Transformata Laplace — Capitolul 7

Definiția 1. Funcția $f: \mathbb{R} \to \mathbb{R}(\mathbb{C})$ astfel încât: i) $f \in L^1_{loc}(\mathbb{R})$ are un număr finit de discontinuități de prima speță; ii) $f(t) = 0$, $\forall t < 0$; iii) există $M > 0$ și $\gamma_0 \in \mathbb{R}$ astfel încât $|f(t)| \leq M \cdot e^{\gamma_0 t}$, $\forall t$.
Atunci f se numește <u>original Laplace</u> sau <u>semnal original</u>. γ_0 se numește indice de creștere.

Definiția 2. Aplicația $F_L: \mathbb{C} \to \mathbb{C}$ definită prin integrala improprie cu parametru
$$F_L(s) = \int_0^\infty f(t) \cdot e^{-st} dt$$
cu $s = \alpha + j\omega \in \mathbb{C}$ se numește <u>transformata Laplace</u> (sau imaginea Laplace) a lui $f(t)$.
Avem corespondența $f(t) \xleftrightarrow{L} F_L(s)$. Se mai folosește notația $F_L(s) = \mathcal{L}\{f(t)\}$.

Definiția 3. Cea mai mică valoare a indicilor de creștere se numește <u>abscisă de convergență</u> și se notează c_f.

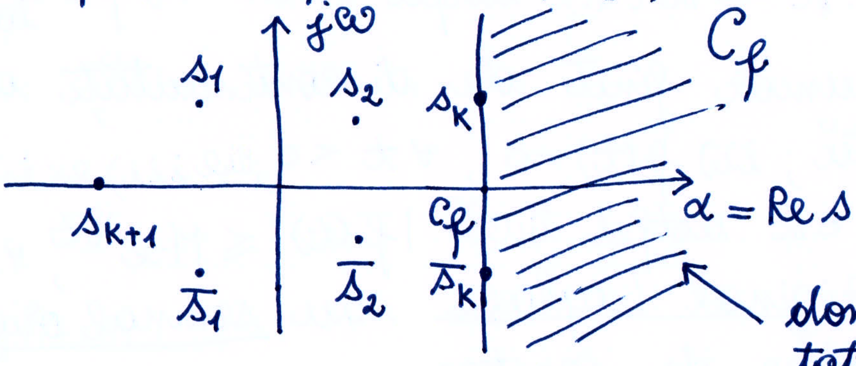
$\boxed{\text{Proprietate}}$ ① $F_L(s)$ este convergentă (bine definită) și olomorfă (derivabilă și dezvoltabilă în serie Taylor) dacă și numai dacă $\operatorname{Re} s > c_f$. Domeniul $C_f = \{\operatorname{Re} s > c_f\}$ se numește <u>domeniu de convergență</u> pentru $F_L(s)$.

② Dacă $s_1, s_2, \dots, s_K, \dots$ sînt singularitățile lui $F_L(s)$, atunci $c_f = \max\{\operatorname{Re} s_1, \operatorname{Re} s_2, \dots, \operatorname{Re} s_K, \dots\}$.

③ i) dacă $\underline{f(t) \in \mathbb{R}}$, atunci $F_L(\bar{s}) = \overline{F_L(s)}$. Într-adevăr
$$\overline{F_L(s)} = \int_0^\infty f(t) \cdot \overline{e^{-\alpha t - j\omega t}} dt = \int_0^\infty f(t) \cdot e^{-\alpha t} \cdot \overline{e^{-j\omega t}} dt =$$
$$= \int_0^\infty f(t) \cdot e^{-\alpha t} \cdot \overline{(\cos\omega t - j\sin\omega t)} dt = \int_0^\infty f(t) \cdot e^{-\alpha t} \cdot e^{j\omega t} dt$$
$$= \int_0^\infty f(t) \cdot e^{-(\alpha - j\omega)t} dt = \int_0^\infty f(t) \cdot e^{-\bar{s}t} dt = F_L(\bar{s}).$$

ii) dacă $F_L(s)$ este rațională cu coeficienți reali, atunci

$F_L(\bar{s}) = \overline{F_L(s)}$ și $F_L(s)$ are un număr finit de perechi de poli complex conjugați: $(s_1, \bar{s}_1)$, $(s_2, \bar{s}_2)$, ..., $(s_K, \bar{s}_K)$, și un nr. finit de poli reali $s_{K+1}, \dots, s_n$



domeniul de convergență; toți poli sînt situați la stînga abscisei de convergență sau pe ea.

Inversa transformatei Laplace

Fie imaginea $F_L(s)$ olomorfă pe $\mathrm{Re}\, s > c_f$. Atunci definim transformata Laplace inversă aplicația

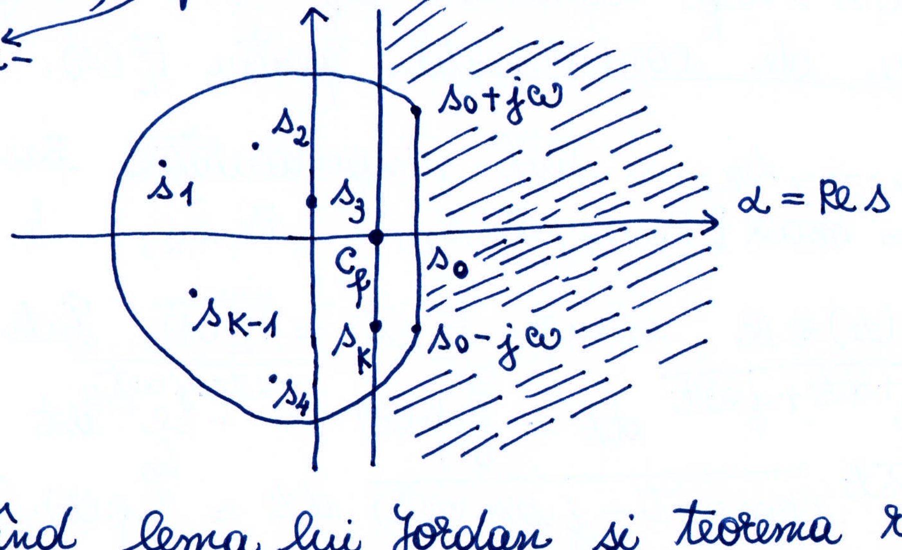
$$\mathcal{L}^{-1}\{F_L(s)\} = f(t) \stackrel{def.}{=} \frac{1}{2\pi j} \int_{s_0 - j\infty}^{s_0 + j\infty} F_L(s) \cdot e^{st}\, ds, \text{ unde}$$

$\boxed{s_0 > c_f}$.

Folosim corespondența: $F_L(s) \xleftrightarrow{L^{-1}} f(t)$, iar $\mathcal{L}^{-1}\{F_L(s)\}$ se numește transformata Laplace inversă a lui $F_L(s)$.

Considerăm conturul Bromwich care conține în interior (toți polii) lui $F_L(s)$:

toate singularitățile



Aplicînd lema lui Jordan și teorema reziduurilor pe conturul Bromwich găsim formula reziduurilor pentru recuperarea originalului

$$f(t) = \{ \sum_K \mathop{\mathrm{Res}}_{s = s_K} F_L(s) \cdot e^{st} \} \cdot \sigma(t),$$ unde s_K = singularitățile lui $F_L(s)$ cu $\mathrm{Re}\, s_K \leq c_f$.

Caz special. Considerăm că $F_L(s)$ este funcție rațională complexă cu coeficienți reali. Recuperăm originalul Laplace astfel:

① $\boxed{\text{Formula I}^{a} \text{ a lui Heaviside.}}$ $F_L(s) = \frac{P(s)}{Q(s)}$ are poli simpli nenuli, atunci:

$$F_L(s) = \frac{P(s)}{Q(s)} \xleftrightarrow{L^{-1}} f(t) = \mathcal{L}^{-1}\{F_L(s)\} = \left\{\sum_k \frac{P(s_k)\cdot e^{s_k t}}{Q'(s_k)}\right\}\cdot\sigma(t).$$

② $\boxed{\text{Formula a II}^{a} \text{ a lui Heaviside}}$ $F_L(s) = \frac{P(s)}{s\cdot Q(s)}$ are $s=0$ pol simplu și s_k poli simpli nenuli.

$$F_L(s) \xleftrightarrow{L^{-1}} f(t) = \left\{\frac{P(0)}{Q(0)} + \sum_k \frac{P(s_k)\cdot e^{s_k t}}{s_k\cdot Q'(s_k)}\right\}\cdot\sigma(t).$$

③ Caz general: $s_1, \dots, s_n$ poli reali cu ordine de multiplicitate $m_1, \dots, m_n \geq 2$ și $s_1, \dots, s_p$ poli complecși cu multiplicitățile $m_1, \dots, m_p \geq 2$. Atunci:

$$f(t) = \mathcal{L}^{-1}\{F_L(s)\} = \left[\sum_{k=1}^{n} \operatorname*{Res}_{s=s_k} F_L(s)\cdot e^{st} + 2\sum_{l=1}^{p} \operatorname{Re}\left\{\operatorname*{Res}_{\substack{s=s_l \\ \operatorname{Im} s_l>0}} F_L(s)\cdot e^{st}\right\}\right]\sigma(t)$$

$$* \operatorname*{Res}_{s=s_k} F_L(s)\cdot e^{st} = \frac{1}{(m_k-1)!}\lim_{s\to s_k}\left[(s-s_k)^{m_k}\cdot F_L(s)e^{st}\right]^{(m_k-1)}.$$

$\boxed{\text{Proprietăți ale transformatei Laplace TL}}$

1. Liniaritatea. $f_1(t) \xleftrightarrow{L} F_1(s)$, $f_2(t) \xleftrightarrow{L} F_2(s)$. Atunci $\forall \alpha_1, \alpha_2 \in \mathbb{C} \Rightarrow \alpha_1 f_1(t) + \alpha_2 f_2(t) \xleftrightarrow{L} \alpha_1 F_1(s) + \alpha_2 F_2(s)$, $\operatorname{Re} s > \max\{c_{f_1}, c_{f_2}\}$

Considerăm $f(t) \xleftrightarrow{L} F_L(s)$, $\operatorname{Re} s > c_f$.

2. Schimbarea de scară: $f(at) \xleftrightarrow{L} \frac{1}{a} F_L\left(\frac{s}{a}\right)$, $\operatorname{Re} s > c_f$, $\forall a > 0$.

3. Întârzierea: $f(t-t_0) \xleftrightarrow{L} e^{-t_0 s}\cdot F_L(s)$, $\forall t_0 > 0$; $\operatorname{Re} s > c_f$.

4. Deplasarea: $e^{s_0 t} f(t) \xleftrightarrow{L} F_L(s-s_0)$, $\forall s_0 \in \mathbb{C}^*$; $\operatorname{Re} s > \operatorname{Re} s_0 + c_f$.

5. Derivarea: i) în timp (a originalului):

$$f^{(n)}(t) \xleftrightarrow{L} s^n F_L(s) - s^{n-1} f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0); \operatorname{Re} s > c_f$$

ii) în frecvență (a imaginii):

$$t^n f(t) \xleftrightarrow{L} (-1)^n\cdot (F_L(s))^{(n)}, \forall n \geq 1, \operatorname{Re} s > c_f.$$

6. Integrarea: i) în timp: $\int_0^t f(x)dx \xleftrightarrow{L} \frac{1}{s}\cdot F_L(s)$, $\operatorname{Re} s > \max\{c_f, 0\}$

ii) imaginii: $\frac{f(t)}{t} \xleftrightarrow{L} \int_s^\infty F_L(p)\,dp$, $\operatorname{Re} s > c_f$.

⑦ <u>Produs de convoluție.</u> $f*g(t)=\int_0^t f(\tau)\cdot g(t-\tau)\,d\tau \xleftrightarrow{L}$ $\xleftrightarrow{L} F_L(s)\cdot G_L(s)$, $\text{Re}\, s>\max\{c_f, c_g\}$.

⑧ $f(t)=\sum_{k=0}^{\infty} f_0(t-kT) \underset{T>0}{\xleftrightarrow{L}} \sum_{k=0}^{\infty} F_0(s)\cdot e^{-skT} = F_0(s)\cdot \sum_{k\geq 0}(e^{-sT})^k =$

$$= \frac{F_0(s)}{1-e^{-sT}}, \quad \text{Re}\, s> c_{f_0}.$$

$\boxed{\text{TL uzuale}}$ ① $<\mathcal{L}\{\delta(t)\}, \varphi(t)> \underset{\text{def.}}{=} <\delta(t), \mathcal{L}\{\varphi(t)\}> = \int_0^\infty \varphi(t)\,dt$

$= <1, \varphi(t)> \Rightarrow$

$\delta(t) \xleftrightarrow{L} 1$.

② $\sigma(t) \xleftrightarrow{L} F_L(s)=\int_0^\infty e^{-st}\,dt = -\lim_{t\to\infty}\frac{1}{s}\cdot e^{-st}+\frac{1}{s}$ este convergentă $\Leftrightarrow \lim_{t\to\infty}\frac{1}{s}\cdot e^{-st}=0 \underset{s=\alpha+j\omega}{\Longleftrightarrow} \lim_{t\to\infty}\frac{1}{\sqrt{\alpha^2+\omega^2}}\cdot e^{-(\text{Re}\, s)t}=0 \Leftrightarrow$

$\text{Re}\, s>0$. Deci: $\boxed{\sigma(t) \xleftrightarrow{L} \frac{1}{s}, \ \text{Re}\, s>0}$.

③ $e^{s_0 t}\sigma(t) \underset{\text{deplasarea}}{\xleftrightarrow{L}} \frac{1}{s-s_0}$, $\text{Re}\, s>\text{Re}\, s_0$, $\forall s_0\in\mathbb{C}$; ④ $\cos(\omega_0 t)\cdot\sigma(t)$

$= \frac{1}{2}\left(e^{j\omega_0 t}\sigma(t)+e^{-j\omega_0 t}\sigma(t)\right) \xleftrightarrow{L} \frac{1}{2}\left(\frac{1}{s-j\omega_0}+\frac{1}{s+j\omega_0}\right)$, $\text{Re}\, s>0$

$\cos(\omega_0 t)\cdot\sigma(t) \xleftrightarrow{L} \frac{s}{s^2+\omega_0^2}$, $\omega_0>0$; $\text{Re}\, s>0$.

⑤ $\sin(\omega_0 t)\cdot\sigma(t) \underset{\omega_0>0}{\xleftrightarrow{L}} \frac{1}{2j}\left(\frac{1}{s-j\omega_0}-\frac{1}{s+j\omega_0}\right)=\frac{\omega_0}{s^2+\omega_0^2}$, $\text{Re}\, s>0$.

⑥ $t\cdot\sigma(t) \underset{\text{der. imaginii}}{\xleftrightarrow{L}} -\left(\frac{1}{s}\right)'=\frac{1}{s^2}$; $t^n\sigma(t) \xleftrightarrow{L} (-1)^n\cdot\left(\frac{1}{s}\right)^{(n)}=\frac{n!}{s^{n+1}}$, $\text{Re}\, s>0$.

⑦ $e^{at}\sigma(t) \underset{a\in\mathbb{R}^*}{\xleftrightarrow{L}} \frac{1}{s-a}$, $\text{Re}\, s>a$; $t^n e^{at}\cdot\sigma(t) \underset{\text{deplasarea}}{\xleftrightarrow{L}} \frac{n!}{(s-a)^{n+1}}$, $n\geq 1$, $\text{Re}\, s>a$.

⑧ $e^{at}\cos(\omega_0 t)\sigma(t) \underset{\text{deplasarea}}{\xleftrightarrow{L}} \frac{s-a}{(s-a)^2+\omega_0^2}$, $\text{Re}\, s>a$, $\omega_0>0$.

$e^{at}\sin(\omega_0 t)\sigma(t) \xleftrightarrow{L} \frac{\omega_0}{(s-a)^2+\omega_0^2}$,

⑨ $t\cdot\sin(\omega_0 t)\sigma(t) \underset{\text{der. im.}}{\xleftrightarrow{L}} -\left(\frac{\omega_0}{s^2+\omega_0^2}\right)'=\frac{2\omega_0 s}{(s^2+\omega_0^2)^2}$, $\text{Re}\, s>0$

$t\cdot\cos(\omega_0 t)\cdot\sigma(t) \xleftrightarrow{L} -\left(\frac{s}{s^2+\omega_0^2}\right)'=\frac{s^2-\omega_0^2}{(s^2+\omega_0^2)^2}$, $\text{Re}\, s>0$.

⑩ $\sinh at\,\sigma(t) \xleftrightarrow{L} \frac{1}{2}\left(\frac{1}{s-a}-\frac{1}{s+a}\right)=\frac{a}{s^2-a^2}$, $a>0$, $\text{Re}\, s>a$.

$\cosh at\,\sigma(t) \xleftrightarrow{L} \frac{1}{2}\left(\frac{1}{s-a}+\frac{1}{s+a}\right)=\frac{s}{s^2-a^2}$, $a>0$, $\text{Re}\, s>a$.

TL **Probleme.**

Cele 2 formule se utilizează în aplicațiile 1-3.

$$f_0(t) \xleftrightarrow{L} F_0(s) \Rightarrow$$

$T>0$

$$f(t) = \sum_{k=0}^{\infty} f_0(t-kT) \xleftrightarrow{L} \frac{F_0(s)}{1-e^{-sT}}$$

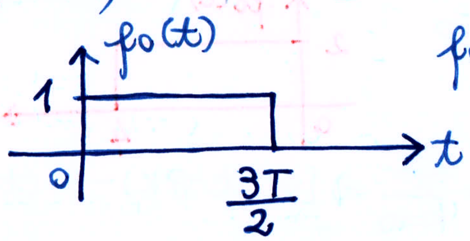
$$\sigma(t)-\sigma(t-t_0) \xleftrightarrow{L} \frac{1}{s} - \frac{1}{s}\cdot e^{-t_0 s}$$

$t_0>0$.

$\{\mathrm{Re}\, s>0\}$

Aplicații:

1.



$f_0(t) \xleftrightarrow{L} F_0(s) = ?$

$$f_0(t) = \sigma(t) - \sigma\left(t-\frac{3T}{2}\right) \Rightarrow$$

$$f_0(t) \xleftrightarrow{L} F_0(s) = \frac{1}{s}\left(1-e^{-s\cdot\frac{3T}{2}}\right)$$

$$f(t) = \sum_{k\geq 0} f_0(t-kT) \xleftrightarrow{L} \frac{F_0(s)}{1-e^{-sT}} = \frac{1-e^{-s\cdot\frac{3T}{2}}}{s(1-e^{-sT})},$$

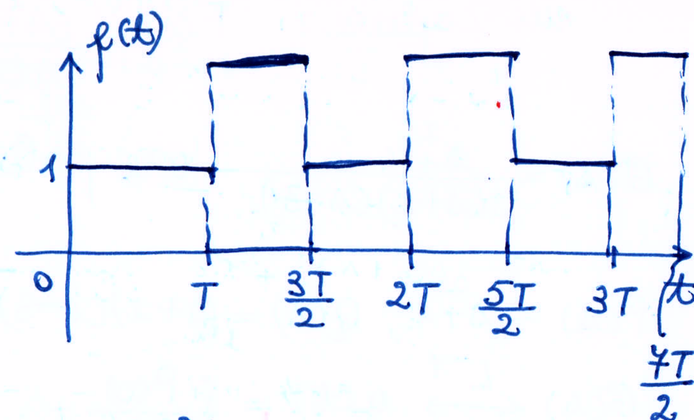
$\mathrm{Re}\, s>0$.

$$f_0(t) \doteq \begin{cases} 1, & t\in\left(0,\frac{3T}{2}\right) \\ 0, & \text{rest} \end{cases}$$

$$f_0(t-T) = \begin{cases} 1, & t\in\left(T,\frac{5T}{2}\right) \\ 0, & \text{rest} \end{cases}$$

$$f_0(t-2T) = \begin{cases} 1, & t\in\left(2T,\frac{7T}{2}\right) \\ 0, & \text{rest} \end{cases}$$

$$f(t) = \begin{cases} 1, & t\in(0,T) \\ 2, & t\in\left(T,\frac{3T}{2}\right) \\ 1, & t\in\left(\frac{3T}{2},2T\right) \\ 2, & t\in\left(2T,\frac{5T}{2}\right) \\ \vdots \end{cases}$$



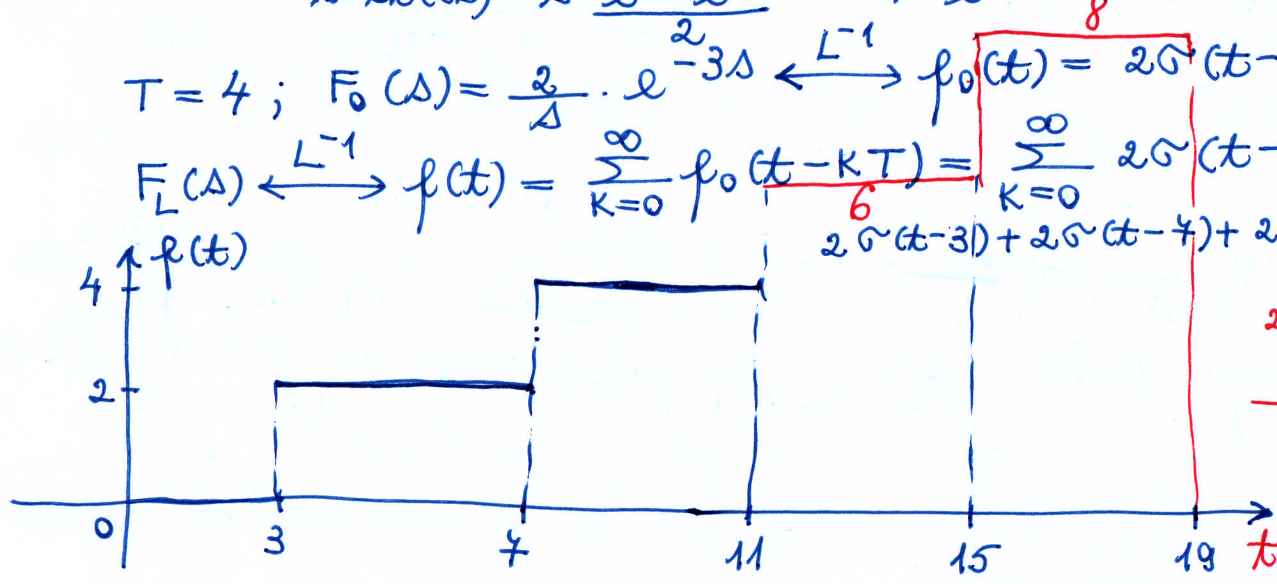
2.

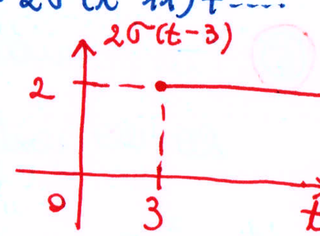
$$F_L(s) = \frac{e^{-s}}{s\cdot \mathrm{sh}(2s)} = \frac{e^{-s}}{s\cdot\frac{e^{2s}-e^{-2s}}{2}} = \frac{\frac{2}{s}\cdot e^{-3s}}{1-e^{-4s}} = \frac{F_0(s)}{1-e^{-4s}}$$

$$T=4;\ F_0(s) = \frac{2}{s}\cdot e^{-3s} \xleftrightarrow{L^{-1}} f_0(t) = 2\sigma(t-3)$$

$$F_L(s) \xleftrightarrow{L^{-1}} f(t) = \sum_{k=0}^{\infty} f_0(t-kT) = \sum_{k=0}^{\infty} 2\sigma(t-4k-3).$$

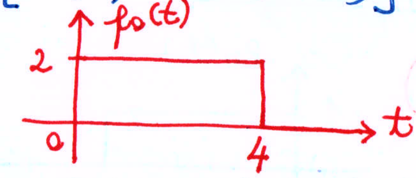
$$2\sigma(t-3)+2\sigma(t-7)+2\sigma(t-11)+\ldots$$





3. $$F_L(s)=\frac{e^{2s}}{s\cdot ch(2s)}=\frac{\frac{2}{s}\cdot e^{+2s}}{e^{2s}+e^{-2s}}=\frac{\frac{2}{s}}{1+e^{-4s}}=\frac{\frac{2}{s}(1-e^{-4s})}{1-e^{-8s}}=$$

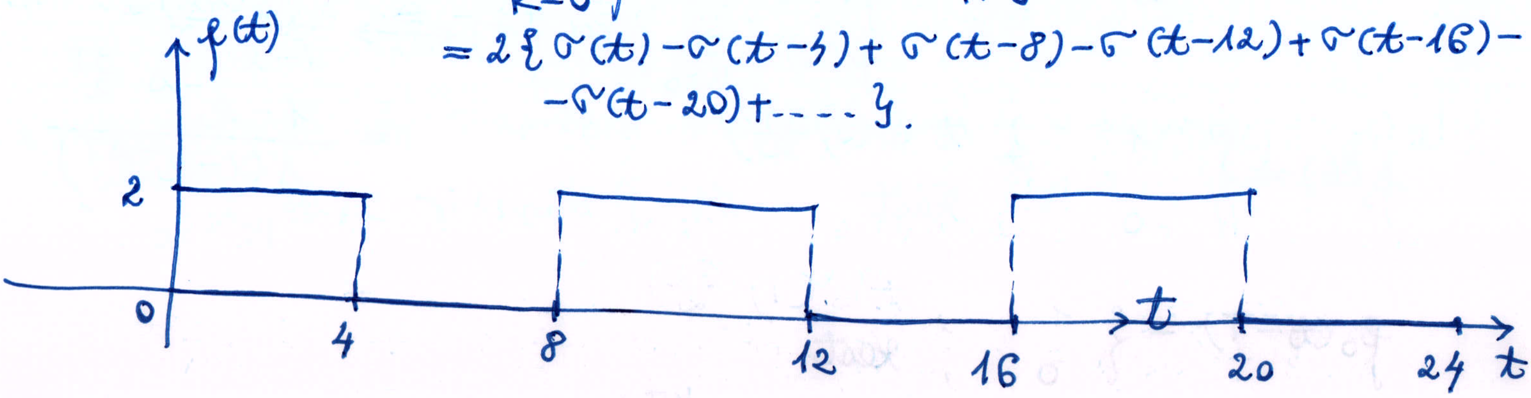
$$=\frac{\frac{2}{s}-\frac{2}{s}\cdot e^{-4s}}{1-e^{-8s}} \Rightarrow \begin{cases} F_0(s)=\frac{2}{s}-\frac{2}{s}\cdot e^{-4s} \xleftrightarrow{L^{-1}} \\ f_0(t)=2[\sigma(t)-\sigma(t-4)] \\ T=8 \end{cases}$$



$$F_L(s)\xleftrightarrow{L^{-1}} f(t)=\sum_{K=0}^{\infty} f_0(t-KT)=$$

$$=\sum_{K=0}^{\infty} f_0(t-8K)=\sum_{K=0}^{\infty} 2[\sigma(t-8K)-\sigma(t-8K-4)]$$

$$=2\{\sigma(t)-\sigma(t-4)+\sigma(t-8)-\sigma(t-12)+\sigma(t-16)-$$
$$-\sigma(t-20)+\dots\}.$$



4. Determinați semnalul original corespunzător imaginii Laplace: $F(s)=\frac{(s^2+1)\cdot e^{-2s}}{s(s+2)(s-3)}=\underbrace{\frac{s^2+1}{s(s+2)(s-3)}}_{G(s)}\cdot e^{-2s}$

$G(s)=\frac{s^2+1}{s(s+2)(s-3)}$ are polii $\begin{matrix} s_1=0 & m_1=1 \\ s_2=-2 & m_2=1 \\ s_3=3 & m_3=1 \end{matrix} \Rightarrow c_f=\max\{0,-2,3\}=3$

$C_f = \text{Re}\, s>3$

$P(s)=s^2+1;\ Q(s)=(s+2)(s-3)=s^2-s-6 \Rightarrow Q'(s)=2s-1.$

$$G(s)\xleftrightarrow{L^{-1}} g(t)\underset{\text{Heaviside II}}{=}\left\{\frac{P(0)}{Q(0)}+\frac{P(-2)\cdot e^{-2t}}{-2\cdot Q'(-2)}+\frac{P(3)\cdot e^{3t}}{3\cdot Q'(3)}\right\}\cdot\sigma(t)=$$

$$=\left\{-\frac{1}{6}+\frac{5e^{-2t}}{-2\cdot(-5)}+\frac{10\cdot e^{+3t}}{3\cdot 5}\right\}\sigma(t)=\left\{-\frac{1}{6}+\frac{1}{2}\cdot e^{-2t}+\frac{2}{3}e^{3t}\right\}\sigma(t).$$

$$F(s)=G(s)\cdot e^{-2s}\underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} g(t-2)=\left\{-\frac{1}{6}+\frac{1}{2}e^{-2(t-2)}+\frac{2}{3}e^{3(t-2)}\right\}\cdot\sigma(t-2).$$

5. Fie imaginea Laplace $F(s)=\frac{s^2+5s+2}{s^2+9}$. a) domeniul de convergență pentru $F(s)$; b) semnalul original $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

a) $s^2+9=0 \Rightarrow s_{1,2}=\pm 3j$ poli simpli pentru $F_L(s)$;
$c_f=\max\{\text{Re}\, s_1, \text{Re}\, s_2\}=0 \Rightarrow F_L(s)$ are domeniul de convergență $C_f=\{\text{Re}\, s>0\}$.

b) $F_L(s)=\dfrac{s^2+5s+2}{s^2+9}=1+\dfrac{5s-7}{s^2+9}$ $G(s)$ are polii simpli $s_{1,2}=\pm 3j$

$P(s)=5s-7$; $Q(s)=s^2+9 \rightarrow Q'(s)=2s.$

Heaviside I:

$$G(s)\overset{L^{-1}}{\longleftrightarrow} g(t)=\left\{\frac{P(3j)\cdot e^{3jt}}{Q'(3j)}+\frac{P(-3j)\cdot e^{-j3t}}{Q'(-3j)}\right\}\cdot\sigma(t)=$$

$$=\left\{\frac{(15j-7)e^{j3t}}{6j}+\frac{(-15j-7)\cdot e^{-j3t}}{-6j}\right\}\cdot\sigma(t)=\left\{\frac{5}{2}\left(e^{j3t}+e^{-j3t}\right)-\right.$$

$$\left.-\frac{7}{6j}\left(e^{j3t}-e^{-j3t}\right)\right\}\sigma(t)=\left(\frac{5}{2}\cdot 2\cos 3t-\frac{7}{6j}\cdot 2j\sin 3t\right)\sigma(t)$$

$$g(t)=\left(5\cos 3t-\frac{7}{3}\sin 3t\right)\sigma(t);\quad 1\overset{L^{-1}}{\longleftrightarrow}\delta(t)\Rightarrow$$

$$F_L(s)=1+G(s)\overset{L^{-1}}{\longleftrightarrow} f(t)=\delta(t)+\left(5\cos 3t-\frac{7}{3}\sin 3t\right)\sigma(t)$$

$\boxed{\text{Obs.}}$ $f_0(t)\overset{F}{\longleftrightarrow}F_0(\omega)$; $f(t)=\sum_{k=0}^{\infty}f_0(t-kT)\overset{F}{\longleftrightarrow}\int_{-\infty}^{\infty}\sum_{k\geq 0}f_0(t-kT)\cdot$

$$\cdot e^{-j\omega t}dt=\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}f_0(t-kT)\cdot e^{-j\omega t}dt \overset{t-kT=y}{=\!=}$$

$$\sum_{k=0}^{\infty}\int_{-\infty}^{\infty}f_0(y)\cdot e^{-j\omega(y+kT)}dy=\sum_{k=0}^{\infty}\left(\int_{-\infty}^{\infty}f_0(y)\cdot e^{-j\omega y}dy\right)\cdot$$

$$\cdot e^{-j(kT)\omega}=F_0(\omega)\cdot\sum_{k\geq 0}\left(e^{-jT\omega}\right)^k=\frac{F_0(\omega)}{1-e^{-jT\omega}}.$$

6) Determinați semnalul original corespunzător imaginii Laplace

$$F(s)=\underbrace{\frac{s^2+2}{(s+4)(s-1)}}_{G(s)}\cdot e^{-5s}.\qquad G(s)=\frac{s^2+2}{s^2+3s-4}=$$

$$\begin{array}{r|l} s^2+2 & s^2+3s-4\\ -s^2-3s+4 & 1\\ \hline /-3s+6 & \end{array}$$

$$G(s)=1+\underbrace{\frac{-3s+6}{s^2+3s-4}}_{H(s)}\qquad H(s)=\frac{-3s+6}{s^2+3s-4}\qquad \begin{array}{l}P(s)=-3s+6\\ Q(s)=s^2+3s-4\rightarrow\\ Q'(s)=2s+3.\end{array}$$

$H(s)$ are polii: $s_1=-4$ $m_1=1$
$s_2=1$ $m_2=1$

Cu Heaviside I avem:

$$H(s)\overset{L^{-1}}{\longleftrightarrow}h(t)=\left\{\frac{P(-4)\cdot e^{-4t}}{Q'(-4)}+\frac{P(1)e^{t}}{Q'(1)}\right\}\cdot\sigma(t)=\left(\frac{18e^{-4t}}{-5}+\frac{3e^{t}}{5}\right)\sigma(t)=$$

$$=\left(-\frac{18}{5}e^{-4t}+\frac{3}{5}e^{t}\right)\sigma(t)=\frac{3}{5}\left(e^{t}-6e^{-4t}\right)\sigma(t).$$

$$G(s)\overset{L^{-1}}{\longleftrightarrow}g(t)=\delta(t)+\frac{3}{5}\left(e^{t}-6e^{-4t}\right)\sigma(t)$$

$$F(s)=G(s)\cdot e^{-5s}\underset{\text{întârziere}}{\overset{L^{-1}}{\longleftrightarrow}}g(t-5)=\delta(t-5)+\frac{3}{5}\left(e^{t-5}-6e^{-4(t-5)}\right)\sigma(t-5)=f(t)$$

(7.) $F(s) = \dfrac{s^2 \cdot e^{-8s}}{(2s+1)(s+5)} = \underbrace{\dfrac{s^2}{2s^2+11s+5}}_{G(s)} \cdot \overset{=4=}{e^{-8s}}$

$G(s) = \dfrac{1}{2} + \underbrace{\dfrac{-\frac{11}{2}s - \frac{5}{2}}{2s^2+11s+5}}_{H(s)}$; $H(s)$ are polii : $s_1 = -\dfrac{1}{2}$ $m_1 = 1$, $s_2 = -5$ $m_2 = 1$

$H(s) = \dfrac{P(s)}{Q(s)} = \dfrac{-\frac{11}{2}s - \frac{5}{2}}{2s^2+11s+5}$, $Q'(s) = 4s+11$.

Heaviside I

$H(s) \overset{L^{-1}}{\longleftrightarrow} h(t) = \left\{ \dfrac{P(s_K) \cdot e^{s_K t}}{Q'(s_K)} \right\} \cdot \sigma(t) =$

$= \left\{ \dfrac{\left(-\frac{11}{2} \cdot \left(-\frac{1}{2}\right) - \frac{5}{2}\right) \cdot e^{-\frac{t}{2}}}{4 \cdot \left(-\frac{1}{2}\right) + 11} + \dfrac{-\frac{11}{2} \cdot (-5) - \frac{5}{2}}{4 \cdot (-5) + 11} \cdot e^{-5t} \right\} \sigma(t)$

$= \left\{ \dfrac{\frac{1}{4} \cdot e^{-\frac{t}{2}}}{9} + \dfrac{25}{-9} e^{-5t} \right\} \cdot \sigma(t) = \dfrac{1}{9}\left(\dfrac{1}{4} e^{-\frac{t}{2}} - 25 e^{-5t}\right)\sigma(t)$.

$g(t) = \dfrac{1}{2}\delta(t) + h(t)$ + întârzierea $\Rightarrow f(t) = \dfrac{1}{2}\delta(t-8) + \dfrac{1}{9}\left(\dfrac{1}{4} e^{-\frac{t-8}{2}} - 25 e^{-5(t-8)}\right) \cdot \sigma(t-8)$.

(8.) $F(s) = \dfrac{(s^2-25)(s+2) \cdot e^{-2s}}{(s+5)(s+1)(s+3)}$, $\operatorname{Re} s > -1$.

$\downarrow$

are polii simpli $s_1 = -5$, $s_2 = -1$, $s_3 = -3$.

$G(s) = \dfrac{s^3+2s^2-25s-50}{s^3+9s^2+23s+15} = 1 - \dfrac{7s^2+48s+65}{s^3+9s^2+23s+15}$

$H(s) = \dfrac{7s^2+48s+65}{s^3+9s^2+23s+15} = \dfrac{P(s)}{Q(s)}$, $Q'(s) = 3s^2+18s+23$.

$h(t) = \mathcal{L}^{-1}\left\{ \dfrac{P(s)}{Q(s)} \right\} = \left\{ \sum_K \dfrac{P(s_K) \cdot e^{s_K t}}{Q'(s_K)} \right\} \cdot \sigma(t) =$

$= \left\{ \dfrac{(7 \cdot (-5)^2 + 48 \cdot (-5) + 65) e^{-5t}}{3 \cdot (-5)^2 + 18 \cdot (-5) + 23} + \dfrac{7 \cdot (-1)^2 + 48(-1) + 65}{3(-1)^2 + 18(-1) + 23} \cdot e^{-t} + \right.$

$\left. + \dfrac{7(-3)^2 + 48(-3) + 65}{3(-3)^2 + 18(-3) + 23} \cdot e^{-3t} \right\} \cdot \sigma(t) = \left(-\dfrac{105}{4} \cdot e^{-5t} + \dfrac{5}{4} e^{-t} + 25 e^{-3t}\right) \cdot \sigma(t)$

$g(t) = \delta(t) - h(t) \Rightarrow f(t) = \delta(t-2) - h(t-2) \Rightarrow$

$f(t) = \delta(t-2) + \left[\dfrac{105}{4} e^{-5(t-2)} - \dfrac{5}{4} e^{-(t-2)} - 25 \cdot e^{-3(t-2)}\right] \cdot \sigma(t-2)$.

$= 3 =$

9. $F(s) = \dfrac{s^2+1}{(s+1)(s+2)} = 1 + \underbrace{\dfrac{-3s-1}{s^2+3s+2}}_{G(s)}$ $\quad G(s) = \dfrac{P(s)}{Q(s)} = \dfrac{-3s-1}{s^2+3s+2}$

$\operatorname{Re} s > -1$ $\quad Q'(s) = 2s+3;\ s_1 = -1\ m_1 = 1;\ s_2 = -2\ m_2 = 1$

$g(t) = \left\{ \sum_k \dfrac{P(s_k)e^{s_k t}}{Q'(s_k)} \right\} \sigma(t) = \left\{ \dfrac{-3\cdot(-1)-1)e^{-t}}{2(-1)+3} + \right.$

$\left. + \dfrac{(-3(-2)-1)e^{-2t}}{2(-2)+3} \right\} \sigma(t) = (2e^{-t} - 5e^{-2t})\cdot \sigma(t).$

$f(t) = \delta(t) + (2e^{-t} - 5e^{-2t})\cdot\sigma(t).$

10. $F(s) = \dfrac{2}{(s-3)(s+3)^2} \xleftrightarrow{L^{-1}} f(t) = \left(\operatorname{Res}_{s=3}\{F(s)\cdot e^{st}\} + \operatorname{Res}_{s=-3}\{F(s)\cdot e^{st}\} \right) \sigma(t) =$

$\operatorname{Re} s > 3$ $\quad = 2\left\{ \dfrac{e^{st}}{(s+3)^2}\Big|_{s=3} + \left(\dfrac{e^{st}}{s-3}\right)'\Big|_{s=-3} \right\}\cdot\sigma(t) =$

$= 2\left\{ \dfrac{e^{3t}}{36} + \dfrac{te^{st}(s-3)-e^{st}}{(s-3)^2}\Big|_{s=-3} \right\}\cdot\sigma(t) =$

$= \left\{ \dfrac{2}{36}e^{3t} + \dfrac{2(-6t-1)e^{-3t}}{36} \right\}\sigma(t) = \dfrac{1}{18}(e^{3t} - e^{-3t} - 6te^{-3t})\sigma(t)$

$= \left[\dfrac{1}{9}\sinh(3t) - \dfrac{t}{3}e^{-3t}\right]\cdot\sigma(t).$

11. $F(s) = \dfrac{s^3-3s^2+3s+3}{s-2} = s^2 - s + 1 + \dfrac{5}{s-2} \xleftrightarrow{L^{-1}} \delta''(t) - \delta'(t) + \delta(t) + 5e^{2t}\sigma(t).$

<u>Temă:</u> $\begin{cases} F(s) = \dfrac{2s^2+20s+45}{(s^2+5s+6)(s+4)} \xleftrightarrow{L^{-1}} ? \\ \operatorname{Re} s > -2. \end{cases}$

12. $F(s) = \dfrac{s^2-s+2}{s(s+1)(s+2)}$, $\quad Q(s) = s^2+3s+2 \to Q'(s) = 2s+3$; $\quad P(s) = s^2-s+2$

$\operatorname{Re} s > 0$ $\quad$ Heaviside II:

$f(t) = \left\{ \dfrac{P(0)}{Q(0)} + \dfrac{P(-1)e^{-t}}{(-1)Q'(-1)} + \dfrac{P(-2)\cdot e^{-2t}}{(-2)\cdot Q'(-2)} \right\}\cdot\sigma(t) = \left\{ 1 + \dfrac{4}{-1\cdot 1}\cdot e^{-t} + \dfrac{8}{-2\cdot(-1)}\cdot e^{-2t} \right\}\sigma(t) \Rightarrow$

$f(t) = (1 - 4e^{-t} + 4e^{-2t})\sigma(t).$

13. a) $F(s) = \dfrac{s-2}{(s^2-4s+5)^2}$; b) $F(s) = \dfrac{2s}{(s^2+4)^2}\cdot e^{-2s}.$

a) $F(s)$ are polii dubli complex conjugați $s^2-4s+5=0 \to s_{1,2} = 2\pm j$, $m_{1,2} = 2$

$F(s) \xleftrightarrow{L^{-1}} f(t) = \mathcal{L}^{-1}\{F(s)\} = \left\{ 2\operatorname{Re}\operatorname{Res}_{s=2+j}\left[\dfrac{(s-2)\cdot e^{st}}{(s-2-j)^2(s-2+j)^2} \right] \right\}\cdot\sigma(t) =$

$$= \left\{ 2\,\mathrm{Re} \cdot \left[\frac{(s-2)\cdot e^{st}}{(s-2+j)^2} \right]' \Bigg|_{s=+2+j} \right\} \cdot \sigma(t) = \left\{ 2\,\mathrm{Re}\, \frac{[e^{st} + t e^{st}(s-2)]\cdot(s-2+j)^2 - (s-2)\cdot e^{st}\cdot 2\cancel{(s-2+j)}}{(s-2+j)^{\cancel{4}\,3}} \Bigg|_{s=2+j} \right\} \cdot \sigma(t) = \left\{ 2\,\mathrm{Re}\, \frac{[e^{(2+j)t} + t\cdot j\cdot e^{(2+j)t}]\,2j - 2j\cdot e^{(2+j)t}}{-8j} \right\} \cdot \sigma(t) =$$

$$= \left\{ 2\,\mathrm{Re}\, \frac{\cancel{2j e^{(2+j)t}} + 2j^2 t e^{2t}(\cos t + j \sin t) - \cancel{2j e^{(2+j)t}}}{-8j} \right\} \cdot \sigma(t) =$$

$$= \left\{ 2\,\mathrm{Re}\, \frac{- t e^{2t} \cos t - 2 t e^{2t} j \sin t}{-8j} \right\} \cdot \sigma(t) =$$

$$= \frac{t}{2} e^{2t} \sin t \cdot \sigma(t). \qquad F(s) \xleftrightarrow{L^{-1}} f(t) = \frac{t}{2}\cdot e^{2t} \sin t \cdot \sigma(t).$$

b). $$G(s) = \frac{2s}{(s^2+4)^2} \xleftrightarrow{L^{-1}} g(t) = \left\{ 2\,\mathrm{Re} \operatorname*{Res}_{s=2j} \frac{2s}{(s^2+4)^2} \cdot e^{+st} \right\} \cdot \sigma(t) =$$

$(s-2j)^2\cdot(s+2j)^2$

$$= \left\{ 2\,\mathrm{Re} \left[\frac{2s\cdot e^{st}}{(s+2j)^2} \right]' \Bigg|_{s=2j} \right\} \cdot \sigma(t) = \left\{ 2\,\mathrm{Re}\, \frac{(e^{st} + st\cdot e^{st})(s+2j)^2 - 2s\cdot e^{st}\cdot\cancel{(s+2j)}}{(s+2j)^{\cancel{4}\,3}} \Bigg|_{s=2j} \right\} \cdot \sigma(t) = 4\left\{ \mathrm{Re}\, \frac{\cancel{4j e^{j2t}} + (2j)\cdot 4j t e^{j2t} - \cancel{4j\cdot e^{j2t}}}{(4j)^3} \right\} \cdot \sigma(t) =$$

$$= \frac{1}{16} \left\{ \mathrm{Re}\, \frac{\overset{-8}{8j^2}\, t (\cos 2t + j \sin 2t)}{-j} \right\} \cdot \sigma(t) =$$

$$= \frac{1}{2} \left\{ \mathrm{Re}\, (t \sin 2t - j t \cos 2t) \right\} \cdot \sigma(t) = \frac{t}{2} \sin 2t \cdot \sigma(t).$$

$$G(s) = \frac{2s}{(s^2+4)^2} \xleftrightarrow{L^{-1}} g(t) = \frac{t}{2} \sin 2t \cdot \sigma(t)$$

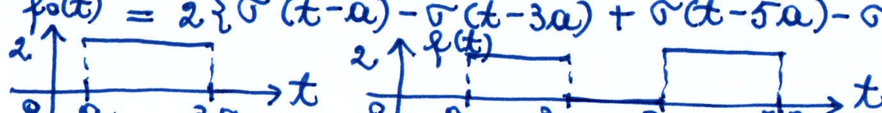
$$F(s) = G(s)\cdot e^{-2s} \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} f(t) = g(t-2) = \frac{1}{2}(t-2) \sin 2(t-2) \cdot \sigma(t-2)$$

(14) $$F_L(s) = \frac{1}{s\,\mathrm{ch}(as)} = \frac{\frac{1}{s}}{\frac{e^{+as} + e^{-as}}{2}} = \frac{2\frac{e^{-as}}{s}}{1 + e^{-2as}} = \frac{\frac{2}{s} e^{-as} (1 - e^{-2as})}{1 - e^{-4as}}$$

$$= \frac{\frac{2}{s} e^{-as} - \frac{2}{s} e^{-2as}}{1 - e^{-4as}} = \frac{F_0(s)}{1 - e^{-4as}}; \quad \left[\begin{array}{l} F_0(s) = \frac{2}{s} e^{-as} - \frac{2}{s} e^{-3as} \xleftrightarrow{L^{-1}} \\ f_0(t) = 2[\sigma(t-a) - \sigma(t-3a)] \\ T = 4a \end{array} \right.$$

$$f(t) = \mathcal{L}^{-1}\{F_L(s)\} = \sum_{k=0}^{\infty} f_0(t - 4ak) = \sum_{k=0}^{\infty} 2[\sigma(t-a-4ak) - \sigma(t-3a-4ak)]$$

$$f_0(t) = 2\{\sigma(t-a) - \sigma(t-3a) + \sigma(t-5a) - \sigma(t-7a) + \dots\}$$



(15) $F(s)=\dfrac{2s^2+20s+45}{(s^2+5s+6)(s+4)}$ o imagine Laplace. Aflați: c_f, C_f și $f(t)=\mathcal{L}^{-1}\{F(s)\}$.

Soluție: $F(s)=\dfrac{2s^2+20s+45}{(s+2)(s+3)(s+4)}=\dfrac{P(s)}{Q(s)}$ are polii:

$$\left.\begin{array}{ll} s_1=-2 & m_1=1 \\ s_2=-3 & m_2=1 \\ s_3=-4 & m_3=1 \end{array}\right| \Rightarrow c_f=\max\{-2,-3,-4\}=-2;\ C_f=\{\operatorname{Re} s>-2\}$$

Aplicăm $\boxed{\text{Heaviside I}}$:

$$f(t)=\left\{\sum_{k=1}^{3}\frac{P(s_k)\cdot e^{s_k t}}{Q'(s_k)}\right\}\cdot\sigma(t);\quad \begin{array}{l} Q(s)=s^3+9s^2+26s+24 \\ Q'(s)=3s^2+18s+26. \end{array}$$

$$f(t)=\left\{\sum_{k=1}^{3}\frac{(2s_k^2+20s_k+45)\cdot e^{s_k t}}{3s_k^2+18s_k+26}\right\}\cdot\sigma(t)=$$

$$=\left(\frac{13}{2}e^{-2t}-3e^{-3t}-\frac{3}{2}e^{-4t}\right)\cdot\sigma(t).$$

$\boxed{\text{Trecerea TL} \rightarrow \text{TF}}$ (de la transformata Laplace la transformata Fourier)

Teoremă.

Considerăm $F_L(s)$ o imagine Laplace a originalului $f(t)$ și corespondența $f(t)\xleftrightarrow{L}F_L(s)$ pentru $\{\operatorname{Re} s>c_f\}$. Atunci:

i) dacă $F_L(s)=G(s)\cdot e^{-as}$ cu $G(s)$ rațională și $\boxed{c_f<0}$ atunci există $F(\omega)=F_L(s)\big|_{s=j\omega}$.

ii) dacă $F_L(s)$ este funcție rațională care are numai poli simpli imaginari $j\omega_k$ și $\boxed{c_f=0}$ atunci există $F(\omega)=F_L(s)\big|_{s=j\omega}+\sum_k \pi a_k\delta(\omega-\omega_k)$

unde $a_k=\operatorname{Res}_{s=j\omega_k} F_L(s)$.

iii) dacă $F_L(s)=G(s)\cdot e^{-as}$, $\boxed{c_f>0}\Rightarrow$ nu există $F(\omega)$.

iv) $f(t)$ semnal original absolut integrabil $\Rightarrow$ există $F_L(s)$,

$F(\omega)$ și $F(\omega) = F_L(s)\big|_{s=j\omega}$.

Aplicații ale teoremei de trecere TL → TF.

(16) Considerăm corespondența următoare:

$f(t) \xleftrightarrow{L} F_L(s) = \dfrac{s}{s^2+3js+10}$. Aflați dacă există $F(\omega)$ și în caz afirmativ calculați $F(\omega)$.

Soluție: Polii lui $F_L(s)$ sînt rădăcinile ecuației:

$$s^2 + 3js + 10 = 0 \Rightarrow s_{1,2} = \frac{-3j \pm \sqrt{-9-40}}{2} \Rightarrow$$

$$\left.\begin{array}{l} s_1 = -5j \\ s_2 = 2j \end{array}\right| \text{ poli simpli imaginari.}$$

$\mathrm{Re}\, s_1 = \mathrm{Re}\, s_2 = 0 \Rightarrow c_f = 0$.

Aplicăm TL → TF ii).

$$\cdot\ a_1 = \operatorname*{Res}_{s=-5j} F_L(s) = \left.\frac{s}{2s+3j}\right|_{s=-5j} = \frac{5}{7}$$

$$\cdot\ a_2 = \operatorname*{Res}_{s=2j} F_L(s) = \left.\frac{s}{2s+3j}\right|_{s=2j} = \frac{2}{7}.$$

Există $F(\omega) = F_L(s)\big|_{s=j\omega} + \pi a_1 \delta(\omega+5) + \pi a_2 \delta(\omega-2) =$

$$= \frac{j\omega}{-\omega^2 + j3\omega + 10} + \frac{\pi}{7}\left[5\delta(\omega+5) + 2\delta(\omega-2)\right].$$

17) $f(t) \xleftrightarrow{L} F_L(s) = \frac{s+1}{s(s+2)(s+4)}$. Determinați dacă există $F(\omega)$.

Polii sunt: $s_1 = 0 \quad m_1 = 1$ $\quad c_f = \max\{Re\, s_1, Re\, s_2, Re\, s_3\} = 0 \Rightarrow$

$s+2=0 \Rightarrow s_2 = -2 \; m_2 = 1$ $\quad c_f = 0 \Rightarrow$ nu putem aplica TL→TF ii) deoarece $F_L(s)$ nu are numai poli simpli imaginari

$s+4=0 \Rightarrow s_3 = -4 \quad m_3 = 1$

Heaviside II

$F_L(s) = \frac{P(s)}{s \cdot Q(s)} = \frac{s+1}{s(s+2)(s+4)}$; $\begin{cases} Q(s) = s^2+6s+8 \\ Q'(s) = 2s+6 \end{cases}$ $\quad [P(s) = s+1.$

$$f(t) = \left\{\frac{P(0)}{Q(0)} + \sum_{k=2}^{3} \frac{P(s_k) \cdot e^{s_k t}}{s_k \cdot Q'(s_k)}\right\} \cdot \sigma(t) =$$

$$= \left\{\frac{1}{8} + \frac{(s_2+1)e^{s_2 t}}{s_2 \cdot (2s_2+6)} + \frac{(s_3+1)e^{s_3 t}}{s_3 \cdot (2s_3+6)}\right\} \cdot \sigma(t) =$$

$$= \left\{\frac{1}{8} + \frac{-e^{-2t}}{-2 \cdot 2} + \frac{-3 \cdot e^{-4t}}{-4 \cdot (-2)}\right\} \sigma(t) \Rightarrow$$

$$f(t) = \frac{1}{8}\sigma(t) + \frac{1}{4} e^{-2t}\sigma(t) - \frac{3}{8} e^{-4t}\sigma(t).$$

$\frac{1}{8} \;\Big|\; \sigma(t) \xleftrightarrow{F} \frac{1}{j\omega} + \pi\delta(\omega)$; $\quad e^{-at}\sigma(t) \underset{a>0}{\xleftrightarrow{F}} \frac{1}{a+j\omega}$

$\frac{1}{4} \;\Big|\; e^{-2t}\sigma(t) \xleftrightarrow{F} \frac{1}{2+j\omega}$

$-\frac{3}{8} \;\Big|\; e^{-4t}\sigma(t) \xleftrightarrow{F} \frac{1}{4+j\omega}$

$$f(t) \xleftrightarrow{F} \frac{1}{8}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right) + \frac{1}{4(2+j\omega)} - \frac{3}{8(4+j\omega)} = F(\omega).$$

18) Examen

Fie imaginea Laplace $F_L(s) = \frac{(s^2-25)(s+2)}{(s^2+6s+5)(s+3)} \cdot e^{-2s}$.

a) determinați domeniul de convergență al lui $F_L(s)$

b) $f(t) = \mathcal{L}^{-1}\{F_L(s)\}$ semnalul original

c) $F(\omega)$ dacă există

Rezolvare:

a) $F_L(s) = \frac{(s-5)(s+2)}{(s+1)(s+3)} \cdot e^{-2s}$ are polii: $s_1 = -1 \quad m_1 = 1$

$s_2 = -3 \quad m_2 = 1$

$c_f = \max\{Re\, s_1, Re\, s_2\} = \max\{-1, -3\} = -1 \Rightarrow C_f = \{Re\, s > -1\}$

b) $F_L(s) = \underbrace{\frac{(s-5)(s+2)}{(s+1)(s+3)}}_{G(s)} e^{-2s}$; $G(s) = \frac{\overbrace{s^2-3s-10}^{=10=}}{s^2+4s+3}$

$$\begin{array}{r|l} s^2-3s-10 & s^2+4s+3 \\ -s^2-4s-3 & 1 \\ \hline / \ -7s-13 & \end{array}$$

$G(s) = 1 + \underbrace{\frac{-7s-13}{s^2+4s+3}}_{\cdot H(s)}$ $\quad H(s) = \frac{-7s-13}{s^2+4s+3} = \frac{P(s)}{Q(s)}$ are polii simpli, nenuli $s_1=-1$ $m_1=1$, $s_2=-3$ $m_2=1$

$Q'(s) = 2s+4$. Cu formula Heaviside II:

$$H(s) = \frac{-7s-13}{s^2+4s+3} = \frac{P(s)}{Q(s)} \xleftrightarrow{L^{-1}} h(t) = \left\{ \frac{P(s_1)\cdot e^{s_1 t}}{Q'(s_1)} + \frac{P(s_2)\cdot e^{s_2 t}}{Q'(s_2)} \right\} \cdot \sigma(t)$$

$$= \left\{ \frac{-7(-1)-13}{2(-1)+4} \cdot e^{-t} + \frac{-7(-3)-13}{2(-3)+4} \cdot e^{-3t} \right\} \sigma(t)$$

** $\boxed{h(t) = \left\{ \sum_{k=1}^{2} \frac{P(s_k)\cdot e^{s_k t}}{Q'(s_k)} \right\} \cdot \sigma(t) = \left\{ \sum_{k=1}^{2} \frac{(-7s_k-13)\cdot e^{s_k t}}{2s_k+4} \right\} \cdot \sigma(t)}$

$h(t) = (-3e^{-t} - 4e^{-3t})\sigma(t)$.

$g(t) = \mathcal{L}^{-1}\{G(s)\} = ?$

$G(s) = 1 + H(s) \xleftrightarrow{L^{-1}} g(t) = \delta(t) + (-3e^{-t} - 4e^{-3t})\sigma(t)$

$F_L(s) = e^{-2s} \cdot G(s) \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} f(t) = g(t-2) = \delta(t-2) - (3e^{-t+2} + 4e^{-3t+6}) \cdot \sigma(t-2)$.

c) $\operatorname{Re} s > -1 = c_f$; $c_f = -1 < 0 \Rightarrow$ există $F(\omega) = F_L(s)\big|_{s=j\omega}$

$$F(\omega) = \frac{(j\omega)^2 - 3j\omega - 10}{(j\omega)^2 + 4j\omega + 3} e^{-j2\omega} = \frac{-\omega^2 - 3j\omega - 10}{-\omega^2 + 4j\omega + 3} \cdot e^{-j2\omega} \Rightarrow$$

$$F(\omega) = \frac{\omega^2 + 3j\omega + 10}{\omega^2 - 4j\omega - 3} \cdot e^{-j2\omega}.$$

Seminar

19. Fie imaginea Laplace $F_L(s) = \frac{s^2}{s^2+9} \cdot e^{-9s}$.

a) c_f ; b) $f(t)$ semnalul original ; c) $F(\omega)$ dacă există.

Rezolvare: a) $F_L(s)$ are polii $s_{1,2} = \pm 3j \Rightarrow c_f = 0 \Rightarrow \{\operatorname{Re} s > 0\}$.

b) $G(s) = \frac{s^2}{s^2+9} = 1 - \frac{9}{s^2+9} \xleftrightarrow{L^{-1}} g(t) = \delta(t) - 3\sin 3t \cdot \sigma(t)$

$F_L(s) = G(s) \cdot e^{-9s} \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} f(t) = g(t-9) = \delta(t-9) - 3\sin 3(t-9) \cdot \sigma(t-9)$.

c) $g(t) = \delta(t) - 3\sin 3t \cdot \sigma(t) \xleftrightarrow{F} G(\omega) = 1 - \frac{9}{9-\omega^2} - \frac{3\pi}{2j}[\delta(\omega-3) - \delta(\omega+3)]$

$f(t) = g(t-9) \xleftrightarrow{F} F(\omega) = e^{-j9\omega} - \frac{9e^{-j9\omega}}{9-\omega^2} - \frac{3\pi}{2j}[\delta(\omega-3) - \delta(\omega+3)] \cdot e^{-j9\omega}$

Altfel, cu TL → TF. ii)

$G(s)$ are polii simpli imaginari $s_{1,2} = \pm 3j$ și $c_f = 0$:

$G(\omega) = G(s)\big|_{s=j\omega} + \pi a_1 \delta(\omega-\omega_1) + \pi a_2 \delta(\omega-\omega_2)$, unde

$a_1 = \operatorname*{Res}_{s=3j} G(s) = \operatorname*{Res}_{s=3j} \frac{s^2}{s^2+9} = \frac{s^2}{2s}\Big|_{s=3j} = \frac{3j}{2}$

$a_2 = \operatorname*{Res}_{s=-3j} G(s) = \frac{s}{2}\Big|_{s=-3j} = \frac{-3j}{2}$.

$G(\omega) = \frac{-\omega^2}{9-\omega^2} + \frac{3\pi j}{2}[\delta(\omega-3) - \delta(\omega+3)] = 1 - \frac{9}{9-\omega^2} - \frac{3\pi}{2j}[\delta(\omega-3) - \delta(\omega+3)]$

$f(t) = g(t-9) \xrightarrow{\text{întîrz.}} F(\omega) = e^{-j9\omega} G(\omega) = e^{-j9\omega} - \frac{9e^{-j9\omega}}{9-\omega^2} - \frac{3\pi}{2j}[\delta(\omega-3) - \delta(\omega+3)]e^{-j9\omega}$

---

20 a) $F_L(s) = \frac{s^2+3s-4}{(s+1)(s-2)(s-3)} \cdot e^{-2s} \xleftrightarrow{L^{-1}} f(t) = ?$ există $F(\omega)$?

Seminar:

b) $F_L(s) = \frac{s^2-4}{(s^2+4)^2} \cdot e^{-3s} \xleftrightarrow{L^{-1}} f(t) = ?$ există $F(\omega) = ?$

c) $F_L(s) = \frac{s+2}{(s^2+4s+8)^2} \cdot e^{-s} \xleftrightarrow{L^{-1}} f(t) = ?$ $\exists\, F(\omega) = ?$

Rezolvare: b) $G(s) = \frac{s^2-4}{(s^2+4)^2} = -\left(\frac{s}{s^2+4}\right)' = -(\mathcal{L}\{\cos 2t \cdot \sigma(t)\})' \overset{\text{derivarea imaginii}}{=}$

$= \mathcal{L}\{t \cdot \cos 2t \cdot \sigma(t)\} \Rightarrow$

$G(s) \xleftrightarrow{L^{-1}} g(t) = t \cdot \cos 2t \cdot \sigma(t)$

$F_L(s) = G(s) \cdot e^{-3s} \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} g(t-3) = f(t) = (t-3)\cos 2(t-3)\sigma(t-3)$.

.. $C_f = \{\operatorname{Re} s > 0\}$, $F_L(s)$ are polii imaginari dubli $s_{1,2} = \pm 2j$.

Nu putem aplica TL → TF.

$g(t) = \cos 2t\, \sigma(t) \xleftrightarrow{F} \frac{1}{j} \cdot \frac{\omega}{\omega^2-\omega_0^2} + \frac{\pi}{2}[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \overset{\omega_0 = 2}{=}$

$= \frac{j\omega}{4-\omega^2} + \frac{\pi}{2}[\delta(\omega-2) + \delta(\omega+2)]$

$g(t) = t\cos 2t\, \sigma(t) \underset{\text{derivarea în frecvență}}{\xleftrightarrow{F}} -\left(\frac{+\omega}{4-\omega^2}\right)' + j\frac{\pi}{2}[\delta'(\omega-2) + \delta'(\omega+2)] =$

$= -\frac{\omega^2+4}{(4-\omega^2)^2} + j\frac{\pi}{2}[\delta'(\omega-2) + \delta'(\omega+2)] = G(\omega)$

$G(\omega) = -\frac{4+\omega^2}{(4-\omega^2)^2} + j\frac{\pi}{2}[\delta'(\omega-2) + \delta'(\omega+2)]$.

$f(t) = g(t-3) \Rightarrow F(\omega) = e^{-j3\omega} G(\omega) = e^{-j3\omega}\left\{-\frac{4+\omega^2}{(4-\omega^2)^2} + j\frac{\pi}{2}[\delta'(\omega-2) + \delta'(\omega+2)]\right\}$

c) $G(s)=\dfrac{s+2}{[(s+2)^2+4]^2} \underset{\text{deplasarea}}{\overset{L^{-1}}{\longleftrightarrow}} e^{-2t}\cdot \mathcal{L}^{-1}\left\{\dfrac{s}{(s^2+4)^2}\right\} =$

$= e^{-2t}\cdot \dfrac{1}{4}\,\mathcal{L}^{-1}\left\{\dfrac{4s}{(s^2+2^2)^2}\right\} = \dfrac{e^{-2t}}{4}\cdot \mathcal{L}^{-1}\left\{-\left(\dfrac{2}{s^2+2^2}\right)'\right\} =$

$= \dfrac{e^{-2t}}{4}\cdot \mathcal{L}^{-1}\left\{-\left(\mathcal{L}\{\sin 2t\cdot\sigma(t)\}\right)'\right\} \underset{\text{imag.}}{\overset{\text{der.}}{=\!=}} \dfrac{e^{-2t}}{4}\,\mathcal{L}^{-1}\{\mathcal{L}\{t\cdot \sin 2t\,\sigma(t)\}\}$

$= \dfrac{e^{-2t}}{4}\cdot t\sin 2t\,\sigma(t) = g(t)$

$F_L(s) = G(s)\cdot e^{-s} \underset{\text{întârzierea}}{\overset{L^{-1}}{\longleftrightarrow}} f(t)= g(t-1) = \dfrac{e^{-2(t-1)}}{4}\cdot (t-1)\sin 2(t-1)\cdot \sigma(t-1).$

$F_L(s)$ are polii $s_{1,2}=-2\pm 2j \Rightarrow c_f = \max\{\operatorname{Re} s_1, \operatorname{Re} s_2\} = -2<0 \Rightarrow f(t)$ are

$F(\omega) \overset{TL\to TF\ i)}{=\!=\!=} F_L(s)\big|_{s=j\omega} = \dfrac{2+j\omega}{(-\omega^2+4j\omega+8)^2}\cdot e^{-j\omega}$

a) $G(s)=\dfrac{s^2+3s-4}{(s+1)(s-2)(s-3)}$ are polii $\begin{matrix} s_1=-1 & m_1=1\\ s_2=2 & m_2=1\\ s_3=3 & m_3=1\end{matrix} \Rightarrow c_f=3 \Rightarrow$

$\Rightarrow \operatorname{Re} s>3 \Rightarrow C_f=\{\operatorname{Re} s>3\} \leftarrow$ domeniul de convergență pt. $F_L(s)$.

$G(s)=\dfrac{P(s)}{Q(s)} = \dfrac{s^2+3s-4}{s^3-4s^2+s+6}$ $\quad Q(s)=(s+1)(s^2-5s+6)= s^3-5s^2+6s + s^2-5s+6 = s^3-4s^2+s+6$

H.I $\quad Q'(s) = 3s^2-8s+1$

$g(t)=\left\{\sum_{k=1}^{3}\dfrac{P(s_k)\cdot e^{s_k t}}{Q'(s_k)}\right\}\cdot\sigma(t) = \left\{\dfrac{(-1)^2+3(-1)-4}{3(-1)^2-8\cdot(-1)+1}\cdot e^{-t} + \dfrac{2^2+3\cdot 2-4}{3\cdot 2^2-8\cdot 2+1}\cdot e^{2t}\right.$

$\left. + \dfrac{3^2+3\cdot 3-4}{3\cdot 3^2-8\cdot 3+1}\cdot e^{3t}\right\}\sigma(t) \Rightarrow$

$g(t) = \left\{\dfrac{-6}{-4}e^{-t} + \dfrac{6}{-3}e^{2t} + \dfrac{14}{4}\cdot e^{3t}\right\}\sigma(t).$

$g(t) = \left(\dfrac{3}{2}e^{-t} - 2e^{2t} + \dfrac{7}{2}e^{3t}\right)\sigma(t)$

$F_L(s) = G(s)\cdot e^{-2s} \underset{\text{întârzierea}}{\overset{L^{-1}}{\longleftrightarrow}} f(t) = g(t-2) = \left(\dfrac{3}{2}e^{-t+2} - 2e^{2t-4} + \dfrac{7}{2}e^{3t-6}\right)\sigma(t-2).$

$\cdot\ c_f = 3>0 \Rightarrow$ nu există $F(\omega)$.

Teorema de trecere TF → TL

TF → TL cu poli numai complecşi

1. $F(\omega)$ = funcţie raţională sau raţională $\cdot e^{-j\alpha\omega}$ cu $\alpha > 0$, transformată Fourier pentru semnalul original $f(t)$.
Atunci există $F_L(s) = F(\omega)\big|_{\omega = \frac{s}{j}}$ TL pentru $f(t) \Leftrightarrow F_L(s)$ este olomorfă în $\{\operatorname{Re} s > 0\}$.
Adică toţi polii lui $F_L(s)$ sînt în $\{\operatorname{Re} s \le 0\}$.

Aplicaţie: Următoarele TF sunt şi TL?

(1) $F(\omega) = \dfrac{\omega - 2}{\omega^2 + 2j\omega + 15} \Rightarrow F_L(s) = F(\omega)\big|_{\omega = \frac{s}{j}} = \dfrac{\frac{s}{j} - 2}{-s^2 + 2s + 15} =$

$= \dfrac{2 + js}{s^2 - 2s - 15}$ are polii $s_1 = 5 \in \{\operatorname{Re} s > 0\} \Rightarrow F_L(s)$ nu
$s_2 = -3$
este olomorfă pe $\{\operatorname{Re} s > 0\} \Rightarrow$ nu există $F_L(s)$.

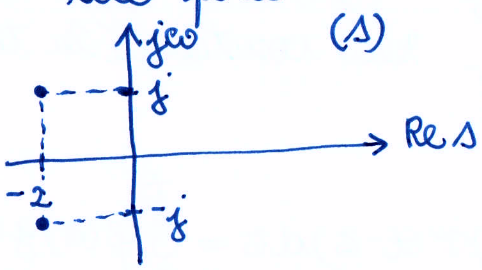
Metoda a II-a: Direct.

$F(\omega) = \dfrac{5-2j}{8} \cdot \dfrac{1}{\omega - 3j} + \dfrac{3+j}{8} \cdot \dfrac{1}{\omega + 5j} = \dfrac{2+5j}{8} \cdot \dfrac{1}{3 + j\omega} + \dfrac{-1+3j}{8} \cdot$
$\cdot \dfrac{1}{-5 + j\omega} \xleftrightarrow{L^{-1}} f(t) = \dfrac{2+5j}{8} \cdot e^{3t}\sigma(-t) + \dfrac{-1+3j}{8} \cdot e^{-5t}\sigma(t)$,
dar nu e semnal cauzal $\Rightarrow$ nu există $F_L(s)$.

(2) $F(\omega) = \dfrac{2 + 2j\omega}{-\omega^2 + 5j\omega + 6} \Rightarrow F_L(s) = \dfrac{2 + 2s}{s^2 + 5s + 6}$ are polii $\left.\begin{matrix} s_1 = -2 \\ s_2 = -3 \end{matrix}\right| \Rightarrow$
$F_L(s)$ este olomorfă pe $\{\operatorname{Re} s > 0\} \Rightarrow$ este TL.

(3) $F(\omega) = \dfrac{2 + j\omega}{-\omega^2 + 4j\omega + 5} \cdot e^{-j\omega} \Rightarrow F_L(s) = F(\omega)\big|_{\omega = \frac{s}{j}} = \dfrac{2+s}{s^2 + 4s + 5} \cdot e^{-s}$

are polii: $s^2 + 4s + 5 = 0 \Rightarrow s_{1,2} = -2 \pm j$ poli simpli $\Rightarrow$
$F_L(s)$ este olomorfă şi deci este pe $\operatorname{Re} s > 0$
TL.



$G(s) = \dfrac{s+2}{(s+2)^2 + 1} \xleftrightarrow{L^{-1}} g(t) = e^{-2t} \cdot \mathcal{L}^{-1}\left\{\dfrac{s}{s^2+1}\right\}$
$= e^{-2t}\cos t \cdot \sigma(t)$

$F_L(s) = G(s) \cdot e^{-s} \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} f(t) = g(t-1) =$
$= e^{-2(t-1)}\cos(t-1) \cdot \sigma(t-1)$

$\cdot\ c_f = \max\{\operatorname{Re} s_1, \operatorname{Re} s_2\} = -2 \overset{<0}{\Rightarrow} F(\omega) = F_L(s)\big|_{s = j\omega} = \dfrac{2 + j\omega}{-\omega^2 + 4j\omega + 5} \cdot e^{-j\omega}$. Se verifică!

(4) $F(\omega) = \dfrac{-5j\omega}{\omega^2 + \omega - 6}$. Există $F_L(s) = ?$

$F_L(s) = F(\omega)\big|_{\omega = \frac{s}{j}} = \dfrac{-5s}{-s^2 - sj - 6} = \dfrac{5s}{s^2 + js + 6} = \dfrac{2}{s - 2j} + \dfrac{3}{s + 3j}$.

Polii lui $F_L(s)$ sunt: $s_1=2j$, $s_2=-3j$. $F_L(s)$ este olomorfă pe $\{\operatorname{Re} s>0\}$. Presupunem că $F_L(s)$ este TL:

$$F_L(s)=\frac{2}{s-2j}+\frac{3}{s+3j}\xleftrightarrow{L^{-1}} f(t)=2e^{2jt}\sigma(t)+3e^{-3jt}\sigma(t).$$

· Calculăm: $f(t)\xleftrightarrow{F}F(\omega)=?$

$$\sigma(t)\xleftrightarrow{F}\frac{1}{j\omega}+\pi\delta(\omega)\Rightarrow e^{2jt}\sigma(t)\xleftrightarrow{F}\frac{1}{j(\omega-2)}+\pi\delta(\omega-2)\quad\Big|\cdot 2$$

$$e^{-j3t}\sigma(t)\xleftrightarrow{F}\frac{1}{j(\omega+3)}+\pi\delta(\omega+3)\quad\Big|\cdot 3$$

$$\boxed{e^{j\omega_0 t}f(t)\xleftrightarrow{F}F(\omega-\omega_0)}$$
deplasarea

$$f(t)\xleftrightarrow{F}F_0(\omega)=\frac{2}{j\omega-2j}+\frac{3}{j\omega+3j}+\pi[2\delta(\omega-2)+3\delta(\omega+3)]=$$

$$=\frac{5j\omega}{-\omega^2-\omega+6}+\pi[2\delta(\omega-2)+3\delta(\omega+3)]=\frac{-5j\omega}{\omega^2+\omega-6}+\pi[2\delta(\omega-2)+3\cdot\delta(\omega+3)]\neq F(\omega).$$

$F_L(s)$ este o imagine Laplace care implică o TF
$F_0(\omega)=\dfrac{-5j\omega}{\omega^2+\omega-6}+\pi[2\delta(\omega-2)+3\delta(\omega+3)]$.

Deci $F(\omega)=\dfrac{-5j\omega}{\omega^2+\omega-6}$ nu este o TF. Nici cond. de a avea numai poli complecși nu este îndeplinită.

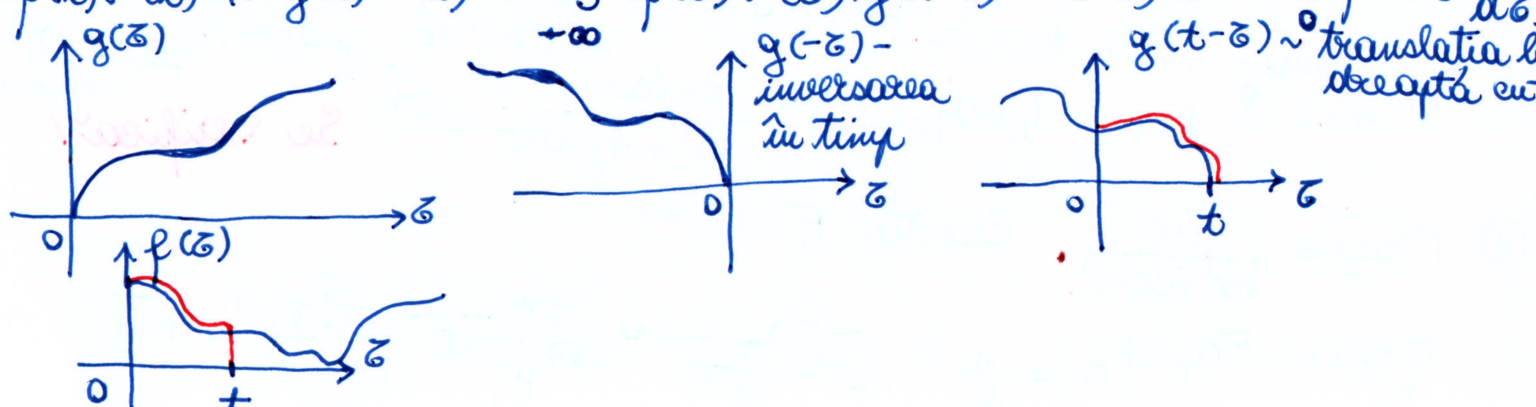
Ecuații diferențiale cu TL

1. $Li'(t)+Ri(t)=\underbrace{E\sigma(t)}_{\text{excitație treaptă}}$, $i(0)=0$.

2. $\dfrac{\partial T}{\partial t}(x,t)+v\dfrac{\partial T}{\partial x}(x,t)=0$, $T(x,0)=0$ și $T(0,t)=T_0\sigma(t)$ ← intrarea constantă în timp
proces termic prin suprafață tubulară.

Convoluția pentru semnale originale

$$f(t)\sigma(t)*g(t)\sigma(t)=\int_{-\infty}^{\infty}f(\tau)\sigma(\tau)g(t-\tau)\sigma(t-\tau)\,d\tau=\int_0^t f(\tau)g(t-\tau)\,d\tau$$



① $L i'(t) + R i(t) = \underbrace{E\sigma(t)}_{\text{exitație treaptă}}$, $i(0)=0$.

$L s\, I(s) + R I(s) = \frac{E}{s} \Rightarrow I(s) = \frac{E}{s(Ls+R)}$ are polii $s_1 = 0$ $m_1=1$, $s_2 = -\frac{R}{L}$ $m_2=1$

$\operatorname{Re} I(s) > 0$. $Q'(s) = L$; $Q(s) = Ls+R$.

Heaviside II:

$$I(s) = \frac{E}{s(Ls+R)} = \frac{P(s)}{sQ(s)} \xleftrightarrow{\mathcal{L}^{-1}} i(t) = \mathcal{L}^{-1}\left\{\frac{E}{s(Ls+R)}\right\} = \left\{\frac{P(0)}{Q(0)} + \right.$$

$$\left. + \frac{P(-\frac{R}{L})e^{-R/Lt}}{(-\frac{R}{L})\cdot L}\right\}\cdot\sigma(t) = \left\{\frac{E}{R} + \frac{E}{-R}\cdot e^{-\frac{R}{L}t}\right\}\cdot\sigma(t)$$

$$i(t) = \frac{E}{R}(1 - e^{-\frac{R}{L}t})\sigma(t) = \frac{E}{R}\sigma(t) - \frac{E}{R}\cdot e^{-\frac{R}{L}t}\sigma(t).$$

② $\frac{\partial T}{\partial t}(x,t) + v\frac{\partial T}{\partial x}(x,t) = 0$ cu $\begin{cases} T(x,0)=0 \\ T(0,t) = T_0\sigma(t) \end{cases}$ ← intrare constantă în timp

? proces termic prin suprafață tubulară

$T(x,t) \xleftrightarrow{L} T(x,s)$

$\frac{\partial T}{\partial t}(x,t) \xleftrightarrow{L} sT(x,s) - \overbrace{T(x,0)}^{0} = sT(x,s)$

$\frac{\partial T}{\partial x}(x,t) \xleftrightarrow{L} \frac{\partial T}{\partial x}(x,s) = T'_x(x,s)$

$\ln|T| = -\frac{s}{v}x + C_0 = \ln e^{-\frac{s}{v}x} + \ln e^{C_0} = \ln(e^{-\frac{s}{v}x}K)$, $e^{C_0} = K$

$sT(x,s) + vT'_x(x,s) = 0 \to v\frac{dT}{dx} = -sT \Rightarrow \frac{dT}{T} = -\frac{s}{v}dx \Rightarrow$

$\ln|T(x,s)| = \ln e^{-\frac{s}{v}x} + \ln K(s) \Rightarrow |T(x,s)| = K(s)\cdot e^{-\frac{s}{v}x} \Rightarrow T(x,s) = \pm K(s)\, e^{-\frac{x}{v}\cdot s} = C_0(s)\, e^{-\frac{x}{v}s}$.

$\cdot T(x,s) = C_0(s) e^{-\frac{x}{v}s} \xleftrightarrow{L^{-1}} ?$

$T(0,t) = T_0\sigma(t) \xleftrightarrow{L^{-1}} \boxed{\frac{T_0}{s} = T(0,s)}$

$T(x,s) = C_0(s)\cdot e^{-\frac{x}{v}s}$

$T(0,s) = \boxed{C_0(s) = \frac{T_0}{s}}$

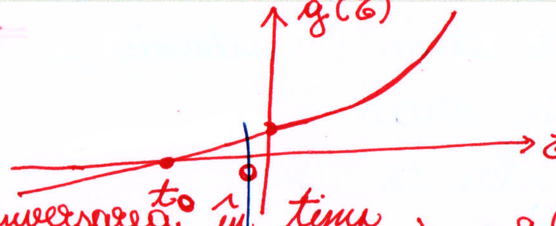
$\Rightarrow T(x,s) = \frac{T_0}{s}\cdot e^{-\frac{x}{v}s}$

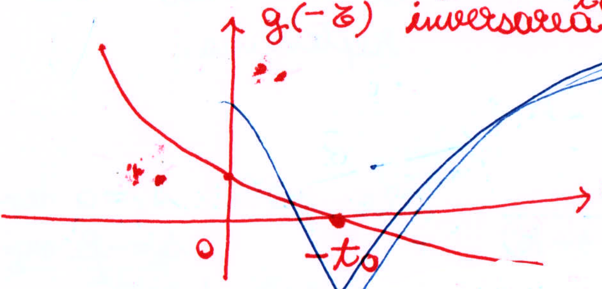
$\frac{T_0}{s} \xleftrightarrow{L^{-1}} T_0\sigma(t)$

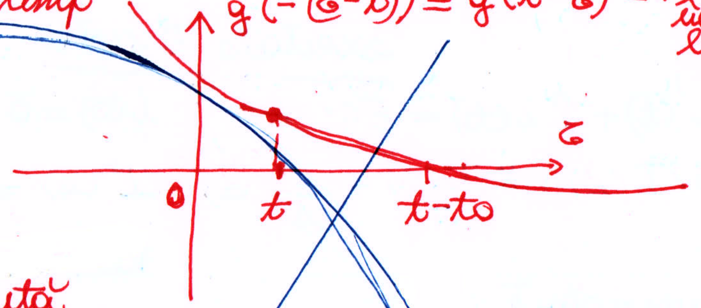
$\frac{T_0}{s}\cdot e^{-\frac{x}{v}s} \underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}} T_0\sigma(t - \frac{x}{v}) \Rightarrow$

$$\boxed{T(x,t) = T_0\sigma(t - \tfrac{x}{v})}$$

$\int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\,d\tau$







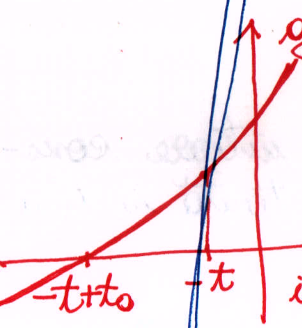
excitație treaptă

$\boxed{Li'(t)+Ri(t)=E\sigma(t), \quad i(0)=0}$

$LsI(s)+RI(s)=\frac{E}{s} \iff (R+Ls)I(s)=\frac{E}{s} \to I(s)=\frac{E}{s(R+Ls)}=$

$=E\left(\frac{A}{s}+\frac{B}{sL+R}\right) \Big|\cdot s,\ s=0 \to \boxed{\frac{E}{R}=EA} \to \boxed{A=\frac{1}{R}}$

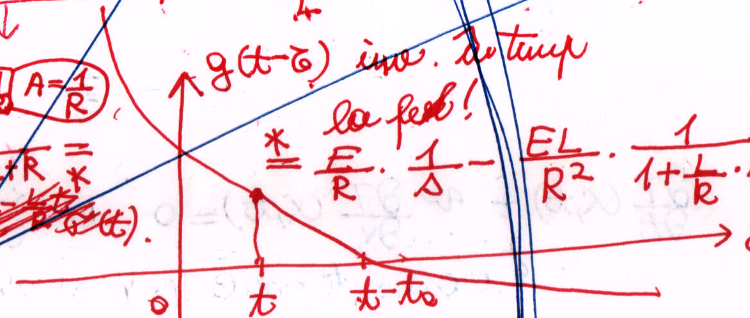
$\Big|\cdot(sL+R),\ s=-\frac{R}{L} \to \frac{E}{-\frac{R}{L}}=EB \to \boxed{B=-\frac{L}{R}}$



$I(s)=\frac{E}{R}\cdot\frac{1}{s}-\frac{LE}{R}\cdot\frac{1}{sL+R}\overset{*}{=}$

~~$i(t)=\frac{E}{R}\sigma(t)-\frac{E}{R}\sigma(t)$~~.

$\overset{*}{=}\frac{E}{R}\cdot\frac{1}{s}-\frac{EL}{R^2}\cdot\frac{1}{1+\frac{L}{R}\cdot s}=$



$i(t)=$

$=\frac{E}{R}\cdot\frac{1}{s}-\frac{E}{R}\cdot\frac{1}{s+\frac{R}{L}}$

· tr. st. cu t → inversarea în timp

sau

·· invers. în timp → transl. la dreapta în timp cu t.

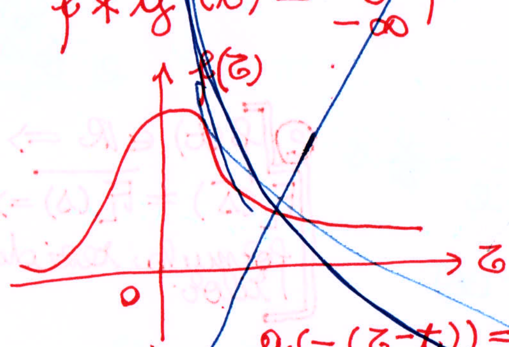
$i(t)=\frac{E}{R}\sigma(t)-\frac{E}{R}\cdot e^{-\frac{R}{L}t}\sigma(t).$

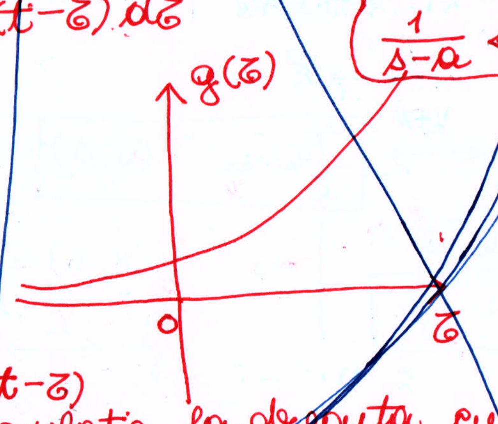
$e^{at}\sigma(t) \xleftrightarrow{L} \frac{1}{s-a}$

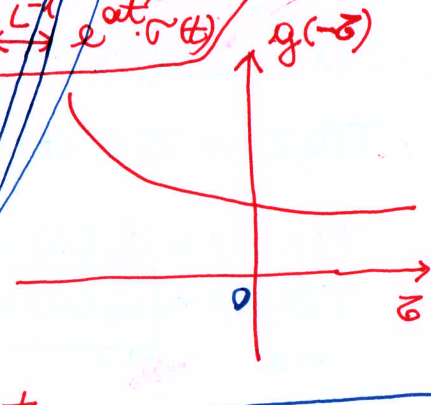
$\frac{1}{s-a} \xleftrightarrow{L^{-1}} e^{at}\sigma(t)$

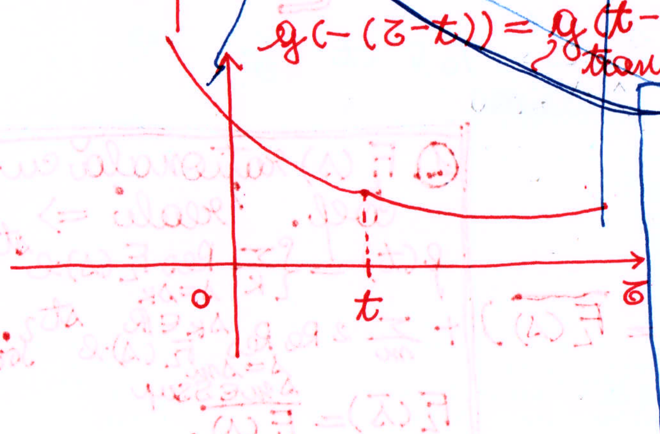
Convoluția

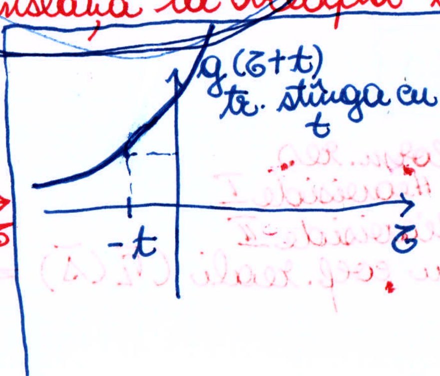
$f*g(t)=\int_{-\infty}^{\infty} f(\tau)\cdot g(t-\tau)\,d\tau$

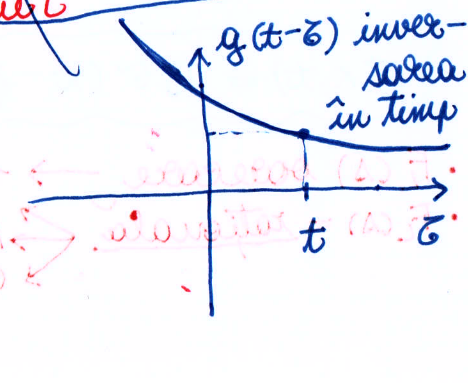












Remarcă ① $F(s)=\frac{P(s)}{Q(s)}\cdot e^{-st}$ cu $P(s)$ și $Q(s)$ polinoame cu coeficienți reali, $s=\alpha+j\omega\in\mathbb{C}$. Atunci $\overline{F(s)}=F(\bar{s})$.

Demonstrație:
$$\overline{F(s)}=\frac{\overline{P(s)}}{\overline{Q(s)}}\cdot\overline{e^{-\alpha t-j\omega t}}=\frac{P(\bar{s})}{Q(\bar{s})}\cdot e^{-\alpha t}\cdot\overline{e^{-j\omega t}}=$$
$$=\frac{P(\bar{s})}{Q(\bar{s})}\cdot e^{-\alpha t}\cdot e^{j\omega t}=\frac{P(\bar{s})}{Q(\bar{s})}\cdot e^{-(\alpha-j\omega)t}=$$
$$=\frac{P(\bar{s})}{Q(\bar{s})}\cdot e^{-\bar{s}t}=F(\bar{s}).$$

De aici se deduce: $F(s)+F(\bar{s})=2\,\mathrm{Re}\,F(s)$.

② Formula Heaviside (I)

Fie imaginea Laplace $F_L(s)=\frac{P(s)}{Q(s)}$ cu coeficienți rea
și cu singularitățile următoare: $s_1,\dots,s_n$ poli reali simpli nenuli și $s_1,\dots,s_p$ poli simpli complex conjugați. Atunci:
$$f(t)=\mathcal{L}^{-1}\{F_L(s)\}=\left\{\sum_{k=1}^{n}\frac{P(s_k)\cdot e^{s_k t}}{Q'(s_k)}+\sum_{l=1}^{p}2\,\mathrm{Re}\,\frac{P(s_l)\cdot e^{s_l t}}{Q'(s_l)}\right\}\cdot\sigma(t).$$

La fel pentru (Heaviside II).

Exemple.

① $F_L(s)=\frac{2s+1}{(s^2+1)(s^2+4)}\cdot e^{-2s}$ o imagine Laplace.

a) $c_f=?$ $C_f=?$ b) $f(t)=\mathcal{L}^{-1}\{F_L(s)\}=?$ c) există $F(\omega)=$

Soluție: a) Poli simpli imaginari $s_{1,2}=\pm j$, $s_{3,4}=\pm 2j\Rightarrow$
$c_f=0$, $C_f=\{\mathrm{Re}\,s>0\}$.

b) Aplicăm formula lui Heaviside I pentru $G(s)=\frac{2s+1}{(s^2+1)(s^2+4)}$
$P(s)=2s+1$, $Q(s)=s^4+5s^2+4\longrightarrow Q'(s)=4s^3+10s$.

$$G(s)=\frac{P(s)}{Q(s)}=\frac{2s+1}{s^4+5s^2+4}\xleftrightarrow{L^{-1}}g(t)=\mathcal{L}^{-1}\left\{\frac{2s+1}{s^4+5s^2+4}\right\}=$$
$$=\left\{2\,\mathrm{Re}\,\frac{P(j)\cdot e^{jt}}{Q'(j)}+2\,\mathrm{Re}\,\frac{P(2j)\cdot e^{j2t}}{Q'(2j)}\right\}\sigma(t)=\left\{2\,\mathrm{Re}\,\frac{(2j+1)e^{jt}}{4j^3+10j}+\right.$$
$$\left.+2\,\mathrm{Re}\,\frac{(4j+1)e^{j2t}}{4\cdot(2j)^3+10\cdot 2j}\right\}\sigma(t)=\frac{1}{6}(2\sin t+2\cos t-\sin 2t-4\cos 2t)\sigma(t)$$

Cu întârzierea: $F_L(s) = G(s)\cdot e^{-2s} \xleftrightarrow[=17=]{L^{-1}} f(t) = g(t-2) =$

$= \frac{1}{6}[2\sin(t-2) + 2\cos(t-2) - \sin 2(t-2) - 4\cos 2(t-2)]\sigma(t-2).$

$G(s) = \frac{2s+1}{(s^2+1)(s^2+4)}$ are $c_g = 0$ și numai poli simpli imaginari

deci, cu TL $\to$ TF avem că există $G(\omega) = G(s)\big|_{s=j\omega} +$

$+\pi a_1\delta(\omega-\omega_1) + \pi a_2\delta(\omega-\omega_2) + \pi a_3\delta(\omega-\omega_3) +$

$+\pi a_4\delta(\omega-\omega_4)$ unde: $s_1 = j = j\omega_1 \to \omega_1 = 1$; $s_2 = -j = j\omega_2 \to$

$\omega_2 = -1$; $s_3 = 2j = j\omega_3 \to \omega_3 = 2$; $s_4 = -2j = j\omega_4 \to \omega_4 = -2$.

$a_1 = \operatorname{Res}_{s=j} G(s) = \operatorname{Res}_{s=j} \frac{2s+1}{s^4+5s^2+4} = \frac{2s+1}{4s^3+10s}\Big|_{s=j} = \frac{1}{3} - \frac{j}{6}.$

$a_2 = \operatorname{Res}_{s=-j} G(s) = \overline{\operatorname{Res}_{s=j} G(s)} = \overline{a_1} = \frac{1}{3} + \frac{j}{6}$; $a_3 = \operatorname{Res}_{s=2j} G(s) =$

$= \frac{2s+1}{4s^3+10s}\Big|_{s=2j} = -\frac{1}{3} + \frac{j}{12}$; $a_4 = \operatorname{Res}_{s=-2j} G(s) = \overline{\operatorname{Res}_{s=2j} G(s)} = \overline{a_3} = -\frac{1}{3} - \frac{j}{12}.$

$$G(\omega) = \frac{2j\omega+1}{(1-\omega^2)(4-\omega^2)} + \frac{\pi}{3}[\delta(\omega-1)+\delta(\omega+1)] - j\frac{\pi}{6}[\delta(\omega-1)-\delta(\omega+1)]$$
$$-\frac{\pi}{3}[\delta(\omega-2)+\delta(\omega+2)] + \frac{\pi j}{12}[\delta(\omega-2)-\delta(\omega+2)].$$

Cu întârzierea avem: $\left.\begin{matrix} g(t) \xleftrightarrow{F} G(\omega) \\ f(t) = g(t-2) \end{matrix}\right| \Rightarrow F(\omega) = G(\omega)\cdot e^{-j2\omega}.$

② $F_L(s) = \frac{s-2}{(s^2+9)(s+3)}\cdot e^{-3s}$. Există $F(\omega) = ?$

Soluție: $F_L(s)$ are polii simpli nenuli: $s_{1,2} = \pm 3j$, $s_3 = -3 \Rightarrow$

$c_f = 0 \Rightarrow G(s) = \frac{s-2}{(s^2+9)(s+3)}$ are $c_g = 0$. Nu putem aplica TL $\to$ TF.

Cu Heaviside I găsim: $G(s) \xleftrightarrow{L^{-1}} g(t) = \frac{1}{18}(4\cos 3t + 2\sin 3t - 5e^{-3t})\sigma(t).$

Folosim TF uzuale și obținem:

$$g(t) \xleftrightarrow{F} G(\omega) = \frac{2j\omega-3}{9(9-\omega^2)} - \frac{5}{18}\cdot\frac{1}{3+j\omega} + \frac{\pi}{9}[\delta(\omega-3)+\delta(\omega+3)] +$$
$$+\frac{\pi}{18j}[\delta(\omega-3)-\delta(\omega+3)].$$

Pe de altă parte, folosind întârzierea la Laplace:

$F_L(s) = G(s)\cdot e^{-3s} \xleftrightarrow{L^{-1}} f(t) = g(t-3).$

Folosind întârzierea la Fourier găsim:

$f(t) = g(t-3) \xleftrightarrow{F} F(\omega) = G(\omega)\cdot e^{-j3\omega}.$

(5.)

$$y''+2y'+2y=\sin t\cdot\sigma\left(t-\frac{\pi}{2}\right)\qquad y(0)=y'(0)=0$$

$$\begin{array}{c|l} 2 & y(t)\xleftrightarrow{L} y(s)\\ 2 & y'(t)\xleftrightarrow{L} s\,y(s)\\ & y''(t)\xleftrightarrow{L} s^2 y(s)\end{array}$$

$\sin t\cdot\sigma(t-\frac{\pi}{2})=$

$$\left.\begin{array}{l}\cos(t-\frac{\pi}{2})\sigma(t-\frac{\pi}{2})\xleftrightarrow{L}(s^2+2s+2)\,y(s)\\ \cos(t-\frac{\pi}{2})\sigma(t-\frac{\pi}{2})\xleftrightarrow{L}\dfrac{s}{s^2+1}\cdot e^{-\frac{\pi}{2}s}\end{array}\right|\Rightarrow y(s)=\frac{s}{(s^2+1)(s^2+2s+2)}\cdot e^{-\frac{\pi}{2}s}$$

$F(s)=\dfrac{s}{(s^2+1)(s^2+2s+2)}$ are polii: $s_{1,2}=\pm j$ $m_{1,2}=1$ $\rightarrow$ cu

$s_{3,4}=-1\pm j$ $m_{3,4}=1$

Heaviside I avem:

$$F(s)=\frac{s}{(s^2+1)(s^2+2s+2)}=\frac{P(s)}{Q(s)}\xleftrightarrow{L^{-1}}f(t)=\mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)(s^2+2s+2)}\right\}=$$

$$=\left\{2\,\mathrm{Re}\,\frac{P(s_1)\cdot e^{s_1t}}{Q'(s_1)}+2\,\mathrm{Re}\,\frac{P(s_3)\cdot e^{s_3t}}{Q'(s_3)}\right\}\cdot\sigma(t)\stackrel{*}{=}$$

$$\frac{P(\bar{s}_1)\cdot e^{\bar{s}_1t}}{Q'(\bar{s}_1)}=\overline{\frac{P(s_1)\cdot e^{s_1t}}{Q'(s_1)}}\Rightarrow\frac{P(\bar{s}_1)\cdot e^{\bar{s}_1t}}{Q'(\bar{s}_1)}+\frac{P(s_1)\cdot e^{s_1t}}{Q'(s_1)}=2\,\mathrm{Re}\,\frac{P(s_1)\cdot e^{s_1t}}{Q'(s_1)},$$

$s_2=\bar{s}_1$. $Q'(s)=(s^2+1)\cdot(2s+2)+2s\cdot(s^2+2s+2)$.

$$\stackrel{*}{=}\left\{2\,\mathrm{Re}\,\frac{\cancel{s_1}\cdot e^{s_1t}}{2\cancel{s_1}(s_1^2+2s_1+2)}+2\,\mathrm{Re}\,\frac{s_3\cdot e^{s_3t}}{(s_3^2+1)(2s_3+2)}\right\}\cdot\sigma(t)=$$

$$=\left\{\cancel{2}\,\mathrm{Re}\,\frac{e^{jt}}{\cancel{2}(\underbrace{-1+2j+2}_{1+2j})}+2\,\mathrm{Re}\,\frac{(-1+j)\,e^{(-1+j)t}}{(-2j+1)\cdot 2j}\right\}\cdot\sigma(t)=\left\{\mathrm{Re}\,\frac{(\cos t+j\sin t)(1-2j)}{5}\right.$$

$$\left.+\cancel{2}\,\mathrm{Re}\;e^{-t}\cdot\frac{(-1+j)\cdot e^{jt}}{\cancel{2}(2+j)}\right\}\cdot\sigma(t)=\left\{\frac{1}{5}\cos t+\frac{2}{5}\sin t+\frac{e^{-t}}{5}\mathrm{Re}\,(-1+j)\cdot\right.$$

$$\left.\cdot(2-j)\,e^{jt}\right\}\cdot\sigma(t)=\left\{\frac{1}{5}\cos t+\frac{2}{5}\sin t+\frac{e^{-t}}{5}\mathrm{Re}\,(-1+3j)(\cos t+j\sin t)\right\}\cdot\sigma(t)=$$

$$=\frac{1}{5}\cos t\cdot\sigma(t)+\frac{2}{5}\sin t\cdot\sigma(t)-\frac{1}{5}e^{-t}\cos t\,\sigma(t)-\frac{3}{5}e^{-t}\sin t\,\sigma(t).$$

$$y(s)=F(s)\cdot e^{-\frac{\pi}{2}s}\underset{\text{întârzierea}}{\xleftrightarrow{L^{-1}}}y(t)=f\left(t-\frac{\pi}{2}\right)$$

$\cdot\ \sin(t-\frac{\pi}{2})=-\cos t$; $\cos(t-\frac{\pi}{2})=\sin t$

$$y(t)=\frac{1}{5}\sin t\,\sigma(t-\tfrac{\pi}{2})-\frac{2}{5}\cos t\,\sigma(t-\tfrac{\pi}{2})-\frac{1}{5}e^{-t+\frac{\pi}{2}}\cdot\sin t\cdot\sigma(t-\tfrac{\pi}{2})$$
$$+\frac{3}{5}e^{-t+\frac{\pi}{2}}\cos t\,\sigma(t-\tfrac{\pi}{2}).$$

⑥ $y''(t-1)+2y'(t-1)+y(t-1)=\cos(t-1)\cdot\sigma(t-1)$ cu $y(0)=y'(0)=0$

ecuație dif. cu argument întârziat

$$2\left|\begin{array}{l} y(t-1)\xleftrightarrow{L} e^{-s}\cdot y(s) \\ y'(t-1)\xleftrightarrow{L} e^{-s}\cdot s\,y(s) \\ y''(t-1)\xleftrightarrow{L} e^{-s}s^2y(s)\end{array}\right. \longrightarrow y'(t-1)\xleftrightarrow{L} e^{-s}\cdot\mathcal{L}\{y'(t)\}= = e^{-s}\cdot s\,y(s).$$

$$\left.\begin{array}{l}\cos(t-1)\sigma(t-1)\xleftrightarrow{L}(s^2+2s+1)e^{-s}y(s)=e^{-s}\cdot(s+1)^2y(s)\\ \cos(t-1)\sigma(t-1)\xleftrightarrow{L} e^{-s}\cdot\mathcal{L}\{\cos t\,\sigma(t)\}=e^{-s}\cdot\frac{s}{s^2+1}\end{array}\right|\Rightarrow$$

$$y(s)=\frac{s}{(s^2+1)(s+1)^2}\xleftrightarrow{L^{-1}}\left\{\operatorname*{Res}_{s=-1}\frac{s\cdot e^{st}}{(s^2+1)(s+1)^2}+2\operatorname{Re}\operatorname*{Res}_{s=j}\frac{s\cdot e^{st}}{(s^2+1)(s+1)^2}\right\}\cdot\sigma(t)$$

$$=\left\{\left(\frac{s\cdot e^{st}}{s^2+1}\right)'\Big|_{s=-1}+2\operatorname{Re}\frac{s\cdot e^{st}}{(s+j)(s+1)^2}\Big|_{s=j}\right\}\cdot\sigma(t)=$$

$$=\left\{\frac{(e^{st}+st\cdot e^{st})(s^2+1)-2s^2e^{st}}{(s^2+1)^2}\Big|_{s=-1}+2\operatorname{Re}\frac{j\,e^{jt}}{2j\underbrace{(1+j)^2}_{2j}}\right\}\sigma(t)=$$

$$=\left\{\frac{\cancel{2e^{-t}}-2te^{-t}-\cancel{2e^{-t}}}{4}+\frac{1}{2}\operatorname{Re}\frac{1}{j}(\cos t+j\sin t)\right\}\cdot\sigma(t)=$$

$$=\boxed{\left(-\frac{t}{2}e^{-t}+\frac{1}{2}\sin t\right)\sigma(t)=y(t)}$$

$$y(t)+\int_0^t\sin(t-\tau)\,y(\tau)\,d\tau=\sigma(t)-(t-2)\sigma(t-2)$$

$$y(t)\xleftrightarrow{L}y(s)$$

$$\int_0^t\sin(t-\tau)\cdot y(\tau)d\tau=\sin t*y(t)\xleftrightarrow{L}\frac{1}{s^2+1}\cdot y(s)$$

$$t\sigma(t)-(t-2)\sigma(t-2)\xleftrightarrow{L}\frac{1}{s}-\frac{1}{s^2}\cdot e^{-2s}$$

$$\left(1+\frac{1}{s^2+1}\right)y(s)=\frac{1}{s}-\frac{1}{s^2}\cdot e^{-2s}\rightarrow y(s)=\underbrace{\frac{s^2+1}{s(s^2+2)}}_{y_1(s)}-\underbrace{\frac{s^2+1}{s^2(s^2+2)}}_{y_2(s)}\cdot e^{-2s}$$

$\cdot$ $$y_1(s)=\frac{P(s)}{s\cdot Q(s)}=\frac{s^2+1}{s(s^2+2)}\xleftrightarrow[H\ II]{L^{-1}}y_1(t)=\left\{\frac{P(0)}{Q(0)}+2\operatorname{Re}\frac{P(j\sqrt2)\cdot e^{j\sqrt2 t}}{j\sqrt2\cdot Q'(j\sqrt2)}\right\}\sigma(t)$$

$$=\left\{\frac{1}{2}+2\operatorname{Re}\frac{(1-2)\cdot e^{j\sqrt2 t}}{j\sqrt2\cdot2j\sqrt2}\right\}\sigma(t)=\left\{\frac12+\frac12\operatorname{Re}(\cos\sqrt2t+j\sin\sqrt2t)\right\}\cdot$$

$$\cdot\sigma(t)=\frac12\sigma(t)+\frac12\cos(\sqrt2t)\cdot\sigma(t).$$

$\cdot\cdot$ $$y_2(s)=\frac{s^2+1}{s^2(s^2+2)}=\frac12\left(\frac{1}{s^2}+\frac{1}{s^2+2}\right)\xleftrightarrow{L^{-1}}y_2(t)=\frac12 t\,\sigma(t)+\frac{1}{2\sqrt2}\sin(\sqrt2t)\,\sigma(t)$$

$$y(t)=y_1(t)-y_2(t-2)=\frac12\sigma(t)+\frac12\cos(\sqrt2t)\cdot\sigma(t)-\frac12(t-2)\sigma(t-2)+\frac{1}{2\sqrt2}\sin(\sqrt2(t-2))\cdot\sigma(t-2)$$

7. $x''' - 2x'' - x' + 2x = 5\sin 2t\,\sigma(t)$, $x(0)=1$, $x'(0)=1$, $x''(0)= -1$

$$\begin{array}{r|l} 2 & x(t) \xleftrightarrow{L} X(s) \\ -1 & x'(t) \xleftrightarrow{L} sX(s)-1 \\ -2 & x''(t) \xleftrightarrow{L} s^2X(s)-s-1 \\ & x'''(t) \xleftrightarrow{L} s^3X(s)-s^2-s+1 \end{array}$$

$$5\sin 2t\,\sigma(t) \xleftrightarrow{L} (s^3-2s^2-s+2)X(s)-s^2+s+4$$

$$5\sin 2t\,\sigma(t) \xleftrightarrow{L} \frac{10}{s^2+4}$$

$$(s-2)(s-1)(s+1)X(s) = s^2-s-4+\frac{10}{s^2+4}$$

$$X(s) = \underbrace{\frac{s^2-s-4}{(s+1)(s-1)(s-2)}}_{X_1(s)} + \underbrace{\frac{10}{(s^2+4)(s-1)(s+1)(s-2)}}_{X_2(s)}$$

$\cdot\ x_1(t) = \left\{ \operatorname*{Res}_{s=-1} \frac{(s^2-s-4)\cdot e^{st}}{(s+1)(s-1)(s-2)} + \operatorname*{Res}_{s=1} \frac{(s^2-s-4)\cdot e^{st}}{(s+1)(s-1)(s-2)} + \operatorname*{Res}_{s=2} \frac{(s^2-s-4)e^{st}}{(s+1)(s-1)(s-2)} \right\}$

$\cdot\sigma(t) = \left\{ \frac{(s^2-s-4)e^{st}}{(s-1)(s-2)}\Big|_{s=-1} + \frac{(s^2-s-4)e^{st}}{(s+1)(s-2)}\Big|_{s=1} + \frac{(s^2-s-4)e^{st}}{(s+1)(s-1)}\Big|_{s=2} \right\}$

$\cdot\sigma(t) = \left\{ \frac{-2e^{-t}}{6} + \frac{-4e^{t}}{-2} + \frac{-2e^{2t}}{3} \right\}\cdot\sigma(t).$

$$\boxed{x_1(t) = \left(-\tfrac{1}{3}e^{-t} + 2e^{t} - \tfrac{2}{3}e^{2t}\right)\sigma(t).}$$

$\cdot\cdot\ x_2(t) = \left\{ \frac{10\cdot e^{st}}{(s^2+4)(s-1)(s-2)}\Big|_{s=-1} + \frac{10\cdot e^{st}}{(s^2+4)(s+1)(s-2)}\Big|_{s=1} + \frac{10e^{st}}{(s^2+4)(s-1)(s+1)}\Big|_{s=2} \right.$

$\left. + 2\,\mathrm{Re}\frac{10e^{st}}{(s+2j)(s^2-1)(s-2)}\Big|_{s=2j} \right\}\cdot\sigma(t) = \left\{ \frac{10e^{-t}}{5\cdot(-2)\cdot(-3)} + \frac{10e^{t}}{5\cdot2\cdot(-1)} + \right.$

$\left. + \frac{10e^{2t}}{8\cdot1\cdot3} + 2\,\mathrm{Re}\frac{10e^{j2t}}{4j\cdot(-5)\cdot 2(j-1)} \right\}\sigma(t) = \left\{ +\tfrac{1}{3}e^{-t} - e^{t} + \tfrac{5}{12}e^{2t} + \right.$

$\left. + \tfrac{1}{2}\,\mathrm{Re}\frac{e^{j2t}}{1+j} \right\}\sigma(t) \overset{*}{=} \left\{ +\tfrac{1}{3}e^{-t} - e^{t} + \tfrac{5}{12}e^{2t} + \tfrac{1}{4}\cos 2t + \tfrac{1}{4}\sin 2t \right\}\sigma(t).$

$\tfrac{1}{2}\,\mathrm{Re}\,\frac{(1-j)}{2}(\cos 2t + j\sin 2t) = \tfrac{1}{4}(\cos 2t + \sin 2t).$

$$x(t) = \left(e^{t} - \tfrac{1}{4}e^{2t} + \tfrac{1}{4}\cos 2t + \tfrac{1}{4}\sin 2t\right)\cdot\sigma(t).$$

⑧ $x'' - 2x' + 5x = e^{t}\cos 2t\,\sigma(t)$, $x(0)=x'(0)=1$.

$$\begin{array}{r|l}
5 & x(t) \overset{L}{\longleftrightarrow} X(s) \\
-2 & x'(t) \overset{L}{\longleftrightarrow} sX(s)-1 \\
 & x''(t) \overset{L}{\longleftrightarrow} s^2X(s)-s-1 \\
\hline
\end{array}$$

$$e^{t}\cos 2t\,\sigma(t) \overset{L}{\longleftrightarrow} (s^2-2s+5)X(s)-s+1 \quad\Big|\Rightarrow [(s-1)^2+4]X(s) = s-1 + \frac{s-1}{(s-1)^2+4}.$$

$$\Big\updownarrow L$$

$$\frac{s-1}{(s-1)^2+4}$$

$$X(s) = \frac{s-1}{(s-1)^2+4} + \frac{s-1}{[(s-1)^2+4]^2} \overset{L^{-1}}{\longleftrightarrow} x(t) = e^{t}\,\mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + e^{t}\,\mathcal{L}^{-1}\Big\{\frac{s}{(s^2+4)^2}\Big\} =$$

$$= e^{t}\cos 2t\,\sigma(t) + \frac{e^{t}}{2}\,\mathcal{L}^{-1}\Big\{-\Big(\frac{1}{s^2+4}\Big)'\Big\} = e^{t}\cos 2t\,\sigma(t) + \frac{e^{t}}{4}\cdot$$

$$\cdot\,\mathcal{L}^{-1}\{-(\mathcal{L}\{\sin 2t\,\sigma(t)\})'\} = e^{t}\cos 2t\,\sigma(t) + \frac{e^{t}}{4}\,\mathcal{L}^{-1}\{\mathcal{L}\{t\cdot\sin 2t\,\sigma(t)\}\} =$$

$$= e^{t}\cos 2t\,\sigma(t) + \frac{e^{t}}{4}\, t\sin 2t\,\sigma(t).$$

$y''(t) + 2y'(t-2) + y(t-4) = t\sigma(t)$, $y(0)=y'(0)=0$.

$$y(t) \overset{L}{\longleftrightarrow} Y(s)$$

$$\begin{array}{r|l}
2 & y'(t-2) \overset{L}{\longleftrightarrow} e^{-2s}\,\mathcal{L}\{y'(t)\} = e^{-2s}s\cdot Y(s) \\
1 & y''(t) \overset{L}{\longleftrightarrow} s^2Y(s) \\
1 & y(t-4) \overset{L}{\longleftrightarrow} e^{-4s}Y(s) \\
\hline
\end{array}$$

$$t\sigma(t) \overset{L}{\longleftrightarrow} Y(s)\cdot(s^2+2s\cdot e^{-2s}+e^{-4s}) = (s+e^{-2s})^2\cdot Y(s)$$

$$\Big\updownarrow L$$

$$\frac{1}{s^2}$$

$$(s+e^{-2s})^2\,Y(s) = \frac{1}{s^2} \Rightarrow Y(s) = \frac{1}{s^2(s+e^{-2s})^2} = \frac{1}{s^4}\cdot\frac{1}{\left(1+\frac{e^{-2s}}{s}\right)^2} =$$

$$= \frac{1}{s^4}\cdot\Big(1+\frac{e^{-2s}}{s}\Big)^{-2} = \frac{1}{s^4}\Big\{\sum_{n=1}^{\infty}\frac{(-2)(-3)(-4)\ldots(-n-1)}{n!}\cdot\frac{e^{-2ns}}{s^n}+1\Big\} =$$

$$= \frac{1}{s^4} + \sum_{n\geq 1}(-1)^n(n+1)\cdot\frac{1}{s^{n+4}}\cdot e^{-2ns}$$

$$\frac{1}{s^4} \overset{L^{-1}}{\longleftrightarrow} \frac{t^3}{3!}\sigma(t);\quad \frac{1}{s^{n+4}} \overset{L^{-1}}{\longleftrightarrow} \frac{t^{n+3}}{(n+3)!}\sigma(t) \Rightarrow \frac{e^{-2ns}}{s^{n+4}} \overset{L^{-1}}{\longleftrightarrow} \frac{(t-2n)^{n+3}}{(n+3)!}\sigma(t-2n)$$

$$y(t) = \frac{t^3}{3!}\sigma(t) + \sum_{n=1}^{\infty}(-1)^n\frac{(n+1)}{(n+3)!}\cdot(t-2n)^{n+3}\sigma(t-2n),\quad t\geq 2n \Rightarrow n\geq\frac{t}{2}$$

$$y(t) = \sum_{n=0}^{[\frac{t}{2}]}(-1)^n\frac{(n+1)}{(n+3)!}\cdot(t-2n)^{n+3}.$$

9) $y'' - 3y' + 2y = f(t)$ dacă $y(0) = 0$, $y'(0) =$ și $f(t) = t\cdot\sigma(t-1)$.

Soluție: $f(t) = (t-1)\sigma(t-1) + \sigma(t-1) \xleftrightarrow{L} F_L(s) = \frac{s+1}{s^2}\cdot e^{-s}$

Ecuația algebrică este:

$$(s^2 - 3s + 2)\,Y(s) = \frac{s+1}{s^2}\cdot e^{-s} \Rightarrow Y(s) = \frac{s+1}{s^2(s-1)(s-2)}\cdot e^{-s}.$$

Aplicăm formula reziduurilor:

$$Y_1(s) = \frac{s+1}{s^2(s-1)(s-2)} \xleftrightarrow{L^{-1}} y_1(t) = \left\{\left[\frac{(s+1)\cdot e^{st}}{s^2 - 3s + 2}\right]'\Bigg|_{s=0} + \right.$$

$$\left. + \frac{(s+1)e^{st}}{s^2(s-2)}\Bigg|_{s=1} + \frac{(s+1)e^{st}}{s^2(s-1)}\Bigg|_{s=2}\right\}\cdot\sigma(t) = \left(\frac{5}{4} + \frac{t}{2} - 2e^{t} + \frac{3}{4}e^{2t}\right)\cdot\sigma(t).$$

$Y(s) = Y_1(s)\cdot e^{-s}$ și cu întârzierea: $y(t) = y_1(t-1) \Rightarrow$

$$y(t) = \left[\frac{3}{4} + \frac{t}{2} - 2e^{t-1} + \frac{3}{4}e^{2(t-1)}\right]\cdot\sigma(t-1).$$