CS-417
COMPUTER SYSTEMS MODELING

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(LECTURE # 17)

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Recap of Lecture # 16

Classification of States of Markov Chain

Re-current, Transient & Absorbing States

Example Problems

Chapter # 5 (Cont'd)

MARKOV CHAINS

PERIODICITY PROPERTIES

- Another useful property of Markov chains is *periodicities.*

- A state *i* is said to be **periodic** with period d_i , if on leaving state *i* return is possible only in a number of steps that is multiple of the integer $d_i > 1$. If the value of $d_i = 1$, state *i* is aperiodic.

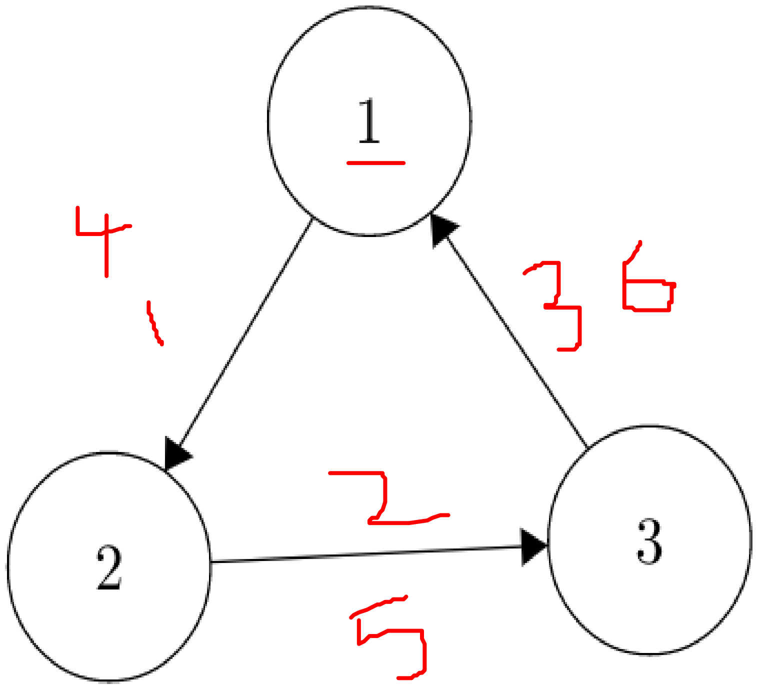
- Each state in a Markov Chain has a period. The period is defined as the greatest common denominator (divisor) of the length of return trips (i.e., number of steps it takes to return), given that you start in that state.

Example Problem 4

Consider the given Markov Chain with 3 states. Determine its period.

Solution:

- We can exactly predict it's movement over time (that is, it is deterministic). Imagine first that we start in State 1.
- We know with certainty that in the next step we go to State 2, and then State 3, and then back to State 1. In fact, we know how long every possible path of return to State 1 is: 3 steps, 6 steps, 9 steps, etc.
- The greatest common devisor of these path lengths is 3, so State 1 (and the other two states) have period 3.



PERIODICITY PROPERTIES (Cont'd)

- Just as recurrence is a class property, it can be shown that periodicity is a class property.

- That is, if state *i* in a class has period *t*, then all states in that class have period *t.*

$d_i = 1$

- In a finite-state Markov chain, recurrent states that are aperiodic are called **ergodic** states.

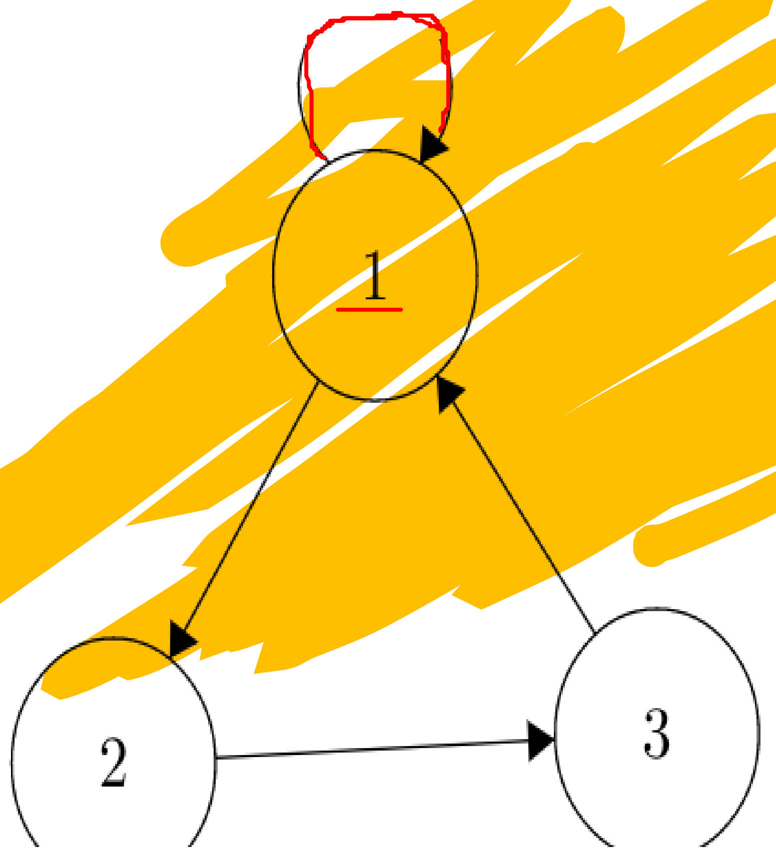
- A Markov chain is said to be *ergodic* if all its states are ergodic states.

Example Problem 5

Consider the given Markov Chain with 3 states. Determine its period.

Solution:

- If we start in State 1, we could automatically just loop back after one period, but we could also go through the whole chain (to State 2, then 3, then 1) in 3 steps.
- So two possible return lengths are 1 and 3 (there are more, of course), and already the greatest common denominator of these path lengths is 1 So State 1 has period one.
- What about State 2? If we start at State 2, we know we will go to State 3 and then State 1. However, there we could stay in State 1 for any period of time before returning to State 2.
- We could return in 3, 4, 5, etc. number of steps. Again, the greatest common denominator here is 1, so State 2 also has period 1. You can check, but the same holds for State 3.



$3 \to 1 \to 2$

$3 \to 1 \to 1 \to 2$

Example Problem 6

Consider the Markov chain that has the following (one-step) transition matrix:

$$
\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{bmatrix}
0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0
\end{bmatrix} \end{array}
$$

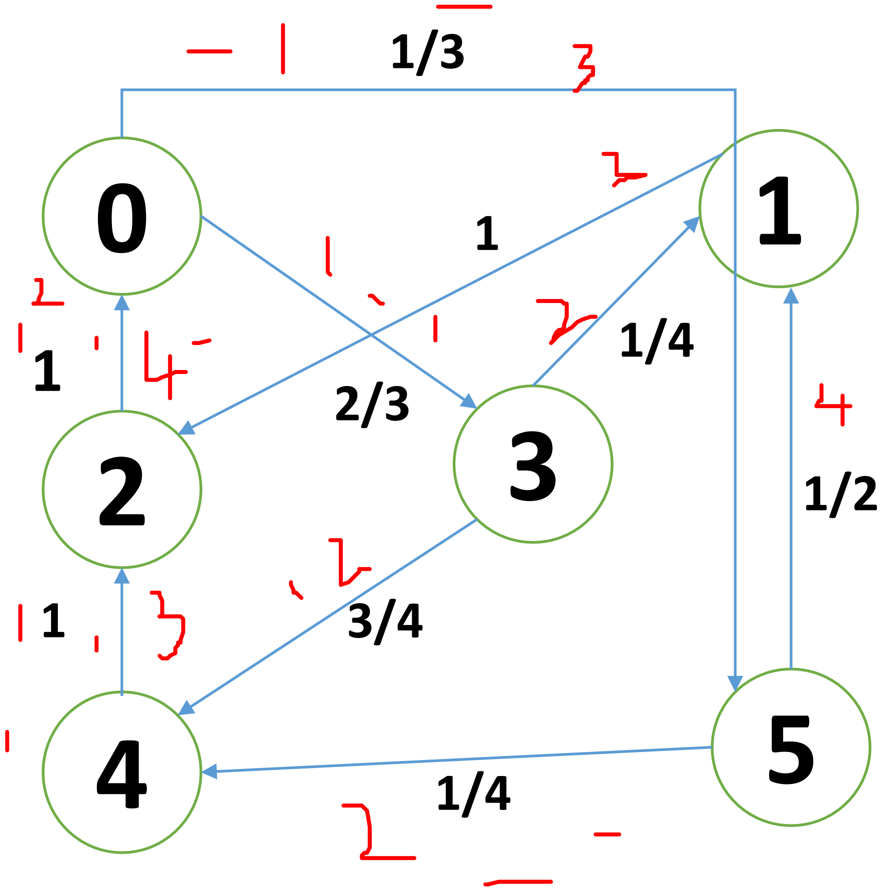
a) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.

b) For each of the classes, determine the period of the states in that class.

c) Also determine whether the given Markov chain is ergodic or not? Provide proper reason.

Solution:

$$\mathbf{P} = \begin{array}{c c} \text{State} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$



First of all draw the state transition diagram.

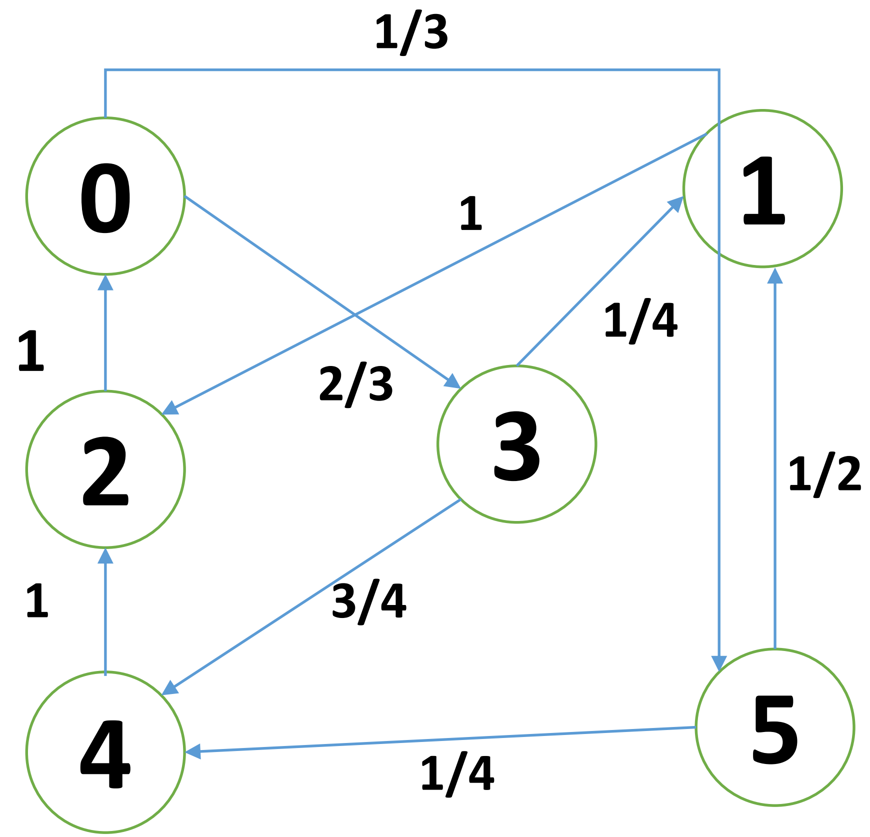
a) All the states are re-current.

b) Period for the states is 4.

c) The chain is not ergodic since the chain is not aperiodic.

Solution:

$$\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{bmatrix} \end{array}$$



First of all draw the state transition diagram.

a) All the states are re-current.

b) Period for the states is 4.

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Task

Consider the Markov chain that has the following (one-step) transition matrix:

$$
\mathbf{P} = \begin{array}{c} \text{State} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}
\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & \frac{4}{5} & 0 & \frac{1}{5} & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{10} & \frac{2}{5} \\ 0 & 0 & 0 & 1 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix} \end{array}
$$

a) Determine the classes of this Markov chain and, for each class, determine whether it is recurrent or transient.

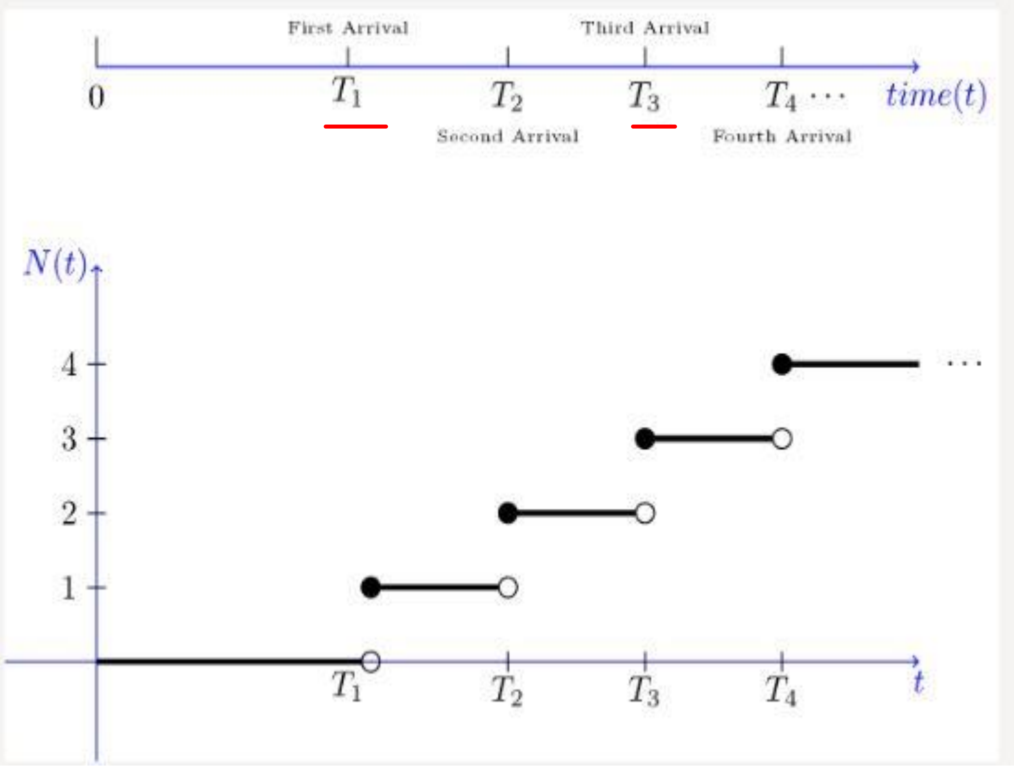
b) For each of the classes, determine the period of the states in that class.

c) Also determine whether the given Markov chain is ergodic or not? Provide proper reason.

d) Is there any absorbing state in the chain ?

Counting Processes (Example of Stochastic Processes)

- In some problems, we count the occurrences of some types of events. In such scenarios, we are dealing with a *counting process*.
- For example, a random process N(t) shows the number of customers who arrive at a supermarket by time t starting from time 0.
- For such a processes, we usually assume N(0)=0, so as time passes and customers arrive, N(t) takes positive integer values.



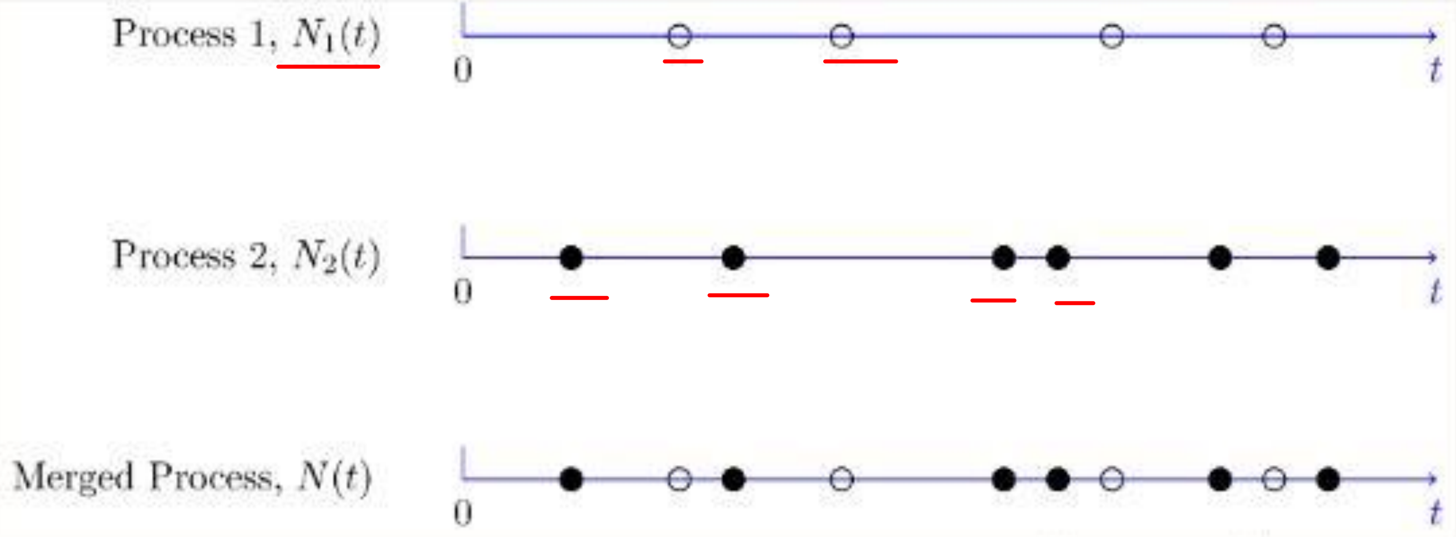
Poisson Process as a Counting Process

- The Poisson process is one of the most widely-used counting processes.
- It is usually used in scenarios where we are counting the occurrences of certain events that appear to happen at a certain rate, but completely at random (without a certain structure).
- For example, suppose that from historical data, we know that earthquakes occur in a certain area with a rate of 2 per month. Other than this information, the timings of earthquakes seem to be completely random.
- Thus, we conclude that the Poisson process might be a good model for earthquakes.

Merging of Poisson Processes

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let us define $N(t) = N_1(t) + N_2(t)$. That is, the random process $N(t)$ is obtained by combining the arrivals in $N_1(t)$ and $N_2(t)$ [redacted]. We claim that $N(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$. To see this, first note that

$$N(0) = N_1(0) + N_2(0)$$
$$= 0 + 0 = 0.$$



Merging of Poisson Processes (Cont'd)

Next, since $N_1(t)$ and $N_2(t)$ are independent and both have independent increments, we conclude that $N(t)$ also has independent increments. Finally, consider an interval of length τ, i.e, $I = (t, t+\tau]$. Then the numbers of arrivals in I associated with $N_1(t)$ and $N_2(t)$ are $Poisson(\lambda_1 \tau)$ and $Poisson(\lambda_2 \tau)$ and they are independent. Therefore, the number of arrivals in I associated with $N(t)$ is $Poisson((\lambda_1 + \lambda_2)\tau)$ (sum of two independent Poisson random variables).

<u>Merging Independent Poisson Processes</u>

Let $N_1(t), N_2(t), \cdots, N_m(t)$ be m independent Poisson processes with rates $\lambda_1, \lambda_2, \cdots, \lambda_m$. Let also

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t), \quad \text{for all } t \in [0, \infty).$$

Then, $N(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2 + \cdots + \lambda_m$.

Splitting of Poisson Processes

Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way [redacted]. For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process ($N_1(t)$), otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,