# Lecture 15
# Chapter # 5 (Cont'd)
# MARKOV CHAINS

**Distribution of $X_t$**

• Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $S = \{1, 2, \ldots, N\}$.

Now each $X_t$ is a random variable, so it has a probability distribution.

• We can write the probability distribution of $X_t$ as an $N \times 1$ vector.

For example, consider $X_0$. Let $\pi$ be an $N \times 1$ vector denoting the probability distribution of $X_0$:

$$\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \pi_2 \\ \vdots \\ \pi_N \end{pmatrix} = \begin{pmatrix} \mathbb{P}(X_0 = 1) \\ \mathbb{P}(X_0 = 2) \\ \vdots \\ \mathbb{P}(X_0 = N) \end{pmatrix}$$

• We will write $X_0 \sim \pi^T$ to denote that the row vector of probabilities is given by the row vector $\pi^T$. Its a colunm vector if i take its transpose it will be a row vector

## Probability distribution of $X_1$

Use the Partition Rule, conditioning on $X_0$:

Probability distribution of a random variable that is representing some state states can be given by a complete vector. so in step 2 we are simply replacing the complete formula by short notation.

$P(X_0 = i) = \pi_i$

$$
\begin{aligned}
\mathbb{P}(X_1 = j) &= \sum_{i=1}^{N} \mathbb{P}(X_1 = j \mid X_0 = i)\boxed{\mathbb{P}(X_0 = i)} \\
&= \sum_{i=1}^{N} p_{ij}\pi_i \quad \textit{by definitions} \\
&= \sum_{i=1}^{N} \pi_i p_{ij} \\
&= \left(\boldsymbol{\pi}^T P\right)_j .
\end{aligned}
$$

If i am provided with the probability distribution vector of $X_0$ then i can determine the probability distribution of $X_1$ by multiplying $X_0$ vector with the transition probability matrix.

This shows that $\mathbb{P}(X_1 = j) = \left(\boldsymbol{\pi}^T P\right)_j$ *for all* $j$.

The row vector $\boldsymbol{\pi}^T P$ is therefore *the probability distribution of* $X_1$:

$$
\boxed{\begin{aligned} X_0 &\sim \boldsymbol{\pi}^T \\ X_1 &\sim \boldsymbol{\pi}^T P. \end{aligned}}
$$

**Probability distribution of $X_2$**

Using the Partition Rule as before, conditioning again on $X_0$:

$$
\mathbb{P}(X_2 = j) = \sum_{i=1}^{N} \mathbb{P}(X_2 = j \mid X_0 = i)\mathbb{P}(X_0 = i) = \sum_{i=1}^{N} \left(P^2\right)_{ij} \pi_i = \left(\boldsymbol{\pi}^T P^2\right)_j .
$$

• Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with $N \times N$ transition matrix P.

• If the probability distribution of $X_0$ is given by the $1\times N$ row vector $\pi^T$, then the probability distribution of $X_t$ is given by the 1

× N row vector $\pi^T P^n$ . That is,

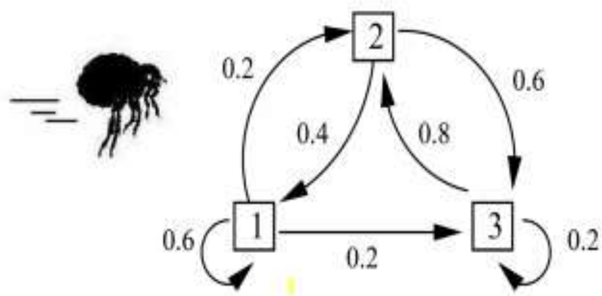
$$X_0 \sim \pi^T \Rightarrow X_t \sim \pi^T P^n$$

Note:

- The distribution of $X_t$ : $Xt \sim \pi^T P^n$
- The distribution of $X_{t+1}$: $X_{t+1} \sim \pi^T P^{n+1}$

## Example Problem

Purpose-flea zooms around the vertices of the given transition diagram. Let $X_t$ be Purpose-flea's state
at time t (t = 0, 1, . . .).




a) Find the transition matrix, P.

$$P = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

b) Find $P(X_2 = 3 | X_0 = 1)$

$P(X_2 = 3 | X_0 = 1) = (P^2)_{13}$

$$(P^2) = \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix} * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$
$$= \begin{bmatrix} 0.44 & 0.28 & 0.28 \\ 0.24 & 0.56 & 0.2 \\ 0.32 & 0.16 & 0.52 \end{bmatrix}$$

$(P^2)_{13} = 0.28$

c) Suppose that Purpose-flea is equally likely to start on any vertex at time 0. Find the probability distribution of $X_1$.
Equally likely meeans every state has same probability & probability of every state will be 1/3

From this info, the distribution of $X_0$ is $\pi^T = (\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3})$

$$X_1 \sim \pi^T P = (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}) * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$
$$X_1 = (\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3})$$

Therefore $X_1$ is also equally likely to be state 1, 2, or 3.

d) Suppose that Purpose-flea begins at vertex 1 at time 0. Find the probability distribution of $X_2$.
it is evident as the time value 0 is provided. & it begins on vertex 1 so probability of vertex 2 & 3 will be 0.
Now, the distribution of $X_0$ is now $\pi^T = (1,0,0)$

$$X_2 \sim \pi^T P = (1\ \ 0\ \ 0) * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix} * \begin{bmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}$$

$X_2 = (0.44\ 0.28\ 0.28)$

$P(X_2 = 1) = 0.44$, $P(X_2 = 2) = 0.28$, $P(X_2 = 3) = 0.28$
there sum should be equal to 1.

**Task**

There are three grocery stores in a small town, say, Store A, B and C. On any given week, 200 people will go to Store A, 120 people will go to Store B and 180 people will go to Store C to do their shopping for the entire week. But people may don't want to visit the same store every time. They typically want to switch stores in the following weeks because of the attractive offers on various items.

So, the following week 80% will go back to Store A and 10% will go to Store B and remaining 10% will go to Store C. Also, 70% will go back to Store B and 20% will now go to Store A and remaining 10% will visit Store C. Moreover,
60% will go back to Store C, 10% will go to Store A and remaining 30% will go to Store C.
Determine the # of customers in these three stores next week.
Draw state transition diagram & transition probability matrix first...

**STEADY-STATE PROBABILITIES**

- The long-run behavior of finite-state Markov chains as reflected by the steady-state probabilities shows that:

- there is a limiting probability that the system will be in each state j after a large number of transitions, and

may be because that particular system is consistent of so many states or sometime it is also possible that there are limited number of states but is actually reaching a certain state after making so many transitions in the other state. So there is a probability that the system will be in state j after large number of transitions & this probability will be independant of initial state.

In an irreducible Markov chain the process can go from any state to any other state whatever be the number of steps it required. It means that all the steps are communicating with one another & we can make transition from any state to any other state. & we are not even concerned about the total number of steps required to perform this operation. i.e. irreducible.

- that this probability is independent of the initial state.

• These properties are summarized below.

For any irreducible ergodic Markov chain, $\lim_{n\to\infty} p_{ij}^{(n)}$ exists and is independent of $i$.

Furthermore,

$$\lim_{n\to\infty} p_{ij}^{(n)} = \pi_j > 0,$$

where the $\pi_j$ uniquely satisfy the following **steady-state equations**

$$\pi_j = \sum_{i=0}^{M} \pi_i p_{ij}, \qquad \text{for } j = 0, 1, \ldots, M,$$

$$\sum_{j=0}^{M} \pi_j = 1.$$

• The $\pi_j$ are called the steady-state probabilities of the Markov chain.

## STEADY-STATE PROBABILITIES

- The term Steady-state probability means that the probability of finding the process in a certain state, say j, after a large number of transitions tends to the value $\pi_j$, independent of the probability distribution of the initial state.
- It is important to note that the steady-state probability **does not imply** that the process settles down into one state.

It is actually the probability of reaching state j from state i after a large number of transitions.

- On the contrary, the process continues to make transitions from state to state, and at any step n the transition probability from state i to state j is still pij.
- The $\pi_j$ can also be interpreted as stationary probabilities (not to be confused with stationary transition probabilities).

## Example

Consider the weather forecast model. Determine its steady-state probabilities:

$$P = \begin{array}{c} \\ \boxed{S} \\ \boxed{R} \end{array} \begin{array}{c} \begin{array}{cc} \boxed{S} & \boxed{R} \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$\pi = \pi P$ ; we already know this formula

$$(\pi_S \quad \pi_R) = (\pi_S \quad \pi_R) \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

; sunny state(S) the rainy state(R) .

this particular vector can be represented by these two terms.

we will multiply these variables with transitional matrix.

$$\Rightarrow 0.7\pi_s + 0.4\,\pi_R = \pi_s \Rightarrow 0.3\pi_s = 0.4\pi_R$$
$$0.3\pi_s + 0.6\,\pi_R = \pi_R \Rightarrow \quad \pi_s = 4/3\pi_R$$

We already know sum of steady state probabilities is 1.

$$\pi_s + \pi_R = 1$$

$$\Rightarrow \pi_R = 3/7 \;\&\; \pi_S = 4/7$$

simply by solving above two eqs we got our result & converting them into %

$\Rightarrow$ *In the long history of this city,*
*43% days are rainy and*
*57% days are sunny!!*

**Task**

A Markov chain has the following transition probability matrix:

$$\begin{pmatrix} 0.3 & - & 0 \\ 0 & 0 & - \\ 1 & - & - \end{pmatrix}$$

a) Fill in the blanks.
b) Compute the steady-state probabilities.

Answers:

$$\boldsymbol{\pi_0 = 0.4167, \pi_1 = 0.29167, \pi_2 = 0.29167}$$