CS-417
COMPUTER SYSTEMS MODELING
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(LECTURE # 16)

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Recap of Lecture # 15

Distribution of X_t & Derivations

Steady-State Probabilities

Example Problems

<u>**Chapter # 5 (Cont'd)**</u>

MARKOV CHAINS

Task

Every day, you take the same street from your home to the university. There are three traffic signals along your way, and you have noticed the following Markov dependence. If you come across a green signal at an intersection, then 70% of time the next signal also happens to be green. However, if you see a red signal, then 60% of time the next signal is also red.

(a) Draw state transition diagram for this Markov chain.

(b) Develop transition probability matrix.

(c) If the first light is red, what is the probability that the third light is green?

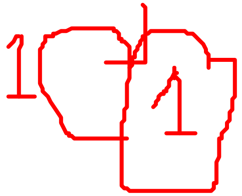
(d) Your classmate Sober has *many* traffic signals between his home and the university. If the first street light is green, what is the probability that the last street light is also green?

CLASSIFICATION OF STATES OF A MARKOV CHAIN

- It is evident that the transition probabilities associated with the states play an important role in the study of Markov chains.
- To further describe the properties of Markov chains, it is necessary to present some concepts and definitions concerning these states.
- State j is said to be **accessible** from state i if $p_{ij}^{(n)} > 0$ for some $n > 0$.
- (Recall that $p_{ij}^{(n)}$ is just the conditional probability of being in state j after n steps, starting in state i.)
- Thus, state j being accessible from state i means that it is possible for the system to enter state j eventually when it starts from state i.

CLASSIFICATION OF STATES OF A MARKOV CHAIN

- In general, a sufficient condition for *all* states to be accessible is that there exists a value of *n* for which $p_{ij}^{(n)} > 0$ for all *i* and *j*.
- If state *j* is accessible from state *i* and state *i* is accessible from state *j*, then states *i* and *j* are said to **communicate.**



- In general,

1. Any state communicates with itself (because $p_{ii}^{(0)} = P\{X_o = i | X_o = i\} = 1$).
2. If state *i* communicates with state *j*, then state *j* communicates with state *i*.

$i \leftrightarrow j$, $j \leftrightarrow k$, $i \leftrightarrow k$

3. If state *i* communicates with state *j* and state *j* communicates with state *k*, then state *i* communicates with state *k*.

- Properties 1 and 2 follow from the definition of states communicating, whereas property 3 follows from the Chapman-Kolmogorov equations.

CLASSIFICATION OF STATES OF A MARKOV CHAIN

- As a result of these three properties of communication,
  - the states may be partitioned into one or more separate **classes**
  - such that those states that communicate with each other are in the same class. (A class may consist of a single state).
- If there is only one class, i.e., all the states communicate, the Markov chain is said to be **irreducible.**

Example Problem 1

Identify the following Markov chain as either irreducible or not irreducible. Give appropriate reason for your answer.

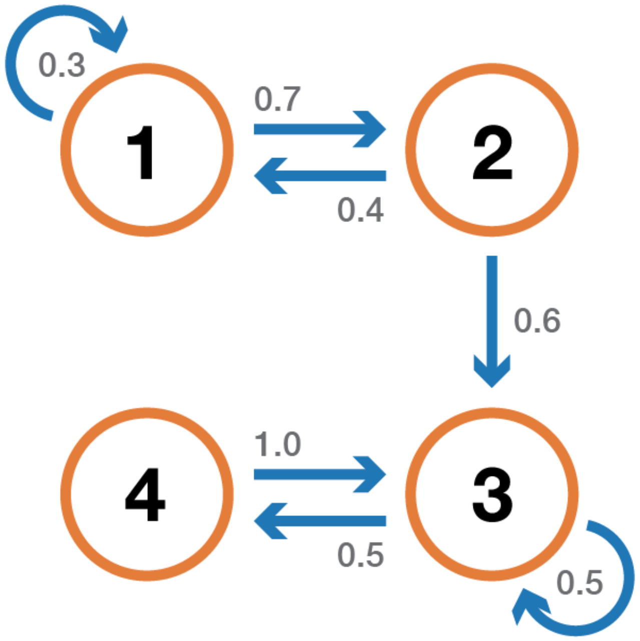
$$P = \begin{array}{c} \begin{matrix} \phantom{\frac{1}{2}} & \phantom{\frac{1}{2}} & 3 & 4 \end{matrix} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

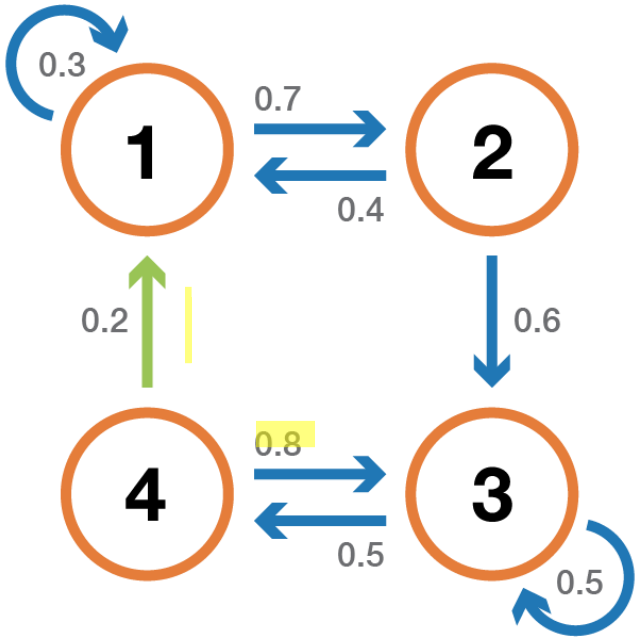
<u>Answer:</u>

The given Markov chain is not irreducible as we can't reach every state for any other state. According to graph theory, the graph of an irreducible Markov chain is strongly connected.

Example Problem 2

Identify the following Markov chains as either irreducible or not irreducible. Give appropriate reason for your answer.





RECURRENT , TRANSIENT & ABSORBING STATES

- A state *i* is said to be a **transient (or non-recurrent)** state if, upon leaving this state, the process *may never return* to this state again.

$i \rightarrow j$

- Therefore, state *i* is transient if and only if there exists a state *j* (*j* ≠ *i*) that is accessible from state *i* but not vice versa, that is, state *i* is not accessible from state *j.*

- Thus, if state *i* is transient and the process visits this state, there is a positive probability (perhaps even a probability of 1) that the process will later move to state *j* and so will never return to state *i.*

- Consequently, a transient state will be visited only a finite number of times.

RECURRENT , TRANSIENT & ABSORBING STATES (Cont'd)

- When starting in state *i,* another possibility is that the process *definitely* will return to this state.
- A state is said to be a **recurrent** state if, upon entering this state, the process *definitely will return* to this state again.
- Therefore, a state is recurrent if and only if it is not transient.
- Since a recurrent state definitely will be revisited after each visit, it will be visited infinitely often if the process continues forever.
- If the process enters a certain state and then stays in this state at the next step, this is considered a *return* to this state.
- Hence, the following kind of state is a special type of recurrent state.
- A state is said to be an **absorbing** state if, upon entering this state, the process *never will leave* this state again.

RECURRENT , TRANSIENT & ABSORBING STATES (Cont'd)

- Therefore, state *i* is an absorbing state if and only if $p_{ii} = 1$.
- Recurrence is a class property.
- That is, all states in a class are either recurrent or transient.
- Furthermore, in a finite-state Markov chain, not all states can be transient.
- Therefore, all states in an irreducible finite-state Markov chain are recurrent.
- Indeed, one can identify an irreducible finite-state Markov chain (and therefore conclude that all states are recurrent) by showing that all states of the process communicate.
- It has already been pointed out that a sufficient condition for *all* states to be accessible (and therefore communicate with each other) is that there exists a value of n for which $\boldsymbol{p_{ij}}^{(n)} > 0$ for all i and j.

Example Problem 3

As another example, suppose that a Markov chain has the following transition matrix:

$$
\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\
\begin{bmatrix}
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} \end{array}
$$

- Note that state 2 is an absorbing state (and hence a recurrent state) because if the process enters state 2 (row 3 of the matrix), it will never leave.
- State 3 is a transient state because if the process is in state 3, there is a positive probability that it will never return.
- The probability is **1/3** that the process will go from state 3 to state 2 on the first step. Once the process is in state 2, it remains in state 2.

Example Problem 3 (Cont'd)

As another example, suppose that a Markov chain has the following transition matrix:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

- State 4 also is a transient state because if the process starts in state 4, it immediately leaves and can never return.
- States 0 and 1 are recurrent states.
- To see this, observe from **P** that if the process starts in either of these states, it can never leave these two states.
- Furthermore, whenever the process moves from one of these states to the other one, it always will return to the original state eventually.