CSM CEP.

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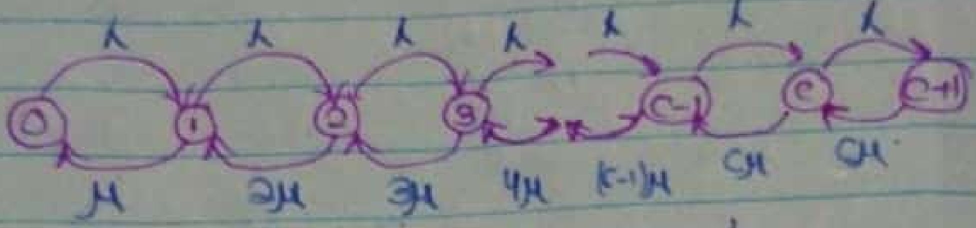
Analysis of M/M/C Queing System

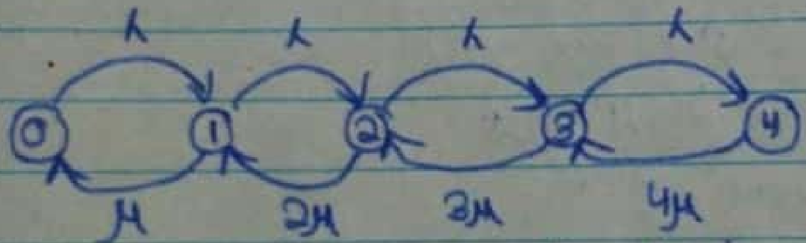


since we know utilisation formula i.e.

$$\rho = \frac{\lambda}{\mu c}$$

If $\rho < 1$ so system is stable only

Since the numbers of servers ie $c=3$ so diagram should be



→ here we know $c=3$ so after 'c' μ (service rate) will be $\mu c = 3\mu$

General Formula

For P_n:-

∴ Rate In = Rate out

→ we will check on state 0, 1 and 2, 3.

At State 0:—

$$P_1 \mu = P_0 \lambda$$

$$P_1 = \frac{P_0 \lambda}{\mu}.$$

but we know that

$$\rho = \frac{\lambda}{\mu c} \Rightarrow \rho c = \frac{\lambda}{\mu}.$$

$$P_1 = P_0 3\rho$$

$$\boxed{P_1 = 3\rho P_0}$$

At State 1:—

Rate In = Rate out

$$P_0 \lambda + P_2 (2\mu) = P_1 \lambda + P_1 \mu$$

$$2\mu P_2 = P_1 \lambda + P_1 \mu - P_0 \mu$$

putting value of P_1

$$2\mu P_2 = \lambda P_0 \cdot 3\rho + \mu \cdot 3\rho P_0 - P_0 \lambda$$

$$2\mu P_2 = P_0 [3\lambda \rho + \mu 3\rho - \lambda]$$

but we know

$$\rho = \frac{\lambda}{3\mu}$$

so putting values.

$$2\mu P_2 = P_0 \left[3\lambda \cdot \frac{\lambda}{3\mu} + 3\mu \cdot \frac{\lambda}{\mu} - \lambda \right]$$

$$2\mu P_2 = \frac{P_0 \lambda^2}{\mu}$$

$$P_2 = \frac{P_0 \lambda^2}{2\mu^2}$$

$$P_2 = \frac{P_0}{2}\left(\frac{\lambda}{\mu}\right)^2$$

$$\boxed{P_2 = \frac{P_0}{2} c^2 \rho^2}$$

At State 2:—

Rate In = Rate out

$$P_3(3\mu) + P_1\lambda = P_2\lambda + P_2(2\mu)$$

$$P_3 \cdot 3\mu = P_2\lambda + 2\mu P_2 - P_1\lambda$$

∴ since we know value of P_1 and P_2

$$3\mu P_3 = \lambda \frac{P_0}{2}\left(\frac{3^2\lambda^2}{3^2\mu^2}\right) + P_0 3^2\left(\frac{\lambda^2}{3\mu^2}\mu\right) - P_0 \frac{3\lambda \cdot \lambda}{3\mu}$$

$$P_3(3\mu) = \frac{P_0\lambda^3}{4\mu^2} + \frac{P_0\lambda^2}{\mu} - \frac{P_0\lambda^2}{\mu}$$

$$P_3 = \frac{P_0}{2\times 3} \cdot \frac{\lambda^3}{\mu^3}$$

$$P_3 = \frac{P_0}{3\times 2} \cdot \frac{\lambda^3}{\mu^3}$$

$$\boxed{P_3 = \frac{P_0}{6} c^3 \rho^3}$$

At state 3:—

Rate in = Rate out

$P_4(3\mu) + P_2\lambda = P_3\lambda + P_3(3\mu)$.

$3\mu P_4 = P_3\lambda + P_3 3\mu - P_2\lambda$ —— (i)

Now put value of P_3 and P_2 in eq (i)

$$\Rightarrow 3\mu P_4 = \frac{P_0}{6}\left(\frac{3^3\lambda^3}{3^3\mu^3}\right)\lambda + \frac{P_0}{6}\left(\frac{3^3\lambda^3}{3^3\mu^3}\right)3\mu -$$

$$\frac{P_0}{2}\left(\frac{3^2\lambda^3}{3^2\mu^2}\right)$$

$$\Rightarrow 3\mu P_4 = \frac{P_0}{6}\frac{\lambda^4}{\mu^3} + \frac{P_0\lambda^3}{2\mu^2} - \frac{P_0\lambda^3}{2\mu^2}$$

$$P_4 = \frac{P_0}{3\times 6}\frac{\lambda^4}{\mu^4}$$

$$P_4 = \frac{P_0}{3\times 6}(CP)^4$$

$$\boxed{P_4 = \frac{P_0}{3\times 6}(CP)^4}$$

At State 4

Rate in = Rate out

$P_0 3\mu + P_3\lambda = P_4\lambda + P_4 3\mu$

putting values of P_3 and P_4

$$3\mu P_0 + \frac{P_0}{6}(3\rho)^3\lambda \; \frac{P_0}{3\times6}(3\rho)^4\lambda + \frac{P_0}{3\times6}(3\rho)^4(3\mu)$$

$$\Rightarrow P_5(3\mu) = \frac{P_0}{3\times6}\left(\frac{3^4\lambda^4}{3^4\mu^4}\right)\lambda + \frac{P_0}{3\times6}\left(\frac{3^4\lambda^4}{3^4\mu^4}\right)3\mu - \frac{P_0}{6}\left(\frac{3^3\lambda^3}{3^3\mu^3}\right)\lambda$$

$$\Rightarrow P_5(3\mu) = \frac{P_0\lambda^5}{3\times6\mu^4} + \frac{P_0\lambda^4}{6\mu^3} - \frac{P_0\lambda^4}{6\mu^3}$$

$$P_5 = \frac{P_0\lambda^5}{3\times6\times3\mu^5}$$

$$P_5 = \frac{P_0}{3^2\times6}(C\rho)^5$$

$$P_5 = \frac{P_0}{3^2\times3\times2}(3\rho)^5.$$

Now finding P_n.

for $0 \le n \le C$ there in our $C = 3$

Which can be written as

$$P_n = \frac{P_0}{cn!}(c\rho)^n$$

so general formula would be;

$$P_n = \frac{P_0}{c^{n-c}\,c!}(c^n\rho^n)$$

$$P_n = \frac{P_0\, c^{-n+c} \cdot c^n \rho^n}{c!}$$

$$\boxed{P_n = \frac{P_0\, \rho^n c^c}{c!}}$$

(7)

• Average Number of Customer In System L_s:-

Since we know that

$$L = \lambda W$$

$$W = W_q + \frac{1}{\mu}$$

$$L = \lambda\left[W_q + \frac{1}{\mu}\right]$$

$$L = \lambda . W_q + \lambda/\mu$$

$$\lambda W_q = L_q$$

$$\therefore L = L_q + c\rho \qquad \because \frac{\lambda}{\mu} = c\rho$$

putting values of L_q

$$\boxed{L = \frac{(c\rho)^c}{c!} \frac{P_0 \, \rho}{(1-\rho)^2} + c\rho}$$

• Average Number of Customers in Queue ∴ L_q:

$$L_q = \sum_{n=0}^{\infty} n P_{n+c}$$

$$L_q = \sum_{n=0}^{\infty} n \frac{P_0 \rho^{n+c} c^c}{c!}$$

$$= \frac{c^c P_0}{c!} \sum_{n=0}^{\infty} n \rho^{n+c}$$

$$L_q = \frac{c^c P_0}{c!} \sum_{n=0}^{\infty} n \rho^n \cdot \rho^c$$

$$L_q = \frac{(c\rho)^c P_0}{c!} \sum_{n=0}^{\infty} n \rho^n$$

but we know

$$\therefore \frac{d}{dx} x^n = n x^{n+1}$$

$$\therefore L_q = \frac{(c\rho)^c P_0}{c!} \rho \frac{d}{d\rho} \left(\sum_{n=0}^{\infty} \rho^n \right)$$

$$L_q = \frac{(c\rho)^c P_0}{c!} \rho \frac{d}{d\rho} \left(\frac{\rho}{1-\rho} \right)$$

$$L_q = \frac{(c\rho)^c P_0}{c!} \rho \left(\frac{1-\rho+\rho}{(1-\rho)^2} \right)$$

9

$$\boxed{L_q = \frac{(c\rho)^c}{c!} P_0 \frac{\rho}{(1-\rho)^2}}$$

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NO: 3 Average number of customers in server.

As we know that:

$$L_s = L - L_q$$

$$= \left[\frac{(c\rho)^c}{c!} \cdot \frac{P_0 \rho}{(1-\rho)^2} + c\rho\right] - \left[\frac{(c\rho)^c}{c!} \cdot \frac{P_0 \rho}{(1-\rho)^2}\right]$$

$$L_s = \frac{(c\rho)^c P_0 \rho}{c!\,(1-\rho)^2}\left[\cancel{1} + \frac{c\rho}{(c\rho)^c P_0 \rho} - \cancel{1}\right]$$

$$L_s = \frac{\cancel{(c\rho)^c P_0 \rho}}{\cancel{c!\,(1-\rho)^2}}\left[\frac{c\rho}{\frac{\cancel{(c\rho)^c P_0 \rho}}{\cancel{c!\,(1-\rho)^2}}}\right]$$

$$\boxed{L_s = c\rho}$$

NO: 4 Probablity of Queyung $\left(P[N \geq C] = \sum_{n=C}^{\infty} P_n\right)$

Using Derived equation of P_n (for $n \geq C$)

$$P_n = P_0 \frac{\rho^n C^c}{c!}$$

Pulling value of P_n we get,

$$P[N \geq C] = \sum_{n=c}^{\infty} P_0 \frac{\rho^n C^c}{c!}$$

$$= P_0 \frac{C^c}{c!} \sum_{n=c}^{\infty} \rho^n$$

$$= P_0 \cdot \frac{C^c}{c!} \sum_{n=c}^{\infty} \left[\rho^c + \rho^{c+1} + \rho^{c+2} + \cdots \infty\right]$$

$$= P_0 \frac{C^c}{c!}\left(\frac{\rho^c}{1-\rho}\right) = \frac{P_0 \cdot (C\rho)^c}{c!\,(1-\rho)}$$

5

Average number of jobs in system

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$$E[n_s] = \sum_{n=1}^{c-1} nP_n + \sum_{n=c}^{\infty} cP_n$$

$$= 1P_0 \frac{(c\rho)}{1!} + 2P_0 \frac{(c\rho)^2}{2!} + \cdots (c-1)P_0 \frac{(c\rho)^{c-1}}{(c-1)!} +$$

$$c(P_c + P_{c+1} + P_{c+2} + \cdots +)$$

$$= c\rho\left(P_0 + P_0 \frac{(c\rho)}{1!} + P_0 \frac{(c\rho)^2}{2!} + \cdots P_0 \frac{(c\rho)^{c-2}}{(C-2)!}\right) + c\rho$$

$$= c\rho\,(P_0 + P_1 + P_2 + \cdots P_{c-2}) + c\rho$$

$$= c\rho\,(1 - P_{c-1} - \rho) + c\rho$$

$$= c\rho - c\rho P_{c-1} + c\rho(1-\rho)$$

$$\therefore \; c\rho(1-\rho) = cP_c = c\rho P_{c-1}$$

$$= c\rho - \cancel{c\rho P_{c-1}} + \cancel{c\rho P_{c-1}}$$

$$\boxed{E[n_s] = c\rho}$$