CS-417
COMPUTER SYSTEMS MODELING

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(LECTURE # 15)

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Recap of Lecture # 14

Formulating Weather Example as Markov Chain

Matrix Revision

Derivation of n-Step Transition Probabilities

Example Problems

Chapter # 5 (Cont'd)

MARKOV CHAINS

Distribution of X_t

- Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $S = \{1, 2, \ldots, N\}$. Now each X_t is a random variable, so it has a probability distribution.
- We can write the probability distribution of X_t as an $N \times 1$ vector. For example, consider X_0. Let π be an $N \times 1$ vector denoting the probability distribution of X_0:

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{pmatrix} = \begin{pmatrix} \mathbb{P}(X_0 = 1) \\ \mathbb{P}(X_0 = 2) \\ \vdots \\ \mathbb{P}(X_0 = N) \end{pmatrix}$$

- We will write $X_0 \sim \pi^T$ to denote that the row vector of probabilities is given by the row vector π^T.

Probability distribution of X_1

Use the Partition Rule, conditioning on X_0:

$$\begin{aligned}
\mathbb{P}(X_1 = j) &= \sum_{i=1}^{N} \mathbb{P}(X_1 = j \mid X_0 = i)\boxed{\mathbb{P}(X_0 = i)} \\
&= \sum_{i=1}^{N} p_{ij}\pi_i \qquad \textbf{\textit{by definitions}} \\
&= \sum_{i=1}^{N} \pi_i p_{ij} \\
&= \left(\boldsymbol{\pi}^T P\right)_j .
\end{aligned}$$

Probability distribution of X_1

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&= \sum_{i=1}^{N} p_{ij}\pi_i \qquad \textit{by definitions} \\
&= \sum_{i=1}^{N} \pi_i p_{ij} \\
&= \left(\boldsymbol{\pi}^T P\right)_j .
\end{aligned}$$

This shows that $\mathbb{P}(X_1 = j) = \left(\boldsymbol{\pi}^T P\right)_j$ *for all* j.

The row vector $\boldsymbol{\pi}^T P$ is therefore *the probability distribution of* X_1:

$$\boxed{\begin{array}{lcl} X_0 & \sim & \boldsymbol{\pi}^T \\ X_1 & \sim & \boldsymbol{\pi}^T P. \end{array}}$$

Probability distribution of X_2

Using the Partition Rule as before, conditioning again on X_0:

$$\mathbb{P}(X_2 = j) = \sum_{i=1}^{N} \mathbb{P}(X_2 = j \mid X_0 = i)\mathbb{P}(X_0 = i) = \sum_{i=1}^{N} \left(P^2\right)_{ij} \pi_i = \left(\boldsymbol{\pi}^T P^2\right)_j .$$

- Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with $N \times N$ transition matrix P.
- If the probability distribution of X_0 is given by the $1 \times N$ row vector π^T, then the probability distribution of X_t is given by the $1 \times N$ row vector $\pi^T P^n$. That is,

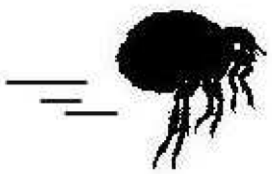
$$X_0 \sim \pi^T \Rightarrow X_t \sim \pi^T P^n$$

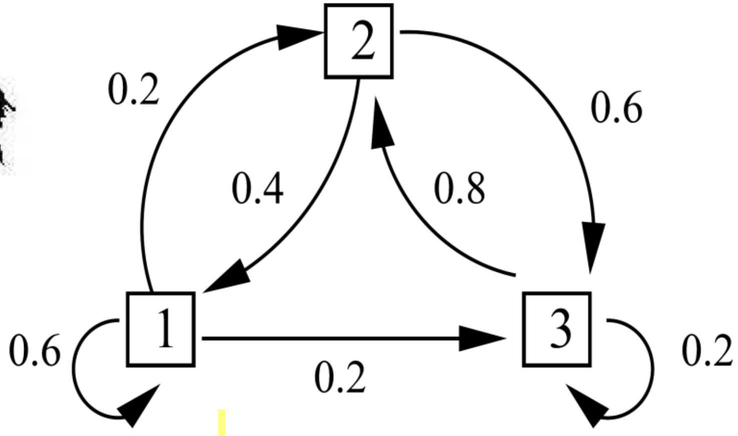
Note:

- The distribution of X_t: $X_t \sim \pi^T P^n$
- The distribution of X_{t+1}: $X_{t+1} \sim \pi^T P^{n+1}$

Example Problem

Purpose-flea zooms around the vertices of the given transition diagram. Let X_t be Purpose-flea's state at time t $(t = 0, 1, \ldots)$.





a) Find the transition matrix, P.

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

b) Find $P(X_2 = 3 \mid X_0 = 1)$

$$P(X_2 = 3 \mid X_0 = 1) = (P^2)_{13}$$

$$(P^2) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix} * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.44 & 0.28 & 0.28 \\ 0.24 & 0.56 & 0.2 \\ 0.32 & 0.16 & 0.52 \end{bmatrix}$$

$$(P^2)_{13} = 0.28$$

c) Suppose that Purpose-flea is equally likely to start on any vertex at time 0. Find the probability distribution of X_1.

From this info, the distribution of X_0 is $\pi^T = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$

$$X_1 \sim \pi^T P = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}) * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

$$X_1 = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$$

Therefore X_1 is also equally likely to be state 1, 2, or 3.

d) Suppose that Purpose-flea begins at vertex 1 at time 0. Find the probability distribution of X_2.

Now, the distribution of X_0 is now $\pi^T = (1,0,0)$

$$X_2 \sim \pi^T P^1 = (1 \quad 0 \quad 0) * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix} * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

$$X_2 = (0.44 \quad 0.28 \quad 0.28)$$

$$P(X_2 = 1) = 0.44, P(X_2 = 2) = 0.28, P(X_2 = 3) = 0.28$$

Task

There are three grocery stores in a small town, say, Store A, B and C. On any given week, 200 people will go to Store A, 120 people will go to Store B and 180 people will go to Store C to do their shopping for the entire week. But people may don't want to visit the same store every time. They typically want to switch stores in the following weeks because of the attractive offers on various items.

So, the following week 80% will go back to Store A and 10% will go to Store B and remaining 10% will go to Store C. Also, 70% will go back to Store B and 20% will now go to Store A and remaining 10% will visit Store C. Moreover, 60% will go back to Store C, 10% will go to Store A and remaining 30% will go to Store C.

Determine the # of customers in these three stores next week.

STEADY-STATE PROBABILITIES

- The long-run behavior of finite-state Markov chains as reflected by the ***steady-state probabilities*** shows that:
  - there is a limiting probability that the system will be in each state *j* after a large number of transitions, and
  - that this probability is independent of the initial state.

- These properties are summarized below.

For any irreducible ergodic Markov chain, $\lim_{n\to\infty} p_{ij}^{(n)}$ exists and is independent of i.

Furthermore,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0,$$

where the π_j uniquely satisfy the following **steady-state equations**

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij}, \qquad \text{for } j = 0, 1, \ldots, M,$$

$$\sum_{j=0}^{M} \pi_j = 1.$$

- The π_j are called the **steady-state probabilities** of the Markov chain.

STEADY-STATE PROBABILITIES

- The term *Steady-state* probability means that the probability of finding the process in a certain state, say *j*, after a large number of transitions tends to the value π_j, independent of the probability distribution of the initial state.
- It is important to note that the steady-state probability ***does not imply*** that the process settles down into one state.
- On the contrary, the process continues to make transitions from state to state, and at any step *n* the transition probability from state *i* to state *j* is still p_{ij}.
- The π_j can also be interpreted as *stationary probabilities* (not to be confused with stationary transition probabilities).

Example

Consider the weather forecast model. Determine its steady-state probabilities:

$$P = \begin{array}{c|cc} & \boxed{S} & \boxed{R} \\ \boxed{S} & 0.7 & 0.3 \\ \boxed{R} & 0.4 & 0.6 \end{array}$$

$\pi = \pi P$

$$(\pi_s \quad \pi_R) = (\pi_s \quad \pi_R)\begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

$$\Longrightarrow 0.7\pi_s + 0.4\,\pi_R = \pi_s \Longrightarrow 0.3\pi_s = 0.4\pi_R$$

$$0.3\pi_s + 0.6\,\pi_R = \pi_R \Longrightarrow \quad \pi_s = 4/3\pi_R$$

$$\pi_s + \pi_R = 1$$

$$\Longrightarrow \pi_R = 3/7 \;\&\; \pi_S = 4/7$$

$\Longrightarrow$ *In the long history of this city, 43% days are rainy and*

57% days are sunny!!

Task

A Markov chain has the following transition probability matrix:

$$\begin{pmatrix} 0.3 & - & 0 \\ 0 & 0 & - \\ 1 & - & - \end{pmatrix}$$

a) Fill in the blanks.

b) Compute the steady-state probabilities.

Answers:

$$\boldsymbol{\pi_0 = 0.4167, \pi_1 = 0.29167, \pi_2 = 0.29167}$$