DERIVATE PARȚIALE

- **Să se calculeze derivatele parțiale de ordinul I pentru funcțiile:**

1. $f(x,y)=\mathrm{e}^{x-y^2}$

2. $f(x,y,z)=\mathrm{e}^{x^2+y^2}\cdot\sin^2 z$

3. $f(x,y)=xy\cdot\operatorname{arctg}\left(\dfrac{x+y}{1-xy}\right)$, $xy\neq 1$

4. $f(x,y,z)=x^{y^z}$, $x,y>0$

- **Să se calculeze derivatele parțiale de ordinul I și II pentru funcțiile:**

5. $f(x,y)=\mathrm{e}^x\cos y$

6. $f(x,y)=x^3+xy$

7. $f(x,y)=\dfrac{x-y}{x+y}$, $(x,y)\neq(0,0)$

8. $f(x,y)=\arccos\dfrac{x}{\sqrt{x^2+y^2}}$, $(x,y)\neq(0,0)$

9. $f(x,y)=\ln\left(x^2+y^2\right)$, $(x,y)\neq(0,0)$

10. $f(x,y)=\dfrac{1}{\sqrt{x^2+y^2}}$, $(x,y)\neq(0,0)$

11. $f(x,y,z)=xyz$

12. $f(x,y,z)=y\sin(x+z)$

- **Să se calculeze derivatele parțiale în punctele specificate:**

13. $f(x,y)=2x^2+xy$, $\dfrac{\partial f}{\partial x}(1,1)$ și $\dfrac{\partial f}{\partial y}(3,2)$

14. $f(x,y)=\mathrm{e}^{\sin xy}$, $\dfrac{\partial f}{\partial x}\left(1,\dfrac{\pi}{2}\right)$ și $\dfrac{\partial f}{\partial y}(1,0)$

15. $f(x,y)=\sqrt{\sin^2 x+\sin^2 y}$, $\dfrac{\partial f}{\partial x}\left(\dfrac{\pi}{4},0\right)$ și $\dfrac{\partial f}{\partial y}\left(\dfrac{\pi}{4},\dfrac{\pi}{4}\right)$

16. $f(x,y)=\ln\left(1+x^2+y^2\right)$, $\dfrac{\partial f}{\partial x}(1,1)$ și $\dfrac{\partial f}{\partial y}(1,1)$

17. $f(x,y)=\sqrt[3]{x^2y}$, $\dfrac{\partial f}{\partial x}(-2,2)$, $\dfrac{\partial f}{\partial y}(-2,2)$ și $\dfrac{\partial^2 f}{\partial x\partial y}(-2,2)$

18. $f(x,y)=xy\ln x$, $x\neq 0$, $\dfrac{\partial^2 f}{\partial x\partial y}(1,1)$ și $\dfrac{\partial^2 f}{\partial y\partial x}(1,1)$

- **Calculați derivatele parțiale de ordinul I pentru funcțiile compuse:**

19. $f(x,y)=\ln\left(u^2+v\right)$, $u(x,y)=e^{x+y^2}$ și $v(x,y)=x^2+y$

20. $f(x,y)=\operatorname{arctg}\dfrac{2u}{v}$, $u(x,y)=x\sin y$ și $v(x,y)=x\cos y$

21. $f(x,y)=\varphi\left(2x\cdot e^y+3y\cdot\sin 2x\right)$

22. $f(x,y)=\varphi(u,v,w)$, $u(x,y)=x\cdot y$, $v(x,y)=x^2+y^2$ și $w(x,y)=2x+3y$

- **Calculați derivatele parțiale de ordinul II** $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$ și $\frac{\partial^2 f}{\partial y^2}$ **pentru funcțiile compuse:**

23. $f(x,y)=\varphi(u,v)$, $u(x,y)=x+y$ și $v(x,y)=x^2+y^2$

24. $f(x,y)=\varphi(u,v)$, $u(x,y)=x^2-y^2$ și $v(x,y)=e^{xy}$

25. $f(x,y)=\ln\left(u^2+v^2\right)$, $u(x,y)=xy$ și $v(x,y)=x^2-y^2$

26. $f(x,y)=\varphi(u,v)$, $u(x,y)=xy$ și $v(x,y)=\frac{x}{y}$

- **Arătați că funcțiile următoare verifică ecuațiile indicate:**

27. $f(x,y)=\varphi\left(\frac{y}{x}\right)$, verifică ecuația $x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=0$

28. $f(x,y,z)=\varphi\left(xy\,,\,x^2+y^2+z^2\right)$, verifică ecuația $xz\frac{\partial f}{\partial x}-yz\frac{\partial f}{\partial y}+\left(y^2-x^2\right)\frac{\partial f}{\partial z}=0$

29. $f(x,y)=y\cdot\varphi\left(x^2-y^2\right)$, verifică ecuația $\frac{1}{x}\frac{\partial f}{\partial x}+\frac{1}{y}\frac{\partial f}{\partial y}=\frac{1}{y^2}f(x,y)$

30. $f(x,y)=e^y\cdot\varphi\left(y\cdot e^{\frac{x^2}{2y^2}}\right)$, verifică ecuația $\left(x^2-y^2\right)\frac{\partial f}{\partial x}+xy\frac{\partial f}{\partial y}=xy\cdot f(x,y)$

31. $f(x,y)=xy\cdot\varphi\left(x^2-y^2\right)$, verifică ecuația $xy^2\frac{\partial f}{\partial x}+x^2y\frac{\partial f}{\partial y}=\left(x^2+y^2\right)f(x,y)$

32. $f(x,y,z)=\frac{xy}{z}\ln x+x\cdot\varphi\left(\frac{x}{y},\frac{z}{x}\right)$ cu $x>0$, $z\neq 0$, verifică ecuația

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}-\frac{xy}{z}-f(x,y,z)=0$$

Indicații și soluții

1. $\dfrac{\partial f}{\partial x}=e^{x-y^2}$; $\dfrac{\partial f}{\partial y}=-2ye^{x-y^2}$

2. $\dfrac{\partial f}{\partial x}=2x\cdot e^{x^2+y^2}\cdot\sin^2 z$; $\dfrac{\partial f}{\partial y}=2y\cdot e^{x^2+y^2}\cdot\sin^2 z$; $\dfrac{\partial f}{\partial z}=2\sin z\cdot\cos z\cdot e^{x^2+y^2}$

3. $\dfrac{\partial f}{\partial x}=y\cdot\operatorname{arctg}\dfrac{x+y}{1-xy}+xy\dfrac{1+y^2}{1+x^2+y^2+x^2y^2}$; $\dfrac{\partial f}{\partial y}=x\cdot\operatorname{arctg}\dfrac{x+y}{1-xy}+xy\dfrac{1+x^2}{1+x^2+y^2+x^2y^2}$

4. $\dfrac{\partial f}{\partial x}=y^z\cdot x^{y^z-1}$ (se derivează ca funcție putere: $\left(x^n\right)'=nx^{n-1}$, unde $n=y^z$); $\dfrac{\partial f}{\partial y}=x^{y^z}\cdot\ln x\cdot z\cdot y^{z-1}$ (se derivează ca funcție exponențială: $\left(a^u\right)'=a^u\ln a\cdot u'$, unde $a=x$ și $u=y^z$); $\dfrac{\partial f}{\partial z}=x^{y^z}\cdot\ln x\cdot y^z\cdot\ln y$

$\dfrac{\partial f}{\partial y}=x^{y^z}\cdot\ln x\cdot z\cdot y^{z-1}$ (se derivează ca funcție exponențială: $\left(a^u\right)'=a^u\ln a\cdot u'$, unde $a=x$ și $u=y^z$)

5. $\dfrac{\partial f}{\partial x}=e^x\cos y$; $\dfrac{\partial f}{\partial y}=-e^x\sin y$; $\dfrac{\partial^2 f}{\partial x^2}=e^x\cos y$; $\dfrac{\partial^2 f}{\partial y^2}=-e^x\cos y$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=-e^x\sin y$

6. $\dfrac{\partial f}{\partial x}=3x^2+y$; $\dfrac{\partial f}{\partial y}=x$; $\dfrac{\partial^2 f}{\partial x^2}=6x$; $\dfrac{\partial^2 f}{\partial y^2}=0$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=1$

7. $\dfrac{\partial f}{\partial x}=\dfrac{2y}{(x+y)^2}$; $\dfrac{\partial f}{\partial y}=\dfrac{-2x}{(x+y)^2}$; $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{-4y}{(x+y)^3}$; $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{4x}{(x+y)^3}$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{2(x-y)}{(x+y)^3}$

8. $\dfrac{\partial f}{\partial x}=-\dfrac{|y|}{\sqrt{x^2+y^2}}$; $\dfrac{\partial f}{\partial y}=\dfrac{x}{x^2+y^2}\cdot\dfrac{|y|}{y}$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=-\dfrac{x^2-y^2}{\left(x^2+y^2\right)^2}\cdot\dfrac{|y|}{y}$; $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{2x\cdot|y|}{\left(x^2+y^2\right)^2}$;

$\dfrac{\partial^2 f}{\partial y^2}=-\dfrac{2x\cdot|y|}{\left(x^2+y^2\right)^2}$

9. $\dfrac{\partial f}{\partial x}=\dfrac{2x}{x^2+y^2}$; $\dfrac{\partial f}{\partial y}=\dfrac{2y}{x^2+y^2}$; $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{2\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$; $\dfrac{\partial^2 f}{\partial y^2}=\dfrac{2\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=-\dfrac{4xy}{\left(x^2+y^2\right)^2}$

10. $\dfrac{\partial f}{\partial x}=-x\left(x^2+y^2\right)^{-\frac{3}{2}}$; $\dfrac{\partial f}{\partial y}=-y\left(x^2+y^2\right)^{-\frac{3}{2}}$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=-3xy\left(x^2+y^2\right)^{-\frac{5}{2}}$;

$\dfrac{\partial^2 f}{\partial x^2}=\left(x^2+y^2\right)^{-\frac{5}{2}}\left(2x^2-y^2\right)$; $\dfrac{\partial^2 f}{\partial y^2}=\left(x^2+y^2\right)^{-\frac{5}{2}}\left(2y^2-x^2\right)$

11. $\dfrac{\partial f}{\partial x}=yz$; $\dfrac{\partial f}{\partial y}=xz$; $\dfrac{\partial f}{\partial z}=xy$; $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial^2 f}{\partial y^2}=\dfrac{\partial^2 f}{\partial z^2}=0$; $\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=z$; $\dfrac{\partial^2 f}{\partial x\partial z}=\dfrac{\partial^2 f}{\partial z\partial x}=y$;

$\dfrac{\partial^2 f}{\partial y\partial z}=\dfrac{\partial^2 f}{\partial z\partial y}=x$

12. $\dfrac{\partial f}{\partial x}=y\cos(x+z)$; $\dfrac{\partial f}{\partial y}=\sin(x+z)$; $\dfrac{\partial f}{\partial z}=y\cos(x+z)$; $\dfrac{\partial^2 f}{\partial x^2}=\dfrac{\partial^2 f}{\partial z^2}=-y\sin(x+z)$; $\dfrac{\partial^2 f}{\partial y^2}=0$;

$\dfrac{\partial^2 f}{\partial x\partial y}=\dfrac{\partial^2 f}{\partial y\partial x}=\cos(x+z)$; $\dfrac{\partial^2 f}{\partial x\partial z}=\dfrac{\partial^2 f}{\partial z\partial x}=-y\sin(x+z)$; $\dfrac{\partial^2 f}{\partial y\partial z}=\dfrac{\partial^2 f}{\partial z\partial y}=\cos(x+z)$

13. $\dfrac{\partial f}{\partial x}=4x+y$, $\dfrac{\partial f}{\partial x}(1,1)=5$; $\dfrac{\partial f}{\partial y}=x$, $\dfrac{\partial f}{\partial y}(3,2)=3$

14. $\frac{\partial f}{\partial x}=e^{\sin xy}y\cdot\cos xy$, $\frac{\partial f}{\partial x}\left(1,\frac{\pi}{2}\right)=0$; $\frac{\partial f}{\partial y}=e^{\sin xy}x\cdot\cos xy$, $\frac{\partial f}{\partial y}(1,0)=1$

15. $\frac{\partial f}{\partial x}=\frac{\sin x\cdot\cos x}{\sqrt{\sin^2 x+\sin^2 y}}$, $\frac{\partial f}{\partial x}\left(\frac{\pi}{4},0\right)=\frac{\sqrt{2}}{2}$; $\frac{\partial f}{\partial y}=\frac{\sin y\cdot\cos y}{\sqrt{\sin^2 x+\sin^2 y}}$, $\frac{\partial f}{\partial y}\left(\frac{\pi}{4},\frac{\pi}{4}\right)=\frac{1}{2}$

16. $\frac{\partial f}{\partial x}=\frac{2x}{1+x^2+y^2}$, $\frac{\partial f}{\partial x}(1,1)=\frac{2}{3}$; $\frac{\partial f}{\partial y}=\frac{2y}{1+x^2+y^2}$, $\frac{\partial f}{\partial y}(1,1)=\frac{2}{3}$

17. $\frac{\partial f}{\partial x}=\frac{2}{3}x^{-\frac{1}{3}}y^{\frac{1}{3}}$, $\frac{\partial f}{\partial x}(-2,2)=-\frac{2}{3}$; $\frac{\partial f}{\partial y}=\frac{1}{3}x^{\frac{2}{3}}y^{-\frac{2}{3}}$, $\frac{\partial f}{\partial y}(-2,2)=\frac{1}{3}$; $\frac{\partial^2 f}{\partial x\partial y}=\frac{2}{9}x^{-\frac{1}{3}}y^{-\frac{2}{3}}$, $\frac{\partial^2 f}{\partial x\partial y}(-2,2)=-\frac{1}{9}$

18. $\frac{\partial f}{\partial x}=y(\ln x+1)$; $\frac{\partial f}{\partial y}=x\ln x$; $\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial^2 f}{\partial y\partial x}=1+\ln x$, $\frac{\partial^2 f}{\partial x\partial y}(1,1)=1$

19. $\frac{\partial u}{\partial x}=e^{x+y^2}$, $\frac{\partial u}{\partial y}=2y\cdot e^{x+y^2}$, $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial y}=1$. Pentru derivatele parțiale de ordinul I ale lui $f(u,v)$ folosim regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2u}{u^2+v}\cdot e^{x+y^2}+\frac{1}{u^2+v}\cdot 2x=\frac{2}{u^2+v}\left(u^2+x\right);$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2u}{u^2+v}\cdot 2ye^{x+y^2}+\frac{1}{u^2+v}\cdot 1=\frac{1}{u^2+v}\left(4u^2y+1\right)$$

20. $\frac{\partial u}{\partial x}=\sin y$, $\frac{\partial u}{\partial y}=x\cos y$, $\frac{\partial v}{\partial x}=\cos y$, $\frac{\partial v}{\partial y}=-x\sin y$. Pentru derivatele parțiale de ordinul I ale lui $f(u,v)$ folosim regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2v}{4u^2+v^2}\cdot\sin y-\frac{2u}{4u^2+v^2}\cdot\cos y=\frac{2\cos y\sin y-2\sin y\cos y}{x\left(3\sin^2 y+1\right)}=0;$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2v}{4u^2+v^2}\cdot x\cos y+\frac{2u}{4u^2+v^2}\cdot x\sin y=\frac{2x^2\left(\cos^2 y+\sin^2 y\right)}{x^2\left(3\sin^2 y+1\right)}=\frac{2}{3\sin^2 y+1}$$

21. Dacă notăm $u(x,y)=2x\cdot e^y+3y\cdot\sin 2x$, avem $f(x,y)=\varphi(u)$.

$\frac{\partial f}{\partial x}=\varphi'(u)\cdot\frac{\partial u}{\partial x}=\varphi'(u)\cdot\left(2e^y+6y\cos 2x\right)$ și $\frac{\partial f}{\partial y}=\varphi'(u)\cdot\frac{\partial u}{\partial y}=\varphi'(u)\cdot\left(2x\cdot e^y+3\sin 2x\right)$.

22. $\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial\varphi}{\partial w}\cdot\frac{\partial w}{\partial x}=y\frac{\partial\varphi}{\partial u}+2x\frac{\partial\varphi}{\partial v}+2\frac{\partial\varphi}{\partial w}$;

$$\frac{\partial f}{\partial y}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial\varphi}{\partial w}\cdot\frac{\partial w}{\partial y}=x\frac{\partial\varphi}{\partial u}+2y\frac{\partial\varphi}{\partial v}+3\frac{\partial\varphi}{\partial w}.$$

23. $\frac{\partial u}{\partial x}=1$, $\frac{\partial u}{\partial y}=1$, $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial y}=2y$.Pentru derivatele parțiale de ordinul I ale lui $f(x,y)$ folosim regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial\varphi}{\partial u}+2x\frac{\partial\varphi}{\partial v};\quad \frac{\partial f}{\partial y}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial\varphi}{\partial u}+2y\frac{\partial\varphi}{\partial v}$$

Pentru simplificarea calculului deriv. parțiale de ord. II, notăm $\frac{\partial\varphi}{\partial u}=g(u,v)$ și $\frac{\partial\varphi}{\partial v}=h(u,v)$ și obținem:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\big(g(u,v)+2x\cdot h(u,v)\big)=\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}\right)+2h(u,v)+2x\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=$$

$$=\frac{\partial^2\varphi}{\partial u^2}+2x\cdot\frac{\partial^2\varphi}{\partial v\partial u}+2\frac{\partial\varphi}{\partial v}+2x\cdot\frac{\partial^2\varphi}{\partial u\partial v}+4x^2\frac{\partial^2\varphi}{\partial v^2};$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\big(g(u,v)+2y\cdot h(u,v)\big)=\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}\right)+2y\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=$$

$$=\frac{\partial^2\varphi}{\partial u^2}+2x\cdot\frac{\partial^2\varphi}{\partial v\partial u}+2y\cdot\frac{\partial^2\varphi}{\partial u\partial v}+4xy\frac{\partial^2\varphi}{\partial v^2};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\big(g(u,v)+2y\cdot h(u,v)\big)=\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y}\right)+2h(u,v)+2y\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial y}\right)=$$

$$=\frac{\partial^2\varphi}{\partial u^2}+2y\cdot\frac{\partial^2\varphi}{\partial v\partial u}+2\frac{\partial\varphi}{\partial v}+2y\cdot\frac{\partial^2\varphi}{\partial u\partial v}+4y^2\frac{\partial^2\varphi}{\partial v^2}.$$

24. $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$, $\frac{\partial v}{\partial x}=y\cdot e^{xy}$, $\frac{\partial v}{\partial y}=x\cdot e^{xy}$. Pentru derivatele parțiale de ordinul I ale lui $f(x,y)$ folosim regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}=2x\frac{\partial\varphi}{\partial u}+ye^{xy}\frac{\partial\varphi}{\partial v};\quad \frac{\partial f}{\partial y}=\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y}=-2y\frac{\partial\varphi}{\partial u}+x\cdot e^{xy}\frac{\partial\varphi}{\partial v}$$

Pentru simplificarea calculului deriv. parțiale de ord. II, notăm $\frac{\partial\varphi}{\partial u}=g(u,v)$ și $\frac{\partial\varphi}{\partial v}=h(u,v)$ și obținem:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\big(2x\cdot g(u,v)+ye^{xy}\cdot h(u,v)\big)=2g(u,v)+2x\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}\right)+y^2e^{xy}h(u,v)+$$

$$+ye^{xy}\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=2\frac{\partial\varphi}{\partial u}+4x^2\frac{\partial^2\varphi}{\partial u^2}+2xye^{xy}\cdot\frac{\partial^2\varphi}{\partial v\partial u}+y^2e^{xy}\frac{\partial\varphi}{\partial v}+2xye^{xy}\cdot\frac{\partial^2\varphi}{\partial u\partial v}+y^2e^{2xy}\frac{\partial^2\varphi}{\partial v^2};$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\big(-2y\cdot g(u,v)+xe^{xy}\cdot h(u,v)\big)=-2y\cdot\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x}\right)+\left(e^{xy}+xye^{xy}\right)h(u,v)+$$

$$+xe^{xy}\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=-4xy\frac{\partial^2\varphi}{\partial u^2}-2y^2e^{xy}\cdot\frac{\partial^2\varphi}{\partial v\partial u}+e^{xy}(1+xy)\frac{\partial\varphi}{\partial v}+2x^2e^{xy}\cdot\frac{\partial^2\varphi}{\partial u\partial v}+xye^{2xy}\frac{\partial^2\varphi}{\partial v^2};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\big(-2yg(u,v)+xe^{xy}h(u,v)\big)=-2g(u,v)-2y\left(\frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y}\right)+x^2e^{xy}h(u,v)+$$

$$+xe^{xy}\left(\frac{\partial h}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial h}{\partial v}\cdot\frac{\partial v}{\partial y}\right)=-2\frac{\partial\varphi}{\partial u}+4y^2\cdot\frac{\partial^2\varphi}{\partial u^2}-2xye^{xy}\frac{\partial^2\varphi}{\partial v\partial u}+x^2e^{xy}\frac{\partial\varphi}{\partial v}+2xye^{xy}\frac{\partial^2\varphi}{\partial u\partial v}+x^2e^{2xy}\frac{\partial^2\varphi}{\partial v^2}.$$

25. $\frac{\partial u}{\partial x}=y$, $\frac{\partial u}{\partial y}=x$, $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial y}=-2y$. Calculăm mai întâi derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2u}{u^2+v^2}y+\frac{2v}{u^2+v^2}2x=\frac{4x^3-2xy^2}{x^4+y^4-x^2y^2};$$

$$\frac{\partial f}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2u}{u^2+v^2}x-\frac{2v}{u^2+v^2}2y=\frac{4y^3-2x^2y}{x^4+y^4-x^2y^2};$$

Derivatele de ordinul al II-lea sunt:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{4x^3-2xy^2}{x^4+y^4-x^2y^2}\right)=\frac{-4x^6+2x^4y^2-2y^6+10x^2y^4}{\left(x^4+y^4-x^2y^2\right)^2};$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{4y^3-2x^2y}{x^4+y^4-x^2y^2}\right)=\frac{-4y^6+2x^2y^4-2x^6+10x^4y^2}{\left(x^4+y^4-x^2y^2\right)^2};$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{4y^3-2x^2y}{x^4+y^4-x^2y^2}\right)=\frac{4xy\left(x^4+y^4-4x^2y^2\right)}{\left(x^4+y^4-x^2y^2\right)^2}$$

26. $\frac{\partial u}{\partial x}=y$, $\frac{\partial u}{\partial y}=x$, $\frac{\partial v}{\partial x}=\frac{1}{y}$, $\frac{\partial v}{\partial y}=-\frac{x}{y^2}$. Calculăm mai întâi derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot y+\frac{\partial \varphi}{\partial v}\cdot\frac{1}{y};\quad \frac{\partial f}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot x-\frac{\partial \varphi}{\partial v}\cdot\frac{x}{y^2};$$

Pentru simplificarea calculului deriv. parțiale de ord. II, notăm $\frac{\partial \varphi}{\partial u}=g(u,v)$ și $\frac{\partial \varphi}{\partial v}=h(u,v)$ și obținem:

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\cdot g(u,v)+\frac{1}{y}h(u,v)\right)=(...)=y^2\frac{\partial^2 \varphi}{\partial u^2}+\frac{\partial^2 \varphi}{\partial u\partial v}+\frac{\partial^2 \varphi}{\partial v\partial u}+\frac{1}{y^2}\frac{\partial^2 \varphi}{\partial v^2}$$

$$\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\cdot g(u,v)-\frac{x}{y^2}h(u,v)\right)=(...)=x^2\frac{\partial^2 \varphi}{\partial u^2}-\frac{x^2}{y^2}\left(\frac{\partial^2 \varphi}{\partial u\partial v}+\frac{\partial^2 \varphi}{\partial v\partial u}\right)+\frac{x^2}{y^4}\frac{\partial^2 \varphi}{\partial v^2}+\frac{2x}{y^3}\cdot\frac{\partial \varphi}{\partial v}$$

$$\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial}{\partial x}\left(xg(u,v)-\frac{x}{y^2}h(u,v)\right)=(...)=\frac{\partial \varphi}{\partial u}+xy\frac{\partial^2 \varphi}{\partial u^2}+\frac{x}{y}\left(\frac{\partial^2 \varphi}{\partial v\partial u}-\frac{\partial^2 \varphi}{\partial u\partial v}\right)-\frac{1}{y^2}\cdot\frac{\partial \varphi}{\partial v}-\frac{x}{y^3}\frac{\partial^2 \varphi}{\partial v^2}$$

27. Notăm $u(x,y)=\frac{y}{x}$, deci $f(x,y)=\varphi(u)$; $\frac{\partial u}{\partial x}=-\frac{y}{x^2}$, $\frac{\partial u}{\partial y}=\frac{1}{x}$; Calculăm derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a funcțiilor compuse:

$\frac{\partial f}{\partial x}=\varphi'(u)\cdot\frac{\partial u}{\partial x}=-\frac{y}{x^2}\cdot\varphi'(u)$; $\frac{\partial f}{\partial y}=\varphi'(u)\cdot\frac{\partial u}{\partial y}=\frac{1}{x}\cdot\varphi'(u)$; Ecuația din enunț este verificată.

28. Notăm $u(x,y,z)=xy$ și $v(x,y,z)=x^2+y^2+z^2$ deci $f(x,y,z)=\varphi(u,v)$, cu $\frac{\partial u}{\partial x}=y$, $\frac{\partial u}{\partial y}=x$, $\frac{\partial u}{\partial z}=0$ și respectiv $\frac{\partial v}{\partial x}=2x$, $\frac{\partial v}{\partial y}=2y$, $\frac{\partial v}{\partial z}=2z$. Calculăm derivatele parțiale de ordinul I ale lui $f(x,y,z)$ folosind regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial \varphi}{\partial u}\cdot y+\frac{\partial \varphi}{\partial v}\cdot 2x;\quad \frac{\partial f}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial \varphi}{\partial u}\cdot x+\frac{\partial \varphi}{\partial v}\cdot 2y;$$

$\frac{\partial f}{\partial z}=\frac{\partial \varphi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial \varphi}{\partial v}\cdot\frac{\partial v}{\partial z}=2z\cdot\frac{\partial \varphi}{\partial v}$. Ecuația din enunț este verificată.

29. Notăm $u(x,y)=x^2-y^2$, deci $f(x,y)=y\cdot\varphi(u)$; $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$; Calculăm derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a produsului a două funcții și regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(y\cdot\varphi(u)\right)=y\cdot\varphi'(u)\cdot\frac{\partial u}{\partial x}=2xy\cdot\varphi'(u);$$

$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(y\cdot\varphi(u)\right)=y'\cdot\varphi(u)+y\cdot\varphi'(u)\cdot\frac{\partial u}{\partial y}=\varphi(u)-2y^2\cdot\varphi'(u)$; Ecuația din enunț este verificată.

30. Notăm $u(x,y)=y\cdot e^{\frac{x^2}{2y^2}}$, deci $f(x,y)=e^y\cdot\varphi(u)$; $\frac{\partial u}{\partial x}=\frac{x}{y}\cdot e^{\frac{x^2}{2y^2}}$, $\frac{\partial u}{\partial y}=e^{\frac{x^2}{2y^2}}\left(1-\frac{x^2}{y^2}\right)$; Calculăm derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a produsului a două funcții și regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(e^y\cdot\varphi(u)\right)=e^y\cdot\varphi'(u)\cdot\frac{\partial u}{\partial x}=\frac{x}{y}e^y\cdot e^{\frac{x^2}{2y^2}}\cdot\varphi'(u);$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(e^y\cdot\varphi(u)\right)=\left(e^y\right)'\cdot\varphi(u)+e^y\varphi'(u)\cdot\frac{\partial u}{\partial y}=e^y\cdot\varphi(u)+e^y\cdot e^{\frac{x^2}{2y^2}}\cdot\varphi'(u)\cdot\left(1-\frac{x^2}{y^2}\right);$$ Ecuația din enunț este verificată.

31. Notăm $u(x,y)=x^2-y^2$, deci $f(x,y)=xy\cdot\varphi(u)$; $\frac{\partial u}{\partial x}=2x$, $\frac{\partial u}{\partial y}=-2y$; Calculăm derivatele parțiale de ordinul I ale lui $f(x,y)$, folosind regula de derivare a produsului a două funcții și regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(xy\cdot\varphi(u)\right)=y\left(x'\cdot\varphi(u)+x\cdot\varphi'(u)\cdot\frac{\partial u}{\partial x}\right)=y\cdot\varphi(u)+2x^2y\cdot\varphi'(u);$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(xy\cdot\varphi(u)\right)=x\left(y'\cdot\varphi(u)+y\cdot\varphi'(u)\cdot\frac{\partial u}{\partial y}\right)=x\cdot\varphi(u)-2xy^2\cdot\varphi'(u);$$ Ecuația din enunț este verificată.

32. $u(x,y,z)=\frac{x}{y}$ și $v(x,y,z)=\frac{z}{x}$ deci $f(x,y,z)=\frac{xy}{z}\ln x+x\cdot\varphi(u,v)$, cu $\frac{\partial u}{\partial x}=\frac{1}{y}$, $\frac{\partial u}{\partial y}=-\frac{x}{y^2}$, $\frac{\partial u}{\partial z}=0$ și respectiv $\frac{\partial v}{\partial x}=-\frac{z}{x^2}$, $\frac{\partial v}{\partial y}=0$, $\frac{\partial v}{\partial z}=\frac{1}{x}$. Calculăm derivatele parțiale de ordinul I ale lui $f(x,y,z)$ folosind regula de derivare a produsului și regula de derivare a funcțiilor compuse:

$$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\left(\frac{xy}{z}\ln x\right)+\frac{\partial}{\partial x}\left(x\cdot\varphi(u,v)\right)=\frac{y}{z}(\ln x+1)+x'\cdot\varphi(u,v)+x\left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial x}\right)=$$
$$=\frac{y}{z}(\ln x+1)+\varphi(u,v)+\frac{x}{y}\cdot\frac{\partial\varphi}{\partial u}-\frac{z}{x}\cdot\frac{\partial\varphi}{\partial v};$$

$$\frac{\partial f}{\partial y}=\frac{\partial}{\partial y}\left(\frac{xy}{z}\ln x\right)+\frac{\partial}{\partial y}\left(x\cdot\varphi(u,v)\right)=\frac{x\ln x}{z}+x\left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial y}\right)=\frac{x\ln x}{z}-\frac{x^2}{y^2}\cdot\frac{\partial\varphi}{\partial u};$$

$$\frac{\partial f}{\partial z}=\frac{\partial}{\partial z}\left(\frac{xy}{z}\ln x\right)+\frac{\partial}{\partial z}\left(x\cdot\varphi(u,v)\right)=-\frac{xy\ln x}{z^2}+x\left(\frac{\partial\varphi}{\partial u}\cdot\frac{\partial u}{\partial z}+\frac{\partial\varphi}{\partial v}\cdot\frac{\partial v}{\partial z}\right)=-\frac{xy\ln x}{z^2}+\frac{\partial\varphi}{\partial v}.$$

Ecuația din enunț este verificată.