Lecture 14
Chapter # 5 (Cont'd)
MARKOV CHAINS

Formulating the Weather Example as a Markov Chain

- Recall that the evolution of the weather in Centerville from day to day has been formulated as a stochastic process $\{X_t\}$ ($t = 0, 1, 2, \ldots$) where

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

$$P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6.$$

probability that today is a dry day & tommorow will be a dry day is 0.8
& probability that today is been raining day but tommorow will be a dry day is given by 0.6

- Furthermore, because these probabilities do not change if information about the weather before today (day t) is also taken into account,

$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 0\} = P\{X_{t+1} = 0 \mid X_t = 0\}$$
$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 1\} = P\{X_{t+1} = 0 \mid X_t = 1\}$$

for $t = 0, 1, \ldots$ and every sequence $k_o, k_1, \ldots, k_{t-1}$.

we dont need this complete formula for every time as markov chain states that future value does'nt depend on past value, it depends only on present value. we just need this portion.

- These equations also must hold if $X_{t+1} = 0$ is replaced by $X_{t+1} = 1$.

Weather Example as a Markov Chain

- (The reason is that states 0 and 1 are mutually exclusive and the only possible states, so the probabilities of the two states must sum to 1.)
- Therefore, the stochastic process has the Markovian property, so the process is a Markov chain.
- Using the notation introduced in this section, the (one-step) transition probabilities are

$$p_{00} = P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$p_{10} = P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6$$

for all $t = 1, 2, \ldots,$ so these are *stationary* transition probabilities.

- Furthermore,

$$p_{00} + p_{01} = 1, \qquad \text{so} \qquad p_{01} = 1 - 0.8 = 0.2,$$
$$p_{10} + p_{11} = 1, \qquad \text{so} \qquad p_{11} = 1 - 0.6 = 0.4.$$

- where these transition probabilities are for the transition from the row state to the column state.

Weather Example as a Markov Chain

- Keep in mind that state 0 means that the day is dry, whereas state 1 signifies that the day has rain, so these transition probabilities give the probability of the state
the weather will be in tomorrow, given the state of the weather today.

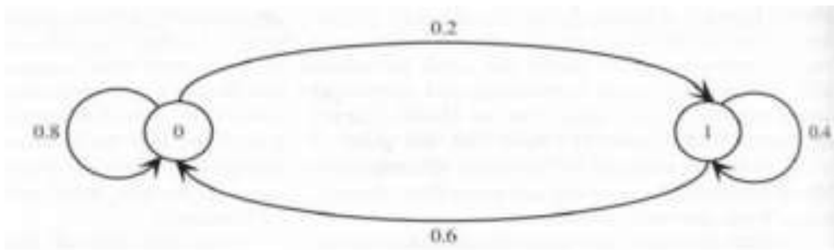
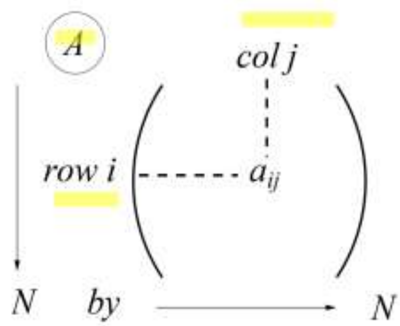


Fig 1: The state transition diagram for the weather example.

- The state transition diagram in Fig 1 graphically depicts the same information provided by the transition matrix.

Matrix Revision

Notation

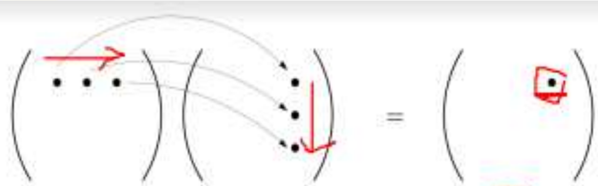


Let A be an $N \times N$ matrix.

We write $A = (a_{ij})$,
i.e. A comprises elements a_{ij}.

The (i, j) element of A is written both as a_{ij} and $(A)_{ij}$
e.g. for matrix A^2 we might write $(A^2)_{ij}$.

Matrix multiplication



Let $A = (a_{ij})$ and $B = (b_{ij})$
be $N \times N$ matrices.

The product matrix is $A \times B = AB$, with elements $(AB)_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$.

Summation notation for a matrix squared

Let A be an $N \times N$ matrix. Then

$$(A^2)_{ij} = \sum_{k=1}^{N} (A)_{ik}(A)_{kj} = \boxed{\sum_{k=1}^{N} a_{ik}a_{kj}.}$$

Pre-multiplication of a matrix by a vector

Let A be an $N \times N$ matrix, and let $\boldsymbol{\pi}$ be an $N \times 1$ column vector: $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_N \end{pmatrix}$.

We can pre-multiply A by $\boldsymbol{\pi}^T$ to get a $1 \times N$ row vector, $\boldsymbol{\pi}^T A = \left((\boldsymbol{\pi}^T A)_1, \ldots, (\boldsymbol{\pi}^T A)_N\right)$, with elements

$$(\boldsymbol{\pi}^T A)_j = \sum_{i=1}^{N} \pi_i a_{ij}.$$

The n-step Transition Probabilities

• Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $S = \{1, 2, \ldots, N\}$.

• Recall that the elements of the transition matrix P are defined as:

$(P)_{ij} = p_{ij} = P(X_1 = j \mid X_0 = i) = P(X_{t+1} = j \mid X_t = i)$ for any t

the future state of this random variable will be j given that its current state is i.

• p_{ij} is the probability of making a transition FROM state i TO state j in a SINGLE step.

Question: what is the probability of making a transition from state i to state j over two steps?

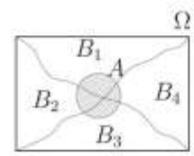
i.e. what is $P(X_2 = j \mid X_0 = i)$?

It means that there must be some intermediate value of random variable at time t=1. i.e. $X_1=k$

means the process the system was at initially at state i then it jumps to state k & then it jumps to state j.

So we can calculate its value by means of 2 step transition probability.

The Partition Theorem



Let $B_1, \ldots, B_m$ form a partition of Ω. Then for any event A,

$$\boxed{\mathbb{P}(A) = \sum_{i=1}^{m} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{m} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}$$

Both formulations of the Partition Theorem are very widely used, but especially the conditional formulation $\sum_{i=1}^{m} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$.

DERIVATION OF 2-STEP TRANSITION PROBABILITY

lecture

We are seeking $\mathbb{P}(X_2 = j \mid X_0 = i)$. Use the *Partition Theorem:*

$$\begin{aligned}
\mathbb{P}(X_2 = j \mid X_0 = i) &= \mathbb{P}_i(X_2 = j) \\
&= \sum_{k=1}^{N} \mathbb{P}_i(X_2 = j \mid X_1 = k)\mathbb{P}_i(X_1 = k) \quad \textit{(Partition Thm)} \\
&= \sum_{k=1}^{N} \mathbb{P}(X_2 = j \mid X_1 = k,\ X_0 = i)\mathbb{P}(X_1 = k \mid X_0 = i) \\
&= \sum_{k=1}^{N} \mathbb{P}(X_2 = j \mid X_1 = k)\mathbb{P}(X_1 = k \mid X_0 = i) \\
&= \sum_{k=1}^{N} p_{kj} p_{ik} \\
&= \sum_{k=1}^{N} p_{ik} p_{kj} \\
&= (P^2)_{ij}.
\end{aligned}$$

The two-step transition probabilities are therefore given by the matrix P^2

$$P(X_2 = j \mid X_0 = i) = P(X_{t+2} = j \mid X_t = i) = (P_2)_{ij} \text{ for any } t$$

They are also termed as Chapman-Kolmogorov equations & provide a method for computing these n-step transition probabilities.

Can you now derive the formula for 3-Steps Transition ?

General Case: n-Step Transition Matrix

• General case: n-step transitions

• The above working extends to show that the n-step transition probabilities are given by the matrix P^n for any n:

$$P(X_t = j \mid X_0 = i) = P(X_{t+n} = j \mid X_t = i) = P^n_{ij} \text{ for any n}$$

• Thus, the n-step transition probability matrix P^n can be obtained by computing the nth power of the one-step transition matrix P.

• ***A Markov Chain is said to be homogenous if all transitions are independent of t.***
It means that if the transition operator for the markov chain does not change across the transition the markov chain is called the time homogenous.
This is a nice property of the time homogenous markov chain that the chain runs for the long time & ultimately the chain will reach an equilibrium that is called the chain stationary distribution.We will assume that all the given transitional probabilities are stationary & they are not going to change only if my markov chain is homogenous or the transitions are independant of time or the transition are not going to change with time only then i can compute the nth step transition matrix. If initial transitional probabilities values are subject to be change then we can not compue the future values.

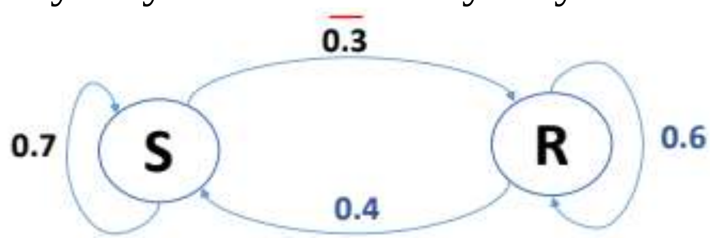
Example Problem
In some town, each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7 whereas rainy day is followed by a sunny day with probability 0.4. It rains on Monday.
Make forecast for Tuesday, Wednesday & Thursday.

If monday is the present state then for tuesday values can be calulated by single step transition probability. & for wednesday values can be calulated by 2 step transition probability & for thursday values can be calulated by 3 step transition probability

Solution:

2-state homogenous Markov Chain is given by the following state transition diagram.

State S for Sunny Day and R for Rainy Day:



Forecast for Tuesday:

$p_{RR} = 1 - 0.4 = 0.6$; will be rain on tuesday

$p_{SR} = 1 - 0.6 = 0.4$; will be sunny day on tuesay

Forecast for Wednesday:

1st Method: Calculating 2-step Transition Probability:

$P^{(2)}_{RS} = ?$

$= P\,[X(2) = S, X(1) = S \mid X(0) = R] + P\,[X(2) = S, X(1) = R \mid X(0) = R]$

$= p_{RS}\, p_{SS} + p_{RR}\, p_{RS}$; we are only interested in the value of present state. X(0) is representing that on monday it was raining.

$= 0.4 * 0.7 + 0.6 * 0.4$

$=0.52$

$P^{(2)}_{RR} = 1 - 0.52 = 0.48$

2nd Method: Calculating 2-step Transition Probability:

get the transitional probability matrix mark the states & simply multiply this matrix by itself to get the result.

$$\begin{array}{c|cc} & S & R \\ \hline S & 0.7 & 0.3 \\ R & 0.4 & 0.6 \end{array}$$

$$P^{(2)} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} * \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

$P_{RS}=0.52$ & $P_{RR}=0.48$

Forecast for Thursday: $P^{(3)}_{RS}$ & $P^{(3)}_{RR}$??

$$P^3 = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix} \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{array}{c} \\ S \\ R \end{array} \begin{array}{cc} S & R \\ \hline 0.583 & 0.417 \\ 0.556 & 0.444 \end{array}$$

Forecast for thursday:

$P^3_{RS} = 0.556$

$P^3_{RR} = 0.444$

Answers:

$P^{(3)}_{RS} = 0.556$

$P^{(3)}_{RR} = 0.444$

Task

A computer system is operating in one of the two modes: restricted and unrestricted. The mode is observed every hour. The probability that it will remain in the same mode during the next hour is 0.3.

(a) Modeling it as a Markov chain, draw its state transition diagram.
(b) Write down the transition probability matrix.
(c) Compute the 3-step transition probability matrix.

(d) If the system is operating in restricted mode at 3pm, what is the probability that it will be in the unrestricted mode at 6pm on the same day?

solve d

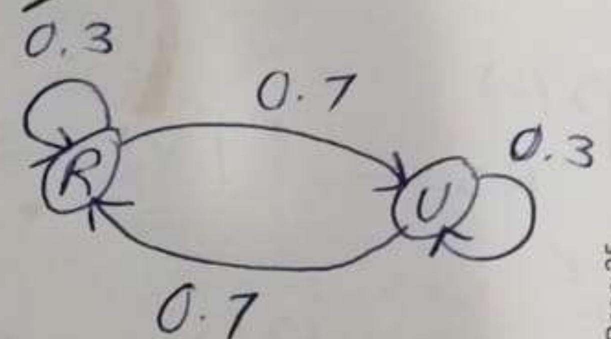
$P_{RR} = 0.3$

$P_{UU} = 0.3$

b)

$$\begin{array}{c|cc} & R & U \\ \hline R & 0.3 & 0.7 \\ U & 0.7 & 0.3 \end{array}$$

a)



c)

$$P^3 = \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix}^3$$

$$= \begin{bmatrix} 0.3 & 0.7 \\ 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.58 & 0.42 \\ 0.42 & 0.58 \end{bmatrix}$$