CS-417
COMPUTER SYSTEMS MODELING

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(LECTURE # 14)

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Recap of Lecture # 13

Stochastic Processes

Structure of Stochastic Processes

Markov Chains

Examples

Chapter # 5 (Cont'd)

MARKOV CHAINS

Formulating the Weather Example as a Markov Chain

- Recall that the evolution of the weather in Centerville from day to day has been formulated as a stochastic process $\{X_t\}$ (t = 0, 1,2, ...) where

$$X_t = \begin{cases} 0 & \text{if day } t \text{ is dry} \\ 1 & \text{if day } t \text{ has rain.} \end{cases}$$

$$P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6.$$

- Furthermore, because these probabilities do not change if information about the weather before today (day t) is also taken into account,

$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 0\} = P\{X_{t+1} = 0 \mid X_t = 0\}$$
$$P\{X_{t+1} = 0 \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = 1\} = P\{X_{t+1} = 0 \mid X_t = 1\}$$

for t = 0, 1, ... and every sequence k_o, k_1, ... , k_{t-1}.

- These equations also must hold if $X_{t+1} = 0$ is replaced by $X_{t+1} = 1$.

Weather Example as a Markov Chain

- (The reason is that states 0 and 1 are mutually exclusive and the only possible states, so the probabilities of the two states must sum to 1.)
- Therefore, the stochastic process has the *Markovian property,* so the process is a Markov chain.
- Using the notation introduced in this section, the (one-step) transition probabilities are

$$p_{00} = P\{X_{t+1} = 0 \mid X_t = 0\} = 0.8,$$
$$p_{10} = P\{X_{t+1} = 0 \mid X_t = 1\} = 0.6$$

for all $t = 1, 2, \ldots$, so these are *stationary* transition probabilities.

- Furthermore,

$$p_{00} + p_{01} = 1, \quad \text{so} \quad p_{01} = 1 - 0.8 = 0.2,$$
$$p_{10} + p_{11} = 1, \quad \text{so} \quad p_{11} = 1 - 0.6 = 0.4.$$

- where these transition probabilities are for the transition *from* the row state *to* the column state.

Weather Example as a Markov Chain

- Keep in mind that state 0 means that the day is dry, whereas state 1 signifies that the day has rain, so these transition probabilities give the probability of the state the weather will be in tomorrow, given the state of the weather today.

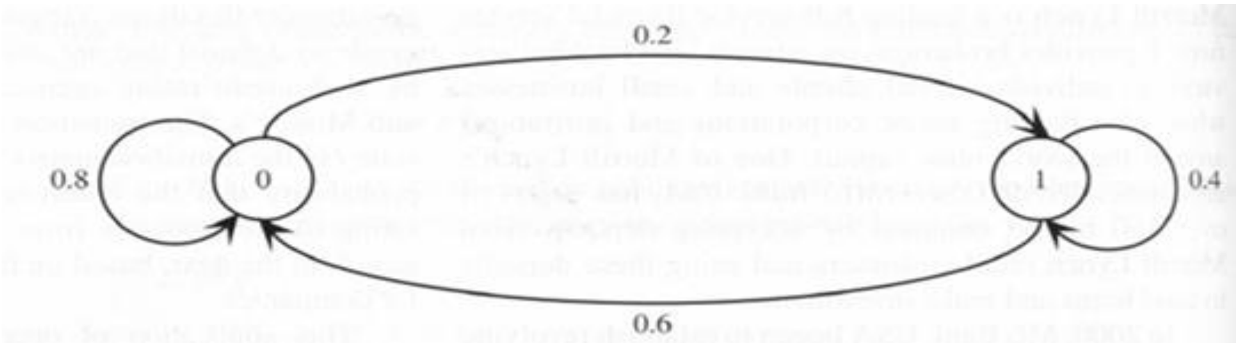


Fig 1: The state transition diagram for the weather example.

- The state transition diagram in Fig 1 graphically depicts the same information provided by the transition matrix.

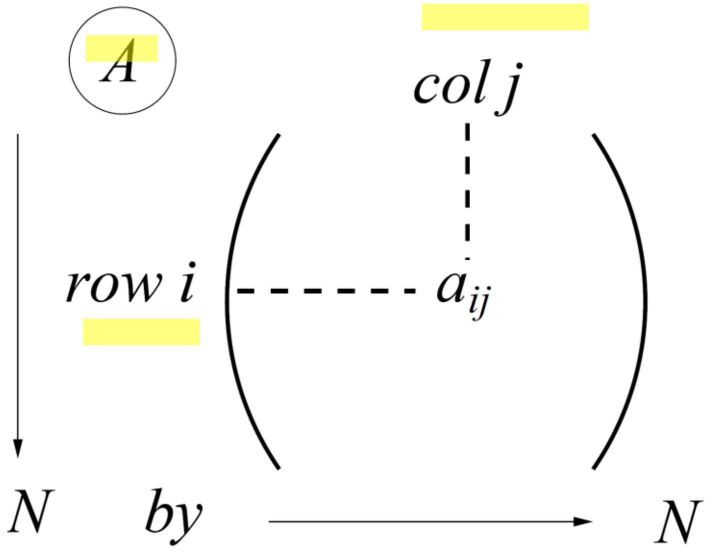
Matrix Revision

Notation

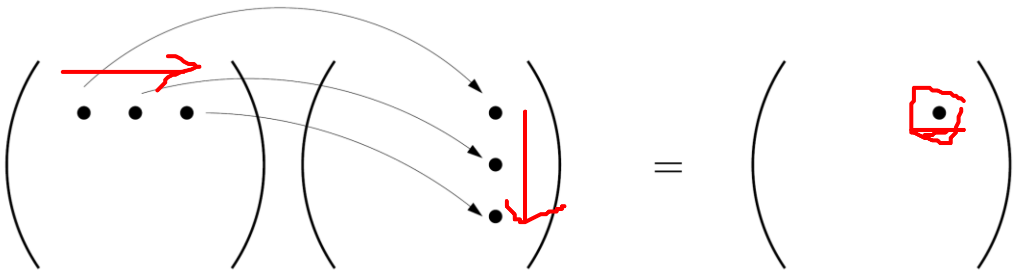
Let A be an $N \times N$ matrix.

We write $A = (a_{ij})$,
i.e. A comprises elements a_{ij}.

The (i, j) element of A is written both as a_{ij} and $(A)_{ij}$
e.g. for matrix A^2 we might write $(A^2)_{ij}$.



Matrix multiplication



Let $A = (a_{ij})$ and $B = (b_{ij})$ be $N \times N$ matrices.

The product matrix is $A \times B = AB$, with elements $(AB)_{ij} = \sum_{k=1}^{N} a_{ik} b_{kj}$.

Summation notation for a matrix squared

Let A be an $N \times N$ matrix. Then

$$(A^2)_{ij} = \sum_{k=1}^{N} (A)_{ik}(A)_{kj} = \sum_{k=1}^{N} a_{ik} a_{kj}.$$

Pre-multiplication of a matrix by a vector

Let A be an $N \times N$ matrix, and let $\boldsymbol{\pi}$ be an $N \times 1$ column vector: $\boldsymbol{\pi} = \begin{pmatrix} \pi_1 \\ \vdots \\ \pi_N \end{pmatrix}$.

We can pre-multiply A by $\boldsymbol{\pi}^T$ to get a $1 \times N$ row vector, $\boldsymbol{\pi}^T A = \left((\boldsymbol{\pi}^T A)_1, \ldots, (\boldsymbol{\pi}^T A)_N\right)$, with elements

$$(\boldsymbol{\pi}^T A)_j = \sum_{i=1}^{N} \pi_i a_{ij}.$$

The n-step Transition Probabilities

- Let $\{X_0, X_1, X_2, \ldots\}$ be a Markov chain with state space $S = \{1, 2, \ldots, N\}$.

- Recall that the elements of the transition matrix P are defined as:

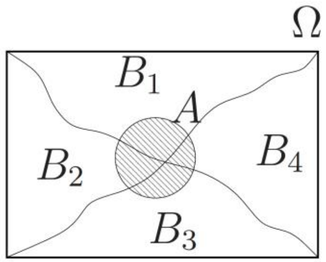
$$(P)_{ij} = p_{ij} = P(X_1 = j \,|X_0 = i) = P(X_{t+1} = j \,|X_t = i) \; \textit{for any } t$$

- p_{ij} is the probability of making a transition FROM state i TO state j in a SINGLE step.

Question: what is the probability of making a transition from state i to state j over two steps? i.e. what is $P(X_2 = j \,|X_0 = i)$?

$X_1 = k$

The Partition Theorem



Let $B_1, \ldots, B_m$ form a partition of Ω. Then for any event A,

$$\boxed{\mathbb{P}(A) = \sum_{i=1}^{m} \mathbb{P}(A \cap B_i) = \sum_{i=1}^{m} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)}$$

Both formulations of the Partition Theorem are very widely used, but especially the conditional formulation $\sum_{i=1}^{m} \mathbb{P}(A \mid B_i)\mathbb{P}(B_i)$.

DERIVATION OF 2-STEP TRANSITION PROBABILITY

We are seeking $\mathbb{P}(X_2 = j \mid X_0 = i)$. Use the *Partition Theorem:*

$$
\begin{aligned}
\mathbb{P}(X_2 = j \mid X_0 = i) \quad &= \quad \mathbb{P}_i(X_2 = j) \\
&= \sum_{k=1}^{N} \mathbb{P}_i(X_2 = j \mid X_1 = k)\mathbb{P}_i(X_1 = k) \qquad \textit{(Partition Thm)} \\
&= \sum_{k=1}^{N} \mathbb{P}(X_2 = j \mid X_1 = k,\ X_0 = i)\mathbb{P}(X_1 = k \mid X_0 = i) \\
&= \sum_{k=1}^{N} \mathbb{P}(X_2 = j \mid X_1 = k)\mathbb{P}(X_1 = k \mid X_0 = i) \\
&= \sum_{k=1}^{N} p_{kj}p_{ik}
\end{aligned}
$$

$$= \sum_{k=1}^{N} p_{ik} p_{kj}$$

$$= (P^2)_{ij}.$$

The two-step transition probabilities are therefore given by the matrix P^2

$$P(X_2 = j \mid X_0 = i) = P(X_{t+2} = j \mid X_t = i) = (P^2)_{ij} \textbf{ for any } t$$

They are also termed as Chapman-Kolmogorov equations & provide a method for computing these n-step transition probabilities.

Can you now derive the formula for 3-Steps Transition ?

General Case: n-Step Transition Matrix

- General case: n-step transitions

- The above working extends to show that the n-step transition probabilities are given by the matrix P^n for any n:

$$P(X_t = j \mid X_0 = i) = P(X_{t+n} = j \mid X_t = i) = P^n_{ij} \text{ for any n}$$

- Thus, the *n*-step transition probability matrix $\mathbf{P}^n$ can be obtained by computing the *n*th power of the one-step transition matrix **P.**
- **A Markov Chain is said to be homogenous if all transitions are independent of t.**

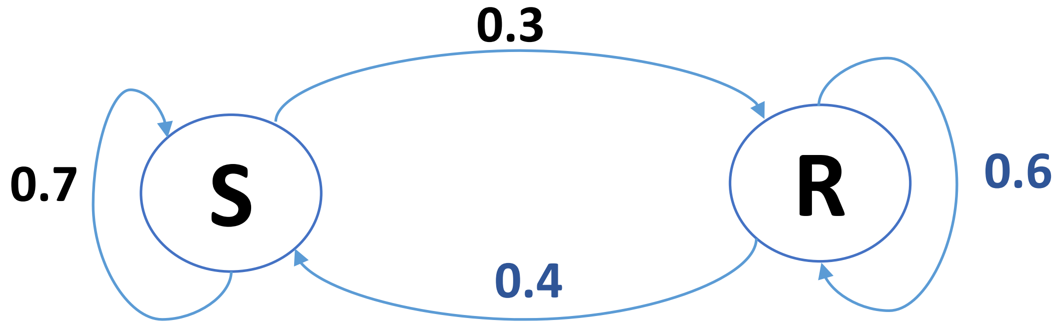
Example Problem

In some town, each day is either sunny or rainy. A sunny day is followed by another sunny day with probability 0.7 whereas rainy day is followed by a sunny day with probability 0.4.

It rains on Monday. Make forecast for Tuesday, Wednesday & Thursday.

Solution:

2-state homogenous Markov Chain is given by the following state transition diagram. State S for Sunny Day and R for Rainy Day:



Forecast for Tuesday: $p_{RR} = 1 - 0.4 = 0.6$

$p_{SR} = 1 - 0.6 = 0.4$

Forecast for Wednesday:

1st Method: Calculating 2-step Transition Probability:

$P^{(2)}_{RS} = ?$

$= P\,[X(2) = S, X(1) = S \mid X(0) = R] + P\,[X(2) = S, X(1) = R \mid X(0) = R]$

$= p_{RS}\, p_{SS} + p_{RR}\, p_{RS}$

$= 0.4 * 0.7 + 0.6 * 0.4$

$= 0.52$

$P^{(2)}_{RR} = 1 - 0.52 = 0.48$

2nd Method: Calculating 2-step Transition Probability:

$$\begin{array}{c} \\ S \\ R \end{array}\begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

$$P^{(2)} = \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} * \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix} = \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

Forecast for Thursday: $P^{(3)}_{RS}$ & $P^{(3)}_{RR}$??

Answers:

$P^{(3)}_{RS} = 0.556$

$P^{(3)}_{RR} = 0.444$

Task

A computer system is operating in one of the two modes: *restricted* and *unrestricted*. The mode is observed every hour. The probability that it will remain in the same mode during the next hour is 0.3.

(a) Modeling it as a Markov chain, draw its state transition diagram.

(b) Write down the transition probability matrix.

(c) Compute the 3-step transition probability matrix.

(d) If the system is operating in restricted mode at 3pm, what is the probability that it will be in the unrestricted mode at 6pm on the same day?