CS-417
COMPUTER SYSTEMS MODELING

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(LECTURE # 13)

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Recap of Lecture # 12

System Availability

Fault, Error and Failures

Software Reliability Vs Hardware Reliability

Reliability Metrics

Reliability Validation

Chapter # 5

MARKOV CHAINS

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STOCHASTIC PROCESSES

Processes that evolve over time in a probabilistic manner.

Mathematically, a **stochastic process** is defined to be an indexed collection of random variables $\{X_t\}$, where the index t runs through a given set T.

- Often T is taken to be the set of non-negative integers, and X_t represents a measurable characteristic of interest at time t.
- e.g., X_t might represent the inventory level of a particular product at the end of week t.
- Stochastic processes are of interest for describing the behavior of a system operating over some period of time.
- State: any one of M + 1 mutually exclusive categories or states possible. For notational convenience, states are labeled 0, 1, …, M.

STRUCTURE OF STOCHASTIC PROCESSES

- Let $\{X_t, t = 0, 1, 2, \ldots, \}$ be a stochastic process that takes on a finite or countable number of possible values.
- This set of possible values of the process will be denoted by the set of nonnegative integers $\{0, 1, 2, \ldots\}$
- If $X_t = i$, then the process is said to be in state i at time t. $X_4 = 3$
- We suppose that whenever the process is in state i, there is a fixed probability P_{ij} that it will next be in state j. $i \xrightarrow{P_{ij}} j$

$$P\{X_{t+1} = j \mid X_t = i, X_{t-1} = i_{t-1}, \ldots, X_1 = i_1, X_0 = i_0\} = P_{ij}$$

for all states $i_0, i_1, \ldots, i_{t-1}, i, j$ and all $t > 0$

STOCHASTIC PROCESSES - Example

- The weather in the town of Centerville can change rather quickly from day to day.
- However, the chances of being dry (no rain) tomorrow are somewhat larger if it is dry today than if it rains today.
- In particular, the probability of
  - being dry tomorrow is **0.8** if it is dry today,
  - but is only **0.6** if it rains today.
- These probabilities do not change if information about the weather before today is also taken into account.
- The evolution of the weather from day to day in Centerville is a stochastic process.
- Starting on some initial day (labeled as day 0), the weather is observed on each day t, for $t = 0, 1, 2, \ldots$.

- The state of the system on day t can be either

$$\text{State } 0 = \text{Day } t \text{ is dry}$$

or

$$\text{State } 1 = \text{Day } t \text{ has rain.}$$

- Thus, for t = 0, 1,2, …, the random variable X_t takes on the values,

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

MARKOV CHAINS

Markov chain: A stochastic process $\{X_t\}$ (t =0, 1, …) with *Markovian property.*

Markovian property says that the conditional probability of any future "event," given any past "event" and the present state $X_t = i$, is ***independent*** of the past event and ***depends only*** upon the present state.

The conditional probabilities $P\{X_{t+1} = j|X_t = i\}$ for a Markov chain are called **one-step transition probabilities.**

- If, for each i and j,

$$P\{X_{t+1} = j|X_t = i\} = P\{X_1 = j|X_o = i\} \text{ for all } t = 0, 1, 2, \ldots,$$

  - then the (one-step) transition probabilities are said to be *stationary.*
  - implies that the transition probabilities do not change over time.

MARKOV CHAINS (Cont'd)

- The existence of stationary (one-step) transition probabilities also implies that, for each i, j, and n (n = 0, 1, 2, …),

$$P\{X_{t+n} = j \mid X_t = i\} = P\{X_n = j \mid X_o = i\}$$

$$\text{for all } t = 0, 1, \ldots.$$

- These conditional probabilities are called ***n*-step transition probabilities.**
- To simplify notation with stationary transition probabilities, let
  - $\boldsymbol{p_{ij} = P\{X_{t+1} = j \mid X_t = i\}}$
  - $\boldsymbol{p_{ij}^{(n)} = P\{X_{t+n} = j \mid X_t = i\}}$
- Thus, the n-step transition probability $\boldsymbol{p_{ij}^{(n)}}$ is just the conditional probability that the system will be in state j after exactly n steps (time units), given that it starts in state i at any time t.
- When n = 1, note that $\boldsymbol{p_{ij}^{(1)} = p_{ij}}$.

MARKOV CHAINS (Cont'd)

- Because the $p_{ij}^{(n)}$ are conditional probabilities, they must be nonnegative, and since the process must make a transition into some state, they must satisfy the properties:

$$p_{ij}^{(n)} \geq 0, \qquad \text{for all } i \text{ and } j;\ n = 0, 1, 2, \ldots,$$

and

$$\sum_{j=0}^{M} p_{ij}^{(n)} = 1 \qquad \text{for all } i;\ n = 0, 1, 2, \ldots.$$

- A convenient way of showing all the n-step transition probabilities is the *n-step transition matrix.*

$$\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\ \begin{bmatrix} p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\ p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)} \end{bmatrix} \end{array}$$

MARKOV CHAINS (Cont'd)

$$
\begin{array}{cc}
 & \begin{array}{cccc} 0 & 1 & \cdots & M \end{array} \\
\mathbf{P}^{(n)} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ \vdots \\ M \end{array} &
\left[\begin{array}{cccc}
p_{00}^{(n)} & p_{01}^{(n)} & \cdots & p_{0M}^{(n)} \\
p_{10}^{(n)} & p_{11}^{(n)} & \cdots & p_{1M}^{(n)} \\
\cdots & \cdots & \cdots & \cdots \\
p_{M0}^{(n)} & p_{M1}^{(n)} & \cdots & p_{MM}^{(n)}
\end{array}\right]
\end{array}
$$

Structure of Transition Matrix

- Note that the transition probability in a particular row and column is for the transition *from* the row state *to* the column state.
- When $n = 1$, we drop the superscript n and simply refer to this as the *transition matrix.*
- The Markov chains to be considered have following properties:-
  1. A finite number of states.
  2. Stationary transition probabilities.
- We also will assume that we know the initial probabilities $P\{X_o = i\}$ for all i.

Example 1 (Forecasting the weather)

Suppose that the chance of rain tomorrow depends on previous weather conditions only through whether or not it is raining today and not on past weather conditions. Suppose also that if it rains today, then it will rain tomorrow with probability α; and if it does not rain today, then it will rain tomorrow with probability β.

If we say that the process is in state 0 when it rains and state 1 when it does not rain, then the preceding is a two-state Markov chain whose transition probabilities are given by:

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \left\| \begin{array}{cc} \alpha & 1-\alpha \\ \beta & 1-\beta \end{array} \right\| \end{array}$$

$P_{00} = \alpha$

$P_{01} = 1-\alpha$

Example 2 (Gary's Mood)

On any given day Gary is either cheerful (C), so-so (S), or glum (G). If he is cheerful today, then he will be C, S, or G tomorrow with respective probabilities 0.5, 0.4, 0.1. If he is feeling so-so today, then he will be C, S, or G tomorrow with probabilities 0.3, 0.4, 0.3. If he is glum today, then he will be C, S, or G tomorrow with probabilities 0.2, 0.3, 0.5.

Letting X_n denote Gary's mood on the nth day, then $\{X_n, n \geq 0\}$ is a three-state Markov chain (state 0 = C, state 1 = S, state 2 = G) with transition probability matrix:

$$\mathbf{P} = \begin{array}{cc} & \begin{array}{ccc} 0 & \quad 1 & \quad 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{Vmatrix} 0.5 & 0.4 & 0.1 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.3 & 0.5 \end{Vmatrix} \end{array} \begin{array}{c} = 1 \\ = 1 \\ = 1 \end{array}$$