Seminar 1 6 oct 2022

1. Fie $x_n = \frac{1}{n}$ $(\forall) n \in \mathbb{N}^*$ Arătați folosind definiția că $\lim_{n \to \infty} x_n$ este 0.

Sol.

$\lim_{n \to \infty} x_n = 0 \Leftrightarrow (\forall) \varepsilon > 0 \ \exists \ n_\varepsilon \in \mathbb{N}^*$ a.î. $(\forall) n \geq n_\varepsilon$ avem $|x_n - 0| < \varepsilon$

Fie $\varepsilon > 0$ căutăm $n_\varepsilon \in \mathbb{N}^*$ a.î. $(\forall) n \geq n_\varepsilon$ avem $|x_n - 0| < \varepsilon$

$|x_n - 0| = |x_n| = \left|\frac{1}{n}\right| \overset{n\in\mathbb{N}^*}{=} \frac{1}{n}$

$\frac{1}{n} < \varepsilon \; /^{-1}$

$\Leftrightarrow n > \frac{1}{\varepsilon}$

$n \ge n_\varepsilon \Rightarrow n > \frac{1}{\varepsilon}$

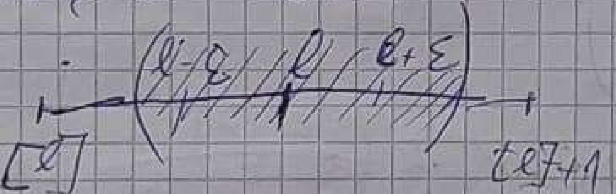
Alegem $n_\varepsilon = \left[\frac{1}{\varepsilon}\right] + 1 \in \mathbb{N}^*$

Avem $n_\varepsilon > \frac{1}{\varepsilon}$ și $(\forall) n \ge n_\varepsilon \overset{\text{tr}}{\Rightarrow} n > \frac{1}{\varepsilon}$

Deci $(\forall) n \ge n_\varepsilon$ avem $|x_n - 0| < \varepsilon$ i.e. $\lim_{n\to\infty} x_n = 0$.

(2) $(x_n)_n \subset \mathbb{Z}$ și $l \in \mathbb{R}$, $\lim_{n\to\infty} x_n = l$

Arătați că $l \in \mathbb{Z}$



$\lim_{n\to\infty} y_n = l \Leftrightarrow (\forall) \varepsilon > 0 \; \exists \, n_\varepsilon \in \mathbb{N}^*$ a.î. $(\forall) n \ge n_\varepsilon$;

$|x_n - l| < \varepsilon$ (atunci)

Pp prin absurd că $l \notin \mathbb{Z}$

$|x_n - l| < \varepsilon \Leftrightarrow -\varepsilon < x_n - l < \varepsilon \; / + l$

$-\varepsilon + l < x_n < \varepsilon + l$

$\Leftrightarrow x_n \in (l - \varepsilon, l + \varepsilon)$

Alegem $\varepsilon > 0$ $[l] < l-\varepsilon$ și $l+\varepsilon < [l]+1$

Un astfel de ε există deoarece

$l-[l] > 0$ și $[l]+1-l > 0$

$l \notin \mathbb{Z}$

$\cancel{\varepsilon < l-[l]}$

$[l] < l-\varepsilon \Leftrightarrow \varepsilon < l-[l]$

$l+\varepsilon < [l]+1 \Leftrightarrow \varepsilon < [l]+1-l$

Putem alege $\varepsilon \in (0, \min(l-[l], [l]+1-l))$

Pt acest ε, $\exists\, n_\varepsilon \in \mathbb{N}$ a.î. $(\forall)\, n \geq n_\varepsilon$ avem

$x_n \in (l-\varepsilon, l+\varepsilon)$

Dar $x_n \in \mathbb{Z}$ $(\forall)\, n \in \mathbb{N}$ și $\cancel{(l-\varepsilon)}$

$(l-\varepsilon, l+\varepsilon) \cap \mathbb{Z} = \emptyset$, contradicție

Prin urmare $l \in \mathbb{Z}$.

$\sqrt[n]{n} \xrightarrow[n\to\infty]{} 1$

Criteriul raportului pt șiruri cu termeni ~~strict~~ >0

Fie $(x_n)_n \subset (0,\infty)$ a.î. $\exists \lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} \overset{not}{=} l \in [0,\infty] = [0,\infty) \cup \{\infty\}$

1. Dacă $l<1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$
2. Dacă ~~lim~~ $l>1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty$
3. Dacă $l=1 \Rightarrow$ acest criteriu nu decide.

3\. Fie $a>0$ Det $\lim\limits_{n\to\infty} n \cdot a^n$

Fie $(x_n)_n = n\cdot a^n$, $n \in \mathbb{N}^*$

Aplicam crit rap. pt $(x_n)_n > 0$

~~Pt x_n~~

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{n+1 \cdot a^{n+1}}{n a^n} =$$

$$= a \lim_{n\to\infty} \frac{n+1}{n} = a$$

I Daca $a<1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

II Dacă $a>1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

III Dacă $a=1$ atunci criteriul nu decide

Fie $a=1$ atunci $x_n = n \cdot 1^n = n \; \forall n \in \mathbb{N}$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} n = \infty$$

Am obținut $\lim\limits_{n\to\infty} x_n = \begin{cases} 0, & a \in (0,1) \\ \infty, & a \in [1,\infty) \end{cases}$

Criteriul radicalului pt șiruri ≥ 0

Fie $(x_n)_n \subset [0,\infty)$ a.î. $\exists \lim\limits_{n\to\infty} \sqrt[n]{x_n} \overset{not}{=} l \in [0,$

1) Dacă $l<1 \Rightarrow \lim\limits_{n\to\infty} x_n = 0$

2) Dacă $l>1 \Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

3) Dacă $l=1$ atunci acest criteriu nu decide

4. Fie $a, b \in (0,\infty)$ det $\lim\limits_{n\to\infty}\left(\frac{a\cdot n^2+5n+3}{bn^2+3n+1}\right)^n$

Sol:

Fie $x_n = \left(\frac{an^2+5n+3}{bn^2+3n+1}\right)^n, n \in \mathbb{N}^*$

Aplicăm criteriul radicalului pt x_n

$\lim\limits_{n\to\infty}\sqrt[n]{x_n} = \lim\limits_{n\to\infty}\frac{an^2+5n+3}{bn^2+3n+1} = \frac{a}{b}$

Dacă $\frac{a}{b}<1$ (i.e $a<b$) $\Rightarrow \lim\limits_{n\to\infty} x_n = 0$

Dacă $\frac{a}{b}>1$ (i.e $a>b$) $\Rightarrow \lim\limits_{n\to\infty} x_n = \infty$

Pt $a=b \Rightarrow$ ~~$\lim\limits_{n\to\infty} x_n = 1$~~

atunci critical nu dea de

$a = b$ $[1^\infty]$

$$\lim_{n\to\infty}\left(\frac{an^2+5n+3}{an^2+3n+1}\right)^n = \lim_{n\to\infty}\left(1+\frac{2n+2}{an^2+3n+1}\right)^n$$

$$= \left[\lim_{n\to\infty}\left(1+\frac{2n+2}{an^2+3n+1}\right)^{\frac{an^2+3n+1}{2n+2}}\right]^{\frac{2n+2}{an^2+3n+1}\cdot n}$$

$$= e^{\lim_{n\to\infty}\frac{2n^2+2}{an^2+3n+1}} = e^{\lim_{n\to\infty}\frac{\cancel{n^2}\left(2+\frac{2}{n^2}\right)}{\cancel{n^2}\left(a+\frac{3}{n}+\frac{1}{n^2}\right)}}$$

$\frac{2}{n^2}\to 0$, $\frac{3}{n}\to\frac{1}{0}$, $\frac{1}{n^2}\to\frac{1}{0}$

$$= e^{\lim_{n\to\infty}\frac{2}{a}} = e^{\frac{2}{a}}$$

~~lim~~

Prop Fie $(x_n)_n \subset (0,\infty)$ a.î. $\exists \lim_{n\to\infty}\frac{x_{n+1}}{x_n} \overset{not}{=} l \in [0,\infty]$ ~~Atunci~~ $\exists \lim_{n\to\infty}\sqrt[n]{x_n} = l$

* Det $\lim_{n\to\infty}\sqrt[n]{n}$

Fie ~~$(x_n)_n$~~ $x_n = n$, $n \in \mathbb{N}^*$

$$\lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\frac{n+1}{n} = 1 \Rightarrow \lim_{n\to\infty}\sqrt[n]{n} = 1$$

prop de mai sus.

6. Fie $x_n = 1+\frac{1}{2}+\dots+\frac{1}{n} - \ln n$ $(\forall) n \in \mathbb{N}^*$

Arat că x_n convergent

Sol.

Arat că x_n monoton și mărginit.

<u>monotonia</u>

Fie $n \in \mathbb{N}^*$

$$x_{n+1} - x_n = \cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}}+\frac{1}{n+1} - \ln(n+1)$$

$$-\left(\cancel{1}+\cancel{\frac{1}{2}}+\dots+\cancel{\frac{1}{n}} - \ln\frac{1}{n}\right)$$

$$= \frac{1}{n+1} - \ln(n+1) + \ln\frac{1}{n}$$

$$= \frac{1}{n+1} - \ln\frac{n+1}{n}$$ <u>LAGRANGE!</u>

$$= \frac{1}{n+1} - \left(\ln(n+1) - \ln n\right)$$

Fie $f_n : [n, n+1] \to \mathbb{R}$

$f_n(x) = \ln x$

f_n cont pe $[n, n+1]$

f_n deriv pe $(n, n+1)$ $\overset{\text{LAGRANGE}}{\Longrightarrow}$ $\exists c_n \in (n, n+1)$

a.î. $f_n'(c_n) = \dfrac{f_n(n+1) - f_n(n)}{n+1-n}$

$\Rightarrow$ $\exists$ ~~$c_n \in (n, n+1)$ a.î. $f_n'(c_n)$~~

$f_n'(c_n) = \dfrac{1}{c_n}$

$$\frac{1}{c_n} = \ln(n+1) - \ln(n)$$

$$c_n \in (n, n+1) \Rightarrow n < c_n < n+1$$

$$\Rightarrow \cancel{\frac{1}{n} > \frac{1}{c_n}} \quad \frac{1}{n+1} \le \frac{1}{c_n} \le \frac{1}{n}$$

$$x_{n+1} - x_n = \frac{1}{n+1} - \frac{1}{c_n} < 0$$

$$\Rightarrow (x_n)_n \downarrow \text{desc} \; (1)$$

<u>Mărginirea</u>: Continuați voi