CS-417
COMPUTER SYSTEMS MODELING

Spring Semester 2020

Batch: 2016-17

(LECTURE # 8)

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Recap of Lecture # 7

Review of Probability – Basic Definitions
Binomial Distribution
Geometric Distribution
Modified Geometric Distribution
Negative Binomial Distribution

Chapter # 3 (Cont'd)

REVIEW OF PROBABILITY THEORY

Commonly used Discrete Probability Distributions

5) Poisson Distribution

- This models occurrences of an event, generally regarded as successes (or failures, depending upon the context) in a given time duration.
- The Poisson pmf is given by:

$$p_X(k) = \begin{cases} \dfrac{\alpha^k e^{-\alpha}}{k!}, & k = 0, 1, 2, \cdots \qquad \alpha > 0 \\ 0, & \text{otherwise} \end{cases}$$

- The parameter α is related to time duration *t* as follows:

$$\alpha = \lambda t$$

where, λ is interpreted as the rate of occurring successes.

Commonly used Discrete Probability Distributions

- When *n* is large and *p* is small, the Poisson pmf (distribution) can also be used as a convenient approximation to the binomial pmf (distribution).

$$\binom{n}{k} p^k q^{n-k} \cong \frac{\alpha^k e^{-\alpha}}{k!}, \qquad \alpha = np$$

- As a rule of thumb, we use it when $n \geq 20$ and $p \leq 0.05$

Example

Queries to a database server arrive at a rate of 12/hour. Calculate the probability that:

a) Exactly six queries will arrive in next 30 min?

b) Three or more queries will arrive in next 15 min?

c) Two, three or four queries will arrive in next 5 minutes?

Solution:

$\alpha = \lambda t$, *here* λ = 12/hour

$\alpha = 12t$

Part a)

t = 0.5 hour

Therefore α = 6.

$$P[X=6] = \frac{e^{-\alpha}\alpha^{k}}{k!} = \frac{e^{-6}6^{6}}{6!} = 0.1606$$

Part b)

α = 12x1/4 = 3

P[X≥3] = 1 – P[X < 3]

$$= 1 - \sum_{k=0}^{2} \frac{e^{-\alpha}\alpha^{k}}{k!}$$

$$= 1 - [e^{-\alpha}\ \{1 + \frac{3}{1!} + \frac{3^2}{2!}\}]$$

= 0.5768

Part c)

α = 12x5/60 = 1

P[X=2] + P[X=3] + P[X=4] = 0.2606

Commonly used Discrete Probability Distributions (Cont'd)

6) Hypergeometric Distribution

- Let X be a random variable with Hypergeometric pmf (distribution) giving number of defectives in a random sample drawn *without replacement* from a lot having certain number of defective components.
- The probability of k defectives in a random sample of m components drawn without replacement from a lot of n components having d defectives can be calculated as:

$$h(k; m, n, d) = \frac{\binom{d}{k}\binom{n-d}{m-k}}{\binom{n}{m}}, \qquad k = 0, 1, 2, \cdots, min\{d, m\}$$

Commonly used Discrete Probability Distributions (Cont'd)

7) Uniform Distribution

- Consider a random variable X that can acquire n different values $\{x_1, x_2, \ldots, x_n\}$.
- The variable X is said to be uniformly distributed if:

$$p_X(x_i) = \begin{cases} \frac{1}{n} & i = 1, 2, \cdots, n \\ 0, & \text{otherwise} \end{cases}$$

- plays an important role in the theory of random numbers and its applications to discrete event simulation.
- In the average-case analysis of programs, it is often assumed that the input data are uniformly distributed over the input space.

Commonly used Discrete Probability Distributions (Cont'd)

8) Constant Distribution

- A random variable *X* is said to have constant distribution if:

$$p_X(x) = \begin{cases} 1, & x = c \\ 0, & \text{otherwise} \end{cases}$$

- This means that every sample point maps to just one real number *c*.

Commonly used Discrete Probability Distributions (Cont'd)

9) Indicator Distribution

- Suppose that an event A partitions the sample space S into two mutually exclusive subsets A and $\bar{A}$. The indicator of event A is a random variable $I_A$defined by:

$$I_A(s) = \begin{cases} 1, & \text{if } s \in A \\ 0, & \text{if } s \in \bar{A} \end{cases}$$

- Then event A occurs if and only if I_A = 1. The pmf of I_A is given by:

$$p_{I_A}(0) = P(\bar{A}) = 1 - \mathrm{P}(A)$$

$$p_{I_A}(1) = P(A)$$

Commonly used Discrete Probability Distributions (Cont'd)

10) Multinomial Distribution

- Multinomial pmf is the generalized binomial pmf.
- There are more than two possible outcomes on each trial.
- Let us define a random vector X = (X_1,X_2,... , X_r) such that X_i gives the number of trials resulting in the *i*th outcome. Let p_i be the probability of *i*th outcome. The joint pmf of *X* is given by:

$$p_X(n_1, n_2, \ldots, n_r) = P[X_1 = n_1, X_2 = n_2, \ldots, X_r = n_r]$$
$$= \frac{n!}{n_1! n_2! \ldots n_r!} p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$$

Task

Q.1) Data packets transmitted by a modem over a phone line form a Poisson process at the rate of 10 packets/sec.

- What is the probability that exactly four packets will be transmitted per second?
- What is the probability that more than three packets will be transmitted per second?
- What is the probability that at least four packets will be transmitted in 2 seconds?

Q.2) The probability of error in the transmission of a bit over a communication channel is 10^{-3}. What is the probability of more than three errors in transmitting a block of 1000 bits?

Answers

Q1)

- 0.0189
- 0.9896
- 0.9999

Q2)

- 0.01898

Exponential Distribution (Continuous Distribution)

- It is used to model the time elapsed between events.
- If the number of arrivals at a service facility during a specified period follows Poisson distribution, then,
  - automatically, the distribution of the time interval between successive arrivals must follow the negative exponential (or, simply, exponential) distribution.
- Specifically, if **λ** is the rate at which Poisson events occur, then the distribution of time between successive arrivals, t, **$f(t) = \lambda e^{-\lambda t}, t > 0$**
- The mean and variance of the exponential distribution are:

$E\{t\} = \frac{1}{\lambda}$	$var\{t\} = \frac{1}{\lambda}$

- The mean $E\{t\}$ is consistent with the definition of λ.

- If λ is the rate at which events occur, then $1/\lambda$ is the average time interval between successive events.

Rare Events

- When two events are extremely unlikely to occur simultaneously or within a very short period of time, they are called rare or Poissonian events, as they are modeled using Poisson distribution.
- Job arrivals to a system, telephone calls, e-mail messages, network breakdowns, virus attacks, software errors are examples of rare events.

Inter-arrival Times of Rare Events are Exponential

- Let *N* be a Poisson random variable denoting number of customers arriving to a system in the interval (0, t]. Hence,

$$P_N(k) = \frac{\alpha^k e^{-\alpha}}{k!}$$

- Let *T* be a random variable denoting interarrival time of customers. Then,

$$P[T > t] = P[N = 0] = \frac{\alpha^0 e^{-\alpha}}{0!} = e^{-\alpha}$$

- where the parameter α = λ*t*

$$P[T > t] = e^{-\lambda t}$$

- which shows that interarrival times are exponentially distributed when arrivals occur according to Poisson distribution

Memoryless Property

- Let T be a random variable denoting service time of a server.

- Suppose that a job currently with the server has already consume service time t.

- We are interested in the probability that the job will stay with the server for additional time s; i.e. we wish to calculate the conditional probability P[T > t + s | T > t]

$$P[T > t + s \mid T > t] = \frac{P[T > t + s]}{P[T > t]} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda t}} = \underline{e^{-\lambda s}}$$

This memoryless property is not for students! It is ONLY for exponential distribution!