Assignment 2

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February 2018

1 Part A

Describe the "*Umbrella domain*" as an HMM:

- What is the set of unobserved variable(s) for a given time-slice t (denoted $\mathbf{X}_t$ in the book)?
- What is the set of observable variable(s) for a given time-slice t (denoted $\mathbf{E}_t$ in the book)?
- Present the *dynamic model* $\mathbf{P}(\mathbf{X}_t|\mathbf{X}_{t-1})$ and the *observation model* $\mathbf{P}(\mathbf{E}_t|\mathbf{X}_t)$ as matrices.
- Which assumptions are encoded in this model? (*Hint:* Read page 568). Are the assumptions reasonable for this particular domain?

The "*Umbrella World*" is a world where a security guard is stationed underground. The guard wishes to infer whether or not it is raining. The guard does this by observing if the director brought with him an umbrella.
From this description of the "*Umbrella World*" we get the following:

- $\mathbf{X}_t = \{Rain_t\}$
- $\mathbf{E}_t = \{Umbrella_t\}$

I have been unable to find anything in the book about *dynamic models*. However, a quick Google search has led me to believe that the *dynamic model* is the same as the *transition model*. (Also, the function given in the exercises is a good indication of the same). Therefore we have:

$$\mathbf{T} = \mathbf{P}(X_t|X_{t-1}) = \begin{pmatrix} 0.7 & 0.3 \\ 0.3 & 0.7 \end{pmatrix}$$

Where $\mathbf{T}_{ij} = P(X_t = j|X_{t-1} = i)$. That is, $\mathbf{T}_{ij}$ is the probability of a transition from state i to state j.

For the *observation model* (or *sensor model*), it is in this case fairly simply to represent it as a matrix. Since we know the value of the evidence variable E_t

at time t, all we have to do is to specify how likely it is that the state causes e_t to appear. Therefore we have that $P(E_t|X_t)$ is:

$$\mathbf{O}_t = \begin{pmatrix} 0.9 & 0 \\ 0 & 0.2 \end{pmatrix}$$

when $Umbrella$ is *true*; and

$$\mathbf{O}_t = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.8 \end{pmatrix}$$

when $Umbrella$ is *false*.

The assumptions encoded in this model are:

- **Markov** assumption,
- **stationary process** assumption, and
- **sensor Markov** assumption

The Markov assumption states that the current state depends on only a *finite fixed number* of previous states. The book defines a *stationary process* as a process of change that is governed by laws that do not themselves change over time. In other words, this means that the probability of rain is the same for all t. Therefore, we end up with a **first-order Markov process**.

Finally, the sensor Markov assumption states that the world in it's current state will provide the current sensor values regardless of any previous sensor values.

There are many factor that has a say for if it will rain or not, but introducing many more sensor/inputs is probably not necessary for this scenario. However, the assumptions do oversimplify the process of deciding if it is raining or not.

2 Part B

Implement filtering using the Forward operation (Equation 15.5 and Equation 15.12). Note that this can be done with simple matrix operations in the HMM.

- Verify your implementation by calculating $\mathbf{P}(X_2|e_{1:2})$, where $e_{1:2}$ is the evidence that the Umbrella was used both on day 1 and day 2. The desired result is (confer the slides from the lecture available on It's Learning) is that the probability of rain at day 2 (after the observations) is 0.883.
- Use your program to calculate the probability of rain at day 5 given the sequence of observations $e_{1:5} = \{Umbrella_1 = true, Umbrella_2 = true, Umbrella_3 = false, Umbrella_4 = true, Umbrella_5 = true\}$. Document your answer by showing all normalized forward messages (in the book the un-normalized forward messages are denoted $f_{1:k}$ for $k = 1, 2, ..., 5$).

Day	*Rain*	$\neg$*Rain*
0	0.5	0.5
1	0.81818	0.18181
2	0.88335	0.11664

Table 1: Normalized forward messages for the two first observations

Day	*Rain*	$\neg$*Rain*
0	0.5	0.5
1	0.81818	0.18181
2	0.88336	0.11664
3	0.19067	0.80933
4	0.73039	0.26921
5	0.86734	0.13266

Table 2: Normalized forward messages for all five observations

2.1 Part C

Implement smoothing using the Forward-Backward algorithm.

- Verify your implementation by calculating $\mathbf{P}(X_1|e_{1:2})$ where $e_{1:2}$ is the evidence that the umbrella was used the first two days (as in Part (b) of the assignment).
- Use your Forward-Backward algorithm to calculate the probability of rain at day 1 given the sequence of observations $e_{1:5} = \{Umbrella1 = true, Umbrella2 = true, Umbrella3 = false, Umbrella4 = true, Umbrella5 = true\}$. Document your answer by showing all backward messages ($b_{k+1:t}$ for $k = 1, 2, ..., 5$).

day	$Rain$	$\neg Rain$
0	0.4593	0.2437
2	0.69	0.41
3	1.0	1.0

Table 3: Backward unnormalized values when running for 2 observations

day	$Rain$	$\neg Rain$
1	0.88335704	0.11664296
2	0.88335704	0.11664296

Table 4: Forward_Backward values when running for 2 observations

day	$Rain$	$\neg Rain$
0	0.04438457	0.02422283
1	0.06611763	0.04550767
2	0.090639	0.150251
3	0.4593	0.2437
4	0.69	0.41
5	1	1

Table 5: Backward unnormalized values when running for all observations

day	$Rain$	$\neg Rain$
1	0.86733889	0.13266111
2	0.82041905	0.17958095
3	0.30748358	0.69251642
4	0.82041905	0.17958095
5	0.86733889	0.13266111

Table 6: Forward_Backward values when running for all observations