GABARAN

Formula Stokes

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EA	3.02	10^{00}
RST	3.02.	10^{00}
EM	7.02	10^{00}

Cursul 13

3 feb

02. februarie

10-12 ee1
12-14 ee2

Fie γ curbă închisă netedă care se sprijină pe o suprafață Σ regulată cu două fețe și fie $\bar{v} \in C^1(\Sigma)$. Atunci: $\oint_\gamma \bar{v}\, d\bar{r} = \iint_\Sigma \operatorname{rot} \bar{v} \cdot \bar{n}\, d\sigma$ unde fața lui Σ rămâne mereu în stânga sensului de parcurgere a curbei γ. Circulația lui $\bar{v}$ de-a lungul lui γ = fluxul rotorului lui $\bar{v}$ prin Σ.

1) $\oint_\gamma \bar{v}\, d\bar{r}$ unde: $\bar{v} = y\bar{i} + z\bar{j} + x\cdot\bar{k}$

$$\gamma: \begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$$

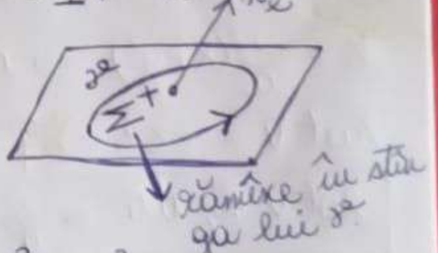
γ parcursă în sens direct (invers acelor de ceasornic).

Soluție: Alegem suprafața Σ porțiunea din planul $x+y+z=0$ decupată de sfera $x^2+y^2+z^2=a^2$, fața exterioară.

$$\Sigma: \Phi(x,y,z) = x+y+z=0 \longrightarrow \bar{n}_e = \frac{\operatorname{grad}\Phi}{\|\operatorname{grad}\Phi\|} = \frac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k})$$

$$z=-x-y \Rightarrow d\sigma = \sqrt{3}\, dx dy.$$

$$\operatorname{rot}\bar{v} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\bar{i}-\bar{j}-\bar{k}$$



$$\gamma: \begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases} \longrightarrow \gamma: \begin{cases} x^2+y^2+x^2+2xy+y^2=a^2 \\ x+y+z=0 \end{cases} \Rightarrow \gamma: \begin{cases} x^2+xy+y^2=\frac{a^2}{2} \\ x+y+z=0 \end{cases}$$

elipsa

$$\left(x+\frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}y\right)^2 = \frac{a^2}{2}; \quad \begin{cases} u = x+\frac{y}{2} \\ v = \frac{\sqrt{3}}{2}y \end{cases} \quad du\,dv = \frac{D(u,v)}{D(x,y)}\,dx dy =$$

$$= \begin{vmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{vmatrix} dx dy = \frac{\sqrt{3}}{2}\, dx dy \rightarrow \boxed{dx dy = \frac{2}{\sqrt{3}}\, du\, dv}$$

$$\operatorname{rot}\bar{v}\cdot \bar{n}_e\, d\sigma = -3\, dx dy.$$

$\oint_\gamma \bar v\, d\bar r = \iint_\Sigma \operatorname{rot}\bar v\cdot \bar n_e\, d\sigma = -3\iint_D dx\,dy$, unde

$D: \left(x+\frac{y}{2}\right)^2+\left(\frac{\sqrt3}{2}y\right)^2 \le \frac{a^2}{2}$ cu schimbarea de variabile

trece în discul $u^2+v^2\le \frac{a^2}{2}$.

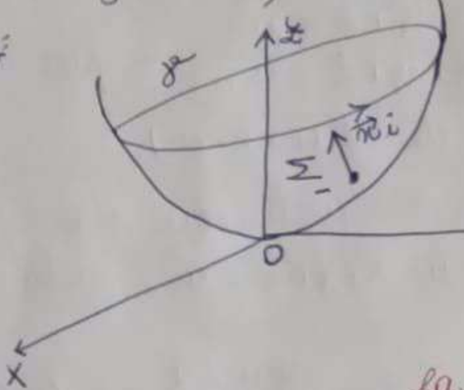
$\oint_\gamma \bar v\, d\bar r = -3\iint_{u^2+v^2\le \frac{a^2}{2}} \frac{2}{\sqrt3}\, du\,dv = -2\sqrt3\cdot$ aria $\left(u^2+v^2\le \frac{a^2}{2}\right) =$

$= -2\sqrt3\cdot \pi\frac{a^2}{2} = -\pi\sqrt3\, a^2.$

(2) $\bar v = (y-z)\bar i + (z-x)\bar j + (x-y)\bar k$

$\gamma: \begin{cases} x+y+z=a \\ x^2+y^2=az,\ a>0 \end{cases}$ $\quad\oint_\gamma \bar v\, d\bar r$, γ parcursă în sens direct.

Soluție:



$\oint_\gamma \bar v\, d\bar r = \iint_{\Sigma_-} \operatorname{rot}\bar v\cdot \bar n_i\, d\sigma$

Σ_- fața interioară a paraboloidului decupată de planul $x+y+z=a$ (Σ_- rămîne la stînga sensului de parcurgere).

$\Sigma_-: \Phi(x,y,z) = az - x^2 - y^2 \Rightarrow \bar n_i = \frac{1}{\sqrt{4x^2+4y^2+a^2}}\left(-2x\bar i - 2y\bar j + a\bar k\right).$

$z = \frac{1}{a}(x^2+y^2) \to d\sigma = \frac{1}{a}\sqrt{4x^2+4y^2+a^2}\, dx\,dy$

$\operatorname{rot}\bar v = -2\bar i - 2\bar j - 2\bar k \Rightarrow$

$\boxed{\operatorname{rot}\bar v\cdot \bar n_i\, d\sigma = \frac{2}{a}\left[2(x+y)-a\right]dx\,dy}$

$\gamma: \begin{cases} \left(x+\frac{a}{2}\right)^2 + \left(y+\frac{a}{2}\right)^2 = \frac{3}{2}a^2 \\ x+y+z=a \end{cases} \Rightarrow D: \left(x+\frac{a}{2}\right)^2 + \left(y+\frac{a}{2}\right)^2 \le \frac{3}{2}a^2$

proiecția pe xOy a domeniului delimitat de γ.

$\oint_\gamma \bar v\, d\bar r = \iint_D \frac{2}{a}\left[2(x+y)-a\right]dx\,dy \overset{*}{=}$

$\begin{cases} x = -\frac{a}{2} + \rho\cos\theta \\ y = -\frac{a}{2} + \rho\sin\theta \end{cases}$ $\quad\overset{*}{=} -2\,\text{aria}(D) + \frac{4}{a}\int_0^{2\pi}\int_0^{\sqrt{\frac{3}{2}}a}\left[\rho^2(\cos\theta+\sin\theta) - a\rho\right]d\rho\, d\theta =$

$= -\cancel{2}\pi\frac{3a^2}{\cancel{2}} - \cancel{4}\cdot 2\pi\cdot\frac{1}{\cancel{2}}\cdot\frac{3}{2}a^2 = -9\pi a^2.$

<u>Sau:</u> alegem Σ^+ fața exterioară a planului $\Rightarrow$

$$\oint_{\gamma} \bar{V}\, d\bar{r} = \iint_{\Sigma^+} \operatorname{rot} \bar{V} \cdot \bar{n}_e \, d\sigma.$$

$\Sigma^+ : x+y+z=a \rightarrow \Phi(x,y,z) = x+y+z-a \Rightarrow \operatorname{grad}\Phi = \bar{i}+\bar{j}+\bar{k}$; $\|\operatorname{grad}\Phi\| = \sqrt{3}$.

$$\left.\begin{array}{l} \bar{n}_e = \frac{1}{\sqrt{3}}(\bar{i}+\bar{j}+\bar{k}) \\ \operatorname{rot}\bar{V} = -2(\bar{i}+\bar{j}+\bar{k}) \end{array}\right| \Rightarrow \left.\begin{array}{l} \operatorname{rot}\bar{V}\cdot\bar{n}_e = -2\sqrt{3} \\ d\sigma = \sqrt{3}\, dx dy \end{array}\right| \Rightarrow \operatorname{rot}\bar{V}\cdot\bar{n}_e\, d\sigma = -6\, dx dy.$$

$$\oint_{\gamma} \bar{V}\, d\bar{r} = \iint_{\Sigma^+} \operatorname{rot}\bar{V}\,\bar{n}_e\, d\sigma = -6 \iint_{D} dx dy = -6\, \text{aria}(D) = -6\cdot\pi\cdot\frac{3}{2}a^2 = \underline{-9\pi a^2}.$$

<u>Direct:</u>

$$\gamma : \begin{cases} \left(x+\frac{a}{2}\right)^2 + \left(y+\frac{a}{2}\right)^2 = \frac{3}{2}a^2 \\ x+y+z=a \end{cases} \rightarrow \text{parametrizarea:}$$

$$\begin{cases} x = -\frac{a}{2} + \sqrt{\frac{3}{2}}\, a\cos\theta \\ y = -\frac{a}{2} + \sqrt{\frac{3}{2}}\, a\sin\theta \\ z = 2a - \sqrt{\frac{3}{2}}\, a(\cos\theta + \sin\theta) \end{cases} \rightarrow \begin{cases} dx = -\sqrt{\frac{3}{2}}\, a\sin\theta\, d\theta \\ dy = \sqrt{\frac{3}{2}}\, a\cos\theta\, d\theta \\ dz = \sqrt{\frac{3}{2}}\, a(\sin\theta - \cos\theta)\, d\theta \end{cases}$$

$\theta \in [0, 2\pi]$

$$\begin{cases} y - z = -\frac{5a}{2} + \sqrt{\frac{3}{2}}\, a(2\sin\theta + \cos\theta) \\ z - x = \frac{5a}{2} - \sqrt{\frac{3}{2}}\, a(\sin\theta + 2\cos\theta) \\ x - y = \sqrt{\frac{3}{2}}\, a(\cos\theta - \sin\theta) \end{cases}$$

$$\oint_{\gamma} \bar{V}\, d\bar{r} = \oint_{\gamma} (y-z)\,dx + (z-x)\,dy + (x-y)\,dz =$$

$$= \int_0^{2\pi} \Big\{\Big[-\frac{5a}{2} + \sqrt{\frac{3}{2}}\, a(2\sin\theta + \cos\theta)\Big]\Big(-\sqrt{\frac{3}{2}}\, a\sin\theta\Big) +$$

$$+ \Big[\frac{5a}{2} - \sqrt{\frac{3}{2}}\, a(\sin\theta + 2\cos\theta)\Big]\Big(\sqrt{\frac{3}{2}}\, a\cos\theta\Big) + \Big(\sqrt{\frac{3}{2}}\, a\Big)^2(\sin\theta - \cos\theta)^2\Big\}\, d\theta =$$

$$= -\int_0^{2\pi} \frac{3}{2}a^2 \cdot \underbrace{2\sin^2\theta}_{1-\cos 2\theta}\, d\theta - \int_0^{2\pi} \frac{3}{2}a^2 \cdot \underbrace{2\cos^2\theta}_{1+\cos 2\theta}\, d\theta - \int_0^{2\pi} + \frac{3}{2}a^2\, d\theta =$$

$$= -\frac{3a^2}{2}\cdot 2\pi - \frac{3a^2}{2}\cdot 2\pi - \frac{3a^2}{2}\cdot 2\pi = \underline{-9\pi a^2}.$$

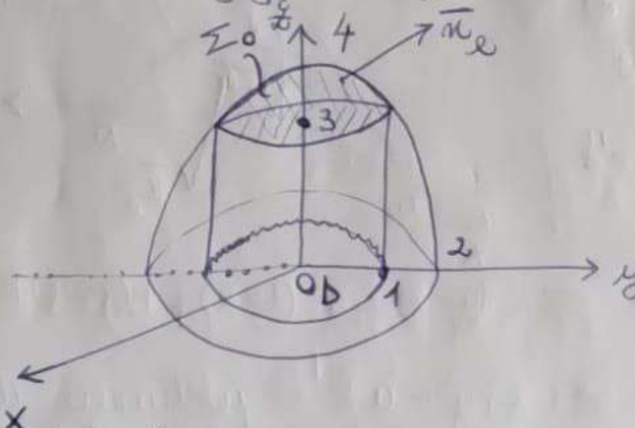
② **Formula Gauss** (flux - divergență)

| Ω mărginit și închis de Σ suprafață netedă, închisă |

Σ $\subset \mathbb{R}^3$ suprafață netedă închisă (care admite plan tangent în fiecare punct al său), Ω domeniul delimitat de Σ, $\bar{v}$ câmp vectorial de clasă C^1 pe $\Omega \cup \Sigma$. Atunci:

$$\mathcal{F}_{\Sigma_+}(\bar{v}) = \iint_{\Sigma_+} \bar{v} \cdot \bar{n}_e \, d\sigma = \iiint_{\Omega} \operatorname{div} \bar{v} \, dx\,dy\,dz.$$

Exemplu. Fie Σ_+ suprafața (închisă, exterioară) determinată de $x^2+y^2 = 4-z$, $z \geq 0$ și $x^2+y^2=1$, $z=3$, $\mathcal{F}_{\Sigma_+}(\bar{v}) = ?$ unde $\bar{v} = x^2\bar{i} + y^2\bar{j} + z^2\bar{k}$.



Σ_0 = porțiunea delimitată de cilindrul $x^2+y^2=1$ în paraboloidul $x^2+y^2=4-z$, $z \geq 0$; $\Sigma = \Sigma_0 \cup \{x^2+y^2=1, z=3\}$

$\operatorname{div} \bar{v} = 2(x+y+z)$.

$$\mathcal{F}_{\Sigma_+}(\bar{v}) = \iint_{\Sigma_+} \bar{v} \cdot \bar{n}_e \, d\sigma = \iiint_{\Omega} \operatorname{div} \bar{v} \, dx\,dy\,dz \stackrel{*}{=}$$

$$\boxed{\Omega : \begin{cases} 3 \leq z \leq 4 - x^2 - y^2 \\ pr_{xOy}\Omega : x^2+y^2 \leq 1 \end{cases}}$$

$$\stackrel{*}{=} \iint_{x^2+y^2\leq 1} \int_3^{4-x^2-y^2} 2(x+y+z)\,dz =$$

$$= \iint_{x^2+y^2\leq 1} 2(x+y)(1-x^2-y^2)\,dx\,dy + \iint_{x^2+y^2\leq 1} [(4-x^2-y^2)^2 - 9]\,dx\,dy =$$

$$= \int_0^1 2\rho^2(1-\rho^2)\,d\rho \int_0^{2\pi} (\sin\theta + \cos\theta)\,d\theta + 2\pi \int_0^1 \rho(4-\rho^2)^2\,d\rho - 9\pi =$$

$(\rho^2-4)^2$

$$= \frac{\pi}{3}(\rho^2-4)^3\Big|_0^1 - 9\cdot\pi = \frac{\pi}{3}(-27+64) - 9\pi = \frac{37}{3}\pi - 18\pi = \frac{10}{3}\pi.$$

(1) $\oint_C \bar v\, d\bar r$; C = curba $\begin{cases} x+y+z=2 \\ x^2+y^2=2x \end{cases}$ $\quad \bar v = x\bar i + y\bar j + \sqrt{x^2+y^2}\,\bar k$

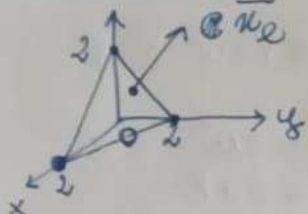
$\boxed{\text{Stokes}}$

Stokes: $\oint_C \bar v\, d\bar r = \iint_{\Sigma_+} \operatorname{rot} \bar v \cdot \bar n_e\, d\sigma$, Σ: porțiunea din planul $x+y+z=2$ decupată de cilindrul $x^2+y^2=2x$.

$\Sigma: \Phi(x,y,z)=0 \iff x+y+z-2=\Phi(x,y,z) \to \bar n_e = \frac{1}{\sqrt3}(\bar i+\bar j+\bar k)$

Alegem orientarea pozitivă a lui C, deci avem Σ_+ fața exterioară.

$z=-x-y \Rightarrow d\sigma = \sqrt3\, dx\, dy$.



$$\operatorname{rot} \bar v = \begin{vmatrix} \bar i & \bar j & \bar k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & \sqrt{x^2+y^2} \end{vmatrix} = \frac{y}{\sqrt{x^2+y^2}}\bar i - \frac{x}{\sqrt{x^2+y^2}}\bar j$$

$\operatorname{rot}\bar v \cdot \bar n_e\, d\sigma = \frac{y-x}{\sqrt{x^2+y^2}}\, dx\,dy \quad \begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$

$D = pr_{xOy}\Sigma: \ x^2+y^2 \le 2x$. $\quad \begin{cases} x \ge 0 \to \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 \le \rho \le 2\cos\theta \end{cases}$

$\oint_C \bar v\, d\bar r = \iint_D \frac{y-x}{\sqrt{x^2+y^2}}\, dx\,dy =$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{\rho^2(-\cos\theta+\sin\theta)}{\rho}\, d\rho\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta(\sin\theta - \cos\theta)\, d\theta$

$= -4\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta$; $\quad t = \cos^2\theta \to \boxed{\theta = \arccos\sqrt t}$

$d\theta = \frac{-1}{2\sqrt t\cdot\sqrt{1-t}}\, dt$;

$\oint_C \bar v\, d\bar r = -4\int_1^0 t^{\frac32} \cdot \frac{-1}{2\sqrt t\cdot \sqrt{1-t}}\, dt = -2\int_0^1 t(1-t)^{-\frac12}\, dt =$

$= -2\, B(2,\frac12) = -2\cdot\frac{\Gamma(2)\Gamma(\frac12)}{\Gamma(\frac52)} = -2\cdot\frac{\sqrt\pi}{\frac32\cdot\frac12\sqrt\pi} = -\frac83$.

(2) $\cancel{\iint_\Sigma (x^2+y^2+z^2)\, d\sigma}$, Σ suprafața închisă: $x^2+y^2+z^2=a^2$, $x\ge0, y\ge0, z\ge0$.

$\mathcal{F}_\Sigma(\bar v) = ?$ $\quad \bar v = x^2\bar i + y^2\bar j + z^2\bar k$

Gauss: $\mathcal{F}_\Sigma(\bar v) = \iint_\Sigma \bar v\cdot\bar n_e\, d\sigma = \iiint_\Omega \operatorname{div}\bar v\, dx\,dy\,dz$; $\ \Omega: x^2+y^2+z^2 \le a^2$, $x\ge0, z\ge0, y\ge0$.

$\mathcal{F}_\Sigma(\bar v) = \iiint_\Omega 2(x+y+z)\, dx\,dy\,dz$.

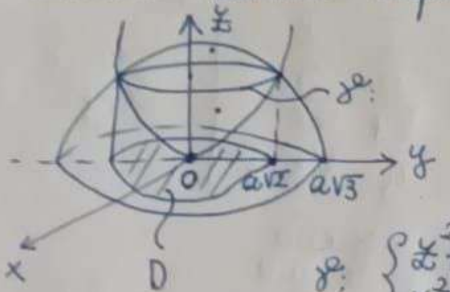
$\begin{cases} x=\rho\cos\theta\sin\varphi \\ y=\rho\sin\theta\sin\varphi \\ z=\rho\cos\varphi \end{cases}$ $\quad \theta\in[0,\frac{\pi}{2}]$, $\varphi\in[0,\frac{\pi}{2}]$, $\rho\in[0,a]$. $\quad dx\,dy\,dz = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi$

$\Phi_{\Sigma}(\vec V) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin^2\varphi\,(\cos\theta\sin\varphi + \sin\theta\sin\varphi + \cos\varphi)\left(\int_0^{a} \rho^3\, d\rho\right) d\theta\, d\varphi =$

$= \left(\int_0^{\frac{\pi}{2}} \underbrace{(\sin\theta+\cos\theta)}_{-\cos\theta+\sin\theta}\, d\theta\right)\left(\int_0^{\frac{\pi}{2}} \underbrace{\sin^2\varphi}_{\frac{1-\cos 2\varphi}{2}}\, d\varphi\right)\cdot\frac{a^4}{4} + \frac{a^4}{4}\cdot\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\, d\varphi =$

$= \underbrace{(\sin\theta-\cos\theta)\Big|_0^{\frac{\pi}{2}}}_{1+1=2}\cdot\frac{\pi}{4}\cdot\frac{a^4}{4} + \frac{\pi a^4}{16}\cdot\underbrace{\sin^2\varphi\Big|_0^{\frac{\pi}{2}}}_{1} = \frac{3\pi a^4}{16}.$

③ Volumul comun suprafețelor: $x^2+y^2+z^2=3a^2$, $x^2+y^2=2az$.



$\frac{x^2+y^2}{2a} \le z \le \sqrt{3a^2-x^2-y^2}$

$\gamma: \begin{cases} x^2+y^2+z^2=3a^2 \\ x^2+y^2 = 2az \end{cases} \Longleftrightarrow$

$\gamma: \begin{cases} z^2+2az-3a^2=0 \\ x^2+y^2=2az \end{cases}$ $\gamma: \begin{cases} x^2+y^2=2a^2 \\ z=a \end{cases}$

$\Omega:$ $\cancel{x^2+y^2+z^2=3a^2}$

$\begin{cases} \frac{x^2+y^2}{2a} \le z \le \sqrt{3a^2-x^2-y^2} \\ pr_{xOy}\,\Omega = D:\ x^2+y^2 \le 2a^2. \end{cases}$

Volumul $Vol(\Omega) = ?$ Ω partea comună al suprafețelor comun:

$Vol(\Omega) = \iint_D \left(\int_{\frac{x^2+y^2}{2a}}^{\sqrt{3a^2-x^2-y^2}} dz\right) dx\,dy = \iint_{x^2+y^2\le 2a^2}\left(\sqrt{3a^2-x^2-y^2} - \frac{x^2+y^2}{2a}\right) dx\,dy$

$= \int_0^{a\sqrt2}\int_0^{2\pi}\left[\rho(3a^2-\rho^2)^{\frac12} - \frac{\rho^3}{2a}\right] d\rho\, d\theta = 2\pi\left[-\frac12\,\frac{(3a^2-\rho^2)^{\frac32}}{\frac32}\Big|_0^{a\sqrt2} - \frac{(a\sqrt2)^4}{8a}\right] =$

$= 2\pi\left[-\frac13 a^3 + \frac13\cdot 3a^2\cdot a\sqrt3 - \frac{a^3}{2}\right] =$

$= 2\pi\left(a^3\sqrt3 - \frac56 a^3\right) = \frac{\pi}{3}a^3(6\sqrt3-5).$

① $\oint_{\gamma} \bar{V}\, d\bar{r}$, $\bar{v} = y\bar{i} + z\bar{j} + x\bar{k}$, $\gamma: \begin{cases} x^2+y^2+z^2=a^2 \\ x+y+z=0 \end{cases}$ ← cerc

$\oint_{\gamma} \bar{v}\, d\bar{r} = \iint_{\Sigma} rot\,\bar{v}\cdot \bar{n}_e\, d\sigma$ Stokes

$rot\,\bar{v} = -\bar{i}-\bar{j}-\bar{k}$; $\Sigma : x+y+z=0 \Rightarrow$

$\bar{n}_e = (\bar{i}+\bar{j}+\bar{k})\cdot\frac{1}{\sqrt{3}}$; $d\sigma = \sqrt{3}\, dx\,dy$.

$\bar{v}\cdot\bar{n}_e\, d\sigma = -3\, dx\,dy$.

$\gamma: \begin{cases} x^2+y^2+z^2=a^2 \\ z=-x-y \end{cases} \Rightarrow \gamma: \begin{cases} 2x^2+2xy+2y^2=a^2 \Leftrightarrow \\ x^2+xy+y^2=\frac{a^2}{2}. \end{cases} \Leftrightarrow \gamma: \left(x+\frac{y}{2}\right)^2+\left(\frac{\sqrt{3}}{2}y\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2.$

$D = pr_{xOy}\Sigma : \left(x+\frac{y}{2}\right)^2+\left(\frac{\sqrt{3}}{2}y\right)^2 \le \left(\frac{a}{\sqrt{2}}\right)^2.$

$\oint_{\gamma} \bar{V}\, d\bar{r} = \iint_{\Sigma} rot\,\bar{v}\cdot\bar{n}_e\, d\sigma = -3\iint_{D} dx\,dy.$

Schimbarea de variabile: $\begin{cases} u = x+\frac{y}{2} \\ v = \frac{\sqrt{3}}{2}y \end{cases} \Rightarrow dx\,dy = \left|\frac{D(x,y)}{D(u,v)}\right| du\,dv$

$= \frac{1}{\left|\frac{D(u,v)}{D(x,y)}\right|}\, du\,dv = \frac{du\,dv}{\begin{vmatrix} 1 & \frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{vmatrix}} = \frac{2}{\sqrt{3}}\, du\,dv.$

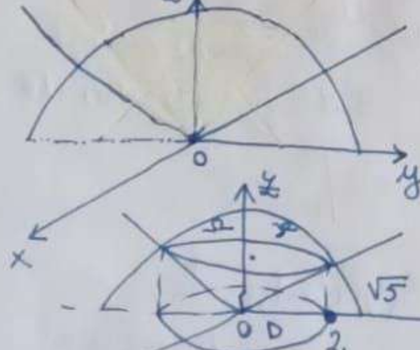
$D \longrightarrow u^2+v^2 \le \left(\frac{a}{\sqrt{2}}\right)^2$

$\oint_{\gamma} \bar{v}\, d\bar{r} = -3\iint_{u^2+v^2\le\left(\frac{a}{\sqrt{2}}\right)^2} \frac{2}{\sqrt{3}}\, du\,dv = -2\sqrt{3}\ \text{aria}\left\{u^2+v^2\le\left(\frac{a}{\sqrt{2}}\right)^2\right\} =$

$= -2\sqrt{3}\cdot\pi\frac{a^2}{2} = -\pi a^2\sqrt{3}.$ definită

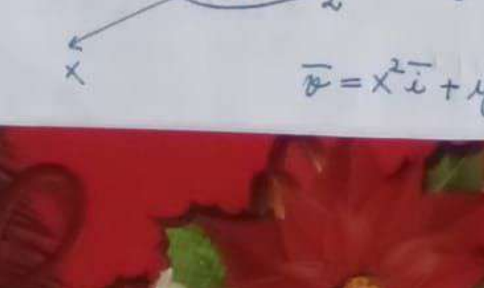
② Considerăm suprafața închisă Σ delimitată de: $2z=\sqrt{x^2+y^2}$ și $5-z=x^2+y^2$. Fie $\bar{v} = x^3\bar{i}+y^3\bar{j}+z\bar{k}$. Calculați $\mathcal{F}_{\Sigma}(\bar{v})$.

$\mathcal{F}_{\Sigma}(\bar{v}) = \iint_{\Sigma} \bar{V}\cdot\bar{n}_e\, d\sigma \underset{Gauss}{=\!=} \iiint_{\Omega} div\,\bar{v}\, dx\,dy\,dz \overset{*}{\Rightarrow}$



$\gamma: \begin{cases} 2z=\sqrt{x^2+y^2} \\ 5-z=x^2+y^2 \end{cases} \Rightarrow \begin{cases} 5-z=4z^2 \\ 2z=\sqrt{x^2+y^2} \end{cases} \Rightarrow$

$\begin{cases} 4z^2+z-5=0,\ z\ge 0 \\ x^2+y^2=4z^2 \end{cases} \Rightarrow \gamma: \begin{cases} z=1 \\ x^2+y^2=4 \end{cases}$ cerc



$\Omega: \begin{cases} \frac{1}{2}\sqrt{x^2+y^2} \le z \le 5-x^2-y^2 \\ pr_{xOy}\Omega : x^2+y^2 \le 4 \end{cases}$

$div\,\bar{v} = 3x^2+3y^2+1 \Rightarrow$

$\bar{v} = x^2\bar{i}+y^2\bar{j}+z\bar{k}.$

$\mathcal{F}_\Sigma(\bar v) = \iint_\Sigma \bar v \cdot \bar n_2 \, d\sigma = \iiint_\Omega [3(x^2+y^2)+1] \, dx\, dy =$

$= \iint_{x^2+y^2\le 4} \left\{ \int_{\frac{1}{2}\sqrt{x^2+y^2}}^{5-x^2-y^2} [3(x^2+y^2)+1] \, dz \right\} dx\, dy =$

$= \iint_{x^2+y^2\le 4} [3(x^2+y^2)+1](5-x^2-y^2-\frac{1}{2}\sqrt{x^2+y^2}) \, dx\, dy = 2\pi \int_0^2 \rho(3\rho^2+1)(5-\frac{\rho}{2}-\rho^2)\, d\rho$

$= 2\pi \int_0^2 (3\rho^3+\rho)(5-\frac{\rho}{2}-\rho^2)\, d\rho = 2\pi \int_0^2 (15\rho^3 - \frac{3}{2}\rho^4 - 3\rho^5 + 5\rho - \frac{\rho^2}{2} - \rho^3)\, d\rho =$

$= 28\pi \cdot \frac{2^4}{4} - 3\pi \cdot \frac{2^5}{5} - \frac{3}{6} 2\pi\, 2^6 + \frac{5}{2} \cdot 2\pi \cdot 2^2 - 2\pi \cdot \frac{1}{6}2^3 =$

$= 4\pi \cdot 2^4 - \frac{96}{5}\pi - 32\pi + 20\pi - \frac{8\pi}{3}.$

$\bar v = x^2 \bar i + y^2 \bar j + z \bar k \to \operatorname{div} \bar v = 2x + 2y + 1 = 2(x+y)+1.$

$\mathcal{F}_2(\bar v) = \iint_{x^2+y^2\le 4} [2(x+y)+1] \int_{\frac{1}{2}\sqrt{x^2+y^2}}^{5-x^2-y^2} dz =$

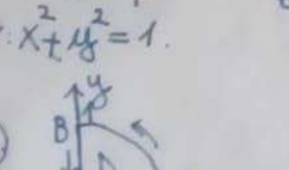
$= \iint_{x^2+y^2\le 4} 2(x+y)(5-x^2-y^2-\frac{1}{2}\sqrt{x^2+y^2})\, dx\,dy + \iint_{x^2+y^2\le 4} (5-x^2-y^2-\frac{\sqrt{x^2+y^2}}{2})\, dx\,dy$

$= \underbrace{\int_0^{2\pi} 2(\cos\theta+\sin\theta)\, d\theta}_{0} \cdot \int_0^2 \rho(5-\rho^2-\frac{\rho}{2})\, d\rho + 2\pi \int_0^2 (5\rho - \rho^3 - \frac{\rho^2}{2})\, d\rho =$

$= 5\pi \rho^2 \Big|_0^2 - \frac{2\pi}{4} \rho^4 \Big|_0^2 - \pi \frac{\rho^3}{3}\Big|_0^2 = 20\pi - 8\pi - \frac{8\pi}{3} = 20\pi - \frac{32}{3}\pi = \frac{28}{3}\pi.$

③ $\oint_\gamma \underbrace{-y^3}_{P} dx + \underbrace{x^3}_{Q} dx = \iint_{x^2+y^2\le 1} 3(x^2+y^2)\, dx\,dy =$ $\boxed{= 3a^2 \pi \frac{\rho^2}{2}\Big|_0^a - \pi \cdot 3 \cdot \frac{\rho^4}{4}\Big|_0^a = \frac{3}{2}\pi a^4 - \frac{3\pi}{4}a^4 = \frac{3\pi}{4}a^4}$

$\gamma: x^2+y^2=1.$ $= 6\pi \int_0^1 \rho^3\, d\rho = \frac{6\pi}{4} = \frac{3\pi}{2}.$

④ 

$\oint_\gamma \underbrace{(x^3-3xy^2)}_{P} dx + \underbrace{(3x^2y-y^3)}_{Q} dy =$

$= \iint_{\substack{x^2+y^2\le 1\\ x\ge 0,\, y\ge 0}} 12xy\, dx\,dy = 12 \int_0^1 \rho^3 d\rho \int_0^{\frac{\pi}{2}} \sin\theta \underbrace{\cos\theta}_{(\sin\theta)'} d\theta =$

$= 3 \cdot \rho^4 \Big|_0^1 \cdot \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}} = \frac{3}{2}.$

⑤ $\gamma = \operatorname{Fr} D$, $D: x^2+y^2-2ay<0,\ y\ge a$

$\oint_\gamma \underbrace{3x^2y}_{P} dx + \underbrace{3xy(2a-y)}_{Q} dy = 3\iint_D (2ay-x^2-y^2)^{1/2} dx\,dy \overset{*}{=}$

$D: x^2+(y-a)^2<a^2,\ y\ge a;$ $\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta + a \end{cases}$ $D \to [0,a]\times[0,\pi]$

$\theta\in[0,2\pi],\ \rho\in[0,a]$

$\overset{*}{=} 3 \int_0^\pi \int_0^a \rho(a^2-\rho^2)\, d\rho = *$

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$\bar{v} = P(x,y)\bar{i} + Q(x,y)\bar{j} \Rightarrow \int_C P\,dx + Q\,dy = \underbrace{\int_C \bar{v}\cdot d\bar{r}}_{\parallel}$

$d\bar{r} = dx\,\bar{i} + dy\,\bar{j}$

$\Downarrow$

$\bar{v}\cdot d\bar{r} = P\,dx + Q\,dy$

$\downarrow$

produs scalar

circulatia câmpului $\bar{v}$ de-a lungul curbei C.

Aplicatie. Folosind Green - Riemann calculati

$$I = \int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} e^{\frac{x^2}{a^2}+\frac{y^2}{b^2}}(-y\,dx + x\,dy).$$

Solutie: $I = \int_{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1} e^{\frac{x^2}{a^2}+\frac{y^2}{b^2}}(-y\,dx + x\,dy) = e\int_C -y\,dx + x\,dy$

$C: \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1.$ $\qquad P(x,y) = -y$ $\qquad Q(x,y) = x$ $\Rightarrow \frac{\partial P}{\partial y} = -1$, $\frac{\partial Q}{\partial x} = 1$ $\Rightarrow$

$D: \frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2.$

$I = \text{Green - Riemann} = e\iint_D 2\,dx\,dy.$

Coordonate polare generalizate: $\begin{cases} x = a\rho\cos\theta & \rho\in[0,1] \\ y = b\rho\sin\theta & \theta\in[0,2\pi] \end{cases}$

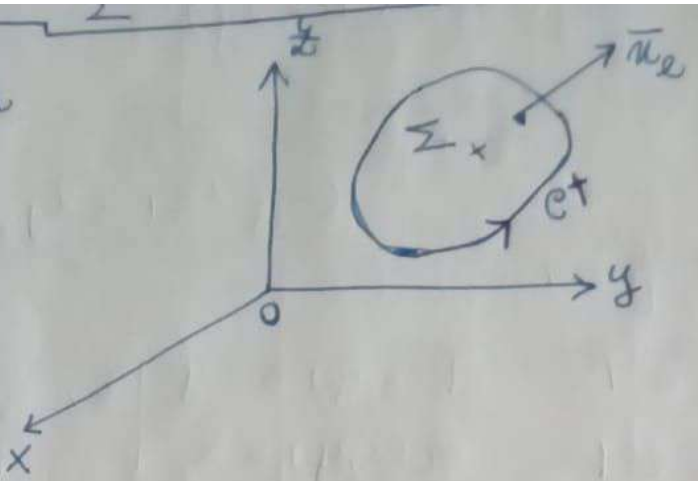
$dx\,dy = ab\rho\,d\rho\,d\theta$

$I = 2e\int_0^1\int_0^{2\pi} ab\rho\,d\rho\,d\theta \overset{\text{Fubini}}{=\!=} ab\left(\int_0^1 2\rho\,d\rho\right)\cdot\left(\int_0^{2\pi} d\theta\right)\cdot e =$

$= eab\cdot \rho^2\Big|_0^1 \cdot 2\pi = 2\pi abe\ ;\ e \simeq 2,7.$

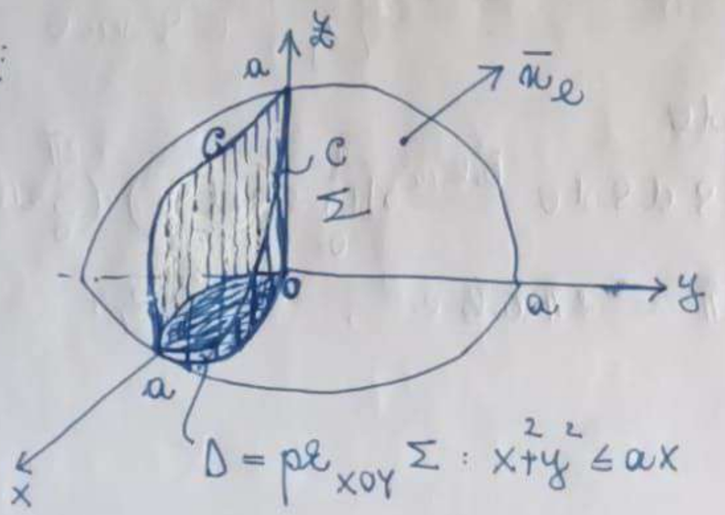
C^+ Σ

circulația



Aplicație: Calculați circulația lui $\bar{v} = y^2\bar{i} + z^2\bar{j} + x^2\bar{k}$ de-a lungul lui C definită prin intersecția suprafețelor $x^2+y^2+z^2=a^2$, $x^2+y^2=ax$, $z \geq 0$; C este parcursă în sens pozitiv relativ la normala exterioară a semisferei superioare.

Soluție:



C trece prin punctul $(a,0,0)$, respectiv prin punctul $(0,0,a)$

Considerăm Σ porțiunea exterioară din semisfera superioară decupată de cilindrul $x^2+y^2=ax$.

$\Sigma: \Phi(x,y,z)=x^2+y^2+z^2-a^2=0 \to \overline{n}_e = \frac{x}{a}\overline{i}+\frac{y}{a}\overline{j}+\frac{z}{a}\overline{k}$

$$\operatorname{rot}\overline{v} = \begin{vmatrix} \overline{i} & \overline{j} & \overline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & z^2 & x^2 \end{vmatrix} = -2(z\overline{i}+x\overline{j}+y\overline{k})$$

$d\sigma = \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy$ (e cunoscut $d\sigma$ pentru sfera $x^2+y^2+z^2=a^2$)

Pe semisfera superioară $z=\sqrt{a^2-x^2-y^2}$, deci:

$$\operatorname{rot}\overline{v}\cdot\overline{n}_e\, d\sigma = -\frac{2}{a}(z\overline{i}+x\overline{j}+y\overline{k})(x\overline{i}+y\overline{j}+z\overline{k})\frac{a\,dxdy}{\sqrt{a^2-x^2-y^2}} =$$

$$= -2\left(x+y+\frac{xy}{\sqrt{a^2-x^2-y^2}}\right)dxdy.$$

Cu Stokes avem: $\int_C \overline{v}\,d\overline{r} = \iint_\Sigma \operatorname{rot}\overline{v}\cdot\overline{n}_e\,d\sigma =$

$$= \iint_{pr_{xOy}\Sigma} -2\left(x+y+\frac{xy}{\sqrt{a^2-x^2-y^2}}\right)dxdy = -2\iint_{x^2+y^2\le ax}(x+y)\,dxdy -$$

$$-2\iint_{x^2+y^2\le ax}\frac{xy}{\sqrt{a^2-x^2-y^2}}dxdy.$$

$\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$ $x\ge 0 \Rightarrow \theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$x^2+y^2\le ax \to \rho\le a\cos\theta \Rightarrow \rho\in[0,a\cos\theta]$

$dxdy=\rho\,d\rho\,d\theta.$

$$\iint_{x^2+y^2\le ax}(x+y)\,dxdy = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\int_0^{a\cos\theta}\rho^2 d\rho\right)(\sin\theta+\cos\theta)\,d\theta =$$

$$= \frac{a^3}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\underbrace{\sin\theta\cdot\cos^3\theta}_{\text{impară}}+\cos^4\theta)\,d\theta = \frac{a^3}{3}\cdot 2\int_0^{\frac{\pi}{2}}(\cos^4\theta)\,d\theta.$$

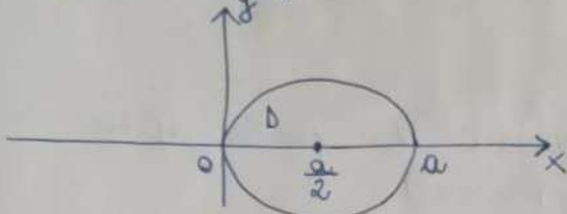
$t=\cos^2\theta \Rightarrow \theta=\arccos\sqrt{t} \Rightarrow d\theta=\frac{-dt}{2\sqrt{t}\cdot\sqrt{1-t}}$; $\left.\begin{matrix}\theta=0\to t=1 \\ \theta=\frac{\pi}{2}\to t=0\end{matrix}\right| \Rightarrow$

$$\int_0^{\frac{\pi}{2}}(\cos^4\theta)\,d\theta = \frac{1}{2}\int_0^1 t^2\cdot t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \frac{1}{2}\int_0^1 t^{\frac{3}{2}}(1-t)^{-\frac{1}{2}}dt =$$

$$= \frac{1}{2} B\left(\frac{5}{2}, \frac{1}{2}\right) = \frac{1}{2} \cdot \frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{\Gamma(3)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2\cdot 2!} = \frac{3\pi}{16}.$$

$$\Rightarrow \iint_{x^2+y^2\le ax} (x+y)\,dx\,dy = \frac{2a^3}{3}\cdot\frac{3\pi}{16} = \frac{\pi a^3}{8}.$$

$\iint_{x^2+y^2\le ax} \frac{xy}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$ o calculăm direct. $x^2+y^2 \le ax \iff \left(x-\frac{a}{2}\right)^2 + y^2 \le \left(\frac{a}{2}\right)^2$



$D: \begin{cases} 0 \le x \le a \\ -\sqrt{ax-x^2} \le y \le \sqrt{ax-x^2} \end{cases}$ simplu în raport cu Oy

$$\iint_{x^2+y^2\le ax} \frac{xy}{\sqrt{a^2-x^2-y^2}}\,dx\,dy = \int_0^a x\left(\underbrace{\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \frac{y}{\sqrt{a^2-x^2-y^2}}\,dy}\right)dx = 0.$$

integrală dintr-o funcție impară în y pe un domeniu simetric față de origine $= 0$

direct

$$\int_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} \frac{y}{\sqrt{a^2-x^2-y^2}}\,dy = -\sqrt{a^2-x^2-y^2}\,\Big|_{-\sqrt{ax-x^2}}^{\sqrt{ax-x^2}} = -\sqrt{a^2-x^2-ax+x^2} + \sqrt{a^2-x^2-ax+x^2} = -\sqrt{a^2-ax} + \sqrt{a^2-ax} = 0.$$

Deci: $\oint_C \bar{v}\,d\bar{r} = -2\cdot\frac{\pi a^3}{8} - 2\cdot 0 = -\frac{\pi a^3}{4}.$

③ Formula lui Gauss-Ostrogradski (Gauss)

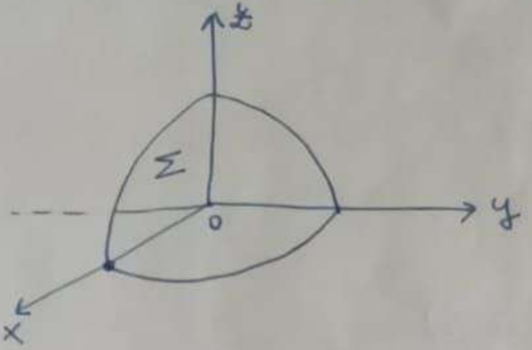
Fie $\Omega \subset \mathbb{R}^3$ compact (domeniu închis și mărginit) cu $\partial\Omega = Fr\,\Omega = \Sigma$ suprafață închisă, netedă.
Fie $\bar{v} = P\bar{i} + Q\bar{j} + R\bar{k}$ câmp vectorial de clasă C^1 pe Ω. Atunci fluxul lui $\bar{v}$ prin Σ este egal cu integrala divergenței lui $\bar{v}$ pe domeniul Ω:

$$\boxed{F_\Sigma(\bar{v}) = \iint_\Sigma \bar{v}\cdot\bar{n}_e\,d\sigma = \iiint_\Omega \operatorname{div}\bar{v}\,dx\,dy\,dz}$$ (formula flux-divergență)

Aplicație.

Fie $\bar{v} = xyz(x\bar{i} + y\bar{j} + z\bar{k})$. Calculați $\mathcal{F}_{\Sigma}(\bar{v})$, unde

$\Sigma = \{(x,y,z) \mid x^2+y^2+z^2 = a^2,\ x \geq 0,\ y \geq 0,\ z \geq 0\}$



Σ închisă

$\Omega: x^2+y^2+z^2 \leq a^2,\ x \geq 0,\ y \geq 0,\ z \geq 0$

cu $Fr\,\Omega = \Sigma$ (Ω domeniul închis de Σ).

$\bar{v} = x^2yz\,\bar{i} + xy^2z\,\bar{j} + xyz^2\,\bar{k}$ câmp vectorial de clasă C^1 pe Ω; $\operatorname{div}\bar{v} = (x^2yz)'_x + (xy^2z)'_y + (xyz^2)'_z = 2xyz + 2xyz + 2xyz = 6xyz$.

Cu formula Gauss:

$$\mathcal{F}_{\Sigma}(\bar{v}) = \iint_{\Sigma} \bar{v}\cdot\bar{n}_e\, d\sigma = \iiint_{\Omega} \operatorname{div}\bar{v}\, dx\,dy\,dz.$$

Coordonate sferice: $\begin{cases} x = \rho\cos\theta\sin\varphi \\ y = \rho\sin\theta\sin\varphi \\ z = \rho\cos\varphi \end{cases}$

$\rho \in [0, a]$

$\theta \in [0, \cancel{2}\frac{\pi}{2}]$ $\left(\begin{matrix} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{matrix}\right)$

$\varphi \in [0, \frac{\pi}{2}]$

$dx\,dy\,dz = \rho^2\sin\varphi\, d\rho\, d\theta\, d\varphi.$

$$\mathcal{F}_{\Sigma}(\bar{v}) = \iiint_{\Omega} 6xyz\, dx\,dy\,dz = 6\int_0^a\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \rho^5 \sin\theta\cos\theta\cdot\cos\varphi\sin^3\varphi\, d\rho\, d\theta\, d\varphi =$$

$$\overset{\text{Fubini}}{=} 6\left(\int_0^a 6\rho^5 d\rho\right)\cdot\left(\int_0^{\frac{\pi}{2}} \sin\theta\cdot\underbrace{\cos\theta}_{(\sin\theta)'}\, d\theta\right)\cdot\left(\int_0^{\frac{\pi}{2}} \sin^3\varphi\cdot\underbrace{\cos\varphi}_{(\sin\varphi)'}\, d\varphi\right) =$$

$$= \rho^6\Big|_0^a \cdot \frac{\sin^2\theta}{2}\Big|_0^{\frac{\pi}{2}} \cdot \frac{\sin^4\varphi}{4}\Big|_0^{\frac{\pi}{2}} =$$

$$= a^6\cdot\frac{1}{2}\cdot\frac{1}{4} = \frac{a^6}{8}.$$

Refaceți figura de la <u>Stokes</u>