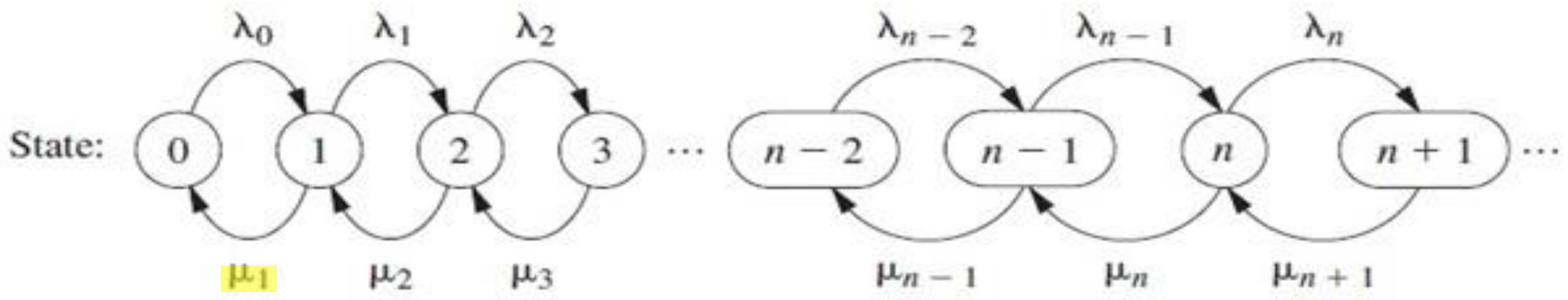

Fig 1: Rate diagram for the birth-and-death process

- To illustrate a balance equation, consider state 0. The process enters this state *only* from state 1.
- Given that the process is in state 1, the mean rate of entering state 0 is μ_1.
- From any *other* state, this mean rate is 0.
- Therefore, the overall mean rate at which the process leaves its current state to enter state 0 (the *mean entering rate*) is

$$\mu_1 P_1 + 0(1 - P_1) = \mu_1 P_1$$

THE BIRTH-AND-DEATH PROCESS (Cont'd)

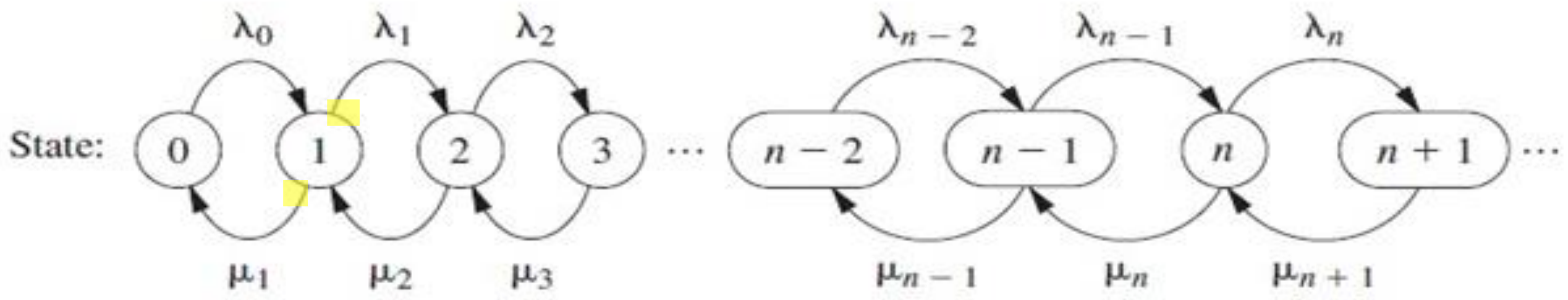

Fig 1: Rate diagram for the birth-and-death process

- By the same reasoning, the *mean leaving rate* must be $\lambda_0 P_0$, so the balance equation for state 0 is

$$\mu_1 P_1 = \lambda_0 P_0$$

- For every other state there are two possible transitions both into and out of the state.
- Therefore, each side of the balance equations for these states represents the *sum* of the mean rates for the two transitions involved.
- Otherwise, the reasoning is just the same as for state 0.

Writing Flow Equations

$\lambda_0 = \lambda_1 = \lambda_2 = \ldots\ldots = \lambda_n = \lambda$

$\mu_0 = \mu_1 = \ldots = \mu_n = \mu$

State 0:

$$\lambda P_0 = \mu P_1$$

$$\Rightarrow \boldsymbol{P_1 = \rho P_0}$$

State 1:

$$\lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1$$

$$\Rightarrow \mu P_2 = (\lambda + \mu)\frac{\lambda}{\mu} P_0 - \lambda P_0$$

$$= [\frac{\lambda}{\mu} + 1 - 1]\, \lambda P_0$$

$$\therefore \boldsymbol{P_2 = \rho^2 P_0}$$

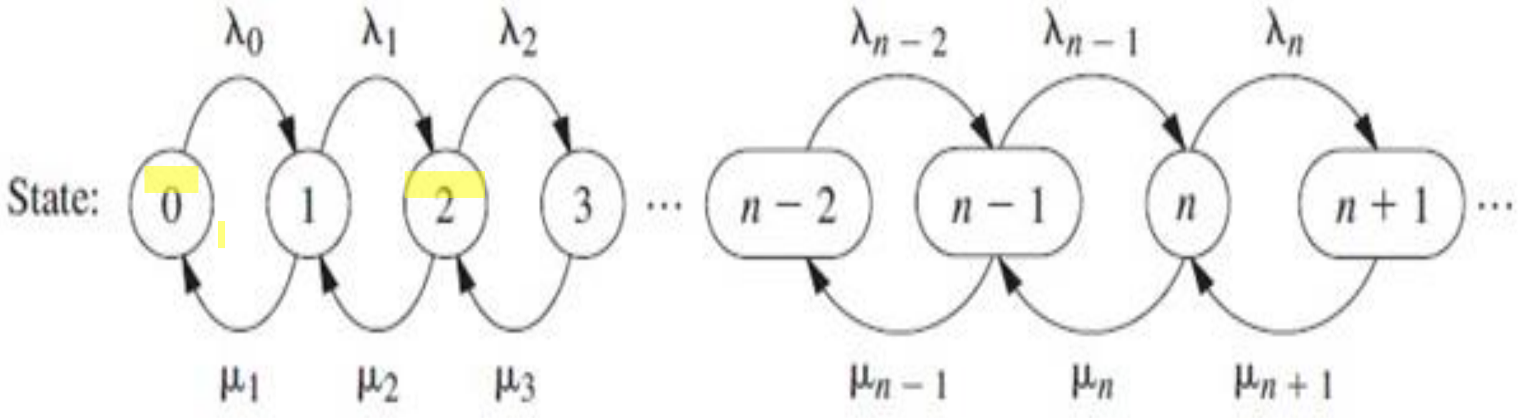

Fig 1: Rate diagram for the birth-and-death process

Writing Flow Equations

So, from these two equations, we can generalize the relation:

$$P_n \quad = \rho^n P_0$$

$$\because \quad \sum_{n=0}^{\infty} P_n = 1$$

$$\Rightarrow P_0 + \rho P_0 + \rho^2 P_0 + \ldots\ldots + \infty = 1$$

$$\Rightarrow P_0 \left(\frac{1}{1-\rho}\right) = 1$$

$$\therefore \quad P_0 \quad = 1 - \rho$$

Q1: Explain what does $\rho = 1 - P_0$ indicate ?

ρ is the probability that server is busy and $1 - P_0$ suggests the probability of at least one customer.

Generalizing, we will get:

$$\therefore \; \boldsymbol{P_n} = \boldsymbol{\rho^n(1 - \rho)}$$