MATEMATICI APLICATE ÎN INGINERIE

Camelia GHELDIU
Mihaela DUMITRACHE

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Cuprins

0.1 Cuvânt înainte

Lucrarea se adresează studenţilor din facultăţile de inginerie: calculatoare, electronică, reţele, energetică, mecanică. Este alcătuită din trei părţi importante: analiză complexă, transformări integrale şi ecuaţile fizicii matematice; fiecare capitol are în componenţă trei părţi: consideraţii teoretice, probleme rezolvate şi probleme propuse.

Autorii au pus accentul pe prezentarea cât mai succintă a noţiunilor de teorie şi pe rezolvarea cât mai schematică a exerciţiilor. Problemele propuse şi rezolvate sunt apropiate ca structură de cerinţele materiilor (cursurilor) inginereşti. Autorii doresc studenţilor interesaţi de Matematici Speciale şi nu numai, o parcurgere plăcută a lucrării.

Autorii

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Capitolul 1

Transformata Laplace (LT). Aplicaţii ale LT în rezolvarea ecuaţiilor diferenţiale.

1.1 Consideraţii teoretice.

Definiţia 1.1 Funcţia $f : \mathbb{R} \to \mathbb{C}$ se numeşte *original* dacă

a) $f(t) = 0$ pentru $t < 0$;

b) f continuă pe porţiuni;

c) $\exists M_f > 0, s_f \in \mathbb{R}$ astfel încât $|f(t)| \leq M_f e^{s_f t}$, $\forall t$.

Notăm

$$\sigma(t) = \begin{cases} 1, t \geq 0 \\ 0, t < 0. \end{cases}$$

funcţia lui Heaviside.

Fie $s \in \mathbb{C}$, Re$s > s_f$. Definim pentru originalul f, aplicaţia

$$\mathcal{L}[f(t)](s) = F(s) = \int_0^\infty f(t)e^{-st}dt$$

unde

$$\mathcal{L}[f(t)](s)$$

este transformata Laplace, iar

$$F(s)$$

este imaginea.

Observaţia 1.2 Aplicaţia care duce fiecare original în imaginea sa se numeşte *transformarea Laplace* şi folosim notaţia:

$$f(t) \overset{\mathcal{L}}{\longleftrightarrow} F(s) = \mathcal{L}[f(t)].$$

Avem

$$|f(t)e^{-st}| \le |f(t)|e^{-t\text{Re } s} \overset{(c)}{\le} M_f e^{-(\text{Re } s - s_f)t}.$$

$$M_f \int_0^\infty e^{-(\text{Re } s - s_f)t}dt = -\frac{M_f}{\text{Re } s - s_f}e^{-(\text{Re } s - s_f)t}\Big|_0^\infty =$$

$$= \frac{M_f}{\text{Re } s - s_f} < \infty$$

rezultă cu criteriul de convergenţă pentru integrale improprii că

$$\int_0^\infty f(t)e^{-st}dt$$

este convergentă, de unde obţinem că $F(s)$ este bine definită.

Definiţia 1.3 Funcţia

$$f(t) = \mathcal{L}^{-1}[F(s)](t)$$

se numeşte *inversa transformatei Laplace.*

Observaţia 1.4 Recuperarea funcţiei original:

a) Dacă

$$F(s) = \frac{P(s)}{Q(s)}$$

este o funcţie raţională; în acest caz descompunem în fracţii simple şi folosim tabelul;

b) Dacă

$$F(s) = \frac{P(s)}{Q(s)} \cdot e^{-as},$$

în acest caz se determină funcţia original pentru $\frac{P(s)}{Q(s)}$, apoi se foloseşte translaţia în t pentru recuperarea semnalului original;

c) Formula Mellin Fourier

$$f(t) = \frac{1}{2\pi i} \int_{s_f - i\infty}^{s_f + i\infty} F(s) e^{st} ds =$$

$$= \sum_k \{\operatorname*{Res}_{s=s_k} F(s) e^{st}\}$$

unde Re $s_k < s_f$, s_k singularităţi izolate ale lui $F(s)$.

Proprietăţi 1.5

1) Liniaritatea

$$\mathcal{L}\left[\alpha f(t)+\beta g(t)\right](s)=$$
$$=\alpha\mathcal{L}\left[f(t)\right](s)+\beta\mathcal{L}\left[g(t)\right](s),\forall\alpha,\beta\in\mathbb{C},$$
$$\text{Re } s>\max\{s_f,s_g\}.$$

2) Asemănarea

$$\mathcal{L}[f(at)](s)\overset{a\geq 0}{=}\int_0^\infty f(at)e^{-st}dt,$$

dar ţinând cont de faptul că $y=at$, $t=y/a$, $dt=\frac{1}{a}dy$, obţinem

$$\int_0^\infty f(y)e^{-\frac{s}{a}y}\frac{dy}{a}=\frac{1}{a}\mathcal{L}[f(t)]\left(\frac{s}{a}\right).$$

3) Deplasarea

$$\mathcal{L}[f(t)e^{\lambda t}](s)=\int_0^\infty f(t)e^{-(s-\lambda)t}dt=\mathcal{L}[f(t)]\left(s-\lambda\right).$$

4) Întârzierea ($b>0$)

$$\mathcal{L}[f(t-b)\sigma(t-b)](s)=\int_b^\infty f(t-b)e^{-st}dt\overset{t-b=y}{=}$$

$$=\int_0^\infty f(y)e^{-sy}e^{-sb}dy=e^{-sb}\mathcal{L}[f(t)]\left(s\right).$$

5) Derivarea originalului. Considerăm $f\in C^1(0,\infty)$.

$$\mathcal{L}[f'(t)](s)=\int_0^\infty f'(t)e^{-st}dt=$$

$$= f(t)e^{-st}|_0^\infty + \int_0^\infty f(t)se^{-st}dt =$$

$$= s\mathcal{L}[f(t)]\,(s) - f(0).$$

Generalizare:

$$\mathcal{L}[f^n(t)](s) =$$

$$= s^n\mathcal{L}[f(t)]\,(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f(0).$$

6) Derivarea imaginii. Cu criteriul comparaţiei integrala

$$\int_0^\infty f(t)e^{-st}dt,$$

este absolut convergentă şi

$$\int_0^\infty tf(t)e^{-st}dt,$$

este uniform convergentă în raport cu s, astfel putem deriva sub semnul integralei în raport cu s, deci

$$(\mathcal{L}[f(t)](s))' =$$

$$= \left(\int_0^\infty f(t)e^{-st}dt\right)'_s = \int_0^\infty \frac{\partial}{\partial s}\left[f(t)e^{-st}\right]dt =$$

$$= -\int_0^\infty tf(t)e^{-st}dt = -\mathcal{L}[tf(t)](s).$$

$$\mathcal{L}[tf(t)](s) = -\left(\mathcal{L}[f(t)](s)\right)'.$$

Generalizare:

$$\mathcal{L}[t^n f(t)](s) = (-1)^n F^n(s).$$

7) Integrarea originalului

$$\mathcal{L}[f(t)](s) = \mathcal{L}\left[\left(\int_0^t f(x)dx\right)'\right](s) \stackrel{(5)}{=}$$

$$= s\mathcal{L}\left[\int_0^t f(x)dx\right](s) - \int_0^0 f(x)dx =$$

$$= s\mathcal{L}\left[\int_0^t f(x)dx\right](s) \Rightarrow$$

$$\mathcal{L}\left[\int_0^t f(x)dx\right](s) = \frac{1}{s}\mathcal{L}[f(t)](s).$$

8) Integrarea imaginii

$$\int_s^\infty F(q)dq =$$

$$= \int_s^\infty \int_0^\infty f(t)e^{-qt}dtdq = \int_0^\infty f(t)\left(\int_s^\infty e^{-qt}dq\right)dt =$$

$$= \int_0^\infty f(t)\frac{e^{-qt}}{-t}|_s^\infty dt = \int_0^\infty \frac{f(t)}{t}e^{-st}dt = \mathcal{L}[\frac{f(t)}{t}](s).$$

Deci

$$\mathcal{L}[\frac{f(t)}{t}](s) = \int_s^\infty \mathcal{L}[f(t)](q)dq.$$

9) Integrarea cu parametru

$$\mathcal{L}\left[\int_{x_1}^{x_2} f(x,t)dx\right](s) = \int_0^\infty \left(\int_{x_1}^{x_2} f(x,t)dx\right)e^{-st}dt =$$

$$= \int_0^\infty \int_{x_1}^{x_2} f(x,t)e^{-st}dxdt = \int_{x_1}^{x_2}\left[\int_0^\infty f(x,t)e^{-st}dt\right]dx =$$

$$= \int_{x_1}^{x_2} \mathcal{L}\left[f(x,t)\right](x,s)dx.$$

10) Produsul de convoluţie

$$f(t) * g(t) = (f * g)(t) = \int_0^t f(t-u)g(u)du = (g * t)(t);$$

$$\mathcal{L}\left[(f * g)\right](s) = F(s) \cdot G(s);$$

$$\mathcal{L}^{-1}\left[F(s) \cdot G(s)\right](t) = f(t) * g(t).$$

Aplicaţii ale transformatei Laplace în rezolvarea problemei Cauchy pentru ecuaţii diferenţiale liniare cu coeficienţi constanţi (exemplificăm cu ecuaţia diferenţială de ordinul al II-lea).

Exemplul 1.6

$$\begin{cases} a_0 y'' + a_1 y' + a_2 y = f(t) \\ y(0) = y_0, y'(0) = y_1. \end{cases}$$

$$\mathcal{L}\left[y(t)\right](s) = Y(s);$$

$$\mathcal{L}\left[y'\right](s) = sY(s) - y_0;$$

$$\mathcal{L}\left[y''\right](s) = s^2 Y(s) - s y_0 - y_1;$$

$$\mathcal{L}\left[f(t)\right](s) = F(s);$$

Se obţine ecuaţia operaţională:

$$a_0\left(s^2 Y(s) - s y_0 - y_1\right) + a_1\left(sY(s) - y_0\right) + a_2 Y(s) = F(s).$$

$$\left(a_0 s^2 + a_1 s + a_2\right) Y(s) = F(s) + a_0 y_0 s + a_0 y_1 + a_1 y_0.$$

Notăm $G(s) = F(s) + a_0 y_0 s + a_0 y_1 + a_1 y_0$ şi obţinem

$$Y(s) = \frac{G(s)}{a_0 s^2 + a_1 s + a_2}.$$

Se aplică transformata Laplace inversă şi rezultă funcţia original:

$$y(t) = \mathcal{L}^{-1}\left[\frac{G(s)}{a_0 s^2 + a_1 s + a_2}\right].$$

Observaţia 1.7

i) Inversa întârzierii.

$$\mathcal{L}^{-1}\left[e^{-sb}F(s)\right](t) = f(t-b)\sigma(t-b).$$

ii) Determinarea imaginii F

a) Calcul direct cu definiţia;

b) Pentru funcţia original: $(0, +\infty)$ fără discontinuităţi se pot folosi proprietăţile transformatei Laplace şi tabelul;

c) Pentru funcţia original "pe ramuri" se foloseşte definiţia sau scrierea cu ajutorul funcţiei Heaviside.

Tabela 1.1: Transformata Laplace.

Nr. crt.	Funcţia original $f(t)$	Transf. Laplace $\mathcal{L}\left[f(t)\right](s)$ sau $F(s)$	Inversa $\mathcal{L}^{-1}\left[F(s)\right](t)$
1)	$1 \cdot \sigma(t)$	$\frac{1}{s}$	$1 \cdot \sigma(t)$
2)	$e^{\lambda t}\sigma(t)$	$\frac{1}{s-\lambda}$	$e^{\lambda t}\sigma(t)$
3)	$t^k\sigma(t), k > -1$	$\frac{\Gamma(k+1)}{s^{k+1}}$	$\frac{t^k}{\Gamma(k+1)}\sigma(t)$
4)	$t^k e^{\lambda t}\sigma(t)$	$\frac{\Gamma(k+1)}{(s-\lambda)^{k+1}}$	$\frac{t^k e^{\lambda t}}{\Gamma(k+1)}\sigma(t)$
5)	$t^n\sigma(t), n \in \mathbb{N}^*$	$\frac{n!}{s^{n+1}}$	$\frac{t^n}{n!}\sigma(t)$
6)	$t^n e^{\lambda t}\sigma(t)$	$\frac{n!}{(s-\lambda)^{n+1}}$	$\frac{t^n}{n!}e^{\lambda t}\sigma(t)$
7)	$\sin \omega t\sigma(t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{\sin \omega t}{\omega}\sigma(t)$
8)	$\cos \omega t\sigma(t)$	$\frac{s}{s^2+\omega^2}$	$\cos \omega t\sigma(t)$
9)	$\sinh at\sigma(t)$	$\frac{a}{s^2-a^2}$	$\frac{\sinh at}{a}\sigma(t)$
10)	$\cosh at\sigma(t)$	$\frac{s}{s^2-a^2}$	$\cosh at\sigma(t)$
11)	$e^{\lambda t}\sin \omega t\sigma(t)$	$\frac{\omega}{(s-\lambda)^2+\omega^2}$	$e^{\lambda t}\frac{\sin \omega t}{\omega}\sigma(t)$
12)	$e^{\lambda t}\cos \omega t\sigma(t)$	$\frac{s-\lambda}{(s-\lambda)^2+\omega^2}$	$e^{\lambda t}\cos \omega t\sigma(t)$
13)	$e^{\lambda t}\sinh at\sigma(t)$	$\frac{a}{(s-\lambda)^2-a^2}$	$e^{\lambda t}\frac{\sinh at}{a}\sigma(t)$
14)	$e^{\lambda t}\cosh at\sigma(t)$	$\frac{s-\lambda}{(s-\lambda)^2-a^2}$	$e^{\lambda t}\cosh at\sigma(t)$
15)	$t\sin \omega t\sigma(t)$	$\frac{2\omega s}{(s^2+\omega^2)^2}$	$t\frac{\sin \omega t}{\omega}\sigma(t)$
16)	$t\cos \omega t\sigma(t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$	$t\cos \omega t\sigma(t)$
17)	$t^n\sin \omega t\sigma(t), n \geq 2$	$\frac{n!}{(s^2+\omega^2)^{n+1}}\mathrm{Im}(s+i\omega)^{n+1}$	$\frac{t^n}{n!}\sin \omega t\sigma(t)$
18)	$t^n\cos \omega t\sigma(t)$	$\frac{n!}{(s^2+\omega^2)^{n+1}}\mathrm{Re}(s+i\omega)^{n+1}$	$t^n\cos \omega t\sigma(t)$
19)	$e^{\lambda t}t\sin \omega t\sigma(t)$	$\frac{2\omega(s-\lambda)}{[(s-\lambda)^2+\omega^2]^2}$	$e^{\lambda t}t\frac{\sin \omega t}{\omega}\sigma(t)$
20)	$e^{\lambda t}t\cos \omega t\sigma(t)$	$\frac{(s-\lambda)^2-\omega^2}{[(s-\lambda)^2+\omega^2]^2}$	$e^{\lambda t}t\cos \omega t\sigma(t)$

1.2 Exerciţii rezolvate.

Exerciţiul 1.8 Calculaţi transformata Laplace pentru următoarele funcţii:

a)
$$f(t) = \cos(\omega t + \theta)\sigma(t);$$

b)
$$\mathcal{L}\left[\sinh 2t \cdot \sin 5t\right](s);$$

c)
$$f(t) = \frac{\sin \omega t}{t}\sigma(t);$$

d)
$$f(t) = \frac{e^{bt} - e^{at}}{2t\sqrt{t}}\sigma(t);$$

e)
$$f(t) = \begin{cases} 1, & t \in (0,1) \\ 2-t, & t \in (1,2) \\ 0, & t > 2. \end{cases}$$

f)
$$f(t) = \begin{cases} \sin t, & t \in (0, 2\pi) \\ 0, & \text{rest.} \end{cases}$$

g)
$$f(t) = \begin{cases} t \cdot e^{-t}, & 1 < t < 3 \\ 0, & \text{rest.} \end{cases}$$

h)
$$f(t) = \int_0^t \frac{\sin x}{x}\sigma(t)dx;$$

i)

$$f(t) = \int_0^t \frac{\cos s}{\sqrt{t-s}} \sigma(t) ds.$$

Soluţie.
a) Folosim definiţia:

$$\mathcal{L}\left[f(t)\right](s) = \int_0^\infty \cos(\omega t + \theta) \left(\frac{e^{-st}}{-s}\right)' dt =$$

$$= \cos(\omega t + \theta) \cdot \frac{e^{-st}}{-s}|_0^\infty + \int_0^\infty \omega \sin(\omega t + \theta) \cdot \frac{e^{-st}}{-s} dt =$$

$$= \frac{\cos\theta}{s} + \omega \sin(\omega t + \theta) \cdot \frac{e^{-st}}{(-s)^2}|_0^\infty -$$

$$- \int_0^\infty \omega^2 \cos(\omega t + \theta) \cdot \frac{e^{-st}}{(-s)^2} dt =$$

$$= \frac{1}{s}\cos\theta - \frac{\omega}{s^2}\sin\theta - \frac{\omega^2}{s^2}\mathcal{L}\left[f(t)\right](s) \Rightarrow$$

$$\left(1 + \frac{\omega^2}{s^2}\right)\mathcal{L}\left[f(t)\right](s) = \frac{1}{s^2}\left(s\cos\theta - \omega\sin\theta\right),$$

dar

$$\left(1 + \frac{\omega^2}{s^2}\right) = \frac{s^2+\omega^2}{s^2},$$

deci

$$\mathcal{L}\left[f(t)\right](s) = \frac{s}{s^2+\omega^2}\cos\theta - \frac{\omega}{s^2+\omega^2}\sin\theta.$$

b)

$$\mathcal{L}\left[\sinh 2t \cdot \sin 5t\sigma(t)\right](s) = \frac{1}{2}\mathcal{L}\left[(e^{2t} - e^{-2t})\sin 5t\sigma(t)\right](s) =$$

$$= \frac{1}{2}\left\{\mathcal{L}\left[e^{2t}\sin 5t\sigma(t)\right](s) - \mathcal{L}\left[e^{-2t}\sin 5t\sigma(t)\right](s)\right\}.$$

Cu teorema deplasării avem:

$$\mathcal{L}\left[\sinh 2t \cdot \sin 5t\sigma(t)\right](s) = \frac{1}{2}\left[\frac{5}{(s-2)^2+25} - \frac{5}{(s+2)^2+25}\right].$$

c) Folosim integrarea imaginii:

$$\mathcal{L}\left[f(t)\right](s) = \int_s^\infty \mathcal{L}\left[\sin\omega t\sigma(t)\right](q)dq =$$

$$= \int_s^\infty \frac{\omega}{q^2+\omega^2}dq = \arctan\frac{q}{\omega}|_s^\infty =$$

$$= \frac{\pi}{2} - \arctan\frac{s}{\omega} = \arctan\frac{\omega}{s}.$$

d)

$$\mathcal{L}\left[f(t)\right](s) = \mathcal{L}\left[\frac{\frac{e^{bt}-e^{at}}{2\sqrt{t}}}{t}\sigma(t)\right](s) =$$

$$= \int_s^\infty \mathcal{L}\left[\frac{e^{bt}-e^{at}}{2t^{\frac{1}{2}}}\sigma(t)\right](q)dq =$$

$$= \frac{1}{2}\int_s^\infty \left\{\mathcal{L}\left[t^{-\frac{1}{2}}e^{bt}\sigma(t)\right](q) - \mathcal{L}\left[t^{-\frac{1}{2}}e^{at}\sigma(t)\right](q)\right\}dq.$$

Cu teorema deplasării avem:

$$= \frac{1}{2}\int_s^\infty \left(\frac{\sqrt{\pi}}{\sqrt{q-b}} - \frac{\sqrt{\pi}}{\sqrt{q-a}}\right)dq.$$

Deoarece

$$\mathcal{L}\left[t^{-\frac{1}{2}}\right](q) = \frac{\Gamma\left(\frac{1}{2}\right)}{q^{\frac{1}{2}}} = \frac{\sqrt{\pi}}{\sqrt{q}},$$

obţinem:

$$\mathcal{L}\left[f(t)\right](s) = \sqrt{\pi}\left(\sqrt{q-b} - \sqrt{q-a}\right)|_s^{\infty} =$$

$$= \sqrt{\pi}\left[\lim_{q\to\infty}\left(\sqrt{q-b} - \sqrt{q-a}\right) - \left(\sqrt{s-b} - \sqrt{s-a}\right)\right] =$$

$$= \sqrt{\pi}\left(\sqrt{s-a} - \sqrt{s-b}\right).$$

$e)$ Folosim definiţia:

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^1 e^{-st}dt + \int_1^2 (2-t)e^{-st}dt =$$

$$= -\frac{1}{s}e^{-st}|_0^1 + (2-t)\frac{e^{-st}}{-s}|_1^2 + \int_1^2 \frac{e^{-st}}{-s}dt =$$

$$= -\frac{1}{s}e^{-s} + \frac{1}{s} + \frac{e^{-s}}{s} + \frac{e^{-st}}{s^2}|_1^2 =$$

$$= \frac{1}{s} + \frac{1}{s^2}(e^{-2s} - e^{-s}).$$

$f)$

$$\mathcal{L}\left[f(t)\right](s) =$$

$$= \mathcal{L}\left[\sin t \cdot \sigma(t)\right](s) - \mathcal{L}\left[\sin((t+2\pi) - 2\pi)\sigma(t-2\pi)\right](s) =$$

$$= \frac{1}{s^2+1} - e^{-2\pi s}\cdot\mathcal{L}\left[\sin(t+2\pi)\sigma(t)\right](s) =$$

$$= \frac{1}{s^2+1} - \frac{e^{-2\pi s}}{s^2+1}.$$

$g)$ Avem:

$$f(t) = \begin{cases} t\cdot e^{-t}, & 1 < t < 3 \\ 0, & \text{rest} \end{cases} = t\cdot e^{-t}(\sigma(t-1) - \sigma(t-3)).$$

$$\mathcal{L}\left[f(t)\right](s) =$$
$$= \mathcal{L}\left[t \cdot e^{-t} u(t-1)\sigma(t-1)\right](s) - \mathcal{L}\left[t \cdot e^{-t} u(t-3)\sigma(t-3)\right](s) =$$
$$= e^{-s}\mathcal{L}\left[(t+1)e^{-t-1}\sigma(t)\right](s) - e^{-3s}\mathcal{L}\left[(t+3)e^{-t-3}\sigma(t)\right](s) =$$
$$= e^{-s-1}\mathcal{L}\left[t\sigma(t)\right](s+1) - e^{-3s-3}\mathcal{L}\left[t \cdot e^{-t}\sigma(t)\right](s) +$$
$$+ e^{-s-1}\mathcal{L}\left[e^{-t}\sigma(t)\right](s) - 3e^{-3s-3}\mathcal{L}\left[e^{-t}\sigma(t)\right](s) =$$
$$= (e^{-s-1} - e^{-3s-3})\frac{1}{(s+1)^2} + (e^{-s-1} - 3e^{-3s-3})\frac{1}{s+1}.$$

h) Folosim integrarea originalului:

$$\mathcal{L}\left[f(t)\right](s) = \frac{1}{s}\mathcal{L}\left[\frac{\sin x}{x}\sigma(t)\right](s) \stackrel{(c)}{=} \frac{1}{s}\arctan\frac{1}{s}.$$

i) Folosim transformata Laplace peste un produs de convoluţie:

$$\mathcal{L}\left[f(t)\right](s) = \mathcal{L}\left[\cos t * \frac{1}{\sqrt{t}}\sigma(t)\right](s) =$$

$$= \mathcal{L}\left[\cos t \cdot \sigma(t)\right](s) \cdot \mathcal{L}\left[\frac{1}{\sqrt{t}}\sigma(t)\right](s) =$$

$$= \frac{s}{s^2+1} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{\sqrt{\pi s}}{s^2+1}.$$

Exerciţiul 1.9 Folosind transformata Laplace calculaţi următoarele integrale improprii:

a) $\int_0^\infty \frac{\sin t}{t} dt$;

b) $\int_0^\infty \frac{\cos bt - \cos at}{t^2} dt$;

c) $\int_0^\infty \frac{\sin tx}{x(x^2+a^2)} dx$.

Soluţie.
a) Aplicăm proprietatea:

$$\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty \mathcal{L}\left[f(t)\right](s)ds$$

$$\int_0^\infty \frac{\sin t}{t}dt = \int_0^\infty \mathcal{L}\left[\sin t \cdot \sigma(t)\right](s)ds =$$

$$= \int_0^\infty \frac{1}{s^2+1}ds = \arctan s|_0^\infty = \frac{\pi}{2}.$$

b) Aceeaşi proprietate ca la a):

$$\int_0^\infty \frac{\cos bt - \cos at}{t^2}dt = \int_0^\infty \mathcal{L}\left[\frac{\cos bt - \cos at}{t}\sigma(t)\right](s)ds$$

folosind integrarea imaginii avem:

$$= \int_0^\infty \int_s^\infty \mathcal{L}\left[\cos bt \cdot \sigma(t) - \cos at \cdot \sigma(t)\right](q)dqds =$$

$$= \int_0^\infty \int_s^\infty \left(\frac{q}{q^2+b^2} - \frac{q}{q^2+a^2}\right)dqds =$$

$$= \frac{1}{2}\int_0^\infty \left[\ln(q^2+b^2) - \ln(q^2+a^2)\right]|_s^\infty ds = \frac{1}{2}\int_0^\infty \ln\frac{s^2+a^2}{s^2+b^2}ds =$$

$$= \frac{1}{2}\int_0^\infty (s)' \ln\frac{s^2+a^2}{s^2+b^2}ds =$$

$$= \frac{1}{2}s\ln\frac{s^2+a^2}{s^2+b^2}|_0^\infty - \frac{1}{2}\int_0^\infty \left(\frac{2s^2}{s^2+a^2} - \frac{2s^2}{s^2+b^2}\right)ds =$$

$$= \int_0^\infty \left(\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2} \right) ds =$$

$$= a \arctan \frac{s}{a}|_0^\infty - b \arctan \frac{s}{b}|_0^\infty = a\frac{\pi}{2} - b\frac{\pi}{2} = \frac{\pi}{2}(a-b).$$

$c)$ Folosim transformata Laplace şi integrarea cu parametru:

$$\mathcal{L}\left[\int_0^\infty \frac{\sin tx}{x(x^2+a^2)} dx \right](s) = \int_0^\infty \frac{\mathcal{L}\left[\sin tx \cdot \sigma(t)\right](s)}{x(x^2+a^2)} dx =$$

$$= \int_0^\infty \frac{x}{x(x^2+a^2)(x^2+s^2)} dx = \int_0^\infty \frac{dx}{(x^2+a^2)(x^2+s^2)} =$$

$$= \frac{1}{s^2-a^2} \int_0^\infty \left(\frac{1}{x^2+a^2} - \frac{1}{x^2+s^2} \right) dx =$$

$$= \frac{1}{s^2-a^2} \left[\frac{1}{a} \arctan \frac{x}{a}|_0^\infty - \frac{1}{s} \arctan \frac{x}{s}|_0^\infty \right] =$$

$$= \frac{1}{s^2-a^2} \left(\frac{\pi}{2a} - \frac{\pi}{2s} \right) =$$

$$= \frac{\pi}{2} \cdot \frac{s-a}{as(s-a)(s+a)} = \frac{\pi}{2a} \cdot \frac{1}{s(s+a)} =$$

$$= \frac{\pi}{2a^2} \left(\frac{1}{s} - \frac{1}{s+a} \right) = \frac{\pi}{2a^2} \mathcal{L}\left[(1-e^{-at})\sigma(t) \right](s) \Rightarrow$$

$$\int_0^\infty \frac{\sin tx}{x(x^2+a^2)} dx = \frac{\pi}{2a^2} \left(1 - e^{-at} \right) \sigma(t).$$

Exerciţiul 1.10 Aflaţi originalul pentru următoarele imagini:

a)

$$F(s) = \frac{27-12s}{(s+4)(s^2+9)};$$

b)

$$F(s) = \frac{1}{(s^2 + a^2)^2};$$

c)

$$F(s) = \frac{e^{-s}}{s^2 + \pi^2} + \frac{e^{-3s}}{s^2 + \pi^2};$$

d)

$$F(s) = \frac{1}{s^2 - 1} - e^{-\pi} \frac{e^{-\pi s}}{s^2 - 1};$$

e)

$$F(s) = \frac{e^{-s}}{s + 2} + \frac{e^{-2s}}{s^2 + 1} + \frac{e^{-3s}}{s^2 + 4s + 5};$$

f)

$$F(s) = \frac{1}{s^4 + 2s^3 + 3s^2 + 2s + 1};$$

g)

$$F(s) = \frac{e^{-s}\left(1 - e^{-s}\right)}{s\left(s^2 + 1\right)};$$

h)

$$F(s) = \frac{s}{(s + 1)^{\frac{5}{2}}};$$

i)

$$F(s) = \frac{2}{(s - 1)^2} + \frac{1}{s\sqrt{s}};$$

j)

$$F(s) = \frac{e^{-\frac{s}{2}}}{(s - 2)^2};$$

k)

$$F(s) = \frac{e^{-\frac{s+2}{3}}}{s^2 + 4s + 13};$$

l)

$$F(s) = \frac{(s-2)e^{-\frac{s-2}{3}}}{(s-2)^2 + 9};$$

m)

$$F(s) = \frac{s - 1 - e^{-s}}{s^2 - 2s + 2}.$$

Soluţie.
a) Descompunem în fracţii simple:

$$F(s) = \frac{27 - 12s}{(s+4)(s^2+9)} = \frac{3}{s+4} - \frac{3s}{s^2+9}$$

şi aplicând transformata Laplace inversă obţinem:

$$f(t) = 3\mathcal{L}^{-1}\left[\frac{1}{s+4}\right](t) - 3\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right](t) =$$

$$= (3e^{-4t} - 3\cos 3t)\sigma(t).$$

b)

$$F(s) = \frac{1}{2a^2} \cdot \frac{-s^2 + a^2 + (s^2 + a^2)}{(s^2+a^2)^2} =$$

$$= \frac{1}{2a^2}\left[\frac{1}{s^2+a^2} - \frac{s^2 - a^2}{(s^2+a^2)^2}\right] \Rightarrow$$

$$f(t) = \frac{1}{2a^2}\left\{\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right](t) - \mathcal{L}^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right](t)\right\} =$$

$$= \left(\frac{1}{2a^3}\sin at - \frac{1}{2a^2}t\cos at\right)\sigma(t).$$

c)

$$F(s) = \frac{e^{-s}}{s^2+\pi^2} + \frac{e^{-3s}}{s^2+\pi^2} \Rightarrow f(t) =?$$

$$F_1(s) = \frac{1}{s^2+\pi^2} \overset{L^{-1}}{\longleftrightarrow} f_1(t) = \frac{1}{\pi}\sin \pi t \cdot \sigma(t);$$

$$F_2(s) = \frac{1}{s^2+\pi^2} \overset{L^{-1}}{\longleftrightarrow} f_2(t) = \frac{1}{\pi}\sin \pi t \cdot \sigma(t);$$

$$F_1(s)e^{-s} \overset{L^{-1}}{\longleftrightarrow} f_1(t-1) =$$

$$= \frac{1}{\pi}\sin \pi(t-1)\sigma(t-1) = -\frac{1}{\pi}\sin \pi t \cdot \sigma(t-1);$$

$$F_2(s)e^{-3s} \overset{L^{-1}}{\longleftrightarrow} f_2(t-3) =$$

$$= \frac{1}{\pi}\sin \pi(t-3)\sigma(t-3) = -\frac{1}{\pi}\sin \pi t \cdot \sigma(t-3);$$

$$f(t) = f_1(t-1)\sigma(t-1) + f_2(t-3)\sigma(t-3) =$$

$$= -\frac{1}{\pi}\sin \pi t(\sigma(t-1) + \sigma(t-3)).$$

Deci

	0		1		3	
$H(t-1)$		0		1		1
$H(t-3)$		0		0		1
$H(t-1)+H(t-3)$		0		1		2

Figura 1.

$$f(t) = \begin{cases} 0, & 0 < t < 1 \\ -\frac{1}{\pi} \sin \pi t, & 1 < t < 3 \\ -\frac{2}{\pi} \sin \pi t, & t > 3. \end{cases}$$

$d)$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{(s^2-1)}\right](t) - e^{-\pi}\mathcal{L}^{-1}\left[\frac{e^{-\pi s}}{s^2-1}\right](t) =$$

$$= \sinh t\sigma(t) - e^{-\pi}\sinh(t-\pi)\sigma(t-\pi) =$$

$$= \begin{cases} \sinh t, & t < \pi \\ \sinh t - e^{-\pi}\sinh(t-\pi), & t \geq \pi. \end{cases}$$

$e)$ Cu relaţia din deplasare

$$\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right](t) = e^{-2t}\sigma(t);$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) = \sin t \cdot \sigma(t);$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+5}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+1}\right](t),$$

folosim teorema deplasării:

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+4s+5}\right](t) = e^{-2t}\mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t) = e^{-2t}\sin t \cdot \sigma(t).$$

Cu relaţia:

$$\mathcal{L}^{-1}\left[e^{-bs}F(s)\right](t) = \mathcal{L}^{-1}\left[F(s)\right](t-b)\sigma(t-b),$$

obţinem:

$$f(t) = e^{-2(t-1)}\sigma(t-1) + \sin(t-2)\sigma(t-2) +$$

$$+e^{-2(t-3)}\sin(t-3)\sigma(t-3).$$

$f)$

$$F(s)=\frac{1}{s^4+2s^3+3s^2+2s+1}=\frac{1}{(s^2+s+1)^2}=$$

$$=\frac{1}{\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}=$$

$$=\frac{2}{3}\cdot\frac{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2-\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}{\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}=$$

$$=\frac{2}{3}\cdot\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}-\frac{2}{3}\cdot\frac{\left(s+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2}{\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}.$$

$$\mathcal{L}^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right](t)\overset{\text{deplasarea}}{=}$$

$$=e^{-\frac{t}{2}}\mathcal{L}^{-1}\left[\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right](t)=$$

$$=\frac{2}{\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sigma(t).$$

$$\mathcal{L}^{-1}\left[\frac{\left(s+\frac{1}{2}\right)^2-\left(\frac{\sqrt{3}}{2}\right)^2}{\left[\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}\right](t) \overset{\text{deplasarea}}{=}$$

$$= e^{-\frac{t}{2}}\mathcal{L}^{-1}\left[\frac{s^2-\left(\frac{\sqrt{3}}{2}\right)^2}{\left[s^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}\right](t) =$$

$$= e^{-\frac{t}{2}}\mathcal{L}^{-1}\left[-\left(\frac{s}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right)'\right](t)$$

folosind derivarea imaginii avem:

$$= e^{-\frac{t}{2}}\mathcal{L}^{-1}\left[\mathcal{L}\left[t\cos\left(\frac{\sqrt{3}}{2}t\right)\sigma(t)\right](s)\right](t) =$$

$$= e^{-\frac{t}{2}}t\cos\left(\frac{\sqrt{3}}{2}t\right)\sigma(t).$$

Astfel,

$$f(t) = \frac{4}{3\sqrt{3}}e^{-\frac{t}{2}}\sin\frac{\sqrt{3}}{2}t\sigma(t) - \frac{2}{\sqrt{3}}e^{-\frac{t}{2}}t\cos\frac{\sqrt{3}}{2}t\sigma(t).$$

g)

$$F(s) = \frac{e^{-s}\left(1-e^{-s}\right)}{s\left(s^2+1\right)} =$$

$$= \frac{e^{-s}}{s\left(s^2+1\right)} - \frac{e^{-2s}}{s\left(s^2+1\right)}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s\left(s^2+1\right)}\right](t) =$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right](t) = (1-\cos t)\sigma(t).$$

folosind "intârzierea" avem:

$$f(t) = \left[1-\cos(t-1)\right]\sigma(t-1) - \left[1-\cos(t-2)\right]\sigma(t-2) =$$

$$= -\sigma(t-2) + \sigma(t-1) + \cos(t-2)\sigma(t-2) - \cos(t-1)\sigma(t-1).$$

Astfel,

$$f(t) = \begin{cases} 0, & t < 1 \\ 1-\cos(t-1), & 1 \le t < 2 \\ \cos(t-2)-\cos(t-1), & t \ge 2. \end{cases}$$

$h)$

$$F(s) = \frac{s}{(s+1)^{\frac{5}{2}}} = \frac{1}{(s+1)^{\frac{3}{2}}} - \frac{1}{(s+1)^{\frac{5}{2}}}.$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+1)^{\frac{3}{2}}}\right](t) - \mathcal{L}^{-1}\left[\frac{1}{(s+1)^{\frac{5}{2}}}\right](t)$$

folosind "deplasarea" avem:

$$f(t) = e^{-t}\mathcal{L}^{-1}\left[\frac{1}{(s)^{\frac{3}{2}}}\right](t) - e^{-t}\mathcal{L}^{-1}\left[\frac{1}{(s)^{\frac{5}{2}}}\right](t).$$

Ştim că:

$$\mathcal{L}\left[t^k\sigma(t)\right] = \frac{\Gamma(k+1)}{s^{k+1}} \rightarrow$$

$$\rightarrow \mathcal{L}^{-1}\left[\frac{1}{s^{k+1}}\right](t) = \frac{t^k}{\Gamma(k+1)}\sigma(t), k > -1.$$

Astfel,

$$f(t) = \left(e^{-t} \frac{\sqrt{t}}{\Gamma\left(\frac{3}{2}\right)} - e^{-t} \frac{t\sqrt{t}}{\Gamma\left(\frac{5}{2}\right)} \right) \sigma(t) =$$

$$= \frac{2}{\sqrt{\pi}} e^{-t} \sqrt{t} \left(1 - \frac{2}{3} t\right) \sigma(t) = 2e^{-t} \sqrt{\frac{t}{\pi}} \left(1 - \frac{2}{3} t\right) \sigma(t).$$

$i)$ Ştim că:

$$\mathcal{L}\left[t^n e^{\lambda t} \sigma(t)\right](s) = \frac{n!}{(s-\lambda)^{n+1}}, n \in \mathbb{N}, \lambda \in \mathbb{C}.$$

Atunci:

I)

$$\frac{1}{(s-\lambda)^{n+1}} \overset{L^{-1}}{\longleftrightarrow} \frac{t^n e^{\lambda t}}{n!} \sigma(t).$$

II)

$$t^k \sigma(t) \overset{L}{\longleftrightarrow} \frac{\Gamma(k+1)}{s^{k+1}}, k > -1.$$

Rezultă

$$\frac{1}{s^{k+1}} \overset{L^{-1}}{\longleftrightarrow} \frac{t^k}{\Gamma(k+1)} \sigma(t).$$

Din liniaritatea lui $\mathcal{L}^{-1}$:

$$f(t) = \mathcal{L}^{-1}\left[\frac{2}{(s-1)^2} + \frac{1}{s\sqrt{s}}\right](t) =$$

$$= 2\mathcal{L}^{-1}\left[\frac{1}{(s-1)^2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s^{1+\frac{1}{2}}}\right](t).$$

Pentru $n = 1$ şi $\lambda = 1$ în relaţia I) avem:

$$\frac{1}{(s-1)^2} \overset{L^{-1}}{\longleftrightarrow} f_1(t) = te^t \sigma(t)$$

şi pentru $k = \frac{1}{2}$ în relaţia $II)$ avem:

$$\frac{1}{s^{1+\frac{1}{2}}} \overset{L^{-1}}{\longleftrightarrow} \frac{t^{\frac{1}{2}}}{\Gamma(1+\frac{1}{2})}\sigma(t) = \frac{\sqrt{t}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\sigma(t) =$$

$$= 2\sqrt{\frac{t}{\pi}}\sigma(t) = f_2(t).$$

Obţinem:

$$f(t) = 2f_1(t) + f_2(t) = \left(2te^t + 2\sqrt{\frac{t}{\pi}}\right)\sigma(t).$$

$j)$

$$F(s) = \frac{e^{-\frac{s}{2}}}{(s-2)^2};$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right](t) = \mathcal{L}^{-1}\left[e^{-\frac{s}{2}}\frac{1}{(s-2)^2}\right](t)$$

folosind "intârzierea" avem:

$$= \mathcal{L}^{-1}\left[\frac{1}{(s-2)^2}\right](t-\frac{1}{2})\sigma(t-\frac{1}{2}) = (t-\frac{1}{2})e^{2(t-\frac{1}{2})}\sigma(t-\frac{1}{2}) =$$

$$= \frac{1}{2}(2t-1)e^{2t-1}\sigma(2t-1).$$

$k)$

$$F(s) = \frac{e^{-\frac{s+2}{3}}}{s^2+4s+13};$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right](t) = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2+9}e^{-\frac{s+2}{3}}\right](t)$$

folosind "deplasarea" avem:

$$= e^{-2t}\mathcal{L}^{-1}\left[\frac{1}{s^2+9}e^{-\frac{s}{3}}\right]$$

folosind "intârzierea" avem:

$$= e^{-2t}\mathcal{L}^{-1}\left[\frac{1}{s^2+9}\right](t-\frac{1}{3})\sigma(t-\frac{1}{3}) =$$

$$= \frac{1}{3}e^{-2t}\sin 3(t-\frac{1}{3})\sigma(t-\frac{1}{3}) =$$
$$= \frac{1}{3}e^{-2t}\sin(3t-1)\sigma(3t-1).$$

l)

$$F(s) = \frac{(s-2)e^{-\frac{s-2}{3}}}{(s-2)^2+9};$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right](t) = \mathcal{L}^{-1}\left[\frac{(s-2)e^{-\frac{s-2}{3}}}{(s-2)^2+9}\right](t)$$

folosind "deplasarea" avem:

$$= e^{2t}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}e^{-\frac{s}{3}}\right]$$

folosind "intârzierea" avem:

$$= e^{2t}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right](t-\frac{1}{3})\sigma(t-\frac{1}{3}) =$$

$$= e^{2t}\cos 3(t-\frac{1}{3})\sigma(t-\frac{1}{3}) =$$
$$= e^{2t}\cos(3t-1)\sigma(3t-1).$$

$m)$

$$F(s) = \frac{s-1-e^{-s}}{s^2-2s+2}.$$

$$F(s) = \frac{s-1}{(s-1)^2+1} - \frac{e^{-s}}{(s-1)^2+1}$$

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right](t) =$$

$$= \mathcal{L}^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] - \mathcal{L}^{-1}\left[\frac{e^{-s}}{(s-1)^2+1}\right]$$

folosind "deplasarea" şi "intârzierea" avem:

$$f(t) = e^t\mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{(s-1)^2+1}\right](t-1)\sigma(t-1) =$$

$$\overset{deplasarea\, in\, al\, doilea\, termen}{=} e^t\cos t\sigma(t) - e^{t-1}\sin(t-1)\sigma(t-1).$$

Exerciţiul 1.11 Rezolvaţi ecuaţiile diferenţiale următoare cu transformata Laplace:

a) $y'' - 4y = \sigma(t-1)$, $y(0) = y'(0) = 0$;

b) $y'' + 4y' + 4y = \sin t \cdot \sigma(t)$, $y(0) = 1$, $y'(0) = 2$;

c) $y(t) + y'(t) - 2\int_0^t y(s)\sin(t-s)ds = (\cos t + \sinh t)\sigma(t)$, $y(0) = 1$.

d) $y''(t) - y(t) = t\sigma(t)$, $y(0) = 1$, $y'(0) = 1$;

e) $y''(t) + 9y(t) = 10e^{-t}\sigma(t)$, $y(0) = 0$, $y'(0) = 0$;

f) $y''(t) - y(t) = g(t)$, $y(0) = y'(0) = 0$;
$g(t) = \begin{cases} t, 0 < t < 1 \\ 0, \text{rest} \end{cases}$

g) $y'' - 2y' - 3y = 0$, $y(1) = -3$, $y'(1) = -17$;

h) $y'' - y = \cosh 2t \cdot \sigma(t)$, $y(0) = y'(0) = 0$;

Soluţie.
a) Aplicăm transformata Laplace şi ţinem cont de liniaritate:

$$\mathcal{L}\left[y''\right](s) - 4\mathcal{L}\left[y\right](s) = \mathcal{L}\left[\sigma(t-1)\right](s).$$

Cu derivarea originalului avem:

$$\mathcal{L}\left[y'\right](s) = s\mathcal{L}\left[y\right](s) - y(0) = sY(s).$$

$$\mathcal{L}\left[y''\right](s) = s\mathcal{L}\left[(y')'\right](s) = s\mathcal{L}\left[y'\right](s) - y'(0) = s^2Y(s).$$

Cu întârzierea:

$$\mathcal{L}\left[\sigma(t-1)\right](s) = e^{-s}\mathcal{L}\left[1\right](s) = \frac{1}{s}e^{-s}.$$

Ecuaţia operaţională este:

$$s^2Y(s) - 4Y(s) = \frac{e^{-s}}{s} \Leftrightarrow$$

$$Y(s) = e^{-s}\frac{1}{s(s^2-4)}.$$

Aplicăm transformata Laplace inversă:

$$y(t) = \mathcal{L}^{-1}\left[e^{-s}\frac{1}{s(s^2-4)}\right](t) = \mathcal{L}^{-1}\left[\frac{1}{s(s^2-4)}\right](t-1)\sigma(t-1).$$

Am folosit formula:

$$\mathcal{L}^{-1}\left[e^{-sb}F(s)\right](t) = \mathcal{L}^{-1}\left[F(s)\right](t-b)\sigma(t-b).$$

Acum:

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2-4)}\right](t) = \frac{1}{4}\mathcal{L}^{-1}\left[\frac{s}{s^2-4} - \frac{1}{s}\right](t) =$$

$$= \frac{1}{4}(\cosh 2t - 1)\sigma(t).$$

Deci

$$y(t) = \frac{1}{4}\left[\cosh 2(t-1) - 1\right]\sigma(t-1).$$

b) Aplicăm transformata Laplace ecuaţiei şi folosim liniaritatea:

$$\mathcal{L}\left[y''\right](s) + 4\mathcal{L}\left[y'\right](s) + 4\mathcal{L}\left[y\right](s) = \mathcal{L}\left[\sin t \cdot \sigma(t)\right](s).$$

Notăm

$$\mathcal{L}\left[y\right](s) = Y(s).$$

Cu derivarea originalului avem:

$$\mathcal{L}\left[y'\right](s) = sY(s) - y(0) = sY(s) - 1.$$

$$\mathcal{L}\left[y''\right](s) = \mathcal{L}\left[(y')'\right](s) = s\mathcal{L}\left[y'\right](s) - y'(0) = s^2Y(s) - s - 2.$$

$$\mathcal{L}\left[\sin t \cdot \sigma(t)\right](s) = \frac{1}{s^2+1}.$$

Ecuaţia operaţională este:

$$s^2Y(s) - s - 2 + 4\left(sY(s) - 1\right) + 4Y(s) = \frac{1}{s^2+1} \Leftrightarrow$$

$$(s+2)^2Y(s) = s + 6 + \frac{1}{s^2+1} = \frac{s^3+6s^2+s+7}{s^2+1} \Rightarrow$$

$$Y(s) = \frac{s^3+6s^2+s+7}{(s+2)^2(s^2+1)}.$$

Cu descompunerea în fracţii simple avem:

$$Y(s) = \frac{a}{s+2} + \frac{b}{(s+2)^2} + \frac{cs+d}{s^2+1}$$

şi folosind metoda coeficienţilor nedeterminaţi obţinem

$$a = \frac{7}{25}, b = \frac{21}{5}, c = \frac{-4}{25}, d = \frac{3}{25}.$$

$$Y(s) = \frac{7}{25} \cdot \frac{1}{s+2} + \frac{21}{5} \cdot \frac{1}{(s+2)^2} - \frac{4}{25} \cdot \frac{s}{s^2+1} + \frac{3}{25} \cdot \frac{1}{s^2+1}.$$

Aplicăm transformata Laplace inversă şi folosim liniaritatea:

$$y(t) = \frac{7}{25} \cdot \mathcal{L}^{-1}\left[\frac{1}{s+2}\right](t) + \frac{21}{5} \cdot \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right](t) -$$

$$-\frac{4}{25} \cdot \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right](t) + \frac{3}{25} \cdot \mathcal{L}^{-1}\left[\frac{1}{s^2+1}\right](t).$$

Ştim că

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+a^2}\right](t) = \cos at \cdot \sigma(t)$$

şi

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+a^2}\right](t) = \frac{\sin at}{a} \cdot \sigma(t)$$

avem:

$$y(t) = \left(\frac{7}{25} \cdot e^{-2t} + \frac{21}{5} \cdot te^{-2t} - \frac{4}{25} \cdot \cos t + \frac{3}{25} \cdot \sin t\right)\sigma(t).$$

c) Folosind liniaritatea transformatei Laplace, ecuaţia operaţională obţinută este:

$$Y(s) + sY(s) - 1 - 2Y(s)\frac{1}{s^2+1} = \frac{s}{s^2+1} + \frac{1}{s^2-1}$$

$$\Rightarrow Y(s) = \frac{s(s^3+s^2+s-1)}{(s^2-1)(s^3+s^2+s-1)} = \frac{s}{s^2-1} \Rightarrow$$
$$y(t) = \cosh t \cdot \sigma(t).$$

d)

$$\mathcal{L}[y''] - \mathcal{L}[y] = \mathcal{L}[t\sigma(t)],$$

dar

$$\mathcal{L}[y] = F(s) \text{ şi } \mathcal{L}[t] = \frac{1}{s^2},$$

deci

$$s^2F(s) - sy(0) - y'(0) - F(s) = \frac{1}{s^2}$$

$$(s^2-1)F(s) = s+1+\frac{1}{s^2} \Rightarrow F(s) = \frac{s}{s^2-1}+\frac{1}{s^2-1}+\frac{1}{s^2(s^2-1)} \Rightarrow$$

$$F(s) = \frac{s}{s^2-1} + \frac{2}{s^2-1} - \frac{1}{s^2} \Rightarrow$$
$$y(t) = (\cosh t + 2\sinh t - t)\sigma(t).$$

f) Ştim că

$$g(t) = \begin{cases} t, 0 < t < 1 \\ 0, \text{rest} \end{cases} = t\left(\sigma(t) - \sigma(t-1)\right)$$

deci

$$(s^2-1)F(s) = \mathcal{L}[t] - \mathcal{L}\left[((t+1)-1)\sigma(t-1)\right] =$$

$$= \frac{1}{s^2} - e^{-s}\mathcal{L}\left[t+1\right] = \frac{1}{s^2} - \frac{1}{s^2}e^{-s} - \frac{1}{s}e^{-s}.$$

Deoarece

$$\mathcal{L}\left[f(t)\sigma(t-b)\right] = e^{-bs}\mathcal{L}\left[f(t+b)\right].$$

Avem

$$F(s)=\underbrace{\frac{1}{s^2(s^2-1)}}_{F_1(s)}-\underbrace{\frac{1}{s^2(s^2-1)}}_{F_2(s)}e^{-s}-\underbrace{\frac{1}{s^2(s^2-1)}}_{F_3(s)}e^{-s}$$

$$\begin{cases} F_1(s)=\frac{1}{s^2-1}-\frac{1}{s^2}\overset{L^{-1}}{\longleftrightarrow}(\sinh t-t)\sigma(t)=f_1(t)\\ F_2(s)=\frac{-1}{s^2(s^2-1)}=\frac{1}{s^2}-\frac{1}{s^2-1}\overset{L^{-1}}{\longleftrightarrow}(t-\sinh t)\sigma(t)=f_2(t)\\ F_3(s)=\frac{-1}{s(s^2-1)}=\frac{1}{s}-\frac{s}{s^2-1}\overset{L^{-1}}{\longleftrightarrow}(1-\cosh t)\sigma(t)=f_3(t)\end{cases}$$

$$\begin{cases} F_2(s)e^{-s}\overset{L^{-1}}{\longleftrightarrow}f_2(t-1)\sigma(t-1)=[t-1-\sinh(t-1)]\sigma(t-1)\\ F_3(s)e^{-s}\overset{L^{-1}}{\longleftrightarrow}f_3(t-1)\sigma(t-1)=[1-\cosh(t-1)]\sigma(t-1).\end{cases}$$

Astfel,

$$y(t)=f_1(t)\sigma(t)+f_2(t-1)\sigma(t-1)+f_3(t-1)\sigma(t-1)=$$

$$=\begin{cases}\sinh t-t, & 0<t<1\\ \sinh t-\sinh(t-1)-\cosh(t-1), & t>1.\end{cases}$$

g) $\widetilde{t}=t-1, \widetilde{y}(\widetilde{t})=y(t)=y(\widetilde{t}+1)=\widetilde{y}(\widetilde{t})$ deci $\widetilde{y}''(\widetilde{t})-2\widetilde{y}'(\widetilde{t})-3\widetilde{y}(\widetilde{t})=0$, $\widetilde{y}(0)=-3$, $\widetilde{y}'(0)=-17$; Astfel,

$$\mathcal{L}\left[\widetilde{y}''\right]-2\mathcal{L}\left[\widetilde{y}'\right]-3\mathcal{L}\left[\widetilde{y}\right]=0.$$

Notăm $F(s)=\mathcal{L}[y]$ şi avem

$$s^2F(s)-s\widetilde{y}(0)-\widetilde{y}'(0)-2\left(sF(s)-\widetilde{y}(0)\right)-3F(s)=0\Rightarrow$$

$$\left(s^2-2s-3\right)F(s)=3s-17+6\Rightarrow$$

$$F(s)=\frac{-3s-11}{(s-3)(s+1)}=\frac{2}{s+1}-\frac{5}{s-3}.$$

Obţinem

$$\widetilde{y}(\widetilde{t})=2e^{-\widetilde{t}}-5e^{3\widetilde{t}},$$

de unde găsim că

$$y(t) = (2e^{-t+1} - 5e^{3(t-1)})\sigma(t) = (2e^{-t+1} - 5e^{3t-3})\sigma(t).$$

h)

$$\mathcal{L}\left[y''\right] - 4\mathcal{L}\left[y\right] = \mathcal{L}\left[\cosh 2t \cdot \sigma(t)\right] \Rightarrow$$
$$(s^2 - 4)F(s) = \frac{s}{(s^2-4)} \Rightarrow$$
$$F(s) = \frac{s}{(s^2-4)^2} = \frac{s}{(s-2)^2(s+2)^2}.$$
$$F(s) = \frac{1}{8}\left(\frac{1}{(s-2)^2} - \frac{1}{(s+2)^2}\right) = \frac{1}{8}\frac{1}{(s-2)^2} - \frac{1}{8}\frac{1}{(s+2)^2}.$$

Folosind deplasarea (sau tabelul), obţinem

$$y(t) = \left(\frac{1}{8}te^{2t} - \frac{1}{8}te^{-2t}\right)\sigma(t) = \left(\frac{t}{4}\sinh 2t\right)\sigma(t).$$

Exerciţiul 1.12 Rezolvaţi ecuaţiile diferenţiale următoare cu transformata Laplace:

a) $y'' + y = 2t\sigma(t)$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, $y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$;

b) $y'' + ky' - 2k^2y = 0$, $y(0) = 2$, $y'(0) = 2k$;

c) $2x'' + 3x' - 2x = 5e^{-2t}\sigma(t)$, $x(0) = 2$, $x'(0) = 5$;

d) $y'' - 3y' + 2y = (4t + e^{3t})\sigma(t)$, $y(0) = 1$, $y'(0) = -1$;

Soluţie.
a) $y'' + y = 2t$, $y\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$, $y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2}$;

$\widetilde{t} = t - \frac{\pi}{4}, t = \widetilde{t} + \frac{\pi}{4}$, deci

$$\widetilde{y}(\widetilde{t}) = y(t) \to$$

$$y'(t) = \widetilde{y}'(\widetilde{t})\frac{d\widetilde{t}}{dt} = \widetilde{y}'(\widetilde{t}) \Rightarrow$$

$$y''(t) = \widetilde{y}''(\widetilde{t})\left(\frac{d\widetilde{t}}{dt}\right)^2 + \widetilde{y}'(\widetilde{t})\frac{d^2\widetilde{t}}{dt^2} = \widetilde{y}''(\widetilde{t}).$$

Astfel,

$$\widetilde{y}'' + \widetilde{y} = 2\widetilde{t} + \frac{\pi}{2}$$

şi

$$\widetilde{y}(0) = y\left(\frac{\pi}{4}\right) = \frac{\pi}{2},\ \widetilde{y}'(0) = y'\left(\frac{\pi}{4}\right) = 2 - \sqrt{2};$$

Obţinem

$$\mathcal{L}\left[\widetilde{y}'\right](s) = sY(s) - \widetilde{y}(0) = sY(s) - \frac{\pi}{2}$$

$$\mathcal{L}\left[\widetilde{y}''\right](s) = \mathcal{L}\left[(\widetilde{y}')'\right](s) = s\mathcal{L}\left[\widetilde{y}'\right](s) - \widetilde{y}'(0) =$$

$$= s^2Y(s) - \frac{\pi}{2}s - 2 + \sqrt{2}.$$

Dar

$$\mathcal{L}\left[\widetilde{t}\right](s) = \frac{1}{s^2}$$

$$\mathcal{L}\left[\frac{\pi}{2}\right](s) = \frac{1}{s}\frac{\pi}{2}$$

$$s^2Y(s) - \frac{\pi}{2}s - 2 + \sqrt{2} + Y(s) = \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s} \Leftrightarrow$$

$$(s^2+1)Y(s) = \frac{\pi}{2}s + (2-\sqrt{2}) + \frac{2}{s^2} + \frac{\pi}{2}\frac{1}{s} \Rightarrow$$

$$Y(s) = \frac{\pi}{2}\frac{s}{s^2+1} + (2-\sqrt{2})\frac{1}{s^2+1} +$$

$$+2\left(\frac{1}{s^2}-\frac{1}{s^2+1}\right)+\frac{\pi}{2}\left(\frac{1}{s}-\frac{s}{s^2+1}\right)\Rightarrow$$

$$Y(s)=\frac{2}{s^2}-\frac{\sqrt{2}}{s^2+1}+\frac{\pi}{2}\frac{1}{s}\Rightarrow$$

$$\widetilde{y}(\widetilde{t})=2\widetilde{t}-\sqrt{2}\sin\widetilde{t}+\frac{\pi}{2}\Rightarrow$$

$$y(t)=\left[2t-\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin t-\frac{1}{\sqrt{2}}\cos t\right)\right]\sigma(t)$$

de unde găsim că

$$y(t)=(2t-\sin t+\cos t)\sigma(t).$$

b) $y''+ky'-2k^2y=0$, $y(0)=2$, $y'(0)=2k$;

$$\mathcal{L}\left[y'\right](s)=sY(s)-y(0)=sY(s)-2$$

$$\begin{aligned}\mathcal{L}\left[y''\right](s)=\mathcal{L}\left[(y')'\right](s)=s\mathcal{L}\left[y'\right](s)-y'(0)=\\=s^2Y(s)-2s-2k.\end{aligned}$$

Deci

$$s^2Y(s)-2s-2k+ksY(s)-2k-2k^2Y(s)=0$$

$$\left(s^2+ks-2k^2\right)Y(s)=2s+4k\Rightarrow$$

$$Y(s)=\frac{2s+4k}{(s-k)(s+2k)}=\frac{2}{s-k}$$

de unde găsim că

$$y(t)=2e^{kt}\sigma(t).$$

c) $2x''+3x'-2x=5e^{-2t}$, $x(0)=2$, $x'(0)=5$;

$$\left(2s^2+3s-2\right)X(s)=\frac{5}{s+2}+4s+16=\frac{4s^2+24s+37}{s+2}\Rightarrow$$

$$X(s)=\frac{4}{s-\frac{1}{2}}-\frac{1}{(s+2)^2}-\frac{2}{s+2}$$

Deci

$$x(t)=\left(4e^{t/2}-te^{-2t}-2e^{-2t}\right)\sigma(t).$$

d) $y''-3y'+2y=(4t+e^{3t})\sigma(t),\ y(0)=1,\ y'(0)=-1;$

$$\mathcal{L}\left[y''\right](s)-3\mathcal{L}\left[y'\right](s)+2\mathcal{L}\left[y\right](s)=\frac{4}{s^2}+\frac{1}{s-3}.$$

Avem

$$\mathcal{L}\left[y''\right](s)=s^2Y(s)-s+1;$$
$$\mathcal{L}\left[y'\right](s)=sY(s)-1;$$
$$\mathcal{L}\left[y\right](s)=Y(s);$$

Deci

$$s^2Y(s)-s+1-3(sY(s)-1)+Y(s)=\frac{4}{s^2}+\frac{1}{s-3}\Leftrightarrow$$

$$\left(s^2-3s+2\right)Y(s)=\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)}\Rightarrow$$

$$Y(s)=\frac{s^4-7s^3+13s^2+4s-12}{s^2(s-3)(s-1)(s-2)}\Rightarrow$$

$$Y(s)=\frac{a}{s-1}+\frac{b}{s-2}+\frac{c}{s-3}+\frac{d}{s^2}+\frac{e}{s}$$

de unde obţinem coeficienţii:

$$a=-\frac{1}{2};b=-4;c=\frac{5}{9};d=2;e=1;$$

şi găsim că

$$y(t) = \left(-\frac{1}{2}e^{t} - 4e^{2t} + \frac{5}{9}e^{3t} + 2t + 1\right)\sigma(t).$$

Exerciţiul 1.13 Rezolvaţi ecuaţia următoare cu transformata Laplace:

$$y(t) + \int_0^t \sin(t-u)y(u)du =$$

$$= t\left[\sigma(t) - \sigma(t-2)\right] + \int_0^t f(u)du - 2\sigma(t-2) =$$

$$= t\sigma(t) - (t-2)\sigma(t-2) + \int_0^t f(u)du - 4\sigma(t-2),$$

unde $f(u)$ este original.

Soluţie.
$a)$ Aplicăm transformata Laplace cu liniaritatea şi notăm:

$$\mathcal{L}\left[y(t)\right](s) = Y(s)$$

deci

$$Y(s) + \mathcal{L}\left[\sin t * y(t)\right](s) = \mathcal{L}\left[t\sigma(t)\right](s) -$$

$$-\mathcal{L}\left[(t-2)\sigma(t-2)\right](s) +$$

$$+\mathcal{L}\left[\int_0^t f(u)du\right](s) - 4\mathcal{L}\left[\sigma(t-2)\right](s).$$

Folosim proprietăţile: produsul de convoluţie, întârzierea şi derivarea originalului, dar şi faptul că:

$$\mathcal{L}\left[t^n\sigma(t)\right](s) = \frac{n!}{s^{n+1}}, n \in \mathbb{N}^*,$$

$$\mathcal{L}\left[\sin t \cdot \sigma(t)\right](s) = \frac{1}{s^2+1},$$

cu care obţinem

$$Y(s) + \frac{Y(s)}{s^2+1} = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{1}{s}F(s) - \frac{4}{s}e^{-2s} \Leftrightarrow$$

$$\frac{s^2+2}{s^2+1}Y(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{4}{s}e^{-2s} + \frac{1}{s}F(s) \Leftrightarrow$$

$$Y(s) = \frac{s^2+1}{s^2(s^2+2)} - \frac{s^2+1}{s^2(s^2+2)}e^{-2s} - 4\frac{s^2+1}{s(s^2+2)}e^{-2s} +$$

$$+F(s)\frac{s^2+1}{s(s^2+2)}$$

Ţinem cont că:

$$\frac{s^2+1}{s^2(s^2+2)} = \frac{1}{2}\left(\frac{1}{s^2} + \frac{1}{s^2+2}\right)$$

şi

$$\frac{s^2+1}{s(s^2+2)} = \frac{1}{2}\left(\frac{1}{s} + \frac{s}{s^2+2}\right).$$

Astfel

$$y(t) = \frac{1}{2} \cdot \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^2+2}\right](t)\sigma(t) -$$

$$-\frac{1}{2} \cdot \mathcal{L}^{-1}\left[\frac{1}{s^2} + \frac{1}{s^2+2}\right](t-2)\sigma(t-2) -$$

$$-2\mathcal{L}^{-1}\left[\frac{1}{s} + \frac{s}{s^2+2}\right](t-2)\sigma(t-2) +$$

$$+\frac{1}{2} \cdot \mathcal{L}^{-1}\left[\mathcal{L}\left[f(t)\right](s) \cdot \mathcal{L}\left[(1+\cos\sqrt{2}t)\sigma(t)\right](s)\right](t) =$$

$$= \frac{1}{2}\left(t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\sigma(t) -$$

$$-\frac{1}{2}\left(t - 2 + \frac{1}{\sqrt{2}}\sin\sqrt{2}(t-2)\right)\sigma(t-2) -$$

$$-2\left(\sigma(t-2) + \cos\sqrt{2}(t-2)\sigma(t-2)\right) +$$

$$+\frac{1}{2}\cdot\mathcal{L}^{-1}\left[\mathcal{L}\left[f(t) * (1 + \cos\sqrt{2}t)\sigma(t)\right](s)\right](t) \Rightarrow$$

$$y(t) = \frac{1}{2}\left(t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\sigma(t) -$$

$$-\frac{1}{2}\left(t - 2 + \frac{1}{\sqrt{2}}\sin\sqrt{2}(t-2)\right)\sigma(t-2) -$$

$$-2\left(\sigma(t-2) + \cos\sqrt{2}(t-2)\sigma(t-2)\right) +$$

$$+\frac{1}{2}\int_0^t f(t-u)(1 + \cos\sqrt{2}u)du.$$

Exerciţiul 1.14 Fie un circuit format dintr-o rezistenţă R, o inductanţă L şi o capacitate C. La momentul t, considerat momentul iniţial $t = 0$, circuitul este conectat la o sursă cu tensiunea electromotoare $e(t)$. Atunci intensitatea curentului $i(t)$ care trece prin circuit satisface ecuaţia:

$$Li'(t) + Ri(t) + \frac{1}{c}\int_0^t i(u)du = e(t), t > 0, i(0) = 0,$$

$i(t) = 0$ pentru $t < 0$ local integrabilă. Aflaţi $i(t)$.

Soluţie.
Aplicăm transformata Laplace cu liniaritatea, derivarea originalului şi integrarea originalului:

$$LsI(s) + RI(s) + \frac{1}{c}\frac{1}{s}I(s) = E(s) \Rightarrow$$

$$I(s) = \frac{s}{Ls^2 + Rs + \frac{1}{c}}E(s).$$

Avem

$$Ls^2 + Rs + \frac{1}{c} = 0 \Rightarrow s_\pm = \frac{-R}{2L} \pm \frac{\sqrt{d}}{2L},$$

unde $d := R^2 - 4\frac{L}{c}$.

$$\frac{s}{Ls^2 + Rs + \frac{1}{c}} = \frac{s}{L(s - s_+)(s - s_-)} = \frac{1}{L}\left(\frac{a}{s - s_+} + \frac{b}{s - s_-}\right) \Rightarrow$$

$$a = \frac{s_+}{s_+ - s_-} = \frac{s_+}{\frac{\sqrt{d}}{L}} = \frac{L}{\sqrt{d}}s_+$$

şi

$$b = -\frac{s_-}{s_+ - s_-} = -\frac{s_-}{\frac{\sqrt{d}}{L}} = -\frac{L}{\sqrt{d}}s_-$$

rezultă

$$I(s) = E(s)\frac{1}{\sqrt{d}}\left(\frac{s_+}{s - s_+} - \frac{s_-}{s - s_-}\right) =$$

$$= \frac{1}{\sqrt{d}}\mathcal{L}\left[s_+e^{s_+t} - s_-e^{s_-t}\right](s) \cdot \mathcal{L}\left[e(t)\right](s) =$$

$$= \mathcal{L}\left[\frac{1}{\sqrt{d}}\left(s_+e^{s_+t} - s_-e^{s_-t}\right) * e(t)\right](s)$$

şi aplicăm transformata Laplace inversă:

$$i(t) = \frac{1}{\sqrt{d}}\left(s_+e^{s_+t} - s_-e^{s_-t}\right) * e(t) \Rightarrow$$

$$i(t) = \frac{1}{\sqrt{d}} \int_0^t \left[s_+ e^{s_+(t-u)} - s_- e^{s_-(t-u)} \right] e(u) du,$$

unde $d := R^2 - 4\frac{L}{c}$.

Exerciţiul 1.15 Să se rezolve ecuaţiile:

a) $x''' + 2x'' + 2x' = \sigma(t)$, $x(0) = x'(0) = x''(0) = 0$;

b) $x''' - 2x'' - x' + 2x = 5 \sin 2t \cdot \sigma(t)$,
$x(0) = 1$, $x'(0) = 1$, $x''(0) = -1$;

c) $x''' + 2x' = \sigma(t-2) - \sigma(t-5)$,
$x(0) = 0$, $x'(0) = -1$, $x''(0) = 2$;

d) $x(t) + 3\int_0^t x(\tau)d\tau = t\sigma(t) - (t-2)\sigma(t-2)$;

e) $x(t) = t\cos 3t \cdot \sigma(t) + \int_0^t \sin 3(t-\tau)x(\tau)d\tau$;

Soluţie.
a) $x''' + 2x'' + 2x' = \sigma(t)$, $x(0) = x'(0) = x''(0) = 0$;

$$\left(s^3 + 2s^2 + 2s + 1\right) X(s) = \frac{1}{s} \Rightarrow$$

$$X(s) = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{s^2+s+1} =$$

$$= \frac{1}{s} - \frac{1}{s+1} - \frac{\sqrt{3}}{2} \frac{\frac{2}{\sqrt{3}}}{\left(s+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}.$$

$$x(t) = \left(1 - e^{-t} - \frac{2}{\sqrt{3}} e^{-\frac{t}{2}} \sin \frac{\sqrt{3}t}{2}\right) \sigma(t).$$

b) $x''' - 2x'' - x' + 2x = 5\sin 2t \cdot \sigma(t)$, $x(0) = 1$, $x'(0) = 1$, $x''(0) = -1$;

$$X(s) = \frac{s^2 - s - 2}{(s-1)(s+1)(s-2)} + \frac{10}{(s-1)(s+1)(s-2)(s^2+4)} =$$

$$= \frac{1}{3}\frac{1}{s+1} + \frac{5}{12}\frac{1}{s-2} + \frac{1}{4}\left(\frac{s}{s^2+4} + \frac{2}{s^2+4}\right) \Rightarrow$$

$$x(t) = \left(\frac{1}{3}e^{-t} + \frac{5}{12}e^{2t} + \frac{1}{4}\cos 2t + \frac{1}{4}\sin 2t\right)\sigma(t).$$

c) $x''' + 2x' = \sigma(t-2) - \sigma(t-5)$, $x(0) = 0$, $x'(0) = -1$, $x''(0) = 2$;

$$\mathcal{L}\left[x'\right](s) = sX(s);$$

$$\mathcal{L}\left[x''\right](s) = s\mathcal{L}\left[x'(t)\right](s) - x'(0) = s^2X(s) + 1;$$

$$\mathcal{L}\left[x'''\right](s) = \mathcal{L}\left[(x'')'(t)\right](s) = s\mathcal{L}\left[x''\right](s) - x''(0) = s^3X(s) + s - 2;$$

$$s^3X(s) + s - 2 + 2sX(s) = \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-5s} \Rightarrow$$

$$\mathcal{L}\left[\sigma(t-2)\right](s) = e^{-2s}\mathcal{L}\left[1\right](s) = \frac{1}{s}e^{-2s};$$

$$\mathcal{L}\left[\sigma(t-5)\right](s) = e^{-5s}\mathcal{L}\left[1\right](s) = \frac{1}{s}e^{-5s};$$

$$X(s)\left(s^3 + 2s\right) = -(s-2) + \frac{1}{s}e^{-2s} - \frac{1}{s}e^{-5s} \Rightarrow$$

$$X(s) = -\frac{s-2}{s(s^2+2)} + \frac{1}{s^2(s^2+2)}e^{-2s} - \frac{1}{s^2(s^2+2)}e^{-5s} =$$

$$X(s) = -\frac{1}{s^2+2} + \frac{2}{s(s^2+2)} +$$
$$+\frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right)e^{-2s} - \frac{1}{2}\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right)e^{-5s}$$

$$x(t) = -\mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right](t) + \mathcal{L}^{-1}\left[\frac{1}{s}\right](t) - \mathcal{L}^{-1}\left[\frac{s}{s^2+2}\right](t) +$$

$$+\frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right)e^{-2s}\right](t) - \frac{1}{2}\mathcal{L}^{-1}\left[\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right)e^{-5p}\right](t)$$

Ţinem cont de relaţiile următoare:

$$\mathcal{L}\left[f(t-b)\sigma(t-b)\right](s) = e^{-sb}\mathcal{L}\left[f(t)\right](s), b > 0$$

din care obţinem

$$\mathcal{L}^{-1}\left[e^{-sb}\mathcal{L}\left[f(t)\right](s)\right](t) = f(t-b)\sigma(t-b)$$

sau, altfel scris

$$\mathcal{L}^{-1}\left[F(s)e^{-sb}\right](t) = \mathcal{L}^{-1}\left[F(s)\right](t-b)\sigma(t-b), b > 0.$$

Ştim că

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right](t) = t\sigma(t),$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2+2}\right](t) = \frac{1}{\sqrt{2}}\sin\sqrt{2}t \cdot \sigma(t),$$

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right)e^{-2s}\right](t) =$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+2}\right](t-2)\sigma(t-2) =$$

$$= \left[t - 2 - \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-2)\right] \sigma(t-2),$$

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s^2} - \frac{1}{s^2+2}\right) e^{-5s}\right](t) =$$

$$= \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+2}\right](t-5)\sigma(t-5) =$$

$$= \left[t - 5 - \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-5)\right] \sigma(t-5),$$

deci

$$x(t) = \left[1 - \frac{1}{\sqrt{2}} \sin \sqrt{2}t - \cos \sqrt{2}t\right] \sigma(t) +$$

$$+ \frac{1}{2}\left[t - 2 - \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-2)\right] \sigma(t-2) -$$

$$- \frac{1}{2}\left[t - 5 - \frac{1}{\sqrt{2}} \sin \sqrt{2}(t-5)\right] \sigma(t-5).$$

d) $x(t) + 3\int_0^t x(\tau)d\tau = t\sigma(t) - (t-2)\sigma(t-2)$; Ştim că

$$\mathcal{L}\left[x(t)\right](s) = X(s),$$

$$\mathcal{L}\left[t\sigma(t)\right](s) = \frac{1}{s^2},$$

$$\mathcal{L}\left[\int_0^t x(\tau)d\tau\right](s) = \frac{1}{s}X(s),$$

$$\mathcal{L}\left[(t-2)\sigma(t-2)\right](s) = e^{-2s}\mathcal{L}\left[t\right](s) = \frac{1}{s^2}e^{-2s}$$

cu care găsim:

$$\left(1 + \frac{3}{s}\right) X(s) = \frac{1}{s^2} - \frac{1}{s^2}e^{-2s} \Rightarrow$$

$$X(s) = \frac{1}{s(s+3)} - \frac{1}{s(s+3)}e^{-2s}.$$

Dar

$$\frac{1}{s(s+3)} = \frac{1}{3}\left(\frac{1}{s} - \frac{1}{s+3}\right),$$

adică

$$x(t) = \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+3}\right](t)\sigma(t) - \frac{1}{3}\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+3}\right](t-2)\sigma(t-2).$$

Cu

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{s+3}\right](t) = (1 - e^{-3t})\sigma(t)$$

vom avea

$$x(t) = \frac{1}{3}(1 - e^{-3t})\sigma(t) - \frac{1}{3}(t - 2 - e^{-3(t-2)})\sigma(t-2).$$

e) $x(t) = t\cos 3t \cdot \sigma(t) + \int_0^t \sin 3(t-\tau)x(\tau)d\tau$;

$$\mathcal{L}\left[\int_0^t \sin 3(t-\tau)x(\tau)d\tau\right](s) =$$

$$= \mathcal{L}\left[x(t) * \sin 3t \cdot \sigma(t)\right](s) = \frac{3X(s)}{s^2+9} \Rightarrow$$

$$X(s) = -\left(\frac{s}{s^2+9}\right)' + \frac{3}{s^2+9}X(s) \Leftrightarrow$$

$$\frac{s^2+6}{s^2+9}X(s) = \frac{s^2-9}{(s^2+9)^2} \Rightarrow$$

$$X(s) = \frac{(s^2-9)(s^2+9)}{(s^2+6)(s^2+9)^2} = \frac{s^2-9}{(s^2+6)(s^2+9)} = \frac{6}{s^2+9} - \frac{5}{s^2+6} \Rightarrow$$

$$x(t) = \left(2\sin 3t - \frac{5}{\sqrt{6}}\sin\sqrt{6}t\right)\sigma(t).$$

Exerciţiul 1.16 Să se rezolve sistemele următoare:

a)
$$\begin{cases} x'' + 2x' + x + y'' + y' + y = \sigma(t), \\ 2x' + 2x + y'' + 2y' = 2t\sigma(t), \end{cases}$$

$x(0) = 0$, $x'(0) = 2$, $y(0) = 1$, $y'(0) = -2$;

b)
$$\begin{cases} x' = y - 7x, \\ y' = -x - 5y, \end{cases}$$

$x(0) = 2$, $y(0) = 1$;

Soluţie.
a)
$$\begin{cases} x'' + 2x' + x + y'' + y' + y = \sigma(t), \\ 2x' + 2x + y'' + 2y' = 2t\sigma(t), \end{cases}$$

$x(0) = 0$, $x'(0) = 2$, $y(0) = 1$, $y'(0) = -2$;

Avem
$$\mathcal{L}\left[x'\right] = sX(s),$$

$$\mathcal{L}\left[x''\right] = \mathcal{L}\left[(x')'\right] = s^2X(s) - 2,$$

$$\mathcal{L}\left[y'\right] = sY(s) - 1,$$

$$\mathcal{L}\left[y''\right] = \mathcal{L}\left[(y')'\right] = s(sY(s) - 1) + 2 = s^2Y(s) - s + 2,$$

$$\mathcal{L}\left[t\right] = \frac{1}{s^2},$$

$$\mathcal{L}\left[1\right] = \frac{1}{s}.$$

$$\begin{cases} s^2X(s) - 2 + 2sX(s) + X(s) + \\ \qquad\qquad +s^2Y(s) - s + 2 + pY(s) - 1 + Y(s) = \frac{1}{s}, \quad \Rightarrow \\ 2pX(s) + 2X(s) + s^2Y(s) - s + 2 + 2sY(s) - 2 = \frac{2}{s^2}, \end{cases}$$

$$\begin{cases} (s+1)^2X(s) + (s^2+s+1)Y(s) = 1 + \frac{1}{s} + s = \frac{s^2+s+1}{s}, \\ 2(s+1)X(s) + s(s+2)Y(s) = \frac{2}{s^2} + s = \frac{s^3+2}{s^2}, \end{cases}$$

$$\triangle = \begin{vmatrix} (s+1)^2 & s^2+s+1 \\ 2(s+1) & s(s+2) \end{vmatrix} =$$

$$= (s+1)\left[s(s^2+3s+2) - 2s^2 - 2s - 2\right] = (s+1)(s-1)(s^2+2s+2).$$

$$\triangle_x = \begin{vmatrix} \frac{s^2+s+1}{s} & s^2+s+1 \\ \frac{s^3+2}{s^2} & s(s+2) \end{vmatrix} =$$

$$= (s^2+s+1)\begin{vmatrix} \frac{1}{s} & 1 \\ \frac{s^3+2}{s^2} & s(s+2) \end{vmatrix} =$$

$$= (s^2+s+1)\left(s+2-s-\frac{2}{s^2}\right) = \frac{2}{s^2}(s+1)(s-1)(s^2+s+1).$$

$$\triangle_y = \begin{vmatrix} (s+1)^2 & \frac{s^2+s+1}{s} \\ 2(s+1) & \frac{s^3+2}{s^2} \end{vmatrix} =$$

$$= \frac{1}{s^2}(s+1)(s-1)(s^3-2s-2).$$

$$X(s) = \frac{\triangle_x}{\triangle} = \frac{2(s+1)(s-1)(s^2+s+1)}{s^2(s+1)(s-1)(s^2+2s+2)} =$$

$$= \frac{2s^2+2s+2}{s^2(s^2+2s+2)} = \frac{s^2}{s^2(s^2+2s+2)} + \frac{s^2+2s+2}{s^2(s^2+2s+2)} \Rightarrow$$

$$X(s) = \frac{1}{s^2} + \frac{1}{(s+1)^2+1}.$$

$$Y(s) = \frac{\triangle_y}{\triangle} = \frac{s^3 - 2s - 2}{s^2(s^2 + 2s + 2)} =$$

$$= \frac{s^3 + s^2}{s^2(s^2 + 2s + 2)} - \frac{s^2 + 2s + 2}{s^2(s^2 + 2s + 2)} = -\frac{1}{s^2} + \frac{s+1}{(s+1)^2 + 1} \Rightarrow$$

$$Y(s) = -\frac{1}{s^2} + \frac{s+1}{(s+1)^2 + 1}.$$

Adică

$$x(t) = (t + e^{-t} \sin t)\sigma(t)$$

şi

$$y(t) = (-t + e^{-t} \cos t)\sigma(t).$$

b) Aplicăm transformata Laplace sistemului diferenţial

$$\begin{cases} \mathcal{L}[x'] = \mathcal{L}[y] - 7\mathcal{L}[x], \\ \mathcal{L}[y'] = -\mathcal{L}[x] - 5\mathcal{L}[y], \end{cases}$$

Cu derivarea originalului avem:

$$\mathcal{L}[x'] = sX(s) - x(0) = sX(s) - 2$$

$$\mathcal{L}[y'] = sY(s) - y(0) = sY(s) - 1.$$

Transformata Laplace a sistemului este

$$\begin{cases} (s+7)X(s) - Y(s) = 2, \\ X(s) + (s+5)Y(s) = 1. \end{cases}$$

Cu regula lui Cramer avem:

$$\triangle = (s+6)^2, \triangle_x = 2s + 11, \triangle_y = s + 5.$$

$$X(s) = \frac{\triangle_x}{\triangle} = \frac{2s+11}{(s+6)^2} = \frac{2}{s+6} - \frac{1}{(s+6)^2}$$

$$Y(s) = \frac{\triangle_y}{\triangle} = \frac{p+5}{(s+6)^2} = \frac{1}{s+6} - \frac{1}{(s+6)^2}.$$

Dar

$$\frac{1}{p-\lambda} \overset{L^{-1}}{\longleftrightarrow} e^{\lambda t}\sigma(t)$$

$$\frac{1}{(p-\lambda)^2} \overset{L^{-1}}{\longleftrightarrow} te^{\lambda t}\sigma(t).$$

În acest caz $\lambda = -6$, deci

$$\begin{cases} x(t) = (2e^{-6t} - te^{-6t})\sigma(t), \\ y(t) = (e^{-6t} - te^{-6t})\sigma(t). \end{cases}$$

Exerciţiul 1.17 a) Dacă f este originalul Laplace şi este de perioadă T atunci:

$$\mathcal{L}\left[f(t)\right](s) = \frac{\int_0^T f(t)e^{-st}dt}{1-e^{-sT}}.$$

Să se exemplifice pentru

$$f(t) = \begin{cases} \sin t, & t \in [0, \pi] \\ 0, & t \in (\pi, 2\pi], \end{cases}$$

periodică de perioadă 2π pe $[0, +\infty)$.

$$\mathcal{L}\left[f(t)\right](s) = \frac{\int_0^{2\pi} f(t)e^{-st}dt}{1-e^{-2\pi s}} =$$

$$= \frac{\int_0^{\pi} \sin te^{-st}dt}{1-(e^{-\pi s})^2}.$$

$$I = \int_0^{\pi} \sin te^{-st}dt = \int_0^{\pi} \sin t \left(\frac{e^{-st}}{-s}\right)' dt =$$

$$= \sin t \frac{e^{-st}}{-s}\Big|_0^\pi - \int_0^\pi \cos t \frac{e^{-st}}{-s} dt =$$

$$= -\int_0^\pi \cos t \left(\frac{e^{-st}}{s^2}\right)' dt = -\cos t \frac{e^{-st}}{s^2}\Big|_0^\pi - \frac{1}{s^2} I =$$

$$= \frac{1}{s^2}\left(e^{-\pi s} + 1\right) - \frac{1}{s^2} I \Rightarrow$$

$$\left(\frac{1}{s^2} + 1\right) I = \frac{1}{s^2}\left(e^{-\pi s} + 1\right) \Rightarrow$$

$$I = \frac{1}{1+s^2}\left(e^{-\pi s} + 1\right).$$

Deci

$$\mathcal{L}\left[f(t)\right](s) = \frac{1+e^{-\pi s}}{(1+s^2)(1+e^{-\pi s})(1-e^{-\pi s})} \Leftrightarrow$$

$$\mathcal{L}\left[f(t)\right](s) = \frac{1}{(1+s^2)(1-e^{-\pi s})}.$$

b) Ecuaţie diferenţială cu argument întârziat:

$$x''(t) - 2x'(t-1) = t\sigma(t),$$

$x(0_+) = x'(0) = 0.$
Folosind derivarea originalului avem:

$$\mathcal{L}\left[x''(t)\right](s) = s^2 X(s) - sX(0) - X'(0) = s^2 X(s).$$

Folosind întârzierea:

$$\mathcal{L}\left[x'(t-1)\right](s) = e^{-s}\mathcal{L}\left[x'(t)\right](s)$$

Folosind din nou derivarea originalului:

$$e^{-s}\mathcal{L}\left[x'(t)\right](s) = se^{-s}X(s).$$

Ecuaţia operaţională:

$$s^2 X(s) - 2se^{-s} X(s) = \frac{1}{s^2}$$

$$X(s)(s^2 - 2se^{-s}) = \frac{1}{s^2}$$

$$X(s) = \frac{1}{s^2 s^2 \left(1 - \frac{2e^{-s}}{s}\right)} = \frac{1}{s^4} \cdot \frac{1}{1 - \frac{2e^{-s}}{s}}.$$

Pentru $\left|\frac{2e^{-s}}{s}\right| < 1$ aplicăm seria geometrică:

$$\frac{1}{1 - \frac{2e^{-s}}{s}} = 1 + \frac{2e^{-s}}{s} + 2^2 \frac{e^{-2s}}{s^2} + \cdots + 2^n \frac{e^{-ns}}{s^n} + \cdots$$

$$X(s) = \frac{1}{s^4} + \frac{2}{s^4} e^{-s} + \frac{2^2}{s^6} e^{-2s} + \cdots + \frac{2^n}{s^{n+4}} e^{-ns} + \cdots$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^{n+1}}\right](s) = \frac{t^n}{n!}\sigma(t), n \in \mathbb{N}.$$

Aplicăm transformata Laplace inversă - recuperarea originalului, folosind întârzierea:

$$x(t) = \frac{t^3}{3!}\sigma(t) + \frac{2}{4!}(t-1)^4\sigma(t-1) + \frac{2^2}{5!}(t-2)^5\sigma(t-2) + \cdots +$$

$$+ \frac{2^n}{(n+3)!}(t-n)^{n+3}\sigma(t-n) + \cdots =$$

$$= \frac{t^3}{3!}\sigma(t) + \sum_{n \geq 1} \frac{2^n}{(n+3)!}(t-n)^{n+3}\sigma(t-n) =$$

$$= \frac{t^3}{3!}\sigma(t) + \sum_{n=1}^{[t]} \frac{2^n}{(n+3)!}(t-n)^{n+3} =$$

$$= \sum_{n=0}^{[t]} \frac{2^n}{(n+3)!}(t-n)^{n+3}.$$

c) Ecuaţie diferenţială:

$$x'' + 2x' + x = \frac{1}{(t+1)e^t}\sigma(t),$$

$x(0) = x'(0) = 0.$

$$\mathcal{L}\left[x''(t)\right](s) = s^2 X(s);$$

$$\mathcal{L}\left[x'(t)\right](s) = sX(s);$$

$$\mathcal{L}\left[x(t)\right](s) = X(s).$$

$$\mathcal{L}\left[x''(t)\right](s) + 2\mathcal{L}\left[x'(t)\right](s) + \mathcal{L}\left[x(t)\right](s) =$$

$$= \mathcal{L}\left[\frac{1}{(t+1)e^t}\sigma(t)\right](s) \Rightarrow$$

$$+2sX(s) + X(s) = \mathcal{L}\left[\frac{1}{(t+1)e^t}\sigma(t)\right](s) \Rightarrow$$

$$(s+1)^2 X(s) = \mathcal{L}\left[\frac{1}{(t+1)e^t}\sigma(t)\right](s) \Rightarrow$$

$$X(s) = \frac{1}{(s+1)^2}\mathcal{L}\left[\frac{1}{(t+1)e^t}\sigma(t)\right](s) =$$

$$= \mathcal{L}\left[te^{-t}\sigma(t)\right](s)\mathcal{L}\left[\frac{1}{(t+1)e^t}\sigma(t)\right](s)$$

$$X(s) = \mathcal{L}\left[te^{-t}\sigma(t) * \frac{1}{(t+1)e^t}\sigma(t)\right](s) \Rightarrow$$

$$x(t) = te^{-t}\sigma(t) * \frac{1}{(t+1)e^t}\sigma(t) = \int_0^t \frac{\tau e^{-\tau}}{(t-\tau+1)e^{t-\tau}}d\tau =$$

$$= e^{-t}\int_0^t \frac{\tau}{t-\tau+1}d\tau = e^{-t}\int_0^t \left(-1 + \frac{t+1}{t-\tau+1}\right)d\tau =$$

$$= e^{-t}\left[-t-(t+1)\ln|t-\tau+1|\big|_0^t\right] =$$
$$= e^{-t}\left[-t+(t+1)\ln(t+1)\right]\sigma(t).$$

Deci
$$x(t) = e^{-t}\left[(t+1)\ln(t+1)-t\right]\sigma(t).$$

d) Ecuaţie diferenţială:

$$x''(t-1)+2x'(t-1)+x(t-1) = \cos t\sigma(t-1),$$

$x(0) = x'(0) = 0.$
Aplicăm derivarea originalului şi întârzierea:

$$\mathcal{L}\left[x''(t-1)\right](s) = e^{-s}\mathcal{L}\left[x''\right](s) = s^2e^{-s}X(s);$$
$$\mathcal{L}\left[x'(t-1)\right](s) = e^{-s}\mathcal{L}\left[x'\right](s) = se^{-s}X(s);$$
$$\mathcal{L}\left[x(t-1)\right](s) = e^{-s}X(s).$$

Astfel,

$$(s^2+2s+1)e^{-s}X(s) = \mathcal{L}\left[\cos t\sigma(t-1)\right](s),$$

$$\mathcal{L}\left[\cos t\cdot\sigma(t-1)\right](s) = e^{-s}\mathcal{L}\left[\cos(t+1)\sigma(t)\right](s) =$$
$$= e^{-s}\mathcal{L}\left[\cos t\cos 1\cdot\sigma(t)-\sin t\sin 1\cdot\sigma(t)\right](s) =$$
$$= e^{-s}\left(\cos 1\frac{s}{s^2+1}-\sin 1\frac{1}{s^2+1}\right) \Rightarrow$$
$$X(s) = \frac{1}{(s+1)^2}\left(\cos 1\frac{s}{s^2+1}-\sin 1\frac{1}{s^2+1}\right) =$$
$$= \cos 1\frac{s}{(s+1)^2(s^2+1)}-\sin 1\frac{1}{(s+1)^2(s^2+1)}.$$

Dar
$$\frac{s}{(s+1)^2(s^2+1)} =$$

$$= \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{1}{(s+1)^2}\right] \overset{L^{-1}}{\longleftrightarrow} \frac{1}{2}\left(\sin t - te^{-t}\right)\sigma(t)$$

şi

$$\frac{-1}{(s+1)^2(s^2+1)} = \frac{1}{2}\left[\frac{1}{s^2+1} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right]$$

$$\overset{L^{-1}}{\longleftrightarrow} \frac{1}{2}\left(\cos t - te^{-t} - e^{-t}\right)\sigma(t).$$

Obţinem:

$$x(t) =$$

$$= \left[\frac{1}{2}\left(\sin t - te^{-t}\right)\cos 1 + \frac{1}{2}\left(\cos t - te^{-t} - e^{-t}\right)\sin 1\right]\sigma(t).$$

e) Ecuaţie diferenţială cu argument întârziat:

$$x''(t) + 2x'(t-2) + x(t-4) = t\sigma(t),$$

$x(0) = x'(0) = 0$.
Aplicăm derivarea originalului şi întârzierea:

$$\mathcal{L}\left[x''(t)\right](s) = s^2 X(s);$$

$$\mathcal{L}\left[x'(t-2)\right](s) = e^{-2s} s X(s);$$

$$\mathcal{L}\left[x(t-4)\right](s) = e^{-4s} X(s).$$

Astfel,

$$(s^2 + 2se^{-2s} + e^{-4s})X(s) = \frac{1}{s^2}$$

$$X(s) = \frac{1}{s^2(s+e^{-2s})^2} = \frac{1}{s^4}\frac{1}{\left(1+\frac{e^{-2s}}{s}\right)^2}.$$

Pentru $\left|\frac{e^{-2s}}{s}\right| < 1$ aplicăm seria binomială:

$$\left(1+\frac{e^{-2s}}{s}\right)^{-2} = 1+\sum_{n=1}^{\infty}\frac{(-2)(-3)\cdots(-1-n)}{n!}\cdot\frac{e^{-2ns}}{s^n} =$$

$$= 1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)!}{n!}\cdot\frac{e^{-2ns}}{s^n} =$$

$$= 1+\sum_{n=1}^{\infty}(-1)^n(n+1)\cdot\frac{e^{-2ns}}{s^n}.$$

$$X(s) = \frac{1}{s^4}+\sum_{n\geq 1}(-1)^n(n+1)\cdot\frac{e^{-2ns}}{s^{n+4}}.$$

Aplicăm întârzierea:

$$x(t) = \frac{t^3}{3!}\sigma(t)+\sum_{n\geq 1}(-1)^n(n+1)\cdot\frac{(t-2n)^{n+3}}{(n+3)!}\sigma(t-2n) \Rightarrow$$

$$x(t) = \sum_{n=0}^{[t]}(-1)^n\frac{(n+1)}{(n+3)!}\cdot(t-2n)^{n+3}.$$

1.3 Exerciţii propuse.

Exemplul 1.18 Calculaţi transformata Laplace pentru:

a)

$$f(t) = (t^2-2)^2\sigma(t);$$

b)

$$f(t) = \cos^2 4t\sigma(t);$$

c)
$$f(t) = e^{-3t} \sin 2t\sigma(t);$$

d)
$$f(t) = t^2 \cos t\sigma(t);$$

e)
$$f(t) = \begin{cases} 1, & t \in (0,1) \\ 2-t, & t \in (1,2) \\ 0, & t > 2. \end{cases}$$
$$= \sigma(t) - \sigma(t-1) + (2-t)(\sigma(t-1) - \sigma(t-2)) =$$
$$= \sigma(t) - (t-1)\sigma(t-1) + (t-2)\sigma(t-2).$$

$$\mathcal{L}(f(t))(s) = F(s) = \frac{1}{s} - e^{-s}\frac{1}{s^2} + e^{-2s}\frac{1}{s^2}.$$

Exerciţiul 1.19 Calculaţi transformatele Laplace pentru următoarele funcţii:

(a) $f(t) = \frac{e^{bt}-e^{at}}{t}\sigma(t);$

(b) $f(t) = \frac{2}{t}(\cos bt - \cos at)\sigma(t);$

(c) $f(t) = \frac{e^{-2t}-1}{t} \sin 3t \cdot \sigma(t);$

(d) $f(t) = \int_0^t (t-s) \sin s ds.$

Exerciţiul 1.20 Folosind transformata Laplace calculaţi integralele improprii:

(a) $\int_0^\infty \frac{e^{-at} \sin bt}{t} dt;$

(b) $\int_0^\infty \frac{\cos bt - \cos at}{t} dt;$

(c) $\int_0^\infty \frac{\sin at - \sin bt}{t} dt$;

(d) $\int_0^\infty \frac{\sin at - \sin bt}{t^2} dt$;

(e) $\int_0^\infty \frac{\sin^3 tx}{x^2} dx$;

(f) $\int_0^\infty \frac{\sin^3 x}{x^2} dx$.

Exerciţiul 1.21 Folosind transformata Laplace rezolvaţi următoarele probleme Cauchy:

(a) $y'' + 4y = \sigma(t-1)$, $y(0) = y'(0) = 0$;

(b) $y'' - y = \cos(t-1)\sigma(t-1)$, $y(0) = y'(0) = 0$;

(c) $y'' + 9y = \sinh(t-1)\sigma(t-1)$, $y(0) = y'(0) = 0$;

(d) $y'' - 3y' + 2y = e^{-2(t-1)}\sigma(t-1)$, $y(0) = y'(0) = 0$;

(e) $y'' + 2y = \sigma(t-2) - \sigma(t-5)$, $y(0) = y'(0) = 0$;

(f) $y'' - 5y' + 6y = e^t\sigma(t)$, $y(0) = -1$, $y'(0) = 1$;

(g) $y'' - 2y' + 2y = t\sigma(t)$, $y(0) = -2$, $y'(0) = 0$;

(h) $y(t) = t\cos 3t \cdot \sigma(t) + \int_0^t (t-s)y(s)ds$;

(i) $y(t) = t\sin 2t \cdot \sigma(t) + \int_0^t \cos 3(t-s)y(s)ds$;

(j) $y(t) - \int_0^t \cosh 2(t-s)y(s)ds = (4 - 4t - 8t^2)\sigma(t)$;

(k) $y(t) + 3\int_0^t y(s)ds = t\sigma(t) - (t-2)\sigma(t-2)$;

(l) $y(t) - 2\int_0^t y(s)ds = \frac{1}{9}(1 - \cos 3t)\sigma(t)$;

(m) $y'' + 3y' + 2y = e^{-3t}\sigma(t)$, $y(0) = -1$, $y'(0) = 2$.

Capitolul 2

Funcţii analitice. Condiţiile Cauchy-Riemann.

2.1 Numere complexe. Funcţii analitice (olomorfe).

2.1.1 Numere complexe.

$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta},$$

$$(x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1).$$

$$r = |z| = \sqrt{a^2 + b^2};\ \theta = \arg z = \arctan\frac{b}{a} + k\pi.$$

$$z^n = r^n e^{in\theta};\ \sqrt[n]{z} = \sqrt[n]{r}e^{i\frac{\theta+2k\pi}{n}}, k \in \{0, 1, ..., n-1\}.$$

$$D_r(z_0) = \{z \in \mathbb{C} |\ |z - z_0| < r\}.$$

Definiţia 2.1 Mulţimea V se numeşte *vecinătate a lui* z_0 dacă $\exists D_r(z_0) \subset V$.

Definiţia 2.2 G se numeşte *mulţime deschisă* dacă $\forall z_0 \in G$, $\exists D_r(z_0) \subset G$ (vecinătate pentru orice punct al ei).

Definiţia 2.3 A se numeşte *mulţime închisă* dacă complementara $\mathcal{C}A =$ este mulţime deschisă.

Definiţia 2.4 G se numeşte *mulţime convexă* dacă $\forall z_1, z_2 \in G, [z_1, z_2] \subset G$, unde

$$[z_1, z_2] = \{z \in \mathbb{C} |\ z = tz_1 + (1-t)z_2, t \in [0,1]\}.$$

Definiţia 2.5 $D \in \mathbb{C}$ se numeşte *domeniu* dacă D este mulţime deschisă şi convexă.

Definiţia 2.6 D domeniu se numeşte *simplu conex* dacă $FrD = \partial D$ este formată dintr-o singură curbă închisă, simplă, fără autointersecţii.

2.1.2 Funcţii analitice (olomorfe).

Fie funcţia

$$f : D \subset \mathbb{C} \to \mathbb{C},\ f(z) = f(x+iy) = u(x,y) + iv(x,y),$$

unde $u, v : D \subset \mathbb{R}^2 \to \mathbb{R}$, $Ref(z) = u(x,y)$,$Imf(z) = v(x,y)$.

Definiţia 2.7 Funcţia $f : D \to \mathbb{C}$, $z_0 \in D$ se numeşte funcţie *derivabilă* (monogenă) în z_0 dacă există

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$

şi este finită.

Definiţia 2.8 Funcţia f se numeşte funcţie *analitică* (olomorfă) în z_0 dacă există V_δ vecinătate a lui z_0 astfel încât există $f'(z)$, în orice $z \in V_\delta$.

Definiţia 2.9 Funcţia f se numeşte funcţie *analitică* pe D dacă este analitică în orice punct din D (putem spune şi derivabilă).

Teorema 2.10 *(Cauchy-Riemann) Funcţia $f : D \subseteq \mathbb{C} \to \mathbb{C}$, $z_0 \in D$, $z_0 = x_0 + iy_0$ este derivabilă în (x_0, y_0) dacă şi numai dacă u, v sunt diferenţiabile în (x_0, y_0) şi sunt îndeplinite condiţiile*

$$\begin{cases} u_x(x_0, y_0) = v_y(x_0, y_0) \\ u_y(x_0, y_0) = -v_x(x_0, y_0). \end{cases} \tag{2.1}$$

Relaţiile (2.1) se numesc condiţiile Cauchy-Riemann.

Teorema 2.11 *(Generalizare Cauchy-Riemann)*

i) Funcţia $f(z) = u + iv$ analitică rezultă:

1. u, v sunt diferenţiabile pe $D \subseteq \mathbb{C}$ şi

2. $\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$

ii) Există u_x, u_y, v_x, v_y continue şi $\begin{cases} u_x = v_y \\ u_y = -v_x, \end{cases}$ atunci funcţia $f(z) = u + iv$ este analitică.

Definiţia 2.12 Funcţia $g(x, y) \in C^1(D)$ *admite conjugată armonică* pe D dacă există $h(x, y) \in C^1(D)$ astfel încât

$$\begin{cases} g_x = h_y \\ g_y = -h_x. \end{cases}$$

Proprietatea 2.13 Dacă funcţia $f(z) = u + iv$ este analitică, atunci $u, v \in C^2(D)$ sunt armonice adică $\triangle u = \triangle v = 0$.

Proprietatea 2.14 Fie $D \subseteq \mathbb{C}$ domeniu simplu conex ($\forall \Gamma \subset D$ curbă închisă, $\triangle$ domeniul delimitat de $\Gamma \Rightarrow \triangle \subset D$). Fie funcţia $g \in C^2(D)$ armonică, atunci ea admite conjugată armonică unic determinată până la o constantă, determinată prin condiţiile Cauchy-Riemann.

Definiţia 2.15 Perechea (g, h) poartă numele de *pereche de funcţii conjugate armonic.*

Teorema 2.16 *O pereche (g, h) de funcţii conjugate armonic determină o funcţie analitică $f = h + ih$ unic determinată până la o constantă.*

Observaţia 2.17 Regulile de derivare pentru funcţiile analitice: sumă, produs, raport, compunere sunt ca în $\mathbb{R}$.

2.2 Funcţii elementare complexe.

2.2.1 Funcţia polinomială.

$$(z^n)' = nz^{n-1}, \forall \in \mathbb{N}^*.$$

Se aplică produsului $(z \cdot z \cdots z)$ regula de derivare de la produs. Polinomul

$$P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

cu coeficienţi complecşi este sumă de funcţii analitice, deci este funcţie analitică şi

$$P'(z) = na_0 z^{n-1} + (n-1)a_1 z^{n-2} + \cdots + a_{n-1}.$$

2.2.2 Funcţia exponenţială.

$$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y) =$$

$$= e^x \cos y + ie^x \sin y,$$

unde

$$\mathrm{Re} e^z = e^x \cos y = u(x, y) \text{ şi } \mathrm{Im} e^z = e^x \sin y = v(x, y).$$

1. $f'(z) = u_x + iv_x = e^x \cos y + ie^x \sin y =$
   $= e^x \cdot e^{iy} = e^{x+iy} = e^z;$

2. $(e^z)' = e^z, \forall z \in \mathbb{C};$

3. $|e^z| = |e^x \cdot e^{iy}| = e^x |\cos y + i \sin y| = e^x \sqrt{\cos^2 y + \sin^2 y} =$
   $= e^x;$

4. $\arg e^z = y;$

5. $e^{z+2\pi i} = e^z \cdot e^{2\pi i} = e^z \cdot (\cos 2\pi + i \sin 2\pi) = e^z, \forall z \in \mathbb{C} \Leftrightarrow$
   $f(z) = e^z$ este funcţie periodică de perioadă 2π;

6. $e^{z_1+z_2} = e^{x_1+x_2+i(y_1+y_2)} = e^{x_1+x_2} \cdot e^{i(y_1+y_2)} =$
   $= e^{x_1} \cdot e^{x_2} [\cos y_1 \cos y_2 - \sin y_1 \sin y_2 +$
   $+ i(\sin y_1 \cos y_2 + \sin y_2 \cos y_1)] =$
   $= e^{x_1}(\cos y_1 + i \sin y_1) \cdot e^{x_2}(\cos y_2 + i \sin y_2) = e^{z_1} \cdot e^{z_2},$
   $\forall z_1, z_2 \in \mathbb{C};$

7. funcţiile trigonometrice reale în funcţie de e^{ix} şi e^{-ix} sunt

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}, x \in \mathbb{R}.$$

2.2.3 Funcţia raţională.

Funcţia

$$f(z) = \frac{P(z)}{Q(z)}$$

este raport de polinoame (care sunt funcţii analitice), deci este funcţie analitică şi

$$f'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q^2(z)},$$

cu z diferit de rădăcinile lui $Q(z)$.

2.2.4 Funcţia multivocă.

Definiţia 2.18 *Funcţia multivocă* este funcţia care ia cel puţin două valori distincte pentru un singur z din domeniul de definiţie.

Exemplu de funcţie multivocă este funcţia logaritm dintr-un număr complex. Fie $z \in \mathbb{C}^*$, definim

$$\mathrm{Ln}z = \{\ln|z| + i(\arg z + 2k\pi) | k \in \mathbb{Z}\}.$$

Dacă îl fixăm pe $k \in \mathbb{Z}$ avem *ramura uniformă* sau determinată notată $\ln_k$ şi definită pe domeniul $D = \mathbb{C}\backslash\{z \in \mathbb{C} | \mathrm{Re}z \leq 0, \mathrm{Im}z = 0\}$ dată de

$$\ln_k z = \ln|z| + i(\arg z + 2k\pi).$$

Dacă avem $k = 0$, obţinem *ramura principală*

$$\ln z = \ln|z| + i\arg z.$$

Pentru orice $k \in \mathbb{Z}$ avem $\ln_k z$ analitică: cu condiţiile Cauchy-Riemann. Pentru $k = 0$ avem

$$\ln z = \ln|z| + i\arg z.$$

$$z = x + iy \Rightarrow |z| = \sqrt{x^2 + y^2}, \arg z = \arctan \frac{y}{x}.$$

$$\text{Re}(\ln z) = u(x, y) = \frac{1}{2} \ln(x^2 + y^2); v(x, y) = \arctan \frac{y}{x}.$$

Obţinem

$$u_x = \frac{x}{x^2 + y^2}; u_y = \frac{y}{x^2 + y^2};$$

$$v_x = \frac{-\frac{y}{x^2}}{1 + \frac{x^2}{y^2}} = \frac{-y}{x^2 + y^2}; v_y = \frac{\frac{1}{x}}{1 + \frac{x^2}{y^2}}.$$

$$f'(z) = (\ln z)' = u_x + iv_x = \frac{x - iy}{(x + iy)(x - iy)} = \frac{1}{z}.$$

$$\text{Ln}z = \{\ln z + i2k\pi | k \in \mathbb{Z}\}.$$

2.2.5 Funcţia putere complexă (aplicaţie multivocă).

$$f(z) = z^\alpha = e^{\alpha \text{Ln} z} = \{e^{\alpha[\ln |z| + i(\arg z + 2k\pi)]} | k \in \mathbb{Z}.\}$$

$$\alpha \in \mathbb{C}^* \backslash \mathbb{Z}.$$

$$f_k(z) = e^{\alpha[\ln |z| + i(\arg z + 2k\pi)]}, k \in \mathbb{Z}.$$

$$f_k'(z) = \frac{\alpha}{z} e^{\alpha[\ln |z| + i(\arg z + 2k\pi)]}, z \in \mathbb{C} \backslash \{z | \text{Re } z \leq 0, \text{Im} z = 0\}.$$

$$f_k(z) = e^{i\alpha 2k\pi} \cdot e^{[\ln |z| + i \arg z]} = e^{i\alpha 2k\pi} \cdot e^{\alpha \ln z} \Rightarrow$$

$$f_k'(z) = e^{i\alpha 2k\pi} \cdot \left(e^{\alpha \ln z}\right)' = \frac{\alpha}{z} e^{i\alpha 2k\pi} \cdot e^{\alpha[\ln |z| + i \arg z]} =$$

$$= \frac{\alpha}{z} e^{\alpha[\ln |z| + i(\arg z + 2k\pi)]} = \frac{\alpha}{z} z^\alpha = \alpha z^{\alpha - 1}.$$

Pentru $\alpha = \frac{1}{n}, n \in \mathbb{N}^*$ avem ramura uniformă

$$f_k(z) = z^{\frac{1}{n}} = \sqrt[n]{z} = \sqrt[n]{|z|} \cdot e^{i \frac{\arg z + 2k\pi}{n}}, k \in \{0, 1, ..., n - 1\}.$$

Funcţia

$$f_k(z) = \sqrt[n]{z} = \{\sqrt[n]{|z|} \cdot e^{i\frac{\arg z + 2k\pi}{n}} | k \in \{0, 1, ..., n-1\}\}$$

are n ramuri (determinări).

2.2.6 Funcţiile trigonometrice complexe (circulare).

1. $\cos z = \frac{e^{iz}+e^{-iz}}{2}$, $\sin z = \frac{e^{iz}-e^{-iz}}{2i}, x \in \mathbb{R}, z \in \mathbb{C} \Rightarrow$
   $e^{iz} = \cos z + i \sin z, z \in \mathbb{C}$

2. $(\cos z)' = i\frac{e^{iz}-e^{-iz}}{2} = -\sin z; (\sin z)' = \cos z;$

3. Sunt periodice, de perioadă $T = 2k\pi$, $k \in \mathbb{Z}$, deoarece e^{iz} este periodică de perioadă $2k\pi$;

4. $e^{i(z_1 \pm z_2)} = \cos(z_1 \pm z_2) + i \sin(z_1 \pm z_2) = e^{iz_1} \cdot e^{\pm iz_2} =$
   $= (\cos z_1 + i \sin z_1)(\cos z_2 \pm i \sin z_2) \Rightarrow$

$$\begin{cases} \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \\ \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1, \\ \sin ix = \frac{e^{-x}-e^{x}}{2i} = i \sinh x, \\ \cos ix = \frac{e^{x}+e^{-x}}{2} = \cosh x, \\ \cosh^2 y - \sinh^2 y = 1, \\ \cosh^2 y + \sinh^2 y = \cosh 2y. \end{cases}$$

5. $\cos z = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy =$
   $= \cos x \cosh y - i \sin x \sinh y;$

6. $\sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y;$

7. $|\cos z| = \sqrt{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y} \overset{\cosh^2 y = \sinh^2 y + 1}{=\!=\!=\!=\!=\!=\!=\!=\!=}$
   $= \sqrt{\cos^2 x + \sinh^2 y}$. Cu relaţiile
$$\begin{cases} \cos^2 x = \frac{1+\cos 2x}{2}, \\ \sinh^2 y = \frac{\cosh 2y - 1}{2}, \end{cases}$$
   obţinem
8. $|\cos z|^2 = \frac{1}{2}(\cos 2x + \cosh 2y) \overset{y\to\infty}{-\!\!\!\longrightarrow} +\infty$.
   Deci $\cos z$ pentru $y \to \infty$ este nemărginită.
9. $|\sin z|^2 = \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \overset{\cosh^2 y = \sinh^2 y + 1}{=\!=\!=\!=\!=\!=\!=\!=\!=}$
   $= \sin^2 x + \sinh^2 y = \frac{1}{2}(1 - \cos 2x + \cosh 2y - 1) =$
   $= \frac{1}{2}(\cosh 2y - \cos 2x) \overset{y\to\infty}{-\!\!\!\longrightarrow} +\infty$.
   Deci $\sin z$ pentru $y \to \infty$ este nemărginită. Pentru y suficient de mare avem $|\sin z|^2 \simeq \frac{e^{2y}}{4}$.
10. Zerourile pentru $\sin z$ şi $\cos z$:
$$\begin{cases} \sin z = 0 \Leftrightarrow e^{2iz} = 1 \\ \Rightarrow 2iz_k \in \mathrm{Ln}1 = \{i2k\pi | k \in \mathbb{Z}\} \Rightarrow z_k = k\pi, \\ \cos z = 0 \Leftrightarrow e^{2iz} = -1 \\ \Rightarrow 2iz_k \in \mathrm{Ln}(-1) = \{i(\pi + 2k\pi) | k \in \mathbb{Z}\} \Rightarrow z_k = \frac{\pi}{2} + k\pi. \end{cases}$$

2.2.7 Funcţiile hiperbolice complexe.

1. $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}, z \in \mathbb{C} \Rightarrow$
   $e^{-z} = \cosh z - \sinh z$, $e^z = \cosh z + \sinh z$;
2. $(\sinh z)' = \cosh z; (\cosh z)' = \sinh z$;
3. $e^{z+2\pi i} = e^z \Rightarrow \sinh(z + 2\pi i) = \sinh z$, $\cosh(z + 2\pi i) = \cosh z, \forall z \in \mathbb{C}$
   $T = 2\pi i$ perioada pricipală pentru sinh, cosh;

4. $e^{z_1+z_2} = \cosh(z_1+z_2) + \sinh(z_1+z_2) = e^{z_1} \cdot e^{z_2} =$
$= (\cosh z_1 + \sinh z_1)(\cosh z_1 + \sinh z_2)$

$$\begin{cases} \sin iz = \frac{e^{-z}-e^{z}}{2i} = i\sinh z, \\ \cos iz = \frac{e^{z}+e^{-z}}{2} = \cosh z, \\ \cosh(z_1+z_2) = \cos i(z_1+z_2) = \\ \qquad = \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 = \\ \qquad = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2, \\ \cosh(z_1-z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2, \\ \sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \sinh z_2 \cosh z_1, \\ \sinh 2z = 2\sinh z \cosh z, \\ \cosh 2z = \cosh^2 z + \sinh^2 z. \end{cases}$$

5. $\cosh z = \cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy =$
$= \cosh x \cos y + i \sinh x \sin y$;

6. $\sinh z = \sinh(x+iy) = \sinh x \cosh y + \cosh x \sinh iy =$
$\sinh x \cos y + i\cosh x \sin y$;

7. $|\cosh z|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y \xlongequal{\cosh^2 y = 1+\sinh^2 y}$
$= \cos^2 y + \sinh^2 x = \frac{1+\cos 2y}{2} + \frac{\cosh x - 1}{2} = \frac{1}{2}(\cosh 2x +$
$\cos 2y) \xrightarrow{x\to\infty} +\infty$.

8. $|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \xlongequal{\cosh^2 y = 1+\sinh^2 y}$
$= \sin^2 y + \sinh^2 x = \frac{1}{2}(\sinh 2x - \cos 2y) \xrightarrow{y\to\infty} +\infty$.
Deci $\sin z$ pentru $y \to \infty$ este nemărginită. Pentru y suficient de mare avem $|\sin z|^2 \simeq \frac{e^{2y}}{4}$.

9. Zerourile pentru $\sinh z$ şi $\cosh z$:

$$\begin{cases} \sinh z = 0 \Leftrightarrow e^{2z} = 1 \\ \Rightarrow 2z_k \in \mathrm{Ln}1 = \{2k\pi i | k \in \mathbb{Z}\} \Rightarrow z_k = k\pi i, \\ \cosh z = 0 \Leftrightarrow e^{2z} = -1 \\ \Rightarrow 2z_k \in \mathrm{Ln}(-1) = \{(\pi + 2k\pi)i | k \in \mathbb{Z}\} \Rightarrow z_k = \left(\frac{\pi}{2} + k\pi\right) i. \end{cases}$$

1. $\sinh z = \frac{e^z - e^{-z}}{2}$, $\cosh z = \frac{e^z + e^{-z}}{2}, z \in \mathbb{C} \Rightarrow$
   $e^{-z} = \cosh z - \sinh z$, $e^z = \cosh z + \sinh z$;

2. $(\sinh z)' = \cosh z; (\cosh z)' = \sinh z;$

3. $e^{z+2\pi i} = e^z \Rightarrow \sinh(z + 2\pi i) = \sinh z$, $\cosh(z + 2\pi i) = \cosh z, \forall z \in \mathbb{C}$
   $T = 2\pi i$ perioada pricipală pentru sinh, cosh;

4. $e^{z_1+z_2} = \cosh(z_1 + z_2) + \sinh(z_1 + z_2) = e^{z_1} \cdot e^{z_2} =$
   $= (\cosh z_1 + \sinh z_1)(\cosh z_1 + \sinh z_2)$

$$\begin{cases} \sin iz = \frac{e^{-z} - e^{z}}{2i} = i \sinh z, \\ \cos iz = \frac{e^{z} + e^{-z}}{2} = \cosh z, \\ \cosh(z_1 + z_2) = \cos i(z_1 + z_2) = \\ \qquad = \cos iz_1 \cos iz_2 - \sin iz_1 \sin iz_2 = \\ \qquad = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2, \\ \cosh(z_1 - z_2) = \cosh z_1 \cosh z_2 - \sinh z_1 \sinh z_2, \\ \sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \sinh z_2 \cosh z_1, \\ \sinh 2z = 2 \sinh z \cosh z, \\ \cosh 2z = \cosh^2 z + \sinh^2 z. \end{cases}$$

5. $\cosh z = \cosh(x + iy) = \cosh x \cosh iy + \sinh x \sinh iy =$
   $= \cosh x \cos y + i \sinh x \sin y;$

6. $\sinh z = \sinh(x + iy) = \sinh x \cosh y + \cosh x \sinh iy = \sinh x \cos y + i \cosh x \sin y;$

7. $|\cosh z|^2 = \cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y \overset{\cosh^2 y = 1 + \sinh^2 y}{=\!=\!=\!=\!=\!=\!=\!=\!=}$
   $= \cos^2 y + \sinh^2 x = \frac{1+\cos 2y}{2} + \frac{\cosh x - 1}{2} = \frac{1}{2}(\cosh 2x + \cos 2y) \xrightarrow{x\to\infty} +\infty.$

8. $|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y \overset{\cosh^2 y = 1 + \sinh^2 y}{=\!=\!=\!=\!=\!=\!=\!=\!=}$
   $= \sin^2 y + \sinh^2 x = \frac{1}{2}(\sinh 2x - \cos 2y) \xrightarrow{y\to\infty} +\infty.$

Tabela 2.1: Formule.

$e^{iz} = \cos z + i \sin z, z \in \mathbb{C};$	$e^z = \cosh z + \sinh z;$
$(\cos z)' = -\sin z;$	$(\cosh z)' = \sinh z;$
$T = 2k\pi;$	$T = 2k\pi i;$
$e^{i(z_1+z_2)} =$	$e^{z_1+z_2} =$
$= e^{iz_1} \cdot e^{iz_2};$	$= e^{z_1} \cdot e^{z_2};$
$\cos(z_1 \pm z_2) =$	$\cosh(z_1 \pm z_2) =$
$= \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2;$	$= \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2;$
$\sin(z_1 \pm z_2) =$	$\sinh(z_1 \pm z_2) =$
$= \sin z_1 \cos z_2 \pm \sin z_2 \cos z_1;$	$= \sinh z_1 \cosh z_2 \pm \sinh z_2 \cosh z_1;$
$\cos z =$	$\cosh z =$
$= \cos x \cosh y - i \sin x \sinh y;$	$= \cosh x \cos y + i \sinh x \sin y;$
$\sin z =$	$\sinh z =$
$= \sin x \cosh y + i \cos x \sinh y;$	$= \sinh x \cos y + i \cosh x \sin y;$
$\lvert \cos z\rvert =$	$\lvert \cosh z\rvert =$
$= \sqrt{\cos^2 x + \sinh^2 y};$	$= \sqrt{\frac{1}{2}(\cosh 2x + \cos 2y)};$
$\lvert \sin z\rvert =$	$\lvert \sinh z\rvert =$
$= \sqrt{\frac{1}{2}(\cosh 2y - \cos 2x)};$	$= \frac{1}{2}(\sinh 2x - \cos 2y);$
Zerourile:	Zerourile:
$z_k = k\pi, z_k = \frac{\pi}{2} + k\pi;$	$z_k = k\pi i, z_k = \left(\frac{\pi}{2} + k\pi\right) i;$
$\sin 2z = 2 \sin z \cos z;$	$\sinh 2z = 2 \sinh z \cosh z;$
$\cos 2z = \cos^2 z - \sin^2 z;$	$\cosh 2z = \cosh^2 z + \sinh^2 z;$
Domeniul pentru $\tan z$:	Domeniul pentru $\tanh z$:
$\mathbb{C}\backslash\{\frac{\pi}{2} + k\pi \mid k \in \mathbb{Z}\};$	$\mathbb{C}\backslash\{\left(\frac{\pi}{2} + k\pi\right) i \mid k \in \mathbb{Z}\};$
$\cot z : \mathbb{C}\backslash\{k\pi \mid k \in \mathbb{Z}\} \to \mathbb{C};$	$\coth z : \mathbb{C}\backslash\{k\pi i \mid k \in \mathbb{Z}\} \to \mathbb{C};$
$\tan z = \frac{\sin z}{\cos z};$	$\tanh = \frac{\sinh z}{\cosh z};$
$\cot z = \frac{\cos z}{\sin z};$	$\coth = \frac{\cosh z}{\sinh z};$
$(\tan z)' = \frac{1}{\cos^2 z};$	$(\tanh z)' = \frac{1}{\cosh^2 z};$
$(\cot z)' = \frac{-1}{\sin^2 z}.$	$(\coth z)' = \frac{-1}{\sinh^2 z}.$

Deci $\sin z$ pentru $y \to \infty$ este nemărginită. Pentru y suficient de mare avem $|\sin z|^2 \simeq \frac{e^{2y}}{4}$.

9. Zerourile pentru $\sinh z$ şi $\cosh z$:

$$\begin{cases} \sinh z = 0 \Leftrightarrow e^{2z} = 1 \\ \Rightarrow 2z_k \in \mathrm{Ln}1 = \{2k\pi i | k \in \mathbb{Z}\} \Rightarrow z_k = k\pi i, \\ \cosh z = 0 \Leftrightarrow e^{2z} = -1 \\ \Rightarrow 2z_k \in \mathrm{Ln}(-1) = \{(\pi + 2k\pi)i | k \in \mathbb{Z}\} \Rightarrow z_k = \left(\frac{\pi}{2} + k\pi\right) i. \end{cases}$$

2.3 Exerciţii rezolvate.

Exerciţiul 2.19 Aflaţi funcţia analitică $f = u + iv$ dacă

$$u(x, y) = e^{x^2-y^2} \cos 2xy.$$

Soluţie. Arătăm că $\triangle u = u_{xx} + u_{yy} = 0$.

$$u_x = 2xe^{x^2-y^2} \cos 2xy - 2ye^{x^2-y^2} \sin 2xy;$$

$$u_y = -2ye^{x^2-y^2} \cos 2xy - 2xe^{x^2-y^2} \sin 2xy;$$

$$u_{xx} = (4x^2+2)e^{x^2-y^2} \cos 2xy - 8xye^{x^2-y^2} \sin 2xy - 4y^2e^{x^2-y^2} \cos 2xy =$$

$$= (4x^2 - 4y^2 + 2)e^{x^2-y^2} \cos 2xy - 8xye^{x^2-y^2} \sin 2xy.$$

$$u_{yy} = (-4x^2 + 4y^2 - 2)e^{x^2-y^2} \cos 2xy + 8xye^{x^2-y^2} \sin 2xy.$$

Cum $\triangle u = 0 \Rightarrow u$ este armonică, rezultă că există v conjugată armonică a lui u, adică:

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$$

Deci

$$v_x = -u_y = 2ye^{x^2-y^2}\cos 2xy + 2xe^{x^2-y^2}\sin 2xy$$

şi integrăm în raport cu x:

$$v(x,y) = \int v_x dx =$$

$$= \int 2ye^{x^2-y^2}\cos 2xy dx + \int 2xe^{x^2-y^2}\sin 2xy dx =$$

$$= \int e^{x^2-y^2}(\sin 2xy)_x dx + \int 2xe^{x^2-y^2}\sin 2xy dx \stackrel{\text{int.părţi}}{=}$$

$$= e^{x^2-y^2}\sin 2xy dx - \int 2xe^{x^2-y^2}\sin 2xy dx +$$

$$+ \int 2xe^{x^2-y^2}\sin 2xy dx + K(y) =$$

$$= e^{x^2-y^2}\sin 2xy dx + K(y).$$

$$v_y = u_x \Leftrightarrow (e^{x^2-y^2}\sin 2xy)_y + K'(y) =$$

$$= 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy$$

$$\Leftrightarrow -2ye^{x^2-y^2}\sin 2xy + 2xe^{x^2-y^2}\cos 2xy + K'(y) =$$

$$= 2xe^{x^2-y^2}\cos 2xy - 2ye^{x^2-y^2}\sin 2xy \Leftrightarrow$$

$$K'(y) = 0 \Rightarrow K(y) = k.$$

Deci

$$v(x,y) = e^{x^2-y^2}\sin 2xy + k,$$

de unde obţinem funcţia f:

$$f(x,y) = u(x,y) + iv(x,y) =$$

$$= e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy + ik =$$
$$= e^{x^2-y^2}(\cos 2xy + i\sin 2xy) + ik = e^{x^2-y^2}e^{i2xy} + ik =$$
$$= e^{x^2-y^2+i2xy} + ik = e^{(x+iy)^2} + ik \overset{z=x+iy}{=} e^{z^2} + ik.$$

Exerciţiul 2.20 Aflaţi funcţia analitică $f = u + iv$ dacă

$$v(x, y) = 3\cosh x \sin y + \cos x \sinh y.$$

Soluţie. Avem:

$$v_x = 3\sinh x \sin y - \sin x \sinh y;$$
$$v_y = 3\cosh x \cos y + \cos x \cosh y;$$
$$v_{xx} = 3\cosh x \sin y - \cos x \sinh y;$$
$$v_{yy} = -3\cosh x \sin y + \cos x \sinh y;$$

Cum $\triangle v = v_{xx} + v_{yy} = 0 \Rightarrow v$ este armonică, rezultă că există u conjugată armonică a lui v, adică:

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$$

Deci

$$u_x = 3\cosh x \cos y + \cos x \cosh y$$

şi integrăm în raport cu x:

$$u(x, y) = \int u_x dx = 3\cos y \int \cosh x dx + \cosh y \int \cos x dx =$$
$$= 3\sinh x \cos y + \sin x \cosh y + K(y) \Rightarrow$$

$$u_y = -3\sinh x \sin y + \sin x \sinh y + K'(y) =$$

$$= -v_x = -3\sinh x \sin y + \sin x \sinh y \Rightarrow$$

$$K'(y) = 0 \Rightarrow K(y) = k.$$

Deci

$$u(x,y) = 3\sinh x \cos y + \sin x \cosh y + k,$$

şi

$$v(x,y) = 3\cosh x \sin y + \cos x \sinh y,$$

de unde obţinem funcţia f:

$$f(x,y) = u(x,y) + iv(x,y) =$$

$$= 3(\sinh x \cos y + i\cosh x \sin y) + (\sin x \cosh y + i\cos x \sinh y) + k =$$

$$= 3\sinh z + \sin z + k.$$

Exerciţiul 2.21 Aflaţi funcţia analitică $f = u + iv$ dacă

$$u(x,y) = 2\sinh x \cos y + 3\sin x \cosh y.$$

Soluţie. Avem:

$$u_x = 2\cosh x \cos y + 3\cos x \cosh y;$$

$$u_y = -2\sinh x \sin y + 3\sin x \sinh y;$$

$$u_{xx} = 2\sinh x \cos y - 3\sin x \cosh y;$$

$$u_{yy} = -2\sinh x \cos y + 3\sin x \cosh y;$$

Cum $\triangle u = u_{xx} + u_{yy} = 0 \Rightarrow u$ este armonică, rezultă că există v conjugată armonică a lui u, adică:

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$$

Deci
$$v_y = 2\cosh x \cos y + 3\cos x \cosh y$$
şi integrăm în raport cu y:
$$v(x,y) = \int v_y dy = 2\cosh x \int \cos y dy + 3\cos x \int \cosh y dy =$$
$$= 2\cosh x \sin y + 3\cos x \sinh y + K(x).$$

$$v_x = 2\sinh x \sin y - 3\sin x \sinh y + K'(x),$$
Dar
$$K'(y) = 0 \Rightarrow K(y) = k \Rightarrow$$
$$v_x = 2\sinh x \sin y - 3\sin x \sinh y + k$$

Deci
$$v(x,y) = 2\cosh x \sin y + 3\cos x \sinh y + k$$
şi
$$u(x,y) = 2\sinh x \cos y + 3\sin x \cosh y,$$
de unde obţinem funcţia f:
$$f(x,y) = u(x,y) + iv(x,y) =$$
$$= 2(\sinh x \cos y + i\cosh x \sin y) + 3(\sin x \cosh y + i\cos x \sinh y) + ik =$$
$$= 2\sinh z + 3\sin z + ik.$$

Exerciţiul 2.22 Să se găsească funcţia $f = u + iv$ analitică dacă se cunoaşte

a) $u(x,y) = \frac{1-x^2-y^2}{(1+x)^2+y^2}$;

b) $v(x,y) = e^{-y}\sin x$;

c) $u(x,y) = \cosh x \cos y$.

Soluţie.
a) Avem:

$$u_x = \frac{-2x[(1+x)^2+y^2] - 2(1+x)(1-x^2-y^2)}{[(1+x)^2+y^2]^2} =$$

$$= \frac{-2x(1+x)^2 - 2xy^2 - 2(1-x^2-y^2+x-x^3-xy^2)}{[(1+x)^2+y^2]^2} =$$

$$= \frac{-2x(1+2x+x^2) - 2xy^2 - 2 + 2x^2 + 2y^2 - 2x + 2x^3 + 2xy^2)}{[(1+x)^2+y^2]^2} =$$

$$= \frac{-2x^2-4x-2+2y^2}{[(1+x)^2+y^2]^2} = -2\frac{(1+x)^2-y^2}{[(1+x)^2+y^2]^2}.$$

$$u_y = \frac{-2y(1+x)^2 - 2y^3 - 2y(1-x^2-y^2)}{[(1+x)^2+y^2]^2} =$$

$$= \frac{-2y-4xy-2yx^2-2y^3-2y+x^2y+3y^3}{[(1+x)^2+y^2]^2} =$$

$$= -4\frac{y(x+1)}{[(1+x)^2+y^2]^2}.$$

La fel se calculează u_{xx}, $u_{yy} \Rightarrow \triangle u = u_{xx} + u_{yy} = 0 \Rightarrow v$ este armonică, rezultă că există v conjugată armonică a lui u, adică:

$$\begin{cases} u_x = v_y \\ u_y = -v_x. \end{cases}$$

Folosim o altă metodă

$$f'(z) = u_x + iv_x = u_x - iu_y = -2\frac{(1+x)^2 - y^2 - 2iy(x+1)}{[(1+x)^2+y^2]^2} =$$

$$= -2\frac{(x+1-iy)^2}{(x+1-iy)^2(x+1+iy)^2} = \frac{-2}{(x+iy+1)^2} \overset{z=x+iy}{=} \frac{-2}{(z+1)^2}.$$

Integrăm, ţinând cont că în complex avem aceleaşi primitive ca în real, şi obţinem

$$f(z) = -2\int \frac{dz}{(z+1)^2} = \frac{2}{(z+1)} + C.$$

b) Ştim că: $v(x,y) = e^{-y}\sin x$. Atunci

$$v_x = e^{-y}\cos x \Rightarrow v_{xx} = -e^{-y}\sin x$$

şi

$$v_y = -e^{-y}\sin x \Rightarrow v_{yy} = e^{-y}\sin x$$

deci $\triangle v = v_{xx} + v_{yy} = 0 \Rightarrow v$ este armonică, rezultă că există u conjugată armonică a lui v, adică:

$$\begin{cases} u_x = v_y = -e^{-y}\sin x \\ u_y = -v_x = -e^{-y}\cos x. \end{cases}$$

Integrăm în raport cu x:

$$u(x,y) = -e^{-y}\int \sin x dx = e^{-y}\cos x + K(y) \Rightarrow$$

$$u_y = -e^{-y}\cos x + K'(y) = -v_x = -e^{-y}\cos x \Rightarrow$$

$$K'(y) = 0 \Rightarrow K(y) = k \Rightarrow u(x,y) = e^{-y}\cos x + k \Rightarrow$$

$$f(z) = u + iv = e^{-y}(\cos x + i\sin x) + k = e^{-y}e^{ix} + k =$$

$$= e^{ix+i^2y} + k = e^{i(x+iy)} + k \Rightarrow$$

$$f(z) = e^{iz} + k.$$

c) Avem: $u(x, y) = \cosh x \cos y$. Atunci

$$u_x = \sinh x \cos y \Rightarrow u_{xx} = \cosh x \cos y$$

şi

$$u_y = -\cosh x \sin y \Rightarrow u_{yy} = -\cosh x \cos y$$

deci $\triangle u = u_{xx} + u_{yy} = 0 \Rightarrow u$ este armonică, rezultă că există v conjugată armonică a lui u, adică:

$$\begin{cases} u_x = v_y = \sinh x \cos y \\ u_y = -v_x = -\cosh x \sin y. \end{cases}$$

Integrăm în raport cu y:

$$v(x, y) = \int v_y dy = \sinh x \int \cos y dy = \sinh x \sin y + K(x) \Rightarrow$$

$$v_x = \cosh x \sin y + K'(x) = -u_y = \cosh x \sin y \Rightarrow$$

$$K'(x) = 0 \Rightarrow K(x) = k \Rightarrow v(x, y) = \sinh x \sin y + k \Rightarrow$$

$$f(z) = u + iv = \cosh x \cos y + i \sinh x \sin y + ik =$$

$$= \cosh x \cosh iy + \sinh x \sinh iy + ik = \cosh(x + iy) + ik \Rightarrow$$

$$f(z) = \cosh z + ik.$$

Exerciţiul 2.23 (funcţii elementare complexe) Rezolvaţi ecuaţia

$$\sin z = 1 + i.$$

Soluţie.

$$e^{iz} - e^{-iz} = 2i(1 + i) = -2 + 2i = -2(1 - i)$$

Înmulţim relaţia cu e^{iz} şi găsim

$$(e^{iz})^2 - 1 = -2(1-i)e^{-iz} \Leftrightarrow (e^{iz})^2 - 1 + 2(1-i)e^{-iz} = 0 \Leftrightarrow$$

$$e^{iz} = \frac{-2(1-i) \pm \sqrt{4(1-i)^2 + 4}}{2} = -1 + i \mp \sqrt{1-2i} \Rightarrow$$

$$iz_k \in Ln(-1 + i \pm \sqrt{1-2i}).$$

Calculăm $\sqrt{1-2i}$. Avem

$$\sqrt{1-2i} = a + ib \Leftrightarrow 1 - 2i = a^2 - b^2 + i2ab \Rightarrow$$

$$\begin{cases} a^2 - b^2 = 1 \\ ab = 1 \end{cases} \Leftrightarrow \begin{cases} b = -\frac{1}{a} \\ a^2 - \frac{1}{a^2} = 1 \end{cases} \Leftrightarrow \begin{cases} a^4 - a^2 - 1 = 0 \\ a^2 = t \end{cases}$$

Rezultă ecuaţia în necunoscuta t:

$$t^2 - t - 1 = 0$$

cu soluţiile:

$$t_{1,2} = \frac{1 \pm \sqrt{5}}{2},$$

de unde obţinem

$$a^2 = t_1 = \frac{1 + \sqrt{5}}{2} > 0,$$

soluţia $t_2 < 0$ şi nu corespunde. Astfel

$$a_{1,2} = \pm\sqrt{\frac{1+\sqrt{5}}{2}} \text{ şi } b_{1,2} = \mp\frac{1}{\sqrt{\frac{1+\sqrt{5}}{2}}} = \mp\sqrt{\frac{\sqrt{5}-1}{2}}.$$

Deci

$$\left.\begin{array}{r} \sqrt{1-2i} = \begin{cases} \sqrt{\frac{1+\sqrt{5}}{2}} - i\sqrt{\frac{\sqrt{5}-1}{2}} \\ -\sqrt{\frac{1+\sqrt{5}}{2}} + i\sqrt{\frac{\sqrt{5}-1}{2}} \end{cases} \\ \mathrm{Ln}z = \{\ln|z| + i(\arg z + 2k\pi) | k \in \mathbb{Z}\} \\ \Rightarrow \mathrm{Ln}z = \{\ln z + i2k\pi | k \in \mathbb{Z}\}. \end{array}\right\} \Rightarrow$$

$$z_k = -i\text{Ln}\left[(-1+i\mp\sqrt{1-2i})\right] =$$

$$= -i\text{Ln}\left[\left(-1\pm\sqrt{\frac{1+\sqrt{5}}{2}}\right)+i\left(1\mp\sqrt{\frac{\sqrt{5}-1}{2}}\right)\right] =$$

$$= -i\left\{\ln\left(\left(-1\pm\sqrt{\frac{1+\sqrt{5}}{2}}\right)+i\left(1\mp\sqrt{\frac{\sqrt{5}-1}{2}}\right)\right)+\right.$$

$$\left.+2k\pi i|k\in\mathbb{Z}\right\}$$

$$\Rightarrow z_k = -i\ln\left(\left(-1\pm\sqrt{\frac{1+\sqrt{5}}{2}}\right)+i\left(1\mp\sqrt{\frac{\sqrt{5}-1}{2}}\right)\right)+$$

$$+2k\pi|k\in\mathbb{Z}.$$

Exerciţiul 2.24 (funcţii elementare complexe) Să se determine mulţimea de definiţie pentru funcţia

$$f(z)=\ln\frac{z+1+i}{z+1-2i}.$$

Soluţie. Funcţia

$$\ln z:\mathbb{C}\backslash\{z\in\mathbb{C}|\text{Re}z\leq 0,\ \text{Im}z=0\}\to\mathbb{R};$$

Deci

$$f(z)=\ln\frac{z+1+i}{z+1-2i}$$

este definită pe mulţimea

$$\mathbb{C}\backslash\{z\in\mathbb{C}|\text{Re}\frac{z+1+i}{z+1-2i}\leq 0,\ \text{Im}\frac{z+1+i}{z+1-2i}=0\}.$$

Determinăm partea reală:

$$\mathrm{Re}\frac{z+1+i}{z+1-2i}=\mathrm{Re}\frac{(x+1)+i(y+1)}{(x+1)+i(y-2)}=$$

$$=\mathrm{Re}\frac{[(x+1)+i(y+1)][(x+1)-i(y-2)]}{(x+1)^2+(y-2)^2}=$$

$$=\frac{(x+1)^2+(y+1)(y-2)}{(x+1)^2+(y-2)^2}\leq 0 \Leftrightarrow$$

$$x^2+2x+1+y^2-y-2\leq 0 \Leftrightarrow (x+1)^2+\left(y-\frac{1}{2}\right)^2\leq\left(\frac{3}{2}\right)^2$$

ecuaţie ce reprezintă interiorul cercului centrat în $\left(-1,\frac{1}{2}\right)$ şi de rază $\frac{3}{2}$ care în complex se scrie

$$\left|z+1-\frac{i}{2}\right|\leq\frac{3}{2}.$$

Determinăm partea imaginară:

$$\mathrm{Im}\frac{z+1+i}{z+1-2i}=0\Leftrightarrow (x+1)(y+1)-(x+1)(y-2)=0\Leftrightarrow$$

$$(x+1)(y+1-y+2)=0\Leftrightarrow$$

$$3(x+1)=0\Rightarrow x=-1\Leftrightarrow \mathrm{Re}z=-1.$$

Deci, domeniul de definiţie pentru funcţia

$$f(z)=\ln\frac{z+1+i}{z+1-2i}$$

este domeniul

$$D=\mathbb{C}\setminus\left\{\left|z+1-\frac{i}{2}\right|\leq\frac{3}{2},\mathrm{Re}z=-1\right\}.$$

Exerciţiul 2.25 (funcţii elementare complexe) Calculaţi $e^{\sqrt{i}}$.
Soluţie.

$$\sqrt{i} = \sqrt{e^{i\frac{\pi}{2}}} = e^{i\frac{\frac{\pi}{2}+2k\pi}{2}}, k \in \{0,1\} \Rightarrow$$

$$\sqrt{i} = \begin{cases} e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} \\ e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}. \end{cases}$$

Deci

$$e^{\sqrt{i}} = \begin{cases} e^{\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}} \\ e^{-\frac{1}{\sqrt{2}}-i\frac{1}{\sqrt{2}}} \end{cases} = \begin{cases} e^{\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right), \\ e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right). \end{cases}$$

Exerciţiul 2.26 (funcţii elementare complexe) Demonstraţi egalităţile următoare

a) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \forall z_1, z_2 \in \mathbb{C}$;

b) $\sinh^2 z = \frac{\cosh 2z - 1}{2}, \forall z \in \mathbb{C}$;

Soluţie.
a) Avem:

$$e^{i(z_1 \pm z_2)} = e^{iz_1}e^{\pm iz_2} \overset{Euler}{=}$$

$$\cos(z_1 \pm z_2) + i\sin(z_1 \pm z_2) = (\cos z_1 + i\sin z_1)(\cos z_2 \mp i\sin z_2) \Leftrightarrow$$

$$\begin{cases} \cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2 \\ \sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2. \end{cases}$$

b) Calculăm:

$$\frac{\cosh 2z - 1}{2} = \frac{\frac{e^{2z}+e^{-2z}}{2} - 1}{2} =$$

$$= \frac{(e^z)^2 + (e^{-z})^2 - 2e^z e^{-z}}{4} =$$

$$= \left(\frac{e^z - e^{-z}}{2}\right)^2 = \sinh^2 z, \forall z \in \mathbb{C}.$$

Exerciţiul 2.27 (funcţii elementare complexe) Găsiţi mulţimea de definiţie pentru funcţia

$$f(z) = \ln \frac{z+i}{iz-1}.$$

Notăm Ln prin ln .

Soluţie.

$$D = \mathbb{C} \backslash \left\{ \mathrm{Re} \frac{z+i}{iz-1} \leq 0,\ \mathrm{Im} \frac{z+i}{iz-1} = 0 \right\}.$$

Deci

$$\frac{z+i}{iz-1} = \frac{x+1+iy}{i(x+iy)-1} = \frac{(x+1+iy)(-y-1-ix)}{(-y-1)^2 + x^2}.$$

Determinăm partea reală:

$$\mathrm{Re} \frac{z+i}{iz-1} = (x+1)(-y-1) + xy \leq 0 \Leftrightarrow -x-y-1 \leq 0 \Leftrightarrow$$

$$x+y+1 \geq 0.$$

Determinăm partea imaginară:

$$\mathrm{Im} \frac{z+i}{iz-1} = 0 \Leftrightarrow -y(y+1) - x(x+1) = 0 \Leftrightarrow x^2 + x + y^2 + y = 0 \Rightarrow$$

$$\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{2}.$$

Adică domeniul de definiţie este

$$D = \mathbb{C} \backslash \left\{ \mathrm{Re} z + \mathrm{Im} z + 1 \geq 0, \left| z - \left(-\frac{1}{2} - \frac{i}{2} \right) \right| = \frac{1}{\sqrt{2}} \right\}.$$

Exerciţiul 2.28 (funcţii elementare complexe) Găsiţi soluţiile ecuaţiei:

$$\cos z = i.$$

Soluţie. Avem:

$$e^{iz} + e^{-iz} = 2i \overset{e^{iz}=t}{=} t^2 - 2it + 1 = 0 \Rightarrow$$

$$t_{1,2} = \frac{2i \mp \sqrt{-8}}{2} = i \mp \sqrt{2}i = i(1 \pm \sqrt{2}).$$

$$e^{iz} = i(1+\sqrt{2}) \Rightarrow iz_k \in \left\{ \ln(1+\sqrt{2}) + i\left(\frac{\pi}{2} + 2k\pi\right) | k \in \mathbb{Z} \right\} \Rightarrow$$

$$z_k = \frac{\pi}{2} + 2k\pi - i\ln(1+\sqrt{2}), k \in \mathbb{Z}$$

şi

$$e^{iz} = i(1-\sqrt{2}) \Rightarrow iz_k \in \left\{ \ln(\sqrt{2}-1) + i\left(\frac{3\pi}{2} + 2k\pi\right) | k \in \mathbb{Z} \right\} \Rightarrow$$

$$z_k = \frac{3\pi}{2} + 2k\pi - i\ln(\sqrt{2}-1), k \in \mathbb{Z}.$$

2.4 Exerciţii propuse.

Exerciţiul 2.29 Determinaţi în fiecare din următoarele cazuri $\mathrm{Re} z$, $\mathrm{Im} z$, $|z|$ şi $\arg z$:

a) $z = \frac{1}{i}$;

b) $z = \frac{1-i}{1+i}$;

c) $z = \frac{2}{1-3i}$;

d) $z = (1 + i\sqrt{3})^3$;

e) $z = (1 + i\sqrt{3})^n$.

Exerciţiul 2.30 Determinaţi în fiecare din următoarele cazuri toţi $z \in \mathbb{C}$ pentru care:

a) $z^3 = 1$;

b) $z^3 = i$;

c) $z^4 = -1$;

d) $z^8 = 1$;

e) $z^2 = 1 - i$;

f) $z^2 = 3 + 4i$;

g) $z^3 = -2 + 2i$;

h) $z^5 = -4 + 3i$.

Exerciţiul 2.31 Explicaţi de ce pentru orice $z \in \mathbb{C}$ au loc relaţiile:

a) $|z_1 + z_2| \leq |z_1| + |z_2|$;

b) $|z_1 - z_2| \geq ||z_1| - |z_2||$.

Exerciţiul 2.32 Verificaţi prin calcul identitatea

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right)$$

şi descoperiţi semnificaţia ei geometrică.

Exerciţiul 2.33 Precizaţi, pentru fiecare din următoarele cazuri, ce mulţime de puncte $P(z)$ din planul complex verifică relaţia:

a) $|z - z_0| < 1$, $|z - z_0| = 1$ şi $|z - z_0| > 1$ pentru $z_0 \in \mathbb{C}$, fixat;

b) $1 < |z| < 2$;

c) $1 < \mathrm{Re}z| < 2$;

d) $1 < \mathrm{Im}z| < 2$;

e) $\frac{\pi}{3} < \arg z < \frac{\pi}{2}$;

f) $|z + 3| = 5$;

g) $|z + 2 + i| < 2$;

h) $|z - 1 - i| > 3$;

i) $2 \leq |z - 2 - 3i| < 4$;

j) $|z - 2| + |z + 2| = 5$;

k) $|z - 2| - |z + 2| > 3$;

l) $|z - i| = |z + 1|$;

m) $|z + 1| = |z - 3| = |z + 4i|$.

Exerciţiul 2.34 Determinaţi funcţiile $u, v : \mathbb{R}^2 \to \mathbb{R}$ pentru fiecare funcţie complexă $f(z)$ dată:

a) $f(z) = \overline{z}$;

b) $f(z) = z^2$;

c) $f(z) = \frac{z-1}{z+i}$;

d) $f(z) = e^z$;

e) $f(z) = \frac{e^z + e^{-z}}{2}$, notată $\cosh z$;

f) $f(z) = \frac{e^z - e^{-z}}{2}$, notată $\sinh z$;

g) $f(z) = \frac{e^{iz} + e^{-iz}}{2}$, notată $\cos z$;

h) $f(z) = \frac{e^{iz} - e^{-iz}}{2i}$, notată $\sin z$;

i) $f(z) = \frac{e^z - e^{-z}}{e^z + e^{-z}}$, notată $\tanh z$;

j) $f(z) = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$, notată $\tan z$.

Exerciţiul 2.35 Determinaţi expresia funcţiei $f(z)$ atunci când se cunosc funcţiile $u(x, y)$ şi $v(x, y)$ (unde $z = x + iy$):

a) $u = x$, $v = -y$;

b) $u = x^2 - y^2$, $v = 2xy$;

c) $u = -2xy$, $v = x^2 - y^2$;

d) $u = e^x \cos y$, $v = e^x \sin y$;

e) $u = e^y \cos x$, $v = -e^y \sin x$;

f) $u = \cos x \sinh y$, $v = \sin x \cosh y$.

Exerciţiul 2.36 Calculaţi sub forma $u + iv$ următoarele expresii:

a) $e^{\pi i}$;

b) $\cos(1+i)$;

c) $\sin 3i$;

d) $\cosh 3\pi i$;

e) $\ln(1+i)$.

Exerciţiul 2.37 Determinaţi valorile lui $z \in \mathbb{C}$ $(z = x + iy)$ pentru care:

a) $z^2 + (2-3i)z - 5 - i = 0$;

b) $e^z = 4 - 3i$;

c) $\sin z = 10$;

d) $\cosh z = -1$.

Exerciţiul 2.38 Determinaţi funcţia analitică $f(z) = u + iv$ în următoarele cazuri:

a) $u = \cos x \cosh y$;

b) $v = \cos x \sinh y$;

c) $u = \sinh x \cos y$;

d) $v = \sinh x \sin y$;

e) $v = e^x \sin y$;

f) $v = \ln(x^2 + y^2)$;

g) $u = x^2 - y^2 + 3x$;

h) $u = 6x - 2y$;

i) $v = 6xy - 6y + 3$;

j) $u = x^2 - y^2 - y$;

k) $u = \frac{1}{2}\ln(x^2 + y^2)$;

l) $u = \frac{x}{2}\ln(x^2 + y^2) - y \arctan\frac{y}{x}$;

m) $u = e^x(x\cos y - y\sin y)$;

n) $u = x^2 - y^2 + \frac{x}{x^2+(y+1)^2}$;

o) $v = e^{-y}\sin x + x^2 + xy - y^2$;

p) $v = \arctan\frac{y}{x} + 2xy$.

Exerciţiul 2.39 (funcţii complexe elementare) Să se aducă la forma $A + iB$ expresiile:

a) e^i;

b) $\sinh 2i$;

c) $\cosh(2 + 3i)$;

d) $\cos(1 - i)$;

e) $\tan(1 - 2i)$;

f) $\ln(-2i)$;

g) $\ln(-3 + 4i)$;

h) $\ln\frac{1-i}{\sqrt{3}+i}$.

Exerciţiul 2.40 (funcţii complexe elementare) Calculaţi:

a) i^{1-i};

b) $(1+i\sqrt{3})^i$;

c) 1^{-i};

d) $|\sin z|$.

Exerciţiul 2.41 (funcţii complexe elementare) Demonstraţi egalităţile:

a) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$;

b) $\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$;

c) $\sin 2z = 2 \sin z \cos z$;

d) $\cosh 2z = \cosh^2 z + \sinh^2 z$;

e) $\sinh^2 z = \frac{\cosh 2z - 1}{2}$;

f) $\cosh^2 z = \frac{\cosh 2z + 1}{2}$.

Exerciţiul 2.42 (funcţii complexe elementare) Să se rezolve ecuaţiile următoare:

a) $e^{i3z} = -1$;

b) $e^{\frac{1}{z^2}} = 1$;

c) $\sin z = 10$;

d) $\tanh z = 2$.

Capitolul 3

Reziduuri. Integrale improprii rezolvate cu reziduuri.

3.1 Consideraţii teoretice.

3.1.1 Integrala complexă.

Integrala complexă se notează:

$$\int_C f(z)dz \tag{3.1}$$

unde

- $f(z)$ este funcţie complexă;

- C este curbă (curbă de integrare, arc) în planul complex;

Reprezentarea parametrică a curbei

$$C: \ z(t) = x(t) + iy(t), \ a \leq t \leq b. \tag{3.2}$$

- Sens pozitiv pentru C - sensul de creştere al argumentului t (C este curbă orientată);

- C este netedă şi simplă - are derivata continuă şi nenulă în orice punct:

$$\dot{z} = \frac{dz}{dt} = \dot{x}(t) + i\dot{y}(t).$$

- C este închisă - punctul terminal coincide cu cel iniţial; atunci integrala complexă se notează:

$$\oint_C f(z)dz.$$

Ipoteze:

(i) Toate curbele de integrare sunt netede pe porţiuni (sunt formate dintr-un număr finit de curbe netede "alipite");

(ii) Funcţia $f(z)$ este continuă.

Proprietăţi 3.1

1. *Liniaritate*:

$$\int_C [\alpha f_1(z) + \beta f_2(z)]dz = \alpha \int_C f_1(z)dz + \beta \int_C f_2(z)dz.$$

2. *Schimbarea sensului* (pentru o aceeaşi curbă C, cu z_0 punct iniţial cu z_1 punct terminal):

$$\int_{z_0}^{z_1} f(z)dz = -\int_{z_1}^{z_0} f(z)dz.$$

3. *Partiţionarea curbei*:

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

unde curbele C_1 şi C_2 formează curba C.

Teorema 3.2 *Fie D un domeniu simplu conex (∂D este o curbă închisă simplă) şi funcţia $f : D \to \mathbb{C}$ analitică pe D, punctele z_0, z_1 din D, atunci oricare ar fi curba C din D care leagă z_0 de z_1, avem:*

$$\int_C f(z)dz = \int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0), \tag{3.3}$$

unde $F'(z) = f(z)$.

Observaţia 3.3 Integrarea lui $f(z)$ este *independentă de curbă.*

Definiţia 3.4 Fie C o curbă netedă pe porţiuni, reprezentată prin $z = z(t)$, $a \leq t \leq b$ şi $f(z)$ continuă pe C, atunci:

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt, \tag{3.4}$$

unde $\dot{z} = \frac{dz}{dt}$.

Observaţia 3.5 Dacă $f(z) = u(x, y) + iv(x, y)$, $C : z = z(t)$, $a \leq t \leq b$ atunci

$$\int_C f(z)dz = \int_a^b f(z(t))\dot{z}(t)dt =$$

$$= \int_C u(x, y)dx - v(x, y)dy + i \int_C u(x, y)dy - v(x, y)dx.$$

Teorema 3.6 *(Teorema integrală a lui Cauchy: T.I.C.) Fie funcţia $f : D \to \mathbb{C}$, D un domeniu simplu conex şi f analitică pe D, atunci oricare ar fi curba C din D curbă închisă, simplă, avem:*

$$\oint_C f(z)dz = 0. \tag{3.5}$$

Observaţia 3.7

1. T.I.C. se poate aplica funcţiilor analitice pentru orice z;

2. Funcţiilor cu singularităţi (puncte unde nu sunt definite sau nu sunt analitice) situate în afara curbei;

3. Există situaţii în care integrarea unor funcţii care au singularităţi în interiorul curbei C va da zero (dar acest lucru se face prin calcul direct, nu prin aplicarea teoremei).

Teorema 3.8 *(Formula integrală a lui Cauchy: F.I.C.) Fie funcţia $f(z)$ analitică pe D, cu D un domeniu simplu conex, atunci pentru orice punct z_0 din D şi orice curbă C din D închisă (contur) ce conţine în interior pe z_0, avem:*

$$\oint_C \frac{f(z)}{z - z_0}dz = 2\pi i f(z_0). \tag{3.6}$$

Generalizare.

1. Dacă funcţia $f(z)$ este analitică pe D atunci ea are derivate de orice ordin pe D, care sunt la rândul lor tot funcţii analitice.

2. Pentru $f(z)$ analitică şi C curbă simplă închisă în D, care conţine punctul z_0, avem:

$$\oint_C \frac{f(z)}{(z - z_0)^{n+1}}dz = \frac{2\pi i}{n!} f^{(n)}(z_0), n = 1, 2, 3, ...$$

3.1.2 Seria Taylor.

Teorema 3.9 *Fie funcţia $f : D \to \mathbb{C}$, analitică pe D, cu D un domeniu simplu conex, $z_0 \in D$. Atunci $\exists \rho > 0$ astfel încât*

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!}(z - z_0), \tag{3.7}$$

*$\forall z \in V_\delta(z_0)$, unde $V_\delta(z_0) = \{|z - z_0| < \rho\}$, adică f se dezvoltă **în serie Taylor** în jurul lui z_0.*

3.1.3 Seria Laurent.

Teorema 3.10 *Fie $W_{r,\rho}(z_0) = \{z \in \mathbb{C} | r < |z - z_0| < \rho\}$ coroana circulară de rază interioară r şi rază exterioară ρ, funcţia $f : W_{r,\rho} \to \mathbb{C}$ analitică . Atunci*

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n, \forall z \in W_{r,\rho}(z_0), \tag{3.8}$$

unde

$$a_n = \frac{1}{2\pi i} \oint_\Gamma \frac{f(z)}{(z - z_0)^{(n+1)}} dz, \ \forall n \in \mathbb{Z},$$

cu Γ curbă închisă (contur) simplă, netedă pe porţiuni ce înconjoară z_0 în coroană.

*Scrierea (7.6) reprezintă dezvoltarea funcţiei f **în serie Laurent** în jurul lui z_0 şi este echivalentă cu*

$$f(z) = \sum_{n=-\infty}^{-1} a_n (z - z_0)^n + \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{3.9}$$

Serii Taylor complexe uzuale:

1)

$$\frac{1}{1-z} = 1 + z + \cdots + z^n + \cdots, \ |z| < 1;$$

2)

$$e^z = 1 + \frac{z}{1!} + \cdots + \frac{z^n}{n!} + \cdots, \ z \in \mathbb{C};$$

3)

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots, \ z \in \mathbb{C};$$

4)

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots, \ z \in \mathbb{C};$$

5)

$$\sin z = \frac{z}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots, \ z \in \mathbb{C};$$

6)

$$\sinh z = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n+1}}{(2n+1)!} + \cdots, \ z \in \mathbb{C};$$

Aplicaţie:

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} \frac{1}{n! z^n} = \cdots + \frac{1}{n! z^n} + \cdots + \frac{1}{2! z^2} + \frac{1}{1! z} + 1.$$

3.1.4 Singularităţi.

Definiţia 3.11 Fie funcţia $f : D \to \mathbb{C}$ analitică pe $D \backslash \{z_0\}$. Punctul z_0 se numeşte *singularitate* (izolată) pentru $f(z)$.

Definiţia 3.12 Un punct în care funcţia f nu este definită sau analitică se numeşte *singularitate* pentru $f(z)$.

Tipul singularităţilor:

1. z_0 este *punct singular eliminabil* dacă:
$$\exists \lim_{z \to z_0} f(z) = ct;$$

2. z_0 se numeşte *pol* dacă:
$$\lim_{z \to z_0} |f(z)| = +\infty;$$

3. z_0 este *punct singular esenţial* dacă:
$$\lim_{z \to z_0} f(z) \text{ nu există.}$$

Observaţia 3.13

1. z_0 este *pol de ordinul* $k \geq 1$ pentru $f(z)$ dacă şi numai dacă:
$$\exists \lim_{z \to z0} (z - z_0)^k f(z) = \text{ct, nenulă;}$$

2. z_0 este *pol de ordinul* $k \geq 1$ pentru $f(z)$ dacă şi numai dacă partea principală a dezvoltării în serie Laurent a lui $f(z)$ în jurul lui z_0 are k termeni nenuli (adică $a_{-k} \neq 0$, $a_{-n} = 0, \forall n > k$).

3. z_0 este *pol de ordinul* $k \geq 1$ pentru $f(z) = \frac{g(z)}{h(z)}$ dacă şi numai dacă $g(z_0) \neq 0$ şi z_0 este rădăcină de ordinul k pentru $h(z)$.

Observaţia 3.14 z_0 este *punct singular esenţial* dacă şi numai dacă partea princilală a dezvoltării în serie Laurent a lui $f(z)$ în jurul lui z_0 are o infinitate termeni nenuli.

3.1.5 Reziduul într-un pol.

Definiţia 3.15 Fie z_0 un pol de ordinul $k \geq 1$ pentru funcţia $f(z)$. Atunci *reziduul* funcţiei $f(z)$ în polul z_0 are expresia următoare

$$\mathrm{Res}_{z=z_0} = \frac{1}{(k-1)!} \lim_{z \to z_0} \left[(z-z_0)^k f(z) \right]^{k-1}. \tag{3.10}$$

Observaţia 3.16 Dacă z_0 este pol simplu ($k = 1$) pentru funcţia $f(z) = \frac{g(z)}{h(z)}$, atunci

$$\mathrm{Res}_{z=z_0} f(z) = \frac{g(z_0)}{h'(z_0)} = \lim_{z \to z_0} \left[(z-z_0) f(z) \right]. \tag{3.11}$$

3.1.6 Teorema reziduurilor.

Observaţia 3.17 Dacă z_0 este singularitate esenţială, reziduul funcţiei $f(z)$ se calculează cu ajutorul dezvoltării în serie Laurent.

$$\mathrm{Res}_{z=z_0} f(z) = a_{-1}, \tag{3.12}$$

adică este egal cu coeficientul lui $\frac{1}{z-z_0}$ din dezvoltarea în serie Laurent a lui $f(z)$ în jurul lui z_0.

Teorema 3.18 *Fie C curbă simplă închisă, parcursă în sens trigonometric şi $f(z)$ o funcţie analitică pe curba C şi în interiorul lui C, cu excepţia unui număr finit de puncte singulare z_1, z_2, ... , z_n situate în interiorul curbei C. Atunci*

$$\oint_C f(z)dz = 2\pi i \sum_{k=1}^{n} Res_{z=z_k} f(z). \tag{3.13}$$

3.1.7 Aplicaţii ale teoremei reziduurilor în calculul unor integrale reale.

1. *Integrale raţionale în* $\sin\theta$ *şi* $\cos\theta$ de forma

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta)d\theta. \tag{3.14}$$

a) Se rezolvă printr-o schimbare de variabilă "consacrată" $z = e^{i\theta}$.

b) Deoarece $\theta \in [0, 2\pi]$, variabila $z = e^{i\theta}$ va avea ca domeniu cercul unitate (curba de ecuaţie $|z| = 1$).

c) Formule de transformare a funcţiei iniţiale $F(\cos\theta, \sin\theta)$ în $f(z)$:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right),$$

$$\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right).$$

d) Diferenţierea notaţiei $z = e^{i\theta}$ conduce la

$$\frac{dz}{d\theta} = ie^{i\theta},$$

adică

$$d\theta = \frac{1}{iz} dz.$$

Deci, integrala iniţială devine

$$J = \int_0^{2\pi} F(\cos\theta, \sin\theta) d\theta = \oint_{|z|=1} f(z) \frac{1}{iz} dz, \tag{3.15}$$

integrală care se rezolvă cu teorema reziduurilor.

2. *Integrale improprii* de forma

$$\int_{-\infty}^{\infty} f(x) dx. \tag{3.16}$$

a) Funcţia $f(x)$ este o funcţie reală, raţională, cu numitorul diferit de zero pentru orice $x \in \mathbb{R}$ (nu are poli pe axa reală) şi gradul numitorului depăşeşte cu cel puţin două unităţi gradul numărătorului.

b) Deoarece $f(x)$ este raţională, atunci funcţia complexă asociată $f(z)$ are un număr finit de poli situaţi în semiplanul superior. Alegem conturul C neted şi simplu, de forma $C = [-R, R] \cup C_0^+$, unde C_0 este semicercul superior cu centrul în origine şi rază R, R suficient de mare ales asfel încât C să conţiină în interior toţi polii din semiplanul superior ai lui f. C este parcurs în sens trigonometric, $C_0 : |z| = R$, $z = Re^{it}$, $t \in [0, \pi]$. Aplicăm teorema reziduurilor pentru $f(z)$ şi conturul C:

$$\oint_C f(z) dz = \int_{-R}^{R} f(x) dx + \int_{C_0^+} f(z) dz =$$

$$= 2\pi i \sum \text{Res} f(z), \tag{3.17}$$

unde reziduurile se iau numai în polii din semiplanul superior.

$$\left| \int_{C_0^+} f(z)dz \right| = \left| \int_0^\pi f(Re^{it}) Rie^{it} dt \right| \leq$$

$$\leq R \int_0^\pi \left| f(Re^{it}) \right| dt \leq R \sup_{t\in[0,\pi]} \left| f(Re^{it}) \right| \xrightarrow{R\to\infty} 0 \tag{3.18}$$

rezultă că pentru $R \to \infty$ avem

$$\int_{C_0^+} f(z)dz \to 0. \tag{3.19}$$

Trecem la limită după $R \to \infty$ în relaţia (3.17):

$$\oint_C f(z)dz = \lim_{R\to\infty} \int_{-R}^R f(x)dx =$$

$$= \int_{-\infty}^\infty f(x)dx = 2\pi i \sum \text{Res} f(z)$$

rezultă

$$\int_{-\infty}^\infty f(x)dx = \oint_C f(z)dz = 2\pi i \sum \text{Res} f(z) \tag{3.20}$$

unde reziduurile sunt din semiplanul superior.

3. *Integrale de tip Fourier.*

$$\int_{-\infty}^\infty f(x) \cos ax dx \tag{3.21}$$

sau

$$\int_{-\infty}^\infty f(x) \sin ax dx \tag{3.22}$$

cu $a > 0$.

a) Funcţia $f(x)$ este o funcţie reală, raţională, cu numitorul diferit de zero pentru orice $x \in \mathbb{R}$ (nu are poli pe axa reală) şi are gradul numitorului mai mare decât gradul numărătorului cu cel puţin o unitate.

b) Se porneşte de la integrala asociată

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx \tag{3.23}$$

şi de la integrala complexă asociată $f(x)e^{iaz}$.

c) Cu teorema reziduurilor avem

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \sum \text{Res}\left[f(z)e^{iaz}\right], \tag{3.24}$$

unde reziduurile se calculează în polii situaţi în partea superioară a planului complex.

c) După calcularea reziduurilor avem

$$\int_{-\infty}^{\infty} f(x)\cos axdx = -2\pi \sum \text{Im Res}\left[f(z)e^{iaz}\right], \tag{3.25}$$

$$\int_{-\infty}^{\infty} f(x)\sin axdx = 2\pi \sum \text{Re Res}\left[f(z)e^{iaz}\right]. \tag{3.26}$$

3.2 Exerciţii rezolvate.

Exerciţiul 3.19 Determinaţi tipul singularităţilor pentru:

a) $f(z) = \frac{z^3+2z-3}{z^2}$, $z_0 = 0$ punct singular;

b) $f(z) = e^{\frac{1}{z-1}}$, $z_0 = 1$ punct singular;

c) $f(z) = \frac{\sin z}{z}$, $z = 0$ punct singular;

d) $f(z) = \frac{\sin z}{z^4}$, $z = 0$ punct singular.

Soluţie.
a) $f(z) = \frac{z^3+2z-3}{z^2}$, $z_0 = 0$ punct singular;

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} \frac{3}{|z|^2} = +\infty$$

rezultă că $z = 0$ este pol dublu.

b) $f(z) = e^{\frac{1}{z-1}}$, $z_0 = 1$ punct singular; Considerăm restricţia lui $f(z)$ la mulţimea $A = \{z \in \mathbb{C} |\ \mathrm{Im} z = 0\}$.

$$\lim_{z\to 1} f_{/A}(z) = \lim_{x\to 1} e^{\frac{1}{x-1}}$$

nu există, deoarece:

$$\lim_{x\to 1, x<1} e^{\frac{1}{x-1}} = e^{-\infty} = 0$$

$$\lim_{x\to 1, x>1} e^{\frac{1}{x-1}} = e^{+\infty} = +\infty$$

deci nu există $\lim_{z\to 1} f_{/A}(z)$ rezultă că nu există $\lim_{z\to 1} f(z)$, adică $z = 1$ este punct singular esenţial.

c) $f(z) = \frac{\sin z}{z}$, $z = 0$ punct singular;

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1$$

adică $z = 0$ este punct singular aparent.

d) $f(z) = \frac{\sin z}{z^4}$, $z = 0$ punct singular.

$$\lim_{z\to 0} |f(z)| = \lim_{z\to 0} |\frac{\sin z}{z} \cdot \frac{1}{z^3}| = 1 \cdot \lim_{z\to 0} \frac{1}{|z|^3} = +\infty$$

rezultă că $z = 0$ este pol de ordinul 3.

Exercițiul 3.20 Determinați tipul singularităților pentru:

a) $f(z) = \frac{\sin 4z}{(z-4)^3 z(z+i)}$, $z_1 = 4$, $z_2 = 0$, $z_3 = -i$ puncte singulare;

b) $f(z) = e^{\frac{1}{z}} \frac{1}{z-1}$, $z_1 = 0$, $z_2 = 1$ puncte singulare.

Soluție.
a) $f(z) = \frac{\sin 4z}{(z-4)^3 z(z+i)}$, $z_1 = 4$, $z_2 = 0$, $z_3 = -i$ puncte singulare;

$$\lim_{z\to 4} f(z)(z-4)^3 = \lim_{z\to 4} \frac{\sin 4z}{z(z+i)} = \frac{\sin 16}{4(4+i)},$$

limita este finită, nenulă, rezultă că $z_1 = 4$ este pol de ordinul 3.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\sin 4z}{z} \frac{1}{(z-4)^3(z+i)} = \frac{4}{-64i} = \frac{i}{16},$$

limita este o constantă, rezultă că $z_2 = 0$ este punct singular eliminabil.

$$\lim_{z\to -i} f(z)(z+i) = \lim_{z\to -i} \frac{\sin 4z}{(z-4)^3 z} = \frac{\sin(-4i)}{(-i)(-i-4)^3} =$$

$$= \frac{-i \sinh 4}{(-i)(-i-4)^3} = \frac{\sinh 4}{(-i-4)^3}$$

limita este o constantă nenulă, deci $z_3 = -i$ este pol de ordinul 1.

b) $f(z) = e^{\frac{1}{z}} \frac{1}{z-1}$, $z_1 = 0$, $z_2 = 1$ puncte singulare.

$$\lim_{z\to 0} f(z) = -\lim_{z\to 0} e^{\frac{1}{z}}.$$

$$A = \{z \in \mathbb{C} | \text{Im} z = 0\} \Rightarrow \lim_{z\to 0} f_{/A}(z) = \lim_{x\to 0} e^{\frac{1}{x}}$$

limita nu există, deoarece:

$$\lim_{x\to 0,\, x<0} e^{\frac{1}{x}} = e^{-\infty} = 0$$

$$\lim_{x\to 0,\, x>0} e^{\frac{1}{x}} = e^{\infty} = 0,$$

deci nu există limita

$$\lim_{z\to 0} e^{\frac{1}{z}}$$

adică, nu există limita

$$\lim_{z\to 0} f(z)$$

de unde obţinem că $z = 0$ este punct singular esenţial.

Acum calculăm limita următoare:

$$\lim_{z\to 1} f(z)(z-1) = \lim_{z\to 1} e^{\frac{1}{z}} = e,$$

deci limita este o constantă nenulă, rezultă că $z = 1$ este pol de ordinul unu.

Exerciţiul 3.21 Determinaţi tipul singularităţii lui $z = 0$ pentru:

$$f(z) = \frac{\sin z}{z^5}$$

şi

$$\mathrm{Res}_{z=0} f(z).$$

Soluţie.
Dezvoltăm în serie Laurent în jurul lui $z = 0$:

$$f(z) = \frac{1}{z^5}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2z+1)!} =$$

$$= \frac{1}{z^5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) =$$

$$= \frac{1}{z^4} - \frac{1}{3!z^2} + \frac{1}{5!} - \frac{z^2}{7!} \cdots, \forall z \in \mathbb{C}$$

deci $z = 0$ este pol de ordinul patru.

$$\mathrm{Res}_{z=0} f(z) = a_{-1} = 0.$$

Exerciţiul 3.22 Calculaţi folosind F.I.C. urmatoarele integrale

$$(a) \qquad \oint_C \frac{e^{2z}}{z(z-2i)^2} dz,$$

unde C este cuprinsă între $|z - i| = 3$ (sens trigonometric) şi $|z| = 1$ (sens invers trigonometric).

Soluţie.
$C = C_1^+ \vee AB^+ \vee C_2^- \vee AB^-$ unde $C_1^+ : |z - i| = 3$ parcurs în sens trigonometric, $C_2^- : |z| = 1$ parcurs în sens invers trigonometric, AB un segment ce leagă un punct $A \in C_1$ de $B \in C_2$ şi cu F.I.C. avem:

$$\oint_C = \frac{\frac{e^{2z}}{z}}{(z-2i)^2} dz = \frac{2\pi i}{1!} \left(\frac{e^{2z}}{z} \right)'_{/z=2i} =$$

$$= 2\pi i \frac{2ze^{2z} - e^{2z}}{z^2}_{/z=2i} = \frac{2\pi i e^{4i}}{-4}(4i - 1) =$$

$$= \frac{\pi e^{4i}}{2}(4 + i) = \frac{\pi}{2}(4 + i)(\cos 4 + i \sin 4).$$

$$(b) \qquad \oint_C \frac{e^z}{(z-1)^2 \cdot (z^2+4)} dz$$

Soluţie.

$$C : |z - 2 + i| = \sqrt{3}$$
$$\underbrace{z = 1}_{\substack{pol\\dublu}}, \underbrace{z = 2i, z = -2i}_{\substack{poli\\simpli}}$$

$$|1 - 2 + i| = |-1 + i| = \sqrt{2} < \sqrt{3} \Rightarrow z = 1 \in C$$
$$|2i - 1 + i| = |-2 + 3i| = \sqrt{4 + 9} = \sqrt{13} > \sqrt{3} \Rightarrow z = 2i \notin C$$
$$|-2i - 2 + i| = |-2 - i| = \sqrt{5} > \sqrt{3} \Rightarrow z = -2i \notin C$$

Cu teorema reziduurilor avem

$$\oint_C \frac{e^z}{(z-1)^2 \cdot (z^2+4)} dz = 2\pi i \operatorname*{Res}_{z=1} f(z) = 2\pi i \cdot \lim_{z \to 1} \left(\frac{e^z}{z^2+4} \right)' =$$

$$= 2\pi i \frac{e^z (z^2+4) - e^z \cdot 2z}{(z^2+4)^2}|_{z=1} = 2\pi i e \frac{5-2}{25} = \frac{6\pi i e}{25}$$

$$(c) \quad \oint_C \frac{tg(\pi z)}{z^3} dz;$$

Soluţie.

$$\lim_{z \to 0} z^2 \cdot f(z) = \lim_{z \to 0} \frac{tg(\pi z)}{z^3} = \pi$$

π constantă nenulă.

$$C : |z + \frac{i}{2}| = 1$$

$z = 0 \in C \left(|\frac{i}{2}| = \frac{1}{2} < 1 \right) \Rightarrow z = 0$ pol de ordin 2

$$\oint\limits_C \frac{tg\,(\pi z)}{z^3} dz = 2\pi i \operatorname*{Res}_{z=0} f\,(z) =$$

$$= 2\pi i \cdot \lim_{z\to 0} \left(\frac{tg\,(\pi z)}{z}\right)' = 2\pi i \lim_{z\to 0} \frac{\frac{z\pi}{\cos^2 \pi z} - tg\pi z}{z^2} =$$

$$= 2\pi i \lim_{z\to 0} \frac{\pi z - \frac{1}{2}\sin 2\pi z}{z^2} =$$

$$\overset{L'Hospital}{=} 2\pi i \lim_{z\to 0} \frac{\pi - \pi\cos 2\pi z}{2z} =$$

$$= 2\pi i \lim_{z\to 0} 2\pi^2 \frac{\sin 2\pi z}{2} = 0.$$

Exerciţiul 3.23 Calculaţi integralele raţionale:

a)

$$\int\limits_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta};$$

b)

$$\int\limits_0^{2\pi} \frac{\sin^2\theta}{5 - 4\sin\theta} d\theta;$$

c)

$$\int\limits_0^{2\pi} \frac{\cos\theta}{13 - 12\cos 2\theta} d\theta;$$

d)

$$\int\limits_0^{2\pi} \frac{d\theta}{7 + 6\cos\theta};$$

e)

$$\int_0^{2\pi} \frac{d\theta}{8 - 2\sin\theta};$$

Soluţie.

$a)$

$$\int_0^{2\pi} \frac{d\theta}{\sqrt{2} - \cos\theta} = \frac{1}{i} \oint_C \frac{1}{z\left(\sqrt{2} - \frac{z^2+1}{2z}\right)} dz =$$

$$= \frac{-2}{i} \oint_C \frac{dz}{z^2 - 2\sqrt{2}z + 1} = \frac{-2}{i} \cdot 2\pi i \underset{z=z_1}{Res} f(z) =$$

$$= -4\pi \frac{1}{2z - 2\sqrt{2}}\Big|_{z=\sqrt{2}-1} = \frac{-4\pi}{2\sqrt{2} - 2 - 2\sqrt{2}} = 2\pi$$

Am folosit substituţia

$$z = e^{i\theta} \Rightarrow ie^{i\theta} d\theta = dz \Rightarrow d\theta = \frac{dz}{iz}; \theta \in [0, 2\pi] \Rightarrow C : |z| = 1$$

Funcţia complexă asociată este:

$$f(z) = \frac{1}{z^2 - 2\sqrt{2}z + 1};$$

Aflăm polii funcţiei $f(z)$:

$$z^2 - 2\sqrt{2}z + 1 = 0 \Rightarrow$$

$$z_{1,2} = \frac{2\sqrt{2} \pm 2}{2} = \sqrt{2} \pm 1 = \begin{cases} z_1 = \sqrt{2} - 1 \in C \\ z_2 = \sqrt{2} + 1 \notin C. \end{cases}$$

$b)$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\sin\theta} d\theta = \frac{1}{i} \oint_C \frac{\left(\frac{z^2-1}{2iz}\right)^2}{5-4\frac{(z^2-1)}{2iz}} \frac{dz}{z} =$$

$$= \frac{1}{4} \oint_C \frac{\left(z^2-1\right)^2}{z^2 \cdot (2z^2-5iz-2)} dz$$

Funcţia

$$f(z) = \frac{\left(z^2-1\right)^2}{z^2 \cdot (2z^2-5iz-2)}$$

are polii

$$\begin{aligned} z_0 &= 0 \in C \,\text{pol dublu}, \\ z_2 &= 2i \notin C \,\text{pol simplu}, \\ z_3 &= \tfrac{i}{2} \in C \,\text{pol simplu}. \end{aligned}$$

Cu teorema reziduurilor avem

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\sin\theta} d\theta = \frac{1}{4} \cdot 2\pi i \left(\underset{z=0}{Res}\, f(z) + \underset{z=\frac{i}{2}}{Res}\, f(z) \right) =$$

$$\cdot \underset{z=0}{Res}\, f(z) = \left[\frac{\left(z^2-1\right)^2}{2z^2-5iz-2} \right]'_{|z=0} = \frac{5i}{4}.$$

$$\cdot \underset{z=\frac{i}{2}}{Res}\, f(z) = \lim_{z\to\frac{i}{2}} \left(z-\frac{i}{2}\right) \frac{\left(z^2-1\right)^2}{2z^2\left(z-\frac{i}{2}\right)(z-2i)} = -\frac{25}{12}i.$$

$$\int_0^{2\pi} \frac{\sin^2\theta}{5-4\sin\theta} d\theta = \frac{\pi i}{2} \left(\frac{5i}{4} - \frac{25i}{12} \right) = \frac{5\pi}{12}.$$

$c)$

$$\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta} d\theta;$$

$$z = e^{i\theta}; \quad \cos 2\theta = \frac{z^2 + \frac{1}{z^2}}{2} = \frac{z^4+1}{2z^2}$$

$$\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta} d\theta = \frac{1}{2i} \oint_C \frac{z^2+1}{13z^2-6z^4-6} dz =$$

$$= -\frac{1}{2i} \oint_C \frac{z^2+1}{6z^4-13z^2+6} dz.$$

Polii sunt: $z_{1,2} = \pm\sqrt{\frac{3}{2}} \notin C$ respectiv $z_{3,4} = \pm\sqrt{\frac{2}{3}} \in C$

$$\cdot \underset{z=z_3=\sqrt{\frac{2}{3}}}{Res} f(z) = \frac{z^2+1}{24z^3-13z\cdot 2}|_{z=+\sqrt{\frac{2}{3}}} = \frac{1+\frac{2}{3}}{\sqrt{\frac{2}{3}}\left(24\cdot\frac{2}{3}-13\cdot 2\right)} =$$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\frac{5}{3}}{-10} = -\frac{1}{6}\sqrt{\frac{3}{2}}$$

$$\cdot \underset{z=-\sqrt{\frac{2}{3}}}{Res} f(z) = \frac{z^2+1}{z(24z^2-26)}|_{z=-\sqrt{\frac{2}{3}}} = \frac{1+\frac{2}{3}}{-\sqrt{\frac{2}{3}}\left(24\cdot\frac{2}{3}-13\cdot 2\right)} =$$

$$= \sqrt{\frac{3}{2}} \cdot \frac{5}{3(16-6)} = \frac{1}{6}\sqrt{\frac{3}{2}}$$

$$\int_0^{2\pi} \frac{\cos\theta}{13-12\cos 2\theta}d\theta = \frac{-1}{2i}\cdot 2\pi i\left(-\frac{1}{6}\sqrt{\frac{3}{2}}+\frac{1}{6}\sqrt{\frac{3}{2}}\right) = 0.$$

d)

$$\int_0^{2\pi} \frac{d\theta}{7+6\cos\theta} = \frac{1}{i}\oint_C \frac{1}{7+3\cdot\frac{z^2+1}{z}}\cdot\frac{dz}{z} = \frac{1}{i}\oint_C \frac{dz}{3z^2+7z+3} =$$

$$= \frac{1}{i}\cdot 2\pi i \underset{z=-\frac{7}{6}+\frac{\sqrt{13}}{6}}{Res} f(z) = 2\pi\frac{1}{\left(3z^2+7z+3\right)'}\Big|_{|z=-\frac{7}{6}+\frac{\sqrt{13}}{6}} =$$

$$= \frac{2\pi}{6z+7}\Big|_{z=\frac{-7+\sqrt{13}}{6}} = \frac{2\pi}{-7+\sqrt{13}+7} = \frac{2\pi}{\sqrt{13}}$$

$$f(z) = \frac{1}{3z^2+7z+3}; C: |z| = 1$$

$$3z^2+7z+3 = 0 \Leftrightarrow z_{1,2} = \frac{-7\pm\sqrt{13}}{6};$$

$$\left|\frac{-7+\sqrt{13}}{6}\right| = \frac{7-\sqrt{13}}{6}$$

$$z_1 = \frac{-7+\sqrt{13}}{6} \in C.$$

e)

$$\int_0^{2\pi} \frac{d\theta}{8-2\sin\theta} = \frac{1}{i}\oint_C \frac{1}{8-\frac{z^2-1}{iz}}\cdot\frac{dz}{z} =$$

$$= \oint_C \frac{dz}{8iz-z^2+1} = -\oint_C \frac{dz}{z^2-8iz-1} = -2\pi i \underset{z=\left(4-\sqrt{15}\right)i}{Res} f(z) =$$

$$= \frac{-\pi}{-\sqrt{15} - i} = \frac{\pi}{\sqrt{15}}$$

unde

$$f(z) = \frac{1}{z^2 - 8iz - 1}, \quad C : |z| = 1$$

$z^2 - 8iz - 10 = 0 \Leftrightarrow z_{1,2} = \dfrac{8i \pm 2\sqrt{15}i}{2} = \left(4 \pm \sqrt{15}\right) i$ poli simpli;

Doar

$$z = \left(4 - \sqrt{15}\right) i \in C.$$

Exerciţiul 3.24 Calculaţi integralele reale improprii:

a)

$$\int_0^\infty \frac{dx}{1 + x^4};$$

b)

$$\int_{-\infty}^\infty \frac{x^2 + 1}{x^4 + 1} dx;$$

c)

$$\int_{-\infty}^\infty \frac{x + 3}{\left(x^2 - 2x + 2\right)^2} dx;$$

d)

$$\oint_C \frac{50z}{z^3 + 2z^2 - 7z + 4} dz;$$

e)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+2x+5)} dx.$$

Soluţie.

(a)

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{1+x^4} \quad \begin{cases} gr.P+2=2<4=gr.Q \\ Q\neq 0, \text{ pe R.} \end{cases}$$

$f(z) = \frac{1}{1+z^4}$ funcţia complexă asociată care are polii:

$$\begin{aligned}
&z^4 = -1 = e^{i\pi} \Rightarrow z_k = e^{i\frac{\pi+2k\pi}{4}}, k\in\{0,1,2,3\} \\
&z_0 = e^{\frac{i\pi}{4}}, z_1 = e^{i\frac{3\pi}{4}} z_1, z_2 \in Ss \\
&z_2 = e^{i\frac{5\pi}{4}} = e^{-i\frac{3\pi}{4}} = e^{i\left(2\pi-\frac{3\pi}{4}\right)} = e^{-i\frac{3\pi}{4}} \\
&z_3 = e^{i\frac{7\pi}{4}} = e^{i\left(2\pi-\frac{\pi}{4}\right)} = e^{-i\frac{\pi}{4}}
\end{aligned}$$

$$\begin{aligned}
&\underset{z=z_0}{Res} f(z) = \tfrac{1}{4z^3}\big|_{z=e^{i\frac{\pi}{4}}} = e^{-i\frac{3\pi}{4}} \\
&\underset{z=z_1}{Res} f(z) = \tfrac{1}{4e^{i\frac{3\pi}{4}}} = \tfrac{e^{-i\frac{\pi}{4}}}{4}
\end{aligned}$$

$$\int_0^{\infty} \frac{dx}{1+x^4} = \frac{1}{2}\cdot 2\pi i\left(\underset{z=z_0}{Res} f(z) + \underset{z=z_1}{Res} f(z)\right) =$$

$$= \frac{\pi i}{4}\left(e^{-i\frac{3\pi}{4}} + e^{-i\frac{\pi}{4}}\right) = \frac{\pi i}{4}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) =$$

$$= -\pi i^2 \cdot \frac{\sqrt{2}}{4} = \frac{\pi}{2\sqrt{2}}$$

(b)

$$\int_{-\infty}^{\infty} \frac{x^2+1}{x^4+1}dx = 2\pi i \left(\underset{z=e^{\frac{i\pi}{4}}}{Res} f(z) + \underset{z=e^{\frac{i3\pi}{4}}}{Res} f(z) \right) =$$

$$= 2\pi i \left(\frac{z^2+1}{4z^3}|_{z=e^{i\frac{\pi}{4}}} + \frac{z^2+1}{4z^3}|_{z=e^{i\frac{3\pi}{4}}} \right) =$$

$$= \frac{2\pi i}{4} \left[(1+i)\, e^{-i\frac{3\pi}{4}} + (1-i)\, e^{-i\frac{\pi}{4}} \right] =$$

$$= \frac{\pi i}{\sqrt{2}\cdot 2} \left[(1+i)(-1-i) + (1-i)(1-i) \right] =$$

$$= \frac{\pi i}{\sqrt{2}\cdot 2} \left[-(1+i)^2 + (1-i)^2 \right] =$$

$$= \frac{\pi i}{2\sqrt{2}} (-2i - 2i) = \pi\sqrt{2} \left(-i^2 \right) = \pi\sqrt{2}$$

$f(z) = \frac{z^2+1}{z^4+1}$ funcţia complexă asociată

(c)

$$\int_{-\infty}^{\infty} \frac{x+3}{(x^2-2x+2)^2} dx$$

$f(z) = \frac{z+3}{(z^2-2z+2)^2}$ funcţia complexă asociată

Gradul numărătorului $+2 = 3 < 4 =$ gradul numitorului

Numitorul nu se anulează în $\mathbb{R}$

Polii lui $f(z)$ sunt $: z^2 - 2z + 2 = 0 \Leftrightarrow z_{1,2} = \frac{2\pm\sqrt{4-8}}{2} = 1 \pm i$ poli dubli

Calculăm rezidul în $z = 1 + i \in Ss$ semiplanul superior

$$\underset{z=1+i}{Res}\, f(z) = \lim_{z\to 1+i}\left[(z-1-i)^2\cdot\frac{z+3}{(z-1-i)^2(z-1+i)^2}\right]' =$$

$$= \frac{(z-1+i)^2 - 2(z+3)(z-1+i)}{(z-1+i)^4}\Big|_{z=1+i} =$$

$$= \frac{(z-1+i) - 2(z+3)}{(z-1+i)^3}\Big|_{z=z+i} = \frac{2i - 2(4+i)}{(2i)^3} = \frac{-8}{-8i} = \frac{1}{i}.$$

Cu teorema reziduurilor avem:

$$\int_{-\infty}^{\infty}\frac{x+3}{(x^2-2x+2)^2}dx = \oint_C f(z)dz = 2\pi i\,\underset{z=1+i}{Res}\, f(z) = \frac{2\pi i}{i} = 2\pi,$$

C = contur neted ce conţine toţi polii lui $f(z)$ din Ss.

(d)

$$\oint_C \frac{50z}{z^3+2z^2-7z+4}dz,$$

$C : |z-2| = 2.$

$$f(z) = \frac{50z}{z^3 - z^2 + 3z^2 - 3z - (4z-4)}$$

dar

$$\begin{array}{l} z^3 - z^2 = z^2(z-1) \\ 3z^2 - 3z = 3z(z-1) \;\Rightarrow \\ (4z-4) = 4(z-1) \end{array}$$

$$f(z) = \frac{50z}{(z-1)(z-1)(z+4)} =$$

$$= \frac{50z}{(z-1)^2(z+4)} \Rightarrow$$

$$\begin{array}{l} z = 1 \text{ pol dublu, } z = 1 \in C \\ z = -4 \text{ pol simplu, } z = -4 \notin C \end{array} \Rightarrow$$

Cu teorema rezuduurilor

$$\oint_C \frac{50z}{(z-1)^2(z+4)} dz = 2\pi i \operatorname*{Res}_{z=1} f(z) =$$

$$= 2\pi i \left(\frac{50z}{z+4}\right)' |_{z=1} = 2\pi i \cdot 50 \left(1 - \frac{4}{z+4}\right)' |_{z=1} =$$

$$= 100\pi i \frac{4}{(z+4)} |_{z=1} = \frac{400\pi i}{5} = 16\pi i.$$

(e)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+4)(x^2+2x+5)} dx =$$

$$= 2\pi i \left(\operatorname*{Res}_{z=2i} f(z) + \operatorname*{Res}_{z=-1+2i} f(z)\right) =$$

$$= 2\pi i \left(\frac{4}{17} + \frac{i}{17} - \frac{4}{17} - \frac{13i}{4 \cdot 17}\right) =$$

$$= -\frac{\pi i^2 \cdot 9}{2 \cdot 17} = \frac{9\pi}{34}.$$

$$\cdot \operatorname*{Res}_{z=2i} f(z) = \frac{z^2}{2z(z^2+2z+5) + (z^2+4)(2z+2)} |_{z=2i} =$$

$$= \frac{-4}{4i(-4+4i+5)} = \frac{-1}{i(1+4i)} =$$

$$= \frac{-1}{-4+i} = \frac{4+i}{17} = \frac{4}{17} + \frac{i}{17}$$

$$\cdot \operatorname*{Res}_{z=-1+2i} f(z) = \frac{(-1+2i)^2}{4i(1-4i-+)} =$$

$$= \frac{1-4i-4}{4i(1-4i)} = \frac{-3-4i}{4(4+i)} = \frac{(-3-4i)(4-i)}{4\cdot 17} =$$

$$= \frac{-16-13i}{4\cdot 17} = -\frac{4}{17} - \frac{13}{4\cdot 17}i.$$

Exerciţiul 3.25 Calculaţi următoarele integrale de tip Fourier:

a)

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{k^2+x^2}dx, \quad \int_{-\infty}^{\infty} \frac{\sin 3x}{k^2+x^2}dx; k > 0;$$

b)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^4+1}dx;$$

c)

$$\int_{0}^{\infty} \frac{\cos ax}{x^4+4b^4}dx, \quad a,\ b > 0;$$

d)

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2-2x+5}dx.$$

Soluţie.

(a)

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{k^2+x^2}dx, \int_{-\infty}^{\infty} \frac{\sin 3x}{k^2+x^2}dx; k>0$$

$f(z) = \frac{e^{i3z}}{k^2+z^2}$ funcţie complexă asociată cu polii simpli $z_{1,2} = \pm ki$. Reţinem $z_1 = ki \in Ss$.

$$\cdot \operatorname*{Res}_{z=ki} f(z) = \frac{e^{i3z}}{2z}|_{z=ik} = \frac{e^{-3k}}{2ik} = -\frac{e^{-3k}}{2k}i$$

$\cdot \int\limits_{-\infty}^{\infty} \frac{\cos 3x}{k^2+x^2}dx = -2\pi Im \operatorname*{Res}_{z=ik} f(z) = -2\pi \cdot \frac{-e^{-3k}}{2k} = \frac{\pi}{k} \cdot e^{-3k}$

$\cdot \cdot \int\limits_{-\infty}^{\infty} \frac{\sin 3x}{k^2+x^2}dx = 2\pi Im \operatorname*{Res}_{z=ik} f(z) = 0.$

(b)

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^4+1}dx;$$

$f(z) = \frac{e^{iz}}{z^4+1}$ funcţia complexă asociată
Polii din Ss sunt:

$$z_1 = e^{i\frac{\pi}{4}}$$
$$z_2 = e^{i\frac{3\pi}{4}}$$

$$\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}} f(z) = \frac{e^{iz}}{4z^3}|_{z=e^{i\frac{\pi}{4}}=\frac{\sqrt{2}}{2}(1+i)} = \frac{e^{i\frac{\sqrt{2}}{2}(1+i)}}{4e^{i\frac{3\pi}{4}}} =$$

$$= \frac{e^{\frac{\sqrt{2}}{2}(-1+i)}}{4} \cdot e^{-i\frac{3\pi}{4}} \overset{Euler}{=} \frac{1}{4} \cdot e^{-\frac{\sqrt{2}}{2}} \cdot e^{\frac{\sqrt{2}}{2}} \cdot \left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) \overset{Euler}{=}$$

$$= -\frac{1}{4\sqrt{2}} e^{-\frac{\sqrt{2}}{2}} (1+i) \left(\cos \frac{\sqrt{2}}{2} + i \sin \frac{\sqrt{2}}{2} \right)$$

$$\Rightarrow \cdot Re \underset{z=e^{\frac{i\pi}{4}}}{Res} f(z) = -\frac{1}{4\sqrt{2}} e^{-\frac{\sqrt{2}}{2}} \left(\cos \frac{\sqrt{2}}{2} - \sin \frac{\sqrt{2}}{2} \right) =$$

$$= \frac{1}{4\sqrt{2}} e^{-\frac{\sqrt{2}}{2}} \left(\sin \frac{\sqrt{2}}{2} - \cos \frac{\sqrt{2}}{2} \right)$$

$$\cdot \cdot \underset{z=e^{i\frac{3\pi}{4}}}{Res} f(z) = \frac{e^{i\left(-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}}{4e^{i\frac{\pi}{4}}} =$$

$$= \frac{e^{-\frac{\sqrt{2}}{2}}}{4\sqrt{2}} \left(\cos \frac{1}{\sqrt{2}} - i \sin \frac{1}{\sqrt{2}} \right) (1-i) \Rightarrow$$

$$\cdot \cdot Re \underset{z=e^{i\frac{3\pi}{4}}}{Res} f(z) = \frac{e^{-\frac{\sqrt{2}}{2}}}{4\sqrt{2}} \left(\cos \frac{\sqrt{2}}{2} - \sin \frac{\sqrt{2}}{2} \right) \Rightarrow$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^4+1} dx = 2\pi \sum Re \underset{z=e^{i\frac{\pi}{4}}}{Res} f(z) =$$

$$= 2\pi \left[Re \underset{z=e^{i\frac{\pi}{4}}}{Res} f(z) + Re \underset{z=e^{i\frac{3\pi}{4}}}{Res} f(z) \right] = 0$$

$$\cdot \cdot \int_{-\infty}^{\infty} \frac{\cos x}{x^4+1} dx = -2\pi \left(Im \underset{z=e^{i\frac{\pi}{4}}}{Res} f(z) + Im \underset{z=e^{i\frac{3\pi}{4}}}{Res} f(z) \right) =$$

$$= -2\pi \left(-\frac{e^{-\frac{\sqrt{2}}{2}}}{4\sqrt{2}} \left(\cos \frac{\sqrt{2}}{2} + \sin \frac{\sqrt{2}}{2} \right) + \right.$$

$$+\frac{e^{-\frac{\sqrt{2}}{2}}}{4\sqrt{2}}\left(-\cos\frac{\sqrt{2}}{2}-\sin\frac{\sqrt{2}}{2}\right)\Bigg)=$$

$$=\pi e^{-\frac{\sqrt{2}}{2}}\left(\sin\frac{\pi}{4}\cos\frac{\sqrt{2}}{2}+\cos\frac{\pi}{4}\sin\frac{\sqrt{2}}{2}\right)=$$

$$=\pi e^{-\frac{\sqrt{2}}{2}}\cdot\sin\left(\frac{\pi}{4}+\frac{\sqrt{2}}{2}\right)$$

(c)

$$\int\limits_{0}^{\infty}\frac{\cos ax}{x^4+4b^4}dx=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax}{x^4+4b^4}dx$$

$f(z)=\frac{e^{iaz}}{z^4+4b^4}$ are polii = rădăcinile ecuaţiei binome

$z^4+4b^4=0 \Leftrightarrow z^4=-4b^4=4b^4e^{i\pi} \Rightarrow$
$z_k=\sqrt{2}b\cdot e^{i\frac{\pi+2k\pi}{4}}, k\in\{0,1,2,3\}$
$z_0=\sqrt{2}b\cdot e^{\frac{i\pi}{4}}, z_1=\sqrt{2}b\cdot e^{i\frac{3\pi}{4}}, z_2=\sqrt{2}b\cdot e^{i\frac{5\pi}{4}}, z_3=\sqrt{2}b\cdot e^{i\frac{7\pi}{4}},$
$z_0, z_1\in Ss; z_2z_3\in Si$

$$\int\limits_{0}^{\infty}\frac{\cos ax}{x^4+4b^4}dx=\frac{1}{2}\int\limits_{-\infty}^{\infty}\frac{\cos ax}{x^4+4b^4}dx=$$

$$=\frac{1}{2}(-2\pi)\left\{Im\left(\underset{z=z_0}{Res}\,f(z)\right)+Im\left(\underset{z=z_1}{Res}\,f(z)\right)\right\}=$$

$$=-\pi Im\left\{\left(\underset{z=z_0}{Res}\,f(z)\right)+\left(\underset{z=z_1}{Res}\,f(z)\right)\right\}=$$

$$=-\pi\left\{Im\left(\underset{z=z_0}{Res}\,f(z)\right)+Im\left(\underset{z=z_1}{Res}\,f(z)\right)\right\}$$

$$\cdot \operatorname*{Res}_{z=z_0} f(z) = \frac{e^{iaz}}{\left(z^4+4b^4\right)'}\Big|_{z=\sqrt{2}b\cdot e^{i\frac{\pi}{4}}} = \frac{e^{ia\sqrt{2}b\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{4\cdot 2\sqrt{2}b^3 e^{i\frac{3\pi}{4}}} =$$

$$= \frac{e^{-ab}}{8\sqrt{2}b^3}\cdot e^{iab}\cdot e^{-i\frac{3\pi}{4}} = -\frac{e^{-ab}}{8\sqrt{2}b^3}\cdot e^{i\left(ab+\frac{\pi}{4}\right)}$$

deoarece

$$e^{-i\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} = -\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} =$$

$$= -e^{i\frac{\pi}{4}} e^{-i\frac{3\pi}{4}} = -e^{i\frac{\pi}{4}}.$$

$$\cdot\cdot \operatorname*{Res}_{z=z_1} f(z) = \frac{e^{iaz}}{4z^3}\Big|_{z=\sqrt{2}b\cdot e^{i\frac{3\pi}{4}}} = \frac{e^{ia\sqrt{2}b\left(-\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}}{4\cdot 2\sqrt{2}b^3 e^{i\frac{9\pi}{4}}} =$$

$$= \frac{e^{-ab}}{8\sqrt{2}b^3}\cdot e^{-iab}\cdot e^{-i\frac{\pi}{4}} = \frac{e^{-ab}}{8\sqrt{2}b^3}\cdot e^{-i\left(ab+\frac{\pi}{4}\right)}$$

deoarece

$$e^{i\frac{9\pi}{4}} = e^{i\frac{\pi}{4}}$$

(iz are perioada π.)

$$\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z) = -\frac{e^{-ab}}{4\sqrt{2}b^3}\underbrace{\left(e^{i\left(ab+\frac{\pi}{4}\right)} - e^{i\left(ab+\frac{\pi}{4}\right)}\right)}_{i\sin\left(ab+\frac{\pi}{4}\right)}$$

$$\Rightarrow Im\left\{\operatorname*{Res}_{z=z_0} f(z) + \operatorname*{Res}_{z=z_1} f(z)\right\} = -\frac{e^{-ab}}{4\sqrt{2}b^3}\cdot\sin\left(ab+\frac{\pi}{4}\right)$$

$$\Rightarrow \int\limits_0^\infty \frac{\cos ax}{x^4+4b^4}dx = -\pi Im\left\{\underset{z=z_0}{Res}\, f(z) + \underset{z=z_1}{Res}\, f(z)\right\} =$$

$$= -\pi \cdot \frac{e^{-ab}}{4\sqrt{2}b^3} \cdot \sin\left(ab+\frac{\pi}{4}\right) = \frac{\pi e^{-ab}}{4\sqrt{2}b^3} \cdot \sin\left(ab+\frac{\pi}{4}\right)$$

$\cdot z_0$ punct singular aparent $\Leftrightarrow (\exists) \lim\limits_{z\to z_0} f(z) =$constantă finită

$\cdot z_0$ pol pentru $f(z) \Leftrightarrow \lim\limits_{z\to z_0} |f(z)| = +\infty$, $f(z) = \frac{\sin z}{z(z-i)^3(z+3)}$

$$\cdot \lim_{z\to i} |f(z)| = \lim_{z\to i} |\frac{\sin z}{z(z+3)}| \cdot \lim_{z\to i} |\frac{1}{(z-i)^3}| =$$

$$= |\frac{\sin i}{i(i+3)}| \cdot \lim_{z\to i} \frac{1}{(z-i)^3} = \frac{sh1}{\sqrt{10}} \cdot \lim_{\substack{x\to 0\\ y\to 1}} \frac{1}{|x+i(y-1)|^3} =$$

$$= \frac{sh1}{\sqrt{10}} \cdot \lim_{\substack{x\to 0\\ y\to 1}} \frac{1}{\left[x^2+(y-1)^2\right]^{\frac{3}{2}}} = \frac{sh1}{\sqrt{10}} \cdot \frac{1}{0^+} = +\infty$$

$\Rightarrow z = i$ pol triplu

$$|x+i(y-1)|^3 = \left[x^2+(y-1)^2\right]^{\frac{3}{2}}$$

$$\sin i = i \cdot sh1 \Rightarrow \frac{\sin i}{i(i+3)} = \frac{i\cdot sh1}{i(i+3)}$$

$$|\frac{sh1}{i+3}| = \frac{|sh1|}{\sqrt{10}} = \frac{sh1}{\sqrt{10}}$$

$$\cdot \cdot f(z) = \frac{1}{z-2i} e^{\frac{1}{z}};$$

$$\lim_{z\to 0} f(z) = -\frac{1}{2i}\lim_{z\to 0} e^{\frac{1}{z}}$$

Facem $z \to 0$ după

$$\begin{cases} x \to 0 \\ y = 0 \end{cases} \Rightarrow \begin{cases} \lim\limits_{\substack{x>0 \\ x\to 0}} e^{\frac{1}{x}} = e^{-\infty} = 0 \\ \lim\limits_{\substack{x>0 \\ x\to 0}} e^{\frac{1}{x}} = e^{\infty} = \infty \end{cases} \Rightarrow \lim_{z\to 0} e^{\frac{1}{z}}$$

SAU:

Considerăm mulţimea:

$$A = \{z \in \mathrm{C} | z = x \in \mathbb{R}\},$$

funcţia $g(z) = e^{\frac{1}{z}}$ şi restricţia ei la $A, g_{/A}^{(z)} = e^{\frac{1}{x}}$

$$\lim_{\substack{x\to 0 \\ x<0}} e^{\frac{1}{x}} = e^{-\infty} = 0 \neq \infty = e^{\infty} = \lim_{\substack{x\to 0 \\ x>0}} e^{\frac{1}{x}}$$

$$\Rightarrow \lim_{x\to 0} e^{\frac{1}{x}} \Rightarrow \lim_{z\to 0} g_{/A}(z) \Rightarrow \lim_{z\to 0} g(z) \Rightarrow$$

$\Rightarrow z = 0$, punct singular esenţial.

(d)

$$\int_{-\infty}^{\infty} \frac{x\sin 2x}{x^2 - 2x + 5} dx$$

$f(z) = \frac{ze^{i2z}}{z^2-2z+5}$ funcţia complexă asociată

$z^2 - 2z + 5 = 0 \Rightarrow z_{1,2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$ poli ordin 1.

$$z = 1 + 2i \in Ss$$

$$\int_{-\infty}^{\infty} \frac{x \sin 2x}{x^2 - 2x + 5} dx = 2\pi Re \underset{z=1+2i}{Res} f(z) =$$

$$= 2\pi \cdot Re \frac{z \cdot e^{i2z}}{2z - 2}|_{z=1+2i} = 2\pi \cdot Re \frac{(1 + 2i)\, e^{2i(1+2i)}}{2 + 4i - 2} \overset{Euler}{=}$$

$$= 2\pi \cdot Re \frac{e^{-4}}{4i} (1 + 2i) (\cos 2 + i \sin 2) =$$

$$= \frac{\pi}{2e^4} Re (2 - i) (\cos 2 + i \sin 2) = \frac{\pi}{2e^4} (2 \cos 2 + \sin 2) .$$

3.3 Exerciţii propuse.

Exerciţiul 3.26 Determinaţi punctele singulare ale următoarelor funcţii complexe (precizaţi şi tipul lor) şi calculaţi reziduurile corespunzătoare:

a)

$$f(z) = \frac{1}{4 + z^2};$$

b)

$$f(z) = \frac{z^2}{z^4 - 1};$$

c)

$$f(z) = \frac{\cos z}{z^6};$$

d)

$$f(z) = \tan z.$$

Exerciţiul 3.27 Folosind teorema reziduurilor calculaţi integralele:

a)

$$\oint\limits_{|z|=1} \frac{\sin z}{z^4} dz;$$

b)

$$\oint\limits_{|z|=\frac{3}{2}} \frac{9z+i}{z^3+z} dz;$$

c)

$$\oint\limits_{|z|=1} \frac{\cosh z}{z^2-3iz} dz;$$

d)

$$\oint\limits_{|z|=1} \frac{1-4z+6z^2}{(z^2+\frac{1}{4})(2-z)} dz;$$

e)

$$\oint\limits_{|z+\frac{i}{2}|=1} \frac{\tan(\pi z)}{(z^3)} dz;$$

f)

$$\oint\limits_{|z|=1} \frac{30z^2-23z+5}{(2z-1)^2(3z-1)} dz;$$

g)

$$\oint_C f(z)dz;$$

unde $f(z) = \frac{ze^{\pi z}}{(z^4-16)+ze^{\frac{\pi}{z}}}$, $C : 9x^2 + y^2 = 9$.

Exerciţiul 3.28 Calculaţi integralele reale:

a)

$$\int_{-\infty}^{\infty} \frac{x}{x^4+1}dx;$$

b)

$$\int_{-\infty}^{\infty} \frac{1}{x^6+1}dx;$$

c)

$$\int_{-\infty}^{\infty} \frac{1}{x^4+16}dx;$$

d)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2-2x+5)^2}dx;$$

e)

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}dx;$$

f)

$$\int_{0}^{2\pi} \frac{1}{37 - 12\cos\theta} d\theta;$$

g)

$$\int_{0}^{2\pi} \frac{1}{5 - 4\sin\theta} d\theta;$$

h)

$$\int_{0}^{2\pi} \frac{1 + 4\cos\theta}{17 - 8\cos\theta} d\theta;$$

i)

$$\int_{-\infty}^{\infty} \frac{\cos 4x}{x^4 + 5x^2 + 4} dx;$$

j)

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{(x^2 + 2x + 2)^2} dx.$$

Capitolul 4

Transformata Fourier (TF). Transformata Fourier discretă (TFD).

4.1 Transformata Fourier

Definiţia 4.1

$$L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} | (L) \int_{-\infty}^{\infty} |f(t)| dt < \infty\}.$$

Observaţia 4.2

1. Dacă f este integrabilă Riemann pe $\mathbb{R}$ atunci f este integrabilă Lebesque pe $\mathbb{R}$;

2. Fie
$$f(t) = \begin{cases} 1, & t \in \mathbb{Q}, \\ 0, & t \notin \mathbb{Q}, \end{cases}$$

atunci f nu este integrabilă Riemann, dar este integrabilă Lebesque;

$$(L)\int_{\mathbb{R}} f(t)dt = 0.$$

3.
$$\left.\begin{array}{l} f \in L^1(\mathbb{R}) \\ g(t) = f(t)e^{-i\omega t}\ (\omega \in \mathbb{R}) \\ |g(t)| = |f(t)| \end{array}\right\} \Rightarrow g \in L^1(\mathbb{R});$$

4. folosind criteriul comparaţiei avem

$$\left.\begin{array}{l} f \in L^1(\mathbb{R}) \\ f(t) = u(t) + iv(t) \end{array}\right\} \Rightarrow u, v \in L^1(\mathbb{R});$$

deoarece

$$|u(t)|, |v(t)| \leq \sqrt{u^2(t) + v^2(t)} = |f(t)|.$$

Definiţia 4.3 Fie $f \in L^1(\mathbb{R})$. Funcţia

$$F : \mathbb{R} \to \mathbb{C}, F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$$

se numeşte *transformata Fourier a lui f*, iar aplicaţia care asociază fiecărui f din $L^1(\mathbb{R})$ transformata sa Fourier se numeşte *transformarea Fourier*

$$f \overset{\mathcal{F}}{\rightsquigarrow} F\ \ (F = \mathcal{F}[f]).$$

Notaţie $F(\omega) = \widehat{f}(\omega)$ dacă $F = \mathcal{F}[f]$.

Observaţia 4.4

1. Funcţia f se numeşte *semnal*; F se numeşte *spectrul* semnalului f sau *funcţia spectrală*;
2. $f(t)$, $t \in \mathbb{R}$ semnal continual;
3. $f : \mathbb{Z} \to \mathbb{C}$ semnal digital;
4. t timp sau spaţiu;
5. $e^{-i\omega t} = \cos \omega t - i \sin \omega t$;
6. $\cos \omega(t+T) = \cos \omega t \Rightarrow \omega T = 2\pi \Rightarrow T = \frac{2\pi}{\omega} \Leftrightarrow \omega = \frac{2\pi}{T} = 2\pi\nu$;
7. $[\omega] = H_z \rightsquigarrow$ frecvenţă.

Exemplul 4.5

1)

$$\Pi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 1, \end{cases}$$

"top-hat", $f \rightsquigarrow$ funcţia dreptunghi.

$$\Pi \in L^1(\mathbb{R}) \Rightarrow \exists \widehat{\Pi}(\omega) = \int_{-\infty}^{\infty} \Pi(t) e^{-i\omega t} dt =$$

$$= \int_{-1}^{1} \Pi(t) e^{-i\omega t} dt = -\frac{1}{i\omega} e^{-i\omega t}|_{-1}^{1} = \frac{e^{-i\omega} - e^{i\omega}}{-i\omega} =$$

$$= 2\frac{\sin \omega}{\omega}.$$

$$\widehat{\Pi}(0) = \int_{-1}^{1} dt = 2 \Rightarrow \widehat{\Pi}(\omega) = 2\mathrm{sa}(\omega),$$

unde *sa* reprezintă funcţia *sinus atenuat* şi este definit astfel:

$$\mathrm{sa}(\omega) = \begin{cases} \frac{\sin\omega}{\omega}, & \omega \neq 0, \\ 1, & \omega = 0. \end{cases}$$

2)

$$\Lambda(t) = \begin{cases} 1 - |t|, & |t| \leq 1, \\ 0, & |t| > 1, \end{cases}$$

$$\widehat{\Lambda}(\omega) = \int_{-\infty}^{\infty} \Lambda(t) e^{-i\omega t} dt = \int_{-1}^{1} (1 - |t|) e^{-i\omega t} dt =$$

$$= \int_{-1}^{1} (1 - |t|) \cos(\omega t) dt - i \int_{-1}^{1} (1 - |t|) \sin(\omega t) dt =$$

$$= 2 \int_{0}^{1} (1 - |t|) \cos(\omega t) dt =$$

$$= 2(1 - t) \frac{\sin\omega t}{\omega} \Big|_0^1 + 2 \int_0^1 \frac{\sin\omega t}{\omega} dt =$$

$$= -\frac{2}{\omega^2} \cos\omega t \Big|_0^1 = \frac{2}{\omega^2}(1 - \cos\omega) = \frac{4}{\omega^2} \sin^2\frac{\omega}{2} =$$

$$= \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} \right)^2.$$

$$\widehat{\Lambda}(0) = \int_{-\infty}^{\infty} \Lambda(t) dt = \int_{-1}^{1} (1 - |t|) dt =$$

$$= 2 \int_0^1 (1 - t) dt = 2 \left(1 - \frac{1}{2} \right) = 1.$$

$$\widehat{\Lambda}(\omega) = \begin{cases} \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}} \right)^2, & \omega \neq 0 \\ 1, & \omega = 0 \end{cases} = \left(\mathrm{sa}\frac{\omega}{2} \right)^2.$$

3)
$$f(t) = e^{-a|t|}, \ \ a > 0.$$
$$F(\omega) = \int_{-\infty}^{0} e^{at} e^{-i\omega t} dt + \int_{0}^{\infty} e^{-at} e^{-i\omega t} dt =$$
$$= \int_{-\infty}^{0} e^{(a-i\omega)t} dt + \int_{0}^{\infty} e^{-(a+i\omega)t} dt =$$
$$= \frac{1}{a-i\omega} e^{(a-i\omega)t} |_{-\infty}^{0} - \frac{1}{a+i\omega} e^{-(a+i\omega)t} |_{0}^{\infty} =$$
$$= \frac{1}{a-i\omega} + \frac{1}{a+i\omega} = \frac{2a}{a^2+\omega^2}.$$

4)
$$f(t) = \begin{cases} 0, & t < 0 \\ e^{-at}, & t \geq 0 \end{cases} \Rightarrow \widehat{F}(\omega) = \frac{1}{a+i\omega}.$$

Proprietatea 4.6 $f \in L^1(\mathbb{R})$, $F = \mathcal{F}[f]$. Atunci

1) F mărginită;

2) $\lim_{\omega \to \pm\infty} F(\omega) = 0$;

3) F uniform continuă pe $\mathbb{R}$;

Proprietatea 4.7 $f, g \in L^1(\mathbb{R})$, $\widehat{f} = \widehat{g} \Rightarrow f = g$ a.p.t. ($f(t) = g(t), \forall t \in \mathbb{R} \setminus A$, măs$(A) = 0$).

Proprietatea 4.8 (teorema de inversare 1) $f \in L^1(\mathbb{R})$, $F = \mathcal{F}[f]$, f cu variaţie mărginită pe orice interval din $\mathbb{R}$ rezultă:

$$\frac{f(t+0) + f(t-0)}{2} = \frac{1}{2\pi} (vp) \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega.$$

Explicaţii: Dacă f cu variaţie mărginită intervalul $[a, b]$ rezultă că $\exists M > 0$ astfel încât

$$\sum_{k=1}^{n} |f(t_k) - f(t_{k-1})| < M$$

pentru orice diviziune

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Observaţia 4.9 f cu variaţie mărginită pe intervalul $[a, b]$ rezultă

a) f mărginită pe $[a, b]$;

b) f integrabilă pe $[a, b]$;

c) eventualele puncte de discontinuitate ale lui f sunt de prima speţă, iar mulţimea acestora este cel mult numărabilă (finită sau numărabilă);

Proprietatea 4.10 (teorema de inversare 2) f continuă, $F \in L^1(\mathbb{R})$ rezultă

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega,$$

în particular,

$$\int_{-\infty}^{\infty} F(\omega) d\omega = 2\pi f(0).$$

Definiţia 4.11 Funcţiile monotone pe porţiuni şi netede pe porţiuni se numesc *funcţii cu variaţie mărginită.*

Exerciţiul 4.12 Să se reprezinte funcţia dreptunghiulară printr-o integrală Fourier.

$$\Pi(t) \rightsquigarrow \widehat{\Pi}(\omega) = 2\mathrm{sa}(\omega),$$

$\mathrm{sa} \notin L^1(\mathbb{R})$, Π monotonă pe porţiuni, rezultă din teorema de inversare 1:

$$\frac{\Pi(t+0)+\Pi(t-0)}{2} = \frac{1}{2\pi}(vp)\int_{-\infty}^{\infty} \widehat{\Pi}(\omega)e^{i\omega t}d\omega =$$

$$= \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} \widehat{\Pi}(\omega)e^{i\omega t}d\omega =$$

$$= \frac{1}{2\pi}\lim_{r\to\infty}\int_{-r}^{r} \frac{2\sin\omega}{\omega}e^{i\omega t}d\omega,$$

dar $e^{i\omega t} = \cos\omega t + i\sin\omega t$, deci

$$= \frac{1}{\pi}\lim_{r\to\infty}\int_{-r}^{r} \frac{\sin\omega}{\omega}\cos\omega t d\omega +$$

$$+ \frac{i}{\pi}\lim_{r\to\infty}\int_{-r}^{r} \frac{\sin\omega}{\omega}\sin\omega t d\omega =$$

$$= \frac{2}{\pi}\lim_{r\to\infty}\int_{0}^{r} \frac{\sin\omega}{\omega}\cos\omega t d\omega =$$

$$= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin\omega}{\omega}\cos\omega t d\omega,$$

adică reprezentarea lui Π printr-o integrală Fourier.

Exerciţiul 4.13 Din exemplul (4.5) punctul 3) avem $f(t) = e^{-a|t|} \Rightarrow F(\omega) = \frac{2a}{\omega^2+a^2}$, f continuă, $F \in L^1(\mathbb{R})$. Atunci cu teorema de inversare avem:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{\omega^2+a^2}(\cos\omega t + i\sin\omega t) =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{2a}{\omega^2+a^2}\cos\omega t d\omega +$$

$$+\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{2a}{\omega^2+a^2}\sin\omega t d\omega =$$

$$= \frac{1}{\pi}\lim_{r\to\infty}\int_{-r}^{r}\frac{2a}{\omega^2+a^2}\cos\omega t d\omega +$$

$$+\frac{i}{\pi}\lim_{r\to\infty}\int_{-r}^{r}\frac{2a}{\omega^2+a^2}\sin\omega t d\omega =$$

$$= \frac{1}{\pi}\int_{0}^{\infty}\frac{2a}{\omega^2+a^2}\cos\omega t d\omega \Rightarrow$$

$$f(t) = \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega \Rightarrow$$

$$f(t) = \frac{2a}{\pi}\int_{0}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega \Rightarrow$$

$$\int_{0}^{\infty}\frac{\cos\omega t}{\omega^2+a^2}d\omega = \frac{\pi}{2a}e^{-a|t|}.$$

Observaţia 4.14 f cu variaţie mărginită intervalul $[a, b]$ rezultă

a) $F(\omega)$ este funcţia spectrală a semnalului f;

b) ω este domeniul de frecvenţe;

c) $F(\omega) = 0$ pentru orice $|\omega| > \Omega$ (f se numeşte semnal cu bandă de frecvenţă limitată).

Proprietatea 4.15 (teorema Plancherel sau teorema energiei) Fie

$$f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \rightsquigarrow (L) \int_{\mathbb{R}} |f(t)|^2 dt < \infty.$$

Rezultă $F \in L^2(\mathbb{R})$ şi

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega.$$

Observaţia 4.16 $L^2(\mathbb{R})$ spaţiu vectorial cu produsul scalar

$$\langle f, g \rangle \overset{def.}{=} \int_{\mathbb{R}} f(t)\overline{g(t)} dt;$$

şi norma

$$\|f\|_{L^2(\mathbb{R})} \overset{def.}{=} \sqrt{\langle f, f \rangle} = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2};$$

$|F(\omega)|^2$ este spectrul de energie al semnalului f.

Exemplul 4.17 $f(t) = e^{-a|t|}$, $a > 0$, $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $F(\omega) = \frac{2a}{\omega^2+a^2}$. Cu teorema energiei avem:

$$2\pi \int_{-\infty}^{\infty} e^{-2a|t|} dt =$$

$$= \int_{-\infty}^{\infty} \left(\frac{2a}{\omega^2 + a^2} \right)^2 d\omega \Rightarrow$$

$$\int_{-\infty}^{\infty} \left(\frac{2a}{\omega^2 + a^2} \right)^2 d\omega = 4\pi \int_0^{\infty} e^{-2at} dt =$$

$$= -\frac{4\pi}{2a} e^{-2at} \Big|_0^{\infty} = \frac{2\pi}{a} \Rightarrow$$

$$2 \cdot 4a^2 \int_0^{\infty} \frac{d\omega}{\left(\omega^2 + a^2\right)^2} = \frac{2\pi}{a} \Rightarrow$$

$$\int_0^{\infty} \frac{d\omega}{\left(\omega^2 + a^2\right)^2} = \frac{\pi}{4a^3}.$$

Proprietatea 4.18 (formula Parseval) Fie $f, g \in L^1(\mathbb{R}) \Rightarrow f \cdot \widehat{g}, \widehat{f} \cdot g \in L^1(\mathbb{R})$ şi are loc relaţia:

$$\int_{-\infty}^{+\infty} f(t) \cdot \widehat{g}(t) dt = \int_{-\infty}^{+\infty} \widehat{f}(t) \cdot g(t) dt.$$

Exerciţiul 4.19 Fie $f(t) = H(t+a) - H(t-a) = \begin{cases} 1, |t| < a \\ 0, |t| \geq a \end{cases}$ şi $g(t) = e^{-at} H(t)$. Avem $\widehat{f}(t) = \frac{\sin at}{t}$ şi $\widehat{g}(t) = \frac{1}{a+it}$. Cu proprietatea (4.18) avem:

$$\int_0^{+\infty} \frac{\sin at}{t} \cdot e^{-at} dt = \int_{-\infty}^{+\infty} f(t) \cdot \frac{1}{a+it} dt =$$

$$= \int_{-a}^{a} \frac{a - it}{a^2 + t^2} dt = 2a \int_0^{a} \frac{dt}{a^2 + t^2} =$$

$$= \frac{2a}{a} \arctan \frac{t}{a} \Big|_0^a = 2 \frac{\pi}{4} = \frac{\pi}{2}.$$

Definiţia 4.20 Dacă f şi g sunt funcţii local integrabile pe $\mathbb{R}$ şi există

$$\int_{-\infty}^{+\infty} f(t - \tau) g(\tau) d\tau \equiv (f * g)(t)$$

atunci $(f * g)(t)$ se numeşte *produsul de convoluţie* al funcţiilor f şi g $(f * g = g * f)$.

Proprietatea 4.21 (reguli de calcul)

1) $\mathcal{F}$ aplicaţie liniară:

$$\mathcal{F}[\alpha f + \beta g] = \alpha\mathcal{F}[f] + \beta\mathcal{F}[g], \alpha, \beta \in \mathbb{C};$$

2) Teorema asemănării (schimbarea de scară):

$$\alpha \in \mathbb{R}^* \Rightarrow \mathcal{F}[f(\alpha t)](\omega) = \frac{1}{|\alpha|}F\left(\frac{\omega}{\alpha}\right),$$

$$f(\alpha t) \rightsquigarrow \frac{1}{|\alpha|}F\left(\frac{\omega}{\alpha}\right), F = \mathcal{F}[f];$$

3) Teorema deplasării:

$$\omega \in \mathbb{R},\ f(t) \cdot e^{i\omega_0 t} \rightsquigarrow F(\omega - \omega_0),$$

deplasarea spectrului unui semnal cu o frecvenţă constantă ω_0: (*modulare*);

$$\mathcal{F}[f(t) \cdot e^{i\omega_0 t}](\omega) = F(\omega - \omega_0);$$

4) Teorema întârzierii (întârziere semnal, deplasare în timp sau translaţie în t):

$$\tau \in \mathbb{R},\ f_\tau(t) = f(t - \tau) \rightsquigarrow e^{-\tau\omega}F(\omega),$$

deviaţie indusă în faza spectrului;

5) $f, g \in L^1(\mathbb{R}) \Rightarrow \exists f * g,\ g * f \in L^1(\mathbb{R})$ şi

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g];$$

6) Teorema de derivare a imaginii: dacă f funcţie continuă şi $tf(t) \in L^1(\mathbb{R})$ atunci:

$$F'(\omega) = -i\mathcal{F}[tf(t)](\omega);$$

7) Teorema de derivare a originalului: dacă f funcţie local netedă pe porţiuni şi $f' \in L^1(\mathbb{R})$ atunci:

$$\mathcal{F}[f'](\omega) = i\omega F(\omega);$$

În general,

$$\mathcal{F}[f^{(k)}](\omega) = (i\omega)^k F(\omega),\ \forall k \in \mathbb{N}^*;$$

8) Proprietatea de simetrie: dacă f funcţie continuă şi $F \in L^1(\mathbb{R})$ atunci:

$$F(t) \overset{F}{\leftrightarrow} 2\pi f(-\omega),$$

unde folosim şi notaţia $f(t) \overset{F}{\leftrightarrow} F(\omega)$ pentru $F(\omega) = \mathcal{F}[f]$.

Exerciţiul 4.22

$$\Pi_a(t) = \begin{cases} 1, |t| \leq a \\ 0, |t| > a. \end{cases}$$

Calculaţi $\Pi_a * \Pi_a(t)$ şi $\mathcal{F}\{\Pi_a * \Pi_a\}$.

Soluţie.

$$\Pi_a * \Pi_a(t) = \int_{-\infty}^{\infty} \Pi_a(\tau) \cdot \Pi_a(t-\tau) d\tau.$$

$$\Pi_a(t-\tau) = \begin{cases} 1, t-a \leq \tau \leq t+a \\ 0, rest. \end{cases}$$

Cazul 1. $t + a < -a \to t < -2a \to$

$$\Pi_a * \Pi_a(t) = \int_{-\infty}^{\infty} 0 d\tau = 0.$$

Cazul 2. $t-a<-a<t+a<a \to -2a<t<0 \to$

$$\Pi_a * \Pi_a(t) = \int_{-a}^{t+a} d\tau = t+2a.$$

Cazul 3. $-a<t-a<a<t+a \to 0<t<2a \to$

$$\Pi_a * \Pi_a(t) = \int_{t-a}^{a} d\tau = 2a-t.$$

Cazul 4. $a<t-a \to t>2a \to \Pi_a * \Pi_a(t) = 0.$

Deci

$$\Pi_a * \Pi_a(t) = \begin{cases} 2a+t, -2a<t<0; \\ 2a-t, 0<t<2a; \\ 0, rest. \end{cases} = \begin{cases} 2a-|t|, |t|<2a; \\ 0, rest. \end{cases}$$

$$\Pi_a(t) \overset{F}{\leftrightarrow} 2asa(a\omega),$$

cu produsul de convoluţie

$$\Pi_a * \Pi_a(t) \overset{F}{\leftrightarrow} 4a^2 sa^2(a\omega).$$

Exerciţiul 4.23 Calculaţi $F[g*f]$ unde $f(t)=e^{-t^2}$ şi $g(t)=e^{-t}\sigma(t)$.

Soluţie.

$$F[g](\omega) = \int_{-\infty}^{\infty} g(t)\cdot e^{-i\omega t}dt =$$

$$= \int_{0}^{\infty} e^{-t}\cdot e^{-i\omega t}dt = \int_{0}^{\infty} e^{-(1+i\omega)t} = -\frac{1}{1+i\omega}\cdot e^{-(1+i\omega)t}|_0^{\infty} = \frac{1}{1+i\omega}$$

$$F\ [f] = ?$$

$$f'(t) = -2t \cdot e^{-t^2} = -2tf(t) \Rightarrow F\ [f'(t)](\omega) = -2F\ [t \cdot f(t)](\omega)$$

$$i\omega F(\omega) = -2iF'(\omega) \Rightarrow \omega F(\omega) = -2F'(\omega) \Rightarrow$$
$$\frac{dF}{F} = -\frac{\omega}{2}d\omega \Rightarrow |F(\omega)| = k \cdot e^{-\frac{\omega^2}{2 \cdot 2}} \Rightarrow$$

$$F(\omega) = \pm k \cdot e^{-\ \frac{\omega^2}{4}} = C \cdot e^{-\ \frac{\omega^2}{4}}$$

$F(\omega) = C \cdot e^{-\frac{\omega^2}{4}}$, C constantă reală nenulă.

$$C = F(0) = \int_{-\infty}^{\infty} e^{-t^2} e^{-i\omega t} dt = \int_{-\infty}^{\infty} e^{-t^2} dt = 2\int_{0}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$

Deci

$$F(\omega) = F[f](\omega) = \sqrt{\pi} \cdot e^{-\frac{\omega^2}{4}} \Rightarrow$$
$$F[g * f](\omega) = F[g](\omega) \cdot F[f](\omega) = \frac{1}{1 + i\omega} \cdot \sqrt{\pi} \cdot e^{-\ \frac{\omega^2}{4}}.$$

Observaţia 4.24

$$F[e^{-t^2}](\omega) = \sqrt{\pi} \cdot e^{-\ \frac{\omega^2}{4}}$$

$$F[e^{-a^2t^2}](\omega) = F[f(at)](\omega) \overset{\text{asemănarea}}{=\!=\!=\!=} \frac{1}{a} F\left(\frac{\omega}{a}\right) = \frac{\sqrt{\pi}}{a} \cdot e^{-\ \frac{\omega^2}{4a^2}}.$$

Exerciţiul 4.25 (Aplicaţie a transformării Fourier pentru ecuaţia căldurii) Ecuaţia căldurii:

$$\begin{cases} \frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0 \\ T(x,0) = f(x) \qquad T(x,t) = ? \\ x \in R,\ t > 0 \end{cases}$$

Notăm

$$F(\omega,t) = \int_{-\infty}^{\infty} T(x,t) \cdot e^{-i\omega x} dx = F(T)$$

transformata Fourier în raport cu variabila x a funcţiei $T(x,t)$.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \Rightarrow F\left[\frac{\partial T}{\partial t}\right] = F\left[\frac{\partial^2 T}{\partial x^2}\right]$$

cu proprietatea (7) derivarea originalului avem:

$$\frac{\partial}{\partial t} F(\omega,t) = (i\omega)^2 F[T]$$

deci $F(\omega,t)$ satisface ecuaţia

$$\frac{\partial F}{\partial t} = -\omega^2 F \Rightarrow$$

$$\frac{\partial F}{F} = -\omega^2 dt \Rightarrow \ln |\ F(\omega,t)\ | = \ln K(\omega) \cdot e^{-\omega^2 t} \Rightarrow$$

$$\Rightarrow F(\omega,t) = \pm K(\omega) \cdot e^{-\omega^2 t} \Rightarrow F(\omega,t) = C(\omega) \cdot e^{-\omega^2 t}$$

$$F(\omega,0) = C(\omega) = \int_{-\infty}^{\infty} f(x) \cdot e^{-i\omega x} dx = F[f(x)](\omega) = \hat{f}(\omega)$$

Deci

$$F(\omega, t) = \hat{f}(\omega) \cdot e^{-\omega^2 t}$$

Din

$$F[e^{-a^2x^2}](\omega) = \frac{\sqrt{\pi}}{a} \cdot e^{-\frac{\omega^2}{4a^2}} = F[e^{-a^2x^2}](\omega)$$

luând $a^2 = \frac{1}{4t}$ rezultă

$$e^{-\omega^2 t} = \frac{1}{2\sqrt{\pi t}} \cdot F[e^{-\frac{x^2}{4t}}](\omega).$$

Deci

$$F(\omega, t) = \hat{f}(\omega) \cdot \frac{1}{2\sqrt{\pi t}} F[e^{-\frac{x^2}{4t}}](\omega) = \frac{1}{2\sqrt{\pi t}} \mathcal{F}\left[f(x) * e^{-\frac{x^2}{4t}}\right](\omega) \Rightarrow$$

$$T(x, y) = \frac{1}{2\sqrt{\pi t}} \int\limits_{-\infty}^{\infty} f(x - y) \cdot e^{-\frac{y^2}{4t}} dy.$$

adică formula Poisson.

4.2 Transformarea Fourier prin sinus şi cosinus

Proprietatea 4.26 Fie $f \in L^1(\mathbb{R})$, atunci

f pară $\Leftrightarrow f(-x) = f(x)\ (\forall)\ x \Rightarrow F(\omega) = 2\int\limits_0^\infty f(t) \cos \omega t \ \mathrm{dt}$

f impară $\Leftrightarrow f(-x) = f(x)\ (\forall) x \Rightarrow F(\omega) = -2i\int\limits_0^\infty f(t) \sin \omega t \ \mathrm{dt}$

Se foloseşte: $e^{-i\omega t} = \cos \omega t - i \sin \omega t$.

Definiţia 4.27 Fie $f \in L^1(0,\infty)$:

$$F_c(\omega) = \int\limits_0^\infty f(t)\cos\omega t \ \mathrm{dt} \rightarrow$$

se numeşte *transformata lui f prin cosinus.*

$$F_s(\omega) = \int\limits_0^\infty f(t)\sin\omega t \ \mathrm{dt} \rightarrow$$

se numeşte *transformata lui f prin sinus.*

Proprietatea 4.28 $f \in L^1(0,\infty)$

1. Fie $\tilde{f}$= prelungirea lui f la $\mathbb{R}$ prin paritate

$$\tilde{f}(x) = \begin{cases} f(x)\ , \ x \geq 0 \\ f(-x), \ x < 0 \end{cases}$$

Reprezentaînd pe $\tilde{f}$ printr-o integrală Fourier avem:

$$\frac{\tilde{f}(t+0)+\tilde{f}(t-0)}{2} = \frac{2}{\pi}\int\limits_0^\infty F_c(\omega)\cos\omega t \ d\omega \ , \ (\forall) \ t > 0$$

de unde : $f(t) = \frac{2}{\pi}\int\limits_0^\infty F_c(\omega)\cos\omega t d\omega$, $t \geq 0 \rightarrow$ Reprezentarea lui f printr-o integrală Fourier in cosinus.

2. Fie $\tilde{f}$ = prelungirea lui f la R prin imparitate

$$\tilde{f}(x) = \begin{cases} f(x)\ , \ x \geq 0 \\ -f(-x)\ , \ x \ < \ 0 \end{cases}$$

$$\frac{\tilde{f}(t+0)+\tilde{f}(t-0)}{2}=\frac{2}{\pi}\int\limits_0^\infty F_s(\omega)\sin\omega t\ d\omega\ ,t>0$$

$\Rightarrow\ f(t)\ =\ \frac{2}{\pi}\int\limits_0^\infty F_s(\omega)\sin\omega t d\omega\ ,\ t\geq 0\ \ \rightarrow$ Reprezentarea lui f printr-o integrală Fourier în sinus.

Exerciţiul 4.29 Să se reprezinte printr-o integrală Fourier în sinus, funcţia:

$$f(t)=\begin{cases}\sin t\ ,\ t\in[0,\pi]\\ 0\ ,\ t>\pi\end{cases}$$

Soluţie. Avem $\widetilde{f}$ prelungirea pe $\mathbb{R}$ a lui f prin imparitate şi găsim: $\widetilde{f}$ este monotonă pe porţiuni pe orice interval compact din $\mathbb{R}$ ($\Rightarrow$ este cu variaţie mărginită pe orice $[a,b]$).

Avem

$$f(t)=\frac{2}{\pi}\int_0^\infty F_s(\omega)\cdot\sin\omega t\ \mathrm{d}\omega,\ t\geq 0$$

$\mathrm{F}_s(\omega)=?$

$$F_s(\omega)=\int\limits_0^\infty f(t)\sin\omega t dt=\int\limits_0^\pi \sin t\cdot\sin\omega t dt=$$

$$=\frac{1}{2}\int\limits_0^\infty[\cos(t-\omega t)-\cos(t+\omega t)]dt=$$

$$=\frac{1}{2}\int\limits_0^\pi[\cos(1-\omega)t-\cos(1+\omega)t]dt=$$

$$\frac{1}{2}\left[\frac{\sin(1-\omega)t}{1-\omega}|_0^\pi - \frac{\sin(1+\omega)t}{1-\omega}|_0^\pi\right] =$$

$$= \frac{1}{2}\left(\frac{\sin\pi\omega}{1-\omega} + \frac{\sin\pi\omega}{1+\omega}\right) = \frac{\sin\pi\omega}{1-\omega^2},\ \omega \neq \pm 1$$

$$\Rightarrow f(t) = \frac{2}{\pi}\int\limits_0^\infty \frac{\sin\pi\omega}{1-\omega^2}\cdot\sin\omega t\ \mathrm{d}\omega.$$

Teorema 4.30 *(Teorema de eşantionare Shannon - Nyquist) Fie funcţia $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, f continuă şi $F(\omega) = 0$, pentru $|\omega| > \Omega$. Atunci*

$$f(t) = \sum_{n=-\infty}^{\infty} f(t_n)\cdot\Omega(t-t_n), \quad t_n = n\frac{\pi}{\omega}.$$

Exerciţiul 4.31

$$f(t) = \begin{cases} 1-|t|\,, t \in [-1,1] \\ 0\ , \text{rest} \end{cases} \quad \mathrm{F}(\omega) = ?$$

Soluţie.

$$F(\omega) = \int\limits_{-\infty}^{\infty} f(t)\cdot\cos\omega t\ \mathrm{dt}\ - i\int\limits_{-\infty}^{\infty} f(t)\cdot\sin\omega t\ \mathrm{dt} =$$

$$= 2\int\limits_0^1 (1-t)\cos\omega t\ \mathrm{dt} = \frac{2}{\omega}\int\limits_0^1 (1-t)\cdot(\sin\omega t)' dt =$$

$$= \frac{2}{\omega}(1-t)\sin\omega t|_0^1 + \frac{2}{\omega}\int\limits_0^1 \sin\omega t\ \mathrm{dt} = \frac{-2}{\omega^2}\cos\omega t|_0^1 =$$

$$= \frac{2}{\omega^2}(1 - \cos\omega) = \left(\frac{\sin\frac{\omega}{2}}{\frac{\omega}{2}}\right)^2, \omega \neq 0.$$

$$F(0) = 2\int_0^1 (1-t)dt = 2 - 1 = 1 \Rightarrow$$

$$F(\omega) = sa^2\frac{\omega}{2}\ ,\ \omega \neq 0.$$

Exercițiul 4.32

$$f(t) = \frac{t}{t^2 + a^2}, a > 0, F(\omega) =?$$

Soluție.

Asociem funcția complexă $f(z) = \frac{z}{z^2+a^2} \cdot e^{-i\omega z}$

$$z^2 + a^2 = 0 \Leftrightarrow z = \pm ia,\ \text{poli simpli}$$

Cu teorema reziduurilor avem:

$$\int_{-\infty}^{\infty} f(x)e^{iax}dx =$$

$$= \begin{cases} 2\pi i \sum\limits_k \operatorname*{Res}\limits_{z=z_k} f(z),\ a > 0 \\ -2\pi i \sum\limits_k \operatorname*{Res}\limits_{z=z_k} f(z),\ a < 0. \end{cases}$$

Calculăm reziduurile în cei doi poli:

$$\operatorname*{Res}_{z=ia} f(z) = \lim_{z \to ia}\ (z - ia) \cdot \frac{z}{(z-ia)(z+ia)} \cdot e^{-i\omega z} =$$

$$= \frac{ia}{2ia} \cdot e^{-i\omega \cdot ia} = \frac{1}{2} \cdot e^{\omega a}$$

$$\operatorname*{Res}_{z=-ia} f(z) = \lim_{z\to -ia} (z+ia)\cdot \frac{z}{(z-ia)(z+ia)}\cdot e^{-i\omega z} =$$

$$= \frac{-ia}{-2ia}\cdot e^{i\omega\cdot ia} = \frac{1}{2}e^{-\omega a}$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}\ \mathrm{dt} = \lim_{r\to\infty}\int_{-r}^{r} f(t)\cdot e^{-i\omega t} dt =$$

$$= \oint_c f(z)dz = \begin{cases} 2\pi i \operatorname*{Res}_{z=ia} f(z)\ , \omega<0 \\ -2\pi i \operatorname*{Res}_{z=-ia} f(z)\ , \omega>0 \end{cases} =$$

$$= \begin{cases} \pi i\cdot e^{\omega a}\ , \omega<0 \\ -\pi i\cdot e^{-\omega a}\ , \omega>0 \end{cases} = -\pi i sgn(\omega)\cdot e^{-|\omega| a}.$$

Exerciţiul 4.33 Fie

$$f(t) = \frac{1}{(1+t^2)^2}.$$

Calculaţi $F(\omega) = ?$

Soluţie. Asociem funcţia complexă $f(z) = \frac{1}{(1+z^2)^2}\cdot e^{-i\omega z}$, $z=\pm i$ poli de ordinul doi.

$$\operatorname*{Res}_{z=i} f(z) = \lim_{z\to i}\left[(z-i)^2\frac{1}{(z-i)^2\cdot(z+i)^2}\cdot e^{-i\omega z}\right]' =$$

$$= \lim_{z\to i}\frac{-i\omega\cdot e^{-i\omega z}\cdot(z+i)^2 - 2e^{i\omega z}\cdot(z+i)}{(z+i)^4} =$$

$$= \frac{-i\omega \cdot e^{-i\omega \cdot i} \cdot 2i - 2e^{-i\omega \cdot i}}{(2i)^3} = \frac{2}{-8i}(\omega - 1) \cdot e^{\omega} =$$

$$= \frac{1}{4i}(1 - \omega) \cdot e^{\omega}.$$

$$\operatorname*{Res}_{z=-i} f(z) = \lim_{z \to -i} \left((z+i)^2 \cdot \frac{1}{(z+i)^2(z-i)^2} \cdot e^{-i\omega z} \right)' =$$

$$\lim_{z \to -i} \frac{e^{-i\omega z} \cdot (-i\omega)(z-i)^2 - 2(z-i) \cdot e^{-i\omega z}}{(z-i)^4} =$$

$$= \frac{e^{-\omega}(-i\omega)(-2i) - 2e^{-\omega}}{(-2i)^3} = -\frac{2(1+\omega)e^{-\omega}}{8i} = -\frac{1}{4i}(1+\omega)e^{-\omega}$$

$$F(\omega) = \int_{-\infty}^{\infty} \frac{1}{(1+t^2)^2} \cdot e^{-i\omega t} dt = \begin{cases} 2\pi i \operatorname*{Res}_{z=i} f(z) \ , \ \omega < 0 \\ -2\pi i \operatorname*{Res}_{z=-i} f(z) \ , \ \omega > 0 \end{cases} =$$

$$= \begin{cases} \frac{\pi}{2}(1-\omega)e^{\omega} \ , \ \omega < 0 \\ \frac{\pi}{2}(1+\omega)e^{-\omega}, \ \omega > 0 \end{cases} = \frac{\pi}{2}\left(1 + |\omega|\right) e^{-|\omega|}.$$

Exerciţiul 4.34 Rezolvaţi ecuaţia intrgrală:

$$\int_0^{\infty} f(t) \cos \omega t \ \mathrm{dt} = \mathrm{g}(\omega) = \begin{cases} 1 - \omega, \ \omega \in [0,1] \\ 0, \ \omega \in \mathbb{R} \backslash [0,1]. \end{cases}$$

Soluţie.

Prelungim prin paritate funcţia $f(t)$ pe $(-\infty, 0)$.

$$2\int_0^{\infty} f(t)\cos\omega t \text{ dt} = 2\text{g}(\omega) \Rightarrow \int_{-\infty}^{\infty} f(t)\cdot\cos\omega t \text{ dt} = 2\text{g}(\omega)$$

$$\int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}dt = \int_{-\infty}^{\infty} f(t)\cos\omega t dt - i\int_{-\infty}^{\infty} f(t)\sin\omega t dt = 2g(\omega)$$

$$\Rightarrow F(\omega) = 2g(\omega) = \begin{cases} 2(1-\omega),\ \omega\in[0,1] \\ 0\ ,\ \text{rest.}\end{cases}$$

Teorema inversării:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega t}d\omega = \frac{1}{\pi}\int_0^1 (1-\omega)\cdot e^{i\omega t}d\omega =$$

$$= \frac{1}{\pi i t}\int_0^1 (1-\omega)\cdot(e^{i\omega t})'d\omega =$$

$$= \frac{1}{\pi i t}(1-\omega)e^{i\omega t}|_0^1 + \frac{1}{\pi i t}\int_0^1 e^{i\omega t}dt = \frac{-1}{\pi i t} + \frac{1}{\pi (it)^2}e^{i\omega t}|_0^1 =$$

$$= -\frac{1}{\pi i t} - \frac{1}{\pi t^2}(e^{it}-1) = \frac{1}{\pi}\left(-\frac{1}{it} - \frac{1}{t^2}e^{it} + \frac{1}{t^2}\right).$$

Exerciţiul 4.35 Să se rezolve ecuaţia integrală:

$$\int_0^{\infty} f(t)\sin\omega t dt = g(\omega) = \begin{cases} e^{\omega} & ,\omega\in[0,2] \\ 0 & ,\text{în rest.}\end{cases}$$

Soluţie.

Prelungim $f(t)$ prin imparitate pe $(-\infty, 0)$.

$$2\int_0^\infty f(t)\cdot\sin\omega t\ \mathrm{dt} = 2\mathrm{g}(\omega) \Rightarrow$$

$$\int_{-\infty}^{\infty} f(t)\sin\omega t dt = 2g(\omega),$$

dar $e^{-i\omega t} = \cos\omega t - i\sin\omega t \Rightarrow$

$$\int_{-\infty}^{\infty} f(t)\cos\omega t dt - i\int_{-\infty}^{\infty} f(t)\cdot\sin\omega t dt = -2ig(\omega) \Rightarrow$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\cdot e^{-i\omega t}dt = -2ig(\omega).$$

$$F(\omega) = \begin{cases} -2ie^{\omega} & ,\omega\in[0,2] \\ 0 & ,\text{în rest.}\end{cases}$$

şi cu teorema inversării:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\cdot e^{i\omega t}d\omega =$$

$$= -\frac{i}{\pi}\int_0^2 e^{\omega}\cdot e^{i\omega t}d\omega = \frac{-i}{\pi}\cdot\int_0^2 e^{(1+it)\omega}d\omega =$$

$$= \frac{-i}{\pi(1+it)}\cdot e^{(1+it)\omega}\Big|_0^2 =$$

$$= \frac{-i}{\pi(1+it)} \cdot (e^{2(1+it)} - 1) = \frac{i}{\pi(1+it)} \cdot (1 - e^{2(1+it)}).$$

Exerciţiul 4.36

$$\int_0^\infty f(t) \cos \omega t \text{ dt} = \begin{cases} 1-\omega & ,\omega \in [0,1] \\ 0 & ,\omega > 1 \end{cases} = g(\omega).$$

Soluţie. Acum folosim reprezentarea lui $f(t)$ printr-o integrală Fourier în cosinus.

$$f(t) = \frac{2}{\pi} \int_0^\infty F_c(\omega) \cdot \cos \omega t \text{ d}\omega$$

unde

$$F_c(\omega) = \int_0^\infty f(t) \cdot \cos \omega t dt =$$

$$= g(\omega) = \begin{cases} 1-\omega & ,\omega \in [0,1] \\ 0 & ,\omega > 1. \end{cases}$$

$$\Rightarrow f(t) = \frac{2}{\pi} \int_0^1 (1-\omega) \cos \omega t d\omega =$$

$$\frac{2}{\pi t}(1-\omega) \sin \omega t|_0^1 + \frac{2}{\pi t} \int_0^1 \sin \omega t d\omega =$$

$$= \frac{-2}{\pi t^2} \cdot \cos \omega t|_0^1 = \frac{-2}{\pi t^2} \cdot (\cos t - 1) = \frac{4 \sin^2 \frac{t}{2}}{\pi t^2}.$$

Exerciţiul 4.37

$$\int_0^\infty f(t)\sin\omega t\ \mathrm{dt} = \mathrm{g}(\omega) = \begin{cases} e^\omega & ,\omega\in[0,2] \\ 0 & ,\omega>2. \end{cases}$$

Soluţie.
Reprezentăm f printr-o integrală Fourier in sinus:

$$f(t) = \frac{2}{\pi}\int_0^\infty F_s(\omega)\cdot\sin\omega t\ \mathrm{d}\omega.$$

$$F_s(\omega) = \int_0^\infty f(t)\sin\omega t\ \mathrm{dt} = \mathrm{g}(\omega) = \begin{cases} e^\omega,\ \omega\in[0{,}2] \\ 0\ ,\ \omega>2 \end{cases}$$

$$\Rightarrow f(t) = \frac{2}{\pi}\int_0^\infty g(\omega)\sin\omega t d\omega = \frac{2}{\pi}\int_0^2 e^\omega\cdot\sin\omega t d\omega =$$

$$= \frac{2}{\pi}\cdot Im\int_0^2 e^\omega\cdot e^{i\omega t}d\omega = \frac{2}{\pi}Im\int_0^2 e^{\omega(1+it)}d\omega =$$

$$= \frac{2}{\pi}\cdot Im\frac{1}{1+it}\cdot e^{\omega(1+it)}|_0^2 = \frac{2}{\pi}Im\frac{e^{2+i2t}-1}{1+it} =$$

$$= \frac{2}{\pi(1+t^2)}Im\frac{e^2(\cos 2t+i\sin 2t)-1}{(1-it)^{-1}} =$$

$$= \frac{2}{\pi(1+t^2)}\cdot[e^2\sin 2t - t(e^2\cos 2t-1)].$$

4.3 Exerciţii propuse

Exerciţiul 4.38 Calculaţi transformatele Fourier ale următoarelor funcţii:

(a) $f(t) = K[\sigma(t-a) - \sigma(t-b)], b > a > 0, K > 0;$

(b) $f(t) = (1 - \frac{|t|}{a})(u(t+2a) - u(t-2a)), a > 0;$

(c) $f(t) = \begin{cases} t - t^2 & , \text{ t} \in (0,1) \\ 0 & , \text{ rest.} \end{cases}$

Exerciţiul 4.39 Calculaţi transformatele Frourier prin sin şi cos pentru:

(a) $f(t) = \begin{cases} 1 & , 0 \leq t < 1 \\ 0 & , t \geq 1; \end{cases}$

(b) $f(t) = e^{-at}$, t $\in (0, \infty), a > 0;$

Exerciţiul 4.40 Rezolvaţi ecuaţiile integrale:

(a) $\int\limits_0^\infty g(u) \sin utdu = \begin{cases} 1, \ 0 \leq t < 1 \\ 2, \ 1 \leq t < 2 \\ 0, \ t > 2; \end{cases}$

(b) $\int\limits_0^\infty g(u) \cos utdu \begin{cases} 1 - \frac{t^2}{2}, \ 0 \leq t < 1 \\ \frac{1}{4} \ , \ 1 \leq t \\ 0 \ , \ t > 1. \end{cases}$

4.4 Transformarea Fourier discretă

Motivare:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-i\omega t} dt \approx \int_{a}^{b} f(t) \cdot e^{-i\omega t} dt \approx$$

$$\approx \sum_{k=0}^{n-1} f(a+kt) \cdot e^{-i\omega(a+kt)} t = t \cdot e^{-i\omega a} \cdot \sum_{k=0}^{n-1} z_k \cdot e^{-i\omega kt}$$

Notăm $\omega_r = r \cdot \frac{2\pi}{nt}$

$$F(\omega_r) \approx t \cdot e^{-i\omega_r a} \sum_{k=0}^{n-1} z_k \cdot e^{-i\frac{2\pi kr}{n}}.$$

Observaţia 4.41 $z^n = 1 \Rightarrow \omega_k = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, k = \overline{0, n-1}$

$$\omega_k = e^{i\frac{2k\pi}{n}} = \left(e^{i\frac{2\pi}{n}}\right)^k$$

$$\omega_1 = e^{i\frac{2\pi}{n}} = \omega \Rightarrow \omega_k = \omega^k$$

Fie $\xi = \overline{\omega} \Rightarrow \xi_k = \xi^k = \overline{\omega}^k = \overline{\omega^k} = \overline{\omega_k}$.
Deci : $\xi_k = \overline{\omega_k} = \xi^k$.

Observaţia 4.42 Fie $z \in \mathbb{C}^n$, $\left.\begin{array}{l} z = (z_0, ..., z_{n-1}) \\ \omega = (\omega_0, ..., \omega_{n-1}) \end{array}\right\} \Rightarrow$ definim: $z \cdot \omega =< z, \omega >= \sum\limits_{k=0}^{n-1} z_k \cdot \overline{\omega_k}$.

Proprietatea 4.43 Fie

$$E_k = \frac{1}{\sqrt{n}}(1, \omega_k, \omega_k^2, ..., \omega_k^{n-1}),$$

$k = \overline{0, n-1}$, atunci

$$\{E_0, E_1, ..., E_{n-1}\}$$

formează bază ortonormală în $\mathbb{C}^n$.

Definiţia 4.44

$$F_k = (1, \omega_k, \omega_k^2, ..., \omega_k^r, ..., \omega_k^{n-1}), k = \overline{0, n-1}$$

şi

$$\hat{z}_k = z \cdot F_k, k = \overline{0, n-1}.$$

Atunci

$$\hat{z} = (\hat{z}_0, \hat{z}_1, ..., \hat{z}_{n-1})$$

reprezintă *trasformata Fourier discretă a lui z* (DFT)(Discret Fourier Transform).

Observaţia 4.45

$$\hat{z}_k = z \cdot F_k = (z_0, z_1, ..., z_{n-1}) \cdot (1, \omega_k, \omega_k^2, ..., \omega_k^{n-1}) =$$

$$= \sum_{r=0}^{n-1} z_r \cdot \overline{\omega_k}^r = \sum_{r=0}^{n-1} z_r \cdot \xi^{kr}.$$

$$\hat{z}_k = \sum_{r=0}^{n-1} \xi^{kr} \cdot z_r, k = \overline{0, n-1}.$$

Introducem matricea $n \times n$ cu componentele $(\xi^{kr})_{k,r=\overline{0,n-1}}$ şi o notăm:

$$\mathbb{F}_n = [\xi^{kr}]_{\substack{k = 0, n-1 \\ r = 0, n-1}}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \xi^3 & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \xi^6 & \xi^{2(n-1)} \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \xi^{n-1} & \xi^{2(n-1)} & \xi^{3(n-1)} & \xi^{(n-1)^2} \end{bmatrix}.$$

Atunci transformata Fourier discretă pentru $z \in \mathbb{C}^n$ este dată de:

$$\begin{bmatrix} \hat{z}_0 \\ . \\ . \\ . \\ \hat{z}_{n-1} \end{bmatrix} = \left[\xi^{kr}\right] \cdot \begin{bmatrix} z_0 \\ . \\ . \\ . \\ z_{n-1} \end{bmatrix}$$

$$\hat{z} = \mathbb{F}_n \cdot z$$

Exerciţiul 4.46 $\mathbb{C}^2$, $\mathbb{F}_2 =?$

Soluţie.

$$z^2 = 1 \Rightarrow z = \pm 1 \Rightarrow$$

$$\Rightarrow \omega = e^{i\frac{2\pi}{2}} = e^{i\pi} = -1 \text{ şi } \xi = \overline{\omega} = -1.$$

$$\mathbb{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & \xi \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exerciţiul 4.47 $\mathbb{C}^4$, $\mathbb{F}_4 =?$ Calculaţi DFT pentru $z = (1, -i, 2, 3i+5)$.

Soluţie.

$$z^4 = 1 \Rightarrow \pm 1, \pm i; \ \omega = e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i$$

$$\Rightarrow \xi = -i.$$

$$\mathbb{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \xi & \xi^2 & \xi^3 \\ 1 & \xi^2 & \xi^4 & \xi^6 \\ 1 & \xi^3 & \xi^6 & \xi^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$

$$\hat{z} = \mathbb{F}_4 \cdot z =$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} 1 \\ -i \\ 2 \\ 3i+5 \end{bmatrix} = \begin{bmatrix} 8+2i \\ -5+5i \\ -2-2i \\ 3-5i \end{bmatrix}$$

Exerciţiul 4.48 $f(t) = \sin t$, $t \in [0, 2\pi]$. Calculaţi DTF pentru $f(t)$.

Soluţie. $\triangle t = \frac{2\pi}{n}, \xi = \overline{\omega} = e^{-i\frac{2\pi}{n}}$.

$$f(k\triangle t) = \sin\frac{k2\pi}{n} = z_k, k = \overline{0, n-1}.$$

$$\hat{z}_k = \sum_{r=0}^{n-1} z_r \cdot \xi^{kr} = \sum_{r=0}^{n-1} \sin\frac{2\pi r}{n} \cdot e^{-i\frac{2\pi kr}{n}} =$$

$$= \sum_{r=0}^{n-1} e^{-i\frac{2kr\pi}{n}} \cdot \frac{e^{i\frac{2\pi r}{n}} - e^{-i\frac{2\pi r}{n}}}{2i} =$$

$$= \frac{1}{2i}\sum_{r=0}^{n-1}\left[e^{i(1-k)\frac{2\pi r}{n}} - e^{-i(1+k)\frac{2\pi r}{n}}\right] =$$

$$= \frac{1}{2i}\left[\sum_{r=0}^{n-1}\left(e^{i(1-k)\frac{2\pi}{n}}\right)^r - \sum_{r=0}^{n-1}\left(e^{-i(1+k)\cdot\frac{2\pi}{n}}\right)^r\right] \overset{k\neq 1, n-1}{=\!=\!=}$$

$$= \frac{1}{2i}\left[\frac{1-e^{i(1-k)2\pi}}{1-e^{i(1-k)\frac{2\pi}{n}}} - \frac{1-e^{-i(1+k)2\pi}}{1-e^{-i(1+k)\frac{2\pi}{n}}}\right] = \frac{1}{2i}\cdot 0 = 0.$$

$$e^{i(1-k)2\pi} = e^{i0} = 1.$$

$$\hat{z}_k = 0 \text{ pentru } k \neq 1, n-1.$$

Pentru $k = 1$ rezultă

$$z_1 = \frac{1}{2i}\left[\sum_{r=0}^{n-1} 1 - \sum_{r=0}^{n-1}\left(e^{-i\frac{4\pi}{n}}\right)^r\right] = \frac{1}{2i}\left(n - \frac{1-1}{1-e^{\frac{-i4\pi}{n}}}\right) = \frac{1}{2i}\cdot n.$$

Pentru $k = n-1$ rezultă

$$z_{n-1} = \frac{1}{2i}\left[\frac{1-e^{-i(n-2)\cdot 2\pi}}{1-e^{-i(n-2)\cdot\frac{2\pi}{n}}} - \sum_{r=0}^{n-1} 1\right] = -\frac{1}{2i}\cdot n.$$

Definiţia 4.49 $\mathbb{F}_n$ inversabilă şi $\mathbb{F}_n^{-1} = \frac{1}{n}\overline{\mathbb{F}_n}$. Atunci avem:

$$\hat{z} = \mathbb{F}_n \cdot z \Leftrightarrow$$
$$z = \mathbb{F}_n^{-1}\cdot\hat{z} = \frac{1}{n}\overline{\mathbb{F}_n}\cdot\hat{z}$$

inversa transformatei Fourier discrete (IDFT).

Capitolul 5

Transformata Z (Laplace discretă). Transformata Laplace în timp discret (TLTD).

Definiţia 5.1 Fie $x : \mathbb{Z} \to \mathbb{C}$ un semnal discret (şir) şi $x(n) = x_n, x \equiv (x_n)_{n\in\mathbb{Z}}$; valorile lui x, x_n se numesc *eşantioane.*

Notăm cu S_d mulţimea semnalelor discrete şi cu S_d^+ mulţimea semnalelor cu suport pozitiv, adică $x_n = 0$ pentru $n < 0$.

Definiţia 5.2 Convoluţie: $x, y \in S_d$ pentru orice $n \in \mathbb{Z}$ avem

$$\{z_n\}_{n\in\mathbb{Z}} = x * y = \sum_{k\in\mathbb{Z}} x_{n-k} \cdot y_k$$

convergentă.

Proprietăţi 5.3 1. $(\forall)\alpha, \beta \in \mathbb{C} \Rightarrow x * (\alpha y + \beta z) = \alpha(x * y) + \beta(x * z)$.

2. Pentru $x, y \in S_d^+ \Rightarrow x * y = y * x$.

3. Semnalul impuls unitate la momentul k: $\delta_k(n)$

$$\delta_k(n) = \begin{cases} 1, \ k = n \\ 0, \ k \neq n \end{cases} = \begin{cases} 1, \ n = k \\ 0, \ n \neq k. \end{cases}$$

$$\delta_0(n) = \begin{cases} 1, \ n = 0 \\ 0, \ n \neq 0 \end{cases} \overset{not.}{=} \delta.$$

4. Semnalul unitate (unitar): $H(n) = \begin{cases} 1, \ n \geq 0 \\ 0, \ n < 0. \end{cases}$

5. $$(x * \delta_k)_n = x_{n-k}, \forall n, k \in \mathbb{Z}$$

   $x * \delta_k$ se numeşte *întârziatul lui x cu k momente*.

5.1 Transformata Z. Definiţie. Proprietăţi. Exemple.

Definiţia 5.4 $D_x \equiv$ mulţimea deschisă maximă de convergenţă a seriei $\sum\limits_{n \in \mathrm{Z}} x_n \cdot z^{-n}$. Fie $x \in S_d, D_x \neq \emptyset$. Atunci

$$X(z) = L(x) = \sum_{n=-\infty}^{\infty} x_n \cdot z^{-n}, z \in D_x$$

se numeşte *transformata Z a semnalului discret x sau transformata Laplace discretă a lui x*.

Observaţia 5.5 Aproximarea transformatei Laplace a lui f - funcţie original cu ajutorul transformării Z.

$$f \in \mathcal{O} \Rightarrow L\left[f\right](s) = \int\limits_0^\infty f(t) \cdot e^{-st} dt =$$

$$= \lim_{n \to \infty} \int\limits_0^{nT} f(t) \cdot e^{-st} dt \overset{T\, suficient\, de\, mic}{\approx}$$

$$\lim_{n \to \infty} \sum_{k=0}^{n-1} f(kT) \cdot e^{-skT} \cdot T \overset{not. x_k = f(kT)}{- - - \to}$$

$$= T \cdot \lim_{n \to \infty} \sum_{k=0}^{n-1} x_k \cdot (e^{sT})^{-k} =$$

$$T \cdot \sum_{k=0}^{\infty} x_k \cdot (e^{sT})^{-k} = T \cdot X(e^{sT}),$$

unde X este transformata Z a lui x:
$X = \mathcal{L}(x)$
$f \in \mathcal{O} \to x \in S_d^+$.

$\mathcal{L}[f](s) = T \cdot X(e^{sT})$
$X(x) = \mathcal{L}(x); x \equiv (x_k)_{k \in \mathbb{Z}} \in S_d^+, x_k = f(kT)$.

Proprietăţi 5.6 (Proprietăţi ale transformatei Z)

1. $x \in S_d \Rightarrow D_x = \emptyset$ sau $D_x =$ disc deschis centrat în 0 sau $D_x =$ coroană circulară deschisă centrată în 0.

2. $x \in S_d^+ \Rightarrow D_x = \emptyset$ sau $D_x =$ exteriorul unui disc centrat în 0.

3. Transformata Z este liniară:

   $$x, y \in S_d, D_x \cap D_y \neq \emptyset \Rightarrow \mathcal{L}(\alpha x + \beta y) = \alpha \mathcal{L}(x) + \beta \mathcal{L}(y)$$

   $\forall \alpha,\ \beta \in \mathbb{C}$, scalari.

4.
$$\left.\begin{array}{l} x, y \in S_d \\ D_x \cap D_y \neq \emptyset \\ \exists x * y \end{array}\right\} \Rightarrow \begin{array}{l} \mathcal{L}(x * y) = \mathcal{L}(x) \cdot \mathcal{L}(y) \\ \text{Vom nota şi astfel:} \\ \text{"}\mathcal{L}[x] = \mathcal{L}(x)\text{".} \end{array}$$

5.
$$\mathcal{L}[nx_n] = -z \cdot X'(z),$$

unde $X(z) = \mathcal{L}[x]$.

6. X funcţie olomorfă pe $B(0; r, R), X = \mathcal{L}[x] \Rightarrow$

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho\in(r,R)} z^{n-1} \cdot X(z)dz, \ \forall n \in \mathbb{Z}.$$

7.
$$\left.\begin{array}{l} X(z) = \frac{A(z)}{B(z)}, grad\ A < \ grad\ B \\ z_1, ..., z_n \ \text{ toate rădăcinile lui } B \\ R = \max\{|z_1|, ..., |z_n|\} \end{array}\right\} \Rightarrow$$

$X(z)$,pentru $|z| > R$ este transformata Z a unui semnal $x \in S_d^+$.

Exemplul 5.7 Fie funcţia unitate $H(n) \to \mathcal{L}[H(n)](z) =$

$$= \sum_{n\in\mathbb{Z}} H(n) \cdot z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots =$$

$$= \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

convergentă $\Leftrightarrow \left|\frac{1}{z}\right| < 1 \Leftrightarrow |z| > 1 \Rightarrow D_x = \{z / |z| > 1\}$

$$X(z) = \frac{z}{z-1}.$$

Exemplul 5.8

$$x_n = H(n) \cdot a^n,\ a \in \mathbb{C}^* \to \sum_{n \in \mathbb{Z}} x_n \cdot z^{-n} = \sum_{n \in \mathbb{N}} a^n \cdot z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

convergentă pentru $\left|\frac{a}{z}\right| < 1 \Leftrightarrow |z| > |a| \Rightarrow$

$$X(z) = \frac{\frac{z}{a}}{\frac{z}{a} - 1} = \frac{z}{z-a},\ \ |z| > |a|\,.$$

$$\mathcal{L}[H(n) \cdot a^n](z) = \frac{z}{z-a}.$$

Exemplul 5.9

$$x_n = nH(n), n \in \mathbb{Z} \to X(z) = \sum_{n \in \mathbb{N}} n \cdot z^{-n}$$

Considerăm seria

$$\sum_{n \in \mathbb{N}} ny^n = y \sum_{n \in \mathbb{N}} ny^{n-1} = y \sum_{n \geq 0} (y^n)' = y \left(\sum_{n \geq 0} y^n\right)'$$

convergentă pentru $|y| < 1 \Rightarrow$

$$\sum_{n \in \mathbb{N}} n \cdot z^{-n}$$

convergentă pentru $|z| > 1$.

Fie

$$S(y) = \sum_{n=1}^{\infty} ny^n = y \left(\sum_{n=0}^{\infty} y^n\right)' =$$

$$y \left(\frac{1}{1-y}\right)' = \frac{y}{(1-y)^2} \Rightarrow$$

$$\Rightarrow X(z) = S\left(\frac{1}{z}\right) = \frac{\frac{1}{z}}{\left(1-\frac{1}{z}\right)^2} = \frac{z}{(z-1)^2}, |z| > 1.$$

$$\mathcal{L}[nH(n)](z) = \frac{z}{(z-1)^2}.$$

Exemplul 5.10

$$x = \delta_k \to \sum_{n\in\mathrm{Z}} (\delta_k)_n \cdot z^{-n} = z^{-k} \Rightarrow$$

$$D_{\delta_k} = \begin{cases} \mathbb{C},\ k \leq 0 \\ \mathbb{C}^*,\ k > 0. \end{cases}$$

$$\mathcal{L}[\delta_k](z) = z^{-k}.$$

Exemplul 5.11

$$\left.\begin{array}{l} \mathcal{L}[x * \delta_k](z) = \mathcal{L}[x_{n-k}](z) \\ \mathcal{L}[x] = X(z) \end{array}\right\} \Rightarrow$$

$$\mathcal{L}[x * \delta_{-k}] = z^{-k} \cdot X(z) \text{ convoluţia.}$$

Altfel:

$$\mathcal{L}[x * \delta_k](z) = \sum_{n\in\mathbb{Z}} (x * \delta_k)_n \cdot z^{-n} =$$

$$\sum_{n\in\mathbb{Z}} x_{n-k} \cdot z^{-n} =$$

$$= z^{-k} \sum_{n\in\mathbb{Z}} x_{n-k} \cdot z^{-(n-k)} = z^{-k} \cdot \sum_{l\in\mathbb{Z}} x_l \cdot z^{-l} = z^{-k} \cdot X(z)$$

Exemplul 5.12

$$x_n = H(n) \cdot \cos(\omega n), n \in \mathbb{Z}, \omega \in \mathbb{R}^*$$

$$x_n = H(n) \cdot \frac{e^{i\omega n} + e^{-i\omega n}}{2} = \frac{1}{2} H(n) \cdot (e^{i\omega})^n + \frac{1}{2} H(n) \cdot (e^{-i\omega})^n$$

$$\Rightarrow \mathcal{L}[x_n](z) \overset{liniaritatea}{=\!=\!=\!=\!=}$$

$$= \frac{1}{2} \mathcal{L}[H(n) \cdot (e^{i\omega})^n](z) + \frac{1}{2} \mathcal{L}[H(n) \cdot (e^{-i\omega})^n](z) =$$

$$= \frac{1}{2} \left(\frac{z}{z - e^{i\omega}} + \frac{z}{z - e^{-i\omega}} \right) =$$

$$= \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1}, |z| > \left| e^{\pm i\omega} \right| = 1.$$

Exemplul 5.13 Analof avem:

$$x_n = H(n) \cdot \sin(\omega n), \omega \neq 0 \Rightarrow$$

$$\mathcal{L}[x_n](z) = \frac{2z\sin\omega}{z^2 - 2z\cos\omega + 1}, |z| > 1.$$

Exerciţiul 5.14 Aflaţi semnalul $(x_n)_{n \in \mathbb{Z}} = x$ dacă se cunoaşte transformata sa Z:

(i)

$$X(z) = \frac{z}{(z-2)(z^2+1)}, |z| > 2;$$

(ii)

$$X(z) = \frac{2z+3}{z^2 - 5z + 6}, |z| > 3;$$

(iii)

$$X(z) = \frac{z}{(z-1)^2(z^2+z-6)}, |z| > 3.$$

Soluţie:

i)

$$X(z) = \frac{A(z)}{B(z)} \overset{P_7}{\Rightarrow} x \in S_d^+ \Rightarrow x_n = 0, n \leq -1.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>2} \frac{z^n}{(z-2)(z+i)(z-i)} dz =$$

$$\overset{teorema\, reziduurilor}{=\!=\!=\!=\!=\!=\!=\!=\!=} \operatorname*{Res}_{z=2} f(z) + 2Re \operatorname*{Res}_{z=i} f(z).$$

Dar

$$\operatorname*{Res}_{z=2} f(z) = \frac{z^n}{(z^2+1)}\Big|_{z=2} = \frac{2^n}{5}$$

şi

$$\operatorname*{Res}_{z=i} f(z) = \frac{z^n}{(z-2)(z+i)}\Big|_{z=i} = \frac{i^n}{(i-2)\cdot 2i} = \frac{1}{2} \cdot \frac{i^n}{-1-2i} =$$

$$= -\frac{1}{2} \cdot \frac{i^n}{1+2i} = -\frac{1}{2} \cdot \frac{i^n}{5}(1-2i) = \frac{1}{10} \cdot i^n(-1+2i) =$$

$$= \begin{cases} \frac{-1+2i}{10}, n = 4k \\ \frac{-2-i}{10}, n = 4k+1 \\ \frac{1-2i}{10}, n = 4k+2 \\ \frac{2+i}{10}, n = 4k+2 \end{cases} \Rightarrow 2Re \operatorname*{Res}_{z=i} f(z) = \begin{cases} \frac{-1}{5}, n = 4k \\ \frac{-2}{5}, n = 4k+1 \\ \frac{1}{5}, n = 4k+2 \\ \frac{2}{5}, n = 4k+3. \end{cases}$$

$$x_n = \frac{2^n}{5} + \begin{cases} -\frac{1}{5}, n = 4k \\ -\frac{2}{5}, n = 4k+1 \\ \frac{1}{5}, n = 4k+2 \\ \frac{2}{5}, n = 4k+3 \end{cases}, n \geq 0.$$

ii) $P_7 \Rightarrow x_n = 0, n < 0.$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{z^{n-1}(2z+3)}{(z-2)(z-3)} dz = \operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=3} f(z).$$

Avem

$$\operatorname*{Res}_{z=2} f(z) = \frac{z^{n-1} \cdot (2z+3)}{z-3}|_{z=2} = -7 \cdot 2^{n-1}$$

şi

$$\operatorname*{Res}_{z=3} f(z) = \frac{z^{n-1}(2z+3)}{z-2}|_{z=3} = 9 \cdot 3^{n-1} = 3^{n+1}.$$

Deci

$$x_n = 3^{n+1} - 7 \cdot 2^{n-1}, n \geq 0.$$

iii)

$$X(z) = \frac{A(z)}{B(z)} \overset{P_7}{\Rightarrow} x_n = 0, n \leq -1.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{z^n}{(z-1)^2(z+3)(z-2)} dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-3} f(z) + \operatorname*{Res}_{z=2} f(z).$$

$$\operatorname*{Res}_{z=1} f(z) = \left[\frac{z^n}{(z+3)(z-2)} \right]'_{|z=1} =$$

$$= \frac{n \cdot z^{n-1}(z+3)(z-2) - z^n(2z+1)}{(z+3)^2 \cdot (z-2)^2}|_{z=1} = \frac{-4n-3}{16}.$$

$$\operatorname*{Res}_{z=-3} f(z) = \frac{z^n}{(z-1)^2 \cdot (z-2)^2}|_{z=-3} = -\frac{1}{80} \cdot (-3)^n.$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{z^n}{(z-1)^2(z+3)}|_{z=2} = \frac{2^n}{5}.$$

Deci

$$x_n = -\frac{4n+3}{16} + \frac{2^n}{5} - \frac{1}{80} \cdot (-3)^n, n \geq 0.$$

Exerciţiul 5.15 Rezolvaţi ecuaţiile de forma $a * x = b$ cu $x \in S_d^+$, unde:

(i)

$$a = \delta_{-2} + \delta_{-1} - 6\delta,\ b_n = nH(n),\ n \in \mathbb{Z};$$

(ii)

$$a = \delta_{-2} - 3\delta_{-1} + 2\delta,\ b_n = 5 \cdot 3^n H(n),\ n \in \mathbb{Z};$$

(iii)

$$a = \delta_{-2} - \frac{5}{2}\delta_{-1} + \delta,\ b_n = \cos(n+1)H(n+1).$$

Soluţie:
i)Folosim convoluţia şi

$$\mathcal{L}(a) \cdot x(z) = \mathcal{L}(b) \Rightarrow$$

$$(z^2 + z - 6)x(z) = \frac{z}{(z-1)^2} \Rightarrow$$

$$x(z) = \frac{z}{(z-1)^2(z-2)(z+3)}, |z| > 3 \Rightarrow$$

$$\Rightarrow x_n = 0, \forall n \leq -1.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{z^n}{(z-1)^2(z-2)(z+3)} dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=-3} f(z).$$

Avem

$$\operatorname*{Res}_{z=1} f(z) = \left[\frac{z^n}{(z-2)(z+3)} \right]'_{|z=1} =$$

$$= \frac{n \cdot z^{n-1}(z-2)(z+3) - z^n(2z+1)}{(z-2)^2(z+3)^2} |_{z=1} =$$

$$= \frac{-4n-3}{4^2} = \frac{-n}{4} + \frac{-3}{4^2} = -\frac{4n+3}{16}.$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{z^n}{(z-1)^2(z+3)} |_{z=2} = \frac{2^n}{5}.$$

$$\operatorname*{Res}_{z=-3} f(z) = \frac{z^n}{(z-1)^2(z-2)} |_{z=-3} = \frac{(-3)^n}{-80} = -\frac{1}{80} \cdot (-3)^n.$$

Deci

$$x_n = -\frac{4n+3}{16} + \frac{2^n}{5} - \frac{1}{80} \cdot (-3)^n, n \geq 0.$$

ii)

$$\mathcal{L}(a)x(z) = \mathcal{L}(b) \Rightarrow (z^2 - 3z + 2)x(z) = \frac{5z}{z-3} \Rightarrow$$

$$X(z) = \frac{5z}{(z-1)(z-2)(z-3)}, |z| > 3 \overset{P_7}{\Rightarrow} x_n = 0$$

pentru $n \leq -1$.

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{5z^n}{(z-1)(z-2)(z-3)} dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) + \operatorname*{Res}_{z=3} f(z) = \frac{5}{2} - 5 \cdot 2^n + \frac{5}{2} \cdot 3^n \Rightarrow$$

$$\Rightarrow x_n = \frac{5}{2}(1 - 2^{n+1} + 3^n), n \geq 0, x_n = 0, \forall n \leq -1$$

Ecuaţia $x * a = b$ se mai scrie:

$$(x * \delta_{-2})_n - 3(x * \delta_{-1})_n + 2(x * \delta)_n = 5 \cdot 3^n H(n), \forall n \in \mathrm{Z} \Rightarrow$$

$$\Rightarrow x_{n+2} - 3x_{n+1} + 2x_n = 5 \cdot 3^n H(n), \forall n \in \mathrm{Z}$$

Calcularea primilor termeni ai şirului:

$$n = -2 \Rightarrow x_0 - 3x_{-1} + 2x_{-2} = 5 \cdot 3^{-2} H(-2), x_0 = 0$$

$$n = -1 \Rightarrow x_1 - 3x_0 + 2x_{-1} = 5 \cdot 3^{-1} H(-1), x_1 = 0$$

iii)

$$\mathcal{L}(a)x(z) = \mathcal{L}(b) \Rightarrow (z^2 - \frac{5}{2}z + 1)x(z) = \mathcal{L}(b).$$

$$b = (b_n)_{n\in \mathrm{Z}}, b_n = \cos(n+1)H(n+1) = \cos(n)H(n) * \delta_{-1} \Rightarrow$$

$$\mathcal{L}(b) = \mathcal{L}(b_n) = \mathcal{L}(\cos(n)H(n)) \cdot \mathcal{L}(\delta_{-1}) =$$

$$= \frac{z(z-\cos 1)}{z^2 - 2z\cos 1 + 1} \cdot z = \frac{z^2(z-\cos 1)}{(z-e^i)(z-e^{-i})}$$

$$\Rightarrow X(z) = \frac{z^2(z-\cos 1)}{\left(z-\frac{1}{2}\right)(z-2)\left(z-e^i\right)\left(z-e^{-i}\right)}, |z| > 2$$

$$\overset{P_7(x\in S_d^+)}{\Rightarrow} x_n = 0, \forall n < 0.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>2} \frac{z^{n+1}(z-\cos 1)}{\left(z-\frac{1}{2}\right)(z-2)\left(z-e^i\right)\left(z-e^{-i}\right)} dz =$$

$$= \operatorname*{Res}_{z=\frac{1}{2}} f(z) + \operatorname*{Res}_{z=2} f(z) + 2Re \operatorname*{Res}_{z=e^i} f(z).$$

Avem

$$\operatorname*{Res}_{z=\frac{1}{2}} f(z) = \frac{z^{n+1}(z-\cos 1)}{(z-2)(z^2-2z\cos 1+1)}\Big|_{z=\frac{1}{2}} =$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{2^n}\left(\frac{1}{2}-\cos 1\right)}{\frac{-3}{2}\left(\frac{1}{4}-\cos 1+1\right)} = -\frac{1}{2^{n-1}} \cdot \frac{1-2\cos 1}{5-4\cos 1}.$$

$$\operatorname*{Res}_{z=2} f(z) = \frac{z^{n+1}(z-\cos 1)}{\left(z-\frac{1}{2}\right)(z^2-2z\cos 1+1)}\Big|_{z=2} =$$

$$= \frac{2^{n+1}(2-\cos 1)}{\frac{3}{2}(4-4\cos 1+1)} = \frac{2^{n+2}(2-\cos 1)}{3(5-4\cos 1)}.$$

$$\operatorname*{Res}_{z=e^i} f(z) = \frac{z^{n+1}(z-\cos 1)}{\left(z-\frac{1}{2}\right)(z-2)\left(z-e^{-i}\right)}\Big|_{z=e^i} =$$

$$= \frac{e^{i(n+1)}(e^i-\cos 1)}{\left(e^i-\frac{1}{2}\right)(e^i-2)\,2i\sin 1} = \frac{e^{i(n+1)}}{(2e^i-1)(e^i-2)} =$$

$$= \frac{e^{i(n+1)}}{(2\cos 1-i\sin 1)(\cos 1-2+i\sin 1)} =$$

$$= \frac{e^{i(n+1)}(2\cos 1+i2\sin 1)(\cos 1-2-i\sin 1)}{\left[(2\cos 1-1)^2+4\sin^2 1\right]\left[(\cos 1-2)^2+\sin^2 1\right]} =$$

$$= \frac{[\cos(n+1)+i\sin(n+1)]}{(5-4\cos 1)(5-4\cos 1)}\cdot$$

$$\cdot\left[(2\cos 1-1)(\cos 1-2)+2\sin^2 1+\right.$$

$$\left.+i(2\sin 1(\cos 1-2)-\sin 1(2\cos 1-1))\right] \Rightarrow$$

$$\Rightarrow Re \operatorname*{Res}_{z=e^i} f(z) =$$

$$= [\cos(n+1)\cdot(4-5\cos 1)-\sin(n+1)\cdot(-3\sin 1)]\cdot$$

$$\cdot\frac{1}{(5-4\cos 1)^2} =$$

$$= \frac{(4-5\cos 1)\cos(n+1)+3\sin(n+1)\sin 1}{(5-4\cos 1)^2}.$$

$$x_n = -\frac{1}{2^{n-1}}\cdot\frac{1-2\cos 1}{5-4\cos 1}+\frac{2^{n+2}}{3}\cdot\frac{2-\cos 1}{5-4\cos 1}+$$

$$+\frac{(4-5\cos 1)\cos(n+1)+3\sin(n+1)\sin 1}{(5-4\cos 1)^2}, n\geq 0.$$

Exerciţiul 5.16 Să se găsească termenul general al şirului:

$$x_{n+2} - 4x_{n+1} + 3x_n = 2H(n),$$

$n \in \mathcal{Z}, x_n \in S_d^+$.

Soluţie: Aplicăm peste relaţia de recurenţă transformata Laplace discretă (transformata Z):

$$(x * \delta_{-2})_n - 4\,(x * \delta_{-1})_n + 3\,(x * \delta)_n = 2H(n), \forall n \in \mathbb{Z} \Rightarrow$$

$$x * (\delta_{-2} - 4\delta_{-1} + 3\delta) = 2H(n) \Rightarrow$$

$$\mathcal{L}(x) \cdot \left(z^2 - 4z + 3\right) = \frac{2z}{z-1}$$

$$\Rightarrow X(z) = \frac{2z}{(z-1)^2(z-3)} \overset{P_7}{\Rightarrow} x \in S_d^+ \Rightarrow x_n = 0, \forall n \leq -1.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{2z^n}{(z-1)^2(z-3)} dz = \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=3} f(z).$$

$$\operatorname*{Res}_{z=1} f(z) = \left(\frac{2z^n}{z-3}\right)'_{|z=1} = \frac{2n \cdot z^{n-1}(z-3) - 2z^n}{(z-3)^2}|_{z=1} =$$

$$= \frac{-n \cdot 4 - 2}{4} = -n - \frac{1}{2}.$$

$$\operatorname*{Res}_{z=3} f(z) = \frac{2z^n}{(z-1)^2}|_{z=3} = \frac{2 \cdot 3^n}{4} = \frac{1}{2} \cdot 3^n.$$

Deci

$$x_n = -n - \frac{1}{2} + \frac{1}{2} \cdot 3^n, n \geq 0, x_n = 0$$

pentru $n \leq -1$.

Calcularea primilor termeni ai şirului:

$$x_{n+2} - 4x_{n+1} + 3x_n = 2H(n)$$

$$n = -2 \Rightarrow x_0 - 4x_{-1} + 3x_{-2} = 2H(-2) \Rightarrow x_0 = 0$$

$$n = -1 \Rightarrow x_1 - 4x_0 + 3x_{-1} = 2H(-1) \Rightarrow x_1 = 1.$$

Exerciţiul 5.17 Rezolvaţi următoarele ecuaţii E.D.F.(ecuaţii cu diferenţe finite) (găsiţi şirurile următoare):

(a)

$$x_{n+2} - 7x_{n+1} + 12x_n = 2 \cdot 5^n, x_0 = 2, x_1 = 1, n \geq 0.$$

Soluţie.

Extindem şirul x_n prin $x_n = 0$ pentru $n \leq -1$. Atunci:

$$x_{n+2} - 7x_{n+1} + 12x_n = \begin{cases} 2 \cdot 5^n, n \geq 0 \\ -13, n = -1 \\ 2, n = -2 \\ 0, n \leq -3 \end{cases} =$$

$$= 2 \cdot 5^n H(n) - 13(\delta_{-1})_n + 2(\delta_{-2})_n = b_n, \forall n \in \mathbb{Z}.$$

Ecuaţia se mai scrie:

$$\left(x * \delta_{-2}\right)_n - 7\left(x * \delta_{-1}\right)_n + 12\left(x * \delta\right)_n = b_n, \forall n \in \mathbb{Z} \Rightarrow$$

$$x * \left(\delta_{-2} - 7\delta_{-1} + 12\delta\right) = b,$$

unde $b \equiv (b_n)_{n \in \mathbb{Z}}$ şi aplicăm ultimei ecuaţii transformata Z.

$$X(z)(z^2-7z+12)=\mathcal{L}(b)=\mathcal{L}(b_n)=$$
$$=2\mathcal{L}(5^nH(n))-13\mathcal{L}(\delta_{-1})+2\mathcal{L}(\delta_{-2})\Rightarrow$$

$$X(z)(z-3)(z-4)=2\frac{z}{z-5}-13z+2z^2=$$
$$=\frac{2z-13z(z-5)+2z^2(z-5)}{z-5}=$$
$$=\frac{2z-13z^2+65z+2z^3-10z^2}{z-5}=\frac{2z^3-23z^2+67z}{z-5}\Rightarrow$$

$$X(z)=\frac{2z^3-23z^2+67z}{(z-3)(z-4)(z-5)},|z|>5\overset{P_7}{\Rightarrow}$$

$x_n=0$ pentru $n<0$.

Pentru $n\geq 0$ rezultă

$$x_n=\frac{1}{2\pi i}\oint\limits_{|z|=\rho>5} z^{n-1}x(z)dz=$$

$$=\frac{1}{2\pi i}\oint\limits_{|z|=\rho>5}\frac{z^n(2z^2-23z+67)}{(z-3)(z-4)(z-5)}dz=$$

$$=\operatorname*{Res}_{z=3}f(z)+\operatorname*{Res}_{z=4}f(z)+\operatorname*{Res}_{z=5}f(z)=$$
$$=\frac{z^n(2z^2-23z+67)}{(z-4)(z-5)}|_{z=3}+\frac{z^n(2z^2-23z+67)}{(z-3)(z-5)}|_{z=4}+$$
$$+\frac{z^n(2z^2-23z+67)}{(z-3)(z-4)}|_{z=5}=$$

$$=\frac{3^n(18-69+67)}{(-1)(-1)}+\frac{4^n(32-92+67)}{1(-1)}+$$

$$+\frac{5^n(50-115+67)}{2(-1)} = 8\cdot 3^n - 7\cdot 4^n - 5^n$$

$$\Rightarrow x_n = 8\cdot 3^n - 7\cdot 4^n - 5^n, n\geq 0.$$

(b)

$$x_{n+2} - 4x_{n+1} + 3x_n = (n+1)\cdot 4^n, n\geq 0, x_0 = 0, x_1 = 1.$$

Extindem x_n cu $x_n = 0$ pentru $n \leq -1$

$$x_{n+2} - 4x_{n+1} + 3x_n = \begin{cases} (n+1)\cdot 4^n & ,n\geq 0 \\ 1 & ,n=-1 \\ 0 & ,n\leq -2 \end{cases}$$

$$= (n+1)\cdot 4^n \cdot H(n) + (\delta_{-1})_n = b_n, \forall n\in\mathbb{Z}.$$

$$(x*\delta_{-2})_n - 4(x*\delta_{-1})_n + 3(x*\delta)_n = (b)_n, \forall n\in\mathbb{Z}, b\equiv (b_n)_{n\in\mathbb{Z}}$$

$$x*(\delta_{-2} - 4\delta_{-1} + 3\delta) = b \Rightarrow$$

$$X(z)\left(\mathcal{L}(\delta_{-2}) - 4\mathcal{L}(\delta_{-1}) + 3\mathcal{L}(\delta)\right) =$$

$$= \mathcal{L}(b) = \mathcal{L}\left((n+1)\cdot 4^n\cdot H(n)\right) + \mathcal{L}(\delta_{-1}) \Rightarrow$$

$$\Rightarrow (z^2 - 4z + 3)X(z) = \frac{4z}{(z-4)^2} + \frac{z}{z-4} + z =$$

$$= z\cdot\frac{4+z-4+z^2-8z+16}{(z-4)^2} =$$

$$= \frac{z(z^2-7z+16)}{(z-4)^2}.$$

$$\underset{|z|>4}{X(z)} = \frac{z(z^2-7z+16)}{(z-1)(z-3)(z-4)^2} \overset{P_7}{\Rightarrow} x_n = 0, n \leq -1.$$

Pentru $n \geq 0$ rezultă

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>4} z^{n-1} X(z) dz =$$

$$= \frac{1}{2\pi i} \oint\limits_{|z|=\rho>4} \frac{z^n(z^2-7z+16)}{(z-1)(z-3)(z-4)^2} dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=3} f(z) + \operatorname*{Res}_{z=4} f(z).$$

Calculăm reziduurile:

$$\operatorname*{Res}_{z=1} f(z) = \frac{z^n(z^2-7z+16)}{(z-3)(z-4)^2}|_{z=1} = \frac{10}{-2\cdot 9} = -\frac{5}{9};$$

$$\operatorname*{Res}_{z=3} f(z) = \frac{z^n(z^2-7z+16)}{(z-1)(z-4)^2}|_{z=3} = \frac{3^n}{2\cdot 1}(9-21+16) = 2\cdot 3^n;$$

$$\operatorname*{Res}_{z=4} f(z) = \left[\frac{z^n(z^2-7z+16)}{(z-1)(z-3)}\right]'_{|z=4} =$$

$$= \frac{\left[n\cdot z^n(z^2-7z+16) + z^n(2z-7)\right](z-1)(z-3)}{(z-1)^2\cdot(z-3)^2}|_{z=4} -$$

$$- \frac{z^n(z^2-7z+16)(z-3+z-1)}{(z-1)^2\cdot(z-3)^2}|_{z=4} =$$

$$= \frac{(n \cdot 4^{n-1}(16 - 28 + 16) + 4^n)(4-1)(4-3)}{3^2 \cdot 1^2} -$$

$$-\frac{4^n(16 - 28 + 16)(8-4)}{3^2 \cdot 1^2} =$$

$$= \frac{(n \cdot 4^n + 4^n) \cdot 3 - 4^{n+2}}{9} =$$

$$= \frac{n}{3} \cdot 4^n - \frac{13}{9} \cdot 4^n.$$

Deci

$$x_n = -\frac{5}{9} + 2 \cdot 3^n - \frac{n}{3} \cdot 4^n - \frac{13}{9} \cdot 4^n, n \geq 0.$$

Exerciţiul 5.18 Şirul lui Fibonacci:

$$\begin{cases} x_{n+2} = x_{n+1} + x_n, n \geq 0 \\ x_0 = 0, x_1 = 1. \end{cases}$$

Soluţie.
Extindem şirul x_n la $x_n = 0, \forall n \leq -1$.

$$x_{n+2} - x_{n+1} - x_n = \begin{cases} 0, n \geq 0 \\ x_1 - x_0 - x_{-1} = 1, n = -1 \\ x_0 - x_{-1} - x_{-2} = 0, n = -2 \\ 0, n \leq -3 \end{cases} =$$

$$= \begin{cases} 0, n \neq -1 \\ 1, n = -1 \end{cases} = (\delta_{-1})_n, \forall n \in \mathbb{Z}.$$

$$x * (\delta_{-2} - \delta_{-1} - \delta) = \delta_{-1} \Rightarrow X(z)(z^2 - z - 1) = z \Rightarrow$$

$$X(z)=\frac{z}{\left(z-\frac{1+\sqrt{5}}{2}\right)\left(z-\frac{1-\sqrt{5}}{2}\right)}, |z|>\frac{1+\sqrt{5}}{2}\overset{P_7}{\Rightarrow} x_n=0, n\le -1$$

$$n\ge 0\Rightarrow x_n=\frac{1}{2\pi i}\oint_{|z|=\rho>\frac{1+\sqrt{5}}{2}} z^{n-1}X(z)dz=$$

$$=\frac{1}{2\pi i}\oint_{|z|=\rho>\frac{1+\sqrt{5}}{2}}\frac{z^n}{\left(z-\frac{1+\sqrt{5}}{2}\right)\left(z-\frac{1-\sqrt{5}}{2}\right)dz}=$$

$$=\operatorname*{Res}_{z=\frac{1+\sqrt{5}}{2}} f(z)+\operatorname*{Res}_{z=\frac{1-\sqrt{5}}{2}} f(z)=$$

$$=\frac{z^n}{z-\frac{1-\sqrt{5}}{2}}\Big|_{z=\frac{1+\sqrt{5}}{2}}+\frac{z^n}{z-\frac{1+\sqrt{5}}{2}}\Big|_{z=\frac{1-\sqrt{5}}{2}}\Rightarrow$$

$$x_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right], \forall n\ge 0.$$

Exerciţiul 5.19

$$x_{n+2}+x_{n+1}-6x_n=n, n\ge 0, x_0=0, x_1=-1$$

Extindem x_n pe $\mathbb{Z}^*$ cu $x_n=0, \forall n\le -1$

$$x_{n+2}+x_{n+1}-6x_n=\begin{cases} n,\ n\ge 0\\ x_1+x_0-6x_{-1}, n=-1\\ 0,\ n\le -2\end{cases}=$$

$$=\begin{cases} n, n\ge 0\\ -1, n=-1\\ 0, n\le -2\end{cases}=$$

$$= n \cdot H(n) - \delta_{-1} = b_n, \forall n \in \mathbb{Z}$$

$$x * (\delta_{-2} + \delta_{-1} - 6\delta) = b, b \equiv (b_n)_{n\in\mathbb{Z}} \Rightarrow$$

$$\Rightarrow X(z)(z^2 + z - 6) = L(b) = L(n \cdot H(n)) - L(\delta_{-1})$$

$$= \frac{z}{(z-1)^2} - z = z \cdot \frac{1 - z^2 + 2z - 1}{(z-1)^2} = \frac{z^2(2-z)}{(z-1)^2}, |z| > 1 \Rightarrow$$

$$X(z) = \frac{z^2(2-z)}{(z-1)^2(z+3)(z-2)} = -\frac{z^2}{(z-1)^2(z+3)}, |z| > 3$$

$$\overset{P_7}{\Rightarrow} x_n = 0, \forall n \leq -1.$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} z^n \frac{-z^2}{(z-1)^2(z+3)} dz =$$

$$= -\frac{1}{2\pi i} \oint\limits_{|z|=\rho>3} \frac{z^{n+1}}{(z-1)^2(z+3)} dz =$$

$$= -\operatorname*{Res}_{z=1} f(z) - \operatorname*{Res}_{z=-3} f(z)$$

$$\operatorname*{Res}_{z=1} f(z) = \left(\frac{z^{n+1}}{z+3}\right)'_{|z=1} = \frac{(n+1) \cdot z^n(z+3) - z^{n+1}}{(z+3)^2}\Big|_{z=1} =$$

$$= \frac{4n+3}{16}$$

$$\operatorname*{Res}_{z=-3} f(z) = \frac{z^{n+1}}{(z-1)^2}\Big|_{z=-3} = \frac{(-3)^{n+1}}{16} = -\frac{3}{16}\cdot(-3)^n$$

$$x_n = \frac{3}{16}\cdot(-3)^n - \frac{4n+3}{16}, n \geq 0.$$

Exerciţiul 5.20

$$x_{n+4} + 2x_{n+3} + 3x_{n+2} + 2x_{n+1} + x_n = 0, \forall n \geq 0$$

$$x_0 = x_1 = 0, x_2 = -1, x_3 = 0.$$

Extindem şirul x_n la $x_n = 0, \forall n \leq -1$

$$x_{n+4} + 2x_{n+3} + 3x_{n+2} + 2x_{n+1} + x_n =$$

$$= \begin{cases} 0, n \geq 0 \\ x_3 + 2x_2 + 3x_1 + 2x_0 + x_{-1}, n = -1 \\ x_2 + 2x_1 + 3x_0, n = -2 \\ x_1 + 2x_0, n = -3 \\ x_0, n = -4 \\ 0, n \leq -5 \end{cases} =$$

$$= \begin{cases} -2, n = -1 \\ -1, n = -2 \\ 0, rest \end{cases} = -2\delta_{-1} - \delta_{-2} = b_n, \forall n \in \mathbb{Z}$$

$$x * (\delta_{-4} + 2\delta_{-3} + 3\delta_{-2} + 2\delta_{-1} + \delta) = b, b \equiv (b_n)_{n\in\mathbb{Z}}$$

$$X(z) = \frac{-z(z+2)}{(z^2+z+1)^2}, |z| > 1 \overset{P_7}{\Rightarrow} x_n = 0, \forall n \leq -1$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>1} \frac{-z(z+2)}{(z^2+z+1)^2} dz = -2Re \operatorname*{Res}_{z=\varepsilon} \frac{z^n(z+2)}{(z-\varepsilon)^2(z-\overline{\varepsilon})^2},$$

unde ε este rădăcina de ordinul trei a unităţii din $Ssup$. $\varepsilon = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ pol de ordin 2.

$$\operatorname*{Res}_{z=\varepsilon} \frac{z^n(z+2)}{(z-\varepsilon)^2(z-\overline{\varepsilon})^2} = \left[\frac{z^n(z+2)}{(z-\overline{\varepsilon})^2}\right]'_{|z=\varepsilon} =$$

$$= \frac{\left[n \cdot z^{n-1}(z+2) + z^n\right](z-\overline{\varepsilon})^2 - 2z^n(z+2)(z-\overline{\varepsilon})}{(z-\overline{\varepsilon})^4}\Big|_{z=\varepsilon} =$$

$$= \frac{1}{(\varepsilon-\overline{\varepsilon})^3}\left[n \cdot \varepsilon^{n-1}(\varepsilon+2)(\varepsilon-\overline{\varepsilon}) + \varepsilon^n(\varepsilon-\overline{\varepsilon}) - 2\varepsilon^n(\varepsilon+2)\right] =$$

$$= \frac{\varepsilon^{n-1}}{(\varepsilon-\overline{\varepsilon})^3}\left[n(\varepsilon+2)(\varepsilon-\overline{\varepsilon}) + \varepsilon(\varepsilon-\overline{\varepsilon}) - 2\varepsilon(\varepsilon+2)\right]$$

$$\left\{\begin{array}{l} \varepsilon \cdot i\sqrt{3} + n \cdot i\sqrt{3} \cdot (\varepsilon+2) - 2\varepsilon^2 - 4\varepsilon = \\ = i\sqrt{3}\left((n+1)\varepsilon + 2n\right) + 2\varepsilon + 2 - 4\varepsilon = \\ = i\sqrt{3}\left[(n+1)\varepsilon + 2n\right] + 2(1-\varepsilon) \\ -\varepsilon^2 = \varepsilon + 1 \text{ şi } \varepsilon - \overline{\varepsilon} = i\sqrt{3}. \end{array}\right.$$

$$\Rightarrow x_n = -2Re\left\{\frac{\varepsilon^{n-1}}{-3} \cdot \left[(n+1)\varepsilon + 2n\right] + \frac{2(1-\varepsilon)}{-i3\sqrt{3}} \cdot \varepsilon^{n-1}\right\}, n \geq 0.$$

Exerciţiul 5.21 Exemplu exerciţiu examen:
a) Cu ajutorul transformatei Z aflaţi şirul: $(x_n)_{n\geq 0}$ definit prin:

$$x_n = x_{n-3} - 3x_{n-2} + 3x_{n-1}, n \geq 3, x_0 = x_1 = 0, x_2 = 1$$

Soluţie: Prelungim şirul x_n cu zero pentru $n < 0$. Avem:

$$x_n - 3x_{n-1} + 3x_{n-2} - x_{n-3} = \begin{cases} 0, n \geq 3 \\ 1, n = 2 \\ 0, n \leq 1 \end{cases} =$$

$$= \begin{cases} 1, n = 2 \\ 0, n \neq 2 \end{cases} = (\delta_2)_n, \forall n \in \mathbb{Z}$$

Ecuaţia se mai scrie cu ajutorul produsului de convoluţie:

$$(x * \delta_0)_n - 3(x * \delta_1)_n + 3(x * \delta_2)_n - (x * \delta_3)_n = (\delta_2)_n, \forall n \in \mathbb{Z} \Rightarrow$$

$$x * (\delta_0 - 3\delta_1 + 3\delta_2 - \delta_3) = \delta_2.$$

Aplicăm transformata Z pentru produsul de convoluţie şi $\mathcal{L}(\delta_k) = z^{-k}$. Avem:

$$X(z)\left(1 - \frac{3}{z} + \frac{3}{z^2} - \frac{1}{z^3}\right) = \frac{1}{z^2} \Rightarrow$$

$$\Rightarrow \underset{|z|>1}{X(z)} = \frac{z}{(z-1)^3} \overset{P_7}{\Rightarrow} x_n = 0, \forall n < 0.$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>1} z^{n-1} X(z) dz = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>1} \frac{z^n}{(z-1)^3} dz =$$

$$\underset{z=1 \text{ pol triplu}}{\overset{th.res}{=}} \frac{1}{2\pi i} \cdot 2\pi i \operatorname*{Res}_{z=1} f(z) =$$

$$= \frac{1}{2!} \lim_{z \to 1} \left[(z-1)^3 \cdot \frac{z^n}{(z-1)^3} \right]'' =$$

$$= \frac{1}{2} n(n-1) \cdot z^{n-2}|_{z=1} = \frac{1}{2}(n^2 - n) \Rightarrow$$

$$\Rightarrow x_n = \frac{n^2 - n}{2}, n \geq 0.$$

b) Aflaţi transformata Z a semnalului:

$$x_n = 2 \cdot 3^n H(n), H(n) = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

$$\mathcal{L}(x_n) = Z\{x_n\} = \sum_{n \in \mathrm{Z}} \frac{2 \cdot 3^n H(n)}{z^n} = 2 \sum_{n=0}^{\infty} \frac{3^n}{z^n} = 2 \sum_{n \geq 0} \left(\frac{3}{z} \right)^n =$$

$$= 2 \cdot \frac{1}{1 - \frac{3}{z}} = \frac{2z}{z-3}, |z| > 3.$$

Exerciţiul 5.22 Rezolvaţi următoarele E.D.F. sau aflaţi şirurile $(x_n)_{n \geq 0}$definite pe următoarele relaţii de recurenţă:

a.

$$10x_n - 3x_{n-1} - x_{n-2} = 4, n \geq 2, x_0 = 2, x_1 = -1;$$

b.

$$6x_n - 5x_{n-1} + x_{n-2} = (0,25)^n, n \geq 2, x_0 = 0, x_1 = 1;$$

c.

$$x_n - 3x_{n-1} + 2x_{n-2} = 4, n \geq 2, x_0 = 5, x_1 = -3;$$

d.
$$x_n - 2x_{n-1} + x_{n-2} = 0, n \geq 2, x_0 = -1, x_1 = 2.$$

Soluţie:
a. Prelungim pe x_n cu 0 pentru $n < 0$.

$$10x_n - 3x_{n-1} - x_{n-2} = \begin{cases} 4, n \geq 2 \\ 10x_1 - 3x_0, n = 1 \\ 10x_0, n = 0 \\ 0, n \leq -1 \end{cases} =$$

$$= \begin{cases} 4, n \geq 2 \\ -16, n = 1 \\ 20, n = 0 \\ 0, n \leq -1 \end{cases} =$$

$$= 4H(n-2) - 16(\delta_1)_n + 20\,(\delta_0)_n\,, \forall n \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow 10\,(x * \delta_0)_n - 3\,(x * \delta_1)_n - (x * \delta_2)_n = b_n \equiv (b_n)_n\,, \forall n \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x * (10\delta_0 - 3\delta_1 - \delta_2) = b$$

aplicăm transformata Z:

$$X(z) \cdot \left(10 - \frac{3}{z} - \frac{1}{z^2}\right) =$$

$$= 4L(H(n-2))(z) - \frac{16}{z} + 20 = \frac{4}{z^2} \cdot \frac{z}{z-1} - \frac{16}{z} + 20 \Leftrightarrow$$

$$\Leftrightarrow X(z) \cdot \frac{10z^2 - 3z - 1}{z^2} =$$

$$= \frac{4z - 16z^2 + 16z + 20z^2 - 20z}{z^2(z-1)} \Rightarrow$$

$$\Rightarrow X(z) = \frac{-16z^2 + 20z^2}{(z-1)(5z+1)(2z-1)} =$$

$$= \frac{4z^2}{(z-1)(5z+1)(2z-1)}, |z| > 1 \stackrel{P_7}{\Rightarrow}$$

$$\Rightarrow x_n = 0, n < 0.$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>1} \frac{4z^{n+1}}{(z-1)(5z+1)(2z-1)} dz =$$

$$= \frac{4}{20\pi i} \oint\limits_{|z|=\rho>1} \frac{z^{n+1}}{(z-1)\left(z+\frac{1}{5}\right)\left(z-\frac{1}{2}\right)} dz =$$

$$= \frac{2\pi i}{5\pi i} \left(\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=\frac{-1}{5}} f(z) + \operatorname*{Res}_{z=\frac{1}{2}} f(z) \right) =$$

$$= \frac{2}{5} \left(\frac{1}{\frac{6}{5} \cdot \frac{1}{2}} + \frac{\left(\frac{-1}{5}\right)^{n+1}}{\left(\frac{-6}{5}\right)\left(\frac{-7}{10}\right)} + \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{-1}{2}\right) \cdot \frac{7}{10}} \right) =$$

$$= \frac{2}{3} + \frac{10}{21} \cdot \left(\frac{-1}{5}\right)^{n+1} - \frac{4}{7} \cdot \frac{1}{2^n}$$

b. $x_n = 0$ pentru $n < 0$.

$$6x_n - 5x_{n-1} + x_{n-2} = \begin{cases} \left(\frac{1}{4}\right)^n, n \geq 2 \\ 6, n = 1 \\ 0, n = 0 \\ 0, n \leq -1 \end{cases} =$$

$$= \begin{cases} \left(\frac{1}{4}\right)^n, n \geq 2 \\ 6, n = 1 \\ 0, n \leq -1 \end{cases} =$$

$$= \left(\frac{1}{4}\right)^n H(n-2) + 6(\delta_1)_n = b_n, \forall n \in \mathbb{Z}.$$

$$6\left(x * \delta_0\right)_n - 5\left(x * \delta_1\right)_n + \left(x * \delta_2\right)_n = (b)_n \equiv b_n, \forall n \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x * (6\delta_0 - 5\delta_1 + \delta_2) = b_n$$

aplicăm transformata Z:

$$(*)\, \mathcal{L}(x) \cdot \left(6 - \frac{5}{z} + \frac{1}{z^2}\right) = \mathcal{L}(b)$$

$$\mathcal{L}(b) = \mathcal{L}\left(\left(\frac{1}{4}\right)^n H(n-2)\right) + 6\mathcal{L}(\delta_1) =$$

$$\overset{\text{întârzierea}}{=\!=\!=\!=\!=} \left(\frac{1}{4}\right)^2 \cdot z^{-2} \cdot \mathcal{L}\left(\left(\frac{1}{4}\right)^n H(n)\right) - \frac{6}{z} =$$

$$= \frac{1}{16z^2} \cdot \frac{z}{z - \frac{1}{4}} + \frac{6}{z} =$$

$$= \frac{1}{4z(4z-1)} + \frac{6}{z} = \frac{24(4z-1)+1}{4z(4z-1)}$$

$$(*) \Rightarrow X(z) \cdot \frac{6z^2 - 5z + 1}{z^2} = \frac{96z - 23}{4z(4z-1)} \Rightarrow$$

$$\Rightarrow X(z) = \frac{z(96z-23)}{4(4z-1)(2z-1)(3z-1)} =$$

$$= \frac{1}{96} \cdot \frac{96z\left(z - \frac{23}{96}\right)}{\left(z - \frac{1}{4}\right)\left(z - \frac{1}{2}\right)\left(z - \frac{1}{3}\right)}$$

$$\Rightarrow X(z) = \frac{z\left(z-\frac{23}{96}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} \overset{P_7}{\Rightarrow} x_n = 0, n < 0.$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=\rho>1} \frac{z^n\left(z-\frac{23}{96}\right)}{\left(z-\frac{1}{4}\right)\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)} dz =$$

$$= \operatorname*{Res}_{z=\frac{1}{4}} f(z) + \operatorname*{Res}_{z=\frac{1}{3}} f(z) + \operatorname*{Res}_{z=\frac{1}{2}} f(z) =$$

$$= \frac{1}{4^n}\left(\frac{1}{4}-\frac{23}{96}\right) \cdot \frac{1}{\left(\frac{1}{4}-\frac{1}{3}\right)\left(\frac{1}{4}-\frac{1}{2}\right)} + \frac{1}{3^n} \cdot \frac{\frac{1}{3}-\frac{23}{96}}{\left(\frac{1}{3}-\frac{1}{4}\right)\left(\frac{1}{3}-\frac{1}{2}\right)} +$$

$$+ \frac{1}{2^n}\left(\frac{1}{2}-\frac{23}{96}\right) \cdot \frac{1}{\left(\frac{1}{2}-\frac{1}{4}\right)\left(\frac{1}{2}-\frac{1}{3}\right)} = \frac{1}{4^{n+1}} - \frac{27}{4} \cdot \frac{1}{3^n} + \frac{25}{2^{n+3}}.$$

c. $x_n = 0, n < 0$

$$x_n - 3x_{n-1} + 2x_{n-2} = \begin{cases} 4, n \geq 2 \\ x_1 - 3x_0, n = 1 \\ x_0, n = 0 \\ 0, n \leq -1 \end{cases} =$$

$$= \begin{cases} 4, n \geq 2 \\ -18, n = 1 \\ 5, n = 0 \\ 0, n \leq -1 \end{cases} =$$

$$= 4H(n-2) - 18(\delta_1)_n + 5(\delta_0)_n = b_n, \forall n \in \mathbb{Z}.$$

$$(x*\delta_0)_n - 3\,(x*\delta_1)_n + 2\,(x*\delta_2)_n = (b)_n, (b)_n \equiv (b)_n \forall n \in \mathbb{Z} \Leftrightarrow$$

$$\Leftrightarrow x*(\delta_0 - 3\delta_1 + 2\delta_2) = b$$

$$X(z)\cdot\left(1-\frac{3}{z}=\frac{2}{z^2}\right) = 4z^{-2}\mathcal{L}(H(n)) - 18z^{-1} + 5 \Leftrightarrow$$

$$\Leftrightarrow X(z)\cdot\frac{z^2-3z+2}{z^2} = \frac{4}{z^2}\cdot\frac{z}{z-1} - \frac{18}{z} + 5$$

$$X(z)\cdot\frac{z^2-3z+2}{z^2} = \frac{4-18z+18+z^2-5z}{z(z-1)}, |z| > 2.$$

Pentru $n \geq 0$ avem

$$x_n = \frac{1}{2\pi i}\int\limits_{|z|=\rho>2}\frac{z^n(5z^2-23z+22)}{(z-1)^2(z-2)}dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) =$$

$$= \lim_{z\to 1}\left[\frac{z^n(5z^2-23z+22)}{z-2}\right]' + \frac{z^n(5z^2-23z+22)}{(z-1)^2}\Big|_{z=2} =$$

$$= \frac{\left[n\cdot z^{n-1}(5z^2-23z+22) + z^n(10z-23)\right](z-2)}{(z-2)^2}\Big|_{z=1} -$$

$$-\frac{z^n(5z^2-23z+22)}{(z-2)^2}\Big|_{z=1} +$$

$$+2^n(20-46+22)=$$

$$=\frac{(4n-13)(-1)-4}{(-1)^2}-2^{n+2}=9-4n-2^{n+2}.$$

Deci:

$$x_n=9-4n-2^{n+2}, n\geq 0.$$

d. $x_n=0, n<0.$

$$x_n-2x_{n-1}+x_{n-2}=\begin{cases}0, n\geq 2\\ x_1-2x_0, n=1\\ x_0, n=0\\ 0, \leq -1\end{cases}=$$

$$=\begin{cases}4, n=1\\ -1, n=0\\ 0, \text{rest}\end{cases}=4(\delta_1)_n-(\delta_0)_n, \forall n\in\mathbb{Z}$$

$$x*(\delta_0-2\delta_{-1}+\delta_{-2})=4\delta_1-\delta_0$$

$$\mathcal{L}(x)\cdot\left(1-\frac{2}{z}+\frac{1}{z^2}\right)=\frac{4}{z}-1\Leftrightarrow X(z)\cdot\frac{(z-1)^2}{z^2}=\frac{4-z}{z}\Rightarrow$$

$$\Rightarrow X(z)=\frac{z(4-z)}{(z-1)^2}, |z|>1\overset{P_7}{\Rightarrow}x_n=0, n<0.$$

Pentru $n\geq 0$ avem

$$x_n=\frac{1}{2\pi i}\oint\limits_{|z|=\rho>1}\frac{z^n(4-z)}{(z-1)^2}dz=\operatorname*{Res}_{z=1}f(z)=$$

$$=\lim_{z\to 1}\left[z^n(4-z)\right]'=\left[n\cdot z^{n-1}(4-z)-z^n\right]|_{z=1}=3n-1.$$

$$x_n=3n-1, n\geq 0.$$

Observaţia 5.23 Cele patru ecuaţii se pot scrie translatat, adică:

$$10x_{n+2} - 3x_{n+1} - x_n = 4,\ n \geq 0,\ x_0 = 2,\ x_1 = 1$$

şi procedăm ca la exerciţiile (5.17)-(5.21).

5.2 Transformata Laplace discretă pentru fruncţia original discretă

Definiţia 5.24 Funcţia

$$f : \mathbb{Z} \to \mathbb{C}$$

se numeşte *funcţia original discretă* dacă:

i) $f(n) = 0$ pentru $n < 0$;

ii) Există $M > 0$ şi $\alpha > 0$ astfel încât

$$|f(n)| \leq M \cdot e^{\alpha n}, \forall n \geq 0.$$

Definiţia 5.25 Se numeşte *transformata Laplace discretă* a funcţiei original discretă f, funcţia de variabilă complexă, notată

$$F^*(q) = \sum_{n=0}^{\infty} f(n) \cdot e^{-nq}.$$

Observaţia 5.26 $F^*(q)$ există $\Leftrightarrow$ conform criteriului comparaţiei:

$$|e^{-nq}| \cdot e^{\alpha n} < 1 \Leftrightarrow e^{(\alpha - Req)n} < 1 \Leftrightarrow$$

$$\Rightarrow Req > \alpha.$$

Observaţia 5.27 Altă notaţie:

$$F^*(q) = D\left[f(n)\right](q).$$

$F^*(q)$ se mai numeşte *imaginea* original discretă.

Formula de inversiune:

$$f(n) = \sum Res\left(F^*(q) \cdot e^{nq}\right)$$

în singularităţile q_k ale lui $F^*(q)$.

Exemplul 5.28

$$f(n) = \begin{cases} a^n, n \geq 0 \\ 0, n < 0 \end{cases} \Rightarrow$$

$$\Rightarrow F^*(q) = \sum_{n=0}^{\infty} a^n \cdot e^{-nq} = \sum_{n\geq 0} \left(\frac{a}{e^q}\right)^n =$$

$$= \frac{1}{1-\frac{a}{e^q}} = \frac{e^q}{e^q - a}, \left|\frac{a}{e^q}\right| < 1 \Leftrightarrow |a| < e^{Req}.$$

Deci:

$$D\left[a^n\right](q) = \frac{e^q}{e^q - a}$$

pentru $e^{Req} > |a|$.

Exemplul 5.29

$$\eta(n) = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases}$$

funcţia lui Heaviside discretă.

Aici: $a = 1$ rezultă

$$D\left[\eta(n)\right](q) = \frac{e^q}{e^q - 1}, e^{Re\, q} > 1.$$

Deci:

$$D\left[\eta(n)\right](q) = \frac{e^q}{e^q - 1}, e^{Re\, q} > 1.$$

Proprietăţi 5.30 Proprietăţi ale transformatei Laplace discretă cu original discret. Reguli de calcul.

1) *Liniaritatea*

$$F^*(q) = D\left[f_1(n) + ... + f_k(n)\right](q) = F_1^*(q) + ... + F_k^*(q).$$

2) *Deplasarea*

$$D\left[f(n) \cdot e^{an}\right](q) = \sum_{n \geq 0} f(n) \cdot e^{-(q-a)n} = D\left[f(n)\right](q - a)$$

Exemplul 5.31

$$D\left[\sin \alpha n\right](q) = \frac{1}{2i}\left(D\left[e^{i\alpha n}\right](q) - D\left[e^{-i\alpha n}\right](q)\right) =$$

$$= \frac{1}{2i}\left(D\left[1\right](q - i\alpha) - D\left[1\right](q + i\alpha)\right) =$$

$$= \frac{1}{2i}\left(\frac{e^{q-i\alpha}}{e^{q-i\alpha} - 1} - \frac{e^{q+i\alpha}}{e^{q+i\alpha} - 1}\right) =$$

$$= \frac{1}{2i} \cdot \frac{e^{2q} - e^{q-i\alpha} - e^{2q} + e^{q+i\alpha}}{e^{2q} - e^{q+i\alpha} - e^{q-i\alpha} + 1} =$$

$$= \frac{e^q}{2i} \cdot \frac{e^{i\alpha} - e^{-i\alpha}}{e^{2q} - e^q(e^{i\alpha} + e^{-i\alpha}) + 1} = \frac{e^q \sin \alpha}{e^{2q} - 2e^q \cos \alpha + 1}.$$

Analog:

$$D\left[\cos \alpha n\right](q) =$$

$$= \frac{e^q(e^q - \cos \alpha)}{e^{2q} - 2e^q \cos \alpha + 1}, e^{Req} > 1.$$

3) *Teorema întârzierii*

$$k \in \mathbb{N}^*, D\left[f(n-k)\right](q) = \sum_{n=0}^{\infty} f(n-k) \cdot e^{-nq} =$$

$$= e^{-kq} \sum_{n=0}^{\infty} f(n-k) \cdot e^{-q(n-k)} =$$

$$= e^{-kq} \sum_{n=0}^{\infty} f(n) \cdot e^{-qn} = e^{-nq} D\left[f(n)\right](q) \Rightarrow$$

$$\Rightarrow D\left[f(n-k)\right](q) = e^{-kq} D\left[f(n)\right](q).$$

$$D\left[f(n+k)\right](q) = \sum_{n=0}^{\infty} f(n+k) \cdot e^{-nq} =$$

$$= e^{kq} \sum_{n=0}^{\infty} f(n+k) \cdot e^{-q(n+k)} =$$

$$= e^{kq} \left[F^*(q) - \sum_{m=0}^{k-1} f(m) \cdot e^{-qm}\right] \Rightarrow$$

$$\Rightarrow D\left[f(n+k)\right](q) = e^{kq}\left(F^*(q) - \sum_{m=0}^{k-1} f(m)\cdot e^{-qm}\right).$$

4) *Derivarea* în raport cu un parametru

$$F^*(q,x) = \sum_{n=0}^{\infty} f(n,x)\cdot e^{-qn}$$

şi există $\frac{\partial f}{\partial x}(n,x)$ astfel încât $\sum\limits_{n\geq 0} \frac{\partial f}{\partial x}(n,x)\cdot e^{-qn}$ converge

$$\Rightarrow \frac{\partial F^*}{\partial x}(q,x) = \sum_{n\geq 0} \frac{\partial f}{\partial x}(n,x)\cdot e^{-qn}.$$

5) *Derivarea imaginii*

$$F^*(q) = \sum_{n=0}^{\infty} f(n)e^{-nq}, k\in\mathbb{N}^* \Rightarrow$$

$$\frac{d^kF^*}{dq^k}(q) = \sum_{n=0}^{\infty}(-1)^k n^k f(n)e^{-nq}.$$

$$\Rightarrow D\left[n^k f(n)\right](q) = (-1)^k \frac{d^kF^*}{dq^k}(q).$$

Exemplul 5.32 Calculaţi transformata Laplace discretă pentru funcţia original discretă:

$$f(n) = n^2.$$

$$F^*(q) = D\left[n^2\cdot 1\right](q) =$$

$$= (-1)^2 \left(D[1](q)\right)_q'' = \left(\frac{e^q}{e^q - 1}\right)'' =$$

$$= \left(1 + \frac{1}{e^q - 1}\right)'' = \left(\frac{-e^q}{(e^q - 1)^2}\right)' =$$

$$= -\frac{e^q(e^q + 1)}{(e^q - 1)^3}, e^{Req} > 1.$$

6) *Teorema de integrare* a transformatei Laplace discrete pentru original discret

$$F^*(s) - f(0) = \sum_{n=1}^{\infty} f(n) e^{-ns}$$

integrăm în raport cu s pe $[q, \infty)$

$$\Rightarrow \int_q^{\infty} [F^*(s) - f(0)]\, ds = \sum_{n=1}^{\infty} f(n) \int_q^{\infty} e^{-ns} ds \Rightarrow$$

$$\Rightarrow D\left[\frac{f(n)}{n}\right](q) = \int_q^{\infty} \left(F^*(s) - f(0)\right) ds, n \geq 1.$$

7) Transformata Laplace discretă a diferenţelor divizate de ordinul k ale originalului

Definiţia 5.33 Funcţia

$$f : \mathbb{N} \to \mathbb{C},$$

defineşte *diferenţele divizate* ale lui f. recursiv:

$$\begin{aligned}
&\Delta^0 f(n) = f(n)\\
&\Delta^1 f(n) = \Delta f(n) = f(n+1) - f(n)\\
&\vdots\\
&\Delta^k f(n) = \Delta^{k-1} f(n+1) - \Delta^{k-1} f(n)
\end{aligned}$$

Observaţia 5.34

$$\Delta^k f(n) = \sum_{m=1}^{k} (-1)^m C_k^m f(n+k-m).$$

$$F^*(\Delta^k f(n))(q) =$$

$$= (e^q - 1)^k \cdot F^*(q) - e^q \sum_{m=0}^{k-1} (e^q - 1)^{k-m-1} \Delta^m f(n).$$

8) Transformata Laplace discretă *pentru un număr finit de funcţii original*

$$g(n) = \sum_{k=0}^{n-1} f(k) \Rightarrow$$

$$D\left[g(n)\right](q) = \frac{F^*(q)}{e^q - 1},$$

unde

$$F^*(q) = D\left[f(n)\right](q).$$

9) Transformata Laplace discretă *a produsului de convoluţie*

$$(f * g)(n) = f(n) * g(n) =$$

$$= \sum_{m=0}^{n} f(n-m)g(m) = g(n) * f(n) = (g * f)(n)$$

produsul de convoluţie discret pentru $f, g : \mathbb{N} \to \mathbb{C}$, funcţii original discrete.

$$D\left[(f * g)(n)\right](q) = F^*(q) \cdot G^*(q).$$

Transformate Laplace discrete uzuale

$$D\left[a^n\right](q) = \frac{e^q}{e^q - a};$$

$$D\left[e^{\alpha n}\right](q) = \frac{e^q}{e^q - e^\alpha};$$

$$D\left[n\right](q) = \frac{e^q}{(e^q - 1)^2};$$

$$D\left[n^2\right](q) = \frac{e^q(e^q + 1)}{(e^q - 1)^3};$$

$$D\left[\sin \beta n\right](q) = \frac{e^q \sin \beta}{e^{2q} - 2e^q \cos \beta + 1};$$

$$D\left[\cos \beta n\right](q) = \frac{e^q(e^q - \cos \beta)}{e^{2q} - 2e^q \cos \beta + 1};$$

$$D\left[sh\beta n\right](q) = \frac{e^q sh\beta}{e^{2q} - 2e^q ch\beta + 1};$$

$$D\left[ch\beta n\right](q) = \frac{e^q(e^q - ch\beta)}{e^{2q} - 2e^q ch\beta + 1};$$

$$D\left[C_k^n e^{\alpha n}\right](q) = \frac{e^{q-\alpha}}{(e^{q-\alpha}-1)^{k+1}} = \frac{e^{q-\alpha}e^{\alpha(k+1)}}{(e^q-e^\alpha)^{k+1}} = \frac{e^{q+\alpha k}}{(e^q-e^\alpha)^{k+1}};$$

$$D\left[C_k^n\right](q) = \frac{e^q}{(e^q-1)^{k+1}};$$

$$D\left[C_k^n a^n\right](q) = \frac{a^k e^q}{(e^q-q)^{k+1}}.$$

Observaţia 5.35 Transformata *Laplace discretă pentru un original discret* se obţine din transformata Z, înlocuind z prin e^{-q}.

5.3 Exerciţii rezolvate

Exerciţiul 5.36 Calculaţi transformata Z a semnalului

$$f_n = \frac{n^2\cdot 2^n}{n^2+4n+3}, n \geq 0, f_n \in S_d^+.$$

Soluţie:

$$\frac{n^2}{n^2+4n+3} = 1 - \frac{4n+3}{n^2+4n+3} = 1 - \frac{4n+3}{(n+1)(n+3)}$$

$$\frac{4n+3}{(n+1)(n+3)} = \frac{a}{n+1} + \frac{b}{n+3} \Rightarrow$$

$$a = \frac{4n+3}{n+3}\Big|_{n=-1} = -\frac{1}{2}$$

$$b = \frac{4n+3}{n+1}\Big|_{n=-3} = \frac{9}{2}$$

$$f_n = \left(1 + \frac{1}{2} \cdot \frac{1}{n+1} - \frac{9}{2} \cdot \frac{1}{n+3}\right) \cdot 2^n + \frac{1}{2} \cdot \frac{2^n}{n+1} - \frac{9}{2} \cdot \frac{2^n}{n+3}.$$

Aplicăm liniaritatea transformatei Z şi avem:

$$Z\{f_n\} = Z\{2^n\} + \frac{1}{2} Z\left\{\frac{2^n}{n+1}\right\} - \frac{9}{2} Z\left\{\frac{2^n}{n+3}\right\}$$

$$Z\{a^n\} = \frac{z}{z-a}\ ,\ |z| > |a|\,;$$

pentru $a = 2$ avem:

$$Z\{2^n\} = \frac{z}{z-2},$$

pentru $|z| > 2$

$$Z\left\{\frac{2^n}{n+1}\right\} = \sum_{n=0}^{\infty} \frac{\frac{2^n}{n+1}}{z^n} = \sum_{n=0}^{\infty} \frac{\left(\frac{2}{z}\right)^n}{n+1} =$$

$$= \frac{z}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{2}{z}\right)^{n+1}}{n+1} = \frac{z}{2} \sum_{n=0}^{\infty} \left(\int_0^{\frac{2}{z}} t^n dt\right) = \frac{z}{2} \int_0^{\frac{2}{z}} \left(\sum_{n=0}^{\infty} t^n\right) dt =$$

$$= \frac{z}{2} \int_0^{\frac{2}{z}} \frac{1}{1-t} dt = -\frac{z}{2} \ln \frac{z}{z-2} \text{ pentru } |z| > 2.$$

Observaţie: Seria $\sum\limits_{n=0}^{\infty} \frac{\frac{2^n}{n+1}}{z^n}$ este convergentă în baza criteriului raportului.

$$\Leftrightarrow \lim_{n\to\infty} \frac{\frac{2^{n+1}}{n+2}}{|z|^{n+1}} \cdot \frac{|z|^n}{\frac{2^n}{n+1}} \Leftrightarrow \frac{2}{|z|} < 1 \Leftrightarrow |z| > 2.$$

$$Z\left\{\frac{2^n}{n+3}\right\} = \frac{1}{8} Z\left\{\frac{2^{n+3}}{n+3}\right\} = \frac{1}{8}\sum_{n=0}^{\infty} \frac{\frac{2^{n+3}}{n+3}}{z^n} =$$

$$= \frac{z^3}{8}\sum_{n=0}^{\infty} \frac{\left(\frac{2}{z}\right)^{n+3}}{n+3} = \frac{z^3}{8}\sum_{n=0}^{\infty}\left(\int_0^{\frac{2}{z}} t^{n+2}dt\right) =$$

$$= \frac{z^3}{8}\int_0^{\frac{2}{z}} \left(\sum_{n=0}^{\infty} t^{n+2}\right) dt =$$

$$= \frac{z^3}{8}\int_0^{\frac{2}{z}} \frac{t^2}{1-t} dt = \frac{z^3}{8}\int_0^{\frac{2}{z}} \left(-t - 1 + \frac{1}{1-t}\right) dt =$$

$$= \frac{z^3}{8}\left[-\frac{t^2}{2}\Big|_0^{\frac{2}{z}} - t\Big|_0^{\frac{2}{z}} - \ln(1-t)\Big|_0^{\frac{2}{z}}\right] =$$

$$= -\frac{z^2}{4} - \frac{z}{4} + \frac{z^3}{8}\ln\frac{z}{z-2}.$$

Deci:

$$Z\{f_n\} = Z\left\{\frac{n^2 \cdot 2^n}{n^2+4n+3}\right\} =$$

$$= \frac{z}{z-2} + \frac{z}{4}\ln\frac{z}{z-2} + \frac{9}{8}z^2 + \frac{9}{8}z - \frac{9}{16}z^3 \ln\frac{z}{z-2}.$$

Metoda a II-a de calcul pentru $Z\left\{\frac{2^n}{n+3}\right\}$ aplicând translaţia la stânga. Mai întai stabilim formula:

$$Z\left\{f_{n+p}\right\}=\sum_{n=0}^{\infty}\frac{f_{n+p}}{z^{n}}=z^{p}\sum_{n=0}^{\infty}\frac{f_{n+p}}{z^{n+p}}=z^{p}\sum_{n=p}^{\infty}\frac{f_{n}}{z^{n}}=$$

$$=z^{p}\left(\sum_{n=0}^{\infty}\frac{f_{n}}{z^{n}}-\sum_{n=0}^{p-1}\frac{f_{n}}{z^{n}}\right)=z^{p}Z\left\{f_{n}\right\}-\sum_{n=0}^{p-1}f_{n}z^{p-n}\Rightarrow$$

$Z\left\{f_{n+p}\right\}=z^{p}Z\left\{f_{n}\right\}-f_{0}z^{p}-f_{1}z^{p-1}-......-f_{p-2}z^{2}-f_{p-1}z\rightarrow$ translaţia la stanga.

$$Z\left\{\frac{2^{n}}{n+3}\right\}=\frac{1}{4}Z\left\{\frac{2^{n+2}}{(n+2)+1}\right\}=$$

$$=\frac{1}{4}z^{2}Z\left\{\frac{2^{n}}{n+1}\right\}-\frac{f_{0}z^{2}}{4}-\frac{f_{1}z}{4}=\frac{1}{4}z^{2}\left(\ln\frac{z}{z-2}\right)\frac{z}{2}-\frac{1}{4}z^{2}-\frac{1}{4}z=$$

$$=\frac{z^{3}}{8}\ln\frac{z}{z-2}-\frac{z^{2}}{4}-\frac{z}{4}\text{ pentru }|z|>2.$$

Observaţie. Translaţia la dreapta:

$$Z\left\{f_{n-p}\right\}=\sum_{n=0}^{\infty}\frac{f_{n-p}}{z^{n}}=z^{-p}\sum_{n=0}^{\infty}\frac{f_{n-p}}{z^{n-p}}=$$

$$=z^{-p}\left(\sum_{n=0}^{p-1}\frac{f_{n-p}}{z^{n-p}}+\sum_{n=p}^{\infty}\frac{f_{n-p}}{z^{n-p}}\right)=z^{-p}Z\left\{f_{n}\right\}+\sum_{n=0}^{p-1}\frac{f_{n-p}}{z^{n}}=$$

$$=z^{-p}Z\left\{f_{n}\right\}+\sum_{n=0}^{p-1}f_{n-p}\cdot z^{-n}=$$

$$=z^{-p}Z\left\{f_{n}\right\}+f_{-p}+f_{1-p}z^{-1}+f_{2-p}z^{-2}+\ldots+f_{-1}z^{1-p},$$

$f_{-p}\,,\ f_{1-p}\,,\ ....\,,\ f_{-1}\rightarrow$ condiţiile iniţiale ale semnalului.

Exerciţiul 5.37 Fie

$$F(z) = \frac{1}{(z-1)^2(z+2)}\,,\ |z| > 2.$$

Aflaţi semnalul discret $x_n \in S_d^+$, $n \in \mathbb{Z}$, $x_n = 0$ pentru $n < 0$ care are pe $F(z)$ ca transformata Z.

<u>Metoda I</u>:

Fie $u_n = 1$, $\forall\, n \in \mathbb{N}$ şi $u_n = 0$, $\forall n \in \mathbb{Z} \setminus \mathbb{N}$.

$$u_{n-p} = \begin{cases} 1 \ , & n \geq p \\ 0 \ , & n < p \end{cases}$$

cu $p \in \mathbb{N}$, fixat.

Fie $x_n \in S_d^+$ un semnal discret şi $p \in \mathbb{N}$ fixat. Atunci:

$$Z\left\{x_{n-p}u_{n-p}\right\} = \sum_{n=0}^{\infty} \frac{x_{n-p}u_{n-p}}{z^n} = \sum_{n=p}^{\infty} \frac{x_{n-p}}{z^n} =$$

$$= \frac{1}{z^p} \sum_{n=p}^{\infty} \frac{f x_{n-p}}{z^{n-p}} = \frac{1}{z^p} \sum_{n=p}^{\infty} \frac{x_n}{z^n} = \frac{1}{z^p} Z\left\{x_n\right\} \Rightarrow$$

aplicăm inversa transformatei Z

$$\Rightarrow Z^{-1}\left(\frac{1}{z^p} Z\left\{x_n\right\}\right) = x_{n-p}\,,\ p \in \mathbb{N}\ \text{ fixat.}$$

$$\frac{1}{(z-1)^2(z+2)} = \frac{a}{(z-1)^2} + \frac{b}{z+2} + \frac{c}{z-1} \Rightarrow a = \frac{1}{z+2}\Big|_{z=1} = \frac{1}{3}\,;$$

$$b = \frac{1}{(z-1)^2}\Big|_{z=-2} = \frac{1}{9} \Rightarrow$$

$$\frac{1}{(z-1)^2(z+2)} = \frac{\frac{1}{3}}{(z-1)^2} + \frac{\frac{1}{9}}{(z+2)} + \frac{c}{(z-1)} =$$

$$= \frac{\frac{(z-1)^2}{9} + \frac{z+2}{3}}{(z-1)^2(z+2)} = \frac{c}{(z-1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{9-z^2-z-7}{9(z+2)(z-1)^2} = \frac{c}{z-1} \Rightarrow \frac{c}{z-1} = -\frac{1}{9}\cdot\frac{1}{z-1} \Rightarrow$$

$$c = -\frac{1}{9}$$

deci

$$F(z) = \frac{1}{3}\cdot\frac{1}{(z-1)^2} + \frac{1}{9}\cdot\frac{1}{z+2} - \frac{1}{9}\cdot\frac{1}{z-1}$$

$$\Rightarrow x_n = \frac{1}{3}Z^{-1}\left(\frac{1}{(z-1)^2}\right) + \frac{1}{9}Z^{-1}\left(\frac{1}{z+2}\right) - \frac{1}{9}Z^{-1}\left(\frac{1}{z-1}\right).$$

$$Z^{-1}\left(\frac{1}{(z-1)^2}\right) = Z^{-1}\left(\frac{1}{z}\cdot\frac{z}{(z-1)^2}\right) =$$

$$= Z^{-1}\left(\frac{1}{z}\cdot Z\left\{n\cdot u(n)\right\}\right) =$$

$$= (n-1)u_{n-1} = \begin{cases} n-1 & , \quad n \geq 1 \\ 0 & , \quad n = 0 \end{cases}$$

$$Z^{-1}\left(\frac{1}{z+2}\right) = Z^{-1}\left(\frac{1}{z}\cdot\frac{z}{z+2}\right) = Z^{-1}\left(\frac{1}{z}\cdot Z\left\{(-2)^n\right\}\right) =$$

$$= (-2)^{n-1} u_{n-1} = \begin{cases} (-2)^{n-1} & , \quad n \geq 1 \\ 0 & , \quad n = 0 \end{cases}$$

$$Z^{-1}\left(\frac{1}{z-1}\right) = Z^{-1}\left(\frac{1}{z} \cdot \frac{z}{z-1}\right) = Z^{-1}\left(\frac{1}{z} \cdot Z\left\{u_n\right\}\right) =$$

$$= u_{n-1} = \begin{cases} 1 \ , & n \geq 1 \\ 0 \ , & n = 0. \end{cases}$$

Deci:

$$\begin{cases} x_n = \frac{n-1}{3} + \frac{(-2)^{n-1}}{9} - \frac{1}{9} = \frac{3n-4+(-2)^{n-1}}{9}, \ n \geq 1 \\ x_n = 0, \ n = 0. \end{cases}$$

Metoda a-II-a:

Aplicăm formula inversei transformatei Z cu reziduuri:

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=3} x^{n-1} \cdot F(z) dz = \frac{1}{2\pi i} \oint\limits_{|z|=3} \frac{z^{n-1}}{(z-1)^2(z+2)} dz \ , \ n \geq 0.$$

Pentru $n \geq 1$ integrantul are 2 poli în $|z| < 3$: $z_1 = -2$ pol de ordin unu, $z_2 = 1$ pol de ordin doi; Notăm integrantul

$$f(z) = \frac{z^{n-1}}{(z-1)^2(z+2)} \Rightarrow$$

cu teorema reziduurilor:

$$x_n = \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-2} f(z).$$

$$\operatorname*{Res}_{z=1} f(z) = \left(\frac{z^{n-1}}{z+2}\right)' \Big|_{z=1} =$$

$$= \left.\frac{(n-1)z^{n-2}\cdot(z+2) - z^{n-1}}{(z+2)^2}\right|_{z=1} = \frac{3(n-1)-1}{9} = \frac{3n-4}{9}$$

$$\operatorname*{Res}_{z=-2} f(z) = \left.\frac{z^{n-1}}{(z-1)^2}\right|_{z=-2} = \frac{(-2)^{n-1}}{9}\ .$$

Deci:

$$x_n = \frac{3n-4+(-2)^{n-1}}{9},\ \text{pentru } n \geq 1.$$

Pentru $n = 0$

$$\Rightarrow x_0 = \frac{1}{2\pi i} \oint\limits_{|z|=3} \frac{1}{z(z-1)^2(z+2)} dz =$$

$$= \operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=-2} f(z) = \operatorname*{Res}_{\xi=0} g(\xi).$$

Unde:

$$f(z) = \frac{1}{z(z-1)^2(z+2)}$$

şi

$$g(\xi) = -\frac{1}{\xi^2}\cdot f\left(\frac{1}{\xi}\right) = -\frac{1}{\xi^2}\cdot\frac{\xi}{\frac{(1-\xi)^2}{\xi^2}\cdot\frac{1+2\xi}{\xi}} = -\frac{\xi^2}{(1-\xi)^2(1+2\xi)}$$

Cum $g(\xi)$ este derivabilă în $\xi = 0 \Rightarrow \xi = 0$ este punct ordinar pentru $g(\xi) \Rightarrow \operatorname*{Res}_{\xi=0} g(\xi) = 0 \Rightarrow x_0 = 0.$

S-a obţinut acelaşi rezultat ca la metoda I-a.

Exerciţiul 5.38 Calculaţi

$$Z^{-1}\left(F(z)\right)$$

unde

$$F(z) = \frac{z^2 \sin\omega}{(z-1)^2(z^2 - 2z\cos\omega + 1)} \text{ pentru } |z| > 1.$$

Soluţie:

<u>Metoda I</u>: descompunerea în fracţii simple şi folosirea transformatelor Z uzuale.

$$F(z) = z^2 \sin\omega \frac{1}{(z^2 - 2z\cos\omega + 1)\,(z^2 - 2z + 1)} =$$

$$= \frac{z\sin\omega}{2 - 2\cos\omega}\left(\frac{1}{z^2 - 2z + 1} - \frac{1}{z^2 - 2z\cos\omega + 1}\right) =$$

$$= \frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{4\sin^2\frac{\omega}{2}} \cdot \frac{z}{(z-1)^2} - \frac{1}{4\sin^2\frac{\omega}{2}} \cdot \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1} =$$

$$= \frac{1}{2}ctg\frac{\omega}{2} \cdot \frac{z}{(z-1)^2} - \frac{1}{4\sin^2\frac{\omega}{2}} \cdot \frac{z\sin\omega}{z^2 - 2z\cos\omega + 1} \Rightarrow$$

$$\Rightarrow x_n = Z^{-1}\left(F(z)\right) =$$

$$= \frac{1}{2}ctg\frac{\omega}{2} \cdot Z^{1}\left(\frac{z}{(z-1)^2}\right) = \frac{1}{4\sin^2\frac{\omega}{2}} \cdot Z^{-1}\left(\frac{z\,sin\,\omega}{z^2 - 2z\cos\omega + 1}\right) \Rightarrow$$

$$\Rightarrow x_n = \left(\frac{1}{2}ctg\frac{\omega}{2}\right) \cdot n - \frac{\sin n\omega}{4\sin^2\frac{\omega}{2}} = \frac{n}{2}ctg\frac{\omega}{2} - \frac{\sin n\omega}{4\sin^2\frac{\omega}{2}}\,,\ n \geq 0.$$

<u>Metoda a II-a</u> cu reziduuri:

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=2} \frac{z^{n+1} \sin \omega}{(z-1)^2 \left(z - e^{i\omega}\right)\left(z - e^{-i\omega}\right)} dz =$$

$$= \operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=e^{i\omega}} f(z) + \operatorname*{Res}_{z=e^{-i\omega}} f(z) =$$

$$= \left(\frac{z^{n+1}}{z^2 - 2z\cos\omega + 1} \right)'_{z=1} \cdot \sin\omega + \frac{z^{n+1} + \sin\omega}{(z-1)^2 (z - e^{-i\omega})} \,|_{z=e^{i\omega}} +$$

$$+ \frac{z^{n+1} + \sin\omega}{(z-1)^2 (z - e^{i\omega})} \,|_{z=e^{-i\omega}} =$$

$$= \frac{\sin\,(n+1)\, z^n (z^2 - 2z\ \cos\omega + 1) - z^{n+1}(2z - \cos\omega)}{\left(z^2 - 2z\ \cos\omega + 1\right)^2} \,|_{z=1} +$$

$$+ \frac{e^{i(n+1)\omega} \cdot \sin\theta}{\left(e^{i\omega} - 1\right)^2 \cdot 2i\sin\theta} + \frac{e^{-i(n+1)\omega} \cdot \sin\theta}{\left(e^{-i\omega} - 1\right)^2 \cdot (e - 2i\sin\theta)} =$$

$$= \sin\omega \frac{[(n+1)\,(2 - 2\cos\omega) - (2 - 2\cos\omega)]}{4\,(1-\cos\omega)^2} +$$

$$+ \frac{1}{2i} \cdot \frac{e^{i(n+1)\omega}}{-4\sin^2\frac{\omega}{2} e^{i\omega}} + \frac{1}{2i} \cdot \frac{e^{-i(n+1)\omega}}{-4\sin^2\frac{\omega}{2} e^{-i\omega}} =$$

$$= \frac{n}{2} \cdot \frac{\sin\omega}{(1-\cos\omega)} - \frac{1}{2i} \cdot \frac{e^{in\omega} - e^{-in\omega}}{4\sin^2\frac{\omega}{2}} =$$

$$= \frac{n}{2} \cdot \frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{2\sin^2\frac{\omega}{2}} - \frac{\sin n\omega}{4\sin^2\frac{\omega}{2}} = \frac{n}{2} ctg \frac{\omega}{2} - \frac{\sin\omega}{4\sin^2\frac{\omega}{2}}\ ,\ n \geq 0.$$

Exerciţiul 5.39 Aflaţi semnalul x_n , $n \geq 1$, pentru care transformata Z este:

$$F(z) = \frac{(z-1)^3 (z - \cos\omega)}{(z+1)(z^2 - 2z\cos\omega + 1)} \ , \ |z| > 1.$$

Soluţie: Merge formula cu reziduuri, cea cu descompunere fiind dificilă!

$$x_n = \frac{1}{2\pi i} \oint\limits_{|z|=2} z^{n-1} \cdot F(z) dz =$$

$$= \frac{1}{2\pi i} \oint\limits_{|z|=2} \frac{z^{n-1} (z-1)^3 (z\cos\omega)}{(z+1)(z - e^{i\omega})(z - e^{-i\omega})} dz =$$

$$= \operatorname*{Res}_{z=-1} f(z) + \operatorname*{Res}_{z=e^{i\omega}} f(z) + \operatorname*{Res}_{z=e^{-i\omega}} f(z) =$$

$$\operatorname*{Res}_{z=-1} f(z) = \lim_{z \to -1} (z+1) \cdot f(z) = \frac{z^{-1} (z-1)^3 (z - \cos\omega)}{z^2 - 2z\cos\omega + 1} \big|_{z=-1} =$$

$$= \frac{(-1)^{n-1} \cdot (-2)^3 (1 - \cos\omega)}{2(1 - \cos\omega)} = 4 \cdot (-1)^n$$

$$\operatorname*{Res}_{z=e^{i\omega}} f(z) = \frac{e^{i(n-1)\omega} (e^{i\omega} - 1)^3 (e^{i\omega} - \cos\omega)}{(e^{i\omega} + 1)(e^{i\omega} - e^{-i\omega})} =$$

$$= \frac{e^{i(n-1)\omega} \cdot (-8i) \sin^3 \frac{\omega}{2} \cdot e^{3i\frac{\omega}{2}}}{2\cos\frac{\omega}{2} \cdot e^{i\frac{\omega}{2}}} = -4i \cdot e^{in\omega} \cdot \frac{\sin^3 \frac{\omega}{2}}{\cos\frac{\omega}{2}}.$$

$$e^{i\omega} - 1 = 2i \sin\frac{\omega}{2} \left(\cos\frac{\omega}{2} + i \sin\frac{\omega}{2} \right) = 2i \sin\frac{\omega}{2} \cdot e^{i\frac{\omega}{2}}$$

$$e^{i\omega}+1=2\cos\frac{\omega}{2}\left(\cos\frac{\omega}{2}+i\sin\frac{\omega}{2}\right)=2\cos\frac{\omega}{2}\cdot e^{i}\frac{\omega}{2}$$

$$\operatorname*{Res}_{z=-e^{i\omega}} f(z)=\operatorname*{Res}_{z=\overline{e^{i\omega}}} f(z)\overset{f\text{ raţională}}{=\!=\!=\!=\!=}\overline{\operatorname*{Res}_{z=e^{i\omega}} f(z)}=$$

$$=\overline{(-4i)\cdot e^{in\omega}\cdot\frac{\sin^3\omega/2}{\cos\omega/2}}=\overline{(-4i)}\cdot\overline{e^{in\omega}}\cdot\overline{\frac{\sin^3\omega/2}{\cos\omega/2}}=$$

$$=4i\cdot e^{-in\omega}\cdot\frac{\sin^3\omega/2}{\cos\omega/2}$$

$$x_n=4\cdot(-1)^n-2\cdot\sin^2\frac{\omega}{2}\cdot tg\frac{\omega}{2}\left(2i\left(e^{in\omega}-e^{-in\omega}\right)\right)\Rightarrow$$

$$x_n=4\,(-1)^n-2\sin^2\frac{\omega}{2}\cdot tg\frac{\omega}{2}\cdot\sin\omega.$$

Exerciţiul 5.40 Calculaţi

$$Z\left\{\frac{a^n}{n+1}\right\}=?\ ,\ a\in\mathbb{C}$$

Soluţie: Aplicăm defniniţia transformatei Z pentru semnalul

$$x_n=\frac{a^n}{n+1}.$$

$$Z\left\{\frac{a^n}{n+1}\right\}=\sum_{n=0}^{\infty}\frac{\frac{a^n}{n+1}}{z^n}=\sum_{n=0}^{\infty}\frac{\left(\frac{a}{z}\right)^n}{n+1}=\frac{z}{a}\cdot\sum_{n=0}^{\infty}\frac{\left(\frac{a}{z}\right)^{n+1}}{n+1}=$$

$$= \frac{z}{a} \cdot \sum_{n=0}^{\infty} \left(\int_0^{\frac{a}{z}} t^n dt \right) = \frac{z}{a} \int_0^{\frac{a}{z}} \left(\sum_{n=0}^{\infty} t^n \right) dt = \frac{z}{a} \int_0^{\frac{a}{z}} \frac{1}{1-t} dt =$$

$$= -\frac{z}{a} \ln(1-t)\Big|_0^{\frac{a}{z}} = -\frac{z}{a} \ln\left(1 - \frac{z}{a}\right) = -\frac{z}{a} \ln \frac{z-a}{z} \Rightarrow$$

$$Z\left\{ \frac{a^n}{n+1} \right\} = \frac{z}{a} \ln \frac{z}{z-a} \quad \text{pentru } |z| > |a|$$

$$\Rightarrow Z^{-1}\left(z \ln \frac{z}{z-a} \right) = \frac{a^{n+1}}{n+1}.$$

pentru $|z| > |a| \to$ aici este convergentă absolut seria:

$$\sum_{n \geq 0} \frac{\frac{a^n}{n+1}}{z^n}.$$

Notăm $y = \frac{1}{z} \Rightarrow$ seria de puteri $\sum\limits_{n \geq 0} \frac{a^n}{n+1} \cdot y^n$; raza ei de convergenţă este $R = \lim\limits_{n \to \infty} \frac{1}{\sqrt[n]{\frac{|a|^n}{n+1}}} = \frac{1}{|a|} \Rightarrow$ seria în y converge pe $|y| < \frac{1}{|a|} \Rightarrow$ seria în z converge pe $\frac{1}{|z|} < \frac{1}{|a|} \Leftrightarrow |z| > |a|$.

Exerciţiul 5.41 Rezolvaţi EDF următoare care satisfac condiţiile corespunzătoare:

$$y_k - 3y_{k-1} + 2y_{k-2} = 2^k, \; k \in \mathbb{N}, \; y_{-1} = -3, \quad y_{-2} = 5.$$

Soluţie:

$$Z\left\{y_{k-1}\right\} = z^{-1}Y(z) - 3;$$

$$Z\left\{y_{k-2}\right\} = z^{-2}Y(z) + z^{-1}y_{-1} + y_{-2} = z^{-2}Y(z) - 3z^{-1} + 5.$$

$$Y(z) - 3\left(\frac{1}{z}Y(z) - 3\right) + 2\left(\frac{1}{z^2}Y(z) - \frac{3}{z} + 5\right) = \frac{z}{z-2} \Leftrightarrow$$

$$\Leftrightarrow \left(1 - \frac{3}{z} + \frac{2}{z^2}\right)Y(z) = -19 + \frac{z}{z-2} = \frac{-18z+38}{z-2} \Leftrightarrow$$

$$\Leftrightarrow Y(z) = \frac{z^2(-18z+38)}{(z-1)(z-2)^2} = 2z\frac{z(-9z+19)}{(z-1)(z-2)^2} \Rightarrow$$

$$\Rightarrow \frac{-9z^2+19z}{(z-1)(z-2)^2} = \frac{a}{z-1} + \frac{b}{(z-2)^2} + \frac{c}{z-2} \Rightarrow$$

$$\Rightarrow a = \frac{-9z^2+19z}{(z-2)^2}\Big|_{z=1} = \frac{10}{(-1)^2} \Rightarrow a = 10$$

$$b = \frac{-9z^2+19z}{z-1}\Big|_{z=2} = \frac{38-36}{2-1} \Rightarrow b = 2$$

$$\frac{c}{z-2} = \frac{-9z^2+19z}{(z-1)(z-2)^2} - \frac{10}{z-1} - \frac{2}{(z-2)^2} =$$

$$= \frac{-9z^2+19z-10z^2+408-40-2z+2}{(z-1)(z-2)^2} =$$

$$= \frac{-19z^2(z^2-3z+2)}{(z-1)(z-2)^2} = \frac{-19}{z-2} \Rightarrow c = -19$$

$$\Rightarrow Y(z) = 20 \cdot \frac{z}{z-1} + 2 \cdot \frac{2z}{(z-2)^2} - 38 \cdot \frac{z}{z-2} \Rightarrow$$

$$y_k = 20u_k + 2k \cdot 2^k = 20u_k + 2^{k+1} \cdot k - 19 \cdot 2^{k+1}.$$

5.4 Probleme propuse

1) Determinaţi semnalul $x \in S_d^+$ a cărui transformată Z este:

a)
$$\frac{z}{(z-1)(z^2+1)};$$

b)
$$\frac{z}{(z-1)^2(z^2+z-6)};$$

c)
$$\frac{z^2+1}{z^2-z+1}.$$

2) Determinaţi şirurile $\{x_n\}_{n\in\mathbb{N}}$ definite prin recurenţă:

a)$x_0 = 4, x_1 = 6, x_{n+2} - 3x_{n+1} + 2x_n = n, n \in \mathbb{N}$

b)$x_0 = 0, x_1 = -1, x_{n+2} + x_{n+1} - 6x_n = n, n \in \mathbb{N}$

c)$x_0 = 0, x_1 = 1, x_{n+2} - 5x_{n+1} + 6x_n = 4 \cdot 5^n$

d)$x_0 = 0, x_1 = 2, x_{n+2} - 3x_{n+1} + 2x_n = 2^n, n \geq 0$

3) Aflaţi transformata Z a semnalului $x_n = e^{-an} \cos \omega n H(n), \omega \in \mathbb{R}, a \in \mathbb{C}$.

Capitolul 6

Serii Fourier. Coarda vibrată finită.

Definiţia 6.1 Seria

$$\frac{a_0}{2} + \sum_{n\geq 1} (a_n \cdot \cos nx + b_n \sin nx)$$

se numeşte *serie trigonometrică*.

Formă complexă a seriei este:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \frac{e^{inx} + e^{-inx}}{2} + b_n \cdot \frac{e^{inx} - e^{-inx}}{2i} \right) =$$

$$= \frac{a_0}{2} + \sum_{n\geq 1} \left[\left(\frac{a_n - ib_n}{2} \right) \cdot e^{inx} + \left(\frac{a_n + ib_n}{2} \right) \cdot e^{-inx} \right] =$$

$$= \sum_{n\in \mathrm{Z}} c_n \cdot e^{inx},$$

unde $c_n = \begin{cases} c_0 = \frac{a_0}{2} \\ c_n = \frac{a_n - ib_n}{2}, n \geq 1 \\ c_{-n} = \frac{a_n + ib_n}{2} = \overline{c_n}, n \geq 1. \end{cases}$

Definiţia 6.2 $f : \mathbb{R} \to \mathbb{R}$, periodică $T = 2 \cdot l$, integrabilă pe $[-l, l]$.

Se definesc *coeficienţii Fourier* ai lui f prin relaţiile:

$$a_n = \frac{1}{l} \cdot \int\limits_{-l}^{l} f(x) \cdot \cos \frac{n\pi x}{l} dx \text{ , } n = 0, 1, 2....$$

$$b_n = \frac{1}{l} \cdot \int\limits_{-l}^{l} f(x) \cdot \sin \frac{n\pi x}{l} dx \text{ ,} n = 1, 2, 3...$$

iar seria trigonometrică

$$\frac{a_0}{2} + \sum_{n \geq 1} \left(a_n \cdot \cos \frac{n\pi x}{l} + b_n \cdot \sin \frac{n\pi x}{l} \right), x \in \mathbb{R}$$

se numeşte *seria Fourier asociată lui f.*

Forma complexă a seriei Fourier

$$\sum_{n \in \mathbb{Z}} c_n \cdot e^{i \cdot \frac{n\pi x}{l}} \quad ,$$

unde

$$c_n = \frac{1}{2l} \cdot \int\limits_{-l}^{l} f(x) \cdot e^{-i \cdot \frac{n\pi x}{l}} dx, \; n \in \mathbb{Z}.$$

Proprietatea 6.3 (P1).

(1) Dacă funcţia f este pară

$$\Rightarrow a_n = \frac{2}{l} \cdot \int_0^l f(x) \cdot \cos \frac{n\pi x}{l} dx, \; n = 0, 1, 2...$$

$$b_n = 0$$

$$\Rightarrow f \sim \frac{a_0}{2} + \sum_{n \geq 1} a_n \cdot \cos \frac{n\pi x}{l}.$$

(2) Dacă funcţia f este impară

$$\Rightarrow a_n = 0$$

$$b_n = \frac{2}{l} \cdot \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx$$

$$\Rightarrow f \sim \sum_{n \geq 1} b_n \cdot \sin \frac{n\pi x}{l}.$$

Proprietatea 6.4 Proprietăţi ale coeficienţilor lui Fourier (P2.):

(1)

$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0;$$

(2)

$$f \in L^2([-l, l]) \Rightarrow \frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{2l} \cdot \int_{-l}^{l} f^2(x)\, dx$$

(Relaţia Parseval-Leapunov).

Proprietatea 6.5 Teorema de reprezentare în serie Fourier a unei funcţii periodice (P3). Considerăm următoarele ipoteze:

(1) Funcţia

$$f : \mathbb{R} \to \mathbb{R}$$

periodică de perioadă $T = 2 \cdot l$

(2) f cu variaţie mărginită pe $[-l, l]$

$$\Rightarrow \frac{f(x+0) + f(x-0)}{2} =$$

$$= \frac{a_0}{2} + \sum_{n\geq 1} \left(a_n \cdot \cos \frac{n\pi x}{l} + b_n \cdot \sin \frac{n\pi x}{l} \right), \forall x \in \mathbb{R}.$$

Atunci seria Fourier asociată lui f este convergentă şi are loc relaţia de mai sus.

(3) Dacă, în plus (faţă de 1), f este continuă pe $\mathbb{R}$, atunci seria Fourier este uniform convergentă pe $\mathbb{R}$ şi

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi x}{l} + b_n \cdot \sin \frac{n\pi x}{l} \right), \forall x \in \mathbb{R}.$$

Observaţia 6.6 1.

$$f_n \xrightarrow[pe\ Df]{UC} f \Rightarrow G_{f_n} \sim G_f$$

pentru n suficient de mare.

2.

$$a_n \cdot \cos \frac{\pi n x}{l} + b_n \cdot \sin \frac{\pi n x}{l}$$

se numeşte *armonică de ordin* n, rezultă că f este sumă de armonice.

3.

$$x \equiv timp \equiv t \Rightarrow \nu_0 = \frac{1}{T} = \frac{1}{2l} \rightarrow \text{frecvenţă};$$

$$\text{armonica } a_n \cdot \cos(2\pi\nu_0 nt) + b_n \cdot \sin(2\pi\nu_0 nt);$$

$n = 1 \rightarrow$ *ton fundamental*;
$n \geq 2 \rightarrow$ *overtones*.

Observaţia 6.7 Avem

$$a\cos\alpha t + b\sin\alpha t = \left(\frac{a}{\sqrt{a^2+b^2}} \cdot \cos\alpha t + \frac{b}{\sqrt{a^2+b^2}} \cdot \sin\alpha t\right).$$

$$\sqrt{a^2+b^2} = \sqrt{a^2+b^2} \cdot \sin(\alpha t + \rho)$$

Deci

$$a_n\cos(2\pi\nu_0 nt) + b_n\sin(2\pi\nu_0 nt) = A_n\sin(2\pi\nu_0 nt + \rho_n),\ n \geq 1$$

unde

$$\rho_n = \arctan\frac{a_n}{b_n},\ n \geq 1.$$

$A_n = \sqrt{a_n^2 + b_n^2} \rightarrow$ *amplitudinea*;
$\cdot \{A_n\}_{n\in\mathbb{N}} \longrightarrow$ se numeşte *spectrul de energie al semnalului* f;
$\cdot \{\nu_0 n\}_{n\in\mathbb{N}} \rightarrow$ spectrul semnalului f;
$A = \cos(2\pi\nu_0 t) \longrightarrow$ *nota A*.

Exerciţiul 6.8 Să se dezvolte în serie Fourier funcţia:

$$f(x) = \begin{cases} 1, & x \in \left(0, \frac{1}{2}\right] \\ -1, & x \in \left(\frac{1}{2}, 1\right] \end{cases},\ T = 1.$$

Cu P3 avem:

$$\frac{f(x+0) + f(x-0)}{2} =$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cdot \cos \frac{n\pi x}{l} + b_n \cdot \sin \frac{n\pi x}{l} \right), \forall x \in \mathbb{R}.$$

Cum f impară

$$\Rightarrow a_n = 0$$

$$b_n = \frac{2}{l} \cdot \int_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx \overset{l=\frac{1}{2}}{=}$$

$$= 4 \cdot \int_0^{\frac{1}{2}} \sin 2n\pi x dx = -\frac{4 \cos 2n\pi x}{2n\pi} \Big|_0^{\frac{1}{2}} =$$

$$= -\frac{2}{n\pi} \left((-1)^n - 1 \right)$$

$$\Rightarrow b_n = \frac{2}{n\pi} \cdot (1 - (-1)^n) = \begin{cases} 0, & n = \text{par} \\ \frac{4}{n\pi}, & n = \text{impar}, \end{cases}$$

deci

$$\frac{f(x+0) + f(x-0)}{2} =$$

$$= \frac{4}{\pi} \cdot \sum_{k=0}^{\infty} \frac{1}{2k+1} \cdot \sin 2\pi (2k+1) x, \ \forall x \in \mathbb{R}$$

dezvoltarea în serie Fourier.

Exemplul 6.9 Coarda vibrantă finită fixă la capete

$$\frac{\partial^2 u}{\partial t^2} - c^2 \cdot \frac{\partial^2 u}{\partial x^2} = 0, \text{ ecuaţia coardei vibrante}$$

$x \in (0, l)$ spaţiu, $t > 0$ timp, $c > 0$ constantă.

$$u(x, 0) = \rho(x) \quad \frac{\partial u}{\partial t}(x, 0) = \Psi(x), \text{ condiţii iniţiale}$$

$$u(0,t) = u(l,t) = 0, \text{ condiţii la limită.}$$

Cu metoda reflexiilor

$$\rho \in C^2([0,l]), \ \psi \in C^1([0,l])$$

unde:

$$\left.\begin{array}{l} \rho(0) = \rho(l) = 0 \\ \psi(0) = \psi(l) = 0 \\ \rho''(0) = \rho''(l) = 0 \end{array}\right\}$$

condiţia de compatibilitate a datelor.

$$\Rightarrow \exists! \ u \in C^2([0,l] \times [0,\infty))$$

soluţie a problemei formulate.

<u>Metoda Fourier</u>

Căutăm funcţia

$$u(x,t) = X(x) \cdot T(t) \neq 0 \Rightarrow \left\{\begin{array}{l} X(x) \neq 0 \\ T(t) \neq 0. \end{array}\right.$$

$$\left.\begin{array}{l} 0 = u(0,t) = X(0) \cdot T(t) \\ 0 = u(l,t) = X(l) \cdot T(t) \end{array}\right| \Rightarrow X(0) = X(l) = 0$$

$$\left.\begin{array}{l} \frac{\partial^2 u}{\partial t^2} = X(x) \cdot \ddot{T}(t) \\ \frac{\partial^2 u}{\partial x^2} = X''(x) \cdot T(t) \end{array}\right| \xrightarrow{\text{ec.}} X\ddot{T} - C^2 X'' \cdot T = 0 \Rightarrow$$

$$\Rightarrow \frac{X''}{X} = \frac{\ddot{T}}{C^2 T} = -\lambda(\text{constantă}).$$

$$\Rightarrow \left\{\begin{array}{l} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{array}\right. \ (\text{problema Sturm-Liouville})$$

Se caută λ astfel încât $\exists$ X soluţie nenulă. Ecuaţia caracteristică este

$$r^2 + \lambda = 0 \Rightarrow r^2 = -\lambda.$$

1. Dacă $\lambda < 0$

$$\Rightarrow r_{1,2} = \pm\sqrt{-\lambda}$$

reale distincte

$$\Rightarrow X(x) = C_1 \cdot e^{\sqrt{-\lambda}x} + C_2 \cdot e^{-\sqrt{-\lambda}x}$$

$$\underset{\text{initiale}}{\overset{\text{cond.}}{\longrightarrow}} C_1 = C_2 = 0 \quad \Rightarrow X = 0$$

nu convine.

2. Dacă $\lambda = 0$

$$\Rightarrow X'' = 0 \Rightarrow$$

$$X(x) = aX + b \underset{\text{initiale}}{\overset{\text{cond.}}{\longrightarrow}} a = b = 0 \Rightarrow X = 0.$$

3. Dacă $\lambda > 0$

$$\Rightarrow r_{1,2} = \pm i\sqrt{\lambda} \Rightarrow$$

$$X(x) = C_1 \cdot \cos\sqrt{\lambda}x + C_2 \cdot \sin\sqrt{\lambda}x$$

$$X(x) = C_1 \cdot \cos\sqrt{\lambda}x + C_2 \cdot \sin\sqrt{\lambda}x$$

$$X(0) = C_1 \Rightarrow C_1 = 0.$$

$$X(l) = C_2 \cdot \sin\sqrt{\lambda}l = 0 \overset{C_2 \neq 0}{\longrightarrow} \sin\sqrt{\lambda}l = 0 \Rightarrow \sqrt{\lambda}l = n\pi$$

$$\Rightarrow \lambda_n = \frac{n^2\pi^2}{l^2}, \quad n \in \mathbb{N}^*$$

$$\Rightarrow X_n\left(x\right) = C_2 \cdot \sin\frac{n\pi x}{l}.$$

Ecuaţia în T devine:

$$\frac{\ddot{T}}{c^2 \cdot T} = -\lambda_n \Rightarrow \ddot{T} + c^2\lambda_n T = 0$$

$$r^2 + c^2 \cdot \lambda_n = 0 \ \Rightarrow$$

$$r_{1,2} = \pm ic\sqrt{\lambda_n} = \pm ic\frac{n\pi}{l}$$

$$\Rightarrow T_n\left(t\right) = A_n' \cdot \cos c\frac{n\pi}{l}t + B_n' \cdot \sin c\frac{n\pi}{l}t.$$

Rezumat:

$$n \in \mathbb{N}^* \rightarrow u_n\left(x,t\right) = X_n\left(x\right) \cdot T_n\left(t\right) =$$
$$= \left(A_n \cdot \cos c\frac{n\pi}{l}t + B_n \cdot \sin c\frac{n\pi}{l}t\right) \cdot \sin\frac{n\pi x}{l};$$

$$u_n \text{ satisface } \begin{cases} \text{ecuaţia undelor} \\ \text{condiţii la limită} \end{cases}$$

$$\Rightarrow \sum_{k=1}^{n} u_k \text{ satisface } \begin{cases} \text{ecuaţia undelor} \\ \text{condiţii la limită.} \end{cases}$$

Oare $\sum\limits_{n=1}^{\infty} u_n$ este soluţie a problemei?

Presupunem $\sum\limits_{k=1}^{\infty} u_n\left(x,t\right)$ convergentă cu suma $\sum\limits_{n\geq 1} u_n\left(x,t\right) = u\left(x,t\right)$.

Mai presupunem că seria $\sum\limits_{n\geq 1} u_n(x,t)$ poate fi derivată de două ori în raport cu $x, t \Rightarrow$ funcţia u satisface şi condiţiile la limită.

Cerem ca u să satisfacă condiţiile iniţiale

$$A_n = b_n = \frac{2}{l} \cdot \int\limits_0^l \rho(x) \cdot \sin\frac{n\pi x}{l} dx,$$

pentru că

$$\rho(x) = u(x,0) = \sum_{n\geq 1} A_n \cdot \sin\frac{n\pi x}{l}$$

$$\rho \in C^2([0,l]) \sim \rho(x) \overset{Fourier}{\longrightarrow} \sum_{n\geq 1} b_n \cdot \sin\frac{n\pi x}{l},$$

$$\frac{\partial u}{\partial t} = \sum_{n\geq 1} \frac{\partial u_n}{\partial t} =$$

$$= \sum_{n\geq 1} \left(-A_n \cdot c\frac{n\pi}{l} \cdot \sin c\frac{n\pi}{l} t + B_n \cdot c\frac{n\pi}{l} \cdot \cos c\frac{n\pi}{l} t \right) \sin\frac{n\pi x}{l}$$

$$\frac{\partial u}{\partial t}(x,0) = \psi(x) \Leftrightarrow$$

$$\sum_{n=1}^{\infty} B_n \cdot c\frac{n\pi}{l} \cdot \sin\frac{n\pi x}{l} = \psi(x) \overset{Fourier}{\longrightarrow} \sum_{n=1}^{\infty} \tilde{b}_n \cdot \sin\frac{n\pi x}{l},$$

$$\tilde{b}_n = \frac{2}{l} \cdot \int\limits_0^l \psi(x) \cdot \sin\frac{n\pi x}{l} dx \Rightarrow \tilde{b}_n = B_n \cdot c\frac{n\pi}{l} \Rightarrow$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} \left(b_n \cdot \cos c\frac{n\pi}{l} t + \frac{l}{n\pi c} \cdot \tilde{b}_n \sin c\frac{n\pi}{l} t \right) \sin\frac{n\pi x}{l}$$

Unde b_n reprezintă coeficientul Fourier în sin ai lui $\rho(x)$, iar $\tilde{b}_n$ reprezintă coeficientul Fourier în sin ai lui $\psi(x)$. Este vorba de dezvoltarea în serie Fourier de sinusuri pentru $\rho(x)$, respectiv $\psi(x)$.

Observaţia 6.10 Cu condiţii suplimentare faţă de cele de la metoda reflexiilor ($\exists\rho'', \psi''-$ continue pe porţiuni) u dat de relaţia de mai sus este soluţie unică a problemei formulate.

6.1 Exerciţii rezolvate

Exerciţiul 6.11 Reprezentaţi grafic folosind lungimea intervalului de definiţie drept perioadă şi calculaţi seria Fourier corespunzătoare.

1)

$$f(x) = \begin{cases} 0, \ x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ 1, \ x \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$

$T = 2\pi \Rightarrow \omega = \frac{2\pi}{T} = 1$ pulsaţia.

Soluţie.

f prelungită este pară $\Rightarrow b_k = 0$.

$$\forall x \neq \frac{\pi}{2} \Rightarrow f \text{ continua în } x \Rightarrow$$

$$\forall x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos(kx).$$

$$\frac{1}{2}\left(f\left(\frac{\pi}{2}+0\right) + f\left(\frac{\pi}{2}-0\right)\right) =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cdot \cos\left(k\frac{\pi}{2}\right).$$

$$a_0 = \frac{2}{T} \cdot \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x)\, dx = \frac{1}{\pi} \cdot \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} dx = 1$$

$$a_k = \frac{2}{T} \cdot \int_{-\frac{\pi}{2}}^{\frac{3\pi}{2}} f(x) \cos(k\omega x)\, dx = \frac{1}{\pi} \cdot \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \cos(kx)\, dx =$$

$$= \frac{1}{\pi k} \cdot \sin kx \Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{1}{\pi k}\left(\sin\frac{3k\pi}{2} - \sin\frac{k\pi}{2}\right) =$$

$$= \frac{2}{\pi k} \sin\frac{k\pi}{2} \cos k\pi = \frac{2}{\pi k}(-1)^k \sin\frac{k\pi}{2}.$$

Dar:

$$(-1)^k \sin\frac{k\pi}{2} = \begin{cases} 0,\ k = \text{par} = 2n+2 \\ (-1)^{n+1},\ k = \text{impar} = 2n+1 \end{cases} \quad n \geq 0.$$

Atunci:

$$\forall \begin{cases} x \in \mathbb{R} \\ x \neq \frac{\pi}{2} \Rightarrow f(x) = \frac{1}{2} + \sum\limits_{n=0}^{\infty} \frac{2}{\pi} \cdot \frac{(-1)^{n+1}}{2n+1} \cdot \cos(2n+1)\, x, \end{cases}$$

$$x = \frac{\pi}{2} \Rightarrow \frac{f\left(\frac{\pi}{2} - 0\right) + f\left(\frac{\pi}{2} + 0\right)}{2} = \frac{1}{2}.$$

Altfel scris:

$$a_k = \begin{cases} 0,\ k = 2n+2 \\ (-1)^{\frac{k+1}{2}},\ k = 2n+1 \end{cases} \quad n \geq 0, k \geq 1 \Rightarrow$$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{2},\ k = 2n \text{ sau } x = \frac{\pi}{2} + n\pi \\ \frac{1}{2} + \frac{2}{\pi} \cdot \sum\limits_{k=1}^{\infty} \frac{(-1)^{\frac{k+1}{2}}}{k} \cos(kx),\ k = 2n+1. \end{cases}$$

<u>Sau</u>: f pară rezultă $b_n = 0, \forall n \geq 1$. $T = 2l = 2\pi \Rightarrow l = \pi$. Pentru $n \geq 1$ avem:

$$a_n = \frac{2}{l} \cdot \int_0^l f(x) \cos(nx)\, dx = \frac{2}{\pi} \cdot \int_{\frac{\pi}{2}}^{\pi} \cos(nx)\, dx =$$

$$= \frac{2}{n\pi} \cdot \sin nx \Big|_{\frac{\pi}{2}}^{\pi} = \begin{cases} \frac{2}{n\pi} (-1)^{n+1},\ n \geq 1, \text{ n impar} \\ 0,\ n \geq 1, \text{ n par.} \end{cases}$$

Metodă mai rapidă cu proprietatea (6.3).

2)

$$f(x) = |x|,$$

$-2 < x < 2$.

Soluţie.

$$f(x) = \begin{cases} -x, -2 < x \leq 0 \\ x, 0 < x < 2 \end{cases}$$

f pară $\Rightarrow b_k = 0$.

$T = 4$, $\omega = \frac{2\pi}{T} = \frac{\pi}{2}$.

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega x).$$

$$a_0 = \frac{2}{T} \int_{-2}^{2} |x|\, dx = \int_0^2 x dx = \frac{x^2}{2} \Big|_0^2 = 2.$$

$$a_k = \frac{2}{T}\int_{-2}^{2} |x| \cos(k\omega x)\, dx = \int_{0}^{2} x \cos\left(\frac{k\pi}{2}x\right) dx =$$

$$= \frac{2}{k\pi}\int_{0}^{2} x \left[\sin\left(\frac{k\pi}{2}x\right)\right]' dx =$$

$$= \frac{2}{k\pi} x \sin\left(\frac{k\pi}{2}x\right)\Bigg|_{0}^{2} - \frac{2}{k\pi}\int_{0}^{2} \sin\left(\frac{k\pi}{2}x\right) dx =$$

$$= \frac{2}{k\pi}\frac{2}{k\pi}\cos\frac{k\pi}{2}x\Bigg|_{0}^{2} = \frac{4}{k^2\pi^2}\left[(-1)^k - 1\right] \Rightarrow$$

$$f(x) = 1 + \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[(-1)^k - 1\right]\cos\left(\frac{k\pi}{2}x\right), \forall x \in \mathbb{R}.$$

3)

$$f(x) = \begin{cases} -1 - x, & -1 < x < 0 \\ 1 - x, & 0 < x < 1. \end{cases}$$

Soluţie.

$$f(-x) = \begin{cases} 1 + x, & -1 < x < 0 \\ x - 1, & 0 < x < 1 \end{cases} = -f(x)$$

$$\Rightarrow f \text{ impară} \Rightarrow \mathrm{a}_k = 0,\ k = 0, 1, 2, ...$$

$$T = 2,\ \omega = \frac{2\pi}{T} = \pi.$$

$$b_k = \frac{2}{T}\int\limits_{-1}^{1} f(x)\cdot\sin(k\omega x)\,dx =$$

$$= 2\int\limits_{0}^{1}(1-x)\cdot\sin(k\pi x)\,dx =$$

$$= -\frac{2}{k\pi}\int\limits_{0}^{1}(1-x)(\cos k\pi x)'\,dx =$$

$$= \frac{2}{k\pi}\cdot(1-x)\cdot\cos k\pi x\,\Bigg|\begin{matrix}0\\-1\end{matrix}\; - \frac{2}{k\pi}\int\limits_{0}^{1}\cos k\pi x\;dx =$$

$$= \tfrac{2}{k\pi} - \tfrac{2}{k\pi}\cdot\tfrac{1}{k\pi}\cdot\sin k\pi x\,\Bigg|\begin{matrix}1\\0\end{matrix}\; = \tfrac{2}{k\pi},\; k = 1, 2, ...$$

$$f(x) = \frac{2}{\pi}\cdot\sum_{k=1}^{\infty}\frac{1}{k}\cdot\sin(k\pi x),\quad (\forall)\, x\in\mathbb{R}.$$

$$f(x) = \sum_{k=1}^{\infty}\frac{2}{k\pi}\cdot\sin(k\pi x),\quad (\forall)\, x\in\mathbb{R}.$$

Observaţia 6.12 Dacă funcţia

$$f : [a, a+T] \to \mathbb{R}$$

este monotonă pe porţiuni, cu un număr finit de discontinuităţi de speţa I a şi periodică de perioadă T, atunci, $(\forall)\, x \in \mathbb{R}$ avem:

$$\frac{f(x+0)+f(x-0)}{2}=\frac{a_0}{2}+\sum_{k=1}^{\infty}\left(a_k\cdot\cos(k\omega x)+b_k\cdot\sin(k\omega x)\right)$$

unde seria din expresia de mai sus este serie Fourier a lui f convergentă pe $\mathbb{R}$.

$$a_k=\frac{2}{T}\int\limits_a^{a+T} f(x)\cdot\cos(k\omega x)\,dx,\ \ k\geq 0$$

$$b_k=\frac{2}{T}\int\limits_a^{a+T} f(x)\cdot\sin(k\omega x)\,dx,\ \ k\geq 1,$$

$$x\in(-\pi,\pi].$$

În plus, dacă f este continuă atunci seria Fourier este uniform convergentă pe $\mathbb{R}$.

Dacă f prelungită este pară $\Rightarrow b_k=0, k=1,2,3,...$ şi dacă f prelungită este impară $\Rightarrow a_k=0, k=0,1,2,....$

Exerciţiul 6.13 Fie

$$T=2\cdot\pi,\ \ f(x)=x^2.$$

a) Determinaţi seria Fourier asociată lui f şi arătaţi că

$$x^2=\frac{\pi^2}{3}+4\sum_{a=1}^{\infty}(-1)^n\cdot\frac{\cos(nx)}{n^2},\ \ x\in[-\pi,\pi].$$

b)

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6},\ \ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12};\ \ \sum_{n\geq 1}\frac{1}{n^4}=\frac{\pi^2}{90}.$$

Soluţie.

a)

$$f \text{ pară} \Rightarrow b_n = 0, \ n = 1, 2, ...$$

$$a_0 = \frac{2}{2\pi} \int\limits_{-\pi}^{\pi} x^2 dx = \frac{2}{\pi} \int\limits_{0}^{\pi} x^2 dx = \frac{2}{\pi} \cdot \frac{x^3}{3} \Bigg| \begin{matrix} \pi \\ 0 \end{matrix} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi} \int\limits_{0}^{\pi} x^2 \cdot \cos \frac{n\pi x}{\pi} dx = \frac{2}{\pi n} \int\limits_{0}^{\pi} x^2 \cdot (\sin(nx))' \, dx =$$

$$= \frac{2}{\pi n} \cdot x^2 \cdot \sin(nx) \Bigg| \begin{matrix} \pi \\ 0 \end{matrix} - \frac{4}{\pi n} \int\limits_{0}^{\pi} x \cdot \sin(nx) dx =$$

$$= \frac{4}{\pi n^2} \cdot x \cdot \cos(nx) \Bigg| \begin{matrix} \pi \\ 0 \end{matrix} -$$

$$- \frac{4}{\pi n^2} \int\limits_{0}^{\pi} \cos(nx) dx = \frac{4}{n^2} \cdot (-1)^n - \frac{4}{\pi n^3} \cdot \sin(nx) \Bigg| \begin{matrix} \pi \\ 0 \end{matrix} \Rightarrow$$

$$a_n = b_n = \frac{4}{n^2} \cdot (-1)^n, \ n = 1, 2, ...$$

$$f(x) \text{ continuă } \Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(nx), \ \ (\forall)\, x \in \mathbb{R}$$

şi seria Fourier e uniform convergentă pe $\mathbb{R}$.

$$\Rightarrow x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos(nx), \quad (\forall)\, x \in \mathbb{R} \to (\forall)\, x \in [-\pi, \pi].$$

Observaţie: "mai riguros" avem:

$$a_n = \frac{4}{n^2} \cdot (-1)^n, n \geq 1;\ a_0 = \frac{2 \cdot \pi^2}{3};\ b_n = 0,\ n \geq 1.$$

seria Fourier asociată:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cdot \cos(nx) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos(nx)$$

$f(x)$ continuă pe $(-\pi, \pi] \Rightarrow f(x) = \frac{\pi^2}{3} + 4\sum_{n\geq 1} \frac{(-1)^n}{n^2} \cdot \cos(nx),$

$(\forall)\, x \in (-\pi, \pi]$ şi din periodicitate avem

$$f(x) = x^2 = \frac{\pi^3}{3} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos(nx), \quad (\forall)\, x \in \mathbb{R}.$$

b)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}; \quad \sum_{n\geq 1} \frac{1}{n^4} = \frac{\pi^2}{90}.$$

Soluţie.

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos(nx), \quad (\forall)\, x \in \mathbb{R}.$$

$$x = \pi \Rightarrow \frac{2 \cdot \pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

$$x = 0 \Rightarrow -\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

Relaţia Parseval-Leapunov:

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) = \frac{1}{l} \int_{-l}^{l} f^2(x)\, dx;\ l = \pi$$

$$\frac{1}{2} \cdot \left(\frac{2 \cdot \pi^2}{3}\right)^2 + \sum_{n=1}^{\infty} \frac{16}{n^4} = \frac{1}{\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{2}{\pi} \cdot \frac{x^5}{5} \Bigg|_{0}^{\pi} = \frac{2 \cdot \pi^4}{5}.$$

$$\frac{2 \cdot \pi^4}{9} + 16 \sum_{n \geq 1} \frac{1}{n^4} = \frac{2 \cdot \pi^4}{5}.$$

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{1}{16} \cdot 2 \cdot \left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{2 \cdot 4}{16 \cdot 5 \cdot 9} \cdot \pi^4 = \frac{\pi^4}{90}.$$

Exerciţiul 6.14 Să se dezvolte în serie Fourier următoarea funcţie:

$$f(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right] \\ 1 - t, & t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

de perioadă $T = 1$.

Soluţie.

Funcţia f este continuă, monotonă pe porţiuni, deci este dezvoltabilă în serie Fourier şi $\forall t \in [0,1]$

$$f(t) = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n \cos 2\pi nt + b_n \sin 2\pi nt)$$

uniform convergentă pe $[0,1]$, iar coeficienţii Fourier sunt:

$$a_0 = 2\int_0^1 f(t)dt = 2\left(\int_0^{\frac{1}{2}} tdt + \int_{\frac{1}{2}}^1 (1-t)\,dt\right) =$$

$$= t^2 \Big|_0^{\frac{1}{2}} + 2t \Big|_{\frac{1}{2}}^1 - t^2 \Big|_{\frac{1}{2}}^1 =$$

$$= \frac{1}{4} + 2\left(1 - \frac{1}{2}\right) - \left(1 - \frac{1}{4}\right) = \frac{5}{4} - \frac{3}{4} = \frac{1}{2}.$$

$$\left.\begin{array}{l} a_n = 2\int_0^1 f(t)\ \cos 2\pi ntdt \\ b_n = 2\int_0^1 f(t)\ \sin 2\pi ntdt \end{array}\right\} \Rightarrow a_n + ib_n = 2\int_0^1 f(t)e^{i2\pi nt}dt =$$

$$= 2\int_0^{\frac{1}{2}} t\ e^{i2\pi nt}dt + 2\int_{\frac{1}{2}}^1 (1-t)e^{i2\pi nt}dt.$$

Deci

$$a_n + ib_n = \frac{1}{i\pi n}\left[\frac{1}{2}e^{i\pi n} - \frac{1}{i2\pi n}e^{i2\pi nt}\Big|_0^{\frac{1}{2}}\right] +$$

$$+\frac{1}{i\pi n}\left[-\frac{1}{2}e^{i\pi n}+\frac{1}{i2\pi n}e^{i2\pi nt}\Big|_{\frac{1}{2}}^{1}\right]=$$

$$=\frac{1}{2\pi^2n^2}(e^{i\pi n}-1)-\frac{1}{2\pi^2n^2}(e^{i2\pi n}-e^{i\pi})=$$

$$=\frac{1}{\pi^2n^2}e^{i\pi n}-\frac{1}{\pi^2n^2}=\frac{1}{\pi^2n^2}[(-1)^n-1]$$

$$=\begin{cases}\frac{-2}{\pi^2(2n-1)^2}, n=2n-1, n\geq 1\\ 0, n=2n.\end{cases}$$

$$a_n=\begin{cases}\frac{-2}{\pi^2(2n-1)^2}, & n=2n-1, n\geq 1\\ 0, & n=2n.\end{cases}$$

$$b_n=0, (\forall)\, n\geq 1.$$

Deci

$$f\left(t\right)=\frac{1}{4}-\frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2\left(2n-1\right)\pi t}{\left(2n-1\right)^2}, (\forall)\, t\in\left[0,1\right].$$

Metoda a-II-a

f este pară $\Rightarrow b_n$, $\forall n\geq 1$. $T=2l=1\Rightarrow l=\frac{1}{2}$ şi $\omega=2\pi$ rezultă

$$a_n=\frac{2}{l}\int_0^l f(t)\cos n\omega t dt=4\int_0^{1/2}t\cos 2n\pi t dt=$$

$$=4\int_0^{1/2}t\left(\frac{\sin 2n\pi t}{2n\pi}\right)'dt\overset{\text{părţi}}{=\!=\!=}-4\int_0^{1/2}\frac{\sin 2n\pi t}{2n\pi}dt=$$

$$=\frac{4}{(2n\pi)^2}\cos 2n\pi t\Big|_0^{\frac{1}{2}}=\frac{1}{\pi^2n^2}[(-1)^n-1],\ n\geq 1.$$

Teorema 6.15 *Dacă*

$$f(t) = \frac{a_0}{2} + \sum_{n\geq 1}\left(a_n \cos\frac{2\pi}{T}nt + b_n \sin\frac{2\pi}{T}nt\right), (\forall)\, t \in \mathbb{R},$$

f este de perioadă T şi seria din dreapta este uniform convergentă pe $\mathbb{R}$, atunci ea este serie Fourier asociată lui $f(t)$.

Exerciţiul 6.16 Să se dezvolte în serie Fourier:

$$f(t) = \frac{a \sin t}{1 - 2a\cos t + a^2},$$

$a \in \mathbb{C}$, $|a| < 1$, $t \in \mathbb{R}$.

Soluţie.

$$1-2a\cos t+a^2 = 1-a\left(e^{it}+e^{-it}\right)+a^2 e^{it}e^{-it} = \left(1-ae^{it}\right)\left(1-ae^{-it}\right).$$

$$f(t) = \frac{a\left(e^{it}-e^{-it}\right)}{2i\left(1-ae^{it}\right)\left(1-ae^{-it}\right)} = \\ = \frac{1}{2i\left(1-ae^{it}\right)} - \frac{1}{2i\left(1-ae^{-it}\right)}.$$

$$\left|ae^{\pm it}\right| = |a| < 1 \Rightarrow \begin{array}{l} \frac{1}{1-ae^{it}} = \sum\limits_{n=0}^{\infty} a^n e^{int} \\ \frac{1}{1-ae^{-it}} = \sum\limits_{n=0}^{\infty} a^n e^{-int}. \end{array}$$

Deci

$$f(t) = \sum_{n\geq 0} a^n \sin nt, (\forall)\, t \in \mathbb{R}, n \geq 1.$$

$$\left.\begin{array}{r} f(t) = \sum\limits_{n=1}^{\infty} a^n \sin nt, (\forall)\, t \in \mathbb{R}; |a^n \sin nt| \leq |a|^n \\ \sum\limits_{n\geq 1} |a|^n \text{convergentă} \end{array}\right\} \Rightarrow$$

$$\Rightarrow \sum_{n=1}^{\infty} a^n \sin nt$$

este uniform convergentă pe $\mathbb{R}$, deci f este dezvoltată în serie Fourier.

Exerciţiul 6.17 Să se dezvolte în serie Fourier

$$f(t) = ln\left(1 - 2acost + a^2\right), a \in \mathbb{R},$$

$|a| < 1$.

Soluţie. Funcţia f are perioda $T = 2\pi$,

$$f'(t) = \frac{2a \sin t}{1 - 2a \cos t + a^2} = 2\sum_{n=1}^{\infty} a^n \sin nt, (\forall)\, t \in \mathbb{R}.$$

Integrăm termen cu termen, deoarece seria din dreapta este uniform convergentă.

$$f(t) = 2\sum_{n\geq 1} a^n \int \sin nt \text{ dt} = -2\sum_{n\geq 1} \frac{a^n}{n} \cos nt, (\forall)\, t \in \mathbb{R}.$$

Seria obţinută este o serie uniform convergentă deoarece:

$$\left|\frac{a^n}{n} \cos nt\right| \leq \frac{|a|^n}{n}, (\forall)\, t \in \mathbb{R}, n \geq 1$$

şi seria

$$\sum_{n=1}^{\infty} \frac{|a|^n}{n}$$

este convergentă cu criteriul raportului. Deoarece seria din dreapta este uniform convergentă pe $\mathbb{R}$, ea reprezintă seria Fourier a lui $f(t)$ pe $\mathbb{R}$.

Exerciţiul 6.18 Să se dezvolte în serie Fourier:

$$f\left(t\right)=e^{\cos x}\cos\left(\sin x\right)_{,}\ g\left(x\right)=e^{\cos x}\sin\left(\sin x\right).$$

Soluţie. Funcţiile $f,\ g$ au perioada $T=2\pi$.

$$f\left(x\right)+ig\left(x\right)=e^{\cos x}\cdot e^{i\sin x}=e^{\cos x+i\sin x}=$$

$$=\sum_{n=0}^{\infty}\frac{\left(\cos x+i\sin x\right)^{n}}{n!}=\sum_{n=0}^{\infty}\frac{\cos nx}{n!}+i\sum_{n=0}^{\infty}\frac{\sin nx}{n!},(\forall)\,x\in\mathbb{R}.$$

Seriile $\sum\limits_{n=0}^{\infty}\frac{\cos nx}{n!},\ \sum\limits_{n=0}^{\infty}\frac{\sin nx}{n!}$ sunt uniform convergente pe $\mathbb{R}$, deoarece $\left|\frac{\cos nx}{n!}\right|\leq\frac{1}{n!},\ \left|\frac{\sin nx}{n!}\right|\leq\frac{1}{n!},\ \sum\limits_{n=0}^{\infty}\frac{1}{n!}=e$ este convergentă $\forall x\in\mathbb{R}$, deci reprezintă seriile Fourier pentru $f(x)$ şi $g(x)$.

$$f(x)=\sum_{n=0}^{\infty}\frac{\cos nx}{n!},\ \forall x\in\mathbb{R}$$

$$g(x)=\sum_{n=0}^{\infty}\frac{\sin nx}{n!},\ \forall x\in\mathbb{R}.$$

Dacă $f\left(t\right)$ continuă pe $[0,T]$, monotonă pe porţiuni, periodică de perioadă T, atunci:

$$f\left(t\right)=\frac{a_{0}}{2}+\sum_{n\geq1}\left(a_{n}\cos nt+b_{n}\sin nt\right),(\forall)t\in\mathbb{R}.$$

Exerciţiul 6.19 Să se dezvolte în serie Fourier:

$$f\left(t\right)=\frac{1}{\cos t}=\sec t=\sec t,t\in\left[-\frac{\pi}{4},\frac{\pi}{4}\right],$$

f este monotonă pe porţiuni, continuă pe $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, este dezvoltabilă în serie Fourier şi de perioadă $T = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$, f este pară, deci

$$b_n = 0, n \geq 1.$$

$$a_0 = \frac{2}{T}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} f(t)dt = \frac{2}{\frac{\pi}{2}}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dt}{\cos t} = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{dt}{\cos t} =$$

$$= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{dt}{\frac{1}{\sqrt{1+tg^2 t}}} =$$

$$= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \sqrt{1+tg^2 t}dt \underbrace{=}_{tgt=y} \frac{8}{\pi}\int_0^1 \frac{dy}{\sqrt{1+y^2}} =$$

$$= \frac{8}{\pi}\ln\left(y+\sqrt{1+y^2}\right)\Bigg|_0^1 = \frac{8}{\pi}\ln\left(1+\sqrt{2}\right);$$

$$a_n = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \frac{\cos 4nt}{\cos t}dt;$$

Dar

$$\frac{\cos 4nt}{\cos t} - \frac{\cos 4(n-1)t}{\cos t} = 2\cos(4n-1)t - 2\cos(4n-3)t.$$

$$a_n - a_{n-1} = \frac{16}{\pi}\left[\int_0^{\frac{\pi}{4}} \cos(4n-1)t\ dt - \int_0^{\frac{\pi}{4}} \cos(4n-3)t\ dt\right] =$$

$$= \frac{16}{\pi} \left[\frac{1}{4n-1} \sin \frac{\pi}{4} (4n-1) - \frac{1}{4n-3} \sin (4n-3) \frac{\pi}{4} \right]$$

$$a_n - a_0 =$$

$$= \frac{16}{\pi} \sum_{k=1}^{n} \left[\frac{1}{4k-1} \sin \frac{\pi}{4} (4k-1) - \frac{1}{4k-3} \sin (4k-3) \frac{\pi}{4} \right] =$$

$$= \frac{16}{\pi} \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{1}{4k-1} - \frac{1}{4k-3} \right) \sin \left(k\pi - \frac{\pi}{4} \right)$$

dar

$$\sin \left(k\pi - \frac{3\pi}{4} \right) = \frac{(-1)^{k+1}}{\sqrt{2}} \ ,$$

deci

$$a_n - a_0 = \frac{16}{\pi\sqrt{2}} \sum_{k=1}^{n} (-1)^{k+1} \left(\frac{1}{4k-1} - \frac{1}{3k-2} - \frac{1}{4k-3} \right) =$$

$$= \frac{16}{\pi\sqrt{2}} \left(\frac{(-1)^{n+1}}{4n-1} - \frac{1}{2} + \frac{(-1)^{n+1}}{4n-3} + \frac{1}{2} \right) =$$

$$= \frac{16}{\pi\sqrt{2}} (-1)^{n+1} \left(\frac{1}{4n-1} - \frac{1}{4n-3} \right)$$

$$\Rightarrow a_n = a_0 + \frac{16}{\pi\sqrt{2}} (-1)^{n+1} \left(\frac{1}{4n-1} - \frac{1}{4n-3} \right), n \geq 1.$$

Exerciţiul 6.20 Să se dezvolte în serie Fourier

$$f(t) = \frac{1}{2+\cos t}.$$

Soluţie. Funcţia f are perioada $T = 2\pi$, continuă pe $\mathbb{R}$ şi monotonă pe porţiuni, deci f este dezvoltabilă în serie Fourier uniform convergentă pe $\mathbb{R}$ şi cum f este pară avem:

$$b_n = 0, (\forall)\, n \geq 1 \Rightarrow f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos nt,$$

unde:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\, dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{dt}{2+\cos t} = \frac{1}{\pi}\oint_C \frac{1}{2+\frac{z^2+1}{2z}}\frac{dz}{iz} =$$

$$= \frac{2}{\pi i}\oint_C \frac{dz}{z^2+4z+1} = \frac{2}{\pi i}\cdot 2\pi i \operatorname*{Res}_{z=z_1} f(z),$$

unde

$$f(z) = \frac{1}{z^2+4z+1}$$

$$z^2+4z+1 = 0 \Rightarrow z_{1,2} = \frac{-4 \pm 2\sqrt{3}}{2} = -2 \pm \sqrt{3},$$

poli simpli

$$z_1 = -2+\sqrt{3} \in C, z_2 \notin C.$$

$$\operatorname*{Res}_{z=z_1} f(z) = \left.\frac{1}{2z+4}\right|_{z=-2+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{\cos nt}{2+\cos t}dt = \frac{1}{\pi}Re\int_{-\pi}^{\pi} \frac{\left(e^{it}\right)^n}{2+\cos t}dt =$$

$$= \frac{1}{\pi}Re\oint_C \frac{z^n}{2+\frac{z^2+1}{2z}}\frac{dz}{iz} = \frac{2}{\pi}Re\frac{1}{i}\cdot\oint_C \frac{z^n}{z^2+4z+1} =$$

$$= \frac{2}{\pi}Re\frac{1}{i}\cdot 2\pi i \operatorname*{Res}_{z=z_1} \frac{z^n}{z^2+4z+1} = 4\cdot \frac{z^n}{2z+4}\Bigg|\ z=-2+\sqrt{3} \ .$$

Deci

$$f(t) = \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty}\left(-2+\sqrt{3}\right)^n \cos nt, (\forall)\, t\in\mathbb{R}.$$

Exerciţiul 6.21 Să se dezvolte în serie Fourier:

$$f(t) = \ln(2+\cos t)$$

$$f(t) = \frac{\sin t}{5+3\cos t}.$$

Soluţie.

$$f'(t) = \frac{-\sin t}{2+\cos t}, |f'(t)| \le 1, (\forall)\, t\in\mathbb{R}$$

f este periodică cu $T = 2\pi \Rightarrow f'$ este dezvoltată în serie Fourier uniform convergentă pe $\mathbb{R}$ şi cum f' este impară rezultă:

$$f'(t) = \sum_{n=1}^{\infty} b_n \sin nt, (\forall)\ t\in\mathbb{R},$$

unde

$$b_n = \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{\sin t \cdot \sin nt}{2 + \cos t} dt$$

cu teorema reziduurilor avem

$$= \frac{-1}{\pi} \int_{-\pi}^{\pi} \frac{\sin t}{2 + \cos t} Im \cdot e^{int} dt = \frac{-1}{\pi} Im \int_{-\pi}^{\pi} \frac{\left(e^{it}\right)^n \sin t}{2 + \cos t} dt =$$

$$= \frac{-1}{\pi} Im \left(\oint_c \frac{\frac{z^2-1}{2iz} \cdot z^n}{2 + \frac{z^2+1}{2z}} \frac{dz}{iz} \right) = \frac{+1}{\pi} Im \left(\oint_c \frac{\left(z^2 - 1\right) z^{n-1}}{z^2 + 4z + 1} dz \right) =$$

$$= \frac{1}{\pi} Im \left(2\pi i Res \ f(z)\right) =$$

$$= 2Im \left(i \cdot \frac{\left(z^2 - 1\right) z^{n-1}}{2z + 4} \Big| z = -2 + \sqrt{3} \right) =$$

$$= \frac{1}{\sqrt{3}} (-1)^{n-1} \left(2 - \sqrt{3}\right)^{n-1} = \frac{\left(\sqrt{3} - 2\right)^{n-1}}{\sqrt{3}\left(6 - 4\sqrt{3}\right)} =$$

$$= \frac{\left(\sqrt{3} - 2\right)^{n-1}}{2\left(3 - 2\sqrt{3}\right)\sqrt{3}} = \frac{\left(\sqrt{3} - 2\right)^{n-1}}{6\left(\sqrt{3} - 2\right)} = \frac{\left(\sqrt{3} - 2\right)^{n-1}}{6}, n \geq 1.$$

Deci

$$f'(t) = \frac{1}{6} \sum_{n \geq 1} \left(\sqrt{3} - 2\right)^{n-1} \sin nt, (\forall)\, t \in \mathbb{R}.$$

Seria Fourier e uniform, convergentă $\Rightarrow$ se poate intergra termen cu termen obţinând:

$$f(t) = -\frac{1}{6}\sum_{n\geq 1}\frac{\left(\sqrt{3}-2\right)^{n-2}}{n}\cos nt, (\forall)\, t\in\mathbb{R}.$$

Seria Fourier a lui $f(t)$ în baza teoremei (6.17) este uniform convergentă pentru că

$$\left|\frac{\left(\sqrt{3}-2\right)^{n-2}}{n}\cos nt\right| \leq \left|\frac{\sqrt{3}-2}{n}\right|^{n-2},$$

iar seria $\sum_{n\geq 1}\frac{(2-\sqrt{3})^{(n-2)}}{n}$ este convergentă cu criteriul raportului.

Dacă $f(t)$ este egală cu o serie trigonometrică uniform convergentă pe $\mathbb{R}$, atunci aceasta este serie Fourier a lui $f(t)$.

Exerciţiul 6.22 Să se dezvolte în serie Fourier:

$$f(t) = sgnt,$$

definită pe $(-\pi,\pi)$ de perioadă $T = 2\pi$. Apoi să se calculeze

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n+1}.$$

Soluţie. Funcţia

$$f(t) = sgt = sgnt$$

definită pe $(-\pi,\pi)$ de perioadă $T = 2\pi$:

$$f(t) = \begin{cases} -1, & t\in(-\pi,0) \\ 0, & t=0 \\ 1, & t\in(0,\pi). \end{cases}$$

Monotonă pe porţiuni şi continuă pe $(-\pi, \pi) / \{0\}$ discontinuitate de speţa I-a, rezultă că f se dezvoltă în serie Fourier pe $(-\pi, \pi)$ şi

$$f(t) = \frac{a_0}{2} + \sum_{n\geq 1} (a_n \cos nt + b_n \sin nt), (\forall)\, t \in (-\pi, \pi) \setminus \{0\}$$

În $t = 0$ suma seriei Fourier este egală cu

$$\frac{f(0-0) + f(0+0)}{2} = \frac{-1+1}{2} = 0.$$

În extremităţi suma seriei Fourier este:

$$S(-\pi) = S(\pi) = \frac{f(\pi+0) + f(\pi-0)}{2} = \frac{-1+1}{2} = 0$$

$$a_0 = \frac{1}{\pi}\left(\int_{-\pi}^{0} dt + \int_{0}^{\pi} dt\right) = \frac{1}{\pi}\left(-t \Big|_{-\pi}^{0} + \pi\right) = 0$$

Funcţia f este impară.

$$\Rightarrow a_n = 0, (\forall)\, n \geq 0.$$

Deci $(\forall)\, t \neq 0$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt;$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \sin nt dt = \frac{-2}{\pi n} \cos nt \Big|_0^{\pi} =$$

$$= \frac{2}{\pi n}\left(1-(-1)^n\right) = \begin{cases} 0, & n = 2n \\ \frac{4}{\pi(2n-1)}, & n = 2n-1,\, n \geq 1. \end{cases}$$

$$\begin{cases} f(t) = \frac{4}{\pi}\sum\limits_{n=1}^{\infty}\frac{\sin(2n-1)}{2n-1}, (\forall)\, t \in (-\pi,\pi)\setminus\{0\} \\ S(0) = 0;\ S(-\pi) = S(\pi) = 0 \end{cases}$$

$$t = \frac{\pi}{2} \Rightarrow 1 = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(n\pi - \frac{\pi}{2}\right)}{2n-1} = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} \Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}.$$

Exerciţiul 6.23 Să se dezvolte în serie Fourier:

$$f(t) = t^2,$$

definită pe $(-\pi,\pi)$ de perioadă $T = 2\pi$. Apoi să se calculeze

$$\sum_{n-1}^{\infty}\frac{1}{n^2}, \quad \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2},$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}, \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}.$$

Soluţie. Funcţia f este continuă şi monotonă pe porţiuni $\Rightarrow (\forall)\, t \in [-\pi,\pi]$ avem

$$f(t) = t^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi}{T}nt + b_n \sin\frac{2\pi}{T}nt\right),$$

iar seria Fourier este uniform convergentă pe $[-\pi,\pi]$.

Cum

$$f(t) = t^2,$$

ea este pară, deci $\Rightarrow b_n = 0$, astfel

$$f(t) = t^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi}{T}nt = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt$$

$$a_n = \frac{2}{T}\int_{-\pi}^{\pi} f(t)\cos nt\ dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\ dt =$$

$$= \frac{2}{T}\int_{0}^{\pi} t^2 \cos nt\ dt =$$

$$= \frac{2}{\pi n}\int_{0}^{\pi} t^2 (\sin nt)'\, dt = \frac{2}{\pi n}\left[t^2 \sin nt \Big|_{0}^{\pi} - 2\int_{0}^{\pi} t \sin nt\ \mathrm{dt} \right] =$$

$$= \frac{-4}{\pi n}\int_{0}^{\pi} t\sin nt\ \mathrm{dt} = \frac{4}{\pi n^2}\int_{0}^{\pi} t\,(\cos nt)'\ dt =$$

$$= \frac{4}{\pi n^2}\left[t\cos nt \Big|_{0}^{\pi} - \int_{0}^{\pi} \cos nt\ dt \right] =$$

$$= \frac{4}{\pi n}\cos n\pi = \frac{4(-1)^n}{n^2}, n \geq 1.$$

$$a_0 = \frac{2}{\pi}\int_{-\pi}^{\pi} \mathrm{t}^2\ dt = \frac{2}{\pi}\int_{0}^{\pi} \mathrm{t}^2\ dt = \frac{2}{3\pi} t^3 \Big|_{0}^{\pi} = \frac{2\pi^2}{3};$$

$$f(t) = t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \cos nt, (\forall)\, t \in [-\pi, \pi]\,.$$

Dezvoltare în serie Fourier pe $[-\pi, \pi]$.

- Facem $t = \pi \Rightarrow$

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6};$$

- Facem $t = 0 \Rightarrow$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12};$$

- Adăugăm cele 2 serii:

$$\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2} = \frac{\pi^2}{6} \cdot \frac{3}{2} = \frac{\pi^2}{4} \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}.$$

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt \,|\cdot t \Rightarrow$$

uniform convergeţa se pastrează

$$t^3 = \frac{\pi^2}{3} t + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} t \cos nt$$

şi integrăm pe $[0, \pi] \Rightarrow$ uniform convergenţa se integrează termen cu termen

$$\int_0^{\pi} t^3 dt = \frac{\pi^2}{3}\int_0^{\pi} t dt + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^{\pi} t\cos nt \text{ dt} =$$

$$= \frac{\pi^2}{3}\cdot\frac{\pi^2}{2} + 4\sum_{n\geq 1}\frac{(-1)^n}{n^3}\cdot\int_0^{\pi} t\left(\sin nt\right)' dt \overset{*}{=}$$

$$\left[t\sin nt\,\Big|\begin{matrix}\pi\\0\end{matrix} - \int_0^{\pi}\sin nt dt\right] = \frac{1}{n}\cdot\cos nt\,\Big|\begin{matrix}\pi\\0\end{matrix} = \frac{(-1)^n-1}{n}$$

$$\overset{*}{=} \frac{\pi^4}{6} + 4\sum_{n=1}^{\infty}\frac{2}{(2n-1)^4} = \frac{\pi^4}{4} \Leftrightarrow 2\cdot 4\sum_{n\geq 1}\frac{1}{(2n-1)^4} =$$

$$= \frac{\pi^4}{2}\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{\pi^4}{12} \Rightarrow$$

$$\sum_{n=1}^{\infty}\frac{2}{(2n-1)^4} = \frac{\pi^4}{96}.$$

Exerciţiul 6.24 Să se dezvolte în serie Fourier:

$$f(t) = |\cos t|\,, t\in[-\pi,\pi]$$

Soluţie. Avem

$$T = 2\pi,\ \omega = \frac{2\pi}{T}, \quad b_n = 0, (\forall)\, n\geq 1.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}|\cos t|\,dt = \frac{2}{\pi}\int_0^{\pi}|\cos t|dt =$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos t dt - \int_{\frac{\pi}{2}}^{\pi} \cos t dt\right) =$$

$$= \frac{2}{\pi}\left(\sin t \Big|_0^{\frac{\pi}{2}} - \sin t \Big|_{\frac{\pi}{2}}^{\pi}\right) = \frac{2}{\pi}(1+1) = \frac{4}{\pi}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} |\cos t| \cos nt dt =$$

$$= \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos t \cos nt dt - \int_{\frac{\pi}{2}}^{\pi} \cos t \cos nt dt\right) =$$

$$= \frac{1}{\pi}\left[\int_0^{\frac{\pi}{2}} [\cos(n+1)t + \cos(n-1)t] dt\right] -$$

$$-\frac{1}{\pi}\int_{\frac{\pi}{2}}^{\pi} [\cos(n+1)t + \cos(n-1)t]\, dt =$$

$$\frac{1}{\pi}\left[\frac{\sin(n+1)t}{n+1}\Big|_0^{\frac{\pi}{2}} + \frac{\sin(n-1)t}{n-1}\Big|_{\frac{\pi}{2}}^{\pi} -\right.$$

$$\left.-\frac{\sin(n+1)t}{n+1}\Big|_0^{\frac{\pi}{2}} - \frac{\sin(n-1)t}{n-1}\Big|_{\frac{\pi}{2}}^{\pi}\right] =$$

$$-\frac{2}{\pi}\left[\frac{\sin(n+1)\frac{\pi}{2}}{n+1} + \frac{\sin(n-1)\frac{\pi}{2}}{n-1}\right]$$

$$\sin(n+1)\frac{\pi}{2} = \sin\left(n\frac{\pi}{2} + \frac{\pi}{2}\right) = \cos\frac{n\pi}{2} = \begin{cases} (-1)^n, & n = 2n \\ 0, & n = 2n+1 \end{cases}$$

$$\sin(n-1)\frac{\pi}{2} = \sin\left(n\frac{\pi}{2} - \frac{\pi}{2}\right) =$$

$$= -\cos\frac{n\pi}{2}\begin{cases} -(-1)^n, & n = 2n \\ 0, & n = 2n+1 \end{cases}$$

$$a_{2n} = \frac{2}{\pi}\left[\frac{(-1)^n}{2n+1} - \frac{(-1)^n}{2n-1}\right] = \frac{2}{\pi}(-1)^n\frac{-2}{4n^2-1} \Rightarrow$$

$$a_{2n} = -\frac{4}{\pi}\frac{(-1)^n}{4n^2-1}, n \geq 1,\ a_{2n+1} = 0$$

f continuă pe $[-\pi, \pi] \Rightarrow (\forall)\, t \in (-\pi, \pi)$, avem:

$$f(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n\geq 1}\frac{(-1)^n}{4n^2-1}\cos 2nt.$$

Exerciţiul 6.25 Să se dezvolte în serie Fourier:

$$f(t) = \frac{\sin t}{5+3\cos t}, t \in [-\pi, \pi], T = 2, \omega = 1.$$

Soluţie. Funcţia f este impară $\Rightarrow a_n = 0, (\forall)\, n \geq 0$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin t}{5+3\cos t}\sin nt dt =$$

$$= \frac{1}{\pi}Im\int_{-\pi}^{\pi}\frac{\sin t}{5+3\cos t}e^{int}dt = \frac{1}{\pi}Im\oint_C\frac{\frac{z^2-1}{2iz}z^{n-1}}{5+\frac{3(z^2+1)}{2z}}\frac{1}{iz}dz =$$

$$= -\frac{1}{\pi}Im\oint_C\frac{(z^2-1)z^{n-1}}{3z^2+10z+3}dz = -\frac{1}{\pi}Im 2\pi i\operatorname*{Res}_{z=-\frac{1}{3}}\frac{(z^2-1)z^{n-1}}{3z^2+10z+3} =$$

$$= -2Imi\frac{(z^2-1)z^{n-1}}{6z+10}|_{z=-\frac{1}{3}} = -2Imi\frac{-\frac{8}{9}\left(-\frac{1}{3}\right)^{n-1}}{8} =$$

$$= \frac{2}{3} \cdot \frac{(-1)^{n-1}}{3^n}, \, n \geq 1$$

f este continuă pe $[-\pi, \pi] \Rightarrow$

$$= -2Res \ f(z) = -2\frac{z^{n-1}(z^2-1)}{6z+10}\Big|_{z=-\frac{1}{3}} =$$

$$= -2\frac{(-1)^{n-1} \frac{1}{3^{n-1}} \cdot \frac{-8}{9}}{10-2} = \frac{2}{3}\frac{(-1)^{n-1}}{3n} = b_n,$$

$(\forall)\, n \geq 1$, f este continuă şi cu teorema de reprezentare în serie Fourier avem:

$$f(t) = \frac{2}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3^n} \sin nt, (\forall)\, t \in (\pi, -\pi).$$

6.2 Probleme propuse

Exerciţiul 6.26 Să se găsească seria Fourier a funcţiei periodice:

$$f(t) = \frac{\pi}{2sh\pi}e^t$$

definită pe intervalul $(\pi, -\pi)$ de perioadă $T = 2\pi$.

Exerciţiul 6.27 Dezvoltaţi în serie Fourier funcţia

$$f(t) = t^2 :$$

a) pe intervalul $(-\pi, \pi]$, de perioadă $T = 2l$;
b) pe perioada $(l, 3l]$, de perioadă $T = 2l$.

Exerciţiul 6.28 Scrieţi seria Fourier pentru funcţia periodică

$$f(t) = \frac{1}{5 - 4\cos t},$$

definită pe intervalul $(0, 2\pi]$, de perioadă $T = 2\pi$.

Exerciţiul 6.29 Scrieţi seria Fourier pentru funcţia :
a)

$$f(t) = \frac{1}{5 + 3\cos t},$$

pe $(0, 2\pi]$, $T = 2\pi$;
b)

$$f(t) = \frac{1}{5 + 3\sin t},$$

pe $(0, 2\pi]$, $T = 2\pi$;

Exerciţiul 6.30 Dezvoltaţi în serie Fourier pe intervalele indicate, următoarele serii periodice:
a) $f(t) = t$ pe $(-\pi, \pi), T = 2\pi$;
b) $f(t) = \pi^2 - t^2$ pe $(-\pi, \pi), T = 2\pi$;
(c) $f(t) = \begin{cases} at, t \in (-\pi, 0) \\ bt, t \in [0, \pi) \end{cases}, T = 2\pi$;
d) $f(t) = |t|$ pe $(-\pi, \pi), T = 2\pi$;
e) $f(t) = e^{at}$,a $\neq 0$, pe $(-\pi, \pi), T = 2\pi$;
f) $f(t) = |\sin t|$, pe $(-\pi, \pi), T = 2\pi$;
g) $f(t) = 10 - t$, pe $(5, 15), T = 10$;
h)$f(t) = \begin{cases} -t - \pi, t \in \left(-\pi, -\frac{\pi}{2}\right) \\ t, \text{ t} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ -t + \pi, \ t \in \left[\frac{\pi}{2}, \pi\right) \end{cases}, T = 2\pi$
i) $f(t) = \frac{1}{2 + \cos t}$ pe $[-\pi, \pi]$;

Exerciţiul 6.31 Să se dezvolte în serie Forier pe $[0, \pi]$ funcţia periodică $f(t)$, de perioadă $T = \pi$:

$$f(t) = \begin{cases} \frac{\pi}{3}, t \in \left[0, \frac{\pi}{3}\right] \\ 0, t \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \\ -\frac{\pi}{3}, t \in \left[\frac{2\pi}{3}, \pi\right]. \end{cases}$$

Capitolul 7

Ecuaţiile fizicii matematice

7.1 Reducerea la forma canonică a ecuaţiilor cu derivate parţiale de ordinul doi. Problema Cauchy. Ecuaţia coardei vibrante.

Prezentăm acum o metodă de reducere la forma canonică a E.D.P de ordinul al II-lea

Noţiuni teoretice

Fie următoarea E.D.P. de ordinul doi:

$$a\left(x,y\right)\frac{\partial^2 u}{\partial x^2}+b\left(x,y\right)\frac{\partial^2 u}{\partial x\partial y}+c\left(x,y\right)\frac{\partial^2 u}{\partial y^2}+f\left(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right)=0,$$

$u\left(x,y\right)$ de clasă C^2, coeficienţii sunt funcţii continue;

Considerăm ecuaţia caracteristică asociată E.D.P.

$$a(x,y)r^2 - b(x,y)r + c(x,y) = 0, (r = y^{'}(x))$$
$$\Delta = b^2 - 4ac.$$

(1) $\Delta > 0 \Rightarrow$ ecuaţia este de tip *hiperbolic.*

Avem două soluţii: $r_2(x,y) \neq r_2(x,y)$.

$$y'(x) = r_1(x,y) \Rightarrow$$

$\xi(x,y) = C_1$ integrală primă

$$y'(x) = r_2(x,y) \Rightarrow$$

$\eta(x,y) = C_2$ integrală primă

Se face schimbarea de funcţie:

$$u(x,y) = \widetilde{u}(\xi,\eta).$$

Rezultă forma canonică

$$\frac{\partial^2 \widetilde{u}}{\partial\xi\partial\eta} + f_1\left(\xi,\eta,\widetilde{u},\frac{\partial\widetilde{u}}{\partial\xi},\frac{\partial\widetilde{u}}{\partial\eta}\right) = 0.$$

(2) $\Delta = 0$ Ecuaţia este de tip *parabolic.*

$$r_2(x,y) = r_2(x,y)$$

$$\frac{dy}{dx} = y' = r_1(x,y) \Rightarrow$$

$\xi(x,y) = C_1$ integrală primă.

Notăm $\eta = x$

$$u(x,y) = \widetilde{u}(\xi,\eta)$$

şi avem forma canonică:

$$\frac{\partial^2 \widetilde{u}}{\partial \eta^2} + f_1\left(\xi,\eta,\widetilde{u},\frac{\partial \widetilde{u}}{\partial \xi},\frac{\partial \widetilde{u}}{\partial \eta}\right) = 0.$$

(3) $\Delta < 0$ Ecuaţia este de tip *eliptic* şi rădăcinile complex conjugate ale ecuaţiei caracteristice

$$r_{1,2}(x,y) = \alpha(x,y) \pm i\beta(x,y) \Rightarrow$$

integralele prime

$$r_{1,2}(x,y) = \xi(x,y) \pm i\eta(x,y) = c_1 \pm c_2 \Rightarrow$$

forma canonică:

$$\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + f_1\left(\xi,\eta,\widetilde{u},\frac{\partial \widetilde{u}}{\partial \xi},\frac{\partial \widetilde{u}}{\partial \eta}\right) = 0.$$

Formule de derivare:

(1)

$$\frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}.$$

(2)

$$\frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}.$$

(3)

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi}\cdot\frac{\partial \eta}{\partial x}\right)\cdot\frac{\partial \xi}{\partial x}$$

$$+\left(\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\eta}{\partial x}+$$

$$+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2}.$$

(4)

$$\frac{\partial^2 u}{\partial x\partial y}=\left(\frac{\partial^2\widetilde{u}}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2\widetilde{u}}{\partial\eta\partial\xi}\cdot\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\xi}{\partial y}$$

$$+\left(\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\eta}{\partial y}+$$

$$+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial x\partial y}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x\partial y}.$$

(5)

$$\frac{\partial^2 u}{\partial y^2}=\left(\frac{\partial^2\widetilde{u}}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial^2\widetilde{u}}{\partial\eta\partial\xi}\cdot\frac{\partial\eta}{\partial y}\right)\cdot\frac{\partial\xi}{\partial y}$$

$$+\left(\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial y}\right)\cdot\frac{\partial\eta}{\partial y}+$$

$$+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial y^2}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial y^2}.$$

7.1.1 Exerciţii rezolvate

Exerciţiul 7.1

$$\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x\partial y}+2\frac{\partial^2 u}{\partial y^2}=0$$

$u:\mathbb{R}^2\to\mathbb{R}$, de clasă C^2. Să se determine:

(i) Forma canonică, tipul ecuaţiei;

(ii) Soluţia generală;

(iii)

$$\begin{cases} u(0,y) = y \\ \frac{\partial u}{\partial x}(0,y) = 3y^2. \end{cases}$$

Soluţie.

(i)

$$r^2 - 3r + 2 = 0 \Rightarrow r_1 = 1, r_2 = 2.$$

$\Delta > 0$ ecuaţie de tip hiperbolic.

$$\begin{array}{l} y' = 1 \Rightarrow y = x + C_1 \Rightarrow y - x = C_1 \\ y' = 2 \Rightarrow y = 2x + C_2 \Rightarrow y - 2x = C_2 \end{array}$$

$$\Rightarrow \begin{cases} \xi(x,y) = y - x \\ \eta(x,y) = y - 2x. \end{cases}$$

Calculăm derivatele parţiale de ordinul doi folosind derivarea funcţiilor compuse de două variabile:

$$\frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot (-1) + \frac{\partial \widetilde{u}}{\partial \eta} \cdot (-2) \Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2} = -\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot (-1) + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} \cdot (-2)\right) +$$

$$+(-2)\left(\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot (-1) + \frac{\partial^2 \widetilde{u}}{\partial \eta} \cdot (-2)\right)$$

$$\frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \Rightarrow$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot (-1) + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} \cdot (-2) + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot (-1) + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot (-2)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 4\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + 4\frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

$$3 \cdot \left| \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \right.$$

$$2 \cdot \left| \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \right.$$

$$-\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0 \Rightarrow \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0$$

este forma canonică.

(ii) Soluţia generală;

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) = 0 \Rightarrow \frac{\partial \widetilde{u}}{\partial \eta} = f_1(\eta) \Rightarrow$$

$$\widetilde{u} = \int f_1(\eta) d\eta = f(\eta) + g(\xi)$$

$$\widetilde{u}(\xi, \eta) = f(\eta) + g(\xi)$$

unde $f, g : \mathbb{R} \to \mathbb{R}$ de clasă C^2

$$\Rightarrow u(x, y) = f(y - 2x) + g(y - x)$$

este soluţia generală.

(iii)

$$\begin{cases} u(0, y) = y \\ \frac{\partial u}{\partial x}(0, y) = 3y^2 \end{cases} \Rightarrow$$

$$\begin{cases} f(y) + g(y) = y \\ -2f'(y) - g'(y) = 3y^2 \end{cases} \Rightarrow$$

$$\begin{cases} f'(y) + g'(y) = 1 \\ -2f'(y) - g'(y) = 3y^2 \end{cases}$$

$$-f'(y) = 1 + 3y^2 \Rightarrow \begin{cases} f(y) = -y - y^3 + C \\ g(y) = 2y + y^3 - C \end{cases}$$

$$\Rightarrow u(x, y) = -(y - 2x) - (y - 2x)^3 + C +$$

$$+2(y - x) + (y - x)^3 - C =$$

$$= y + (y - x - y + 2x) \cdot$$

$$\cdot (y^2 - 2xy + x^2 + y^2 + 2x^2 - 3xy + y^2 + 4x^2 - 4xy) =$$

$$= y + x(3y^2 - 9xy + 6x^2) \Rightarrow$$

$$u(x, y) = y + 3x(y^2 - 3xy + 2x^2) = y + 3x(x - y)(2x - y).$$

Exerciţiul 7.2

$$x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + 2y \frac{\partial u}{\partial y} = x^5 y^2.$$

(i) Forma canonică, tipul ecuaţiei;

(ii) Soluţia generală;

Soluţie.

(i) Ecuaţia caracteristică

$$x^2 \cdot r^2 + 2xyr + y^2 = 0$$

$\Delta = 0 \Rightarrow$ ecuaţie de tip parabolic.

$$(xr + y)^2 = 0 \Rightarrow r = -\frac{y}{x}$$

$$y' = -\frac{y}{x} <\Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Rightarrow |xy| = K_0 \Rightarrow$$
$$xy = \pm K_0 = C \Rightarrow xy = C$$

$$\begin{cases} \xi = xy \\ \eta = x \end{cases} \qquad u(x,y) = \widetilde{u}(\xi,\eta) \sim \text{ schimbarea de funcţie.}$$

$$\frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot y + \frac{\partial \widetilde{u}}{\partial \eta}$$
$$\frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot x$$
$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot y + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) \cdot y + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot y + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}.$$

$$\cdot \frac{\partial^2 u}{\partial x \partial y} = \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot y + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) \cdot x + \frac{\partial \widetilde{u}}{\partial \xi}$$
$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot x\right) \cdot x$$
$$\frac{\partial^2 u}{\partial y^2} = x^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2}.$$

Deci:

$$x^2 \cdot \left| \frac{\partial^2 \widetilde{u}}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot y^2 + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot y + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \right.$$

$$(-2xy) \cdot \left| \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot xy + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot x + \frac{\partial^2 \widetilde{u}}{\partial \xi} \right.$$

$$y^2 \cdot \left| \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot x^2 \right.$$

$$(2y) \cdot \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot x$$

$$x^5 y^2 = \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot x^2 \Leftrightarrow \frac{\partial^2 \widetilde{u}}{\partial \eta^2} = x^3 y^2;$$

$$\xi = xy, \eta = x \Rightarrow$$

$$x^3 y^2 = \xi^2 \cdot \eta \Rightarrow$$

$$\Rightarrow \frac{\partial^2 \widetilde{u}}{\partial \eta^2} = \xi^2 \cdot \eta$$

este forma canonică.

(ii) Integrăm forma canonică:

$$\frac{\partial}{\partial \eta} \left(\frac{\partial \widetilde{u}}{\partial \eta} \right) = \xi^2 \eta \Rightarrow \frac{\partial \widetilde{u}}{\partial \eta} = \xi^2 \cdot \frac{\eta^2}{2} + f(\xi) \Rightarrow$$

$$\widetilde{u}(\xi, \eta) = \frac{1}{6} \xi^2 \cdot \eta^3 + \eta f(\xi) + g(\xi)$$

Înlocuim

$$\begin{cases} \xi = xy \\ \eta = x \end{cases} \Rightarrow$$

soluţia generală

$$u(x, y) = \frac{1}{6} x^5 y^2 + x f(xy) + g(xy),$$

unde $f, g : \mathbb{R} \to \mathbb{R}$ de clasă C^2.

Exerciţiul 7.3

$$\frac{\partial^2 u}{\partial x^2} + 4x\frac{\partial^2 u}{\partial x \partial y} + 4x^2\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y} = 6y$$

(i) Forma canonică, tipul ecuaţiei;

(ii) Soluţia generală;

(iii)

$$\begin{cases} u(2, y) = \frac{3}{2}y^2 \\ \frac{\partial u}{\partial x}(2, y) = -24. \end{cases}$$

Soluţie:

(i) Ecuaţia caracterisitcă

$$r^2 - 4xr + 4x^2 = 0 \Rightarrow$$

$\Delta = 0 \Rightarrow$ ecuaţia este de tip parabolic.

$$(r - 2x)^2 = 0 \Rightarrow r = y'(x) = 2x \Rightarrow dy = 2xdx$$

$$\Rightarrow y - x^2 = c \Rightarrow \begin{cases} \xi = y - x^2 \\ \eta = x. \end{cases}$$

$$\Rightarrow u(x, y) = \widetilde{u}(\xi, \eta).$$

Calculăm derivatele parţiale

$$\frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \tilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi} \cdot (-2x) + \frac{\partial \tilde{u}}{\partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \tilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial \tilde{u}}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \left[\frac{\partial^2 \tilde{u}}{\partial \xi^2} \cdot (-2x) + \frac{\partial^2 \tilde{u}}{\partial \eta \partial \xi}\right] \cdot (-2x) +$$

$$+\frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} \cdot (-2x) + \frac{\partial^2 \tilde{u}}{\partial \eta^2} - 2\frac{\partial \tilde{u}}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial \xi^2} \cdot (-2x) + \frac{\partial^2 u}{\partial \eta \partial \xi}$$

$$\frac{\partial u}{\partial y} = \frac{\partial^2 \tilde{u}}{\partial \xi^2}$$

Deci:

$$1 \cdot /\frac{\partial^2 u}{\partial x^2} = 4x^2\frac{\partial^2 \tilde{u}}{\partial \xi^2} - 4x\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial \tilde{u}}{\partial \xi}$$

$$4x \cdot /\frac{\partial^2 u}{\partial x \partial y} = (-2x) \cdot \frac{\partial^2 \tilde{u}}{\partial \xi^2} + \frac{\partial^2 \tilde{u}}{\partial \xi^2 \partial \eta}$$

$$4x^2 \cdot /\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \tilde{u}}{\partial \xi^2}$$

$$2 \cdot /\frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \xi}$$

$$\left.\begin{array}{l} 6y = \frac{\partial^2 \tilde{u}}{\partial \eta^2} \Rightarrow \frac{\partial^2 \tilde{u}}{\partial \eta^2} = 6y \\ \xi = y - x^2 \\ \eta = x \Rightarrow y = \xi + \eta^2 \end{array}\right\} \Rightarrow$$

$$\frac{\partial^2 \tilde{u}}{\partial \eta^2} = 6\xi + 6\eta^2$$

adică forma canonică.

(ii) Integrăm în raport cu η

$$\frac{\partial}{\partial\eta}\left(\frac{\partial\tilde{u}}{\partial\eta}\right) = 6\xi + 6\eta^2 \Rightarrow$$

$$\frac{\partial\tilde{u}}{\partial\eta} = 6\xi\eta + 2\eta^3 + f(\xi) \Rightarrow$$

$$\tilde{u}(\xi,\eta) = 3\xi\eta^2 + \frac{\eta^4}{2} + \eta f(\xi) + g(\xi)$$

$$u(x,y) = 3x^2(y-x^2) + \frac{x^4}{2} + xf(y-x^2) + g(y-x^2) \Rightarrow$$

$$u(x,y) = 3x^2y + \frac{5}{2}x^4 + xf(y-x^2) + g(y-x^2)$$

cu $f, g \in C^2,\ f, g : \mathbb{R} \to \mathbb{R}$.

Avem

$$\frac{\partial u}{\partial x} = 6xy - 10x^3 + f(y-x^2) - 2x^2 f'(y-x^2) - 2x \cdot g'(y-x^2).$$

(iii)

$$\begin{cases} u\,(2,y) = 12y - 40 + 2f\,(y-4) + \\ \qquad\qquad +g\,(y-4) = \frac{3}{2}y^2 \\ \frac{\partial u}{\partial x}\,(2,y) = 12y - 80 + f\,(y-4) - 8f'\,(y-4) - \\ \qquad\qquad -4g'\,(y-4) = -24 \end{cases} \Leftrightarrow$$

$$\begin{cases} 2f'\,(y-4) + g'\,(y-4) = 3y - 12 \\ f\,(y-4) - 4\,(2f'(y-4) + g'(y-4)) = 56 - 12y \end{cases}$$

$$\begin{cases} f\,(y-4) = 4\,(3y-12) - 12y + 56 = 8 \\ g\,(y-4) = \frac{3}{2}y^2 - 12y + 40 - 16 = \frac{3}{2}y^2 - 12y + 24 = \\ \qquad = \frac{3}{2}\,(y^2 - 8y + 16) = \frac{3}{2}\,(y-4)^2\,. \end{cases}$$

Fie

$$t = y - 4 \Rightarrow \begin{cases} f(t) = 8 \\ g(t) = \frac{3}{2}t^2 \end{cases} \Rightarrow$$

$$u\left(x, y\right) = 3x^2y - \frac{5}{2}x^4 + 8x + \frac{3}{2}\left(y - x^2\right)^2 =$$

$$= 3x^2y - \frac{5}{2}x^4 + 8x + \frac{3}{2}y^2 - 3x^2y + \frac{3}{2}x^4 \Rightarrow$$

$$u\left(x, y\right) = \frac{3}{2}y^2 - x^4 + 8x.$$

Observaţia 7.4 Pentru a calcula mai uşor derivatele parţiale de ordinul doi introducem operatorii

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial x};$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}.$$

Exerciţiul 7.5 Să se reducă la forma canonică ecuaţia:

$$4y^2\frac{\partial^2 u}{\partial x^2} - e^{2x}\frac{\partial^2 u}{\partial y^2} = 0.$$

Soluţie:

$$\begin{cases} a = 4y^2 \\ b = 0 \\ c = -e^{2x} \end{cases} \Rightarrow \triangle = b^2 - 4ac = 4y^2e^{2x} > 0 \text{ tip parabolic.}$$

$$4y^2\left(\frac{dy}{dx}\right)^2 - e^{2x} = 0 \Leftrightarrow \frac{dy}{dx} = \pm\sqrt{\frac{e^{2x}}{4y^2}} \Rightarrow$$

$$\begin{cases} \frac{dy}{dx} = \sqrt{\frac{e^{2x}}{4y^2}} \\ \frac{dy}{dx} = -\sqrt{\frac{e^{2x}}{4y^2}} \end{cases} \Leftrightarrow \begin{cases} \frac{dy}{dx} = \frac{e^x}{2y} \\ \frac{dy}{dx} = -\frac{e^x}{2y} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2ydy = e^x dx \\ 2ydy = -e^x dx \end{cases} \Leftrightarrow \begin{cases} e^x - y^2 = c_1 \\ e^x + y^2 = c_2 \end{cases} \text{– integralele prime.}$$

Facem schimbarea de variabile şi de funcţie:

$$\begin{cases} \xi = \xi(x,y) = e^x - y^2 \\ \eta = \eta(x,y) = e^x + y^2 \end{cases} \qquad \widetilde{u}(\xi,\eta) = u(x,y).$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = e^x \left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \right) \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = (-2y) \left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta} \right) \end{cases} \Rightarrow$$

$$\begin{cases} \frac{\partial}{\partial x} = e^x \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \right) \\ \frac{\partial}{\partial y} = (-2y) \left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \right) \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[e^x \left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] =$$

$$= e^x \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) + e^x \left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= e^x \left[e^x \left(\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right)\right)\right] +$$

$$+ e^x \left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= e^{2x}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) + e^x \left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right).$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[(-2y)\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] =$$

$$= (-2y)\frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) - 2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= (-2y)\cdot(-2y)\left[\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] -$$

$$-2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= 4y^2\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) - 2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right).$$

Ecuaţia devine:

$$4y^2 e^{2x}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) + 4y^2 e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) -$$

$$-4y^2 e^{2x}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) + 2e^{2x}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) = 0 \Leftrightarrow$$

$$8y^2 e^{2x}\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \left(2y^2 - e^x\right)e^x\frac{\partial \widetilde{u}}{\partial \eta} + \left(2y^2 + e^x\right)e^x\frac{\partial \widetilde{u}}{\partial \xi} = 0 \,\Big|\ : e^x \Leftrightarrow$$

$$8y^2 e^x\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \left(2y^2 + e^x\right)\frac{\partial \widetilde{u}}{\partial \xi} + \left(2y^2 - e^x\right)\frac{\partial \widetilde{u}}{\partial \eta} = 0.$$

$$\begin{cases} \xi = e^x - y^2 \\ \eta = e^x + y^2 \end{cases} \Rightarrow \begin{cases} e^x = \frac{\xi+\eta}{2} \\ y^2 = \frac{\eta-\xi}{2} \end{cases} \Rightarrow \text{ ecuaţia devine:}$$

$$2\left(\eta^2 - \xi^2\right)\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \left(\eta - \xi + \frac{\xi+\eta}{2}\right)\cdot\frac{\partial \widetilde{u}}{\partial \xi} +$$

$$+\left(\eta - \xi - \frac{\xi+\eta}{2}\right)\cdot\frac{\partial \widetilde{u}}{\partial \eta} = 0 \Leftrightarrow$$

$$2\left(\eta^2 - \xi^2\right)\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{3\eta - \xi}{2}\cdot\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\eta - 3\xi}{2}\cdot\frac{\partial \widetilde{u}}{\partial \eta} = 0.$$

Facem schimbare de variabilă şi de funcţie:

$$\begin{cases} \rho = \xi + \eta \\ \sigma = \xi - \eta \end{cases} \qquad u_1(\rho, \sigma) = \widetilde{u}(\xi, \eta) \Rightarrow$$

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial \xi} = \frac{\partial u_1}{\partial \rho} \cdot \frac{\partial \rho}{\partial \xi} + \frac{\partial u_1}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \xi} = \frac{\partial u_1}{\partial \rho} + \frac{\partial u_1}{\partial \sigma} \\ \frac{\partial \widetilde{u}}{\partial \eta} = \frac{\partial u_1}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \eta} + \frac{\partial u_1}{\partial \rho} \cdot \frac{\partial \rho}{\partial \eta} = \frac{\partial u_1}{\partial \rho} - \frac{\partial u_1}{\partial \sigma} \end{cases} \Rightarrow$$

$$\begin{cases} \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \rho} + \frac{\partial}{\partial \sigma} \\ \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \rho} - \frac{\partial u}{\partial \sigma} \end{cases}$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = \frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial u_1}{\partial \rho} - \frac{\partial u_1}{\partial \sigma}\right) =$$

$$= \frac{\partial}{\partial \rho}\left(\frac{\partial u_1}{\partial \rho} - \frac{\partial u_1}{\partial \sigma}\right) + \frac{\partial}{\partial \sigma}\left(\frac{\partial u_1}{\partial \rho} - \frac{\partial u_1}{\partial \sigma}\right) =$$

$$= \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \rho \partial \sigma} + \frac{\partial^2 u_1}{\partial \rho \partial \sigma} - \frac{\partial^2 u_1}{\partial \sigma^2} = \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2}.$$

Ecuaţia devine:

$$(-2)\,\rho \cdot \sigma \left(\frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2}\right) + \left(\frac{3\eta - \xi}{2} + \frac{\eta - 3\xi}{2}\right)\frac{\partial u_1}{\partial \rho} +$$

$$+ \left(\frac{-\eta + 3\xi}{2} + \frac{3\eta - \xi}{2}\right)\frac{\partial u_1}{\partial \sigma} = 0$$

$$(-2)\,\rho \cdot \sigma \left(\frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2}\right) - 2\sigma \cdot \frac{\partial u_1}{\partial \rho} + \rho \cdot \frac{\partial u_1}{\partial \sigma} = 0.$$

Exerciţiul 7.6 Să se rezolve ecuaţia:

$$4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 4u = 2e^{x-y}.$$

Soluţie:

$$\left\{\begin{array}{l} a=4 \\ b=-4 \\ c=1 \end{array}\right. \Rightarrow \triangle = b^2-4ac=16-16=0$$

$\Rightarrow$ ecuaţie de tip parabolic. Ecuaţia caracteristicilor este:

$$4\left(\frac{\partial y}{\partial x}\right)^2+4\frac{\partial y}{\partial x}+1=0 \Rightarrow \frac{\partial y}{\partial x}=\frac{-1}{2} \Leftrightarrow 2dy=-dx \Leftrightarrow$$

$$2y^2+x=c \text{ integrala primă.}$$

Facem schimbarea de variabilă:

$$\left\{\begin{array}{l} \xi=x+2y \\ \eta=x \end{array}\right.$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi,\eta)=u(x,y) \Rightarrow$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x}=\frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=\frac{\partial \widetilde{u}}{\partial \xi}+\frac{\partial \widetilde{u}}{\partial \eta} \\ \frac{\partial u}{\partial y}=\frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=2\frac{\partial \widetilde{u}}{\partial \xi} \end{array}\right\} \Rightarrow \text{operatorii :}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial x}=\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y}=2\frac{\partial}{\partial \xi} \end{array}\right. \Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l} \frac{\partial^2}{\partial x^2}=\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)^2=\frac{\partial^2}{\partial \xi^2}+2\frac{\partial^2}{\partial \xi\partial \eta}+\frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x\partial y}=2\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\right)\cdot\frac{\partial}{\partial \xi}=2\frac{\partial^2}{\partial \xi^2}+2\frac{\partial^2}{\partial \xi\partial \eta} \\ \frac{\partial^2}{\partial y^2}=4\frac{\partial^2}{\partial \xi^2}. \end{array}\right.$$

Atunci ecuaţia devine:

$$4\left(\frac{\partial^2}{\partial\xi^2}+2\frac{\partial^2}{\partial\xi\partial\eta}+\frac{\partial^2}{\partial\eta^2}\right)\cdot\widetilde{u}-8\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\xi\partial\eta}\right)\cdot\widetilde{u}+4\frac{\partial^2\widetilde{u}}{\partial\xi^2}-$$

$$-6\left(\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right)\cdot\widetilde{u}+6\frac{\partial\widetilde{u}}{\partial\xi}-4\widetilde{u}=2e^{\frac{3\eta-\xi}{2}}\Leftrightarrow$$

$$4\frac{\partial^2\widetilde{u}}{\partial\eta^2}-6\frac{\partial\widetilde{u}}{\partial\eta}-4\widetilde{u}=2e^{\frac{3\eta-\xi}{2}}\quad|:2\quad\Leftrightarrow$$

$$2\frac{\partial^2\widetilde{u}}{\partial\eta^2}-3\frac{\partial\widetilde{u}}{\partial\eta}-2\widetilde{u}=e^{\frac{3\eta-\xi}{2}}.$$

Pentru a simplifica ecuaţia facem schimbarea de funcţie:

$$\widetilde{u}\left(\xi,\eta\right)=v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}\ \Rightarrow\ \text{ecuaţia devine:}$$

$$\left.\begin{array}{l}\frac{\partial\widetilde{u}}{\partial\eta}=\frac{\partial v}{\partial\eta}\cdot e^{\alpha\xi+\beta\eta}+\beta\cdot v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}\\[2ex]\frac{\partial^2\widetilde{u}}{\partial\eta^2}=\frac{\partial^2 v}{\partial\eta^2}\cdot e^{\alpha\xi+\beta\eta}+2\beta\frac{\partial v}{\partial\eta}\cdot e^{\alpha\xi+\beta\eta}+\beta^2\cdot v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}\end{array}\right\}\Rightarrow$$

$$2\frac{\partial^2 v}{\partial\eta^2}\cdot e^{\alpha\xi+\beta\eta}+4\beta\frac{\partial v}{\partial\eta}\cdot e^{\alpha\xi+\beta\eta}+2\beta^2\cdot v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}-$$

$$-3\frac{\partial v}{\partial\eta}\cdot e^{\alpha\xi+\beta\eta}-3\beta\cdot v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}-2v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}=e^{\frac{3\eta-\xi}{2}}$$

$$2\frac{\partial^2 v}{\partial\eta^2}\cdot e^{\alpha\xi+\beta\eta}+\left(4\beta-3\right)\cdot\frac{\partial v}{\partial\eta}\cdot e^{\alpha\xi+\beta\eta}+$$

$$+\left(2\beta^2-3\beta-2\right)v\left(\xi,\eta\right)\cdot e^{\alpha\xi+\beta\eta}=e^{\frac{3\eta-\xi}{2}}$$

Pentru a simplifica ecuaţia impunem: $2\beta^2-3\beta-2=0$

$$\Rightarrow\beta_{1,2}=\frac{3\mp\sqrt{9+16}}{4}=\frac{3\mp5}{4}\Rightarrow\left\{\begin{array}{l}\beta_1=\frac{-1}{2},\\ \beta_2=2.\end{array}\right.$$

Alegem $\alpha = 0,\ \beta = \frac{-1}{2} \Rightarrow$ ecuaţia devine:

$$2\frac{\partial^2 v}{\partial \eta^2} \cdot e^{\frac{-\eta}{2}} - 5\frac{\partial v}{\partial \eta} \cdot e^{\frac{-\eta}{2}} = e^{\frac{3\eta-\xi}{2}} \Leftrightarrow$$

$$2\frac{\partial^2 v}{\partial \eta^2} - 5\frac{\partial v}{\partial \eta} = e^{2\eta-\frac{\xi}{2}} \Leftrightarrow$$

$$\frac{\partial}{\partial \eta}\left(2\frac{\partial v}{\partial \eta} - 5v\right) = e^{2\eta-\frac{\xi}{2}} \Rightarrow$$

$$\Rightarrow 2\frac{\partial v}{\partial \eta} - 5v = \frac{1}{2}e^{2\eta-\frac{\xi}{2}} + \phi_1(\xi)$$

- ecuaţie afină (ecuaţie liniară neomogenă). Îi asociem ecuaţia liniară omogenă.

$$2\frac{\partial \overline{v}}{\partial \eta} - 5\overline{v} = 0 \Leftrightarrow 2\frac{d\overline{v}}{\overline{v}} = 5 \cdot d\eta \Leftrightarrow \ln \overline{v} = \frac{5}{2}\eta + \ln \phi_2(\xi) \Rightarrow$$

$$\Rightarrow \ln \overline{v}(\xi,\eta) = \frac{5}{2}\eta + \ln \phi_2(\xi) \Leftrightarrow \overline{v}(\xi,\eta) = e^{\frac{5\eta}{2}} \cdot \phi_2(\xi).$$

Căutăm (efectuând metoda variaţiei constantelor) soluţie de forma:

$$v(\xi,\eta) = e^{\frac{5\eta}{2}} \cdot \phi_2(\xi,\eta)$$

şi ecuaţia devine:

$$2 \cdot \frac{5}{2} \cdot e^{\frac{5\eta}{2}} \cdot \phi_2(\xi,\eta) + 2 \cdot e^{\frac{5\eta}{2}} \cdot \frac{\partial \phi_2}{\partial \eta} - 5 \cdot e^{\frac{5\eta}{2}} \cdot \phi_2(\xi,\eta) =$$

$$= \frac{1}{2} \cdot e^{2\eta-\frac{\xi}{2}} + \phi_1(\xi) \Leftrightarrow$$

$$2 \cdot \frac{\partial \phi_2}{\partial \eta} = \frac{1}{2} \cdot e^{\frac{-\xi+\eta}{2}} + \phi_1(\xi) \cdot e^{\frac{-5\eta}{2}} \Rightarrow$$

$$\phi_2(\xi,\eta) = \frac{-1}{2}\cdot e^{\frac{-\xi+\eta}{2}} - \frac{1}{5}\cdot\phi_1(\xi)\cdot e^{\frac{-5\eta}{2}} + \phi_3(\xi) \Rightarrow$$

$$v(\xi,\eta) = e^{\frac{5\eta}{2}}\left(\frac{-1}{2}\cdot e^{\frac{-\xi}{2}-\frac{\eta}{2}} - \frac{1}{5}\cdot\phi_1(\xi)\cdot e^{\frac{-5\eta}{2}} + \phi_3(\xi)\right) =$$

$$= \frac{-1}{2}\cdot e^{2\eta-\frac{\xi}{2}} - \frac{1}{5}\cdot\phi_1(\xi) + e^{\frac{5\eta}{2}}\cdot\phi_3(\xi)$$

$$\widetilde{u}(\xi,\eta) = v(\xi,\eta)\cdot e^{\frac{-\eta}{2}} = \frac{-1}{2}\cdot e^{\frac{3\eta}{2}-\frac{\xi}{2}} - \frac{1}{5}\cdot e^{\frac{-\eta}{2}}\phi_1(\xi) + e^{2\eta}\cdot\phi_3(\xi).$$

Notăm: $\Phi(\xi) = \phi_1(\xi)$ şi $\Psi(\xi) = \phi_3(\xi) \Rightarrow$

$$\widetilde{u}(\xi,\eta) = \frac{-1}{2}\cdot e^{\frac{3\eta}{2}-\frac{\xi}{2}} - \frac{1}{5}\cdot e^{\frac{-\eta}{2}}\cdot\Phi(\xi) + e^{2\eta}\cdot\Psi(\xi)$$

Revenim la notaţii:

$$\begin{cases} x = \eta \\ 2y + x = \xi \end{cases} \Rightarrow \frac{3\eta-\xi}{2} = \frac{3x-x-2y}{2} = x - y \Rightarrow$$

soluţia generală a ecuaţiei este:

$$u(x,y) = \frac{-1}{2}\cdot e^{x-y} - \frac{1}{5}\cdot e^{\frac{-x}{2}}\cdot\Phi(x+2y) + e^{2x}\cdot\Psi(x+2y).$$

Exerciţiul 7.7 Să se rezolve problema:

$$\begin{cases} 4y^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} + 2x\frac{\partial u}{\partial x} = 0 \\ u(x,1) = f(x) \\ \frac{\partial u}{\partial y}(x,1) = g(x). \end{cases}$$

Soluţie:

$$\left.\begin{array}{l} a = 4x^2 \\ b = 0 \\ c = -y^2 \end{array}\right\} \Rightarrow \triangle = b^2 - 4ac = 16x^2y^2 > 0$$

$\Rightarrow$ ecuaţie de tip hiperbolic. Ecuaţia caracteristicilor este:

$$4x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0 \Rightarrow \frac{dy}{dx} = \pm\frac{y}{2x} \Rightarrow \frac{dy}{dx} = \frac{y}{2x} \text{ şi } \frac{dy}{dx} = \frac{-y}{2x} \Rightarrow$$

integralele prime sunt:

$$\frac{dy}{dx} = \frac{y}{2x} \Rightarrow 2\ln y = \ln x + \ln C_0 \Leftrightarrow \ln y^2 = \ln C_0 \cdot x \Rightarrow$$

$$\Rightarrow y^2 = C_0 \cdot x \Rightarrow \frac{y^2}{x} = C_0 \Rightarrow$$

prima integrală primă este:

$$\frac{y^2}{x} = C_0 \text{ sau } \frac{x}{y^2} = C_1$$

$$\frac{dy}{y} = \frac{-dx}{2x} \Leftrightarrow 2\ln y = \ln\frac{1}{x} + \ln C_2 \Rightarrow xy^2 = C_2.$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi(x,y) = \frac{x}{y^2} \\ \eta(x,y) = xy^2 \end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi,\eta) = u(x,y) \Rightarrow$$

$$\left\{\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{1}{y^2}\cdot\frac{\partial \widetilde{u}}{\partial \xi} + y^2\frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{-2x}{y^3}\cdot\frac{\partial \widetilde{u}}{\partial \xi} + 2xy\frac{\partial \widetilde{u}}{\partial \eta} \end{array}\right. \xrightarrow{operatorii}$$

$$\begin{cases} \frac{\partial}{\partial x} = \frac{1}{y^2} \cdot \frac{\partial}{\partial \xi} + y^2 \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = \frac{-2x}{y^3} \cdot \frac{\partial}{\partial \xi} + 2xy \frac{\partial}{\partial \eta}. \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + y^2 \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{1}{y^2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + y^2 \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{1}{y^2}\left(\frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + y^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) + y^2\left(\frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + y^2 \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right)$$

$$= \frac{1}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + y^4 \frac{\partial^2 \widetilde{u}}{\partial \eta^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-2x}{y^3} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{-2x}{y^3} \cdot \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) + 2x \frac{\partial \widetilde{u}}{\partial \eta} =$$

$$= \frac{-2x}{y^3}\left(\frac{-2x}{y^3} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2xy \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) +$$

$$+ 2xy\left(\frac{-2x}{y^3} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + 2xy \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) +$$

$$+ \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2x \frac{\partial \widetilde{u}}{\partial \eta} =$$

$$= \frac{4x^2}{y^6} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - \frac{8x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + 4x^2 y^2 \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2x \frac{\partial \widetilde{u}}{\partial \eta}.$$

Ecuaţia devine:

$$\frac{4x^2}{y^4}\cdot\frac{\partial^2\widetilde{u}}{\partial\xi^2}+8x^2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+4x^2y^4\frac{\partial^2\widetilde{u}}{\partial\eta^2}-\frac{4x^2}{y^4}\cdot\frac{\partial^2\widetilde{u}}{\partial\xi^2}+8x^2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}-$$

$$-4x^2y^4\frac{\partial^2\widetilde{u}}{\partial\eta^2}-\frac{6x}{y^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}-2xy^2\frac{\partial\widetilde{u}}{\partial\eta}+\frac{2x}{y^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}+2xy^2\frac{\partial\widetilde{u}}{\partial\eta}=0$$

$$16x^2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}-4\frac{x}{y^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\,\Big|:4x^2\Rightarrow 4\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}-\frac{1}{xy^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\Leftrightarrow$$

$$\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}-\frac{1}{4\eta}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\Leftrightarrow\frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\widetilde{u}\right)=0\Rightarrow$$

$$\frac{\partial\widetilde{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\widetilde{u}=\phi(\eta)\Leftrightarrow$$

- ecuaţie afină, căreia îi ataşăm ecuaţia liniară:

$$\frac{\partial\overline{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\overline{u}=0-\text{ecuaţie cu variabile separabile}\Rightarrow$$

$$\frac{\partial\overline{u}}{\overline{u}}=\frac{1}{4\eta}\partial\eta\Rightarrow\ln\overline{u}=\frac{1}{4}\ln\eta+\ln\Phi(\xi)\Rightarrow$$

$$\overline{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi).$$

Căutăm soluţie de forma:

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi,\eta);$$

Introducând în ecuaţie avem:

$$\frac{1}{4\sqrt[4]{\eta^3}}\Phi(\xi,\eta)+\sqrt[4]{\eta}\cdot\frac{\partial\Phi}{\partial\eta}-\frac{1}{4\sqrt[4]{\eta^3}}\Phi(\xi,\eta)=\phi(\eta)\Rightarrow$$

$$\frac{\partial\Phi}{\partial\eta}=\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\Rightarrow\Phi(\xi,\eta)=\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta+\psi(\xi)\Rightarrow$$

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi,\eta)=\sqrt[4]{\eta}\left(\psi(\xi)+\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta\right)=$$

$$=\sqrt[4]{\eta}\cdot\psi(\xi)+\phi_0(\eta),\ \phi_0(\eta)=\sqrt[4]{\eta}\cdot\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta$$

Deci:

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\psi(\xi)+\phi_0(\eta)\Rightarrow u(x,y)=$$

$$=\sqrt[4]{xy^2}\cdot\psi\left(\frac{x}{y^2}\right)+\phi_0\left(xy^2\right).$$

Condiţiile iniţiale sunt:

$$\begin{cases} u(x,1)=f(x)\\ \frac{\partial u}{\partial y}(x,1)=g(x).\end{cases}$$

Ele devin:

$$\begin{cases} \sqrt[4]{x}\cdot\psi(x)+\phi_0(x)=f(x)\\ \left[\sqrt[4]{xy^2}\cdot\psi'\left(\frac{x}{y^2}\right)\cdot\frac{-2x}{y^3}+\right.\\ \left.+\frac{1}{4}\cdot\frac{2xy}{\sqrt[4]{x^3y^6}}\cdot\psi\left(\frac{x}{y^2}\right)+2xy\phi_0'\left(xy^2\right)\right]|_{y=1}=g(x)\end{cases}\Leftrightarrow$$

$$\begin{cases} \sqrt[4]{x}\cdot\psi(x)+\phi_0(x)=f(x)\\ \\ -2x\sqrt[4]{x}\cdot\psi'(x)+\frac{\sqrt[4]{x}}{2}\cdot\psi(x)+2x\phi_0'(x)=g(x)\end{cases}\Rightarrow$$

$$\begin{cases} \phi_0(x)=f(x)-\sqrt[4]{x}\cdot\psi(x)\\ \\ \phi_0'(x)=f'(x)-\frac{1}{4\sqrt[4]{x^3}}\psi(x)-\sqrt[4]{x}\cdot\psi'(x)\end{cases}\Rightarrow$$

$$-2x\sqrt[4]{x}\cdot\psi'(x)+\frac{\sqrt[4]{x}}{2}\cdot\psi(x)+$$

$$+2x\cdot f'(x)-\frac{\sqrt[4]{x}}{2}\cdot\psi(x)-2x\sqrt[4]{x}\cdot\psi'(x)=g(x)\Rightarrow$$

$$-4x\sqrt[4]{x}\cdot\psi'(x)=g(x)-2x\cdot f'(x)\Rightarrow$$

$$\psi'(x)=\frac{-1}{4x^{\frac{5}{4}}}g(x)+\frac{1}{2x^{\frac{1}{4}}}f'(x)=\frac{x^{\frac{-5}{4}}}{4}\Big(2x\cdot f'(x)-g(x)\Big)\Rightarrow$$

$$\begin{cases}\psi(x)=\int_{x_0}^{x}\frac{t^{\frac{-5}{4}}}{4}\big(2t\cdot f'(t)-g(t)\big)\,dt+C\\ \phi_0(x)=f(x)-\sqrt[4]{x}\left[\int_{x_0}^{x}\frac{t^{\frac{-5}{4}}}{4}\big(2t\cdot f'(t)-g(t)\big)\,dt+C\right].\end{cases}$$

Soluţia ecuaţiei este:

$$u(x,y)=\sqrt[4]{xy^2}\left[\int_{x_0}^{\frac{x}{y^2}}\frac{t^{\frac{-5}{4}}}{4}\Big(2t\cdot f'(t)-g(t)\Big)\,dt+C\right]+f\left(xy^2\right)-$$

$$-\sqrt[4]{xy^2}\left[\int_{x_0}^{xy^2}\frac{t^{\frac{-5}{4}}}{4}\Big(2t\cdot f'(t)-g(t)\Big)\,dt+C\right]=$$

$$=\sqrt[4]{xy^2}\cdot\int_{x_0}^{\frac{x}{y^2}}\frac{t^{\frac{-5}{4}}}{4}\Big(2t\cdot f'(t)-g(t)\Big)\,dt+$$

$$+\sqrt[4]{xy^2}\cdot\int_{xy^2}^{x_0}\frac{t^{\frac{-5}{4}}}{4}\Big(2t\cdot f'(t)-g(t)\Big)\,dt+f\left(xy^2\right)\Rightarrow$$

$$u(x,y)=\sqrt[4]{xy^2}\cdot\int_{xy^2}^{\frac{x}{y^2}}\frac{t^{\frac{-5}{4}}}{4}\Big(2t\cdot f'(t)-g(t)\Big)\,dt+f\left(xy^2\right)\Leftrightarrow$$

$$u(x,y)=\frac{\sqrt[4]{xy^2}}{4}\cdot\int_{xy^2}^{\frac{x}{y^2}}t^{\frac{-5}{4}}\cdot\Big(2t\cdot f'(t)-g(t)\Big)\,dt+f\left(xy^2\right).$$

Exerciţiul 7.8 Să se aducă la forma canonică ecuaţia:

$$\frac{\partial^2u}{\partial x^2}-2\sin x\cdot\frac{\partial^2u}{\partial x\partial y}+\left(2-\cos^2x\right)\frac{\partial^2u}{\partial y^2}=0.$$

Soluţie:

$$\begin{cases} a = 1 \\ b = -2\sin x \\ c = 2 - \cos^2 x \end{cases}$$

şi

$$\triangle = b^2 - 4ac = 4\sin^2 x - 8 + 4\cos^2 x = -4 < 0 \Rightarrow$$

ecuaţia este de tip eliptic.
Ecuaţia caracteristicilor este:

$$\left(\frac{dy}{dx}\right)^2 + 2\sin x \frac{dy}{dx} + \left(2 - \cos^2 x\right) = 0 \Rightarrow$$

$$\frac{dy}{dx} = \frac{-2\sin x \mp 2i}{2} \Rightarrow \frac{dy}{dx} = -\sin x \mp i \Leftrightarrow$$

$$dy = (-\sin x \mp i)\, dx \Rightarrow \int dy = \int (-\sin x \mp i)\, dx \Leftrightarrow$$

$$\Leftrightarrow y = \cos x \pm ix + C \Leftrightarrow (y - \cos x) \mp ix = C \Rightarrow$$

$$\begin{cases} \xi = y - \cos x \\ \eta = x \end{cases} \quad \text{şi } \widetilde{u}\,(\xi, \eta) = u\,(x, y) \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \sin x \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = \sin x \cdot \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\sin x \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \sin x \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} =$$

$$\sin x \left(\sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) +$$

$$+ \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} =$$

$$= \sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) = \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) = \frac{\partial^2 \widetilde{u}}{\partial \xi^2}$$

Ecuaţia devine:

$$\sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} -$$

$$-2 \sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} +$$

$$+2 \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - \cos^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} = 0$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} = 0.$$

Exerciţiul 7.9 Să se aducă la forma canonică ecuaţia şi să se rezolve problema Cauchy:

$$\begin{cases} x^2\frac{\partial^2 u}{\partial x^2} - y^2\frac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{y=1} = x^2, \ \frac{\partial u}{\partial y}\Big|_{y=1} = 2x. \end{cases}$$

Soluţie:

$$\begin{cases} a = x^2 \\ b = 0 \\ c = -y^2 \end{cases}$$

şi

$$\triangle = b^2 - 4ac = 4x^2y^2 > 0 \Rightarrow$$

ecuaţia este de tip hiperbolic.

Ecuaţia caracteristicilor:

$$x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0 \Rightarrow \frac{dy}{dx} = \pm\frac{y}{x} \Leftrightarrow$$

$$\begin{cases} \frac{dy}{dx} = \frac{-y}{x} \\ \frac{dy}{dx} = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} \int \frac{dy}{y} = \int \frac{-dx}{x} \\ \int \frac{dy}{y} = \int \frac{dx}{x} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \ln y = \ln\frac{1}{x} + \ln C_1 \\ \ln y + \ln C_2 = \ln x \end{cases} \Leftrightarrow \begin{cases} \ln xy = \ln C_1 \\ \ln\frac{x}{y} = \ln C_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} xy = C_1 \\ \frac{x}{y} = C_2 \end{cases} - \text{integrale prime.}$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi = \xi(x, y) = xy \\ \eta = \eta(x, y) = \frac{x}{y} \end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi(x, y), \eta(x, y)) = u(x, y).$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = y \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{1}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = x \cdot \frac{\partial \widetilde{u}}{\partial \xi} - \frac{x}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = y \cdot \frac{\partial}{\partial \xi} + \frac{1}{y} \cdot \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = x \cdot \frac{\partial}{\partial \xi} - \frac{x}{y^2} \cdot \frac{\partial}{\partial \eta} \end{cases}$$

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \bigg| \cdot x^2 \\ \\ \frac{\partial^2 u}{\partial y^2} = x^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2 \frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{x^2}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + 2 \frac{x}{y^3} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \bigg| \cdot y^2 \end{cases} \Rightarrow$$

$$0 = x^2 \cdot \frac{\partial^2 u}{\partial x^2} - y^2 \cdot \frac{\partial^2 u}{\partial y^2} =$$

$$= x^2 y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - x^2 y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} +$$

$$+ 2x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - 2 \frac{x}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \Leftrightarrow$$

$$4x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - 2 \frac{x}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \bigg| : 4x^2 \Rightarrow$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{1}{2 \underbrace{xy}_{\xi}} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \Leftrightarrow \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{1}{2\xi} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \Leftrightarrow$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{1}{2\xi} \cdot \widetilde{u} \right) = 0 \Rightarrow$$

$$\frac{\partial \widetilde{u}}{\partial \xi} - \frac{1}{2\xi} \cdot \widetilde{u} = \phi_0 (\xi) .$$

Rezolvăm ecuaţia omogenă ataşată:

$$\frac{\partial \overline{u}}{\partial \xi} = \frac{1}{2\xi}\overline{u} \Leftrightarrow \frac{\partial \overline{u}}{\overline{u}} = \frac{\partial \xi}{\xi} \Rightarrow \overline{u}\left(\xi, \eta\right) = \phi_1\left(\eta\right) \cdot \sqrt{\xi} \Rightarrow$$

căutăm $\widetilde{u}$ de forma:

$$\widetilde{u}\left(\xi, \eta\right) = \phi_1\left(\xi, \eta\right) \cdot \sqrt{\xi}.$$

Înlocuim în ecuaţia neomogenă:

$$\frac{\partial \widetilde{u}}{\partial \xi} = \frac{\partial \phi_1}{\partial \xi} \cdot \sqrt{\xi} + \frac{1}{2\sqrt{\xi}} \cdot \phi_1\left(\xi, \eta\right)$$

şi ecuaţia devine:

$$\frac{\partial \phi_1}{\partial \xi} \cdot \sqrt{\xi} + \frac{1}{2\sqrt{\xi}} \cdot \phi_1\left(\xi, \eta\right) - \frac{1}{2\sqrt{\xi}} \cdot \phi_1\left(\xi, \eta\right) = \phi_0\left(\xi\right) \Rightarrow$$

$$\frac{\partial \phi_1}{\partial \xi} = \frac{1}{\sqrt{\xi}} \cdot \phi_0\left(\xi\right) \Rightarrow \phi_1\left(\xi, \eta\right) = \phi_2\left(\xi\right) + \phi_3\left(\eta\right),$$

unde am notat:

$$\phi_2\left(\xi\right) = \int \phi_0\left(\xi\right) \cdot \frac{1}{\sqrt{\xi}} d\xi.$$

$$\phi_1\left(\xi, \eta\right) = \phi_2\left(\xi\right) + \phi_3\left(\eta\right) \Rightarrow \widetilde{u}\left(\xi, \eta\right) = \sqrt{\xi} \cdot \phi_2\left(\xi\right) + \sqrt{\xi} \cdot \phi_3\left(\eta\right).$$

Revenim la notaţiile:

$$\begin{cases} \xi = xy \\ \eta = \frac{x}{y} \end{cases} \Rightarrow$$

$$\begin{cases} u\left(x, y\right) = \sqrt{xy} \cdot \phi_2\left(xy\right) + \sqrt{xy} \cdot \phi_3\left(\frac{x}{y}\right) \\ \frac{\partial u}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}} \cdot \phi_2\left(xy\right) + x\sqrt{xy} \cdot \phi_2'\left(xy\right) + \\ \qquad + \frac{1}{2}\sqrt{\frac{x}{y}} \cdot \phi_3\left(\frac{x}{y}\right) - \frac{x\sqrt{xy}}{y^2} \cdot \phi_3'\left(\frac{x}{y}\right). \end{cases}$$

Impunem lui u condiţiile iniţiale:

$$\left\{\begin{array}{l} u(x,1) = \sqrt{x}\left[\phi_2(x) + \phi_3(x)\right] = x^2 \\ \frac{\partial u}{\partial y}(x,1) = \frac{1}{2}\sqrt{x}\cdot\phi_2(x) + x\sqrt{x}\cdot\phi_2^{'}(x) + \\ \qquad\qquad + \frac{1}{2}\sqrt{x}\cdot\phi_3(x) - x\sqrt{x}\cdot\phi_3^{'}(x) = 2x | : \frac{\sqrt{x}}{2} \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{l} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ \phi_2(x) + 2x\cdot\phi_2^{'}(x) + \phi_3(x) - 2x\cdot\phi_3^{'}(x) = 4\sqrt{x} \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ 2x\left(\phi_2^{'}(x) - \phi_3^{'}(x)\right) = 4\sqrt{x} - x\sqrt{x} \end{array}\right. \Leftrightarrow$$

$$\left\{\begin{array}{l} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ \phi_2^{'}(x) - \phi_3^{'}(x) = \frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2} \end{array}\right. \Rightarrow$$

$$\left\{\begin{array}{l} \phi_2^{'}(x) + \phi_3^{'}(x) = \frac{3}{2}x^{\frac{1}{2}} \\ \phi_2^{'}(x) - \phi_3^{'}(x) = \frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2} \end{array}\right. \Rightarrow$$

$$\phi_2^{'}(x) = \left(\frac{2}{\sqrt{x}} + \sqrt{x}\right) : 2 = \frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{2} \Rightarrow$$

$$\phi_2(x) = 2\sqrt{x} + \frac{1}{2}\cdot\frac{2}{3}x^{\frac{3}{2}} + C_0 = 2\sqrt{x} + \frac{1}{3}x\sqrt{x} + C_0$$

$$2\phi_3^{'}(x) = 2\sqrt{x} - \frac{2}{\sqrt{x}} \Rightarrow$$

$$\phi_3^{'}(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow \phi_3(x) = \frac{2}{3}x^{\frac{3}{2}} - 2\sqrt{x} + C_1.$$

Din egalitatea: $\phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \Rightarrow x^{\frac{3}{2}} + C_0 + C_1 = x^{\frac{3}{2}} \Rightarrow C_0 + C_1 = 0.$

Deci:

$$u(x,y) = \sqrt{xy}\left[\phi_2(xy) + \phi_3\left(\frac{x}{y}\right)\right] =$$

$$= \sqrt{xy}\left[2\sqrt{xy} + \frac{1}{3}xy\sqrt{xy} + C_0 + \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{3}{2}} - 2\sqrt{\frac{x}{y}} + C_1\right] =$$

$$= 2xy + \frac{x^2y^2}{3} + \frac{2}{3}\cdot\frac{x^2}{y} - 2x$$

$$u(x,y) = \frac{x^2y^2}{3} + \frac{2}{3}\cdot\frac{x^2}{y} + 2xy - 2x.$$

Exerciţiul 7.10 Să se aducă la forma canonică ecuaţia:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x\partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{x=0} = y^2, \quad \frac{\partial u}{\partial x}\big|_{x=0} = 1. \end{cases}$$

Soluţie:

$$\begin{cases} a = 1 \\ b = 2 \\ c = -3 \end{cases}$$

şi

$$\triangle = b^2 - 4ac = 16 > 0 \Rightarrow$$

ecuaţia este de tip hiperbolic.

Ecuaţia caracteristicilor este:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} - 3 = 0 \Rightarrow \begin{cases} \frac{dy}{dx} = 3 \\ \frac{dy}{dx} = -1 \end{cases} \Rightarrow \begin{cases} 3x - y = C_1 \\ x + y = C_2 \end{cases} \Rightarrow$$

facem schimbarea de variabilă

$$\begin{cases} \xi = 3x - y \\ \eta = x + y \end{cases}$$

şi de funcţie

$$u(x,y) = \widetilde{u}(\xi(x,y), \eta(x,y)) \Rightarrow$$

$$\left\{\begin{array}{l}\frac{\partial u}{\partial x}=\frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}+\frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}=3\frac{\partial \widetilde{u}}{\partial \xi}+\frac{\partial \widetilde{u}}{\partial \eta}\\ \frac{\partial u}{\partial y}=\frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}+\frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}=-\frac{\partial \widetilde{u}}{\partial \xi}+\frac{\partial \widetilde{u}}{\partial \eta}\end{array}\right. \Rightarrow \left\{\begin{array}{l}\frac{\partial}{\partial x}=3\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial y}=-\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}\end{array}\right. \Rightarrow$$

$$\left\{\begin{array}{l}\frac{\partial^2 u}{\partial x^2}=9\frac{\partial^2 \widetilde{u}}{\partial \xi^2}+6\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}+\frac{\partial^2 \widetilde{u}}{\partial \eta^2}\\ \frac{\partial^2 u}{\partial x\partial y}=-3\frac{\partial^2 \widetilde{u}}{\partial \xi^2}+2\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}+\frac{\partial^2 \widetilde{u}}{\partial \eta^2}\\ \frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 \widetilde{u}}{\partial \xi^2}-2\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}+\frac{\partial^2 \widetilde{u}}{\partial \eta^2}\end{array}\right.$$

Ecuaţia devine:

$$9\frac{\partial^2 \widetilde{u}}{\partial \xi^2}+6\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}+\frac{\partial^2 \widetilde{u}}{\partial \eta^2}-6\frac{\partial^2 \widetilde{u}}{\partial \xi^2}+4\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}+2\frac{\partial^2 \widetilde{u}}{\partial \eta^2}-$$

$$-3\frac{\partial^2 \widetilde{u}}{\partial \xi^2}+6\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}-3\frac{\partial^2 \widetilde{u}}{\partial \eta^2}=0 \Leftrightarrow$$

$$16\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}=0 \Rightarrow \frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}=0 \Rightarrow \frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right)=0 \Rightarrow \frac{\partial \widetilde{u}}{\partial \eta}=\phi_0\left(\eta\right) \Rightarrow$$

$$\widetilde{u}\left(\xi,\eta\right)=\int \phi_0\left(\eta\right)d\eta=\phi_1\left(\eta\right)+\phi_2\left(\xi\right)$$

$$\Leftrightarrow \left\{\begin{array}{l}u\left(x,y\right)=\phi_2\left(3x-y\right)+\phi_1\left(x+y\right)\\ \frac{\partial u}{\partial x}\left(x,y\right)=3\phi_2'\left(3x-y\right)+\phi_1'\left(x+y\right).\end{array}\right.$$

Condiţiile iniţiale devin:

$$\left\{\begin{array}{l}\phi_2\left(-y\right)+\phi_1\left(y\right)=y^2\\ 3\phi_2'\left(-y\right)+\phi_1'\left(y\right)=1\end{array}\right. \xrightarrow{\text{derivăm prima ecuaţie}}$$

$$\left\{\begin{array}{l}-\phi_2'\left(-y\right)+\phi_1'\left(y\right)=2y|\cdot 3\\ 3\phi_2'\left(-y\right)+\phi_1'\left(y\right)=1\end{array}\right. \Rightarrow$$

$$4\phi_1'\left(y\right)=1+6y \Rightarrow \phi_1'\left(y\right)=\frac{1}{4}+\frac{3}{2}y \Rightarrow \phi_1\left(y\right)=\frac{y}{4}+\frac{3}{4}y^2+C_0$$

$$4\phi_2'\left(-y\right)=1-2y \Rightarrow \phi_2'\left(-y\right)=\frac{1}{4}-\frac{y}{2} \xrightarrow{\text{facem } y\to -y}$$

$$\phi_2'(y) = \frac{1}{4} + \frac{y}{2} \Rightarrow \phi_2(y) = \frac{y}{4} + \frac{y^2}{4} + C_1.$$

Ecuaţia $\phi_2(-y) + \phi_1(y) = y^2$ devine:

$$-\frac{y}{4} + \frac{(-y)^2}{4} + C_1 + \frac{y}{4} + \frac{3y^2}{4} + C_0 = y^2 \Leftrightarrow$$

$$y^2 + C_1 + C_0 = y^2 \Rightarrow C_1 + C_0 = 0 \Rightarrow$$

$$u(x,y) = \frac{1}{4}(3x-y) + \frac{(3x-y)^2}{4} + C_1 + \frac{x+y}{4} + \frac{(x+y)^2}{2} + C_0 =$$

$$= x + \frac{1}{4}\left[(3x-y)^2 + (x+y)^2\right] =$$

$$= x + \frac{1}{4}\left(9x^2 - 6xy + y^2 + 3x^2 + 6xy + 3y^2\right) =$$

$$= x + \frac{1}{4}\left(12x^2 + 4y^2\right) = 3x^2 + y^2 + x$$

$$u(x,y) = 3x^2 + y^2 + x.$$

Observaţia 7.11 Ca aplicaţie la reducerea la forma canonică prezentăm problema Cauchy pentru ecuaţia undelor.

Exerciţiul 7.12 Rezolvaţi următoarea P.C. pentru ecuaţia undelor cu reducerea la forma canonică:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, x \in \mathbb{R}, t \geq 0 \\ u(x,0) = e^{2x} \rightarrow \text{ forma iniţială a coardei} \\ \frac{\partial u}{\partial t}(x,0) = \sin x \rightarrow \text{ viteza coardei la t = 0.} \end{cases}$$

Ecuaţia undelor P.C.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t), \quad a > 0 \\ u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = u_1(x). \end{cases}$$

Soluţie.
$u(x,t)$ funcţia necunoscută

$$r^2 - 4 = 0 \Rightarrow r_1 = 2, r_2 = -2 \Rightarrow$$

$$x' = 2 \;\text{ şi } x' = -2 \Rightarrow$$

$$C_1 = x - 2t = \xi \;\text{ şi } C_2 = x + 2t = \eta.$$

Deci:

$$\begin{cases} \xi = x - 2t \\ \eta = x + 2t \end{cases} \qquad u(x,t) = \widetilde{u}(\xi,\eta).$$

Derivatele parţiale ale lui $u(x,t)$ sunt:

$$\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot (-2) + \frac{\partial \widetilde{u}}{\partial \eta} \cdot 2 \Rightarrow$$

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{\partial \widetilde{u}}{\partial \xi^2} \cdot (-2) + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} \cdot 2\right] \cdot (-2) + \left[\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot (-2) + \frac{\partial \widetilde{u}}{\partial \eta^2}\right] \cdot 2$$

$$\frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot 1 + \frac{\partial \widetilde{u}}{\partial \eta} \cdot 1$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}.$$

Deci:

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 8\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + 4\frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

$$(-4) \cdot | \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

$$-16\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0 \Rightarrow$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0 \text{ forma canonică.}$$

Integrăm ecuaţia obţinută

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) = 0 \Rightarrow \frac{\partial \widetilde{u}}{\partial \eta} = f_1(\eta) \Rightarrow$$

$$\widetilde{u}(\xi, \eta) = \int f_1(\eta) d\eta + g(\xi) \Rightarrow$$

$$\widetilde{u}(\xi, \eta) = f(\eta) + g(\xi) \Rightarrow u(x, t) = f(x + 2t) + g(x - 2t),$$

unde $f, g : \mathbb{R} \to \mathbb{R}$ de clasă C^2.

Din condiţiile iniţiale aflăm f şi g.

Avem

$$\frac{\partial u}{\partial t} = 2f'(x + 2t) - 2g'(x - 2t).$$

Condiţiile iniţiale

$$\begin{cases} u(x, 0) = f(x) + g(x) = e^{2x}(*) \\ \frac{\partial u}{\partial t}(x, 0) = 2f'(x) - 2g'(x) = \sin x \end{cases}$$

Derivăm relaţia $(*)$:

$$\Rightarrow \begin{cases} f'(x) + g'(x) = 2e^{2x} \\ f'(x) - g'(x) = \frac{1}{2}\sin x \end{cases} \Rightarrow$$

$$f'(x) = e^{2x} + \frac{1}{4}\sin x \Rightarrow$$

$$\begin{cases} f(x) = \frac{1}{2}e^{2x} - \frac{1}{4}\cos x + C_1 \\ g(x) = \frac{1}{2}e^{2x} + \frac{1}{4}\cos x - C_1 \end{cases} \Rightarrow$$

$$u(x, t) = \frac{1}{2}\left(e^{2x+4t} + e^{2x-4t}\right) + \frac{1}{4}\left[\cos(x - 2t) - \cos(x + 2t)\right] =$$

$$= e^{2x} \cdot \frac{e^{4t} + e^{-4t}}{2} + \frac{1}{4} \cdot 2\sin(2t) \cdot \sin x = e^{2x} \cdot ch(4t) + \frac{1}{2} \cdot \sin x \cdot \sin(2t).$$

$$u(x, t) = e^{2x} \cdot ch(4t) + \frac{1}{2} \cdot \sin x \cdot \sin(2t)$$

unde s-a folosit relaţia

$$\cos\alpha - \cos\beta = 2\sin\frac{\beta-\alpha}{2}\cdot\sin\frac{\beta+\alpha}{2}.$$

Exerciţiul 7.13 Rezolvaţi următoarea P.C. pentru ecuaţia undelor cu reducerea la forma canonică:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = x\cdot t \\ u(x,0) = x \\ \frac{\partial u}{\partial t}(x,0) = x^3. \end{cases}$$

Soluţie. Procedăm la fel

$$\begin{cases} \xi = x - 2t \\ \eta = x + 2t \end{cases}$$

$$u(x,t) = \widetilde{u}(\xi,\eta)$$

şi forma canonică este

$$\left.\begin{array}{l} -16\frac{\partial^2 \widetilde{u}}{\partial\xi\partial\eta} = x\cdot t \\ x = \frac{\xi-\eta}{2}; t = \frac{\eta-\xi}{4} \end{array}\right\} \Rightarrow$$

$$\frac{\partial^2 \widetilde{u}}{\partial\xi\partial\eta} = \frac{\xi^2-\eta^2}{2^7}$$

o integrăm

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}\right) = \frac{1}{2^7}\left(\xi^2-\eta^2\right) \Rightarrow$$

$$\widetilde{u} = \frac{1}{2^7}\left(\frac{\xi^3\eta - \xi\eta^3}{3}\right) + f(\eta) + g(\xi) =$$

$$= \frac{1}{3\cdot 2^7}\xi\eta\cdot\left(\xi^2-\eta^2\right) + f(\eta) + g(\xi) \Rightarrow$$

$$\Rightarrow u(x,t) = \frac{1}{3\cdot 2^7}\left(x^2-4t^2\right)\cdot(-4t)\cdot 2x + f(x+2t) + g(x-2t) \Rightarrow$$

$$\Rightarrow u(x,t) = f(x+2t)+g(x-2t)+\frac{1}{3\cdot 2^4}(4xt^3-tx^3),\ \ f,g\in C^2(\mathbb{R}).$$

$$\frac{\partial u}{\partial t} = 2f'(x+2t) - 2g'(x-2t) + \frac{1}{3\cdot 2^4}(4x\cdot t^2 - x^3) =$$

$$= 2f'(x+2t) - 2g'(x-2t) + \frac{x\cdot t^2}{12} - \frac{x^3}{48}$$

$$\left\{\begin{array}{l} u(x,0) = x \\ \frac{\partial u}{\partial t}(x,0) = x^3 \end{array}\right. \Rightarrow \left\{\begin{array}{l} f(x)+g(x) = x \\ 2f'(x) - 2g'(x) - \frac{x^3}{48} = x^3 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} f(x) = \frac{x}{2} + \frac{49}{8\cdot 96}\cdot x^4 + C \\ g(x) = \frac{x}{2} - \frac{49}{8\cdot 96}\cdot x^4 - C \end{array}\right. \Rightarrow$$

$$u(x,t) = \frac{x+2t}{2} + \frac{x-2t}{2} +$$

$$+\frac{49}{8\cdot 96}\left[(x+2t)^4 - (x-2t)^4\right] + \frac{1}{48}(4xt^3 - tx^3) \Rightarrow$$

$$u(x,t) = x + x^3 t + \frac{25}{6}xt^3.$$

Observaţie: Dacă ecuaţia undelor este:

$$\frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0,$$

avem:

$$r^2 - a^2 = 0 \Rightarrow r_1 = a, r_2 = -a$$

$$x'(t) = a \Rightarrow x - at = C_1$$

$$x'(t) = -a \Rightarrow x + at = C_2.$$

$$\left\{\begin{array}{l} \xi = x - at \\ \eta = x + at \end{array}\right. \qquad u(x,t) = \widetilde{u}(\xi,\eta)$$

$$\frac{\partial u}{\partial t} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot (-a) + \frac{\partial \widetilde{u}}{\partial \eta} \cdot a$$

$$\frac{\partial^2 u}{\partial t^2} = \left[\frac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot (-a) + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} \cdot a\right] \cdot (-a) +$$

$$+ \left[\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot (-a) + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot a\right] \cdot a =$$

$$= a^2 \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2a^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + a^2 \frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}$$

$$-(a^2) \cdot | \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \xi} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}$$

şi

$$-4a^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = 0$$

este forma canonică.

Exerciţiul 7.14

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 2(x+t)e^{x-t}, x \in \mathbb{R}, t \geq 0 \\ u(x,0) = 2x^2 - \frac{x^2}{4} e^x \\ \frac{\partial u}{\partial t}(x,0) = \left(\frac{x^2}{4} - \frac{x}{2} + 2\right) e^x. \end{cases}$$

Soluţie.

$$a = 1 \Rightarrow \begin{cases} \xi = x - t \\ \eta = x + t \end{cases} \qquad u(x,t) = \widetilde{u}(\xi, \eta)$$

$$x = \frac{\xi + \eta}{2}, \quad t = \frac{\eta - \xi}{2}$$

Forma canonică este:

$$-4\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta} = 2\eta e^{\xi}.$$

$$\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta} = -\frac{\eta}{2}e^{\xi} \Rightarrow \frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}\right) = -\frac{\eta}{2}e^{\xi} \Rightarrow$$

$$\left(\frac{\partial\widetilde{u}}{\partial\eta}\right) = -\frac{\eta}{2}e^{\xi} + f_1(\eta) \Rightarrow$$

$$\widetilde{u}(\xi,\eta) = -\frac{\eta^2}{4}e^{\xi} + f(\xi) + g(\eta) \Rightarrow$$

$$u(x,t) = \frac{-(x+t)^2}{4}e^{x-t} + f(x-t) + g(x+t),$$

unde $f, g \in C^2(\mathbb{R})$.

$$\frac{\partial u}{\partial t} = -\frac{x+t}{2}e^{x-t} + \frac{(x+t)^2}{4}e^{x-t} - f'(x-t) + g'(x+t)$$

$$\begin{cases} u(x,0) = -\frac{x^2}{4}e^x + f(x) + g(x) = 2x^2 - \frac{x^2}{4}e^x \\ \frac{\partial u}{\partial t}(x,0) = -\frac{x}{2}e^x + \frac{x^2}{4}e^x - f'(x) + g'(x) = \left(\frac{x^2}{4} - \frac{x}{2} + 2\right)e^x \end{cases}$$

$$\Rightarrow \begin{cases} f'(x) + g'(x) = 4x \\ -f'(x) + g'(x) = 2e^x \end{cases} \Rightarrow g'(x) = 2x + e^x \Rightarrow$$

$$\begin{cases} g(x) = x^2 + e^x + C_1 \\ f(x) = x^2 - e^x - C_1 \end{cases} \Rightarrow$$

$$u(x,t) = -\frac{(x+t)^2}{4}e^{x-t} + (x-t)^2 - e^{x-t} - C_1 + (x+t)^2 + e^{x+t} + C_1 \Rightarrow$$

$$u(x,t) = -\frac{(x+t)^2}{4}e^{x-t} + 2e^x sht + 2(x^2+t^2).$$

Exerciţiul 7.15

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 2xt, \ x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \ln(1+x^2) \\ \frac{\partial u}{\partial t}(x,0) = 2e^x. \end{cases}$$

Soluţie.

$$a = 1 \Rightarrow \begin{cases} \xi = x - t \\ \eta = x + t \end{cases} \qquad u(x,t) = \widetilde{u}(\xi,\eta).$$

$$-4\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = \frac{\eta^2 - \xi^2}{2} \Rightarrow$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = \frac{\xi^2 - \eta^2}{8} \rightarrow \text{ forma canonică.}$$

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) = \frac{1}{8}(\xi^2 - \eta^2) \text{ prin integrare}$$

$$u(\xi,\eta) = \frac{\xi\eta}{24}(\xi^2 - \eta^2) + f(\xi) + g(\eta) \ f, g \in C^2(\mathbb{R})$$

$$u(x,t) = \frac{xt}{6}(t^2 - x^2) + f(x-t) + g(x+t)$$

$$\frac{\partial u}{\partial t} = \frac{xt^2}{2} - \frac{x^3}{6} - f'(x-t) + g'(x+t)$$

$$\begin{cases} u(x,0) = f(x) + g(x) = \ln(1+x^2) \\ \frac{\partial u}{\partial t}(x,0) = -\frac{x^3}{6} - f'(x) + g'(x) = 2e^x \end{cases} \Rightarrow$$

$$\begin{cases} f'(x) + g'(x) = \frac{2x}{1+x^2} \\ -f'(x) + g'(x) = \frac{x^3}{6} + 2e^x \end{cases} \Rightarrow$$

$$\Rightarrow g'(x) = \frac{x}{1+x^2} + \frac{x^3}{12} + e^x \Rightarrow$$

$$\begin{cases} g(x) = \frac{1}{2}\ln(1+x^2) + \frac{x^4}{48} + e^x + C \\ f(x) = \frac{1}{2}\ln(1+x^2) - \frac{x^4}{48} - e^x - C \end{cases} \Rightarrow$$

$$\Rightarrow u(x,t) = \frac{1}{2}\ln\left[1+(x-t)^2\right] - \frac{(x-t)^4}{48} - e^{x-t} - C+$$

$$+\frac{1}{2}\ln\left[1+(x+t)^2\right] + \frac{(x+t)^4}{48} + e^{x+t} + C =$$

$$= \frac{xt}{6}\left(t^2 - x^2\right) + \frac{xt}{6}\left(t^2 + x^2\right) +$$

$$+\frac{1}{2}\ln\left(1 + x^4 + t^4 + 2x^2 + 2t^2 - 2x^2 \cdot t^2\right) + e^x(e^t - e^{-t}) =$$

$$= \frac{xt^3}{3} + 2e^x sht + \frac{1}{2}\ln(1 + x^4 + t^4 + 2x^2 + 2t^2 - 2x^2 \cdot t^2).$$

Exerciţiul 7.16 Rezolvaţi E.D.P. cu date pe curbe incluse în D:

$$\begin{cases} 4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} = 0 \\ u\left|_{y+x=0}\right. = 1, \ \frac{\partial u}{\partial y}\left|_{y+x=0}\right. = x \end{cases}$$

Soluţie.
$a = 4$, $b = 12$, $c = 9 \Rightarrow \delta = b^2 - 4ac = 144 - 4 \cdot 36 = 0$ tip parabolic

Ecuaţia caracteristică : $4r^2 - 12r + 9 = 0 \Rightarrow r_{1,2} = +\frac{3}{2}$ rădăcina dublă

$y' = \frac{dy}{dx} = +\frac{3}{2} \Rightarrow 3x - 2y = c$ integrală primă

$$\begin{cases} \xi = 3x - 2y \\ \eta = x, \end{cases} \qquad u(x,y) = \tilde{u}(\xi,\eta) \Rightarrow$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \tilde{u}}{\partial \xi} \cdot 3 + \frac{\partial \tilde{u}}{\partial \eta} \\ \frac{\partial u}{\partial y} = \frac{\partial \tilde{u}}{\partial \xi} \cdot 2 \end{cases}$$

$$\begin{array}{l} 4/\frac{\partial^2 u}{\partial x^2} = 9 \cdot \frac{\partial^2 \tilde{u}}{\partial \xi} + 6 \cdot \frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \tilde{u}}{\partial \eta^2} \\ 12/\frac{\partial^2 u}{\partial x \partial y} = -6 \cdot \frac{\partial^2 \tilde{u}}{\partial \xi^2} - 2 \cdot \frac{\partial^2 \tilde{u}}{\partial \xi \partial \eta} \\ 9/\frac{\partial^2 u}{\partial y^2} = 4 \cdot \frac{\partial^2 \tilde{u}}{\partial \xi^2} \end{array}$$

Adunăm membru cu membru

$$0 = 4\frac{\partial^2 \tilde{u}}{\partial \eta^2} \Rightarrow \frac{\partial^2 \tilde{u}}{\partial \eta^2} = 0 \Rightarrow$$

$$\frac{\partial}{\partial \eta}(\frac{\partial \tilde{u}}{\partial \eta}) = 0 \Rightarrow \frac{\partial \tilde{u}}{\partial \eta} = f(\xi) \Rightarrow$$

$$\tilde{u}(\xi, \eta) = \eta f(\xi) + g(\xi) \Rightarrow$$

$$\begin{cases} u(x,y) = x \cdot f(3x - 2y) + g(3x - 2y) \\ \frac{\partial u}{\partial y} = -2xf'(3x - 2y) - 2g'(3x - 2y) \end{cases}$$

cu $f, g \in C^1(\mathbb{R})$

Datele pe curba $x + y = 0 \Leftrightarrow y = -x$ se scriu:

$$\begin{cases} u(x,-x) = x \cdot f(5x) + g(5x) = 1 \\ \frac{\partial u}{\partial y}(x,-x) = -2xf'(5x) - 2g'(5x) = x \end{cases}$$

Prin derivare

$$\begin{cases} f(5x) + 5(x \cdot f'(5x) + g'(5x)) = 1 \\ x \cdot f'(5x) + g'(5x) = -\frac{x}{2} \end{cases} \Rightarrow$$

$$\begin{cases} f(5x) = 1 + \frac{5x}{2} \Rightarrow f(t) = 1 + \frac{t}{2} \\ g(5x) = 1 - x(1 + \frac{5x}{2}) = 1 - \frac{1}{5} \cdot 5x(1 + \frac{5x}{2}) \end{cases}$$

$$\Rightarrow g(t) = 1 - \frac{t}{5}(1 + t)$$

$$u(x,y) = x(1 + \frac{3x-2y}{2}) + 1 - \frac{3x-2y}{5} \cdot (1 + 3x - 2y) =$$

$$= x + \frac{3x^2 - 2xy}{2} + 1 - \frac{3x-2y}{5} - \frac{(3x-2y)^2}{5} =$$

$$= \frac{1}{10}[10(x+1) + 5(3x^2 - 2xy) - 2(3x-2y) - 2(3x-2y)^2] =$$

$$= \frac{1}{10}[10x + 10 + 15x^2 - 10xy - 6x + 4x - 18x^2 + 24xy - 8y^2] =$$

$$= \frac{1}{10}(14xy - 3x^2 - 8y^2 + 4x + 4y + 10).$$

7.1.2 Exerciţii propuse.

Exerciţiul 7.17 Rezolvaţi PC pentru ecuaţia coardei vibrante:

(1)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 2(x^2 + 4t^2), & x \in \mathbb{R}, t \geq 0 \\ u(x,0) = 24x^2 \\ \frac{\partial u}{\partial t}(x,0) = 2x^3 - x; \end{cases}$$

(2)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = \sin 2x, & x \in \mathbb{R}, t \geq 0 \\ u(x,0) = e^x \\ \frac{\partial u}{\partial t}(x,0) = -x^2; \end{cases}$$

(3)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = x^2 - t^2, x \in \mathbb{R}, t \geq 0 \\ u(x,0) = 12x \\ \frac{\partial u}{\partial t}(x,0) = x^3; \end{cases}$$

(4)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^x, x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \sin x \\ \frac{\partial u}{\partial t}(x,0) = x + \cos x; \end{cases}$$

(5)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{-x} + t \cdot \cos(x \cdot t), x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \sin x \\ \frac{\partial u}{\partial t}(x,0) = \cos x; \end{cases}$$

(6)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = e^{x^2 - t^2} \cdot (x^2 - t^2) + x^2 + xt + t^2, x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \frac{x^4}{24} + \sin x + \frac{1}{4} e^{x^2} (\ln |x| - 1) \\ \frac{\partial u}{\partial t}(x,0) = \frac{1}{4} e^{x^2} + 2x + 1 + e^{-x}; \end{cases}$$

(7)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = x^2 - t^2 \ x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \cos x \\ \frac{\partial u}{\partial t}(x,0) = x^2. \end{cases}$$

Exerciţiul 7.18 Determinaţi forma canonică pentru:

$$\frac{\partial^2 u}{\partial x^2} - 2e^x \cdot \frac{\partial^2 u}{\partial x \partial y} + (e^{2x} - 1) \cdot \frac{\partial^2 u}{\partial y^2} - e^x \cdot \frac{\partial u}{\partial y} = 0.$$

Exerciţiul 7.19 P.Cauchy pentru E.D.P.:

(1)

$$\begin{cases} y^2\frac{\partial^2 u}{\partial x^2} - 2y\cdot\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{y}\cdot\frac{\partial u}{\partial y} = 0 \\ u(x,-1) = 2, \frac{\partial u}{\partial y}(x,-1) = x \end{cases}$$

(2)

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - 2e^x\cdot\frac{\partial^2 u}{\partial x\partial y} + (e^{2x}-1)\cdot\frac{\partial^2 u}{\partial y^2} - e^x\cdot\frac{\partial u}{\partial y} = 0 \\ u(0,y) = \cos y, \frac{\partial u}{\partial x}(0,y) = \sin y \end{cases}$$

(3)

$$\begin{cases} 2\frac{\partial^2 u}{\partial x^2} - 7\frac{\partial^2 u}{\partial x\partial y} + 3\frac{\partial^2 u}{\partial y^2} = 0 \\ u\left|_{y+3x=0}\right. = \cos x,\, u\left|_{2y+x=0}\right. = 1+5x. \end{cases}$$

7.2 Problema Cauchy pentru ecuaţia căldurii. Formula lui Poisson.

Ecuaţia căldurii este

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial u}{\partial t}$$

cu condiţia iniţială $u(x,0) = f(x)$.

MSV:

$$\begin{cases} u(x,t) = X(x)T(t) \to \frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T'(t)}{T(t)} = k \\ T'(t) - ka^2T(t) = 0 \to T(t) = ce^{ka^2t} \\ X''(x) - kX(x) = 0 \to X(x) = c_1\cos\lambda x + c_2\sin\lambda x, \\ \qquad\qquad k<0 \text{ convine }, k = -\lambda^2,\ \lambda > 0. \end{cases}$$

$$u(x,t,\lambda) = \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right]e^{-\lambda^2a^2t}$$

$$u(x,t)=\int_0^\infty u(x,t,\lambda)d\lambda.$$

Condiţia iniţială:

$$\int_0^\infty u(x,0,\lambda)d\lambda = f(x) \rightarrow$$

$$\int_0^\infty \left[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right]d\lambda = f(x)$$

Integrala Fourier care o reprezintă pe $f(x)$:

$$f(x)=\frac{1}{\pi}\int_0^\infty d\lambda\int_{-\infty}^\infty f(\tau)\cos\lambda(x-\tau)d\tau =$$

$$=\frac{1}{\pi}\int_0^\infty \left[\cos\lambda x\int_{-\infty}^\infty f(\tau)\cos\lambda\tau d\tau +\right.$$

$$\left.+\sin\lambda x\int_{-\infty}^\infty f(\tau)\sin\lambda\tau d\tau\right]d\lambda \Rightarrow$$

$$A(\lambda)=\frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\cos\lambda\tau d\tau$$

$$B(\lambda)=\frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\sin\lambda\tau d\tau$$

Deci

$$u(x,t,\lambda)=\frac{1}{\pi}\left[\int_{-\infty}^\infty f(\tau)\cos\lambda(x-\tau)d\tau\right]e^{-\lambda^2a^2t} \Rightarrow$$

$$u(x,t)=\frac{1}{\pi}\int_0^\infty d\lambda\int_{-\infty}^\infty f(\tau)\cos\lambda(x-\tau)e^{-\lambda^2a^2t}d\tau =$$

$$=\frac{1}{\pi}\int_{-\infty}^\infty f(\tau)\left[\int_0^\infty e^{-\lambda^2a^2t}\cos\lambda(x-\tau)d\lambda\right]d\tau.$$

$$\int_0^\infty e^{-ax^2}\cos bx dx = \frac{1}{2}\frac{\sqrt{\pi}}{\sqrt{a}}e^{-\frac{b^2}{4a}} \rightarrow \text{ Poisson}$$

de unde obţinem

$$b = x - \tau \text{ şi } a = a^2 t$$

şi avem

$$\int_0^\infty e^{-\lambda^2 a^2 t}\cos\lambda(x-\tau)d\lambda = \frac{1}{2}\cdot\frac{\sqrt{\pi}}{a\sqrt{t}}\cdot e^{-\frac{(x-\tau)^2}{4a^2t}} \rightarrow$$

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^\infty f(\tau)\cdot e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau \leftarrow \text{ formula Poisson.}$$

Cazul neomogen

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + g(x,t) \\ u(x,0) = f(x) \end{cases}$$

$$u_h(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^\infty f(\tau)\cdot e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau.$$

$$\begin{cases} \frac{\partial\widetilde{u}}{\partial t}(x,t,s) = a^2\frac{\partial^2\widetilde{u}}{\partial x^2}(x,t,s) \\ \widetilde{u}(x,0,s) = g(x,s) \end{cases}$$

$$\Rightarrow \widetilde{u}(x,t,s) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^\infty g(\tau,s)\cdot e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau$$

$$u_p(x,t) = \int_0^t \widetilde{u}(x,t-s,s)ds =$$

$$= \int_0^t \frac{1}{2a\sqrt{\pi(t-s)}}\left[\int_{-\infty}^\infty g(\tau,s)\cdot e^{-\frac{(x-\tau)^2}{4a^2(t-s)}}d\tau\right]ds.$$

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^\infty f(\tau)\cdot e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau +$$

$$+\int_0^t \frac{1}{2a\sqrt{\pi(t-s)}}\left[\int_{-\infty}^{\infty} g(\tau,s)\cdot e^{-\frac{(x-\tau)^2}{4a^2(t-s)}}d\tau\right]ds.$$

Relaţia

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\tau)\cdot e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau+$$

$$+\int_0^t \frac{1}{2a\sqrt{\pi(t-s)}}\left[\int_{-\infty}^{\infty} g(\tau,s)\cdot e^{-\frac{(x-\tau)^2}{4a^2(t-s)}}d\tau\right]ds$$

este formula Poisson din principiul lui Duhamel.

7.2.1 Exerciţii rezolvate

În cele ce urmează vom folosi formula lui Poisson pentru PC a ecuaţiei căldurii

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} u_0(y)\cdot e^{-\frac{(x-y)^2}{4a^2t}}dy+$$

$$+\int_0^t\int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-s)}}f(y,s)\cdot e^{-\frac{(x-y)^2}{4a^2(t-s)}}dyds.$$

Exerciţiul 7.20 Să se rezolve problema Cauchy:

$$u_t=4u_{xx}+t+e^t,\ u|_{t=0}=2.$$

Soluţie.

$$a=2,\ u_0(y)=2,\ f(x,t)=t+e^t.$$

$$\int_{-\infty}^{\infty} 2e^{-\frac{(x-y)^2}{4a^2t}}dy=4a\sqrt{t}\int_{-\infty}^{\infty} e^{-\tau^2}d\tau=4a\sqrt{\pi t},$$

din substituţia

$$\tau = \frac{x-y}{2a\sqrt{t}} \Rightarrow \frac{-dy}{2a\sqrt{t}} = d\tau \Rightarrow dy = -2a\sqrt{t}d\tau.$$

Deci

$$\frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} 2e^{-\frac{(x-y)^2}{4a^2 t}} dy = 2.$$

Acum

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{2a\sqrt{\pi(t-s)}}(s+e^s) \cdot e^{-\frac{(x-y)^2}{4a^2(t-s)}} dyds =$$

$$\overset{\tau = \frac{x-y}{2a\sqrt{t-s}}}{\underset{dy = -2a\sqrt{t-s}d\tau}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=\!=}} \int_0^t \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (s+e^s) \cdot e^{-\tau^2} d\tau ds =$$

$$= \frac{1}{\sqrt{\pi}} \left[\int_0^t (s+e^s)ds\right] \left[\int_{-\infty}^{\infty} e^{-\tau^2} d\tau\right] =$$

$$= \left(\frac{s^2}{2} + e^s\right) |_0^t = \frac{t^2}{2} + e^t - 1.$$

Astfel,

$$u(x,t) = \frac{t^2}{2} + e^t + 1.$$

Exerciţiul 7.21 Să se rezolve problema Cauchy:

$$u_t = u_{xx} + 3t^2,\ u|_{t=0} = \sin x.$$

Soluţie.

$$a = 1,\ u_0(x) = \sin x,\ f(x,t) = 3t^2.$$

$$I_1 = \int_{-\infty}^{\infty} \sin y e^{-\frac{(x-y)^2}{4a^2 t}} dy = \int_{-\infty}^{\infty} 2a\sqrt{t} \sin(x - 2a\sqrt{t}\tau) e^{-\tau^2} d\tau$$

Ştim că

$$\sin(x - 2a\sqrt{t}\tau) = \sin x \cos(2a\sqrt{t}\tau) - \sin(2a\sqrt{t}\tau) \cos x.$$

Avem

$$\int_{-\infty}^{\infty} \sin(2a\sqrt{t}\tau) e^{-\tau^2} d\tau = 0.$$

Deci

$$I_1 = 4a\sqrt{t} \sin x \int_0^{\infty} \cos(2a\sqrt{t}\tau) e^{-\tau^2} d\tau.$$

$$F(\alpha) = \int_0^{\infty} \cos(2\alpha\tau) e^{-\tau^2} d\tau \Rightarrow$$

$$F'(\alpha) = \int_0^{\infty} -2\tau \sin(2\alpha\tau) e^{-\tau^2} d\tau =$$

$$= \int_0^{\infty} \sin(2\alpha\tau) \left(e^{-\tau^2}\right)' d\tau =$$

$$= \sin(2\alpha\tau) e^{-\tau^2} |_0^{\infty} - \int_0^{\infty} 2\alpha \cos(2\alpha\tau) e^{-\tau^2} d\tau \Rightarrow$$

$$F'(\alpha) = -2\alpha F(\alpha) \Rightarrow \frac{dF}{F} = -2\alpha d\alpha \Rightarrow$$

$$F(\alpha) = Ce^{-\alpha^2},\ F(0) = \int_0^{\infty} e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2} \Rightarrow$$

$$\int_0^{\infty} \cos(2\alpha\tau) e^{-\tau^2} d\tau = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}.$$

$$\alpha = a\sqrt{t} \Rightarrow I_1 = 4a\sqrt{t} \sin x \frac{\sqrt{\pi}}{2} e^{-a^2 t} =$$

$$= 2a\sqrt{t\pi} e^{-a^2 t} \sin x \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4a^2 t}} \sin y dy \overset{a=1}{=\!=} e^{-t} \sin x.$$

$$I_2 = \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} 3s^2 e^{-\frac{(x-y)^2}{4(t-s)}} dyds \overset{\frac{x-y}{2\sqrt{t-s}}=\tau}{=\!=\!=\!=\!=}$$

$$= \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} \cdot 3s^2 e^{-\tau^2} 2\sqrt{t-s} d\tau ds =$$

$$= \left(\int_0^t 3s^2 ds \right) \left(\int_{-\infty}^{\infty} \frac{e^{-\tau^2}}{\sqrt{\pi}} d\tau \right) =$$

$$= s^3 \big|_0^t = t^3.$$

Astfel,

$$u(x,t) = e^{-t} \sin x + t^3.$$

Exerciţiul 7.22 Să se rezolve problema Cauchy:

$$u_t = u_{xx} + e^{-t} \cos x, \;\; u\big|_{t=0} = \cos x.$$

Soluţie.

$$a = 1, \; u_0(x) = \cos x, \; f(x,t) = e^{-t} \cos x.$$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \cos y dy +$$

$$+ \int_0^t \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-s)}} e^{-s} e^{-\frac{(x-y)^2}{4(t-s)}} \cos y dy ds.$$

Calculăm

$$\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \cos y dy \overset{\frac{x-y}{2\sqrt{t}}=\tau}{=\!=\!=\!=\!=}$$

$$\int_{-\infty}^{\infty} e^{-\tau^2} 2\sqrt{t} \cos(x - 2\sqrt{t}\tau) d\tau =$$

$$= 2\sqrt{t}\int_{-\infty}^{\infty} e^{-\tau^2}\cos x\cos(2\sqrt{t}\tau)d\tau = 2\sqrt{t}(\cos x)\cdot 2F(\sqrt{t}) =$$

$$= 4\sqrt{t}\frac{\sqrt{\pi}}{2}e^{-t}\cos x = 2\sqrt{\pi t}e^{-t}\cos x.$$

deoarece am folosit integrala Poisson:

$$F(\alpha) = \int_{-\infty}^{\infty} e^{-\tau^2}\cos(2\alpha\tau)d\tau = \sqrt{\pi}e^{-\alpha^2}.$$

Deci

$$\frac{1}{\sqrt{\pi t}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}}\cos y dy = e^{-t}\cos x.$$

Analog

$$\int_0^t\int_{-\infty}^{\infty}\frac{1}{2\sqrt{\pi(t-s)}}e^{-s}e^{-\frac{(x-y)^2}{4(t-s)}}\cos y dy ds =$$

$$\overset{\frac{x-y}{2\sqrt{t-s}}=\tau}{=\!=\!=\!=\!=}\int_0^t\int_{-\infty}^{\infty}\frac{e^{-s}}{\sqrt{\pi}}e^{-\tau^2}\cos(x-2\sqrt{t-s}\tau)d\tau ds =$$

$$\frac{1}{\sqrt{\pi}}\int_0^t e^{-s}\cos x\int_{-\infty}^{\infty} e^{-\tau^2}\cos(2\sqrt{t-s}\tau)d\tau ds =$$

$$\frac{1}{\sqrt{\pi}}\sqrt{\pi}\cos x\int_0^t e^{-s}e^{-(t-s)}ds = e^{-t}\cos x\int_0^t ds = te^{-t}\cos x.$$

Astfel,

$$u(x,t) = (t+1)e^{-t}\cos x.$$

7.2.2 Exerciţii propuse

Exerciţiul 7.23 Să se rezolve problema Cauchy:

$$u_t = u_{xx} + e^t\sin x,\ u|_{t=0} = \sin x.$$

Exerciţiul 7.24 Să se rezolve problema Cauchy:

$$u_t = u_{xx} + \sin t,\ u|_{t=0} = e^{-x^2}.$$

7.3 Problema mixtă pentru ecuaţia coardei vibrante şi ecuaţia căldurii. Metoda separării variabilelor

7.3.1 Problema mixtă la ecuaţia coardei vibrante

Cazul vibraţiilor forţate (întreţinute de o forţă care acţionează în fiecare punct al coardei).

Principiul lui Duhamel combinat cu metoda separării variabilelor.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), 0 < x < l, t \geq 0 \\ u(x,0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x), 0 \leq x \leq l \\ u(0,t) = u(l,t) = 0, t \geq 0 \end{cases} \tag{7.1}$$

$u(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} - a^2 \frac{\partial^2 u_h}{\partial x^2} = 0, 0 < x < l, t \geq 0 \\ u_h(x,0) = f(x), \frac{\partial u_h}{\partial t}(x,0) = g(x) \\ u_h(0,t) = u_h(l,t) = 0 \end{cases} \tag{7.2}$$

Relaţia (7.2) reprezintă problema vibraţiior libere şi cu metoda separării variabilelor avem:

$u_h(x,t) = \sum\limits_{n=1}^{\infty} (A_n \cos \frac{n\pi at}{l} + B_n \sin \frac{n\pi at}{l}) \cdot \sin \frac{n\pi x}{l}$unde:

$$A_n = \frac{2}{l} \int\limits_0^l f(x) \cdot \sin \frac{n\pi x}{l} dx$$

$$B_n = \frac{2}{n\pi a} \int\limits_0^l g(x) \cdot \sin \frac{n\pi x}{l} dx, n \geq 1$$

$$\begin{cases} \frac{\partial^2 u_p}{\partial t^2} - a^2 \frac{\partial^2 u_p}{\partial x^2} = f(x,t), 0 < x < l \\ u_p(x,0) = 0, \frac{\partial u_p}{\partial t}(x,0) = 0 \\ u_p(0,t) = u_p(l,t) = 0. \end{cases} \tag{7.3}$$

Rezolvarea lui (7.3) se face cu principiul lui Duhamel.

Considerăm problema vibraţiilor libere:

$$\begin{cases} \frac{\partial^2 \tilde{u}}{\partial t^2}(x,t,s) - a^2 \frac{\partial^2 \tilde{u}}{\partial x^2}(x,t,s) = 0, 0 \leq x \leq l \\ \tilde{u}(x,0,s) = 0, \frac{\partial \tilde{u}}{\partial t}(x,0,s) = f(x,s) \\ \tilde{u}(0,t,s) = \tilde{u}(l,t,s) = 0 \end{cases} \tag{7.4}$$

pe care o rezolvăm cu metoda separării variabilelor căutând soluţie de forma:

$$\tilde{u}(x,t,s) = \sum_{n=1}^{\infty} B_n(s) \cdot \sin \frac{n\pi at}{l} \cdot \sin \frac{n\pi x}{l} \tag{7.5}$$

(deoarece $\tilde{u}(x,0,s) = 0$, coeficientul lui $\cos \frac{n\pi at}{l}$ este nul)

Folosim a doua condiţie iniţială pentru găsirea lui $B_n(s)$.

$$\frac{\partial \tilde{u}}{\partial t}(x,0,s) = \sum_{n=1}^{\infty} \frac{n\pi a}{l} B_n(s) \cdot \sin \frac{n\pi x}{l} = f(x,s) =$$

$$= \sum_{n \geq 1} C_n(s) \cdot \sin \frac{n\pi x}{l}$$

serie Fourier de sinusuri pentru $f(x,s)$ unde:

$$C_n(s) = \frac{2}{l} \int_0^l f(x,s) \cdot \sin \frac{n\pi x}{l} dx, n \geq 1.$$

Din identificare avem:

$$B_n(s) = \frac{2}{n\pi a}\int_0^l f(x,s)\cdot\sin\frac{n\pi x}{l}dx, n \geq 1. \tag{7.6}$$

Folosind principiul lui Duhamel avem:

$$u_p(x,t) = \int_0^t \tilde{u}(x, t-s, s)ds.$$

7.3.2 Problema mixtă la ecuaţia căldurii

Metoda separării variabilelor şi principiul lui Duhamel.

$$\begin{cases} \frac{\partial u}{\partial t} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t), 0 \leq x \leq l \\ u(x,0) = f(x) \\ u(0,t) = u(l,t) = 0 \end{cases} \tag{7.7}$$

$$u(x,t) = u_h(x,t) + u_p(x,t)$$

$$\begin{cases} \frac{\partial u_h}{\partial t} - a^2\frac{\partial^2 u_h}{\partial x^2} = 0 \\ u_h(x,0) = f(x) \\ u_h(0,t) = u_h(l,t) = 0 \end{cases} \tag{7.8}$$

Cu M.S.V. $\Rightarrow$

$$u_h(x,t) = \sum_{n=1}^{\infty} A_n e^{-(\frac{n\pi a}{l})^2 t}\sin\frac{n\pi x}{l},$$

$A_n = \frac{2}{l}\int\limits_0^l f(x)\sin\frac{n\pi x}{l}dx, n \geq 1$

$$\begin{cases} \frac{\partial u_p}{\partial t} - a^2\frac{\partial^2 u_p}{\partial x^2} = f(x,t) \\ u_p(x,0) = 0 \\ u_p(0,t) = u_p(l,t) = 0. \end{cases}$$

Fie

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t}(x,t,s) - a^2 \frac{\partial^2 \tilde{u}}{\partial x^2}(x,t,s) = 0 \\ \tilde{u}(x,0,s) = f(x,s) \\ \tilde{u}(0,t,s) = \tilde{u}(l,t,s) = 0 \end{cases}$$

şi cu M.S.V. avem: $\tilde{u}(x,t,s) = \sum\limits_{n=1}^{\infty} A_n(s) e^{-(\frac{n\pi a}{l})^2 t} \sin \frac{n\pi x}{l}$, unde

$$\tilde{u}(x,0,s) = \sum_{n=1}^{\infty} A_n(s) \sin \frac{n\pi x}{l} = f(x,s) = \sum_{n=1}^{\infty} a_n(s) \sin \frac{n\pi x}{l}$$

seria Fourier trunchiată

$$a_n(s) = \frac{2}{l} \int\limits_0^l f(x,s) \cdot \sin \frac{n\pi x}{l} dx$$

iar

$$A_n(s) = \frac{2}{l} \int\limits_0^l f(x,s) \cdot \sin \frac{n\pi x}{l} dx, n \geq 1$$

din identificarea coeficienţilor.

Cu principiul lui Duhamel:

$$u_p(x,t) = \int\limits_0^t \tilde{u}(x,t-s,s) ds$$

Observaţia 7.25

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x,t), t \geq 0 \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

$$u(x,t) = u_h(x,t) + u_p(x,t)$$

Cu formula lui D'Alembert (se deduce din forma canoică):

$$u_h(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] + \frac{1}{2a}\int\limits_{x-at}^{x+at} g(\tau)d\tau.$$

Fie

$$\begin{cases} \frac{\partial^2 \tilde{u}}{\partial t^2}(x,t,s) - a^2 \frac{\partial^2 \tilde{u}}{\partial x^2}(x,t,s) = 0 \\ \tilde{u}(x,0,s) = 0, \frac{\partial \tilde{u}}{\partial t}(x,0,s) = f(x,s). \end{cases}$$

Cu D'Alembert

$$\tilde{u}(x,t,s) = \frac{1}{2a}\int\limits_{x-at}^{x+at} f(\tau,s)d\tau.$$

Cu principiul lui Duhamel

$$u_p(x,t) = \int\limits_0^t u(x,t-s,s)ds \Rightarrow$$

$$u_p(x,t) = \frac{1}{2a}\int\limits_0^t \left[\int\limits_{x-a(t-s)}^{x+a(t-s)} f(\tau,s)d\tau\right] ds.$$

Deci, formula lui D'Alembert ($n = 1$):

$$u(x,t) = \frac{1}{2}[f(x+at) + f(x-at)] +$$

$$+\frac{1}{2a}\int_{x-at}^{x+at} g(\tau)d\tau+\frac{1}{2a}\int_{0}^{t}\left[\int_{x-a(t-s)}^{x+a(t-s)} f(\tau,s)d\tau\right]ds$$

pentru ecuaţia neomogenă.

(**) Problema Cauchy unidimensională pentru ecuaţia căldurii neomogene

$$\begin{cases} \frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=f(x,t), x\in\mathbb{R}, t\geq 0\\ u(x,0)=f(x)\end{cases}$$

$$u(x,t)=u_h(x,t)+u_p(x,t)$$

Soluţia ecuaţiei omogene

$$u_h(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\tau)e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau,$$

formula lui Poisson stabilită la Transformata Fourier

$$\begin{cases} \frac{\partial\tilde{u}}{\partial t}(x,t,s)-a^2\frac{\partial^2\tilde{u}}{\partial x^2}(x,t,s)=0\\ \tilde{u}(x,0,s)=f(x,s)\end{cases}\quad \text{Poisson}$$

$$\tilde{u}(x,t,s)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\tau,s)e^{-\frac{(x-\tau)^2}{4a^2t}}d\tau$$

şi cu principiul lui Duhamel obţinem soluţia particulară:

$$u_p(x,t)=\int_0^t \tilde{u}(x,t-s,s)ds=$$

$$= \int_0^t \frac{1}{2a\sqrt{\pi(t-s)}} \left[\int_{-\infty}^{\infty} f(\tau, s) e^{-\frac{(x-\tau)^2}{4a^2(t-s)}} d\tau \right] ds$$

Atunci, formula lui Poisson ($n = 1$)

$$u(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} f(\tau) \cdot e^{-\frac{(x-\tau)^2}{4a^2 t}} d\tau +$$

$$+ \int_0^t \frac{1}{2a\sqrt{\pi(t-s)}} \left[\int_{-\infty}^{\infty} f(\tau, s) \cdot e^{-\frac{(x-\tau)^2}{4a^2(t-s)}} d\tau \right] ds.$$

7.3.3 Coarda vibrantă finită (oscilaţii libere)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \ x \in (0, l),\, t > 0,\, c > 0 \text{ constantă} \\ u(x, 0) = \varphi(x) \text{ - forma iniţială a coardei} \\ \frac{\partial u}{\partial t}(x, 0) = \psi(x) \text{ - viteza iniţială} \\ u(0, t) = u(l, t) \text{ - încastrarea la capete} \end{cases}$$

$\varphi \in C^2([0, l])$, $\psi \in C^1([0, l])$.

$$\left. \begin{array}{l} \varphi(0) = \varphi(l) = 0 \\ \varphi(0) = \psi(l) = 0 \\ \varphi''(0) = \varphi''(l) = 0 \end{array} \right\} \text{condiţiile de compatibilitate a datelor}$$

rezultă ca $\exists! u \in C^2([0, l] \times [0, \infty))$ soluţie a problemei.

Soluţie. Metoda Fourier-Bernoulli:

$$\left. \begin{array}{l} u(x,t) = X(x)T(t) \text{ - nenule} \\ 0 = u(0,t) = X(0)T(t) \\ 0 = u(l,t) = X(l)T(t) \end{array} \right\} \Rightarrow X(0) = X(l) = 0.$$

$$\left\{\begin{array}{l}\frac{\partial^2 u}{\partial t^2} = X(x)\ddot{T}(t) \\ \frac{\partial^2 u}{\partial x^2} = X''(x)T(t)\end{array}\right. \overset{\text{ecuaţia}}{\Rightarrow} X(x)\ddot{T}(t) - c^2 X''(x)T(t) = 0 \Rightarrow$$

$$\frac{X''(x)}{X(x)} = \frac{\ddot{T}(t)}{c^2 T(t)} = -\lambda(\text{ constantă}) \Rightarrow$$

$$\left\{\begin{array}{l} X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{array}\right. \Rightarrow \begin{array}{l}\text{Problema Sturm - Liouville} \\ \text{se caută } \lambda \text{ astfel încât} \\ \text{există soluţie nenulă.}\end{array}$$

Ecuaţia caracteristică: $r^2 + \lambda = 0$ rezultă

1. $\lambda < 0 \to r_{1,2} = \pm\sqrt{-\lambda} \in \mathbb{R}$, distincte

$$\Rightarrow \left.\begin{array}{l} X(x) = C_1 e^{\sqrt{-\lambda}x} + C_2 e^{-\sqrt{-\lambda}x} \\ X(0) = X(l) = 0 \end{array}\right\} \Rightarrow$$

$$\Rightarrow C_1 = C_2 = 0 \Rightarrow X \equiv 0.$$

2. $\lambda = 0 \to X''(x) = 0 \to X(x) = ax + b \overset{\text{condiţii iniţiale}}{\Rightarrow}$
   $\Rightarrow X \equiv 0.$

3. $\lambda > 0 \to r_{1,2} = \pm i\sqrt{\lambda} \to$

$$\to \begin{array}{l} X(x) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x \\ X(0) = 0 \to C_1 = 0 \\ X(l) = 0 \to \sin\sqrt{\lambda}l = 0 \to \sqrt{\lambda}l = n\pi,\ n \in \mathbb{Z}^* \to \\ \lambda_n = \left(\frac{n\pi}{l}\right)^2,\ n \geq 1 \to X_n(x) = C_2 \sin\frac{n\pi x}{l}. \end{array}$$

$$\ddot{T}_n(t) + \left(\frac{n\pi C}{l}\right)^2 T_n(t) = 0 \to$$

$$T_n(t) = a_n \cos\frac{n\pi Ct}{l} + b_n \sin\frac{n\pi Ct}{l}$$

$$u_n(x,t) = \left(A_n \cos\frac{n\pi Ct}{l} + B_n \sin\frac{n\pi Ct}{l}\right)\sin\frac{n\pi x}{l},\ n \geq 1.$$

Cu metoda suprapunerii

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi Ct}{l} + B_n \sin\frac{n\pi Ct}{l}\right)\sin\frac{n\pi x}{l}$$

$$u(x,0) = \sum_{n\geq 1} A_n \sin\frac{n\pi x}{l} = \varphi(x) \to A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n\geq 1} B_n \frac{n\pi C}{l}\sin\frac{n\pi x}{l} = \psi(x) \to$$

$$\to B_n = \frac{2}{n\pi C}\int_0^l \psi(x)\sin\frac{n\pi x}{l}dx.$$

Aplicaţie:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = 0,\ t \geq 0,\ \ x \in [0,1],\ a > 0 \\ u(0,t) = u(1,t) = 0,\ \forall t \geq 0 \\ u(x,0) = x^4 - 2x^3 + x \\ \frac{\partial u}{\partial t}(x,0) = \sin^3(\pi x) \end{cases}$$

Ecuaţia căldurii (difuzia)

$$\begin{cases} \frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0,\ \ x \in [0,l],\ t > 0 \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = \varphi(x) \end{cases}$$

$\varphi \in C^2([0,l])$, $\psi \in C^1([0,l])$.

$\varphi(0) = \varphi(l) = 0$ $\quad$ $\varphi \in C^2([0,l])$ problema are soluţie unică
$\varphi''(0) = \varphi''(l) = 0$ $\quad$ $u \in C^2([0,l] \times [0,\infty))$

Analog:

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-\left(\frac{n\pi C}{l}\right)^2 t} \sin\frac{n\pi x}{l}$$

$$A_n = \frac{2}{l}\int_0^l \varphi(x)\sin\frac{n\pi x}{l}dx.$$

Coarda vibrantă (cazul neomogen)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\frac{\partial^2 u}{\partial x^2} = f(x,t),\ 0 \le x \le l,\ t > 0 \\ u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \\ u(0,t) = u(l,t) = 0. \end{cases}$$

$$u(x,t) = u_h(x,t) + u_p(x,t)$$

$$u_h(x,t) = \sum_{n\ge 1}\left(A_n\cos\frac{n\pi at}{l} + B_n\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx;$$

$$B_n = \frac{2}{n\pi a}\int_0^l g(x)\sin\frac{n\pi x}{l}dx.$$

Conform principiului lui Duhamel:

$$u_p(x,t) = \int_0^t \widetilde{u}(x,t-s,s)ds.$$

$$\widetilde{u}(x,t,s) = \sum_{0}^{\infty} B_n(s)\sin\frac{n\pi at}{l}\sin\frac{n\pi x}{l}$$

$$B_n(s) = \frac{2}{n\pi a}\int_0^l f(x,s)\sin\frac{n\pi x}{l}dx.$$

$$u(x,t)=\sum_{n\geq 1}\left(A_n\cos\frac{n\pi at}{l}+B_n\sin\frac{n\pi at}{l}\right)\sin\frac{n\pi x}{l}+$$

$$+\sum_{n\geq 1}\left[\int_0^t B_n(s)\sin\frac{n\pi a}{l}(t-s)ds\right]\sin\frac{n\pi x}{l}.$$

$$u(x,t)=$$

$$=\sum_{n\geq 1}\left\{\int_0^l\left[\frac{2}{l}f(\xi)\cos\frac{n\pi at}{l}+\frac{2}{n\pi a}g(\xi)\sin\frac{n\pi at}{l}\right]\sin\frac{n\pi\xi}{l}d\xi\right\}\cdot$$

$$\cdot\sin\frac{n\pi x}{l}+$$

$$+\sum_{n\geq 1}\frac{2}{n\pi a}\left\{\int_0^t\left[\int_0^l f(\xi,s)\sin\frac{n\pi\xi}{l}d\xi\right]\sin\frac{n\pi a}{l}(t-s)ds\right\}\cdot$$

$$\cdot\sin\frac{n\pi x}{l}.$$

Ecuaţia căldurii (cazul neomogen)

$$\begin{cases}\frac{\partial u}{\partial t}-a^2\frac{\partial^2 u}{\partial x^2}=f(x,t),\ 0<x<l,\, t>0\\ u(x,0)=f(x)\\ u(0,t)=u(l,t)=0\end{cases}$$

$$u(x,t)=\sum_{n=1}^{\infty}A_n e^{-\left(\frac{n\pi a}{l}\right)^2 t}\sin\frac{n\pi x}{l}+$$

$$+\sum_{n=1}^{\infty}\left[\int_0^t B_n(s)e^{-\left(\frac{n\pi a}{l}\right)^2(t-s)}ds\right]\sin\frac{n\pi x}{l};$$

$$A_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$$

$$B_n(s) = \frac{2}{l}\int_0^l f(x,s)\sin\frac{n\pi x}{l}dx.$$

7.3.4 Exerciţii rezolvate

Exerciţiul 7.26

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x^2} = 0,\ 0 \le x \le 1,\ t \ge 0 \\ u(x,0) = x(1-x),\ \frac{\partial u}{\partial t}(x,0) = 0 \\ u(0,t) = u(1,t) = 0,\ t \ge 0. \end{cases}$$

$$u(x,t) = \sum_{n=2}^{\infty} \frac{4}{(n\pi)^3} \cdot (1-(-1)^n) \cdot \cos(2n\pi t) \cdot \sin(n\pi x).$$

Soluţie.

$$a = 2,\ L = 1 \Rightarrow$$

$$u(x,t) = \sum_{n=1}^{\infty} (A_n \cos 2n\pi t + B_n \sin 2n\pi t) \cdot \sin n\pi x$$

$$f(x) = x(1-x) = x - x^2 = \sum_{n=1}^{\infty} b_n \sin n\pi x \Rightarrow$$

$$b_n = 2\int_0^1 (x-x^2)\sin n\pi x dx =$$

$$= -\frac{2}{n\pi}(\cos n\pi x)\cdot(x-x^2)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (\cos n\pi x)\cdot(1-2x)dx =$$

$$= \frac{2\sin n\pi x}{n^2\pi^2} \cdot (1-2x)|_0^1 + \frac{4}{n^2\pi^2}\int\limits_0^1 \sin n\pi x dx =$$

$$= \frac{-4}{n^3\pi^3} \cdot \cos n\pi x|_0^1 = \frac{4}{n^3\pi^3}[1-(-1)^n] =$$

$$= \begin{cases} 0, \ n = 2k \\ \frac{8}{\pi^3(2k+1)^3}, \, n = 2k+1, \, k \geq 0 \end{cases}$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin n\pi x \Rightarrow A_n = b_n, \, \forall n \geq 1$$

$$f(x) = u(x,0)$$

$$g(x) = 0 \Rightarrow \beta_n = 0 \Rightarrow B_n = 0$$

$$u(x,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty} \frac{\cos(4k+2)\pi t}{(2k+1)^3} \cdot \sin(2k+1)\pi x.$$

Exerciţiul 7.27

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = 0, 0 < x < \pi, t \geq 0 \\ u(0,t) = u(\pi,t) = 0, t \geq 0 \\ u(x,0) = \sin x - 2\sin 2x + \sin 3x \\ \frac{\partial u}{\partial t}(x,0) = 6\sin 3x - 7\sin 5x. \end{cases}$$

Soluţie.

$$a = 3, \, L = \pi$$

$$u(x,t) = \sum_{n=1}^{\infty}(A_n \cos\frac{3n\pi t}{\pi} + B_n \sin\frac{3n\pi t}{\pi})\sin nx =$$

$$= \sum_{n=1}^{\infty} (A_n \cos 3nt + B_n \sin 3nt) \sin nx$$

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin nx = \sin x - 2\sin 2x + \sin 3x$$

$$A_1 = 1,\ A_2 = -2,\ A_3 = 1,\ A_n = 0,\ n \geq 4$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} 3nB_n \sin nx = 6\sin 3x - 7\sin 5x \Rightarrow$$

$$9B_3 = 6 \Rightarrow 15B_5 = -7$$

$$B_3 = \frac{2}{3},\ B_5 = -\frac{7}{15},\ B_n = 0,\ n \neq 3,5$$

$$u(x,t) = \cos 3t \cdot \sin x - 2\cos 6t \cdot \sin 2x +$$

$$+ (\cos 9t + \frac{2}{3}\sin 9t)\sin 3x - \frac{7}{15}\sin 15t \cdot \sin 5x.$$

Exerciţiul 7.28

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < \pi,\ ,t \geq 0 \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = x^2 - \pi x. \end{cases}$$

Soluţie.

$$a = 1,\ L = \pi \Rightarrow$$

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cdot e^{-n^2 t} \cdot \sin \frac{n\pi x}{\pi} =$$

$$= \sum_{n=1}^{\infty} A_n \cdot e^{-n^2 t} \cdot \sin nx$$

$$u(x,0) = f(x) = x^2 - \pi x \Rightarrow$$

$$A_n = \frac{2}{\pi}\int\limits_0^{\pi} (x^2 - \pi x)\sin n dx =$$

$$= \frac{-2\cos nx}{n\pi}\cdot (x^2 - \pi x)\Big|_0^{\pi} + \frac{2}{n\pi}\int\limits_0^{\pi} \cos nx \cdot (2x-\pi)dx =$$

$$= \frac{2\sin nx}{\pi n^2}\cdot (2x-\pi)\Big|_0^{\pi} - \frac{2}{\pi n^2}\int\limits_0^{\pi} \sin nx\cdot 2dx =$$

$$= \frac{4}{\pi n^3}\cos nx\Big|_0^{\pi} = \frac{-4}{\pi n^3}[1-(-1)^n] =$$

$$= \begin{cases} \frac{-8}{\pi(2k+1)^3}, k \geq 0 \\ 0, \ n = 2k \end{cases}$$

$$u(x,t) = \frac{-8}{\pi}\sum_{k=0}^{\infty} e^{-(2k+1)^2 t}\cdot \frac{\sin(2k+1)x}{(2k+1)^3}.$$

Exerciţiul 7.29

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2x + 1 + \sinh t, 0 < x < 1, \\ u(x,0) = 0, \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Soluţie.

$$u(x,t) = \sum_{n=1}^{\infty}\left[\int_0^t B_n(s)e^{-(n\pi)^2(t-s)}ds\right]\sin n\pi x$$

$$B_n(s) = 2\int_0^1 (-2x + 1 + \sinh s)\sin n\pi x dx =$$

$$= -2(-2x + 1 + \sinh s)\frac{\cos n\pi x}{n\pi}|_0^1 - 4\int_0^1 \frac{\cos n\pi x}{n\pi}dx =$$

$$= \frac{-2}{n\pi}\left[1 - (-1)^n\right]\sinh s + \frac{2}{n\pi}\left[1 + (-1)^n\right].$$

$$\int_0^t B_n(s)e^{-(n\pi)^2 t + (n\pi)^2 s}ds =$$

$$= \frac{2}{n\pi}e^{-(n\pi)^2 t}\left[1 - (-1)^n\right]\frac{1}{2}\int_0^t \left[e^{[(n\pi)^2+1]s} - e^{[(n\pi)^2+1]s}\right]ds +$$

$$+\frac{2}{n\pi}e^{-(n\pi)^2 t}\left[1 + (-1)^n\right]\int_0^t \left[e^{(n\pi)^2 s}\right]ds =$$

$$= \frac{1 - (-1)^n}{n\pi}e^{-(n\pi)^2 t}\left\{\frac{e^{[(n\pi)^2+1]s}}{(n\pi)^2+1}|_0^t - \frac{e^{[(n\pi)^2-1]s}}{(n\pi)^2-1}|_0^t\right\} +$$

$$+\frac{2}{(n\pi)^3}\left[1 + (-1)^n\right]e^{-(n\pi)^2 t}\left[e^{(n\pi)^2 t} - 1\right] =$$

$$= \frac{1 - (-1)^n}{n\pi}\left[\frac{e^t - e^{-(n\pi)^2 t}}{(n\pi)^2+1} - \frac{e^{-t} - e^{-(n\pi)^2 t}}{(n\pi)^2-1}\right] +$$

$$+\frac{2}{(n\pi)^3}\left[1 + (-1)^n\right]\left[1 - e^{-(n\pi)^2 t}\right] \Rightarrow$$

$$\int_0^t B_n(s)e^{-(n\pi)^2(t-s)}ds =$$

$$= \begin{cases} \frac{1}{(n\pi)^3}\left[1 - e^{-(2n\pi)^2 t}\right], \ n = 2, \ n \geq 1, \\ \frac{2}{(2n+1)\pi}\left[\frac{e^t}{((2n+1)\pi)^2+1} - \frac{e^{-t}}{((2n+1)\pi)^2-1} - \frac{2((2n+1)\pi)^2 e^{-((2n+1)\pi)^2 t}}{((2n+1)\pi)^4-1}\right], \\ \qquad n = 2n+1, \ n \geq 0. \end{cases}$$

$$u(x,t) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{(n\pi)^3}\left[1 - e^{-(2n\pi)^2 t}\right]\sin 2n\pi x +$$

$$+2\sum_{n=0}^{\infty}\frac{1}{(2n+1)\pi}\left[\frac{e^t}{((2n+1)\pi)^2+1}-\frac{e^{-t}}{((2n+1)\pi)^2-1}-\right.$$

$$\left.-\frac{2((2n+1)\pi)^2e^{-((2n+1)\pi)^2t}}{((2n+1)\pi)^4-1}\right]\sin(2n+1)\pi x.$$

Exerciţiul 7.30

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}+e^{-2t}\sin\pi x,\ 0<x<1, \\ u(x,0)=\sin 2\pi x+3\sin 3\pi x, \\ \frac{\partial u}{\partial t}(x,0)=2\sin 2\pi x+\sin 3\pi x, \\ u(0,t)=u(1,t)=0. \end{cases}$$

Soluţie. Determinăm $u_h(x,t)$:

$$u_h(x,t)=\sum_{n=1}^{\infty}(A_n\cos 3n\pi t+B_n\sin 3n\pi t)\sin n\pi x$$

$$u_h(x,0)=\sum_{n\geq 1}A_n\sin n\pi x=\sin 2\pi x+3\sin 3\pi x\Rightarrow$$

$$A_2=1,\ A_3=3,\ A_n=0,\ \text{în rest.}$$

$$\frac{\partial u_h}{\partial t}(x,0)=\sum_{n\geq 1}3n\pi B_n\cdot\sin n\pi x=2\sin 2\pi x+\sin 3\pi x\Rightarrow$$

$$3\cdot 2\pi B_2=2\Rightarrow B_2=\frac{1}{3\pi},$$

$$9\pi B_3=1\Rightarrow B_3=\frac{1}{9\pi},$$

$$B_n=0,\ \text{în rest.}$$

$$u_h(x,t)=\left(\cos 6\pi t+\frac{1}{3\pi}\sin 6\pi t\right)\sin 2\pi x+$$

$$+\left(3\cos 9\pi t + \frac{1}{9\pi}\sin 9\pi t\right)\sin 3\pi x.$$

Determinăm $u_p(x,t)$:

$$u_p(x,t) = \sum_{n\geq 1}\left\{\int_0^t B_n(s)\sin 3n\pi(t-s)ds\right\}\sin n\pi x.$$

$$B_n(s) = \frac{2}{3n\pi}\int_0^1 e^{-2s}\sin \pi x \sin n\pi x dx = \begin{cases} \frac{e^{-2s}}{3\pi},\ n=1, \\ 0, n\geq 2.\end{cases}$$

$$\int_0^t B_1(s)\sin 3\pi(t-s)ds = \frac{1}{3\pi}\int_0^t e^{-2s}\sin 3\pi(t-s)ds.$$

$$I = \int_0^t e^{-2s}\sin 3\pi(t-s)ds =$$

$$= e^{-2s}\frac{\cos 3\pi(t-s)}{3\pi}|_0^t + 2\int_0^t e^{-2s}\frac{\cos 3\pi(t-s)}{3\pi}ds =$$

$$= \frac{1}{3\pi}(e^{-2t} - \cos 3\pi t) + 2\int_0^t e^{-2s}\left(\frac{-\sin 3\pi(t-s)}{9\pi^2}\right)' ds =$$

$$= \frac{1}{3\pi}(e^{-2t} - \cos 3\pi t) + \frac{2}{9\pi^2}\sin 3\pi t - \frac{4}{9\pi^2}I \Rightarrow$$

$$\left(1+\frac{4}{9\pi^2}\right)I = \frac{1}{9\pi^2}\left[3\pi e^{-2t} - 3\pi\cos 3\pi t + 2\sin 3\pi t\right]$$

Deci

$$\int_0^t B_1(s)\sin 3\pi(t-s)ds =$$

$$= \frac{1}{3\pi(4+9\pi^2)}(3\pi e^{-2t} - 3\pi\cos 3\pi t + 2\sin 3\pi t).$$

Astfel, $u_p(x,t)$ este:

$$u_p(x,t) = \frac{1}{3\pi(4+9\pi^2)}(3\pi e^{-2t} - 3\pi\cos 3\pi t + 2\sin 3\pi t)\sin \pi x,$$

iar $u(x,t)$ este:

$$u(x,t)=\left(\cos 6\pi t+\frac{1}{3\pi}\sin 6\pi t\right)\sin 2\pi x+$$

$$+\left(3\cos 9\pi t+\frac{1}{9\pi}\sin 9\pi t\right)\sin 3\pi x+$$

$$+\frac{1}{3\pi(4+9\pi^2)}(3\pi e^{-2t}-3\pi\cos 3\pi t+2\sin 3\pi t)\sin \pi x.$$

Exerciţiul 7.31

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=9\frac{\partial^2 u}{\partial x^2}+t^2+t+1,\ 0<x<1,\\ u(x,0)=\sin 2\pi x,\\ \frac{\partial u}{\partial t}(x,0)=x-x^2,\\ u(0,t)=u(1,t)=0.\end{cases}$$

Soluţie. Determinăm $u_h(x,t)$:

$$u_h(x,t)=\sum_{n=1}^{\infty}(A_n\cos 3n\pi t+B_n\sin 3n\pi t)\sin n\pi x$$

$$u_h(x,0)=\sum_{n\geq 1}A_n\sin n\pi x=\sin 2\pi x\Rightarrow A_2=1.$$

$$B_n=\frac{2}{3n\pi}\int_0^1(x-x^2)\sin n\pi x dx=$$

$$=\frac{2}{3n\pi}(x-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right)\Big|_0^1-\frac{2}{3n\pi}\int_0^1(1-2x)\left(-\frac{\cos n\pi x}{n\pi}\right)dx=$$

$$=-\frac{2}{3n\pi}\int_0^1(1-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)'dx=$$

$$=\frac{2}{3n\pi}\int_0^1\frac{2\sin n\pi x}{n^2\pi^2}dx=\frac{-4}{3n^3\pi^3}\frac{\cos n\pi x}{n\pi}\Big|_0^1=\frac{4(1-(-1)^n)}{3n^4\pi^4}.$$

$$u_h(x,t) = \cos 6\pi t \sin 2\pi x + \frac{4}{3\pi^4} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4} \sin 3n\pi t \sin n\pi x.$$

Determinăm $u_p(x,t)$:

$$u_p(x,t) = \sum_{n\geq 1} \left\{ \int_0^t B_n(s) \sin 3n\pi(t-s) ds \right\} \sin n\pi x.$$

$$B_n(s) = \frac{2}{3n\pi} \int_0^1 (s^2+s+1) \sin n\pi x dx =$$

$$= \frac{-2}{3n^2\pi^2}(s^2+s+1)\cos n\pi x|_0^1 = \frac{2}{3n^2\pi^2}(1-(-1)^n)(s^2+s+1).$$

$$\int_0^t \frac{2}{3n^2\pi^2}(1-(-1)^n)(s^2+s+1)\sin 3n\pi(t-s)ds =$$

$$= \frac{2(1-(-1)^n)}{3n^2\pi^2} \int_0^t (s^2+s+1)\sin 3n\pi(t-s)ds.$$

Calculăm integrala

$$I = \int_0^t (s^2+s+1)\sin 3n\pi(t-s)ds.$$

$$I = \int_0^t (s^2+s+1)\left[\frac{\cos 3n\pi(t-s)}{3n\pi}\right]' ds =$$

$$= (s^2+s+1)\frac{\cos 3n\pi(t-s)}{3n\pi}|_0^t - \int_0^t (2s+1)\frac{\cos 3n\pi(t-s)}{3n\pi}ds =$$

$$= \frac{1}{3n\pi}(t^2+t+1-\cos 3n\pi t) - \int_0^t (2s+1)\left[\frac{-\sin 3n\pi(t-s)}{(3n\pi)^2}\right]' ds =$$

$$= \frac{1}{3n\pi}(t^2+t+1-\cos 3n\pi t) - (2s+1)\left[\frac{-\sin 3n\pi(t-s)}{(3n\pi)^2}\right]|_0^t +$$

$$+2\int_0^t \left[\frac{-\sin 3n\pi(t-s)}{(3n\pi)^2}\right] ds =$$

$$= \frac{1}{3n\pi}(t^2+t+1-\cos 3n\pi t) - \frac{\sin 3n\pi t}{(3n\pi)^2} - \frac{2\cos 3n\pi(t-s)}{(3n\pi)^3}\Big|_0^t =$$

$$= \frac{1}{3n\pi}(t^2+t+1-\cos 3n\pi t) - \frac{\sin 3n\pi t}{(3n\pi)^2} - \frac{2(1-\cos 3n\pi t)}{(3n\pi)^3}.$$

Deci

$$\int_0^t B_n(s)\sin 3n\pi(t-s)ds = \frac{2(1-(-1)^n)}{(3n\pi)^3}\cdot$$

$$\cdot\left[t^2+t+1-\cos 3n\pi t - \frac{\sin 3n\pi t}{3n\pi} - \frac{2(1-\cos 3n\pi t)}{(3n\pi)^2}\right].$$

Astfel, $u_p(x,t)$ este:

$$u_p(x,t) = \frac{2}{9\pi^3}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^3}\cdot$$

$$\cdot\left[t^2+t-\frac{\sin 3n\pi t}{3n\pi}+(1-\frac{2}{(3n\pi)^2})(1-\cos 3n\pi t)\right]\sin n\pi x.$$

7.3.5 Exerciţii propuse

Exerciţiul 7.32 Rezolvaţi problema mixtă:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, x\geq 0, 0\leq y\leq l \\ u(x,0)=0, u(x,l)=0 \\ \lim\limits_{x\to\infty} u(x,y)=0, u(0,y)=y(y-l). \end{cases}$$

Exerciţiul 7.33 Integraţi ecuaţiile coardei vibrante:

a)

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0, x \in (0, l) \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = (x-l)\sin x \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$$

b)

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = x + t, x \in (0, 2), t > 0 \\ u(0,t) = u(2,t) = 0, t > 0 \\ u(x,0) = 2x - x^2 \\ \frac{\partial u}{\partial t}(x,0) = \sin \frac{\pi x}{2}. \end{cases}$$

7.4 Problema Dirichlet interioară pentru cerc. Metoda separării variabilelor.

Problema Dirichlet pentru interiorul cercului

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \text{ în } \ x^2 + y^2 < R^2 \to \text{ec. lui Laplace} \\ u(x,y) = f(x,y) \text{ pe } x^2 + y^2 = R^2. \end{cases}$$

Se trece la coordonate polare:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow u(x,y) = u(\rho\cos\theta, \rho\sin\theta) = \widetilde{u}(\rho,\theta)$$

Ecuaţia lui Laplace în coordonate polare:

$$\frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}(\rho\frac{\partial\tilde{u}}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial\tilde{u}^2}{\partial\theta^2} = 0 \Leftrightarrow \frac{\partial\tilde{u}^2}{\partial\rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial\tilde{u}^2}{\partial\theta^2} + \frac{1}{\rho}\cdot\frac{\partial\tilde{u}}{\partial\rho} = 0$$

Cu M.S.V. căutăm soluţie de forma:

$$u = \mathrm{Z}(\rho)\Phi(\theta) \Rightarrow \quad \begin{cases} \Phi''(\theta) + \lambda\Phi(\theta) = 0 \\ \rho \cdot \frac{d}{d\rho}(\rho\frac{d\mathrm{Z}}{d\rho}) - \lambda \mathrm{Z} = 0 \end{cases}$$

$$\tilde{u}(\rho, \theta + 2\pi) = \tilde{u}(\rho, \theta) \Rightarrow \Phi(\theta + 2\pi) = \Phi(\theta) \Rightarrow$$

$$\Rightarrow \sqrt{\lambda} = n \in \mathrm{Z} \Rightarrow \lambda_n = n^2, n \in \mathrm{N} \Rightarrow$$

$$\Rightarrow \Phi_n(\theta) = A_n \cos n\theta + B_n \sin n\theta, n \geq 0.$$

Pentru $\lambda = n^2$ ecuaţia în $\mathrm{Z}(\rho)$ devine:

$$\rho^2 \frac{d^2\mathrm{Z}}{d\rho^2} + \rho\frac{d\mathrm{Z}}{d\rho} - n^2\mathrm{Z} = 0$$

ecuaţia Euler.
Facem substituţia $\rho = e^t$ pentru că $\rho > 0$ şi substituţia de funcţie $\tilde{Z}(t) = \mathrm{Z}(\rho) \Rightarrow \rho\frac{d\mathrm{Z}}{d\rho} = \tilde{\mathrm{Z}}'(t)$.

Analog

$$\rho^2 \frac{d^2\mathrm{Z}}{d\rho^2} = \tilde{\mathrm{Z}}''(t) - \tilde{\mathrm{Z}}'(t)$$

ecuaţia devenind:

$$\begin{aligned} &\tilde{\mathrm{Z}}''(t) - \tilde{\mathrm{Z}}'(t) + \tilde{\mathrm{Z}}'(t) - n^2\tilde{\mathrm{Z}}(t) = 0 \\ &\tilde{\mathrm{Z}}''(t) - n^2\tilde{\mathrm{Z}}(t) = 0. \end{aligned}$$

Ecuaţia caracteristică:

$$r^2 - n^2 = 0 \Rightarrow r_{1,2} = \pm n \Rightarrow \tilde{\mathrm{Z}}_n(t) = ae^{nt} + be^{-nt} \Rightarrow$$

$$\mathrm{Z}_n(\rho) = a\rho^n + b\rho^{-n}, n \geq 1$$

pentru

$$n = 0 \Rightarrow \mathrm{Z}_0(\rho) = C_0 \ln\rho + C_1.$$

Pentru că

$$\lim_{\rho\to 0}\ln\rho=-\infty,\ \lim_{\rho\to 0}\rho^{-n}=\infty,$$

pentru problema Dirichlet interioară luăm:

$$\mathrm{Z}_n(\rho)=a\rho^n, n\geq 1, \mathrm{Z}_0(\rho)=C_1\Rightarrow$$

$$\tilde{u}(\rho,\theta)=\sum_{n=0}^{\infty}\rho^n(A_n\cos n\theta+B_n\sin\theta).$$

Condiţia pe frontieră este:

$$u(R,\theta)=\sum_{n=0}^{\infty}R^n(A_n\cos n\theta+B_n\sin n\theta)= \tag{7.9}$$
$$=f(R\cos\theta,R\sin\theta)=\tilde{f}(\theta)$$

dezvoltăm pe $\tilde{f}(\theta)$ în serie Fourier pe $[0,2\pi]$, funcţie de perioadă $T=2\pi$.

$$\tilde{f}(\theta)=\sum_{n\geq 0}(a_n\cos n\theta+b_n\sin n\theta),$$

unde

$$a_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{f}(\theta)\cdot\cos n\theta d\theta, n\geq 1,$$

$$a_0=\frac{1}{2\pi}\int_0^{2\pi}\tilde{f}(\theta)d\theta,$$

şi

$$b_n=\frac{1}{\pi}\int_0^{2\pi}\tilde{f}(\theta)\cdot\sin n\theta d\theta, n\geq 1$$

Din relaţia(7.9) avem:

$$\sum_{n\geq 0} R^n(A_n\cos n\theta + B_n\sin n\theta) = \sum_{n\geq 0}(a_n\cos n\theta + b_n\sin n\theta)$$

şi din identificare avem:

$$\begin{cases} A_n = \frac{1}{\pi R^n}\int\limits_0^{2\pi}\tilde{f}(\theta)\cdot\cos n\theta d\theta, n\geq 1,\ A_0 = \frac{1}{2\pi}\int\limits_0^{2\pi}\tilde{f}(\theta)d\theta, \\ B_n = \frac{1}{\pi R^n}\int\limits_0^{2\pi}\tilde{f}(\theta)\cdot\sin n\theta d\theta, n\geq 1. \end{cases}$$

Observaţia 7.34 Dacă membrul drept al condiţiei pe frontieră se poate ,,liniariza"- adică se poate scrie în termenii $\sin n\theta$, $\cos n\theta$, coeficienţii A_n, B_n se deduc prin identificare.

Altfel, se calculează integralele.

Se revine la notaţiile: $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow u(x,y).$

7.4.1 Exerciţii rezolvate

Exerciţiul 7.35

$$\begin{cases} \Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ în } x^2+y^2<4 \\ u(x,y) = x^3+3x^2y+x+y^2 \text{ pe } x^2+y^2=4 \end{cases}$$

Soluţie.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow u(x,y) = u(\rho\cos\theta,\rho\sin\theta) = \tilde{u}(\rho,\theta)$$

$$\tilde{u}(\rho,\theta) = \sum_{n\geq 0}\rho^n(A_n\cos n\theta + B_n\sin n\theta)$$

$$u(2\cos\theta, 2\sin\theta) = \tilde{u}(2,\theta) = \sum_{n=0}^{\infty} 2^n (A_n \cos n\theta + B_n \sin n\theta) =$$

$$= 8\cos^3\theta + 24\cos^2\theta\sin\theta + 2\cos\theta + 4\sin^2\theta$$

,,Liniarizăm” membrul drept:

$$\cos^3\theta = \frac{3}{4}\cos\theta + \frac{1}{4}\cos 3\theta;$$

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$8\cos^3\theta + 24\sin\theta - 24\sin^3\theta + 2\cos\theta + 4\sin^2\theta =$$

$$= 6\cos\theta + 2\cos 3\theta + 24\sin\theta - 18\sin\theta + 6\sin 3\theta + 2\cos\theta + 2 - 2\cos 2\theta =$$

$$= 2 + 8\cos\theta + 6\sin\theta - 2\cos 2\theta + 2\cos 3\theta + 6\sin 3\theta$$

Identificarea:

$2^0(A_0\cos 0 + B_0\cdot 0) = 2 \Rightarrow A_0 = 2$

$2^1(A_1\cos\theta + B_1\sin\theta) = 8\cos\theta + 6\sin\theta \Rightarrow A_1 = 4, B_1 = 3$

$2^2(A_2\cos 2\theta + B_2\sin 2\theta) = -2\cos 2\theta \Rightarrow A_2 = -\frac{1}{2}, B_2 = 0$

$2^3(A_3\cos 3\theta + B_3\sin 3\theta) = 2\cos 3\theta + 6\sin 3\theta \Rightarrow A_3 = \frac{1}{4}, B_3 = \frac{3}{4}$

$A_n = B_n = 0, n \geq 4$

$$\tilde{u}(\rho,\theta) = 2 + 4\rho\cos\theta + 3\sin\theta - \frac{\rho^2}{2}\cdot\cos 2\theta + \frac{1}{4}\rho^3\cos 3\theta + \frac{3}{4}\rho^3\sin 3\theta$$

dar

$$\rho^2\cos^2\theta - \rho^2\sin^2\theta = x^2 - y^2$$

$$(x+iy)^3 = \rho^3\cos 3\theta + i\rho^3\sin 3\theta \Rightarrow \quad \begin{cases} \rho^3\cos 3\theta = x^3 - 3xy^2 \\ \rho^3\sin 3\theta = -y^3 + 3x^2y \end{cases}$$

$$u(x,y) = 2 + 4x + 3y - \frac{1}{2}(x^2 - y^2) + \frac{1}{4}(x^3 - 3xy^2 + 9x^2y - 3y^3).$$

Exerciţiul 7.36

$$\begin{cases} \Delta u(x,y) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ în } x^2+y^2<9 \\ u(x,y) = y^3 + x^2 - y^2 + x \text{ pe } x^2+y^2=9. \end{cases}$$

Soluţie.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow$$

$$\begin{cases} u(x,y) = u(\rho\cos\theta, \rho\sin\theta) = \tilde{u}(\rho,0),\ R=3 \\ \tilde{u}(\rho,\theta) = \sum\limits_{n=0}^{\infty} \rho^n (A_n\cos n\theta + B_n \sin n\theta) \end{cases}$$

$\tilde{u}(3,\theta) = 27\sin^3\theta + 9\cos^2\theta - 9\sin^2\theta + 3\cos\theta \leftarrow$ liniarizăm condiţia pe frontieră în coordonate polare

$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

$$\cos^2\theta - \sin^2\theta = \cos 2\theta$$

$$\sum_{n=0}^{\infty} 3^n (A_n\cos n\theta + B_n\sin n\theta) =$$

$$= \tilde{u}(3,\theta) = \frac{81}{4}\sin\theta - \frac{27}{4}\sin 3\theta + 9\cos 2\theta + 3\cos\theta$$

Identificare:

$$\begin{array}{l} A_0 = 0 \\ 3A_1 = 3 \Rightarrow A_1 = 1 \\ 3B_1 = \frac{81}{4} \Rightarrow B_1 = \frac{27}{4} \\ 9A_2 = 9 \Rightarrow A_2 = 1, B_2 = 0 \\ 27A_3 = 0 \Rightarrow A_3 = 0 \\ 27B_3 = \frac{-27}{4} \Rightarrow B_3 = \frac{-1}{4} \\ A_n = B_n = 0, n \geq 4 \end{array}$$

$$\tilde{u}(\rho,\theta) = \rho\cos\theta + \frac{27}{4}\rho\sin\theta + \rho^2\cos 2\theta - \frac{1}{4}\rho^3\sin 3\theta.$$

Dar $\rho^2\cos^2\theta - \rho^2\sin\theta = x^2 - y^2$

$$(x+iy)^3 = \rho^3\cos 3\theta + i\rho^3\sin 3\theta \Rightarrow \rho^3\sin 3\theta = 3x^2y - y^3$$

$$u(x,y) = x + \frac{27}{4}y + x^2 - y^2 - \frac{3}{4}x^2y + \frac{1}{4}y^3.$$

Exerciţiul 7.37 <u>Aplicaţie P.Dirichlet</u>

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ în } x^2+y^2<4 \\ u(x,y) = x^3 + xy + y^2 \text{ pe } x^2+y^2=4. \end{cases}$$

Soluţie.

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \Rightarrow$$

$$\begin{cases} u(x,y) = u(\rho\cos\theta, \rho\sin\theta) = \tilde{u}(\rho,\theta) \\ \tilde{u}(\rho,\theta) = \sum\limits_{n\geq 0} \rho^n(A_n\cos n\theta + B_n\sin n\theta) \end{cases}$$

$$u(2\cos\theta, 2\sin\theta) = \sum_{n=0}^{\infty} 2^n(A_n\cos n\theta + B_n\sin n\theta) =$$

$$= 8\cos^3\theta + 4\sin\theta\sin\theta + 4\sin^2\theta \leftarrow \text{ liniarizăm.}$$

Dar $u(2\cos\theta, 2\sin\theta) = \tilde{u}(2,\theta)$

$$\sum_{n=0}^{\infty} 2^n(A_n\cos n\theta + B_n\sin n\theta) = 2 - 2\cos 2\theta + 2\sin 2\theta + 6\cos\theta + 2\cos 3\theta$$

$$\cos^3\theta = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos 3\theta$$
$$\sin^3\theta = \tfrac{3}{4}\sin\theta - \tfrac{1}{4}\sin 3\theta$$

$$\sum_{n=0}^{\infty} 2^n(A_n\cos n\theta + B_n\sin n\theta) = 2+6\cos\theta - 2\cos\theta + 2\sin 2\theta + 2\cos 3\theta$$

Identificarea:

$2^0(A_0 + B_0 0) = 2 \Rightarrow A_0 = 2$
$2^1(A_1\cos\theta + B_1\sin\theta) = 6\cos\theta \Rightarrow A_1 = 3, B_1 = 0$
$2^2(A_2\cos 2\theta + B_2\sin 2\theta) = -2\cos 2\theta + 2\sin 2\theta \Rightarrow A_2 = -\frac{1}{2}, B_2 = \frac{1}{2}$
$2^3(A_3\cos 3\theta + B_3\sin 3\theta) = 2\cos 3 \Rightarrow A_3 = \frac{1}{4}, B_3 = 0$
$A_n = 0, B_n = 0, n \geq 4$

$$\tilde{u}(\rho,\theta) = 2 + 3\rho\cos\theta - \frac{\rho^2}{2}\cos 2\theta + \frac{\rho^2}{2}\sin 2\theta + \frac{1}{4}\rho^3\cos 3\theta.$$

Dar

$$\rho\cos\theta = x$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\rho^2 = x^2 + y^2$$

$$u(x,y) = 2 + 3x - \frac{x^2}{2} + \frac{y^2}{2} + xy + x^3 - \frac{3}{4}x(x^2+y^2).$$

Exerciţiul 7.38 Să se rezolve problema Dirichlet pe

$$D = \{(x,y)/0 \leq x \leq 1, 0 \leq y \leq 1\}$$

dată astfel:

$$\Delta u = 0,\ u(x,0) = 0,\ u(x,1) = x,\ u(0,y) = 0$$

şi

$$u(1,y) = \sin\frac{\pi y}{2}.$$

Soluţie.

Facem substituţia $u(x,y) = u_1(x,y) + u_2(x,y)$ unde

$$\begin{cases} \Delta u_1 = 0, \text{ pe } [0,1]\times[0,1] \\ u_1(x,0) = 0,\ u_1(x,1) = x \\ u_1(0,y) = u_1(1,y) = 0 \end{cases}$$

pe

$$\begin{cases} \Delta u_2 = 0, \text{ pe } [0,1]\times[0,1] \\ u_2(x,0) = u_2(x,1) = 0 \\ u_2(0,y) = 0, u_2(1,y) = \sin\frac{\pi y}{2} \end{cases}$$

Aplicăm M.S.V. pentru cele 2 probleme.

$u_1(x,y) = X(x)\cdot T(y)$ a.î.

$$X''(x)\cdot T(y) + X(x)\cdot T''(y) = 0 \Rightarrow$$

$$\frac{X''(x)}{X(x)} = -\frac{T''(y)}{T(y)} = -\lambda \Rightarrow$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ T''(y) - \lambda T(y) = 0 \\ X(0) = X(1) = 0. \end{cases}$$

Din:

$$\begin{cases} X''(x) + \lambda X(x) = 0 \\ X(0) = X(1) = 0 \\ T(0) = 0. \end{cases} \Rightarrow$$

$$\begin{cases} \lambda_k = +(k\pi)^2, k = 1,2,\dots \\ X_k(x) = a_k\cdot\sin(k\pi x), k \geq 1. \end{cases}$$

Apoi:

$$\begin{cases} T_k''(y) - (k\pi)^2 T_k(y) = 0 \\ T_k(0) = 0, k = 1, 2, \ldots \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} T_k(y) = c_k \cdot e^{k\pi y} + d_k \cdot e^{-k\pi y} \\ T_k(0) = 0 = c_k + d_k \end{cases}$$

$$\Rightarrow T_k(y) = 2c_k \cdot \text{sh}(k\pi y) = d_k \text{sh}(k\pi y),\ k \geq 1$$

Deci

$$u_1^k(x, y) = A_k \cdot \sin(k\pi x) \cdot \text{sh}(k\pi y)$$
$$u_1(x, y) = \sum_{k=1}^{\infty} A_k \cdot \sin(k\pi x) \cdot \text{sh}(k\pi y)$$

Aflăm A_k din condiţia $u_1(x, 1) = x \Rightarrow$

$$\sum_{k=1}^{\infty} A_k \cdot \sin k\pi x \cdot \text{sh}k\pi = x \Rightarrow$$

$$A_k \cdot \text{sh}k\pi = \frac{2}{1} \int_0^1 x \cdot \sin k\pi x dx =$$

$$= -2x \cdot \frac{\cos k\pi x}{k\pi} \Big|_0^1 + 2 \int_0^1 \frac{\cos k\pi x}{k\pi} dx =$$

$$= -\frac{2}{k\pi}[(-1)^k - 1] = \frac{2}{k\pi}[1 - (-1)^k].$$

Deci:

$$A_n = \frac{2}{\pi} \cdot \frac{1 - (-1)^n}{n} \cdot \frac{1}{\text{sh}n\pi},$$

deci soluţia pentru prima problemă Dirichlet este

$$u_1(x,y) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \cdot \frac{\text{sh}n\pi y}{\text{sh}n\pi} \cdot \sin(n\pi x)$$

Aplicăm tot M.S.V. pentru cea de-a doua problemă:

$$u_2(x,y) = X(x) \cdot Y(y)$$

unde: $-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} = -\lambda$

Unde din condiţii:

$$\begin{cases} Y''(y) + \lambda Y(y) = 0 \\ Y(0) = Y(1) = 0 \end{cases}$$

respectiv

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ X(0) = 0. \end{cases}$$

Din problema lui $Y(y)$ obţinem

$$\lambda_n = (n\pi)^2, n = 1, 2, \ldots$$

şi deci

$$Y_n(y) = \alpha_n \cdot \sin(n\pi y), n = 1, 2, \ldots$$

Din problema lui $X_n(x) \Rightarrow$

$$X_n(x) = \beta_n e^{n\pi x} + \gamma_n e^{-n\pi x}$$
$$X_n(0) = 0 \Rightarrow -\beta_n = \gamma_n \Rightarrow X_n(x) = \sigma_n \text{sh}(n\pi x)$$

$u_2^n(x,y) = B_n \cdot \text{sh}(n\pi x) \cdot \sin(n\pi y)$ şi căutăm soluţia de forma:

$$u_2(x,y) = \sum_{n=1}^{\infty} B_n \cdot \text{sh}(n\pi x) \cdot \sin(n\pi y).$$

Din condiţia $u_2(1, y) = \sin \frac{\pi y}{2}$ găsim:

$$\sum_{n=1}^{\infty} B_n \cdot \mathrm{sh}(n\pi) \cdot \sin n\pi y = \sin \frac{\pi y}{2}$$

$$B_n \cdot \mathrm{sh} n\pi = 2 \int\limits_0^1 \sin n\pi y \cdot \sin \frac{\pi y}{2} dy =$$

$$= \int\limits_0^1 [\cos(n - \frac{1}{2})\pi y - \cos(n + \frac{1}{2})\pi y] dy =$$

$$= \frac{1}{n - \frac{1}{2}} \cdot \sin(n - \frac{1}{2})\pi y \,\Big|_0^1 - \frac{1}{n + \frac{1}{2}} \cdot \sin(n + \frac{1}{2})\pi y \,\Big|_0^1 =$$

$$= \frac{2\sin(n\pi - \frac{\pi}{2})}{2n - 1} - \frac{2\sin(n\pi + \frac{\pi}{2})}{2n + 1} =$$

$$= 2(-1)^{n+1} \left(\frac{1}{2n - 1} + \frac{1}{2n + 1} \right) = \frac{4n(-1)^{n+1}}{4n^2 - 1} \Rightarrow$$

$$B_n = \frac{4n(-1)^{n-1}}{4n^2 - 1} \cdot \frac{1}{\mathrm{sh} n\pi}, n = 1, 2, \ldots$$

$$u_2(x, y) = 2 \sum_{n=1}^{\infty} \frac{2n(-1)^{n-1}}{4n^2 - 1} \cdot \frac{\mathrm{sh}(n\pi x)}{\mathrm{sh} n\pi} \cdot \sin(n\pi y).$$

Deci:

$$u(x, y) = \frac{2}{\pi} \left(\sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n} \cdot \frac{\mathrm{sh} n\pi y}{\mathrm{sh} n\pi} \cdot \sin n\pi x \right) +$$

$$+2\sum_{n=1}^{\infty}\frac{2n(-1)^{n-1}}{4n^2-1}\cdot\frac{\mathrm{sh}(n\pi x)}{\mathrm{sh}n\pi}\cdot\sin n\pi y.$$

7.4.2 Exerciţii propuse

Exerciţiul 7.39 Rezolvaţi următoarele P.D.

a)

$$\begin{cases}\Delta u=0 \text{ în } x^2+y^2<1\\ u(x,y)=\frac{1}{10+6x} \text{ pe } x^2+y^2=1\end{cases}$$

b)

$$\begin{cases}\Delta u(x,y)=0 \text{ în } x^2+y^2<1\\ u(x,y)=x^3+y^3 \text{ pe } x^2+y^2=1.\end{cases}$$

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