# Lecture 16
# Chapter # 5 (Cont'd)
# MARKOV CHAINS

**Task**

Every day, you take the same street from your home to the university. There are three traffic signals along your way, and you have noticed the following Markov dependence. If you come across a green signal at an intersection, then 70% of time the next signal also happens to be green. However, if you see a red signal, then 60% of time the next signal is also red.

(a) Draw state transition diagram for this Markov chain.

(b) Develop transition probability matrix.

(c) If the first light is red, what is the probability that the third light is green?

(d) Your classmate Sober has many traffic signals between his home and the university. If the first street light is green, what is the probability that the last street light is also green?

Solve

## CLASSIFICATION OF STATES OF A MARKOV CHAIN

- It is evident that the transition probabilities associated with the states play an important role in the study of Markov chains. Whenever we perform any problem solution we first draw the state transition diagram & we try to identify all the associate probabilities & w put them in there appropriate places after that we draw the probability transition matrix.
- To further describe the properties of Markov chains, it is necessary to present some concepts and definitions concerning

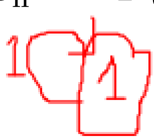
these states.

- State j is said to be **accessible** from state i if $p_{ij}^{(n)} > 0$ for some n > 0. This n represents the number of steps for the transition probability matrix.
- (Recall that $p_{ij}^{(n)}$ is just the conditional probability of being in state j after n steps, starting in state i.)
- Thus, state j being accessible from state i means that it is possible for the system to enter state j eventually when it starts from state i.

It means that if right now i am in state i eventually after either 1 step or 2 step or n step i will go to state j.

- In general, a sufficient condition for all states to be accessible is that there exists a value of n for which $p_{ij}^{(n)} > 0$ for all i and j.
- If state j is accessible from state i and state i is accessible from state j, then states i and j are said to **communicate**.
- In general,

1. Any state communicates with itself (because $p_{ii}^{(0)} = P\{X_o = i|X_o = i\} = 1$).




It means state is communicating with itself i.e. $P_{11}$ . It means it is not communicatig to any other state i.e. it is not making transition to any other state. It is only communicating to itself with 100% probability.

2. If state i communicates with state j, then state j communicates with state i.
3. If state i communicates with state j and state j communicates with state k, then state i communicates with state k.

$i \leftrightarrow j$ , $j \leftrightarrow k$ , $i \leftrightarrow k$ This is the famous transitivity property. If state i communicates with state j & state j communicates with state k then eventually state i also communicates with state k.

▪ Properties 1 and 2 follow from the definition of states communicating, whereas property 3 follows from the Chapman-Kolmogorov equations.

▪ As a result of these three properties of communication,

- the states may be partitioned into one or more separate **classes**
- such that those states that communicate with each other are in the same class.

(A class may consist of a single state).

▪ If there is only one class, i.e., all the states communicate, the Markov chain is said to be **irreducible**.

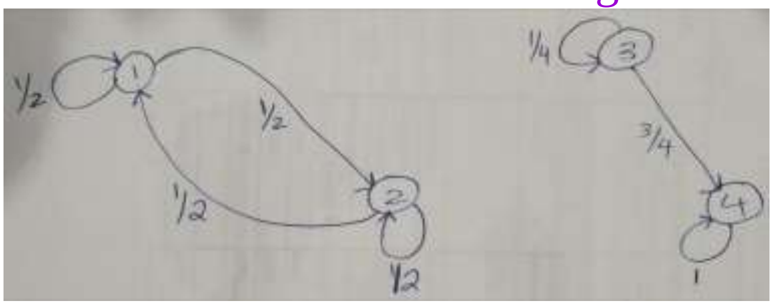
If we want to study various properties of a Markov chain we can analyze the state transition diagram by partitioning it into one or more classes. If we can only make a single class it means that every state can communicate to every other state or starting from any one state we can reach to any other state. It means that all the states are communicating with one another state & such a Markov chain is called *irreducible.*

***Example Problem 1***

Identify the following Markov chain as either irreducible or not irreducible. Give appropriate reason for your answer.

$$P = \begin{array}{c} \begin{array}{cccc} \phantom{\frac{1}{2}} & \phantom{\frac{1}{2}} & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

First make State Transition diagram



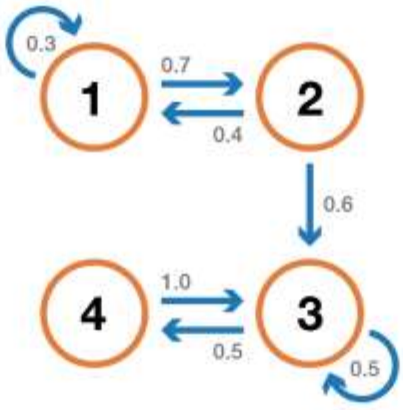

Even without drawing state transition diagram you can see the 0 values in state transition matrix shows that these state are not at all communicating with one another.
(like state 1 can not go to state 3 & 4 & so on.)
So here there are separate classes & we can actually divide it into 2 seperate classes means that the Markov chain is not reducible cause we can not reach every other state with any state.

Answer:
The given Markov chain is not irreducible as we can't reach every state for any other state. According to graph theory, the graph of an irreducible Markov chain is strongly connected.

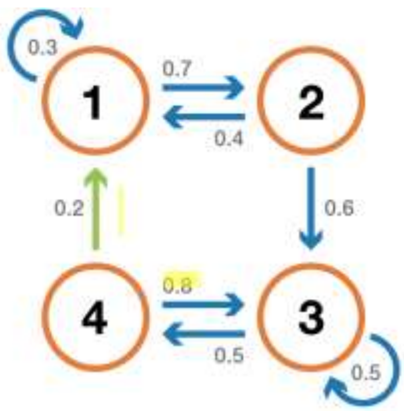
Recall: If in a graph every state is connected to every other state, it is said to be a strongly connected graph.

***Example Problem 2***
Identify the following Markov chains as either irreducible or not

irreducible. Give appropriate reason for your answer.



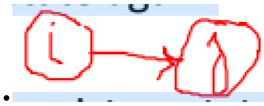

The graph is *not strongly connected* once we are in state 3 we can not go back to state 2, neither from state 3 nor from state 4. So it is not *irreducible.*




In this graph, even from state 4 i can go to state 1, state 2 & state 3, from state 3 i can go to state 4, state 1 & then eventually to state 2. From state 1 i can directly go to state 2 & then  state 3 & then state 4. & From state 2 i can go on state 1 & 3 directly & then to state 4 indirectly. So it is actually representing a *strongly connected graph* i.e. the Markov chain is *irreducible.*

## RECURRENT , TRANSIENT & ABSORBING STATES

- A state i is said to be a ***transient (or non-recurrent)*** state if, upon leaving this state, the process may never return to this state again.

- Therefore, state i is transient if and only if there exists a state j ( j ≠ i) that is accessible from state i but not vice versa, that is, state i is not accessible from state j.  I can actually go from state i to state j.

State j is accesible from state i but vice versa is not true. There is no way to go back on state i from state j. If we ever find such case It means that *state i is the transient state.*

- Thus, if state i is transient and the process visits this state, there is a positive probability (perhaps even a probability of 1) that the process will later move to state j and so will never return to state i. i.e. it is also called non-recurring state.
- Consequently, a transient state will be visited only a finite number of times.
- When starting in state i, another possibility is that the process definitely will return to this state.
- A state is said to be a *recurrent* state if, upon entering this state, the process definitely will return to this state again.
- Therefore, a state is recurrent if and only if it is not transient.
- Since a recurrent state definitely will be revisited after each visit, it will be visited infinitely often if the process continues forever.
- If the process enters a certain state and then stays in this state at the next step, this is considered a *return to this state.*
- Hence, the following kind of state is a special type of recurrent state.
- A state is said to be an ***absorbing*** state if, upon entering this

state, the process never will leave this state again.
In case of **transient** state, there's a positive probability that the process will not return to this state.
& in case of **recurrent** state,   we are deifinite that the process will return to the same state again.
& the last is the **absorbing** state & it says that if we have entered this state then the process will now not leave this state again or it will either absorb in the same state for the rest of its lifetime.

- Therefore, state i is an ***absorbing*** state if and only if $p_{ii} = 1$.
- Recurrence is a class property.
- That is, all states in a class are either recurrent or transient. It means that either they are actually   communicating to each & everyone & revisit the states or they can not revisit the left state.
- Furthermore, in a finite-state Markov chain, not all states can be transient.
- Therefore, all states in an irreducible finite-state Markov chain are recurrent. as the graph of irreducible Markov chain is strongly connected & every state is communicating with every other state.
- Indeed, one can identify an irreducible finite-state Markov chain (and therefore conclude that all states are recurrent) by showing that all states of the process communicate.
- It has already been pointed out that a sufficient condition for all states to be accessible (and therefore communicate with each other) is that there exists a value of n for which $p_{ij}^{(n)} > 0$ for all i and j.

***Example Problem 3***

As another example, suppose that a Markov chain has the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- Note that state 2 is an absorbing state (and hence a recurrent state) because if the process enters state 2 (row 3 of the matrix), it will never leave. Its also a special case of recurrent state as after certain point in time the state is actually visiting itself again.
- State 3 is a transient state because if the process is in state 3, there is a positive probability that it will never return. We can move to state 2 from state 3 but once we are leaving state 3 we can not return. So the probability that a process can go from state 3 to state 2 is 1/3. But once the process is in state 2 it will remain there. It can not go back to state 3 i.e. state 3 is transient.
- The probability is 1/3 that the process will go from state 3 to state 2 on the first step. Once the process is in state 2, it remains in state 2.

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- State 4 also is a transient state because if the process starts in state 4, it immediately leaves and can never return.
- States 0 and 1 are recurrent states. State 0 & 1 are

communicaing with each other

- To see this, observe from P that if the process starts in either of these states, it can never leave these two states.
- Furthermore, whenever the process moves from one of these states to the other one, it always will return to the original state eventually. If i start with state 0 i will go to state 1 & from state 1 with the probability of 1/2 i can go back to state 0.