04.01.2022

Seminar 1 $V = \{v_0, v_1, v_2, \dots\} = \{v_n \mid n \in \mathbb{N}\}$

Formule $\left\langle \begin{array}{l} \neg\varphi \\ \varphi \to \psi \end{array}\right.$

Liste $\left\langle \begin{array}{l} \text{vidă} \\ [H \mid T] \end{array}\right.$ Arbori $\left\langle \begin{array}{l} \text{vid} \\ \text{arb}(R, T, V) \end{array}\right.$

φ	ψ	χ
phi	psi	chi

Principiul de inducție pe formule

Principiul de ind pe $\mathbb{N}$

$$\frac{\begin{array}{ll} \text{I} & P(0) \\ \text{II} & \forall n\,(P(n) \to P(n+1)) \end{array}}{\forall n\, P(n)}$$

$\rightsquigarrow$ Principiul de recursie

A, $a \in A$, $g: A \to A$

$\exists!\ f: \mathbb{N} \to A$

a.î. $f(0) = a$

$\forall n\ f(n+1) = g(f(n))$

$\mathbb{N} \left\langle \begin{array}{l} 0 \\ s(n) \end{array}\right.$

Ind: I. $\forall v \in V\ P(v)$

II. $\forall \varphi\ (P(\varphi) \to P(\neg\varphi))$

III. $\forall \varphi, \psi\ (P(\varphi) \wedge P(\psi) \to P(\varphi \to \psi))$

Rec: A, $G_0: V \to A$

$\rightsquigarrow G_\neg: A \to A$, $G_\to: A \times A \to A$

$\exists!\ F$: Formule $\to A$ a.î.

(i) $\forall v \in V$, $F(v) = G_0(v)$

(ii) $\forall \varphi$, $F(\neg\varphi) = G_\neg(F(\varphi))$

(iii) $\forall \varphi, \psi\ F(\varphi \to \psi) = G_\to(F(\varphi), F(\psi))$

Semantică

$\{0, 1\}$ $\neg: \{0,1\} \to \{0,1\}$

x	$\neg x$
0	1
1	0

x	y	$x \to y$
0	0	1
0	1	1
1	0	0
1	1	1

Th. $\forall e: V \to \{0,1\}$
$\exists!\ e^+: Form \to \{0,1\}$
a.î. (i) $\forall v \in V,\ e^+(v) = e(v)$
(ii) $\forall \varphi\ e^+(\neg \varphi) = \neg^b e^+(\varphi)$
(iii) $\forall \varphi, \psi\ e^+(\varphi \to \psi) = e^+(\varphi) \to^b e^+(\psi)$

dem $A = \{0,1\}$
$G_0 = e$
$G_\neg = \neg^b$
$G_\to = \to^b$

$e^+ = F$

$e: V \to \{0,1\}$
$\varphi, \psi \in Form$
$e \vDash \varphi \iff e^+(\varphi) = 1$ („e satisface φ")
$\vDash \varphi \iff \forall e$ avem $e \vDash \varphi$ („φ e tautologie")
$\varphi \vDash \psi \iff \forall e$ dc $e \vDash \varphi$ atunci $e \vDash \psi$ („din φ se
$\iff \forall e\ \ e^+(\varphi) \le e^+(\psi)$ ~~de~~ se deduce semantic ψ")

φ satisfiabilă $\iff \exists\ e$ a.î. $e \vDash \varphi$
φ nesatisfiabilă $\iff \nexists\ e$ cu $e \vDash \varphi$
$\iff \forall e\ \ e \nvDash \varphi$
$\varphi \sim \psi \iff \forall e$ avem $[e \vDash \varphi \iff e \vDash \psi]$
$\iff \forall e$ avem $e^+(\varphi) = e^+(\psi)$

(2) $\forall \varphi, \psi, \chi \in Form$
(i) $\psi \vDash \varphi \to \psi$
(ii) $\varphi \to (\psi \to \chi) \sim (\varphi \wedge \psi) \to \chi$
$\forall a \in \{0,1\}$
BOOL
$1 \to a$ $\qquad a \to 1 = 1$
$0 \to a = 1$ $\qquad a \to 0 = \neg a$
$1 \wedge a = a$ $\qquad 0 \wedge a = 0$

Sol. (i)

$\psi \vDash \varphi \to \psi$

Tb. să arătăm că pt $\forall e$ cu $e \vDash \psi$ avem $e \vDash \varphi \to \psi$

Fie e cu $e \vDash \psi \Rightarrow e^+(\psi) = 1$

Vrem $e \vDash \varphi \to \psi \Rightarrow e^+(\varphi \to \psi) = 1$

$$e^+(\varphi \to \psi) = e^+(\varphi) \xrightarrow{b} e^+(\psi)$$

$$= e^+(\varphi) \xrightarrow{b} 1 = 1$$

$$\boxed{x \to 1 = 1}$$

(ii) $\varphi \to (\psi \to \chi) \sim (\varphi \wedge \psi) \to \chi$

$\forall e$ in $e^+(\varphi \to (\psi \to \chi)) = e^+((\varphi \wedge \psi) \to \chi)$

Fie $e : V \to \{0, 1\}$

$M.S = e^+(\varphi) \xrightarrow{b} e^+(\psi \to \chi) = e^+(\varphi) \to (e^+(\psi) \to e^+(\chi))$

$\varphi \wedge \psi = \neg(\varphi \to \neg\psi)$

x	y	$x \wedge_b y$
0	0	0
0	1	0
1	0	0
1	1	1

$\forall\ \varphi, \psi$

$e^+(\varphi \wedge \psi) = e^+(\varphi) \wedge_b e^+(\psi)$

$M.D = e^+(\varphi \wedge \psi) \xrightarrow{b} e^+(\chi)$

I. $e^+(\varphi) = 0$

$M.S = 0 \to (e^+(\psi) \to e^+(\chi)) = 1$

$M.D = (0 \wedge e^+(\psi)) \to e^+(\chi)$

$= 0 \to e^+(\chi) = 1$

II. $e^+(\varphi) = 1$

$M.S = 1 \to (e^+(\psi) \to e^+(\chi))$

$= e^+(\psi) \to e^+(\chi)$

$M.D = 1 \wedge e^+(\psi) \to e^+(\chi)$

$= e^+(\psi) \to e^+(\chi)$

$M.S = M.D$

③ Găsiți câte un model pt.

(i) $v_0 \to v_2$

(ii) $v_0 \wedge v_3 \wedge \neg v_4$

$e \vDash \varphi$ ("e este model pt φ")

Sol:

i) Vrem e a.î. $e \vDash v_0 \to v_2$

$e : V \to \{0, 1\}$

$V = \{v_0, v_1, \dots\}$

$$e(v) := \begin{cases} 0, & v = v_0 \\ 1, & v \neq v_0 \end{cases}$$

$$e^+(v_0 \to v_2) = e^+(v_0) \to e^+(v_2) = e(v_0) \to e(v_2) = 0 \to 1 = 1$$

ii) Vrem $e \vDash v_0 \wedge v_3 \wedge \neg v_4 \iff e^+(v_0 \wedge v_3 \wedge \neg v_4) = 1$

$$\forall v \in V \quad e(v) := \begin{cases} 0, & v = v_4 \\ 1, & \text{altfel} \end{cases}$$

$$e^+(v_0 \wedge v_3 \wedge \neg v_4) = e^+(v_0) \wedge e^+(v_3) \wedge \neg e^+(v_4)$$
$$= e(v_0) \wedge e(v_3) \wedge \neg e(v_4)$$
$$= 1 \wedge 1 \wedge \neg 0 = 1$$

④ $\forall \varphi$

$\neg \varphi$ nesat $\Rightarrow \varphi$ tautologie

Sol: $\neg \varphi$ nesat $\iff$ nu $\exists\, e$ a.î. $e \vDash \neg \varphi$

$\iff \forall e$ avem $e \nvDash \neg \varphi$

$\iff \forall e \quad e^+(\neg \varphi) \neq 1$

$\iff \forall e \quad e^+(\neg \varphi) = 0$] [0, 1]

$\iff \forall e \quad \neg e^+(\varphi) = 0$

$\iff \forall e \quad e^+(\varphi) = 1 \iff \varphi$ taut

(5) Confirmați sau infirmați:

(a) $\forall \varphi, \psi \in Form$ $\varphi \wedge \psi$ taut $\Leftrightarrow$ φ, ψ taut

(b) $\forall \varphi, \psi \in Form$ $\varphi \vee \psi$ taut $\Leftrightarrow$ φ taut sau ψ taut

$\varphi \vee \psi = \neg \varphi \to \psi$

Sol: a) $\varphi \wedge \psi$ taut $\Leftrightarrow \forall e,\ e^+(\varphi \wedge \psi) = 1$

$\Leftrightarrow \forall e,\ e^+(\varphi) \wedge e^+(\psi) = 1$

$\Leftrightarrow \forall e [e^+(\varphi) = 1$ și $e^+(\psi) = 1]$

$\Leftrightarrow [\forall e\ e^+(\varphi) = 1]$ și $[\forall e\ e^+(\psi) = 1]$

$\Leftrightarrow\ \models \varphi$ și $\models \psi$

b) Fals.

$\varphi = v_0$

$\psi = \neg v_0$

$\models \varphi \vee \psi \Leftrightarrow\ \models v_0 \vee \neg v_0 \Leftrightarrow \forall e$ ev. $e^+(v_0 \vee \neg v_0) = 1$

$\Leftrightarrow \forall e\ e^+(v_0) \vee e^+(\neg v_0) = 1$ (A)

$\models \varphi \Leftrightarrow\ \models v_0 \Leftrightarrow \forall e$ ev. $e^+(x_0) = 1$

$\forall e$ ev. $e^+(x_0) = 1$

Fals. întrucât $e = 0 \to e(v_0) = 0$

Deci $\not\models \varphi$

Analog $\not\models \psi$ Deci e falsă că [$\models \varphi$ sau $\models \psi$]

(11) (i) Exemple numărabile

A s.n. numărabilă $\Leftrightarrow \exists f: A \to \mathbb{N}$ bijecție

inf.

num	nenum
$\mathbb{N}$	$\mathbb{R}$
$\mathbb{Z}$	$[0,1]$
$\mathbb{Q}$	
$\mathbb{N} \times \mathbb{N}$	

$\mathbb{N}$	0	1	2	3	4	...
$\mathbb{Z}$	0	1	-1	2	-2	...

$\mathbb{Q} = \mathbb{Z} \times \mathbb{N}^* /\sim$

$(m,n) \sim (p,q)$

$\Leftrightarrow mq = np$

$\frac{2}{3} = (2,3) = \{(2,3), (4,6), \ldots$

$V \subseteq$ Fo

num

$\mathbb{Q} \sim S \subset \mathbb{R}$
$= \{(2, 3, \ldots)$

P.P. $\exists f : \mathbb{N} \to [0, 1]$ bij

0	0, 3 7 9 4 1
1	0, 3 1 8 2 5
2	0, 1 2 1 3 4
3	0, 9 7 1 5 2
	0, 6 2 2 6 …

Seq = siruri finite

Expr = Seq(Sym)   Sym = V ∪ {1, →}

= $Sym^0 \cup Sym^1 \cup Sym^2 \ldots$

$Sym^2 = Sym \times Sym$
$\sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

Analog $\forall \sim Sym^n \sim \mathbb{N}$

Expr $\sim \{\lambda\} \cup \mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \cup \mathbb{N} \cup \ldots$
$\sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N}$

$V \subseteq Form \subseteq Expr$