Multimi și funcții   Curs 2

PROP: $A \underset{f'}{\overset{f}{\rightleftarrows}} B \underset{g'}{\overset{g}{\rightleftarrows}} C$, A, B, C multimi
f, f', g, g' funcții

Atunci:

1) f, g inj $\Rightarrow g \circ f$ injectivă
2) f, g surj $\Rightarrow g \circ f$ surjectivă
3) f, g bij $\Rightarrow g \circ f$ bij
4) $g \circ f$ inj $\Rightarrow f$ inj
5) $g \circ f$ surj $\Rightarrow g$ surj
6) g inj, $g \circ f = g \circ f' \Rightarrow f = f'$
7) f surj, $g \circ f = g' \circ f \Rightarrow g = g'$

Dem: 4) Def: $X \xrightarrow{h} Y$, $X' \xrightarrow{h'} Y$

$h = h' \Leftrightarrow \begin{cases} X = X' \\ Y = Y' \end{cases}$ și $(\forall) x \in X$ $h(x) = h'(x)$

Fie $x = y \in A$ ai $f(x) = f(y) \in B$
$\Rightarrow g(f(x)) = g(f(y)) \Leftrightarrow (g \circ f)(x) = (g \circ f)(y)$
$\Downarrow$ inj $\Rightarrow x = y$

Deci f inj

5) $(\forall)\ z \in C\ (\exists)\ x \in A$ a.î. $(g \circ f)(x) = z$
$\Leftrightarrow g(f(x)) = z$

$(\forall)\ z \in C\ (\exists)\ f(x) \in B$ a.î. $g(f(x)) = z$ $\Leftrightarrow$ g surj

6) $(\forall)\ x \in A$ vrem să demonstrăm că
$f(x) = f'(x)$

Fie $x \in A$ arbitrar $\underset{"="}{\Rightarrow} (g \circ f)(x) = (g \circ f')(x)$

$(g\circ f')(x) \Leftrightarrow g(f(x)) = g(f'(x)) \overset{g}{\underset{inj}{\Rightarrow}}$

$f(x) = f'(x)$

Temă 5, 7

Def: Fie A mulțime $id_A = 1_A = \mathbb{1}_A : A \to A$

$id_A(x) = x \quad (\forall) x \in A$

Def: Fie $A \xrightarrow{f} B$ s.n. funcție inversabilă dc. și numai dc. $(\exists)\ g: B \to A$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[rr, bend right, "g\circ f = 1_A"'] & B \arrow[r, "g"] \arrow[rr, bend left, "f\circ g = 1_B"] & A \arrow[r] & B \end{tikzcd}$$

Teoremă $f: A \to B$ bijectivă dc și numai dc este inversabilă

$\boxed{\text{OBS: } g \text{ este unic}}$

Dem "$\Leftarrow$"

OBS id_X este bijectivă $(\forall) X$

$g\circ f = 1_A \Rightarrow \begin{cases} f \text{ inj} \\ g \text{ surj} \end{cases}$

$f\circ g = 1_B \Rightarrow \begin{cases} g \text{ inj} \\ f \text{ surj} \end{cases}$

$g\circ f = 1_A$ bijectivă $\xrightarrow{PROP} \begin{cases} f \text{ inj} \\ g \text{ surj} \end{cases}$

$f\circ g = 1_B \xrightarrow{PROP} \begin{cases} f \text{ surj} \\ g \text{ inj} \end{cases}$

"$\Rightarrow$"

f bij $\Leftrightarrow (\forall) y \in B\ \exists!\ x \in A$ a.î. $f(x) = y$

Definim $g: B \to A \quad g(y) = x$

$(g\circ f)(x) = g(f(x)) = g(y) = x \Rightarrow g\circ f = 1_A$

$(f\circ g)(y) = f(g(y)) = f(x) = y \Rightarrow f\circ g = 1_B$

Exemple:

① $\mathbb{R} \underset{\ln}{\overset{\exp}{\rightleftarrows}} (0, +\infty)$

$\exp(x) = e^x$

$x \longmapsto e^x$

$\ln : (0, +\infty) \to \mathbb{R}$, $\ln(y) = x$

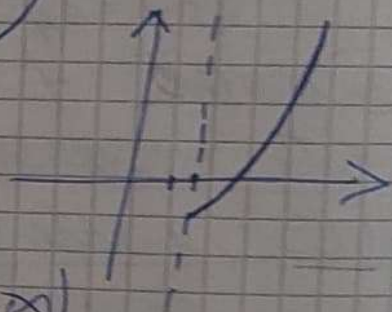
$y \to \ln(y)$

$\ln(e^x) = x$

$e^{\ln y} = y$

② $f(x) = x^2 - 3x + 2 = (x-1)(x-2)$

$f : \left[\underset{\displaystyle -\frac{b}{2a}}{\underset{\shortparallel}{\frac{3}{2}}}, +\infty\right) \to \left[\underset{\displaystyle -\frac{\Delta}{4a}}{\underset{\shortparallel}{-\frac{1}{4}}}, +\infty\right)$



$x = \frac{3}{2}$ este axa de simetrie

$g = f^{-1} : [-1/4, +\infty) \to [3/2, +\infty)$

$x^2 - 3x + 2 = y \Rightarrow x^2 - 3x + 2 - y = 0$

$$x_{1,2} = \frac{3 \pm \sqrt{9 - 4(2-y)}}{2} = \frac{3 \pm \sqrt{1+4y}}{2}$$

$$x = \frac{3 + \sqrt{1+4y}}{2}$$

$$g(y) = \frac{3 + \sqrt{1+4y}}{2}$$

$$g(x) = \frac{3 + \sqrt{1+4x}}{2}$$ $\quad \mathrm{Im} f = [-1/4, +\infty)$

Dem. că $f \circ g = id_{[-1/4, \infty)}$, $g \circ f = id_{[3/2, \infty)}$

imagine directă și imagine inversă a unei funcții

Fie $X \subseteq A$; $Y \subseteq B$ $A \xrightarrow{f} B$

Def imagine directă a lui X prin f

$$f(X) = \{ y \in B \mid (\exists) x \text{ a.î. } f(x) = y \}$$
$$= \{ f(x) \mid x \in X \} \subseteq B$$

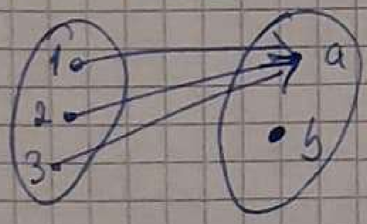
~~Def: preimaginea~~

$\operatorname{Im} f = f(A)$ = imaginea mulțimii A prin f (imaginea lui A)

Def: preimaginea unei mulțimi (imaginea inversă a unei submulțimi din B)

$$f^{-1}(Y) = \{ x \in A \mid f(x) \in Y \}$$

Exemplu



$A \xrightarrow{f} B$

$\{1, 2, 3\}$ $\{a, b\}$

$f(1) = f(2) = f(3) = a$

$\operatorname{Im} f = f(\{1,2,3\}) = \{a\} \subset B$

$f^{-1}(\{a\}) = \{1, 2, 3\} = f^{-1}(B)$

$f^{-1}(\{b\}) = \emptyset$

PROP: Fie $f: A \to B$; $X_1, X_2 \subseteq A$

$Y_1, Y_2 \subseteq B$

1) $X_1 \subseteq X_2 \subseteq A \Rightarrow f(X_1) \subseteq f(X_2) \subseteq B$

2) $Y_1 \subseteq Y_2 \subseteq B \Rightarrow f^{-1}(Y_1) \subseteq f^{-1}(Y_2) \subseteq A$

3) $Y_1, Y_2 \subseteq B \Rightarrow f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$

$f^{-1}(Y_1 \cup Y_2) = f^{-1}(Y_1) \cup f^{-1}(Y_2)$

5) $X_1, X_2 \subseteq A \Rightarrow f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$

6) $X_1, X_2 \subseteq A \Rightarrow f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$

! <u>Ptr</u> f <u>injectivă</u> avem $f(X_1 \cap X_2) \supseteq f(X_1) \cap f(X_2)$

deci ptr f inj avem "="

<u>Rem</u>

6) Exemplu "$\supseteq$" NU ESTE adevarata ptr

$f: \{1,2,3\} \to \{a,b\}$ $f(1) = f(2) = f(3) = a$

$X_1 = \{1,2\}$

$X_2 = \{3\}$

~~$f(X_1$~~ $X_1 \cap X_2 = \emptyset \Rightarrow f(\emptyset) = \emptyset$

$f(X_1) = \{a\} = f(X_2) \Rightarrow f(X_1) \cap f(X_2) = \{a\}$

$\neq \emptyset$

=

$X_1 \cap X_2 \subseteq X_1 \overset{①}{\Longrightarrow} f(X_1 \cap X_2) \subseteq f(X_1)$
$\subseteq X_2 \quad f(X_1 \cap X_2) \subseteq f(X_2) \Big| \Rightarrow$

$f(X_1 \cap X_2) \subseteq f(X_1) \cap f(X_2)$

f inj. Fie $y \in f(X_1) \cap f(X_2) \Leftrightarrow$
$\Leftrightarrow \exists\, y \in f(X_1) \wedge y \in f(X_2)$
$\Leftrightarrow \exists\, x_1 \in X_1$ a.î. $f(x_1) = y \wedge$
$\exists\, x_2 \in X_2$ a.î. $f(x_2) = y$
$f(x_1) = f(x_2) (= y) \overset{f \text{ inj}}{\Longrightarrow} x_1 = x_2$

$x_1 = x_2 = x \in X_1 \cap X_2$
$X_1 \quad X_2 \qquad f(x) = y$

$\Rightarrow y \in \cancel{f(X_1) \cap f(X_2)}\ f(X_1 \cap X_2)$

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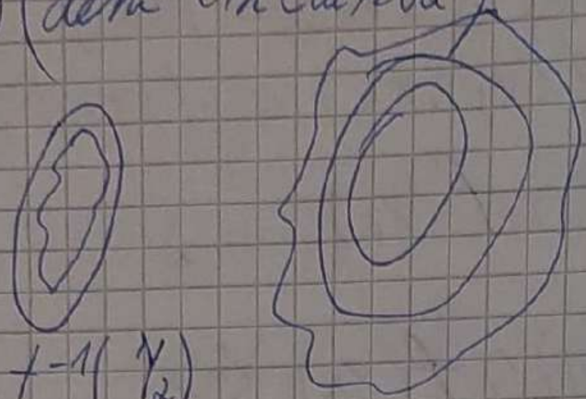
7) $f^{-1}(f(X)) \supseteq X$ cu egalitate pt f inj
8) $f(f^{-1}(Y)) \subseteq Y$ cu egalitate ptr f surj

Dem 3: Fie $x \in f^{-1}(Y_1 \cap Y_2) \Leftrightarrow \underline{f(x) \in Y_1 \cap Y_2}$
$f(x) \in Y_1$ și $f(x) \in Y_2$
$\Leftrightarrow x \in f^{-1}(Y_1) \wedge x \in f^{-1}(Y_2)$
$\Leftrightarrow x \in f^{-1}(Y_1) \cap f^{-1}(Y_2)$

② (dem intuitivă)



$f^{-1}(Y_2)$

__Teoremă__

Fie A mulțime U.a.s.e

1) $|A| < \infty$

2) $(\forall)$ $f: A \to A$ injectivă $\Rightarrow$ f surj (bij)

3) $(\forall)$ $f: A \to A$ surj $\Rightarrow$ f inj (bij)

Fie $A = \mathbb{N} = \{0, 1, 2, 3 \dots\}$

$f: \mathbb{N} \to \mathbb{N}, f(n) = n+1$

$g: \mathbb{N} \to \mathbb{N}, g(n) = \begin{cases} 0, & n = 0 \\ n-1, & n \geq 1 \end{cases}$ $g(0) = g(1) = 0$ NU E bij

$(g \circ f)(n) = n \quad (\forall) n \in \mathbb{N}$

$g \circ f = id_{\mathbb{N}}$

$\Downarrow$

bij $\Rightarrow$ f inj

f NU e surj

$0 \notin Im(f) = f(\mathbb{N})$

$\begin{cases} f \text{ inj} \\ g \text{ surj} \end{cases}$

Def. Două mulțimi s.n echipotente ~~dt~~ $\Leftrightarrow$

$\exists f: A \to B$ bij

Notație $A \simeq B$

OBS

1) $A \simeq A$ $(\exists)$ $id_A : A \to A$ bij (reflexivitate)

$id_A(x) = x$

2) $A \simeq B$ atunci $B \simeq A$ (simetrică)

$A \xrightarrow[bij]{f} B$ atunci $\exists g = f^{-1} : B \to A$
g bij

3) $A \underset{f}{\simeq} B \underset{g}{\simeq} C \Rightarrow A \underset{\substack{g\circ f \\ bij}}{\simeq} C$ (tranzivitate)

$\mathbb{N}$ = mulțime numărabilă - putem număra el.

Def: A s.n numărabilă dacă numai dc

$A \simeq \mathbb{N}$.

$(\exists)$ $f: \mathbb{N} \to A$ bij

$$A = \{ \underset{\substack{\shortparallel \\ f(0)}}{a_0}, \underset{\substack{\shortparallel \\ f(1)}}{a_1}, \underset{\substack{\shortparallel \\ f(2)}}{a_2}, \dots \}$$

PROP Fie A, B mult numarabile $\Rightarrow$

$A \cup B$ este numarabila

Dem:

$A = \{a_0, a_1, a_2 \dots\}$

$B = \{b_0, b_1, b_2 \dots\}$

$A \cup B = \{a_0, b_0, a_1, b_1, a_2, b_2 \dots\}$

COR Fie $A_1 \dots A_k$ multimi numarabile
(COROLAR=CONSECINTA) $\Rightarrow A_1 \cup \dots \cup A_k$

Exemplu: $\mathbb{N}, \mathbb{Z} = \{0, -1, 1, 2, -2 \dots\}$ numarabile

OBS: $(\forall)$ submultime infinita a unei multimi numarabile este numarabila.

$A = \{a_0, a_1, \dots\}$; $B = \{b_{i_0}, b_{i_2} \dots \}$

$B \subset A$

$|B| = \infty$

PROP $\mathbb{N} \times \mathbb{N}$ numarabila

$\{(a, b) \mid a, b \in \mathbb{N}\}$

Dem: $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

$f(a, b) = 2^a(2b+1) - 1$

Tema f este ~~bij~~ surj

$f(a_1,b_1) = f(b_1,b_2) \Leftrightarrow$

$2^{a_1}(2b_1+1) - \cancel{1} = 2^{a_2}(2b_2+1) - \cancel{1}$

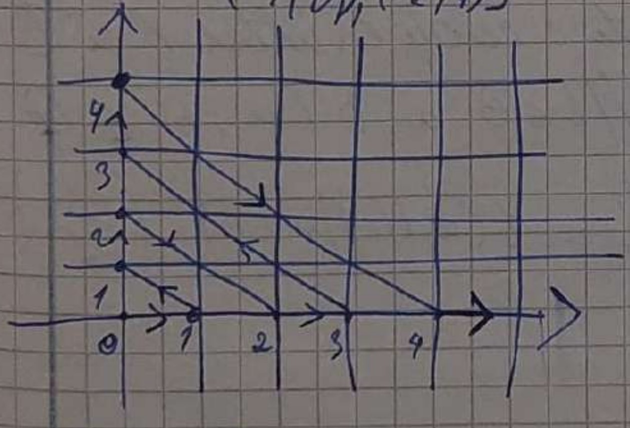
$\underbrace{2b_1+1}_{\text{impar}} = \underbrace{2^{a_2-a_1}}_{2^{a_1-a_2} = 1}(2b_2+1)$

~~$2b_1+1 = 2b_1+1 \Rightarrow$~~ $a_1 = a_2 = 0 \Rightarrow a_1 = a_2$

$2b_1+1 = 2b_2+1 \Rightarrow b_1 = b_2$

Deci $(a_1,b_1) = (a_2,b_2)$ f inj

Sau $\mathbb{N} \times \mathbb{N} = \{(0,0), (1,0), (0,1), (0,2), (1,1), (2,0)$
$(3,0), (2,1)\}$



PROP Dacă A, B numarabile atunci $A \times B$ este numarabila.

Dem: $A \xrightarrow{f} \mathbb{N}$ inj
$B \xrightarrow{g} \mathbb{N}$

$B \xrightarrow[(f,g)]{g} \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$

$(a,b) \longmapsto 2^a(2b+1) - 1$

$(x,y) \longmapsto (f(x), g(y))$

T_1 (Cantor)

$\mathbb{R}$ este o mulțime ne-numărabilă)

T_2 (Cantor)

$\forall$ A mulțime

$\nexists\ f: A \to \mathcal{P}(A)$ surjectivă

părțile lui A

În particular $A \nsim \mathcal{P}(A)$

Relații

$\boxed{\text{Seminar 2}}$