Lecture 22
Chapter # 6 (contd.)
FUNDAMENTALS OF QUEUING MODELS

M/M/1/K System (Finite Buffer Capacity)

The **M/M/1/k** queue is a *short hand* notation for the **M/M/1/k/∞/FIFO** queue.

• M = Arrival process is Poisson
• M = Service (departure) process is Exponential
• 1 = There is 1 server in system
• k = Queue capacity; the $(k+1)^{th}$ arriving client will be *rejected*
• ∞ = *Infinite population size* (the *arrival process* will be *unaffected* by the number of clients already in the system)
• FIFO = First In First Out Service

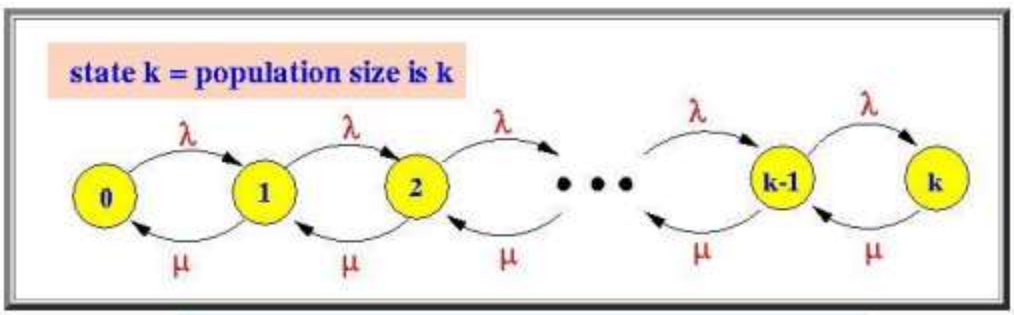

Fig 1: Rate diagram for the M/M/1/K System

Same as M/M/1 birth & death process but the only modification of the M/M/1 queue is that now the total number of customers is limited to k. & if a customer arrives when the queue is full, it is dropped.

So here the equilibrium equations will remain same because the number of servers is 1

We will reuse that particular equation;

Starting with: $P_n = \rho^n P_0$; n =0, 1, 2, ..., k

$$\sum_{n=0}^{k} P_n = 1$$

The only difference is that there will be k instead of infinity.

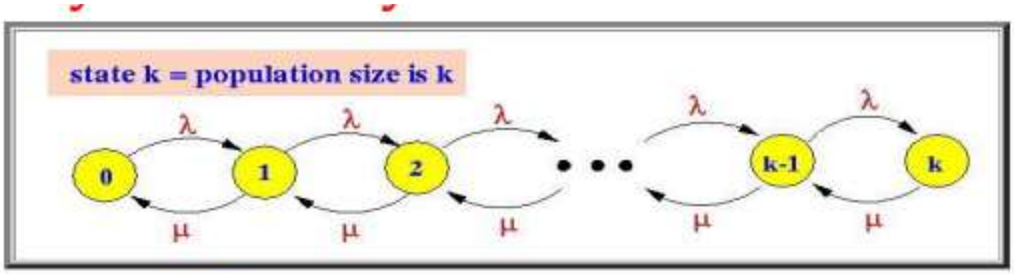

Fig 1: Rate diagram for the M/M/1/K System

Starting with: $P_n = \rho^n P_0$; n = 0 , 1, 2, ……., k

$$P_0 + \rho P_0 + \rho^2 P_0 + \ldots\ldots + \rho^k P_0 = 1$$

$$\Rightarrow \frac{1-\rho^{k+1}}{1-\rho} P_0 = 1$$

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

$$\Rightarrow P_0 = \frac{1-\rho}{1-\rho^{k+1}}$$

$$\therefore P_n = \rho^n \frac{1-\rho}{1-\rho^{k+1}}$$

We have replaced the formula of P_n with the above summation formula where n is 0 till k.
Taking P_0 as common so it becomes infinite geometric series.
The first term or value of a =1.
Ratio is given by ρ
The value of n is k+1 because we are starting our index from 0 till k so total number of elements will be k+1.

Because of the limited system capacity, we do not require λ < μ.

If λ = μ (i. e. ρ = 1):

We do not require the value of arrival rate < service rate
Rather the both could be equal i.e. ρ = 1
In this particular case the values of P_n & ρ will be changed;

$$\sum_{n=0}^{k} P_n = 1$$

$$\sum_{n=0}^{k} \rho^n P_0 = 1$$

$$\Rightarrow (1+1+1+ \ldots\ldots\ldots + 1)\, P_0 = 1$$

$$\Rightarrow P_0 = \frac{1}{1+k} \Rightarrow P_n = \frac{1}{1+k}$$

ρ is replaced by 1.
This particular sequence is a finite arithmetic sequence.
The formula of sum of finite arithmetic sequence is;

$$S = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$= \frac{k+1}{2}\left(2 + \cancel{(k)0}\right)$$

The d=0

M/M/1/K Analysis (Task)

Q) Derive the equation to determine the total number of customers in the system.

Hint: Start with $L = E[N] = \sum_{n=0}^{k} n\, P_n$

Refer to lecture of M/M/1 queueing system & derive it

Result for verification:

$$L = \frac{\rho}{1-\rho} - (k+1)\frac{\rho^{k+1}}{1-\rho^{k+1}}$$

M/M/1/K Analysis (Task)

Total Number of Customers when ρ = 1,

$$L = \sum_{n=0}^{k} n\left(\frac{1}{1+k}\right)$$

$$L = \frac{1}{1+k} \sum_{n=0}^{k} n$$

$$L = \frac{1}{1+k} \cdot \frac{k}{2}(k+1)$$

$$L = \frac{k}{2}$$

Starting with same formula but we replaced the value of P_n with this 1/(1+k) formula & the value of ρ=1.
Take out 1/(1+k) and opening summation i.e. n=0 till k.
Applying formula of sum of finite arithmetic sequence.

Average Number of customers in the Service & in the Queue:

Probability of number of customers in the service given that total number of customers is 0. If there are no customer in the queueing system than there will be no customer in the service facility.

$$L_s = E[N_s] = P[N_s \mid \overset{0}{N = 0}]\ \overset{P_0}{P[N = 0]} + \overset{1}{P[N_s \mid N > 0]}\ \overset{1-P_0}{P[N > 0]}$$

$$= 0 \times P_0 + 1 \times (1 - P_0)$$

$$= L_s = 1 - P_0$$

number of customers in the queue;

$L_q = L - L_s$

$\quad = L - (1 - P_0)$

Effective Arrival Rate (λ_a)

Consider one example; imagine a small store with a single counter & area for browsing & here can be only one person at the counter at a time & no one leaves without buying something. The system is roughly actually performing the Entrance, Browsing, the Checkout & the Exit.
If we talk about the stable system, the rate at which people enter the store is actually the rate at which they arrive at the store. This is also called the arrival rate & the rate at which they exit it will be call the exit rate. So, if the arrival rate exceeds the exit rate then it will be unstable system.
But the latest law tells that the average number of customers in the stores (L) is given by the effective arrival rate times the average time the customers spend in the store i.e. $L = \lambda W$
We can actually apply Little's Law to the system;
As the store has limited size it will no become unstable it will start to overflow & any new arriving customer will simply be rejected. This is the difference between arrival rate & the effective arrival rate.
The arrival rate roughly corresponds to rate at which the customer **arrives** at the store but the effective arrival rate corresponds to the rate at which customers actually **enters** the stores.
The system with the infinite size & no laws leads to are equal*. It means that for the M/M/1 system the arrival rate & the effective arrival rate were equal. But for the finite buffer capacity & the finite space system these two are different.
Here we will use effective arrival rate.

Customers are turned away when there are k customers in the system:

$$\lambda_a = \lambda\,(1 - \mathbf{P_k})$$

Probability of acceptance
Probability of k customers in the system

Using Little's Law,

$$W = \frac{L}{\lambda_a} = \frac{L}{\lambda\,(1 - Pk)}$$

Waiting time in the queue;

$$W_q = \frac{L_q}{\lambda\,(1 - Pk)}$$

Server Utilization (U):

For balancing M/M/1/K system: Effective arrival rate = effective departure rate

$$\lambda (1 - P_k) = \mu(1 - P_0)$$

Probability that the server is busy is given by:

$$\Rightarrow \mathbf{U} = 1 - P_0 = \frac{\lambda}{\mu}(1 - P_k)$$

$$\Rightarrow U = \rho(1 - P_k)$$

Probability distribution of arrival rate

$\lambda = 6$ /sec

Example Problem 1

Packets arrive at a router according to Poisson distribution at an average rate of 6 per second. The router is serving packets at an average rate of 8 packets per second with exponential distribution of service time. However, the buffer has capacity for only 8 packets. Calculate: Probability distribution of service rate

K=8

router is the server we are talking about only 1 server i.e. the case of M/M/1/k system

a) average response time of packets

b) average number of packets dropped if a total of 5000 packets approach for the service

c) average number of packets in the router

d) average number of packets in the buffer

Solve!

Answers:

a) 0.4081 sec

b) 100 packets

c) 2.4

d) 1.665

Example Problem 2

Consider the following single-server queue: the inter-arrival time is exponentially distributed with a mean of 10 minutes and the service time is also exponentially distributed with a mean of 8 minutes, find out:

listen to solve

(i) mean wait in the queue,

(ii) mean number in the queue,

(iii) the mean wait in the system,

(iv) mean number in the system and

(v) proportion of time the server is idle.

Answers:

i) 3.2
ii) 32 mins
iii) 40 mins
iv) 4
v) 0.2

Task Assignment (To be submitted)

See last slides of lec 22 ne solution of Cep