Lecture 17
Chapter # 5 (Cont'd)
MARKOV CHAINS

PERIODICITY PROPERTIES

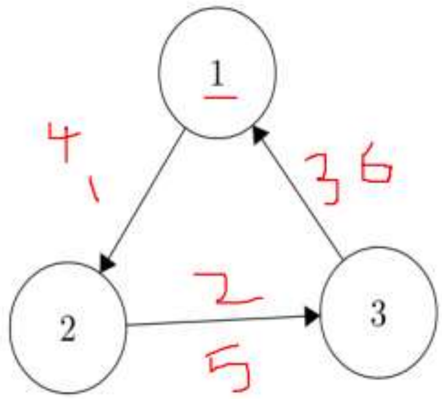
- Another useful property of Markov chains is *periodicities*.
- A state i is said to be *periodic* with period di, if on leaving state i return is possible only in a number of steps that is multiple of the integer $d_i > 1$. If the value of $d_i = 1$, state i is aperiodic.

periodic state & aperiodic state.
If the value of $d_i = 1$ then the state is said to be aperiodic.
& If upon leaving a certain state i I want to reach back to state i after passing through various transitions & other states, it should only be in the number of steps i.e. multiple of the integer d_i & should be greater than 1. Only in this partcular case the state would be called a periodic state.

- Each state in a Markov Chain has a period. The period is defined as the greatest common denominator (divisor) of the length of return trips (i.e., number of steps it takes to return), given that you start in that state.

This GCD value will help us in the calculation of the period of the state & if this GCD value is equal to 1 then the state will be called aperiodic.

Example Problem 4
Consider the given Markov Chain with 3 states. Determine its period.



Every state is communicating with every other state.

Solution:

• We can exactly predict it's movement over time (that is, it is deterministic). Imagine first that we start in State 1. & I want to reach back to state 1 so how many transitions i need to cover? i.e. 3 steps ; step1 then step2 then step3.
What if i want to return back to state 1 again so next step would be step 4 the step5 then step6. So in the 2^{nd} trip with 6 steps overall we can reach back to state 1. then in 3rd trip there will be 9 steps & so on.
we can start to any of the state & will rturn to it in the same way.
So 3 6 9 & so on all are divisible by 3 So there GCD will be 3 i.e. the period of state 1 & state2 & state3 will be 3.

• We know with certainty that in the next step we go to State 2, and then State 3, and then back to State 1. In fact, we know how long every possible path of return to State 1 is: 3 steps, 6 steps, 9 steps, etc.

• The greatest common devisor of these path lengths is 3, so State 1 (and the other two states) have period 3.

PERIODICITY PROPERTIES (Cont'd)

▪ Just as recurrence is a class property, it can be shown that periodicity is a class property.

▪ That is, if state i in a class has period t, then all states in that class have period t.

So in the previous example we can consider that all the states (state1 2 & 3) are actually representing a single class, & of one state in that particular class is having the period value equals to t & in our previous example its period value was 3 so we can verify all the other states in the same class will have the same period value.

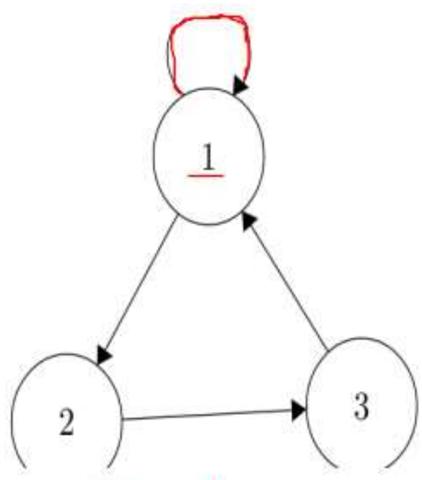
▪ In a finite-state Markov chain, recurrent states that are aperiodic are called ***ergodic*** states.

aperiodic i.e. if the value of period is 1 (d_i = 1) & that state is recurrent so its a ergodic state.

▪ A Markov chain is said to be *ergodic* if all its states are ergodic states.

Example Problem 5

Consider the given Markov Chain with 3 states. Determine its period.



Solution:

• If we start in State 1, we could automatically just loop back after one period, but we could also go through the whole chain (to State 2, then 3, then 1) in 3 steps.
• So two possible return lengths are 1 and 3 (there are more, of course), and already the greatest common denominator of these path lengths is 1 So State 1 has period one.

• What about State 2? If we start at State 2, we know we will go to State 3 and then State 1. However, there we could stay in State 1 for any period of time before returning to State 2.

$3 \to 1 \to 2$

$3 \to 1 \to 1 \to 2$ state 3

• We could return in 3, 4, 5, etc. number of steps. Again, the greatest common denominator here is 1, so State 2 also has period 1. You can check, but the same holds for State 3.

<u>Example Problem 6</u>

Consider the Markov chain that has the following (one-step) transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

for your convenience you can draw state transition diag.

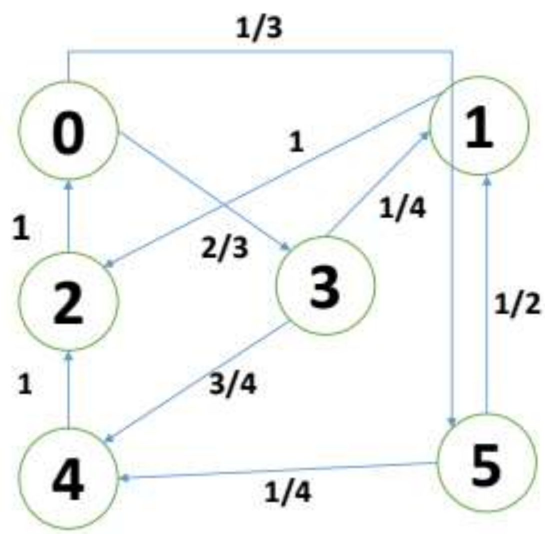
a) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.

b) For each of the classes, determine the period of the states in that class.
c) Also determine whether the given Markov chain is ergodic or not? Provide proper reason.

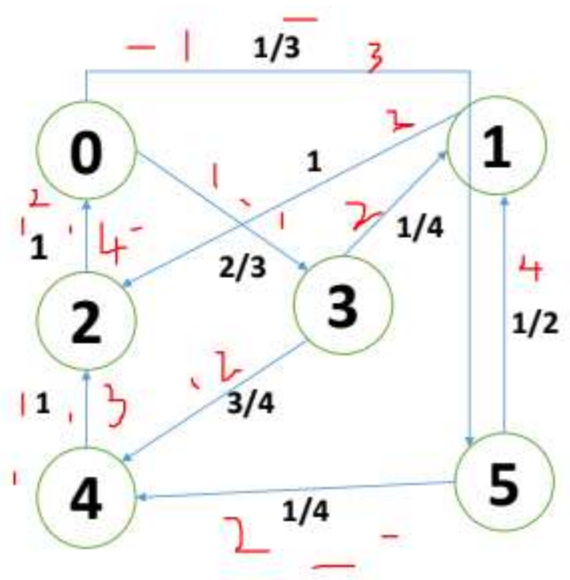
Solution:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$

First of all draw the state transition diagram.



whether the states are transient or recurrent?
For the recurrent states we have to see if lets suppose i am leaving any particular state let say state 0 if i will be able to reach back to that particular state in certain steps if this is the case then this state is said to be recurrent.
& this proerty has to be checked for all the states.



From state0 then state5 then state4 then state2 then back on state0.
AND state0 then state3 then state4 then state2 then back on state0.
AND state0 then state3 then state1 then state2 then back on state0.
I have checked all the paths for state 0. (all takes 4 steps)

Now lets check for state 1;
state1 to state2 then state0 then to state5 then to state1. (again 4 steps)
state1 to state2 then state0 then to state3 then to state1. (again 4 steps)

Now lets check for state 2;
state2 to state0 then state3 then to state4 then to state2. (again 4 steps)
state2 to state0 then state5 then to state4 then to state1. (again 4 steps)
state 2 is also recurrent.

Now lets check for state 3;
state3 to state4 then state2 then to state0 then to state3. (again 4 steps)
state3 to state1 then state2 then to state0 then to state3. (again 4 steps)

Now lets check for state 4;
state4 to state2 then state0 then to state5 then to state4. (again 4 steps)
state4 to state2 then state0 then to state3 then to state4. (again 4 steps)
similary for state5..

a) All the states are re-current.
b) Period for the states is 4. once we identify that the class for particular Markov chain is recurrent then period of all the state is same i.e. 4
c) The chain is not ergodic since the chain is not aperiodic.

<u>Task</u>

Consider the Markov chain that has the following (one-step) transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}$$

a) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.
b) For each of the classes, determine the period of the states in

that class.
c) Also determine whether the given Markov chain is ergodic or not? Provide proper reason.
d) Is there any absorbing state in the chain ?

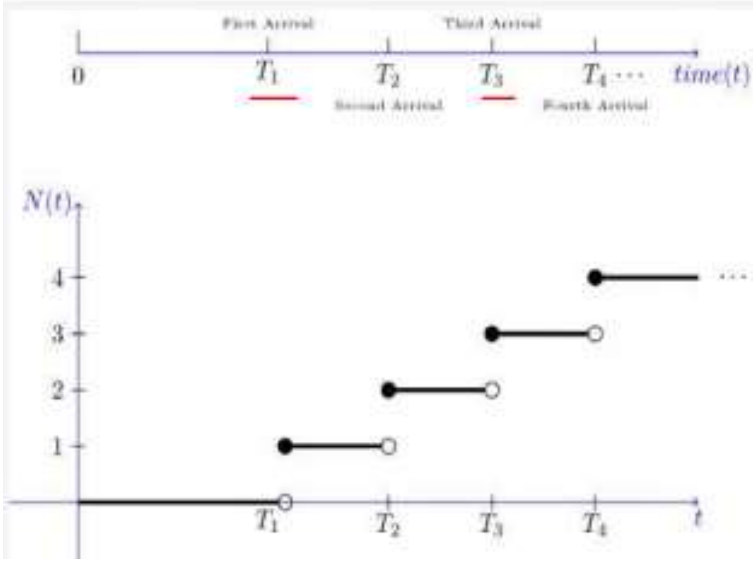
Counting Processes (Example of Stochastic Processes)

• In some problems, we count the occurrences of some types of events. In such scenarios, we are dealing with a counting process.
• For example, a random process N(t) shows the number of customers who arrive at a supermarket by time t starting from time 0.

so if this particular random process is reponsible for counting the total number of customers present at supermarket in a certain interval starting from 0 till t. So this process is called a counting process & is one of the classification of the stochastic process.

• For such a processes, we usually assume N(0)=0, so as time passes and customers arrive, N(t) takes positive integer values.

here on x-axis we got time values, & at y-axis we have got this process value & we are only taking the positive integers over here since we are maintaing a count taht could'nt be negative.
Till time instance T_1 when there was no arrival the value of count was 0.
From this timeline, we are understanding that we got our first arrival at time T_1, so at this particular instance i have actually increased this value. Now the value of N(t) =1.
This count will continue till we get the 2^{nd} arrival.
Now the 2^{nd} arrival is at time T_2 & at T_2 the count value increased to 2. & so on



Poisson Process as a Counting Process

• The Poisson process is one of the most widely-used counting processes.

• It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate, but completely at random (without a certain structure).

• For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random.

• Thus, we conclude that the Poisson process might be a good model for earthquakes.

Merging of Poisson Processes

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let us define $N(t) = N_1(t) + N_2(t)$. That is, the random process $N(t)$ is obtained by combining the arrivals in $N_1(t)$ and $N_2(t)$ [redacted]. We claim that $N(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. To see this, first note that

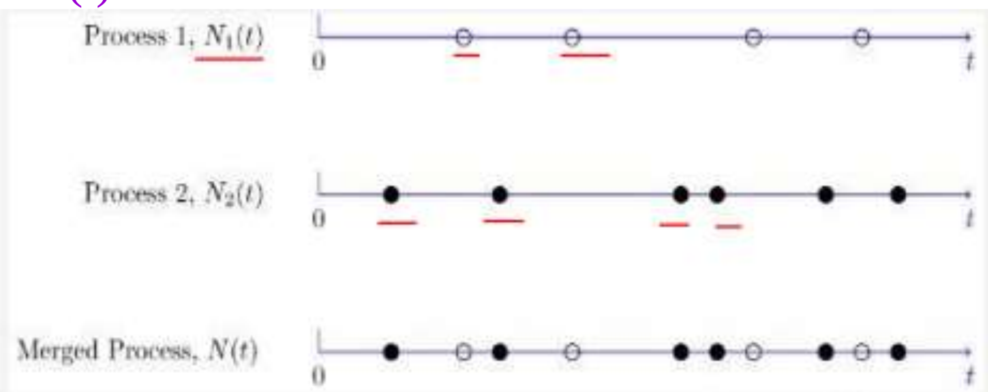
$$N(0) = N_1(0) + N_2(0)$$
$$= 0 + 0 = 0.$$

N(0) would be equal to 0 cause for every counting process we will assume that at time t=0 there's no customers.

There are 4 arrivals in process 1.

There are 6 arrivals in process 2.

& if we merge these 2 processes then we will get this merged process N(t).



Here the arrival rates are simply added.

Next, since $N_1(t)$ and $N_2(t)$ are independent and both have independent increments, we conclude that $N(t)$ also has independent increments. Finally, consider an interval of length τ, i.e, $I = (t, t+\tau]$. Then the numbers of arrivals in I associated with $N_1(t)$ and $N_2(t)$ are $Poisson(\lambda_1\tau)$ and $Poisson(\lambda_2\tau)$ and they are independent. Therefore, the number of arrivals in I associated with $N(t)$ is $Poisson((\lambda_1+\lambda_2)\tau)$ (sum of two independent Poisson random variables).

Merging Independent Poisson Processes

Let $N_1(t), N_2(t), \cdots, N_m(t)$ be m independent Poisson processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_m$. Let also

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t), \quad \text{for all } t \in [0, \infty).$$

Then, $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_m$.

Splitting of Poisson Processes

listen

Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way [redacted]. For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,