CS-417
COMPUTER SYSTEMS MODELING

Spring Semester 2020

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(LECTURE # 22)

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Recap of Lecture # 21

M/M/1 Queue Analysis (Birth-Death Process)
PASTA Theorem
Example Problems

Chapter # 6 (Cont'd)

FUNDAMENTALS OF QUEUING MODELS

M/M/1/K System (Finite Buffer Capacity)

The **M/M/1/k** queue is a **short hand** notation for the **M/M/1/k/∞/FIFO** queue.

- **M = Arrival process** is **Poisson**
- **M = Service (departure) process** is **Exponential**
- **1 =** There is **1 server** in system
- **k = Queue capacity**; **the (k+1)th arriving client** will be ***rejected***
- ∞ = **Infinite population size** (the **arrival process** will be **unaffected** by the number of clients already in the system)
- **FIFO =** First In First Out Service

M/M/1/K System Analysis

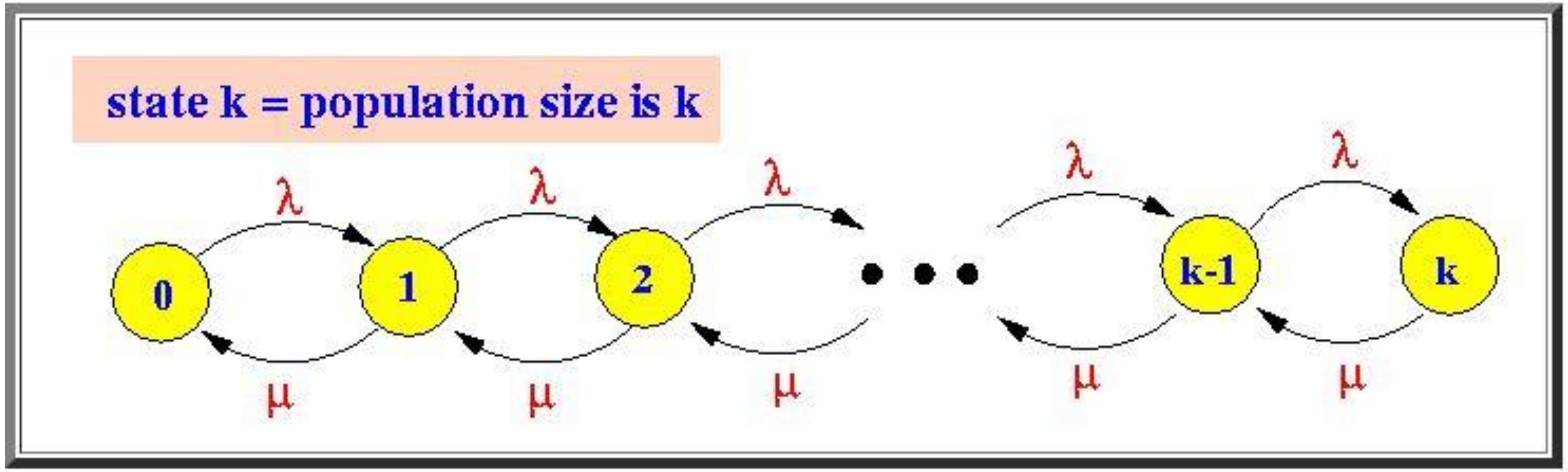


Fig 1: Rate diagram for the M/M/1/K System

Starting with: $\boldsymbol{P_n} = \boldsymbol{\rho^n P_0}$ **; n =0 , 1, 2, ……., k**

$$\sum_{n=0}^{k} P_n = 1$$

M/M/1/K System Analysis

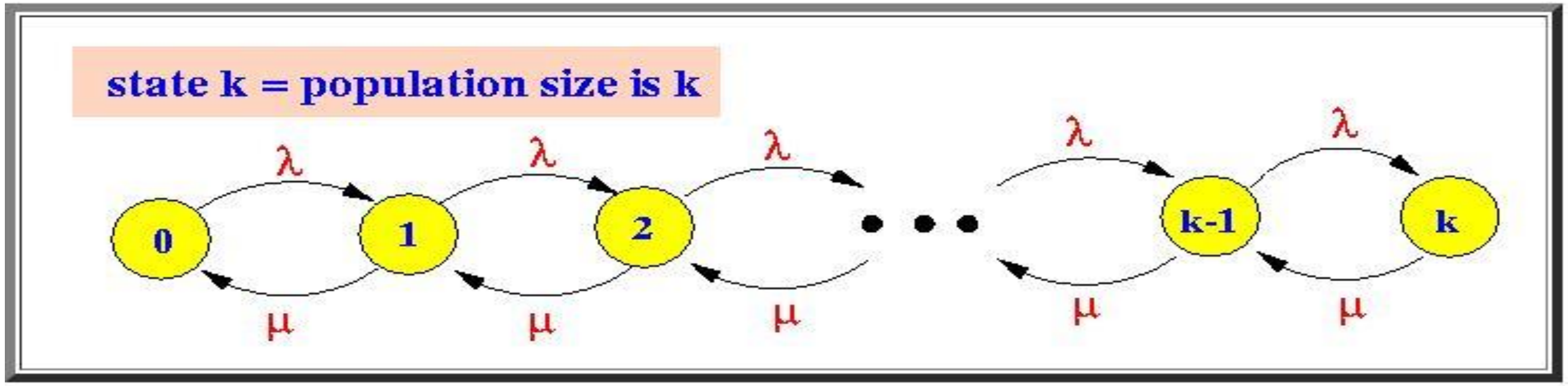


Fig 1: Rate diagram for the M/M/1/K System

Starting with: $P_n = \rho^n P_0$; n =0 , 1, 2, ……., k

$$P_0 + \rho P_0 + \rho^2 P_0 + \ldots\ldots + \rho^k P_0 = 1$$

$$\Rightarrow \frac{1-\rho^{k+1}}{1-\rho} P_0 = 1$$

$$\Rightarrow P_0 = \frac{1-\rho}{1-\rho^{k+1}}$$

$$\therefore P_n = \rho^n \frac{1-\rho}{1-\rho^{k+1}}$$

$$S = a\left(\frac{1-r^n}{1-r}\right)$$

M/M/1/K System Analysis

Because of the limited system capacity, we do not require $\lambda < \mu$.

If $\lambda = \mu$ $(i.e.\ \boldsymbol{\rho = 1})$:

$$\sum_{n=0}^{k} P_n = 1$$

$$\sum_{n=0}^{k} \boldsymbol{\rho^n P_0} = \mathbf{1}$$

$$\Rightarrow (1+1+1+\ \ldots\ldots\ldots + 1)\ \boldsymbol{P_0} = 1$$

$$\Rightarrow\ \boldsymbol{P_0} = \frac{1}{1+k} \Rightarrow\ \boldsymbol{P_n} = \frac{1}{1+k}$$

$$S = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$= \frac{k+1}{2}\left(2 + \cancel{(k)0}\right)$$

M/M/1/K Analysis (Task)

Q) Derive the equation to determine the total number of customers in the system.

Hint: Start with $L = E[N] = \sum_{n=0}^{k} n\, P_n$

Result for verification:

$$L = \frac{\rho}{1-\rho} - (k+1)\frac{\rho^{k+1}}{1-\rho^{k+1}}$$

M/M/1/K Analysis (Task)

Total Number of Customers when $\rho = 1$,

$$L = \sum_{n=0}^{k} n\left(\frac{1}{1+k}\right)$$

$$L = \frac{1}{1+k} \sum_{n=0}^{k} n$$

$$L = \frac{1}{1+k} . \frac{k}{2} (k+1)$$

$$L = \frac{k}{2}$$

M/M/1/K Analysis

Average Number of customers in the Service & in the Queue:

$$L_s = E[N_s] = \overset{0}{P[N_s \mid N = 0]}\ \overset{P_0}{P[N = 0]} + \overset{1}{P[N_s \mid N > 0]}\ \overset{1 - P_0}{P[N > 0]}$$

$$= 0 \text{ x } P_0 + 1 \text{ x } (1 - P_0)$$

$$L_s = 1 - P_0$$

$$L_q = L - L_s$$

$$= L - (1 - P_0)$$

M/M/1/K Analysis

Effective Arrival Rate (λ_a)

Customers are turned away when there are k customers in the system:

$$\lambda_a = \lambda\,(1 - \boldsymbol{P_k})$$

Using Little's Law,

$$W = \frac{L}{\lambda_a} = \frac{L}{\lambda\,(1 - Pk)}$$

$$W_q = \frac{L_q}{\lambda\,(1 - Pk)}$$

M/M/1/K Analysis

Server Utilization (U):

For balancing M/M/1/K system:

$$\lambda\,(1 - \boldsymbol{P_k}) = \mu(1 - \boldsymbol{P_0})$$

Probability that the server is busy is given by:

$$\Rightarrow \mathbf{U} = 1 - \boldsymbol{P_0} = \frac{\lambda}{\mu}(1 - \boldsymbol{P_k})$$

$$\Rightarrow \mathbf{U} = \rho(1 - \boldsymbol{P_k})$$

Example Problem 1

Packets arrive at a router according to Poisson distribution at an average rate of 6 per second. The router is serving packets at an average rate of 8 packets per second with exponential distribution of service time. However, the buffer has capacity for only 8 packets. Calculate:

a) average response time of packets

b) average number of packets dropped if a total of 5000 packets approach for the service

c) average number of packets in the router

d) average number of packets in the buffer

Answers:

a) *0.4081 sec*
b) *100 packets*
c) *2.4*
d) *1.665*

Example Problem 2

Consider the following single-server queue: the inter-arrival time is exponentially distributed with a mean of 10 minutes and the service time is also exponentially distributed with a mean of 8 minutes, find out:

(i) mean wait in the queue,

(ii) mean number in the queue,

(iii) the mean wait in the system,

(iv) mean number in the system and

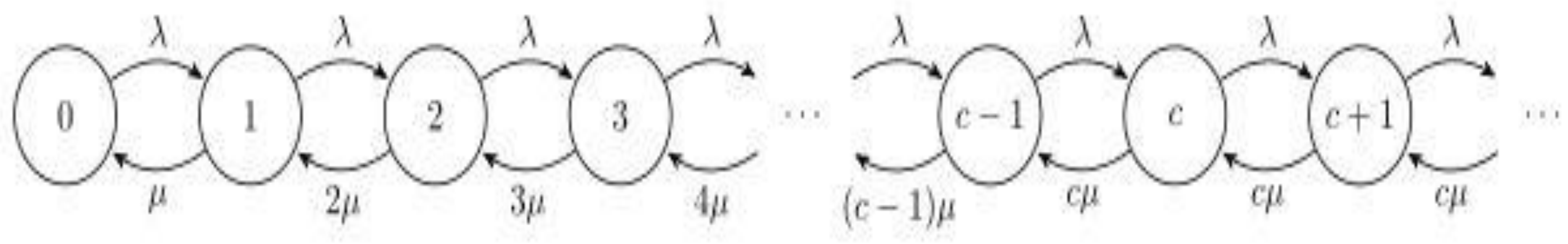
(v) proportion of time the server is idle.

Answers:

i) *3.2*
ii) *32 mins*
iii) *40 mins*
iv) *4*
v) *0.2*

Task Assignment *(To be submitted)*

Perform the analysis of M/M/C Queuing System where C indicates the number of identical servers. The rate diagram for this system is given by:



Server Utilization, $\rho = \dfrac{\lambda}{c\mu} < 1$ for stability.

Consider the value of c = 3.

Provide the flow equations and determine the generalized results of the following:

- *Average number of customers in the system.* ($L = E[N] = \sum_{n=1}^{\infty} nP_n$)
- *Average number of customers in the queue.* ($L_q = \sum_{n=0}^{\infty} nP_{n+c}$)
- *Average number of customers in the server.*
- *Probability of Queuing.* ($P[N \geq C] = \sum_{n=c}^{\infty} P_n$)
- *Average number of busy servers.*

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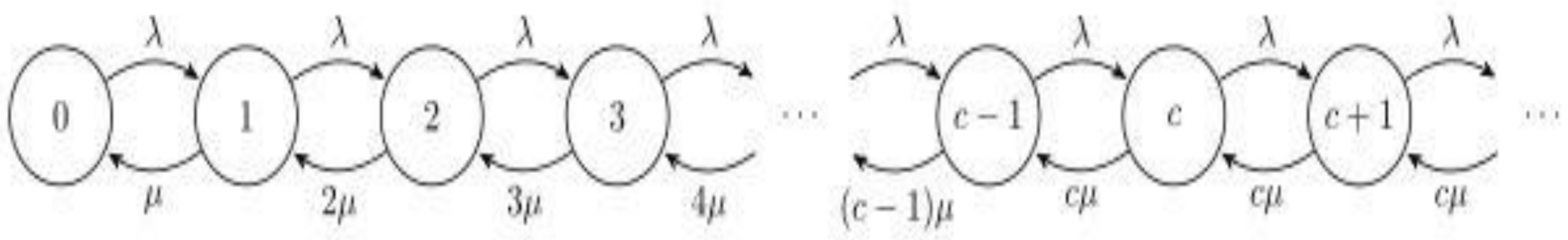

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