Integrala dublă

Fie dreptunghiul sau intervalul bidimensional (i.b.)

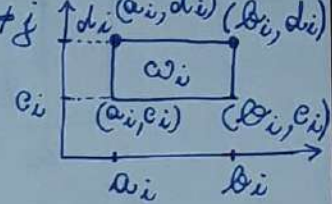
$$\omega_i = [a_i, b_i] \times [c_i, d_i]$$

$$\mathring{\omega}_i = \omega_i \setminus \{ [a_i, b_i] \times \{c_i\} \cup [a_i, b_i] \times \{d_i\} \cup \{a_i\} \times [c_i, d_i] \cup \{b_i\} \times [c_i, d_i] \}.$$

interiorul lui ω_i

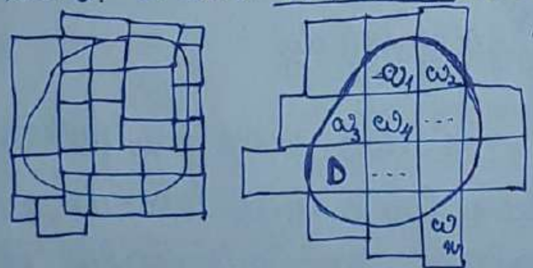
O familie $\{\omega_i\}_{i\in I}$ formează o <u>partiție</u> a lui $\mathbb{R}^2$ dacă:

i) $\mathbb{R}^2 = \bigsqcup_{i\in I} \omega_i$ ii) $\mathring{\omega}_i \cap \mathring{\omega}_j = \emptyset$ pentru $\forall i \neq j$



Fie $D \subset \mathbb{R}^2$ mărginit ($\exists M > 0$, $\forall (x,y) \in \mathbb{R}^2 \to \|(x,y)\|_2 = \sqrt{x^2+y^2} \leq M$)

Fie $\Delta = \{\omega_k\}_{1\leq k\leq n}$ o mulțime de i.b. care au puncte comune cu D. Δ s.n. <u>diviziune</u> a lui D.



. $\dim \omega_k = \text{aria } \omega_k = (b_k - a_k)(d_k - c_k)$

. $\nu(\Delta) = \max\limits_{1\leq k\leq n} \dim \omega_k$

$\uparrow$

norma lui Δ

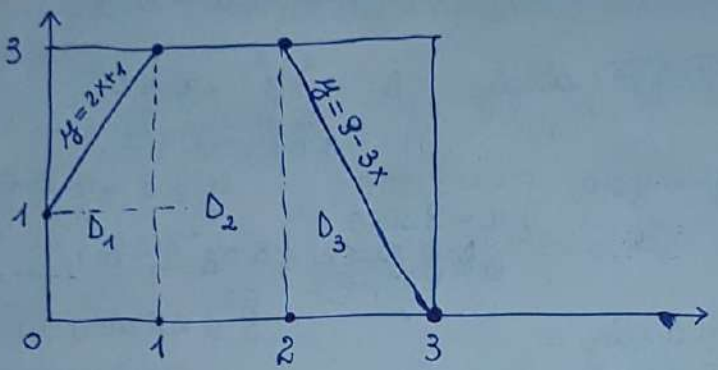
$f: D \longrightarrow \mathbb{R}$ mărginită ($\exists M > 0 : |f(x,y)| \leq M$, $\forall (x,y) \in D$).

Fie $\Delta = \{\omega_k\}_{1\leq k\leq n}$, $(\xi_k, \eta_k) \in \omega_k$ puncte arbitrar alese - puncte intermediare. Definim:

$$S_\Delta(f) = \sum_{k=1}^{n} f(\xi_k, \eta_k) \cdot \text{aria } \omega_k \leftarrow$$ suma Riemann asoc. lui f pentru diviziunea Δ și punctele intermediare (ξ_k, η_k).

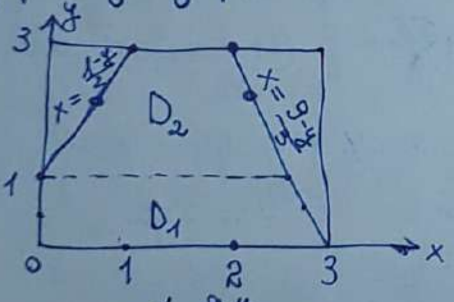
<u>Def:</u> Dacă $\lim\limits_{\nu(\Delta)\to 0} S_\Delta(f)$ există, este finită și nu depinde de alegerea punctelor intermediare, atunci f este integrabilă pe D și $\lim\limits_{\nu(\Delta)\to 0} S_\Delta(f) = \iint\limits_D f(x,y)\,dx\,dy$.

(1) $\iint_D f(x,y)\,dx\,dy$; D: $[0,3]\times[0,3]$, $y+3x\le 9$, $y\le 2x+1$



$D_1: \begin{cases} 0\le x\le 1 \\ 0\le y\le 2x+1 \end{cases}$ $D_2: [1,2]\times[0,3]$ $D_3: \begin{cases} 2\le x\le 3 \\ 0\le y\le 9-3x \end{cases}$

$$\iint_D f = \int_0^1\left(\int_0^{2x+1} f(x,y)\,dy\right)dx + \int_1^2\left(\int_0^3 f(\ )\,dy\right)dx + \int_2^3\left(\int_0^{9-3x} f(x,y)\,dy\right)dx$$

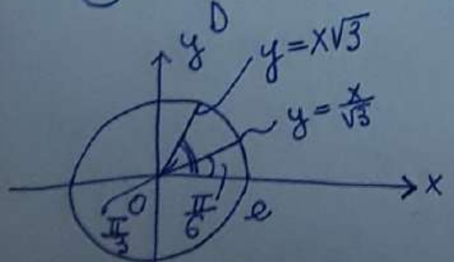


$D_1: \begin{cases} 0\le y\le 1 \\ 0\le x\le \frac{9-y}{3} \end{cases}$

$D_2: \begin{cases} 1\le y\le 3 \\ \frac{1-y}{2}\le x\le \frac{9-y}{3} \end{cases}$

$$\iint_D f = \int_0^1\left(\int_0^{\frac{9-y}{3}} f(\ )\,dx\right)dy + \int_1^3\left(\int_{\frac{1-y}{2}}^{\frac{9-y}{3}} f(\ )\,dx\right)dy.$$

(2) $\iint_D \ln(1+x^2+y^2)\,dx\,dy$ $D: \begin{cases} x^2+y^2=e^2 \\ y=x\sqrt{3} \\ x=y\sqrt{3} \end{cases}$, $x>0$



$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $\rho\in[0,e]$ $\theta\in\left[\frac{\pi}{6},\frac{\pi}{3}\right]$ $dx\,dy=\rho\,d\rho\,d\theta$

$$\iint_D \ln(1+x^2+y^2)\,dx\,dy = \int_0^e\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \rho\ln(1+\rho^2)\,d\theta\,d\rho = \left[\int_0^e \rho\ln(1+\rho^2)\,d\rho\right]\cdot\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta\right) = \frac{\pi}{6\cdot 2}\int_0^e (1+\rho^2)'\ln(1+\rho^2)\,d\rho$$

$$\underbrace{\left(\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} d\theta\right)}_{\frac{\pi}{6}}$$

$$= \frac{\pi}{12}(1+\rho^2)\ln(1+\rho^2)\Big|_0^e -$$

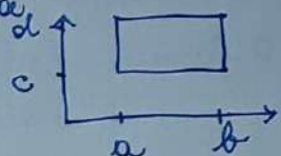
$\boxed{\text{FUBINI}}$ $\int_a^b\int_c^d f(x)\cdot g(y)\,dx\,dy = \left(\int_a^b f(x)\,dx\right)\cdot\left(\int_c^d g(y)\,dy\right).$

· Liniaritatea: $\iint_D (\alpha f + \beta g)\, dx dy = \alpha \iint_D f + \beta \iint_D g$

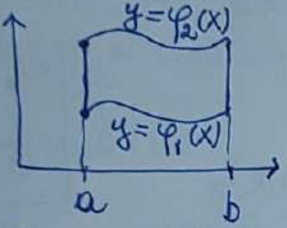
· Aditivitatea: $\iint_{D_1 \cup D_2} f = \iint_{D_1} f + \iint_{D_2} g$; $\mathring{D}_1 \cap \mathring{D}_2 = \emptyset$.

aria $D = \iint_D dx dy$.

<u>Calcul</u>: ① $\iint_{[a,b]\times[c,d]} f = \int_a^b [\int_c^d f(\ )dy]\,dx = \int_c^d [\int_a^b f(\ )dx]\,dy$



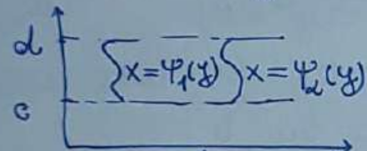
② D:



$D: \begin{cases} a \le x \le b \\ \varphi_1(x) \le y \le \varphi_2(x) \end{cases}$

$\iint_D f = \int_a^b (\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy)\,dx.$

③



$D: \begin{cases} c \le y \le d \\ \psi_1(y) \le x \le \psi_2(y) \end{cases}$

$\iint_D f = \int_c^d (\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx)\,dy.$

D se descompune în 2: se duc ∥ Oy prin punctele de pe Ox, ∥ care nu taie frontiera lui D decât într-un punct.

<u>Schimbarea</u> de <u>variabilă</u> la <u>integrala</u> <u>dublă</u>.

<u>TH.</u> Fie transformarea: $\begin{cases} x = x(\rho,\theta) \\ y = y(\rho,\theta) \end{cases}$ cu $(x(\rho,\theta), y(\rho,\theta)): D_1 \subset \mathbb{R}^2 \to D \subset \mathbb{R}^2$.

Avem: i) $(x(\rho,\theta), y(\rho,\theta))$ bijectivă

ii) $x(\rho,\theta)$ și $y(\rho,\theta)$ continue, au der. parțiale continue + mărginite.

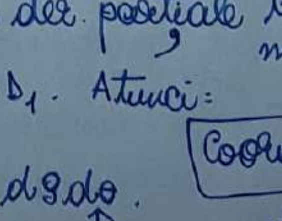
iii) $\mathcal{J} = \frac{D(x,y)}{D(\rho,\theta)} = \begin{vmatrix} x'_\rho & x'_\theta \\ y'_\rho & y'_\theta \end{vmatrix} \ne 0$ pe D_1. Atunci:

iacobianul transf.

$\boxed{\text{Coord. polare}}$

$\iint_D f(x,y)\,dx dy = \iint_{D_1} f(x(\rho,\theta), y(\rho,\theta))\,|\mathcal{J}|\,d\rho d\theta.$

$dx dy = \rho\, d\rho\, d\theta$



<u>EX.</u> i) $x^2+y^2 \le r^2 \to \begin{cases} x = \rho\cos\theta & \rho \in [0,r] \\ y = \rho\sin\theta & \theta \in [0,2\pi] \end{cases}$

ii) $(x-x_0)^2 + (y-y_0)^2 \le r^2 \to \begin{cases} x = x_0 + \rho\cos\theta \\ y = y_0 + \rho\sin\theta \end{cases}$

iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1 \to \begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \end{cases} \to \begin{cases} \rho \in [0,1] \\ \theta \in [0,2\pi] \end{cases}$ $\boxed{dx dy = ab\rho\, d\rho d\theta}$

coord. polare generalizate.

① $\mathcal{A}(D) = \iint_D dx\,dy$; $D: (x-2y+3)^2 + (3x+4y-1)^2 \le 100$

$\begin{cases} u = x-2y+3 \\ v = 3x+4y-1 \end{cases}$ $\quad dx\,dy = \left|\dfrac{D(x,y)}{D(u,v)}\right| du\,dv = \dfrac{du\,dv}{\left|\dfrac{D(u,v)}{D(x,y)}\right|} =$

$= \dfrac{du\,dv}{\left|\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix}\right|} = \dfrac{1}{10}\, du\,dv.$ $\quad D \longrightarrow D_1 : u^2+v^2 \le 100$

$\mathcal{A}(D) = \iint_{u^2+v^2\le 10^2} \frac{1}{10}\, du\,dv = \frac{1}{10}\, \text{aria}(u^2+v^2 \le 10^2) = 10\pi.$

② $D = \{(x,y)\in\mathbb{R}^2 \mid x\ge 0,\ y\ge 0\}$, $f(x,y) = e^{-2(x^2+y^2)}$

$\iint_D e^{-2x^2-2y^2}\, dx\,dy = \int_0^\infty e^{-2x^2}dx \cdot \int_0^\infty e^{-2y^2}dy = \frac{1}{2}\left[\int_0^\infty e^{-t^2}dt\right]^2$

$2x^2 = \cancel{y^2}\, t^2 \Rightarrow x = \frac{1}{\sqrt{2}}\, t \Rightarrow dx = \frac{1}{\sqrt{2}}\, dt \qquad = \frac{1}{2}\cdot\left(\frac{\sqrt{\pi}}{2}\right)^2 = \frac{\pi}{8}.$

③ $\iint_D e^{x^2+y^2}\, dx\,dy$, $D: x^2+y^2 \le 1.$

$\int_0^1 \int_0^{2\pi} \rho\cdot e^{\rho^2}\, d\theta\, d\rho = \left(\int_0^1 \rho\cdot e^{\rho^2} d\rho\right)\cdot\left(\int_0^{2\pi} d\theta\right) = \pi\cdot e^{\rho^2}\Big|_0^1 = \pi(e-1).$

④ $\iint_D (x^2+y^2)\, dx\,dy$, $D: x^2+y^2 \le a^2,\ -\frac{x}{\sqrt{3}} \le y \le \frac{x}{\sqrt{3}}$

$\Downarrow$

$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad x \ge 0 \Rightarrow \cos\theta \ge 0 \Rightarrow \quad x \ge 0.$

$\rho \ge 0$; $\theta \in [-\pi, \pi]$ $\quad \left.\begin{matrix} \theta\in[-\frac{\pi}{2}, \frac{\pi}{2}] \\ \theta\in[-\pi,\pi] \end{matrix}\right| \Rightarrow \underline{\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]}.$

$-\frac{x}{\sqrt{3}} \le y \le \frac{x}{\sqrt{3}} \Big| :x \Rightarrow -\frac{1}{\sqrt{3}} \le \frac{y}{x} = tg\,\theta \le \frac{1}{\sqrt{3}}$

$tg\,\theta$ este s.c. pe $[-\frac{\pi}{2}, \frac{\pi}{2}]$ $\Big| \Rightarrow \theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$

$x^2+y^2 \le a^2 \to \rho^2 \le a^2 \to \rho \in [0, a]$

$\iint_D (x^2+y^2)\, dx\,dy = \int_0^a \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \rho^3\, d\theta\, d\rho = \left(\int_0^a \rho^4 d\rho\right)\left(\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta\right) = \frac{\pi}{3}\frac{a^5}{5} = \frac{\pi}{15}a^5.$

$$-\frac{\pi}{2\cdot 6}\int_0^e (1+\rho^2)\cdot\frac{2\rho}{1+\rho^2}\,d\rho = \frac{\pi}{12}(1+e^2)\ln(1+e^2) - \frac{\pi}{12}\cdot \rho^2\Big|_0^e =$$

$$= \frac{\pi}{12}\left[(1+e^2)\ln(1+e^2) - e^2\right].$$

③ $\iint_D \left(2+\sqrt{1+x^2+y^2}\right)dx\,dy \qquad D:\ x^2+y^2-2y \le 0$

$$x^2+(y-1)^2 \le 1.$$

$x^2+y^2 \le 2y \to y \ge 0;\quad \begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}\quad \theta\in[0,2\pi]$

$y \ge 0 \Rightarrow \theta \in [0,\pi]$.

$x^2+y^2 \le 2y$ în coord. polare

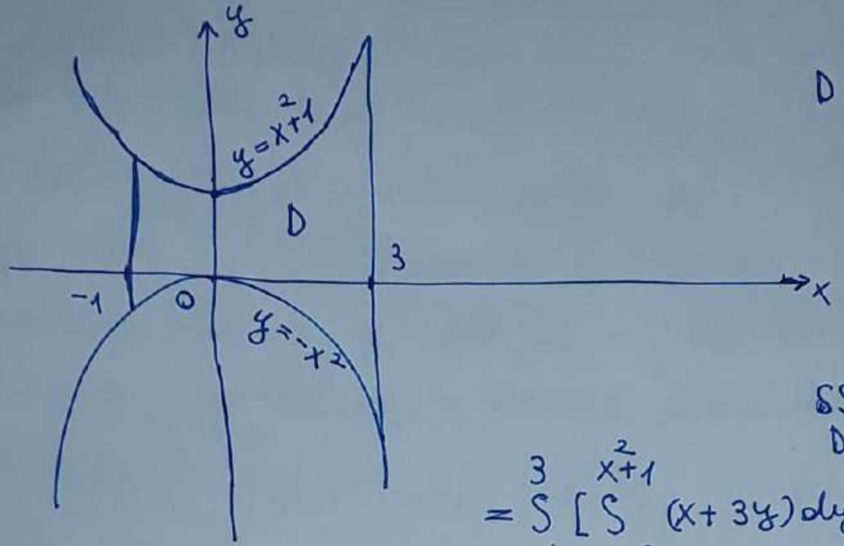
$\rho^2 \le 2\rho\sin\theta \Rightarrow \rho\in[0, 2\sin\theta]$

$$\iint_D \left(2+\sqrt{1+x^2+y^2}\right)dx\,dy =$$

$$= 2\,\text{aria}\,D + \int_0^{\pi}\Big(\int_0^{2\sin\theta} \underbrace{\underbrace{\rho\sqrt{1+\rho^2}}_{\frac{1}{2}(1+\rho^2)'(1+\rho^2)^{\frac{1}{2}}}}_{\frac{1}{2}\cdot\frac{(1+\rho^2)^{3/2}}{\frac{3}{2}}}\,d\rho\Big)\,d\theta = 2\pi + \int_0^{\pi}\frac{1}{3}(1+\rho^2)^{\frac{3}{2}}\Big|_0^{2\sin\theta}\,d\theta$$

$$= 2\pi + \frac{1}{3}\int_0^{\pi}\left[(1+4\sin^2\theta)^{\frac{3}{2}} - 1\right]d\theta$$

$$= \frac{5\pi}{3} + \int_0^{\pi}(1+4\sin^2\theta)^{\frac{3}{2}}\,d\theta.$$



$$D:\begin{cases}-1\le x\le 3\\ -x^2\le y\le x^2+1\end{cases}$$

$$\iint_D (x+3y)\,dx\,dy =$$

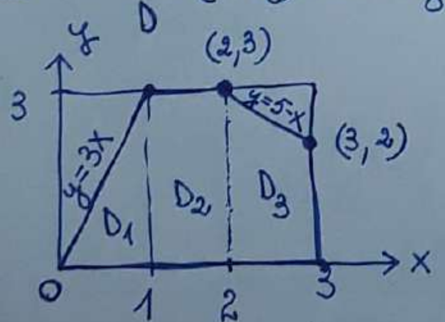
$$=\int_{-1}^{3}\left[\int_{-x^2}^{x^2+1}(x+3y)\,dy\right]dx=\int_{-1}^{3}\Big[x(2x^2+1)+\frac{3}{2}\left[(x^2+1)^2-x^4\right]\Big]dx=$$

$$=\int_{-1}^{3}\left[2x^3+x+\frac{3}{2}(2x^2+1)\right]dx=\int_{-1}^{3}\left(2x^3+x+3x^2+\frac{3}{2}\right)dx=\frac{1}{2}x^4\Big|_{-1}^{3}+\frac{1}{2}x^2\Big|_{-1}^{3}+$$

$$+x^3\Big|_{-1}^{3}+\frac{3}{2}\cdot 4=\frac{1}{2}(81-1)+\frac{1}{2}(9-1)+$$

$$+27+1+6=40+4+34=78.$$

⑧ $\iint_D (x+y)\,dx\,dy$; $D:\ x,y\in[0,3],\ y\le 3x,\ x+y\le 5.$



$$D_1:\begin{cases}0\le x\le 1\\ 0\le y\le 3x\end{cases}\qquad D_2:\begin{cases}1\le x\le 2\\ 0\le y\le 3\end{cases}\qquad D_3:\begin{cases}2\le x\le 3\\ 0\le y\le 5-x\end{cases}$$

$$\cdot\ \iint_{D_1}(x+y)\,dx\,dy=\int_0^1\left(\int_0^{3x}(x+y)\,dy\right)dx=$$

$$=\int_0^1\left(3x^2+\frac{9x^2}{2}\right)dx=x^3\Big|_0^1+\frac{3}{2}\cdot x^3\Big|_0^1=1+\frac{3}{2}=\frac{5}{2}.$$

$$\cdot\cdot\ \iint_{D_2}(x+y)\,dx\,dy=\int_1^2\left(\int_0^3(x+y)\,dy\right)dx=\int_1^2\left(3x+\frac{9}{2}\right)dx=\frac{3}{2}\cdot x^2\Big|_1^2+\frac{9}{2}=$$

$$=\frac{9}{2}+\frac{9}{2}=9.$$

$$\cdot\cdot\cdot\ \iint_{D_3}(x+y)\,dx\,dy=\int_2^3\left(\int_0^{5-x}(x+y)\,dy\right)dx=$$

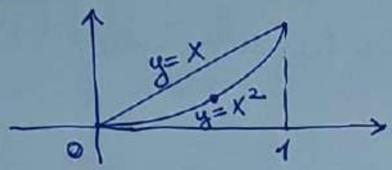
$$=\int_2^3\left(x(5-x)+\frac{(5-x)^2}{2}\right)dx=\int_2^3\left(\cancel{5x}-x^2+\frac{25}{2}-\cancel{5x}+\frac{x^2}{2}\right)dx=\int_2^3\left(\frac{25}{2}-\frac{x^2}{2}\right)dx=$$

$$=\frac{25}{2}-\frac{x^3}{6}\Big|_2^3=\frac{25}{2}-\frac{1}{6}\underbrace{(27-8)}_{19}=\frac{75-19}{6}=\frac{56}{6}=\frac{28}{3}.$$

(5) D: mărginit de curbele: $y = x$, $y = x^2$;

$$\iint_D (3x-y+2)\,dx\,dy$$

$$D: \begin{cases} 0 \le x \le 1 \\ x^2 \le y \le x \end{cases}$$

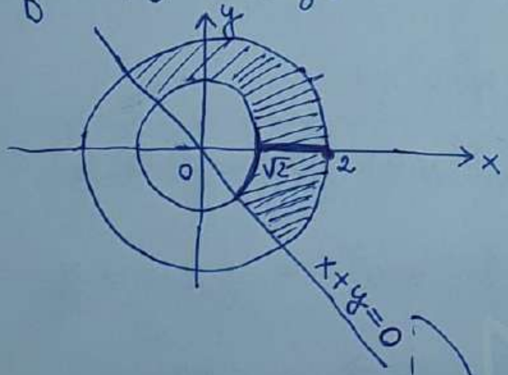


$$\iint_D (3x-y+2)\,dx\,dy = \int_0^1 \left[\int_{x^2}^{x} (3x-y+2)\,dy\right] dx =$$

$$= \int_0^1 \left[3x(x-x^2) - \frac{1}{2}(x^2-x^4) + 2(x-x^2)\right] dx = \int_0^1 \left(3x^2 - 3x^3 - \frac{x^2}{2} + \frac{x^4}{2} + 2x - 2x^2\right) dx = \int_0^1 \left(\frac{x^2}{2} + \frac{x^4}{2} - 3x^3 + 2x\right) dx = \frac{1}{6} + \frac{1}{10} - \frac{3}{4} + 1 =$$

$$= {}^{10/}\frac{1}{6} + {}^{6/}\frac{1}{10} + {}^{15/}\frac{1}{4} = \frac{31}{60}.$$

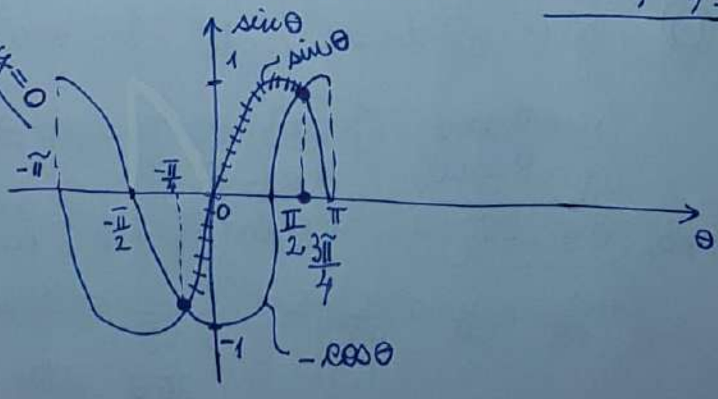
(6) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$; D: $2 \le x^2+y^2 \le 4$, $x+y \ge 0$.

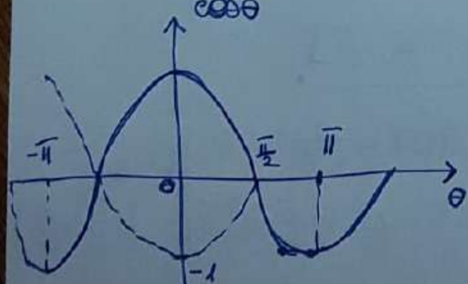


$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases} \qquad \theta \in [-\pi, \pi] \qquad \rho \in [\sqrt{2}, 2]$$

$\sin\theta + \cos\theta \ge 0 \Leftrightarrow$

$\sin\theta \ge -\cos\theta \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{3\pi}{4}\right]$





$$\iint_D \sqrt{x^2+y^2}\,dx\,dy = \left(\int_{\sqrt{2}}^{2} \rho^2\,d\rho\right)\left(\int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\theta\right) = \frac{\pi}{3}\,\rho^3\Big|_{\sqrt{2}}^{2} = \frac{\pi(8-2\sqrt{2})}{3} =$$

$$= \frac{2\pi}{3}(4-\sqrt{2}).$$

(7) $\iint_D (x+3y)\,dx\,dy$; D: mărginit de $y = x^2+1$, $y = -x^2$, $x = -1$, $x = 3$.