$\Rightarrow x_n$ convergent. $\left(\frac{1}{2} < 1 \Rightarrow$ CC raport $\Rightarrow \sum_n x_n$ e conv $\right)$

Seminar 2

$\boxed{1}$ Det $\overline{\lim} x_n$ și $\underline{\lim} x_n$, precizând dacă există limita lui x_n, unde

ⓐ $x_n = 1 + 2 \cdot (-1)^{n+1} + 3(-1)^{\frac{n(n+1)}{2}}$ $(\forall) n \in \mathbb{N}$

Sol

I $\cancel{n = 4k}$ $x_{4n} = 1 + 2 \cdot (-1)^{4n+1} + 3(-1)^{\frac{4n(4n+1)}{2}}$

$= 1 + 2 \cdot (-1) + 3 = 2 \xrightarrow{n\to\infty} 2$

II $x_{4n+1} = 1 + 2 \cdot (-1)^{4n+2} + 3(-1)^{\frac{(4n+1)(4n+2)}{2}}$

$= 1 + 2 + 3(-1)^{\frac{(4n+1) \cdot 2(2n+1)}{2}}$

$= 3 + (-3) = 0 \xrightarrow{n\to\infty} 0$

III $x_{4n+2} = 1 + 2 \cdot (-1)^{4n+3} + 3(-1)^{\frac{(4n+2)(4n+3)}{2}}$

$= 1 - 2 - 3 = -4 \xrightarrow{n\to\infty} -4$

IV $x_{4n+3} = 1 + 2(-1)^{4(n+1)} + 3(-1)^{\frac{(4n+3)(4n+4)}{2}}$

$= 1 + 2 + 3 = 6 \xrightarrow{n\to\infty} 6$

$\mathbb{N} = 4\mathbb{N} \cup (4\mathbb{N}+1) \cup 4\mathbb{N}+2 \cup (4\mathbb{N}+3)$

$\Rightarrow \cancel{L}(x \quad \mathcal{L}(x_n) = \{2, 0, -4, 6\}$

(se considera $\pm\infty$)

$\overline{\lim}\, x_n = 6$

$\underline{\lim}\, x_n = -4$

$\overline{\lim}\, x_n \neq \underline{\lim}\, x_n \Rightarrow \nexists \lim\limits_{n\to\infty} x_n$

(6) $x_n = \sin \frac{n\pi}{3}$ $(\forall) n \in \mathbb{N}$

Sol:

$x_{3n} = \sin \frac{3n\pi}{3} = \sin n\pi = 0 \xrightarrow[n\to\infty]{} 0$

$x_{3n+1} = \sin \frac{(3n+1)\pi}{3} = \sin\left(n\pi + \frac{\pi}{3}\right)$

$= \underbrace{\sin n\pi \cdot \cancel{\sin}\cos \frac{\pi}{3}}_{0} + \underbrace{\cos n\pi}_{(-1)^n} \cdot \underbrace{\cancel{\cos}\sin \frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}$

$= (-1)^n \cdot \frac{\sqrt{3}}{2}$

~~x_{6n}~~

$x_{3\cdot 2n+1} = x_{6n+1} = (-1)^{2n} \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{\sqrt{3}}{2}$

$x_{3(2n+1)+1} = x_{6n+4} = (-1)^{2n+1} \cdot \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}}{2}$

$x_{3n+2} = \sin \frac{(3n+2)\pi}{3} = \sin\left(n\pi + \frac{2\pi}{3}\right)$

$= \underbrace{\sin n\pi \cdot \cos \frac{2\pi}{3}}_{0} + \sin \frac{2\pi}{3} \cdot \cos n\pi$

$= (-1)^n \cdot \frac{\sqrt{3}}{2}$

$$x_{6n+2} = \frac{\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{\sqrt{3}}{2}$$

$$x_{6n+5} = \frac{-\sqrt{3}}{2} \xrightarrow[n\to\infty]{} \frac{-\sqrt{3}}{2}$$

$$L(x_n) = \left\{ \pm\frac{\sqrt{3}}{2}, 0 \right\} \quad N = 6N \cup (6N+1) \cup \ldots \cup (6N+$$

$$\overline{\lim}\, x_n = \frac{\sqrt{3}}{2}$$

$$\underline{\lim}\, x_n = -\frac{\sqrt{3}}{2} \quad \Big| \Rightarrow \nexists \lim_{n\to\infty} x_n$$

(c) $x_n = \dfrac{n \cos \frac{n\pi}{2}}{n^2+1}$ $(\forall)\, n \in N$

Sol: Vom arata că $\lim_{n\to\infty} x_n = 0$

$$-1 \le \cos\frac{n\pi}{2} \le 1 \quad \Big|\cdot \frac{n}{n^2+1} \ (\forall)\, n \in N$$

$$\frac{n}{n^2+1} \le \frac{n}{n^2+1} \cdot \cos\frac{n\pi}{2} \le \frac{n}{n^2+1}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$0$$

CC $\lim_{n\to\infty} x_n = 0 \Rightarrow \underline{\lim}\, x_n = \overline{\lim}\, x_n = 0$ □

Ut natura seriei de mai jos și precizați dacă este conv sau divergentă!

$$\cancel{\sum_{n=1}^{\infty}\frac{n}{(n+1)!} = \sum_{n=1}^{\infty}\frac{n}{(n-1)!\cdot n(n+1)}}$$

$$\cancel{= \sum_{n=1}^{\infty}\frac{1}{(n-1)!\cdot(n+1)}}$$

$$\sum_{n=1}^{\infty}\frac{n}{(n+1)!}$$

$$x_n = \frac{n}{(n+1)!}\ (\forall)\, n \in \mathbb{N}^*$$

$$S_n = x_1 + \dots + x_n = \frac{1}{2!} + \frac{2}{3!} + \dots \frac{n}{(n+1)!}$$

$$= \frac{1+1-1}{2!} + \frac{2+1-1}{3!} + \dots + \frac{n+1-1}{(n+1)!}$$

$$= \frac{2}{2!} - \frac{1}{2!} + \frac{3}{3!} - \frac{1}{3!} + \dots \frac{n+1}{(n+1)!} -$$

$$\frac{1}{(n+1)!} = 1 - \frac{1}{2!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{3!} - \frac{1}{(n+1)!}$$

$$= 1 - \frac{1}{(n+1)!}\ (\forall)\, n \in \mathbb{N}$$

$$\lim_{n\to\infty} S_n = 1 \in \mathbb{R} \Rightarrow S_n \text{ conv}$$

$$\Rightarrow \sum_{n=1} x_n \text{ conv}$$

3. Studiați convergența (sau natura) seriilor

(a) $\sum_{n=1}^{\infty} \frac{a^n}{\sqrt[n]{n}}$, $a>0$

Sol $x_n = \frac{a^n}{\sqrt[n]{n}}$ $(\forall)\, n \in \mathbb{N}^*$

Aplicăm C raportului.

$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{a^{n+1}}{\sqrt[n+1]{n+1}} \cdot \frac{\sqrt[n]{n}}{a^n}$

$= \lim_{n\to\infty} a \cdot \frac{\sqrt[n]{n}}{\sqrt[n+1]{n+1}} = a$

CC criteriului rap avem:

I $a < 1 \Rightarrow$ ~~$\lim_{n\to\infty} \sum_{n=1} x_n = 0$~~ $\sum_{n=1} x_n$ conv

II $a > 1 \Rightarrow$ $\sum_{n=1} x_n$ div

III $a = 1$

$x_n = \frac{1}{\sqrt[n]{n}} = \frac{1}{1} = 1 \neq 0 \Rightarrow$ criteriul nec de divergență

$\lim_{n\to\infty} x_n = 1$ $\sum_{n=1} x_n$ div

(b) $\sum_{n=0}^{\infty} \frac{1}{2^n+n}$

$x_n = \frac{1}{2^n+n}$

$\frac{1}{2^n+n} \leq \frac{1}{2^n}$ $(\forall) n \in \mathbb{N}$

$y_n = \frac{1}{2^n}$

$x_n \leq y_n$ $(\forall) n \in \mathbb{N}$

$\sum_{n=0} y_n = \sum_{n=0} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ (conv seria geometrică cu $q = \frac{1}{2}$)

CC criteriului de comp cu inegalități

$\Rightarrow \sum_{n=0} x_n$ conv

(c) $\sum_{n=1}^{\infty} \left(\frac{an^2+5n+3}{2n^2+3n+1}\right)^n$

(criteriul radicalului)

Fie $a=2 \Rightarrow x_n = \left(\frac{2n^2+5n+3}{2n^2+3n+1}\right)^{n}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(1+\frac{2n+2}{2n^2+3n+1}\right)^{n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{2n+2}{2n^2+3n+1}\right)^{\frac{2n^2+3n+1}{2n+2}}\right]^{\frac{2n+2}{2n^2+3n+1}\cdot n}$$

$$= e^{\lim\limits_{n\to\infty}\frac{2n^2+2n}{2n^2+3n+1}} = e^1 = e \neq 0$$ criteriul nul de div

$\Rightarrow \sum\limits_{n=1} x_n$ div

d) $\sum\limits_{n=1}^{\infty} \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$

Sol

$x_n = \frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}$ $(\forall) n \in \mathbb{N}^*$

$y_n = \frac{\sqrt{n^2}}{\sqrt{n^3}}$ $(\forall) n \in \mathbb{N}^*$

$$\lim_{n\to\infty}\frac{x_n}{y_n} = \lim_{n\to\infty}\frac{\sqrt{n^2+1}}{\sqrt{n^3+1}}\cdot\frac{\sqrt{n^3}}{\sqrt{n^2}} = 1 \in (0,\infty)$$

e criteriului de comparatie cu limita cele două serii au ac ~~limita~~ convergentă

$$\sum_{n=1}^{\infty} y_n = \sum_{n=1}^{\infty} \frac{\sqrt{n^2}}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$$ divergentă (serie armonică generalizată cu $\alpha = \frac{1}{2}$)

Deci $\sum_{n} x_n$ div.