Capitolul 6

Transformata Fourier

Curs 1 de TF

$f:\mathbb{R}\to\mathbb{R}$ cu $\int_{-\infty}^{\infty}|f(t)|\,dt<+\infty$. Atunci $f\in L^1(\mathbb{R})$.

Def. $F:\mathbb{R}\to\mathbb{C}$, $F(\omega)\overset{def.}{=}\int_{-\infty}^{\infty}f(t)\cdot e^{-j\omega t}\,dt$ s.n. transformata Fourier a lui $f(t)$ și scriem:

$f(t)\overset{F}{\longleftrightarrow}F(\omega)$ sau $F(\omega)=\mathcal{F}\{f(t)\}=\hat{f}(\omega)$;

$t\to$ timp
$\omega\to$ frecvență

Obs. Fie $u\in L^1_{loc}(\mathbb{R})\iff u$ este integrabilă pe orice compact din $\mathbb{R}$.

Fie $\varphi\in C_0^\infty(\mathbb{R})\iff\varphi\in C^\infty(\mathbb{R})$ (indefinit derivabilă pe $\mathbb{R}$: are derivată de orice ordin continuă și $\operatorname{supp}\varphi=\overline{\{t\in\mathbb{R}\mid\varphi(t)\neq0\}}$ este compact (închis și mărginit).

$\varphi\in C_0^\infty(\mathbb{R})$ s.n. funcții test.

Definim: $\underbrace{\langle u,\varphi\rangle}_{\text{distribuție sau funcție generalizată}}\overset{not.}{=}\int_{-\infty}^{\infty}u(t)\cdot\varphi(t)\,dt$, $\forall\varphi\in C_0^\infty(\mathbb{R})$

distribuție sau funcție generalizată.

$\cdot\ \langle u(at+b),\varphi(t)\rangle=\int_{-\infty}^{\infty}\varphi(t)\cdot u(at+b)\,dt=\frac{1}{|a|}\int_{-\infty}^{\infty}u(t)\cdot\varphi\left(\frac{t-b}{a}\right)dt=$

$=\langle u(t),\frac{1}{|a|}\varphi\left(\frac{t-b}{a}\right)\rangle$.

$\cdot\cdot\ \langle u^{(n)}(t),\varphi(t)\rangle=(-1)^n\langle u(t),\varphi^{(n)}(t)\rangle$.

Ex ① $\langle\delta(t),\varphi(t)\rangle=\varphi(0)$; ② $\langle\sigma(t),\varphi(t)\rangle=\int_0^\infty\varphi(t)\,dt$.

$\cdot\ \langle\sigma'(t),\varphi(t)\rangle=-\langle\sigma(t),\varphi'(t)\rangle=-\int_0^\infty\varphi'(t)\,dt=-\underbrace{\lim_{t\to\infty}\varphi(t)}_{0\ (\varphi\in C_0^\infty)}+\varphi(0)=$

$=\langle\delta(t),\varphi(t)\rangle\Rightarrow$

$\boxed{\sigma'(t)=\delta(t)}$ $\cdot\cdot\ \langle\hat{u}(\omega),\varphi(\omega)\rangle=\langle u(\omega),\hat{\varphi}(\omega)\rangle$.

$\cdots\ \langle\hat{\delta}(\omega),\varphi(\omega)\rangle\overset{def.}{=}\langle\delta(\omega),\hat{\varphi}(\omega)\rangle=\hat{\varphi}(0)=\int_{-\infty}^{\infty}\varphi(t)\,dt=\langle1,\varphi(\omega)\rangle$

$\Rightarrow\boxed{\hat{\delta}(\omega)=1}$ $\delta(t)\overset{F}{\longleftrightarrow}1$; $\langle\delta(-t),\varphi(t)\rangle=\langle\delta(t),\varphi(-t)\rangle=$

$=\varphi(0)=\langle\delta(t),\varphi(t)\rangle$

$\Downarrow$

$\boxed{\delta(-t)=\delta(t)}$

$\delta(t)=$ impulsul Dirac
$\sigma(t)=$ fcția unitate / Heaviside

Proprietăți ale transformatei Fourier

1 $f_1(t)\overset{F}{\longleftrightarrow}F_1(\omega)$ $f_2(t)\overset{F}{\longleftrightarrow}F_2(\omega)$

$\alpha f_1(t)+\beta f_2(t)\overset{F}{\longleftrightarrow}\alpha F_1(\omega)+\beta F_2(\omega)$, $\forall\alpha,\beta\in\mathbb{C}$

$f(t) \xleftrightarrow{F} F(\omega)$

2. Deplasarea în frecvență

$$f(t)\cdot e^{j\omega_0 t} \xleftrightarrow{F} F(\omega-\omega_0), \quad \omega_0 \in \mathbb{R}$$

3. Întârzierea în timp

$$f(t-t_0) \xleftrightarrow{F} e^{-jt_0\omega}\cdot F(\omega), \quad \text{cu } t_0 \in \mathbb{R}$$

4. Schimbarea scalei

$$f(at) \xleftrightarrow{F} \frac{1}{|a|}F\left(\frac{\omega}{a}\right), \quad \forall a \in \mathbb{R}^*$$

5. Convoluția. $f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(t-\tau)\cdot f_2(\tau)\,d\tau.$

- $f_1(t) * f_2(t) \xleftrightarrow{F} F_1(\omega)\cdot F_2(\omega)$
- $f_1(t)\cdot f_2(t) \xleftrightarrow{F} \frac{1}{2\pi} F_1(\omega) * F_2(\omega)$

6. Transformarea timpului

$$f(at-t_0) \xleftrightarrow{F} \frac{1}{|a|}F\left(\frac{\omega}{a}\right)\cdot e^{-jt_0\frac{\omega}{a}}$$

$$\int_{-\infty}^{\infty} f(at-t_0)\cdot e^{-j\omega t}\,dt = \frac{1}{|a|}\int_{-\infty}^{\infty} f(y)\cdot e^{-j\omega\left(\frac{y}{a}+\frac{t_0}{a}\right)}\,dy =$$

$$= \frac{1}{|a|}\left(\int_{-\infty}^{\infty} f(y)\cdot e^{-j\frac{\omega}{a}y}\,dy\right)\cdot e^{-j\frac{t_0}{a}\omega} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)\cdot e^{-j\frac{t_0}{a}\omega}.$$

7. Diferențierea: – în timp: $f^{(n)}(t) \xleftrightarrow{F} (j\omega)^n F(\omega)$

– în frecvență: $t^n\cdot f(t) \xleftrightarrow{F} j^n F^{(n)}(\omega)$.

8. Integrarea în timp:

$$\int_{-\infty}^{t} f(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega}F(\omega) + \pi F(0)\delta(\omega)$$

9. Inversa transformatei Fourier TF

$$F(\omega) \xleftrightarrow{F^{-1}} f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{j\omega t}\,d\omega.$$

10. Simetria $\quad f(t) \xleftrightarrow{F} F(\omega);\ \boxed{F(t) \xleftrightarrow{F} 2\pi f(-\omega)}$

Aplicații

① $f(t) = e^{-at}\sigma(t) \underset{a>0}{\xleftrightarrow{F}} F(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t}\,dt =$

$$= \frac{-1}{a+j\omega}\cdot e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

② $f(t) = e^{at}\sigma(-t) \underset{a>0}{\xleftrightarrow{F}} F(\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t}\,dt = \frac{1}{a-j\omega}.$

$$\sigma(t)+\sigma(-t)=1 \Rightarrow \boxed{\sigma(-t)=1-\sigma(t)}$$

3. $f(t)=t\cdot e^{-at}\sigma(t) \underset{\text{diferentierea în frecvență}}{\overset{F}{\longleftrightarrow}} j\cdot\left(\mathcal{F}\{e^{-at}\sigma(t)\}\right)' = j\left(\frac{1}{a+j\omega}\right)' =$

$$= \frac{-j^2}{(a+j\omega)^2} = \frac{1}{(a+j\omega)^2}.$$

4. $\delta(t) \overset{F}{\longleftrightarrow} 1$ cu simetria

$1 \overset{F}{\longleftrightarrow} 2\pi\delta(-\omega) = 2\pi\delta(\omega)$

$e^{j\omega_0 t} \underset{\text{deplasarea}}{\overset{F}{\longleftrightarrow}} 2\pi\delta(\omega-\omega_0)$

$\cdot \cos\omega_0 t = \frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t}) \overset{F}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$

$\cdot\cdot \sin\omega_0 t = \frac{1}{2j}(e^{j\omega_0 t}-e^{-j\omega_0 t}) \overset{F}{\longleftrightarrow} \pi j[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)]$

5. $\sigma'(t)=\delta(t) \Rightarrow \sigma(t)=\int_{-\infty}^{t}\delta(\tau)d\tau \underset{\text{integrarea în timp}}{\overset{F}{\longleftrightarrow}} \frac{1}{j\omega}\mathcal{F}\{\delta(t)\} + \pi\underbrace{\hat{\delta}(0)}_{1}\cdot\delta(\omega)$

$$= \frac{1}{j\omega}+\pi\delta(\omega)$$

$$\boxed{\sigma(t) \overset{F}{\longleftrightarrow} \pi\delta(\omega)+\frac{1}{j\omega}}$$

$\sigma(t)\cdot e^{j\omega_0 t} \underset{\text{deplasarea}}{\overset{F}{\longleftrightarrow}} \pi\delta(\omega-\omega_0)+\frac{1}{j(\omega-\omega_0)}$

$\cdot \cos(\omega_0 t)\sigma(t) = \frac{1}{2}(e^{j\omega_0 t}+e^{-j\omega_0 t})\cdot\sigma(t) \overset{F}{\longleftrightarrow} \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + \frac{1}{2j}\left(\frac{1}{\omega-\omega_0}+\frac{1}{\omega+\omega_0}\right) \Rightarrow$

$\cdot \cos(\omega_0 t)\cdot\sigma(t) \overset{F}{\longleftrightarrow} \frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)] + \frac{j\omega}{\omega_0^2-\omega^2}$

$\cdot\cdot \sin(\omega_0 t)\sigma(t) \overset{F}{\longleftrightarrow} j\frac{\pi}{2}[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2}$

$e^{j\omega_0 t}\sigma(t) \overset{F}{\longleftrightarrow} \pi\delta(\omega-\omega_0)+\frac{1}{j(\omega-\omega_0)}$

$e^{-j\omega_0 t}\sigma(t) \overset{F}{\longleftrightarrow} \pi\delta(\omega+\omega_0)+\frac{1}{j(\omega+\omega_0)}$

$(e^{j\omega_0 t}-e^{-j\omega_0 t})\sigma(t) \overset{F}{\longleftrightarrow} \pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)] + \frac{1}{j}\cdot\frac{+2\omega_0}{\omega^2-\omega_0^2}$

$\sin\omega_0 t\cdot\sigma(t) \overset{F}{\longleftrightarrow} j\frac{\pi}{2}[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] + \frac{\omega_0}{\omega_0^2-\omega^2}.$

6. $\sigma(t) \xleftrightarrow{F} \pi\delta(\omega) + \frac{1}{j\omega}$

$$\underset{\text{întârzierea}}{\sigma\left(t-\frac{\pi}{2}\right) \xleftrightarrow{F}} \left(\pi\delta(\omega) + \frac{1}{j\omega}\right)\cdot e^{-j\frac{\pi}{2}\omega}$$

$$E\left(\sigma(t) - \sigma\left(t-\frac{\pi}{2}\right)\right) \xleftrightarrow{F} \left(\pi\delta(\omega)+\frac{1}{j\omega}\right)\left(1-e^{-j\frac{\pi}{2}\omega}\right).$$

$$\| \quad \begin{cases} E & , t\in\left(0,\frac{\pi}{2}\right) \\ 0 & , \text{rest}. \end{cases}$$

7. $f(t) = e^{-t^2} \xleftrightarrow{F} F(\omega)$

$f'(t) = -2t\,e^{-t^2}$

$$\left.\begin{array}{l} f'(t) \xleftrightarrow{F} j\omega F(\omega) \\ -2t\cdot e^{-t^2} \xleftrightarrow{F} -2jF'(\omega) \end{array}\right| \Rightarrow \begin{array}{l} j\omega F(\omega) = -2jF'(\omega) \\ F'(\omega) = -\frac{\omega}{2}F(\omega) \Leftrightarrow \end{array}$$

$$\frac{dF}{F} = -\frac{\omega}{2}\,d\omega \Rightarrow \ln|F(\omega)| = -\frac{\omega^2}{4}+K \Rightarrow F(\omega) = \left(\pm e^{K}\right)\cdot e^{-\frac{\omega^2}{4}} = C\cdot e^{-\frac{\omega^2}{4}}.$$

$$\left.\begin{array}{l} F(0) = \int_{-\infty}^{\infty} e^{-t^2}\,dt = \sqrt{\pi} \\ F(\omega) = C\cdot e^{-\omega^2/4} \end{array}\right| \begin{array}{l} \to C = \sqrt{\pi} \to \\ f(t) = e^{-t^2} \xleftrightarrow{F} \sqrt{\pi}\cdot e^{-\omega^2/4} = F(\omega). \end{array}$$

8. $$g(t) = e^{-\frac{t^2}{2\sigma^2}} = f\left(\frac{t}{\sigma\sqrt{2}}\right) \underset{\substack{a=\frac{1}{\sigma\sqrt{2}} \\ \downarrow \\ \text{schimbarea scalei}}}{\xleftrightarrow{F}} G(\omega) = \frac{1}{\frac{1}{\sigma\sqrt{2}}}F\left(\frac{\omega}{\frac{1}{\sigma\sqrt{2}}}\right) =$$

$$= \sigma\sqrt{2}\,F(\sigma\sqrt{2}\,\omega) = \sigma\sqrt{2}\cdot\sqrt{\pi}\cdot e^{-\frac{2\sigma^2\omega^2}{4}}$$

$$g(t) = e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{F} G(\omega) = \sqrt{2\pi}\,\sigma\, e^{-\frac{\sigma^2\omega^2}{2}}.$$

$$\boxed{e^{-\frac{t^2}{2\sigma^2}} \xleftrightarrow{F} \sigma\sqrt{2\pi}\cdot e^{-\frac{\sigma^2\omega^2}{2}}}$$

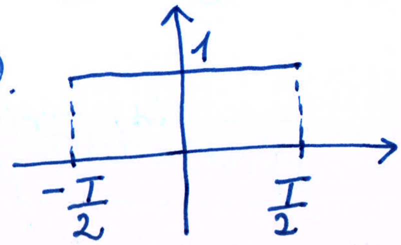
Aplicații.

1. $P_T(t)=\begin{cases}1, & |t|\le T\\ 0, & rest\end{cases}$ semnal poartă

$$F(\omega)=\int_{-\infty}^{\infty} P_T(t)\cdot e^{-j\omega t}\,dt=\int_{-T}^{T} e^{-j\omega t}\,dt = -\frac{1}{j\omega}\cdot e^{-j\omega t}\Big|_{-T}^{T} =$$

$$= -\frac{1}{j\omega}\cdot(\underbrace{e^{-j\omega T}-e^{j\omega T}}_{-2j\sin(\omega T)}) = 2T\,\frac{\sin(\omega T)}{\omega T} = 2T\,\mathrm{sinc}(\omega T)$$

Deci: $P_T(t)\xleftrightarrow{F} p_T(\omega)=2T\,\mathrm{sinc}(\omega T)$.



2. Avem simetria:

$F(t)\xleftrightarrow{F} 2\pi f(-\omega)$

$g(t)=P_T(t)\xleftrightarrow{F} G(\omega)=2T\,\mathrm{sinc}(\omega T)$

Cu simetria: $G(t)\xleftrightarrow{F} 2\pi g(-\omega)$

$$2T\,\mathrm{sinc}(tT)\xleftrightarrow{F} 2\pi P_T(-\omega)$$

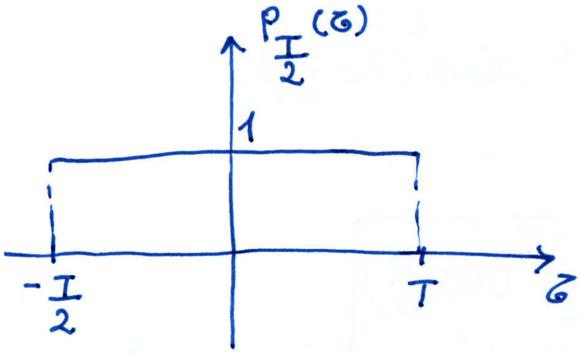
$$2T\,\frac{\sin(tT)}{tT}\xleftrightarrow{F} 2\pi P_T(-\omega)\ \Big/ : 2\pi$$

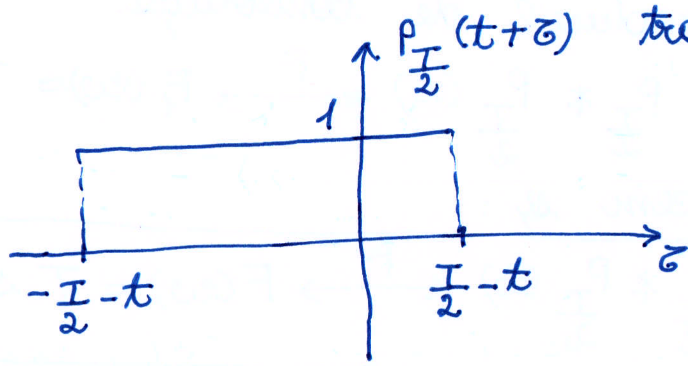
$$\frac{\sin(tT)}{\pi t}\xleftrightarrow{F} P_T(\omega)$$

Deci: $\boxed{\frac{\sin(tT)}{\pi t}\xleftrightarrow{F} P_T(\omega)}$

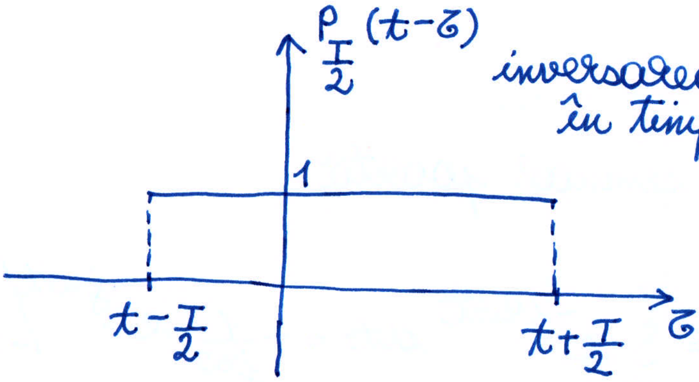
3. Calculați $f_1(t)=P_{\frac{T}{2}}(t)*P_{\frac{T}{2}}(t)$ și $F_1(\omega)$.

$$f_1(t)=\int_{-\infty}^{\infty} P_{\frac{T}{2}}(t-\tau)\cdot P_{\frac{T}{2}}(\tau)\,d\tau.$$

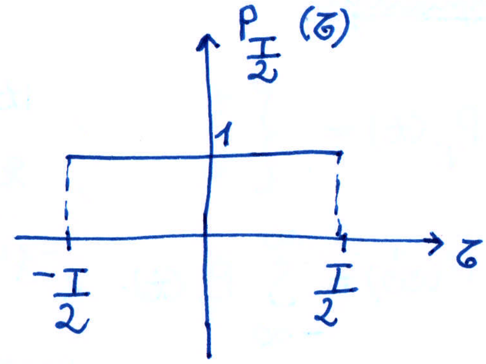




translația la stânga cu t



inversarea în timp



I $\quad t+\frac{T}{2} < -\frac{T}{2} \rightarrow \boxed{t<-T} \rightarrow f_1(t) = P_{\frac{T}{2}} * P_{\frac{T}{2}}(t) = 0$

II $\quad t-\frac{T}{2} < -\frac{T}{2} < t+\frac{T}{2} < \frac{T}{2} \rightarrow \boxed{-T<t<0} \rightarrow f_1(t) = \int_{-\frac{T}{2}}^{t+\frac{T}{2}} d\tau =$

$= t+T.$

$\boxed{f_1(t) = t+T}$; $\quad f_1(-T) = 0, \quad f_1(0) = 0+T = T.$

III $\quad -\frac{T}{2} < t-\frac{T}{2} < t+\frac{T}{2} < \frac{T}{2}$ caz imposibil;

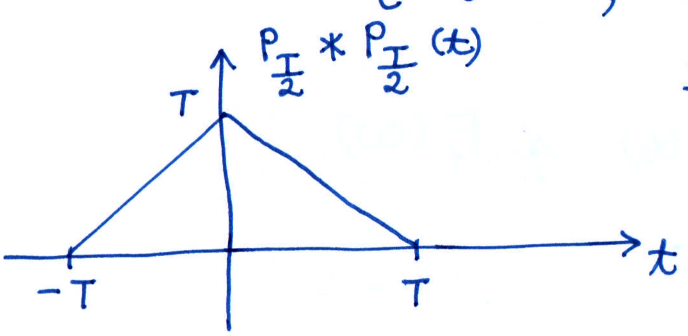
$-\frac{T}{2} < t-\frac{T}{2} < \frac{T}{2} < t+\frac{T}{2} \Rightarrow \boxed{0<t<T} \rightarrow f_1(t) = \int_{t-\frac{T}{2}}^{\frac{T}{2}} d\tau =$

$= T-t.$

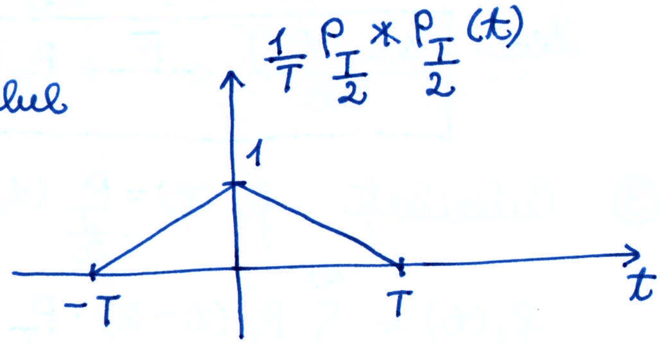
$\boxed{f_1(t) = T-t} \quad f_1(0) = T; \quad f_1(T) = 0.$

IV $\quad \frac{T}{2} < t-\frac{T}{2} \Rightarrow \boxed{T<t} \Rightarrow f_1(t) = 0.$

Deci: $P_{\frac{T}{2}} * P_{\frac{T}{2}}(t) = \begin{cases} T+t, & -T<t<0 \\ T-t, & 0<t<T \\ 0, & \text{rest.} \end{cases} = \begin{cases} T-|t|, & |t|<T \\ 0, & \text{rest} \end{cases}.$



semnalul $\longrightarrow$



Cu produsul de convoluție:

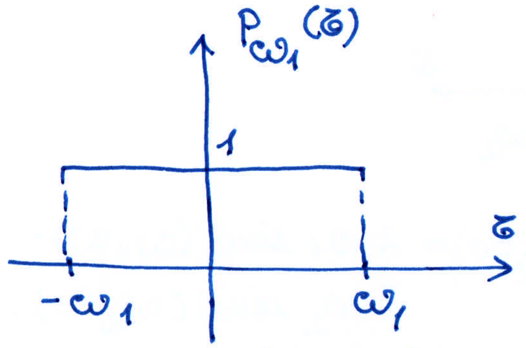
$$f_1(t) = P_{\frac{T}{2}} * P_{\frac{T}{2}}(t) \xleftrightarrow{F} F_1(\omega) = T^2 \operatorname{sinc}^2\left(\omega \frac{T}{2}\right).$$

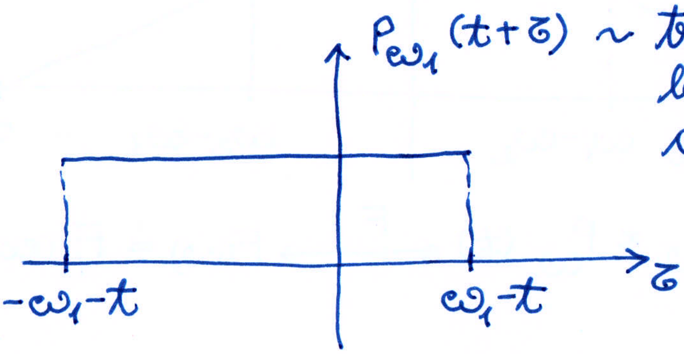
Deducem și:

$$\boxed{\frac{1}{T} P_{\frac{T}{2}} * P_{\frac{T}{2}}(t) \xleftrightarrow{F} F(\omega) = T \operatorname{sinc}^2\left(\omega \frac{T}{2}\right)}$$

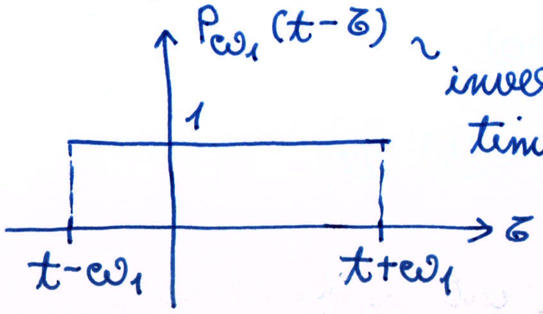
④ Calculați $f(t) = P_{\omega_1} * P_{\omega_2}(t)$ și $F(\omega)$.

$\omega_1 < \omega_2$

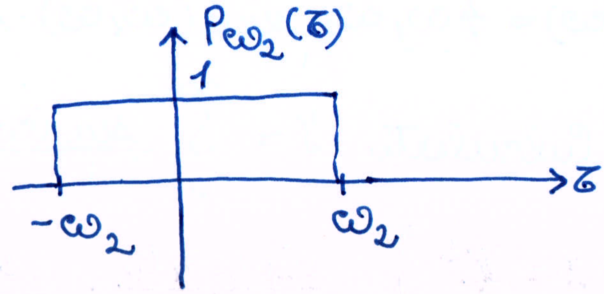




~ translația la stînga cu t



~ inversarea timpului



• $t+\omega_1 < -\omega_2 \Rightarrow \boxed{t < -\omega_1 - \omega_2} \rightarrow P_{\omega_1} * P_{\omega_2}(t) = 0$

• $t-\omega_1 < -\omega_2 < t+\omega_1 < \omega_2 \rightarrow \boxed{-\omega_1 - \omega_2 < t < \omega_1 - \omega_2} \rightarrow$

$$P_{\omega_1} * P_{\omega_2}(t) = \int_{-\omega_2}^{t+\omega_1} d\tau = t + \omega_1 + \omega_2 .$$

$$P_{\omega_1} * P_{\omega_2}(-\omega_1 - \omega_2) = 0 ; \quad P_{\omega_1} * P_{\omega_2}(\omega_1 - \omega_2) = 2\omega_1 .$$

• $-\omega_2 < t-\omega_1 < t+\omega_1 < \omega_2 \rightarrow \boxed{\omega_1 - \omega_2 < t < \omega_2 - \omega_1} \rightarrow$

$$P_{\omega_1} * P_{\omega_2}(t) = \int_{t-\omega_1}^{t+\omega_1} d\tau = 2\omega_1 .$$

$$P_{\omega_1} * P_{\omega_2}(\omega_1 - \omega_2) = P_{\omega_1} * P_{\omega_2}(\omega_2 - \omega_1) = 2\omega_1 .$$

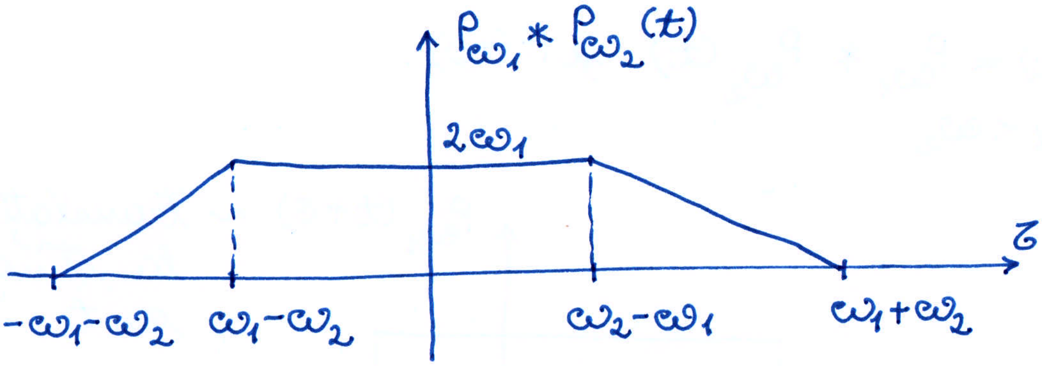
• $-\omega_2 < t-\omega_1 < \omega_2 < t+\omega_1 \rightarrow \boxed{\omega_2 - \omega_1 < t < \omega_1 + \omega_2} \rightarrow$

$$P_{\omega_1} * P_{\omega_2}(t) = \int_{t-\omega_1}^{\omega_2} d\tau = \omega_2 + \omega_1 - t \rightarrow$$

$$P_{\omega_1} * P_{\omega_2}(\omega_2 - \omega_1) = 2\omega_1 ; \quad P_{\omega_1} * P_{\omega_2}(\omega_1 + \omega_2) = 0 .$$

• $\omega_2 < t - \omega_1 \rightarrow \boxed{\omega_1 + \omega_2 < t} \rightarrow P_{\omega_1} * P_{\omega_2}(t) = 0 .$

Deci:
$$P_{\omega_1} * P_{\omega_2}(t) = \begin{cases} \omega_1 + \omega_2 + t & , t \in (-\omega_1 - \omega_2 , \omega_1 - \omega_2) \\ 2\omega_1 & , t \in (\omega_1 - \omega_2 , \omega_2 - \omega_1) \\ \omega_1 + \omega_2 - t & , t \in (\omega_2 - \omega_1 , \omega_1 + \omega_2) \\ 0 & , \text{rest} . \end{cases}$$



$$f(t) = P_{\omega_1} * P_{\omega_2}(t) \xleftrightarrow{F} F(\omega) = F_1(\omega)\cdot F_2(\omega) = 2\omega_1 \operatorname{sinc}(\omega_1\omega)\cdot 2\omega_2 \operatorname{sinc}(\omega_2\omega).$$

$$F(\omega) = 4\omega_1\omega_2 \operatorname{sinc}(\omega_1\omega)\cdot\operatorname{sinc}(\omega_2\omega).$$

⑤ Calculați $\displaystyle I = \int_0^\infty \frac{\sin(\omega_1 t)\cdot\sin(\omega_2 t)}{t^2}\,dt.$

$$I = \frac{\pi^2}{2}\int_{-\infty}^{\infty} \underbrace{\frac{\sin(\omega_1 t)}{\pi t}}_{f_1(t)}\cdot\underbrace{\frac{\sin(\omega_2 t)}{\pi t}}_{f_2(t)}\,dt = \frac{\pi^2}{2}\cdot F(0),$$

unde $F(\omega) = \mathcal{F}\{f_1\cdot f_2\} = \frac{1}{2\pi}(F_1 * F_2)(\omega)$ cu: $\begin{cases} F_1(\omega) = \mathcal{F}\{f_1 \\ F_2(\omega) = \mathcal{F}\{f_2 \end{cases}$

Cu aplicația ② avem:

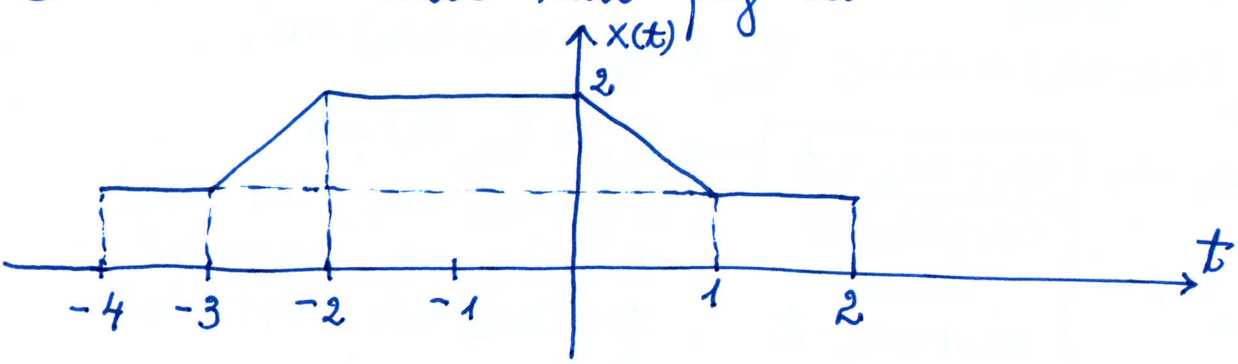
$$f_1(t) = \frac{\sin(\omega_1 t)}{\pi t} \xleftrightarrow{F} F_1(\omega) = P_{\omega_1}(\omega)$$
$$f_2(t) = \frac{\sin(\omega_2 t)}{\pi t} \xleftrightarrow{F} F_2(\omega) = P_{\omega_2}(\omega)$$

Deci:

$$F(0) = \frac{1}{2\pi}(F_1 * F_2)(0) = \frac{1}{2\pi}(P_{\omega_1} * P_{\omega_2})(0) = \frac{1}{2\pi}\cdot 2\omega_1 = \frac{\omega_1}{\pi}.$$

$$I = \frac{\pi^2}{2}F(0) = \frac{\pi^2}{2}\cdot\frac{\omega_1}{\pi} = \frac{\pi\omega_1}{2}.$$

⑥ Fie semnalul din figură



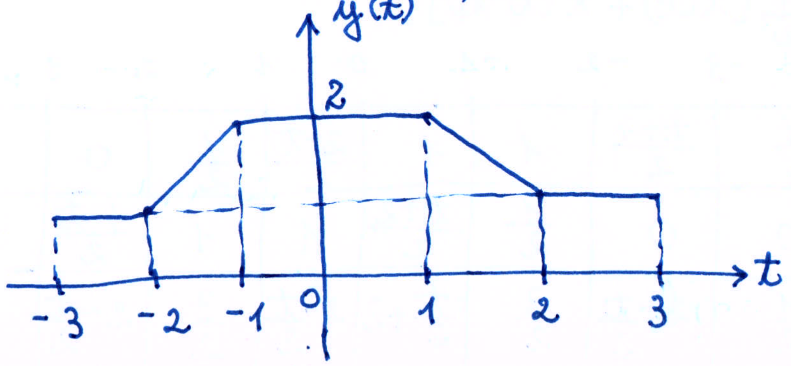
a) $\arg X(\omega) = ?$

b) $X(0) = ?$

c) $I_1 = \int_{-\infty}^{\infty} X(\omega)\,d\omega$, $I_2 = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$; d) $\mathcal{F}^{-1}\{\mathrm{Re}\, X(\omega)\} = ?$

<u>Soluție:</u>

a) $y(t) = x(t-1)$ ~ translația la dreapta cu o unitate



$y(t)$ este par $\longrightarrow$ $Y(\omega) \in \mathbb{R}$

Întârzierea : $y(t) = x(t-1) \xleftrightarrow{F} Y(\omega) = e^{-j\omega} X(\omega)$ $\Big| \longrightarrow$

$0 = \arg Y(\omega) = \arg e^{-j\omega} \cdot X(\omega) = -\omega + \arg X(\omega) \longrightarrow$

$\boxed{\arg X(\omega) = \omega}$

b) $X(\omega) = e^{j\omega} Y(\omega) \Rightarrow X(0) = Y(0) = \int_{-\infty}^{\infty} \underbrace{y(t)}_{par} dt = 2\int_0^{\infty} y(t)\,dt =$

$= 2\left(\int_0^1 2\,dt + \int_1^2 (3-t)\,dt + \int_2^3 dt\right) = 7.$

c) Formula de inversiune:

· $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\cdot e^{j\omega t} d\omega \Rightarrow I_1 = \int_{-\infty}^{\infty} X(\omega)\,d\omega = 2\pi\, x(0) =$

$\underset{\text{grafic}}{\overset{\text{din}}{=}}$ $2\pi \cdot 2 = 4\pi.$

·· $I_2 = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = ?$

$X(\omega) = e^{j\omega} Y(\omega) \Rightarrow |X(\omega)|^2 = Y^2(\omega) \Rightarrow$

$I_2 = \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \int_{-\infty}^{\infty} Y^2(\omega)\,d\omega \underset{\text{energiei}}{\overset{\text{teorema}}{=}} 2\pi \int_{-\infty}^{\infty} \underbrace{|y(t)|^2}_{par} dt =$

$= 4\pi \int_0^{\infty} y^2(t)\,dt = 4\pi\left[\int_0^1 4\,dt + \int_1^2 (3-t)^2 dt + \int_2^3 dt\right] = \frac{88}{3}\pi.$

d) Din punctul a) avem:

$X(\omega) = e^{j\omega} Y(\omega) = Y(\omega)\cdot \cos\omega + j\, Y(\omega) \sin\omega \Rightarrow$

$$\Rightarrow \operatorname{Re} X(\omega) = y(\omega)\cdot\cos\omega = \frac{1}{2}y(\omega)\cdot\left(e^{j\omega}+e^{-j\omega}\right) =$$

$$= \frac{1}{2}\underbrace{y(\omega)\cdot e^{j\omega}}_{X(\omega)}\left(1+e^{-j2\omega}\right) = \frac{1}{2}\left[X(\omega)+e^{-j2\omega}X(\omega)\right]$$

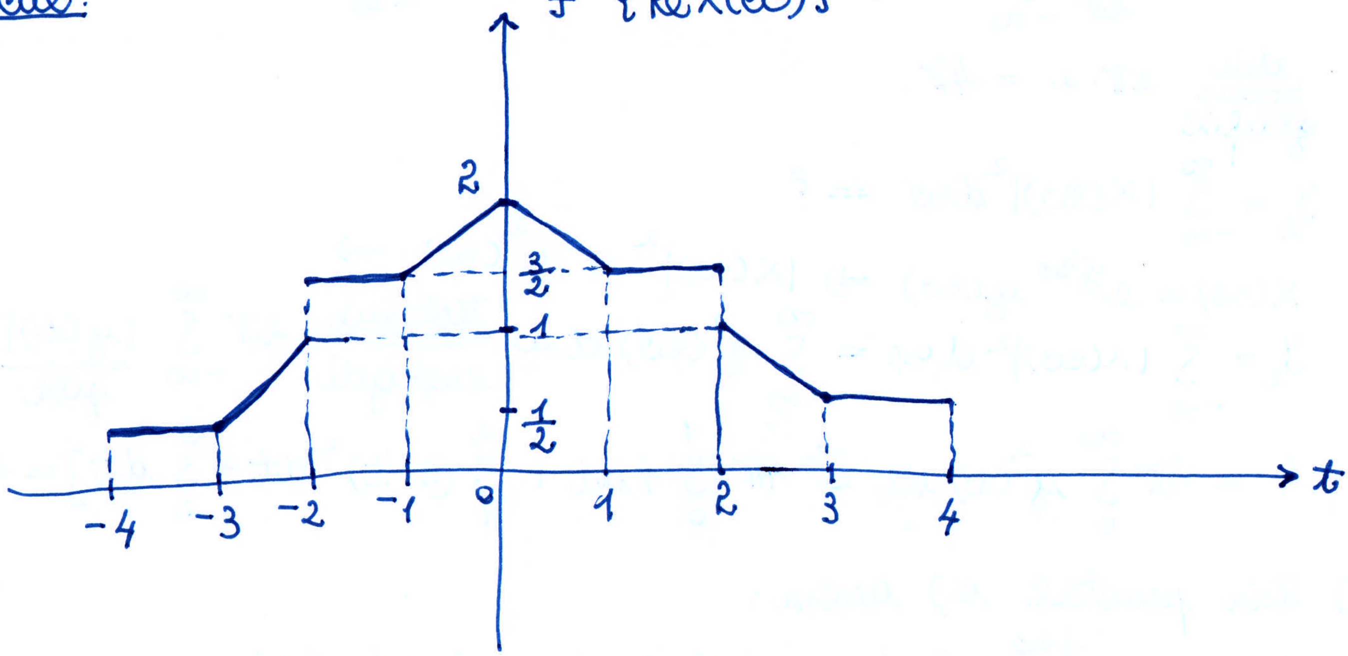
$$\Rightarrow \mathcal{F}^{-1}\{\operatorname{Re} X(\omega)\} = \frac{1}{2}\mathcal{F}^{-1}\{X(\omega)\} + \frac{1}{2}\mathcal{F}^{-1}\{e^{-j2\omega}X(\omega)\} \xlongequal[\text{rea}]{\text{întârzie-}}$$

$$= \frac{1}{2}\left[x(t)+x(t-2)\right].$$

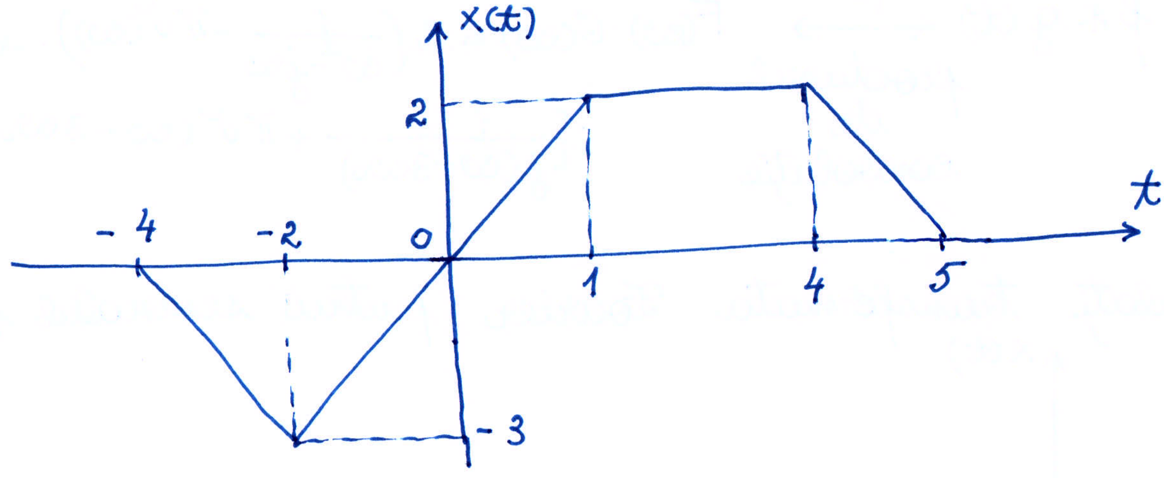
t		-4	-3	-2	-1	0	1	2	3	4
$\frac{1}{2}x(t)$	0	$\frac{1}{2}$	$\frac{t+4}{2}$	1	1	$\frac{2-t}{2}$	$\frac{1}{2}$	0	0	0
$\frac{1}{2}x(t-2)$	0	0	0	$\frac{1}{2}$	$\frac{t+2}{2}$	1	1	$\frac{4-t}{2}$	$\frac{1}{2}$	0
$\mathcal{F}^{-1}\{\operatorname{Re}X(\omega)\}$	0	$\frac{1}{2}$	$2+\frac{t}{2}$	$\frac{3}{2}$	$\frac{t}{2}+2$	$2-\frac{t}{2}$	$\frac{3}{2}$	$2-\frac{t}{2}$	$\frac{1}{2}$	0

$$\mathcal{F}^{-1}\{\operatorname{Re}X(\omega)\} = \begin{cases} \frac{1}{2}, & -4<t<-3 \\ 2+\frac{t}{2}, & -3<t<-2 \\ \frac{3}{2}, & -2<t<-1 \\ 2+\frac{t}{2}, & -1<t<0 \\ 2-\frac{t}{2}, & 0<t<1 \\ \frac{3}{2}, & 1<t<2 \\ 2-\frac{t}{2}, & 2<t<3 \\ \frac{1}{2}, & 3<t<4 \\ 0, & \text{rest} \end{cases}$$

<u>Graficul:</u>



7



Calculați $X(\omega)$.

Soluție: Fie semnalele

$$f(t) = \frac{1}{2}(P_1 * P_1)(t) \longrightarrow x_1(t) = -3 f(t+2)$$
$$g(t) = (P_{\frac{1}{2}} * P_2)(t) \longrightarrow x_2(t) = 2 g\left(t - \frac{5}{2}\right)$$

$$x(t) = x_1(t) + x_2(t) \xleftrightarrow{F} X(\omega) = X_1(\omega) + X_2(\omega)$$

$$x_1(t) \underset{T=2}{\xleftrightarrow{F}} X_1(\omega) = -3 \cdot e^{j2\omega} \cdot 2\, \mathrm{sinc}^2(\omega) = -6 e^{j2\omega} \mathrm{sinc}^2(\omega)$$

$$x_2(t) \underset{\omega_1 = \frac{1}{2},\ \omega_2 = 2}{\xleftrightarrow{F}} X_2(\omega) = 2 \cdot e^{-j\frac{5}{2}\omega} \cdot 4 \cdot \frac{1}{2} \cdot 2\, \mathrm{sinc}\left(\frac{\omega}{2}\right) \cdot \mathrm{sinc}(2\omega) =$$
$$= 8 e^{-j\frac{5}{2}\omega}\, \mathrm{sinc}\left(\frac{\omega}{2}\right) \cdot \mathrm{sinc}(2\omega).$$

Deci:

$$x(t) = x_1(t) + x_2(t) \xleftrightarrow{F} X(\omega) = X_1(\omega) + X_2(\omega) =$$
$$= -6 e^{j2\omega} \mathrm{sinc}^2(\omega) + 8 e^{-j\frac{5}{2}\omega} \cdot$$
$$\cdot\, \mathrm{sinc}\left(\frac{\omega}{2}\right) \cdot \mathrm{sinc}(2\omega).$$

8 Fie $f(t) = (2 e^{-t+2} - 2)\sigma(t-2)$

$g(t) = e^{j3\omega_0 t}\sigma(t)$. Aflați $F\{(f*g)(t)\}$.

Soluție: Avem:

$$e^{-(t-2)}\sigma(t-2) \underset{\substack{\text{întîrzierea} \\ a=1}}{\xleftrightarrow{F}} \frac{1}{1+j\omega} \cdot e^{-j2\omega}$$

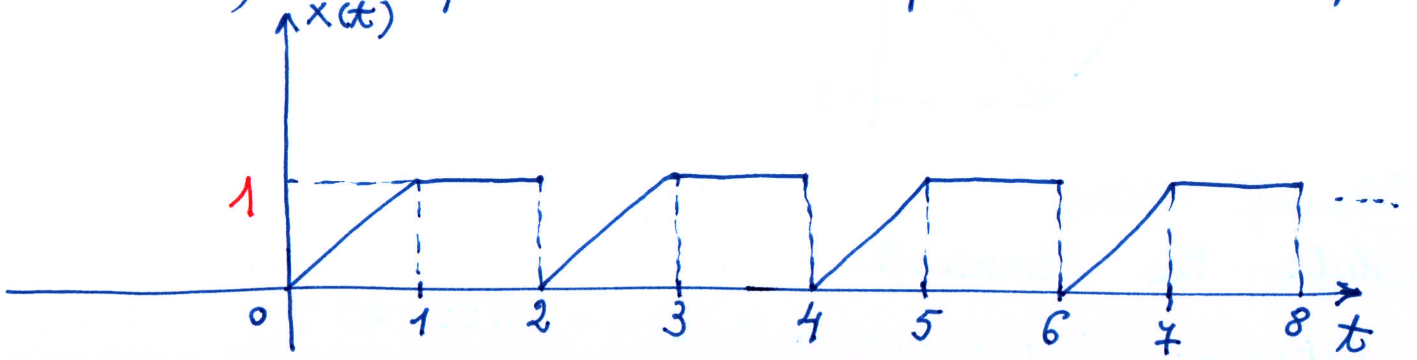
$$\sigma(t-2) \xleftrightarrow{F} \left[\frac{1}{j\omega} + \pi\delta(\omega)\right] \cdot e^{-j2\omega}$$

$$f(t) \xleftrightarrow{F} F(\omega) = 2\left(\frac{1}{1+j\omega} - \frac{1}{j\omega} - \pi\delta(\omega)\right) \cdot e^{-j2\omega}$$

$$g(t) \xleftrightarrow{F} G(\omega) = \frac{1}{j(\omega - 3\omega_0)} + \pi\delta(\omega - 3\omega_0).$$

Atunci: $f * g(t) \xleftrightarrow{\;F\;} F(\omega)\cdot G(\omega) = 2\left(\frac{1}{\omega^2 - j\omega} - \pi\delta(\omega)\right)\cdot e^{-j2\omega}\cdot$
produsul de convoluție
$\cdot\left[\frac{1}{j(\omega-3\omega_0)} + \pi\delta(\omega-3\omega_0)\right].$

⑨ Calculați transformata Fourier pentru semnalul periodic



Soluție: $X(\omega) = \int_0^{\infty} x(t)\cdot e^{-j\omega t}\,dt = \sum_{K=0}^{\infty}\int_{KT}^{(K+1)T} x(t)\cdot e^{-j\omega t}\,dt =$

$= \sum_{K=0}^{\infty}\Big\{\underbrace{\int_{KT}^{KT+\frac{T}{2}} t\cdot e^{-j\omega t}\,dt}_{I_1} + \underbrace{\int_{KT+\frac{T}{2}}^{(K+1)T} e^{-j\omega t}\,dt}_{I_2}\Big\}$

$\cdot\; I_1 = \int_{KT}^{KT+\frac{T}{2}} t\cdot\left(\frac{e^{-j\omega t}}{-j\omega}\right)' dt \overset{\text{părți}}{=} KT(e^{-j\omega\frac{T}{2}}-1)\cdot\frac{e^{-j\omega KT}}{-j\omega} +$

$+ \frac{\frac{T}{2}}{-j\omega}\cdot e^{-j\omega\frac{T}{2}}\cdot e^{-j\omega KT} + \frac{1}{\omega^2}(e^{-j\omega\frac{T}{2}}-1)\cdot e^{-j\omega KT}$

$\cdot\; I_2 = \frac{e^{-j\omega KT}}{-j\omega}\cdot e^{-j\omega\frac{T}{2}}(e^{-j\omega\frac{T}{2}}-1).$

$T = 2 \Rightarrow I_1 + I_2 = \frac{2}{-j\omega}K(e^{-j2\omega})^K(e^{-j\omega}-1) + \frac{e^{-j2\omega}}{-j\omega}\cdot(e^{-j2\omega})^K +$
$+ \frac{1}{\omega^2}(e^{-j\omega}-1)(e^{-j2\omega})^K$

$X(\omega) = \frac{2}{j\omega}(1-e^{-j\omega})\sum_{K=1}^{\infty}K(e^{-j2\omega})^K - \frac{1}{j\omega}\cdot e^{-j2\omega}\sum_{K=0}^{\infty}(e^{-j2\omega})^K +$
$+ \frac{1}{\omega^2}(e^{-j\omega}-1)\cdot\frac{1}{1-e^{-j2\omega}}.$

Folosim: $\sum_{K=0}^{\infty}(e^{-j2\omega})^K = \frac{1}{1-e^{-j2\omega}}$; $\sum_{K=1}^{\infty}K\cdot(e^{-j2\omega})^K = \frac{e^{-j2\omega}}{(1-e^{-j2\omega})^2}$

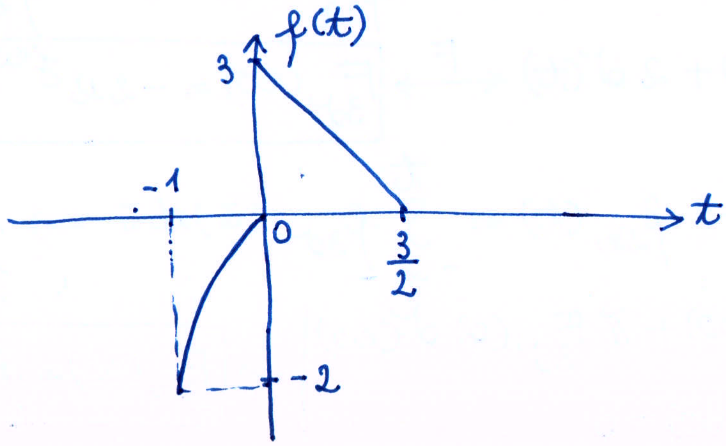
Obținem:
$X(\omega) = \frac{2}{j\omega}(1-e^{-j\omega})\cdot\frac{e^{-j2\omega}}{(1-e^{-j2\omega})^2} - \frac{1}{j\omega}\cdot\frac{e^{-j2\omega}}{1-e^{-j2\omega}} + \frac{1}{\omega^2}\cdot\frac{(e^{-j\omega}-1)}{1-e^{-j2\omega}} =$

$= \frac{1}{j\omega}\cdot\frac{e^{-j2\omega}}{(1+e^{-j\omega})^2} - \frac{1}{\omega^2}\cdot\frac{1}{1+e^{-j\omega}}.$

(10) Folosind metoda derivării și integrării succesive calculați TF pentru semnalul

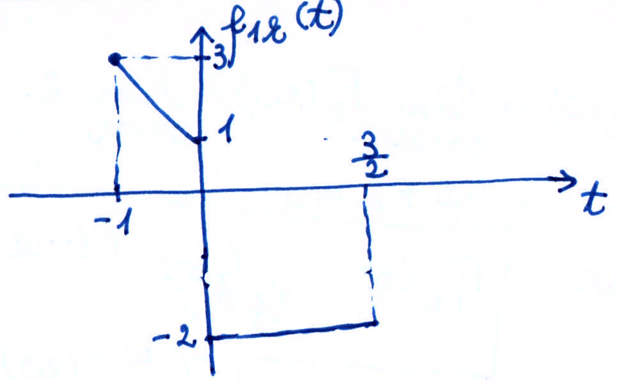
$$f(t)=\begin{cases} t-t^2, & t\in[-1,0) \\ 3-2t, & t\in[0,\frac{3}{2}] \\ 0, & \text{rest}. \end{cases}$$

Soluție:



$$f_1(t)=f'(t)=\underbrace{f_{1r}(t)}_{\text{componenta regulată}}+\underbrace{f_{1\delta}(t)}_{\text{componenta distribuțională}}$$

$$f_{1r}(t)=\begin{cases} 1-2t, & t\in(-1,0) \\ -2, & t\in(0,\frac{3}{2}) \end{cases}$$



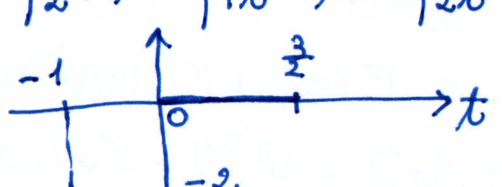
Folosim relația:

$$\boxed{\begin{array}{l} f \text{ are } a_1,\dots,a_n \text{ discontinuități} \Rightarrow \\ f_{1\delta}(t)=\sum_{k=1}^{n} S_f(a_k)\,\delta(t-a_k), \\ \text{unde: } S_f(a_k)=f(a_k+0)-f(a_k-0). \end{array}}$$

$$f_{1\delta}(t)=S_f(-1)\,\delta(t+1)+S_f(0)\,\delta(t)+S_f(\tfrac{3}{2})\,\delta(t-\tfrac{3}{2}) \rightarrow$$

$$f_{1\delta}(t)=-2\delta(t+1)+3\delta(t) \xleftrightarrow{F} \boxed{F_{1\delta}(\omega)=-2e^{j\omega}+3}$$

$$f_2(t)=f'_{1r}(t)=f_{2r}(t)+f_{2\delta}(t);\quad f_{2r}(t)=\begin{cases} -2, & t\in(-1,0) \\ 0, & t\in(0,\frac{3}{2}) \end{cases}$$



$$f_{2\delta}(t) = S_{f_{1r}}(-1)\,\delta(t+1) + S_{f_{1r}}(0)\cdot\delta(t) + S_{f_{1r}}\left(\tfrac{3}{2}\right)\delta\left(t-\tfrac{3}{2}\right) \longrightarrow$$

$$f_{2\delta}(t) = 3\,\delta(t+1) - 3\,\delta(t) + 2\,\delta\left(t-\tfrac{3}{2}\right) \xleftrightarrow{F} \boxed{F_{2\delta}(\omega) = 3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}}$$

- $$f_3(t) = f'_{2r}(t) = \overbrace{f_{3r}(t)}^{0} + f_{3\delta}(t) = f_{3\delta}(t)$$

$$f_{3\delta}(t) = S_{f_{2r}}(-1)\cdot\delta(t+1) + S_{f_{2r}}(0)\cdot\delta(t) + S_{f_{2r}}\left(\tfrac{3}{2}\right)\delta\left(t-\tfrac{3}{2}\right) =$$

$$= -2\,\delta(t+1) + 2\,\delta(t) \xleftrightarrow{F} \boxed{F_{3\delta}(\omega) = -2e^{j\omega} + 2.}$$

- $f'_{2r}(t) = f_{3\delta}(t) \rightarrow f_{2r}(t) = \int_{-\infty}^{t} f_{3\delta}(\tau)\,d\tau \Rightarrow$ cu integrarea în timp

$$\left.\begin{aligned} F_{2r}(\omega) &= \frac{1}{j\omega}F_{3\delta}(\omega) + \pi F_{3\delta}(0)\,\delta(\omega) \\ F_{3\delta}(0) &= \lim_{\omega\to 0} F_{3\delta}(\omega) = 0 \end{aligned}\right| \Rightarrow \boxed{F_{2r}(\omega) = \frac{2}{j\omega}\left(1-e^{j\omega}\right)}$$

- $$F_2(\omega) = F_{2r}(\omega) + F_{2\delta}(\omega) = \frac{2}{j\omega}\left(1-e^{j\omega}\right) + 3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}.$$

$$f'_{1r}(t) = f_2(t) \Rightarrow f_{1r}(t) = \int_{-\infty}^{t} f_2(\tau)\,d\tau \xleftrightarrow{F} F_{1r}(\omega) = \frac{1}{j\omega}F_2(\omega) + \pi F_2(0)\,\delta(\omega).$$

$$F_2(0) = \lim_{\omega\to 0} F_2(\omega) = \lim_{\omega\to 0} 2\,\frac{1-e^{j\omega}}{j\omega} + \lim_{\omega\to 0}\left(3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}\right) =$$

$$= -2 + 2 = 0.$$

Deci: $$\boxed{F_{1r}(\omega) = \frac{2}{(j\omega)^2}\left(1-e^{j\omega}\right) + \frac{1}{j\omega}\left(3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}\right)}$$

$$F_1(\omega) = F_{1r}(\omega) + F_{1\delta}(\omega) = \frac{2}{(j\omega)^2}\left(1-e^{j\omega}\right) + \frac{1}{j\omega}\left(3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}\right) - 2e^{j\omega} + 3.$$

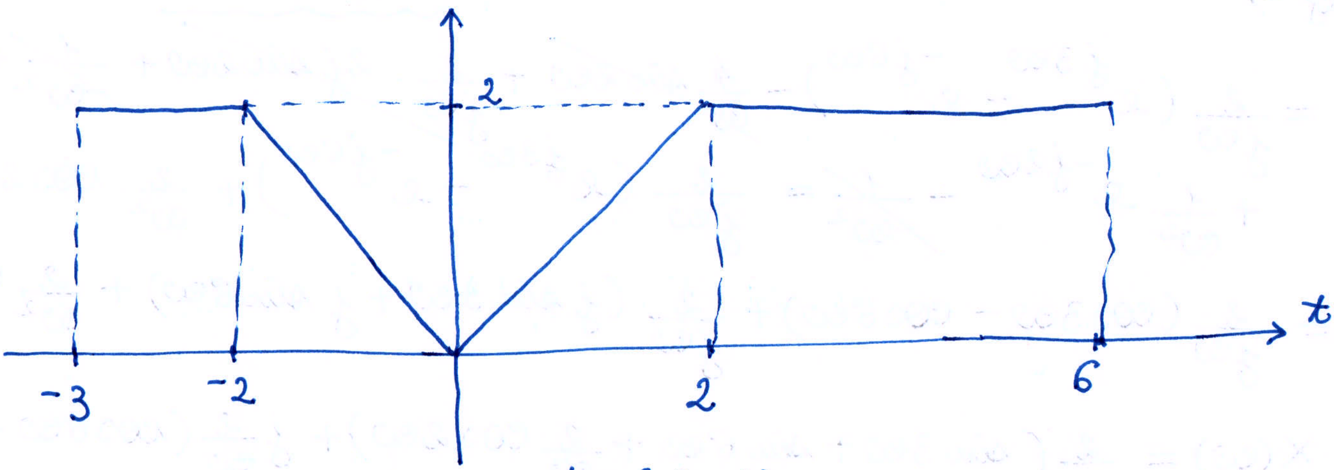
- $$\lim_{\omega\to 0} F_1(\omega) = 3 + 2\lim_{\omega\to 0}\frac{1-e^{j\omega} + j\omega\cdot e^{-j\frac{3}{2}\omega}}{(j\omega)^2} = 3 + 2\lim_{\omega\to 0}\frac{-je^{j\omega} + je^{-j\frac{3}{2}\omega} - j^2\frac{3}{2}\omega\, e^{-j\frac{3}{2}\omega}}{2j\omega} = 3 - \frac{3}{2} + \lim_{\omega\to 0}\frac{e^{-j\frac{3}{2}\omega} - e^{j\omega}}{j\omega} =$$

$$= \frac{3}{2} + \lim_{\omega\to 0}\frac{-j\cdot\frac{3}{2}\cdot e^{-j\frac{3}{2}\omega} - je^{j\omega}}{j} = \frac{3}{2} - \frac{3}{2} - 1 = -1 = F_1(0).$$

- $$f'(t) = f_1(t) \Rightarrow f(t) = \int_{-\infty}^{t} f_1(\tau)\,d\tau \xleftrightarrow{F} F(\omega) = \frac{1}{j\omega}F_1(\omega) + \pi F_1(0)\,\delta(\omega)$$

- $$F(\omega) = \frac{2}{(j\omega)^3}\left(1-e^{j\omega}\right) + \frac{1}{(j\omega)^2}\left(3e^{j\omega} - 3 + 2e^{-j\frac{3}{2}\omega}\right) + \frac{1}{j\omega}\left(3 - 2e^{j\omega}\right) - \pi\,\delta(\omega).$$

(11) Calculați TF pentru semnalul ilustrat mai jos



Soluție: $x(t) = \begin{cases} 2 & , t \in (-3,-2) \\ at+b & , t \in (-2,0) \\ ct+d & , t \in (0,2) \\ 2 & , t \in (2,6) \end{cases}$

Din continuitate aflăm a, b, c și d:

$f(-2) = 2 \rightarrow -2a+b = 2$, $f(0) = 0 \rightarrow b = 0$ $\Big| \rightarrow a = -1,\ b = 0 \Big| \rightarrow f(t) = -t$ pentru $t \in (-2,0)$.

$g(0) = 0 \rightarrow d = 0$, $g(2) = 2 \rightarrow 2c = 2 \rightarrow c = 1$ $\Big| \rightarrow g(t) = t$ pentru $t \in (0,2)$

$x(t) = \begin{cases} 2 & , t \in (-3,-2) \\ -t & , t \in (-2,0) \\ t & , t \in (0,2) \\ 2 & , t \in (2,6) \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt = \int_{-3}^{-2} 2 e^{-j\omega t} dt - \int_{-2}^{0} t \cdot e^{-j\omega t} dt + \int_{0}^{2} t \cdot e^{-j\omega t} dt + 2\int_{2}^{6} e^{-j\omega t} dt =$$

$$= -\frac{2}{j\omega} \cdot e^{-j\omega t}\Big|_{-3}^{-2} - \frac{2}{j\omega} \cdot e^{-j\omega t}\Big|_{2}^{6} - \int_{-2}^{0} t \cdot \left(\frac{e^{-j\omega t}}{-j\omega}\right)' dt + \int_{0}^{2} t \left(\frac{e^{-j\omega t}}{-j\omega}\right)' dt$$

$$= \frac{2}{j\omega}\left(e^{j3\omega} - e^{j2\omega} - e^{-j6\omega} + e^{-j2\omega}\right) + \frac{1}{j\omega} t \cdot e^{-j\omega t}\Big|_{-2}^{0} - \frac{1}{j\omega} t e^{-j\omega t}\Big|_{0}^{2}$$

$$- \int_{-2}^{0} \frac{e^{-j\omega t}}{j\omega} dt + \frac{1}{j\omega} \int_{0}^{2} e^{-j\omega t} dt = \frac{2}{j\omega}\left(e^{j3\omega} - e^{-j6\omega}\right) -$$

$$- \frac{2}{j\omega} \cdot (-2j \sin 2\omega) + \frac{2}{j\omega} \cdot e^{j2\omega} - \frac{2}{j\omega} \cdot e^{-j2\omega} + \frac{1}{(j\omega)^2} \cdot e^{-j\omega t}\Big|_{-2}^{0} -$$

$$-\frac{1}{(j\omega)^2} e^{-j\omega t}\Big|_0^2 = \qquad \underbrace{\frac{4}{\omega}\sin 2\omega}$$

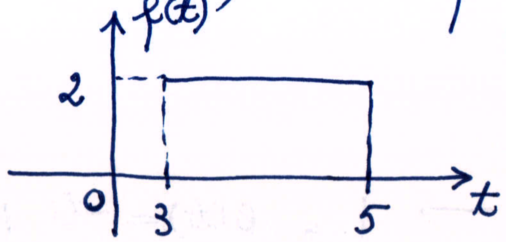
$$= \frac{2}{j\omega}\left(e^{j3\omega} - e^{-j6\omega}\right) - \cancel{\frac{4}{\omega}\sin 2\omega} + \cancel{\frac{2}{j\omega}\cdot 2j\sin 2\omega} + \cancel{\frac{1}{-\omega^2}} + \frac{1}{+\omega^2}\cdot e^{j2\omega} +$$

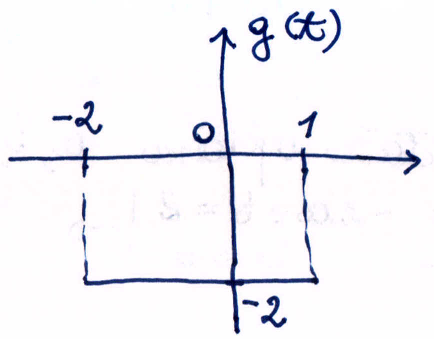
$$+ \frac{1}{\omega^2}\cdot e^{-j2\omega} - \cancel{\frac{1}{-\omega^2}} = \frac{2}{j\omega}\left(e^{j3\omega} - e^{-j6\omega}\right) + \frac{2}{\omega^2}\cdot\cos 2\omega =$$

$$= \frac{2}{j\omega}(\cos 3\omega - \cos 6\omega) + \frac{2}{j\omega}(j\sin 3\omega + j\sin 6\omega) + \frac{2}{\omega^2}\cos 2\omega.$$

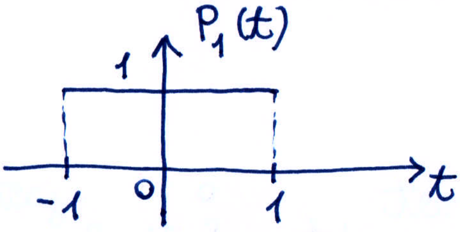
$$X(\omega) = \frac{2}{\omega}\left(\sin 3\omega + \sin 6\omega + \frac{2}{\omega}\cos 2\omega\right) + j\frac{2}{\omega}(\cos 6\omega - \cos 3\omega)$$

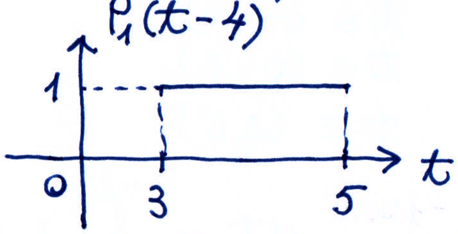
(12) Calculați transformata Fourier pentru $f * g(t)$ unde

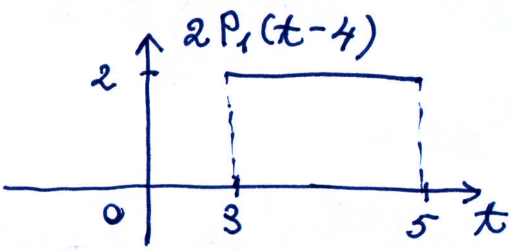




Soluție: Pentru $f(t)$ avem $2T = 5-3 = 2 \Rightarrow \boxed{T = 1}$.
Considerăm semnalul poartă $P_1(t)$ și translația la dreapta cu t_0;
$1 + t_0 = 5 \Rightarrow \boxed{t_0 = 4}$







$f(t) = 2P_1(t-4)$.

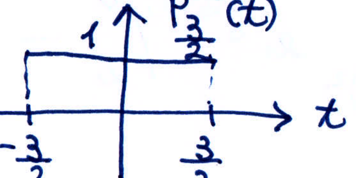
$P_1(t) \xleftrightarrow{F} 2\,\mathrm{sinc}(\omega)$

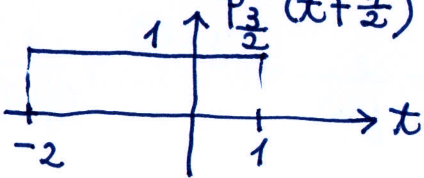
$P_1(t-4) \xleftrightarrow{F} 2e^{-j4\omega}\,\mathrm{sinc}(\omega)$

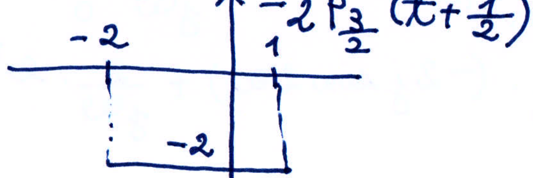
întârzierea

$f(t) = 2P_1(t-4) \xleftrightarrow{F} F(\omega) = 4e^{-j4\omega}\,\mathrm{sinc}(\omega)$

. Pentru $g(t)$ avem: $2T = 1-(-2) = 3 \Rightarrow \boxed{T = \frac{3}{2}}$
Translația este la stânga cu t_0; $\frac{3}{2} - t_0 = 1 \Rightarrow \boxed{t_0 = \frac{1}{2}}$







$g(t) = -2\, P_{\frac{3}{2}}\left(t+\frac{1}{2}\right)$; $\quad P_{\frac{3}{2}}(t) \xleftrightarrow{F} 2\cdot\frac{3}{2}\, \mathrm{sinc}\left(\frac{3\omega}{2}\right) = 3\,\mathrm{sinc}\left(\frac{3\omega}{2}\right)$

$$P_{\frac{3}{2}}\left(t+\frac{1}{2}\right) \xleftrightarrow[\text{întârzierea}]{F} 3e^{j\frac{\omega}{2}}\, \mathrm{sinc}\left(\frac{3\omega}{2}\right)$$

$$g(t) = -2P_{\frac{3}{2}}\left(t+\frac{1}{2}\right) \xleftrightarrow{F} -6e^{j\frac{\omega}{2}}\,\mathrm{sinc}\left(\frac{3\omega}{2}\right) = G(\omega).$$

Aplicăm convoluția în timp:

$$f * g(t) \xleftrightarrow{F} F(\omega)\cdot G(\omega) = -24e^{-j\frac{7\omega}{2}}\,\mathrm{sinc}(\omega)\cdot \mathrm{sinc}\left(\frac{3\omega}{2}\right).$$

(13) $f(t) = K[\sigma(t-a) - \sigma(t-b)]$, unde: $b>a>0$ și $K>0$.

$F(\omega) = ?$

$f(t) = \begin{cases} K, & t\in[a,b) \\ 0, & \text{rest}. \end{cases}$ este semnal tip poartă.

$2T = b-a \Rightarrow T = \frac{b-a}{2}$; translația este la dreapta cu t_0

$\frac{b-a}{2} + t_0 = b \Rightarrow t_0 = b - \frac{b-a}{2} = \frac{a+b}{2}$.

$$\begin{tikzcd}
P_{\frac{b-a}{2}}(t) \arrow[r] \arrow[d, leftrightarrow, "F"] & P_{\frac{b-a}{2}}\left(t-\frac{a+b}{2}\right) \arrow[r] \arrow[d, leftrightarrow, "F"] & K\cdot P_{\frac{b-a}{2}}\left(t-\frac{a+b}{2}\right) = f(t) \arrow[d, leftrightarrow, "F"] \\
(b-a)\,\mathrm{sinc}\left(\frac{\omega(b-a)}{2}\right) \arrow[r] & e^{-j\frac{(a+b)\omega}{2}}(b-a)\,\mathrm{sinc}\left(\frac{b-a}{2}\omega\right) \arrow[r] & F(\omega) = Ke^{-j\frac{a+b}{2}\omega}\cdot \mathrm{sinc}\frac{(b-a)\omega}{2}
\end{tikzcd}$$

$$F(\omega) = K\cdot e^{-j\frac{a+b}{2}\omega}\,\mathrm{sinc}\left(\frac{b-a}{2}\omega\right).$$

(14) $f(t) = \left(1-\frac{|t|}{a}\right)[\sigma(t+2a) - \sigma(t-2a)] \xleftrightarrow{F} ?$

Soluție: $f(t) = \begin{cases} 1-\frac{|t|}{a}, & |t| < 2a \\ 0, & \text{rest} \end{cases}$

$$f(t) = \left(2-\frac{|t|}{a}\right)[\sigma(t+2a) - \sigma(t-2a)] - [\sigma(t+2a) - \sigma(t-2a)] =$$
$$= f_1(t) - f_2(t).$$

$$\cdot\ f_1(t) = 2\cdot \frac{1}{2a}(2a-|t|)[\sigma(t+2a)-\sigma(t-2a)] =$$

$$= 2\cdot\begin{cases}\frac{1}{2a}(2a-|t|) &, |t|<2a\\ 0 &, \text{rest}\end{cases} = 2\cdot\frac{1}{2a}P_a * P_a(t).$$

$$f_1(t) = 2\cdot\frac{1}{2a}P_a * P_a(t) \xleftrightarrow{F} F_1(\omega) = 2\cdot 2a\ \text{sinc}^2(\omega a) =$$
$$= 4a\ \text{sinc}^2(a\omega).$$

$$\cdot\cdot\ f_2(t) = \begin{cases}1 &, |t|<2a\\ 0 &, \text{rest}\end{cases} = P_{2a}(t) \xleftrightarrow{F} F_2(\omega) = 4a\ \text{sinc}(2a\omega)$$

$$f(t) = f_1(t) - f_2(t) \xleftrightarrow{F} F(\omega) = F_1(\omega) - F_2(\omega) = 4a\ \text{sinc}^2(a\omega) -$$
$$- 4a\ \text{sinc}(2a\omega).$$

(15.) $$f(t) = \begin{cases}t-t^2 &, t\in(0,1)\\ 0 &, \text{rest}\end{cases} \xleftrightarrow{F} ?$$

$$F(\omega) = \int_0^1 (t-t^2)\cdot e^{-j\omega t}\,dt \overset{\text{părți}}{=} -\frac{1}{j\omega}(t-t^2)\cdot e^{-j\omega t}\Big|_0^1 +$$

$$+\frac{1}{j\omega}\int_0^1 (1-2t)\cdot e^{-j\omega t}\,dt \overset{\text{părți}}{=} -\frac{1}{(j\omega)^2}(1-2t)\cdot e^{-j\omega t}\Big|_0^1 + \frac{1}{(j\omega)^2}\int_0^1 -2e^{-j\omega t}\,dt =$$

$$= \frac{1}{\omega^2}[-e^{-j\omega}-1] + \frac{2}{(j\omega)^3}\cdot e^{-j\omega t}\Big|_0^1 = -\frac{1}{\omega^2}\cos\omega - \frac{1}{\omega^2} +$$

$$+ j\cdot\frac{1}{\omega^2}\sin\omega - \frac{2}{j\omega^3}(\cos\omega - j\sin\omega - 1) =$$

$$= -\frac{1}{\omega^2}\cdot 2\cos^2\frac{\omega}{2} + \frac{4}{\omega^3}\sin\frac{\omega}{2}\cos\frac{\omega}{2} + j\cdot\frac{1}{\omega^2}2\sin\frac{\omega}{2}\cos\frac{\omega}{2} +$$

$$+ \frac{2}{j\omega^3}\cdot 2\sin^2\frac{\omega}{2} = -\frac{2}{\omega^2}\cos\frac{\omega}{2}\cdot\left(\cos\frac{\omega}{2} - j\sin\frac{\omega}{2}\right) +$$

$$+ \frac{4}{\omega^3}\sin\frac{\omega}{2}\left(\cos\frac{\omega}{2} - j\sin\frac{\omega}{2}\right) =$$

$$= -\frac{2}{\omega^2}\cdot e^{-j\frac{\omega}{2}}\cdot\left(\cos\frac{\omega}{2} - \frac{2}{\omega}\cdot\sin\frac{\omega}{2}\right).$$

(16) $f(t) = (2e^{-t+2} - 2)\sigma(t-2)$, $g(t) = e^{3j\omega_0 t}\sigma(t)$.

$f * g(t) = ?$

<u>Soluție:</u>

$$f*g(t) = \int_2^t (2e^{-\tau+2} - 2)\cdot e^{j3\omega_0(t-\tau)}\, d\tau =$$

$$= 2e^{j3\omega_0 t}\left\{\int_2^t e^2 \cdot e^{-(1+j3\omega_0)\tau} d\tau - \right.$$

$$\left. - \int_2^t e^{-j3\omega_0\tau} d\tau\right\} = 2e^{j3\omega_0 t}\left\{\frac{-e^2}{1+j3\omega_0}\cdot e^{-(1+j3\omega_0)\tau}\Big|_2^t + \right.$$

$$\left. + \frac{1}{j3\omega_0}\cdot e^{-j3\omega_0\tau}\Big|_2^t\right\} =$$

$$= 2e^{j3\omega_0 t}\left\{-\frac{1}{1+j3\omega_0}\cdot e^{2-t-j3\omega_0 t} + \frac{1}{1+j3\omega_0}\cdot e^{2-2-j6\omega_0} + \right.$$

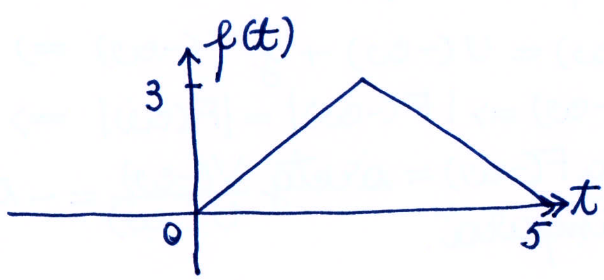
$$\left. + \frac{1}{j3\omega_0}\cdot e^{-j3\omega_0 t} - \frac{1}{j3\omega_0}\cdot e^{-j6\omega_0}\right\} =$$

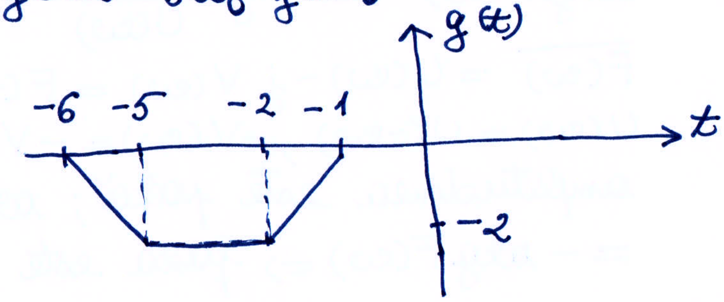
$$= \frac{-2}{1+j3\omega_0}\cdot e^{-t+2} + \frac{2}{1+j3\omega_0}\cdot e^{j3\omega_0(t-2)} + \frac{2}{j3\omega_0} - \frac{2}{j3\omega_0}\cdot e^{j3\omega_0(t-2)} =$$

$$= \frac{-2}{1+j3\omega_0}\cdot e^{-t+2} + \frac{2}{j3\omega_0} + \frac{2}{9\omega_0^2 - j3\omega_0}\cdot e^{j3\omega_0(t-2)} \quad \text{pentru } t>2.$$

$$f*g(t) = \left\{\frac{-2}{1+j3\omega_0}\cdot e^{-t+2} + \frac{2}{j3\omega_0} + \frac{2}{9\omega_0^2 - j3\omega_0}\cdot e^{j3\omega_0(t-2)}\right\}\sigma(t-2)$$

(17)





$f*g(t) \xleftrightarrow{F} ?$

· $2T = 5 \rightarrow T = \frac{5}{2}$; translația la dreapta cu t_0:

$\frac{5}{2} + t_0 = 5 \Rightarrow t_0 = \frac{5}{2} \Rightarrow$

$f(t) = 3\cdot\frac{1}{T} P_{\frac{T}{2}} * P_{\frac{T}{2}}(t = 3\cdot\frac{2}{5} P_{\frac{5}{4}} * P_{\frac{5}{4}}(t - \frac{5}{2}) \xleftrightarrow{F} F(\omega) =$

$= 3\, e^{-j\frac{5}{2}\omega} \cdot \frac{5}{2}\, \mathrm{sinc}^2\left(\frac{5\omega}{4}\right) \Rightarrow F(\omega) = \frac{15}{2} e^{-j\frac{5}{2}\omega}\, \mathrm{sinc}^2\left(\frac{5\omega}{4}\right).$

$2(\omega_2+\omega_1) = -1-(-6) = 5 \rightarrow$ basa mare

$2(\omega_2-\omega_1) = -2+5 = 3 \rightarrow$ basa mică

$\rightarrow \begin{cases} \omega_2+\omega_1 = \frac{5}{2} \\ \omega_2-\omega_1 = \frac{3}{2} \end{cases} \Rightarrow$

$\boxed{\omega_2 = 2}$, $\boxed{\omega_1 = \frac{1}{2}}$

Trapezul se translatează la stânga cu: $\omega_1+\omega_2 - t_0 = -1$

$\Rightarrow \frac{5}{2} - t_0 = -1 \rightarrow \boxed{t_0 = \frac{7}{2}}$

$g(t) = -2\, P_{\frac{1}{2}} * P_2\left(t+\frac{7}{2}\right) \xleftrightarrow{F} G(\omega) = -2\cdot 4\cdot \frac{1}{2}\cdot 2\, \mathrm{sinc}\left(\frac{\omega}{2}\right)\cdot e^{j\frac{7\omega}{2}} \cdot \mathrm{sinc}(2\omega) =$

$= -8\, \mathrm{sinc}\left(\frac{\omega}{2}\right) \mathrm{sinc}(2\omega)\cdot e^{j\frac{7\omega}{2}}$

$G(\omega) = -8 e^{j\frac{7\omega}{2}}\, \mathrm{sinc}\left(\frac{\omega}{2}\right)\cdot \mathrm{sinc}(2\omega).$

$f * g(t) \xleftrightarrow{F} F(\omega)\cdot G(\omega) = -60\, e^{j\omega}\, \mathrm{sinc}^2\left(\frac{5\omega}{4}\right) \mathrm{sinc}\left(\frac{\omega}{2}\right) \mathrm{sinc}(2\omega)$

<u>Remarcă</u>. 1. $\overline{F(\omega)} = \int_{-\infty}^{\infty} f(t)\cdot e^{j\omega t}\, dt = F(-\omega)$ de unde rezultă că este suficient să cunoaștem valorile lui $F(\omega)$ pentru $\omega > 0$.

2. $F(\omega) = U(\omega) + j V(\omega) = |F(\omega)|\cdot e^{j \arg F(\omega)}$

$|F(\omega)| = [U^2(\omega) + V^2(\omega)]^{1/2} \rightarrow$ modulul sau amplitudinea spectrului în frecvență $(F(\omega))$

$\arg F(\omega) = \mathrm{arctg} \frac{V(\omega)}{U(\omega)} \rightarrow$ faza spectrului $F(\omega)$.

$\overline{F(\omega)} = U(\omega) - j V(\omega) = F(-\omega) = U(-\omega) + j V(-\omega) \Rightarrow$

$U(\omega) = U(-\omega)$, $-V(\omega) = V(-\omega) \Rightarrow |F(-\omega)| = |F(\omega)| \Rightarrow$

amplitudinea este pară; $\arg F(-\omega) = \mathrm{arctg} \frac{V(-\omega)}{U(-\omega)} = -\mathrm{arctg} \frac{V(\omega)}{U(\omega)}$

$= -\arg F(\omega) \Rightarrow$ faza este impară.

3. <u>Teorema conservării energiei</u>

$\int_0^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_0^{\infty} |F(\omega)|^2 d\omega.$