MATEMATICI SPECIALE

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Cuprins

0.1 Cuvânt înainte

Cartea de faţă cuprinde programa matematică predată în cadrul cursului de matematici speciale susţinut studenţilor facultăţilor cu profil mecanic şi electric din Universitatea din Piteşti.

În carte sunt prezentate rezultate importante din teoria funcţiilor de o variabilă complexă, a funcţiilor speciale, precum şi cele mai utile metode de aplicare în practică a transformărilor integrale.

Ultimul capitol al cărţii tratează aspectele iniţiale ale teoriei ecuaţiilor cu derivate parţiale de ordinul al doilea, fiind completat de un număr mare de probleme rezolvate, foarte utile în aplicaţiile inginereşti atât mecanice cât şi electrice.

Autorii

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Capitolul 1

Analiză complexă

1.1 Corpul numerelor complexe

1.1.1 Construcţia numerelor complexe

Fie $\mathbb{R}$ corpul numerelor reale. Pe produsul cartezian $\mathbb{R} \times \mathbb{R} = \{(x, y) | x, y \in \mathbb{R}\}$ (notat şi cu $\mathbb{C}$) introducem două operaţii: *adunarea* $(+)$ şi *înmulţirea* $(\cdot)$ definite prin: dacă $z = (x, y)$ şi $z' = (x', y') \in \mathbb{C}$ atunci:
$z + z' = (x + x', y + y')$ şi $z \cdot z' = (x \cdot x' - y \cdot y', x \cdot y' + x' \cdot y)$.

1. **Adunarea** are proprietăţile:

   (a) $(z + z') + z'' = z + (z' + z''), (\forall) z, z', z'' \in \mathbb{C}$ (asociativitatea)

   (b) $z + z' = z' + z, (\forall) z, z' \in \mathbb{C}$ (comutativitatea)

   (c) pentru $0 = (0, 0) \in \mathbb{C}$ avem $z + 0 = 0 + z = z, (\forall) z \in \mathbb{C}$ (existenţa elementului neutru)

   (d) $(\forall) z = (x, y) \in \mathbb{C}, (\exists) -z = (-x, -y) \in \mathbb{C}$ astfel încât $z + (-z) = (-z) + z = 0$ (existenţa elementului opus)

2. **Înmulţirea** are proprietăţile:

   (a) $(z \cdot z') \cdot z'' = z \cdot (z' \cdot z''), (\forall) z, z', z'' \in \mathbb{C}$(asociativitatea)

   (b) $z \cdot z' = z' \cdot z, (\forall) z, z' \in \mathbb{C}$ (comutativitatea)

   (c) pentru $1 = (1, 0) \in \mathbb{C}$ avem $z \cdot 1 = 1 \cdot z = z, (\forall) z \in \mathbb{C}$ (existenţa elementului neutru sau unitate)

   (d) $(\forall) z \in \mathbb{C}^*, z = (x, y) \in \mathbb{C} \backslash \{0\}, (\exists) z^{-1} = \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right) \in \mathbb{C}$ (notat şi $\frac{1}{z}$) astfel încât $z \cdot z^{-1} = z^{-1} \cdot z = 1$ (existenţa elementului invers).

3. **Înmulţirea este distributivă faţă de adunare.** $z \cdot (z' + z'') = z \cdot z' + z \cdot z''$ şi $(z + z') \cdot z'' = z \cdot z'' + z' \cdot z'', (\forall) z, z', z'' \in \mathbb{C}$.

Deci $(\mathbb{C}, +, \cdot)$ este corp comutativ.

Mulţimea numerelor complexe se extinde prin introducerea unui singur punct la infinit, notat cu simbolul ∞.

Notăm $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ extinderea.

Definim: $z + \infty = \infty + z = \infty, (\forall) z \in \mathbb{C}$ şi $z \cdot \infty = \infty \cdot z = \infty, (\forall) z \in \mathbb{C} \backslash \{0\}$. Fără a considera că există ∞^{-1} vom defini: $\frac{z}{\infty} = 0, (\forall) z \in \mathbb{C}$ şi $\frac{z}{0} = \infty, (\forall) z \in \mathbb{C} \backslash \{0\}$.

Considerăm că $\infty \cdot \infty = \infty$; nu se definesc operaţiile $\infty + \infty$, $\infty - \infty$, $0 \cdot \infty$.

Observaţia 1.1 Fie $i = (0, 1)$, atunci:

1. $i^2 = -1$ şi $(x, y) = x + iy$

2. pentru $z = (x, y) \in \mathbb{C}$ avem scrierea algebrică $z = x + iy$.

3. notăm $x = Rez$(partea reală a lui z) şi $y = Imz$(partea imaginară a lui z)

Definiţia 1.2 Fie $z = x + iy \in \mathbb{C}$, atunci: numărul complex $\bar{z} = x - iy$ se numeşte *conjugatul* numărului complex z) şi numărul real pozitiv $|z| = \sqrt{x^2 + y^2}$ se numeşte *modulul* numărului complex z.

Observaţia 1.3 Proprietăţile conjugatului:

1. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$; $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$, $(\forall) z_1, z_2 \in \mathbb{C}$

2. $\bar{\bar{z}} = z, (\forall) z \in \mathbb{C}$

3. $z = \bar{z} \Leftrightarrow Imz = 0 \Leftrightarrow z \in \mathbb{R}$

Observaţia 1.4 Proprietăţile modulului:

1. $|z| = 0 \Leftrightarrow z = 0$; $|z \cdot z'| = |z| \cdot |z'|$; $||z| - \mid z'|| \leq |z + z'| \leq |z| + |z'|, (\forall) z, z' \in \mathbb{C}$

2. $|z| = |\bar{z}|$, $|z|^2 = z \cdot \bar{z}$.

1.1.2 Reprezentarea geometrică a numerelor complexe

Fie π un plan în care fixăm un sistem de axe ortogonale (xOy). Fiecărui număr complex $z = x + iy$ îi corespunde punctul $M(x, y)$.

1. M se numeşte imaginea geometrică a numărului complex z;

2. z se numeşte afixul punctului M;

3. Funcţia $z = x + iy \to M(x, y)$ este o bijecţie între $\mathbb{C}$ şi π;

4. π se numeşte planul complex;

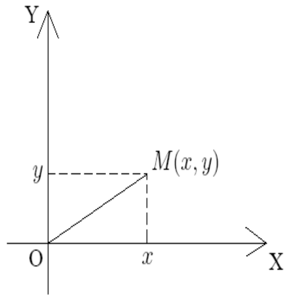

Figura 1.

5. z poate fi reprezentat şi prin vectorul legat $\overline{OM}$.

Observaţia 1.5 Fie $z = x + iy \in \mathbb{C}\setminus\{0\}$ şi M imaginea sa geometrică. Atunci:

1. $|z| = |OM|$
2. simetricul lui M faţă de axa Ox este punctul M' de afix $\bar{z}$;
3. simetricul lui M faţă de originea axelor O are afixul $-z$.

Observaţia 1.6 Fie z şi $z' \in \mathbb{C}$ ale căror imagini geometrice sunt M şi M'. Fie $S \in \pi$ astfel încât $OMSM'$ să fie paralelogram. Atunci:

1. S este imaginea geometrică a numărului complex $z + z'$.
2. $\overline{MM'}$ este convergent cu vectorul asociat numărului $z-z'$.

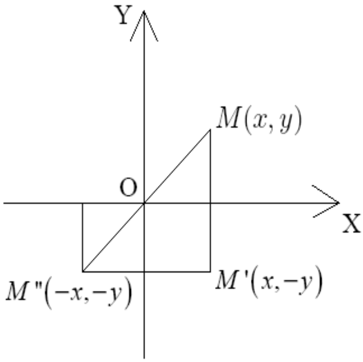

Figura 2.

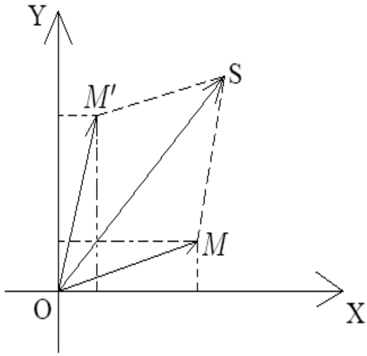

Figura 3.

1.1.3 Reprezentarea trigonometrică şi exponenţială a numerelor complexe

Fie $z = x + iy \in \mathbb{C}\setminus\{0\}$ şi M imaginea sa geometrică. Notăm:

1. $r = |OM| = |z|$ raza polară a imaginii lui z;

2. $\theta = \sphericalangle(\overrightarrow{Ox}, \overrightarrow{OM}) \in [0, 2\pi)$ argumentul redus al lui z, notat şi $\arg z$.

Atunci:

$$x = r\cos\theta,\ y = r\sin\theta,$$

unde

$$\begin{cases} z = r(\cos\theta + i\sin\theta) \text{ (forma trigonometrică)} \\ z = r\cdot e^{i\theta} \text{ (forma exponenţială).} \end{cases}$$

Din dezvoltarea în serie de puteri pentru $\sin\theta$, cosθ, e$^{i\theta}$ avem: $\cos\theta + i\sin\theta = e^{i\theta}$ (formula lui Euler).

Observaţia 1.7 Deoarece funcţiile sin, cos au perioada 2π,

$$z = r(\cos t + i\sin t),\ (\forall)t \in \arg z = \{\arg z + 2\pi : k \in \mathbb{Z}\}.$$

Observaţia 1.8 Dacă $z_1 = r_1(\cos t_1 + i\sin t_1)$, $z_2 = r_2(\cos t_2 + i\sin t_2)$, atunci:

1. $z_1\cdot z_2 = r_1r_2(\cos(t_1+t_2) + i\sin(t_1+t_2))$
2. $z_1^n = r_1^n(\cos nt_1 + i\sin nt_2),\ (\forall)n \in \mathbb{N}$
3. $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(t_1-t_2) + i\sin(t_1-t_2))$

Observaţia 1.9 Pentru $n \in \mathbb{N}$, $n \geq 2$, $z \in \mathbb{C}$ date ecuaţia $z^n = z$ are n rădăcini distincte:

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\arg z + 2k\pi}{n} + i\sin\frac{\arg z + 2k\pi}{n}\right),$$

$k \in \{0, 1, 2, ..., n-1\}$.

Exerciţii:

1. Să se calculeze $(1+i)^{2012}$.

2. Pentru $n \in \mathbb{N}^*$ să se rezolve ecuaţia $\left(\frac{z-1}{z+1}\right)^n = -1$.

$$\frac{z_k - 1}{z_k + 1} = \cos\frac{\pi + 2k\pi}{n} + i\sin\frac{\pi + 2k\pi}{n},\ k \in \{0, 1, 2, ..., n-1\}$$

$$z_k = \frac{1 + \cos\frac{\pi+2k\pi}{n} + i\sin\frac{\pi+2k\pi}{n}}{1 - \cos\frac{\pi+2k\pi}{n} - i\sin\frac{\pi+2k\pi}{n}} =$$

$$= \frac{2\cos\frac{\pi+2k\pi}{n}\left(\cos\frac{\pi+2k\pi}{n} + i\sin\frac{\pi+2k\pi}{n}\right)}{-2i\sin\frac{\pi+2k\pi}{n}\left(\cos\frac{\pi+2k\pi}{n} + i\sin\frac{\pi+2k\pi}{n}\right)} =$$

$$= i\mathrm{ctg}\frac{\pi + 2k\pi}{n},\ k \in \{0, 1, 2, ..., n-1\}.$$

1.1.4 Metrica pe $\mathbb{C}$. Mulţimi deschise. Vecinătăţi. Domenii.

Aplicaţia $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$, $d(z, z') = |z - z'|$ satisface condiţiile:

1. $d(z, z') = 0 \Leftrightarrow z = z'$;

2. $d(z, z') = d(z', z),\ (\forall)z, z' \in \mathbb{C}$;

3. $d(z, z'') \leq d(z, z') + d(z', z''),\ \forall)z, z', z'' \in \mathbb{C}$.

Deci d este o metrică şi $(\mathbb{C}, d)$ este un spaţiu metric.

Definiţia 1.10 Pentru z_0 şi $r > 0$ definim: $D_r(z_0) = \{z \in \mathbb{C} | d(z, z_0) < r\}$ *discul* centrat în z_0 de rază r.

Definiţia 1.11 Mulţimea $G \subset \mathbb{C}$ se numeşte *mulţime deschisă* dacă:

$$(\forall)z \in \mathbb{C}, (\exists)r > 0 \text{ astfel încât } D_r(z) \subset G.$$

Definiţia 1.12 Fie $a \in \mathbb{C}$, $V \subset \mathbb{C}$ se numeşte *vecinătate* a lui a dacă $(\exists) G \subset \mathbb{C}$ mulţime deschisă astfel încât $a \in G \subset V$. Notăm $\mathcal{V}(a)$ (mulţimea) familia vecinătăţilor punctului a.

Definiţia 1.13 Punctul a este *punct interior* pentru A dacă $a \in \mathcal{V}(a)$. $\mathring{A}$= interiorul lui A= mulţimea punctelor interioare ale lui A.

Definiţia 1.14 Punctul a este *punct de aderenţă* pentru A daca $(\forall) V \in \mathcal{V}(a)$, avem: $V \cap A \neq \varphi$. $\bar{A}$ =închiderea lui A.

Definiţia 1.15 Punctul a este *punct de acumulare* pentru A dacă $(\forall) V \in \mathcal{V}(a) \Rightarrow (V \backslash \{a\}) \cap A \neq \varphi$. $A' =$ mulţimea tuturor punctelor de acumulare.

Propoziţia 1.16 *A deschisă $\Leftrightarrow A = \overset{\circ}{A}$; A închisă $\Leftrightarrow A = \bar{A}$; A compactă $\Leftrightarrow$ A mărginiţă şi închisă; A compactă în $\mathbb{C}_\infty \Leftrightarrow A$ închisă.*

Definiţia 1.17 $A \subset \mathbb{C}$; A este neconvexă dacă $(\exists) A_1, A_2 \in \mathbb{C} : A_1 \neq \Phi \neq A_2$ astfel încât $A = A_1 \cup A_2$ şi $\bar{A}_1 \cap A_2 = A_1 \cap \bar{A}_2 = \Phi$.

Definiţia 1.18 A este *conexă* dacă nu este neconexă; A este *domeniu* dacă este deschisă şi conexă.

1.2 Funcţii complexe de variabilă complexă

1.2.1 Definiţii

Definiţia 1.19 Fie $D \subset \mathbb{C}$ şi $f : D \to \mathbb{C}$ se numeşte *funcţie complexă de variabilă complexă.*

Definiţia 1.20 f poate fi privită ca o funcţie de variabilă $z = x + iy \in D$ sau ca o funcţie de două variabile independente $(x, y) \in D : f(z) = f(x, y) = u(x, y) + iv(x, y)$ unde:

$$\begin{matrix} u(x,y) = Ref(z) \\ v(x,y) = Imf(z) \end{matrix} \leftarrow$$

funcţii reale de variabilă complexă.

Definiţia 1.21 Pentru $z_0 \in D'$ spunem că f are limită în punctul z_0 dacă $(\exists)l \in C_\infty$ astfel încât $(\forall)V \in \vartheta(l), (\exists)U \in \vartheta(z_0)$ cu proprietatea: $f(U \cap (D \backslash \{z_0\})) \subset V (\Leftrightarrow (\forall)z \in U \cap (D \backslash \{z_0\}) \Rightarrow f(z) \in V.$

Scriem: $\lim\limits_{z \to z_0} f(z) = l$. Remarcăm că limita există indiferent cum tinde z la z_0.

Definiţia 1.22 f continuă în $z_0 \in D \Leftrightarrow (\forall)V \in \vartheta(f(z_0)), (\exists)U \in \vartheta(x_0)$ astfel încât $f(U \cap D) \subset V \Leftrightarrow (\forall)z \in U \cap D \Rightarrow f(z) \in V$.

Observaţia 1.23 Dacă $z_0 = x_0 + iy_0 \in D'$, $l = l_1 + il_2 \in \mathbb{C}$ atunci f are limită în $(x_0, y_0) \Leftrightarrow (\forall)\varepsilon > 0, (\exists)\delta_\varepsilon$ astfel încât $|f(z) - l| < \varepsilon, (\forall)z \in D \backslash \{z_0\}$ cu $|z - z_0| < \delta_\varepsilon \Leftrightarrow (\forall)\varepsilon > 0, (\exists)\delta_\varepsilon > 0$ astfel încât $|u(x, y) - l_1| < \varepsilon$ şi $|v(x, y) - l_2| < \varepsilon, (\forall)(x, y) \in D \backslash \{(x_0, y_0)\}$ cu $\|(x, y) - (x_0, y_0)\| < \delta_\varepsilon \Leftrightarrow$

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = l_1 \text{ şi } \lim_{(x,y) \to (x_0,y_0)} v(x,y) = l_2.$$

Observaţia 1.24 Fie $z_0 \in D$, atunci f este continuă în $z_0 \Leftrightarrow (\exists) \lim\limits_{z \to z_0} f(z) = f(z_0) \Leftrightarrow (\exists) \lim\limits_{(x,y) \to (x_0,y_0)} u(x, y) = u(x_0, y_0)$ şi $\lim\limits_{(x,y) \to (x_0,y_0)} v(x, y) = v(x_0, y_0) \Leftrightarrow u$ şi v sunt continue în $(x_0 y_0)$.

Observaţia 1.25 1. Considerăm funcţiile $f, g : D \to \mathbb{C}, z_0 \in D', \lambda \in \mathbb{C}$ şi presupunem că există $l_1 = \lim\limits_{z \to z_0} f(z),\ l_2 = \lim\limits_{z \to z_0} g(z)$. Atunci:

i) Dacă $l_1 \neq \infty,\ l_2 \neq \infty \Rightarrow f + g$ are limită în z_0 şi

$$\lim_{z \to z_0} (f + g)(z) = l_1 + l_2.$$

ii) $l_1 \neq \infty$ sau $(l_1 = \infty$şi $\lambda \neq 0) \Rightarrow \lambda f$ are limită în z_0 şi

$$\lim_{z \to z_0} (\lambda f)(z) = \lambda l_1.$$

iii) $l_1, l_2 \neq 0 \cdot \infty \Rightarrow f \cdot g$ nu are limită în z_0 şi

$$\lim_{z \to z_0} (f \cdot g)(z) = l_1 \cdot l_2.$$

iv) $l_2 \neq 0$ şi nu avem cazul $l_1 = l_2 = \infty \Rightarrow$

$$(\exists) \lim_{z \to z_0} \left(\frac{f}{g}\right)(z) = \frac{l_1}{l_2}.$$

2. Analog pentru continuitate; f şi g continue în $z_0 \Rightarrow$

$$(f + g),\ \lambda f,\ f \cdot g,\ \frac{f}{g}, (g(z_0 \neq 0))$$

sunt continue în z_0.

1.2.2 Derivabilitate. Condiţiile Cauchy-Riemann.

Definiţia 1.26 Fie $f : D \to \mathbb{C},\ z_0 \in \overset{\circ}{D}$. f se numeşte *derivabilă (olomorfă) în* z_0 dacă există $\lim\limits_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$.

Observaţia 1.27 f se numeşte *olomorfă pe* D dacă este olomorfă în orice punct al lui D. Funcţia $f' : D \to \mathbb{C}$ se numeşte *derivata lui* f pe D.

Observaţia 1.28 f' are aceiaşi formă ca în cazul real; avem aceleaşi reguli de derivare:

$$\begin{array}{l} (f+g)' = f' + g'; \\ (\lambda f)' = \lambda f'; \\ (f \cdot g)' = f' \cdot g + f \cdot g'; \\ \left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}; \\ (f \circ g)' = (f' \circ g) \cdot g'. \end{array}$$

Teorema 1.29 *(Teorema de reprezentare Cauchy-Riemann)* $f : D \subset \mathbb{C} \to \mathbb{C},\ f(z) = u(x,y) + iv(x,y),\ z_0 \in \overset{\circ}{D}$. *$f$ este derivabilă (olomorfă) în $z_0 \Leftrightarrow u$ şi v sunt diferenţiabile în (x_0, y_0) şi satisfac condiţiile Cauchy-Riemann:*

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases}$$

Demonstraţie. "$\Rightarrow$" f olomorfă în z_0 implică:

$$(\exists) \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0).$$

Deci limita există indiferent cum tinde z la z_0.

Fie $z_0 = x_0 + iy_0$ şi $z = x + iy \in D$ cu $z \neq z_0$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} =$$

$$= \lim_{z \to z_0} \frac{u(x,y) - u(x_0,y_0) + i(v(x,y) - v(x_0,y_0))}{(x - x_0) + i(y - y_0)}.$$

Presupunem că $z \to z_0$ pe oparalelă cu axa reală $\Rightarrow y = y_0$ şi $x \to x_0$ ceea ce implică existenţa limitelor:

$$\begin{aligned} f'(z_0) &= \lim_{x\to x_0} \tfrac{u(x,y_0)-u(x_0,y_0)}{x-x_0} + i \lim_{x\to x_0} \tfrac{v(x,y_0)-v(x_0,y_0)}{x-x_0} = \\ &= \tfrac{\partial u}{\partial x}(x_0,y_0) + i\tfrac{\partial v}{\partial x}(x_0,y_0). \end{aligned} \tag{1.1}$$

Analog , presupunem că $z \to z_0$ după o paralelă cu axa imaginară Oy, atunci: $\begin{cases} x = x_0 \\ y \to y_0. \end{cases}$

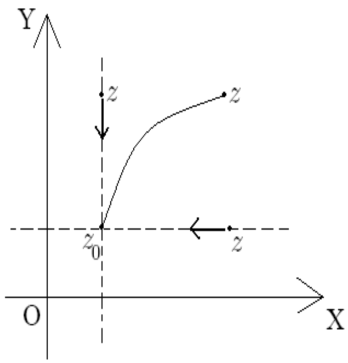

Figura 4.

$$\begin{aligned} f'(z_0) &= \lim_{y\to y_0} \frac{u(x_0,y)-u(x_0,y_0)+i(v(x_0,y)-v(x_0,y_0))}{i(y-y_0)} = \\ &= \frac{1}{i}\frac{\partial u}{\partial y}(x_0,y_0) + \frac{\partial v}{\partial y}(x_0,y_0) = \\ &= \frac{\partial v}{\partial y}(x_0,y_0) - i\frac{\partial u}{\partial y}(x_0,y_0) \end{aligned} \tag{1.2}$$

Din (1.1) şi (1.2) deducem:

$$\begin{cases} \frac{\partial u}{\partial x}(x_0,y_0) = \frac{\partial v}{\partial y}(x_0,y_0) \\ \frac{\partial u}{\partial y}(x_0,y_0) = -\frac{\partial v}{\partial x}(x_0,y_0). \end{cases}$$

Deci am obţinut condiţiile Cauchy-Riemann.
Avem:

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) =$$
$$= \frac{1}{i}\left(\frac{\partial u}{\partial y}(x_0, y_0) + i\frac{\partial v}{\partial y}(x_0, y_0)\right).$$

"$\Leftarrow$" Presupunem că u şi v sunt diferenţiabile în (x_0, y_0), deci există $\frac{\partial u}{\partial x}(x_0, y_0)$, $\frac{\partial u}{\partial y}(x_0, y_0)$, $\frac{\partial v}{\partial x}(x_0, y_0)$ şi $\frac{\partial v}{\partial y}(x_0, y_0)$ şi în plus satisfac condiţiile Cauchy-Riemann. Scriem teorema creşterilor finite pentru u şi v în punctul (x_0, y_0):

$$\begin{aligned}
&u(x, y) - u(x_0, y_0) = \\
&= \tfrac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \tfrac{\partial u}{\partial y}(x_0, y_0)(y - y_0) + \\
&+ \alpha_1(x, y)(x - x_0) + \alpha_2(x, y)(y - y_0),
\end{aligned}$$

$$\begin{aligned}
&v(x, y) - v(x_0, y_0) = \\
&= \tfrac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + \tfrac{\partial v}{\partial y}(x_0, y_0)(y - y_0) + \\
&+ \beta_1(x, y)(x - x_0) + \beta_2(x, y)(y - y_0),
\end{aligned}$$

unde: $\lim\limits_{(x,y)\to(x_0,y_0)} \alpha_i(x, y) = \lim\limits_{(x,y)\to(x_0,y_0)} \beta_i(x, y) = 0,\ i = \overline{1, 2}.$

Calculăm:

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{u(x, y) - u(x_0, y_0) + i(v(x, y) - v(x_0, y_0))}{(x - x_0) + i(y - y_0)} =$$

$$= \frac{\frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0)}{(x - x_0) + i(y - y_0)} +$$
$$+ \frac{\alpha_1(x, y)(x - x_0) + \alpha_2(x, y)(y - y_0)}{(x - x_0) + i(y - y_0)} +$$
$$+ \frac{i\frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + i\frac{\partial v}{\partial y}(x_0, y_0)(y - y_0)}{(x - x_0) + i(y - y_0)} +$$

$$+\frac{i\beta_1(x,y)(x-x_0)+i\beta_2(x,y)(y-y_0)}{(x-x_0)+i(y-y_0)}=$$

$$=\frac{(x-x_0)+i(y-y_0)}{(x-x_0)+i(y-y_0)}\cdot\frac{\partial u}{\partial x}(x_0,y_0)+$$

$$+\frac{i[(x-x_0)\frac{\partial v}{\partial x}(x_0,y_0)-i(y-y_0)\frac{\partial v}{\partial x}(x_0,y_0)]}{(x-x_0)+i(y-y_0)}+$$

$$+\frac{x-x_0}{z-z_0}(\alpha_1(x,y)+i\beta_1(x,y))+\frac{y-y_0}{z-z_0}(\alpha_2(x,y)+i\beta_2(x,y))=$$

$$=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)+\frac{x-x_0}{z-z_0}(\alpha_1(x,y)+i\beta_1(x,y))+$$

$$+\frac{y-y_0}{z-z_0}(\alpha_2(x,y)+i\beta_2(x,y)) \tag{1.3}$$

$$\begin{matrix}\frac{|x-x_0|}{|z-z_0|}\leq 1\\ \frac{|y-y_0|}{|z-z_0|}\end{matrix}\quad\Rightarrow$$

$$\begin{matrix}\frac{|x-x_0|}{|z-z_0|}\cdot|\alpha_1(x,y)+i\beta_1(x,y)|\leq|\alpha_1(x,y)+i\beta_1(x,y)|\\ \frac{|y-y_0|}{|z-z_0|}\cdot|\alpha_2(x,y)+i\beta_2(x,y)|\leq|\alpha_2(x,y)+i\beta_2(x,y)|\end{matrix}\quad\Rightarrow$$

$$\Rightarrow\begin{matrix}\lim\limits_{z\to z_0}\frac{x-x_0}{z-z_0}\cdot\alpha_1(x,y)+i\beta_1(x,y)=0\\ \lim\limits_{z\to z_0}\frac{y-y_0}{z-z_0}\cdot\alpha_2(x,y)+i\beta_2(x,y)=0\end{matrix}$$

Deci în relaţia (1.3) trecem la limită după $z\to z_0\Rightarrow$ $\lim\limits_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\frac{\partial u}{\partial x}(x_0,y_0)+i\frac{\partial v}{\partial x}(x_0,y_0)\Rightarrow f$ derivabilă în z_0 şi $f'(z_0)=\frac{\partial u}{\partial x}(x_0,y_0)+\frac{\partial v}{\partial x}(x_0,y_0)$. Demonstraţia este încheiată. $\square$

$$f \text{ olomorfă pe } D\Rightarrow\left\{\begin{matrix}\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}\end{matrix}\right. \text{ şi } f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}.$$

Remarca 1.30 $f'(z) = \frac{\partial u}{\partial x}(x,y)+i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)-i\frac{\partial u}{\partial y}(x,y) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) = \frac{\partial v}{\partial y}(x,y) + i\frac{\partial v}{\partial x}(x,y)$.

Remarca 1.31 $f = u+iv$ este olomorfă pe D şi $u, v \in C^2(D)$. Atunci u şi v sunt armonice: $\Delta u = \Delta v = 0$.

$$\Delta u = \frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) =$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) = 0.$$

Remarca 1.32 Dacă cunoaştem u(respectiv v) o funcţie armonică pe D putem determina o funcţie v(respectiv u) astfel încât $f = u + iv$ să fie olomorfă pe D.
$dv(x,y) = \frac{\partial v}{\partial x}(x,y)dx + \frac{\partial v}{\partial y}(x,y)dy \to$ formă diferenţială închisă (exactă), deci putem integra pe orice drum între punctele (x,y) şi (x_0,y_0) obţinând:

$$v(x,y) - v(x_0,y_0) = \int_{x_0}^{x} \frac{\partial v}{\partial x}(t,y_0)dt + \int_{y_0}^{y} \frac{\partial v}{\partial y}(x,t)dt =$$
$$= -\int_{x_0}^{x} \frac{\partial u}{\partial y}(t,y_0)dt + \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,t)dt.$$

Exemplul 1.33 Să se arate că $f(z) = e^z$ este derivabilă şi să se calculeze derivata.

Exemplul 1.34 Determinaţi o funcţie olomorfă $f(z) = u(x,y)+iv(x,y)$, ştiind că: $u(x,u) = \frac{1-x^2-y^2}{(1+x)^2+y^2}$ şi $f(1) = 0$.

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial u}{\partial x}(x,y) - i\frac{\partial u}{\partial y}(x,y) =$$

$$= -2\frac{(1+x)^2 - y^2}{[(1+x)^2+y^2]^2} + i\frac{4y(1+x)}{[(1+x)^2+y^2]^2} =$$

$$= -2\frac{(1+x-iy)^2}{(1+x+iy)^2(1+x-iy)^2} = \frac{-2}{(1+z)^2} \Rightarrow$$

prin integrare primitivele funcţiilor elementare în z sunt asemănătoare celor în variabila reală x.

$$f(z) = \frac{2}{1+z} + C$$

şi din $f(1) = 0 \Rightarrow \frac{2}{1+1} + C = 0 \Rightarrow C = -1$, deci

$$f(z) = \frac{2}{1+z} - 1 = \frac{1-z}{1+z}.$$

Observaţia 1.35 Prezentăm o altă demonstraţie pentru teorema Cauchy-Riemann:

Teorema 1.36 *Fie* $f : D \subset \mathbb{C} \to \mathbb{C}, z_0 \in \overset{\circ}{D}$.

Funcţia $f(z) = u(x,y) + iv(x,y)$ *este derivabilă în* $(x_0, y_0) = z_0 = x_0 + iy_0 \Leftrightarrow u, v$ *sunt diferenţiabile în* (x_0, y_0) *şi în acest punct sunt îndeplinite condiţiile Cauchy-Riemann:*

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0) \end{cases}$$

Demonstraţie.

" $\Rightarrow$ " Fie $f'(z_0) = a + ib$ şi $h : D\backslash\{z_0\} \to \mathbb{C}, h(z) = \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)$.

Notăm $h_1 := Reh$ şi $h_2 := Imh$.

f derivabilă în$z_0 \Rightarrow \lim\limits_{z \to z_0} h(z) = 0 \Rightarrow$

$$\lim_{(x,y)\to(x_0,y_0)} h_1(x,y) = 0, \quad \lim_{(x,y)\to(x_0,y_0)} h_2(x,y) = 0. \tag{1.4}$$

Relaţia: $f(z) - f(z_0) = h(z)(z - z_0) + f'(z_0)(z - z_0)$ este echivalentă cu:

$$u(x,y) + iv(x,y) - u(x_0,y_0) - iv(x_0,y_0) =$$

$$= [h_1(x,y)+ih_2(x,y)][x+iy-x_0-iy_0]+(a+ib)(x+iy-x_0-iy_0).$$

Separăm partea reală de partea imaginară şi obţinem:

$$\begin{cases} u(x,y) - u(x_0,y_0) = h_1(x,y)(x-x_0) - h_2(x,y)(y-y_0) + \\ \qquad +a(x-x_0) - b(y-y_0) \\ v(x,y) - v(x_0,y_0) = h_1(x,y)(y-y_0) - h_2(x,y)(x-x_0) + \\ \qquad +a(y-y_0) - b(x-x_0) \end{cases} \Leftrightarrow$$

$$\frac{u(x,y) - u(x_0,y_0) - [a(x-x_0) - b(y-y_0)]}{\|(x,y) - (x_0,y_0)\|} =$$

$$= h_1(x,y)\frac{x-x_0}{\|(x,y)-(x_0,y_0)\|} - h_2(x,y)\frac{y-y_0}{\|(x,y)-(x_0,y_0)\|}$$

$$\frac{v(x,y) - v(x_0,y_0) - [a(y-y_0) - b(x-x_0)]}{\|(x,y) - (x_0,y_0)\|} = \tag{1.5}$$

$$= h_1(x,y)\frac{y-y_0}{\|(x,y)-(x_0,y_0)\|} - h_2(x,y)\frac{x-x_0}{\|(x,y)-(x_0,y_0)\|}$$

Cum

$$\frac{|x-x_0|}{\|(x,y)-(x_0,y_0)\|} = \frac{|x-x_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \leq 1$$

$$\frac{|y-y_0|}{\|(x,y)-(x_0,y_0)\|} = \frac{|y-y_0|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} \leq 1$$

folosind (1.4) şi (1.5) găsim:

$$\begin{cases} \lim\limits_{(x,y)\to(x_0,y_0)} \frac{u(x,y)-u(x_0,y_0)-[a(x-x_0)-b(y-y_0)]}{\|(x,y)-(x_0,y_0)\|} = 0 \\ \lim\limits_{(x,y)\to(x_0,y_0)} \frac{v(x,y)-v(x_0,y_0)-[a(y-y_0)-b(x-x_0)]}{\|(x,y)-(x_0,y_0)\|} = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow u, v$ diferenţiabile în (x_0, y_0) şi $\underbrace{du(x_0, y_0)}_{\text{aplicaţie liniară şi continuă}}$

$$du(x_0, y_0)(x - x_0, y - y_0) =$$

$$= \frac{\partial u}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial u}{\partial y}(x_0, y_0)(y - y_0) =$$

$$= a(x - x_0) - b(y - y_0) \Rightarrow$$

$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) = a;\ \frac{\partial u}{\partial y}(x_0, y_0) = -b.$$

Analog:

$$dv(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial v}{\partial y}(x_0, y_0)(y - y_0) =$$

$$= b(x - x_0) + a(y - y_0) \Rightarrow$$

$$\Rightarrow \frac{\partial v}{\partial x}(x_0, y_0) = b;\ \frac{\partial v}{\partial y}(x_0, y_0) = a.$$

Deci:

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0) \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases}$$

" $\Leftarrow$ " Fie $a = \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0)$ şi $b = -\frac{\partial u}{\partial y}(x_0, y_0) = \frac{\partial v}{\partial x}(x_0, y_0)$.
Fie $g_1, g_2 : D \setminus \{z_0\} \to \mathbb{R}$ definite prin:

$$g_1(x, y) = \frac{u(x, y) - u(x_0, y_0) - [a(x - x_0) - b(y - y_0)]}{\|(x, y) - (x_0, y_0)\|} \frac{n!}{r!\,(n - r)!}$$

$$g_2(x, y) = \frac{v(x, y) - v(x_0, y_0) - [b(x - x_0) + a(y - y_0)]}{\|(x, y) - (x_0, y_0)\|}.$$

Cum u şi v sunt diferenţiabile în $(x_0, y_0) \Rightarrow$

$$\lim_{(x,y)\to(x_0,y_0)} g_1(x,y) = \lim_{(x,y)\to(x_0,y_0)} g_2(x,y) = 0. \tag{1.6}$$

Avem:

$$\begin{cases} u(x,y) - u(x_0,y_0) = a(x-x_0) - b(y-y_0) + \\ \qquad\qquad + g_1(x,y)\,\|(x,y)-(x_0,y_0)\| \\ v(x,y) - v(x_0,y_0) = b(x-x_0) + a(y-y_0) + \\ \qquad\qquad + g_2(x,y)\,\|(x,y)-(x_0,y_0)\| \end{cases}$$

şi înmulţim cu i a doua relaţie, după aceea o adunăm la prima relaţie obţinând:

$$f(z) - f(z_0) =$$

$$= (a+ib)(z-z_0) + [g_1\underbrace{(x,y)}_{=z} + g_2\underbrace{(x,y)}_{=z}]\underbrace{\|(x,y)-(x_0,y_0)\|}_{=|z-z_0|}$$

echivalent cu:

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - (a+ib)\right| = |g_1(z) + g_2(z)|$$

şi cu relaţia (1.6) găsim:

$$\lim_{z\to z_0}\left|\frac{f(z)-f(z_0)}{z-z_0} - (a+ib)\right| = 0,$$

deci f este derivabilă şi în z_0.

$$f'(z_0) = a + ib = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0).$$

$\square$

1.3 Funcţii elementare şi integrala complexă

1.3.1 Definiţii

Definiţia 1.37 *Funcţia polinomială*

$$f : \mathbb{C} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} a_k z^{n-k}$$

cu $a_k \in \mathbb{C}$ pentru $0 \leq k \leq n$, $a_0 \in \mathbb{C}^*$; atunci f este derivabilă şi $f'(z) = \sum_{k=0}^{n-1} a_k \cdot (n-k) \cdot z^{n-k-1}$.

Definiţia 1.38 *Funcţia raţională*

$$f : \mathbb{C}\backslash\{z_1, z_2, z_3, ..., z_s\} \to \mathbb{C},\ f(z) = \frac{\sum_{k=0}^{n} a_k z^{n-k}}{\sum_{k=0}^{m} b_k z^{m-k}},$$

unde $\{z_1, z_2, z_3, ..., z_s\}$ sunt rădăcinile numitorului $a_k, b_k \in \mathbb{C}$ pentru $1 \leq k \leq n$ şi $1 \leq k \leq m$. f este derivabilă şi

$$f'(z) = \frac{\left[\sum_{k=0}^{n-1} a_k \cdot (n-k) \cdot z^{n-k-1}\right]\left[\sum_{k=0}^{m} b_k z^{m-k}\right]}{\left[\sum_{k=0}^{m} b_k z^{m-k}\right]^2} -$$

$$- \frac{\left[\sum_{k=0}^{n} a_k z^{n-k}\right]\left[\sum_{k=0}^{m-1} b_k \cdot (m-k) \cdot z^{m-k-1}\right]}{\left[\sum_{k=0}^{m} b_k z^{m-k}\right]^2}.$$

Definiţia 1.39 *Funcţia exponenţială*

$$f : \mathbb{C} \to \mathbb{C},\ f(z) = e^z$$

f este derivabilă şi $f'(z) = e^z$.

Definiţia 1.40 *Funcţia logaritmică*
Se defineşte ca inversa funcţiei exponenţiale. Pentru $z \in \mathbb{C} \backslash \{0\}$ rezolvăm ecuaţia:

$$\left.\begin{array}{l} e^w = z = |z| \cdot e^{i \arg z} \\ w = u + iv \end{array}\right\} \Rightarrow e^u \cdot e^{iv} = |z| \cdot e^{i(\arg z + 2k\pi)}, k \in \mathbb{Z}$$

$$\Rightarrow \left\{\begin{array}{l} e^u = |z| \\ v = \arg z + 2k\pi, k \in \mathbb{Z} \end{array}\right. \Rightarrow \left\{\begin{array}{l} u = \ln |z| \\ v = \arg z + 2k\pi, k \in \mathbb{Z}. \end{array}\right.$$

Mulţimea

$$\mathrm{Ln}\, z = \{\ln |z| + i(\arg z + 2k\pi) | k \in \mathbb{Z}\}$$

se numeşte *logaritmul numărului complex z.*

Pentru $h \in \mathbb{Z}$ fixat, funcţia

$$f_k : \mathbb{C} \backslash T \to \mathbb{C}, T = \{z \in \mathbb{C} | Imz = 0, Rez \geq 0\},$$

definită prin

$$f_k(z) = \ln |z| + i(\arg z + 2k\pi)$$

se numeşte *ramura continuă a logaritmului.*

Funcţia

$$\ln : \mathbb{C} \backslash T \to \mathbb{C},\ \ln z = \ln |z| + i(\arg z + 2k\pi)$$

se numeşte *ramura principală a logaritmului.*

Restricţia

$$f : \{z \in \mathbb{C} | Imz \in (0, 2\pi)\} \to \mathbb{C} \backslash T,\ f(z) = e^z$$

are ca inversă corestricţia

$$\ln : \mathbb{C} \backslash T \to \{z \in \mathbb{C} | Imz \in (0, 2\pi)\};$$

deci ln este derivabilă şi

$$(\ln z)' = (f^{-1}(z))' = \frac{1}{f'(\ln z)} = \frac{1}{e^{\ln z}} = \frac{1}{z}.$$

Definiţia 1.41 *Funcţiile circulare*

$$\cos, \sin : \mathbb{C} \to \mathbb{C}, \cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

sunt derivabile şi $\begin{cases} (\cos z)' = -\sin z \\ (\sin z)' = \cos z; \end{cases}$

$$\mathrm{tg} : \mathbb{C} \setminus \left\{ \frac{\pi}{2} + k\pi | k \in \mathbb{Z} \right\} \to \mathbb{C},\ \mathrm{tg}\ z = \frac{\sin z}{\cos z}, (\mathrm{tg}\ z)' = \frac{1}{\cos^2 z};$$

$$\mathrm{ctg} : \mathbb{C} \setminus \{k\pi | k \in \mathbb{Z}\} \to \mathbb{C},\ \mathrm{ctg}\ z = \frac{\cos z}{\sin z}, (\mathrm{ctg}\ z)' = \frac{-1}{\sin^2 z}.$$

Definiţia 1.42 *Funcţiile hiperbolice*

$$\mathrm{ch}, \mathrm{sh} : \mathbb{C} \to \mathbb{C}, \mathrm{ch}\ z = \frac{e^{z} + e^{-z}}{2},\ \mathrm{sh}\ z = \frac{e^{z} - e^{-z}}{2}$$

sunt derivabile şi $\begin{cases} (\mathrm{ch}\ z)' = \mathrm{sh}\ z \\ (\mathrm{sh}\ z)' = \mathrm{ch}\ z; \end{cases}$

$$\mathrm{th} : \mathbb{C} \setminus \{z \in \mathbb{C} | \mathrm{ch}\ z = 0\} \to \mathbb{C},\ \mathrm{th}\ z = \frac{\mathrm{sh}\ z}{\mathrm{ch}\ z}, (\mathrm{th}\ z)' = \frac{1}{\mathrm{ch}^2 z};$$

$$\mathrm{cth} : \mathbb{C} \setminus \{z \in \mathbb{C} | \mathrm{sh}\ z = 0\} \to \mathbb{C},\ \mathrm{cth}\ z = \frac{\mathrm{ch}\ z}{\mathrm{sh}\ z}, (\mathrm{cth}\ z)' = \frac{-1}{\mathrm{sh}^2 z}.$$

Definiţia 1.43 *Funcţia putere complexă*
Pentru $z \in \mathbb{C} \backslash \{0\}$ şi $\alpha \in \mathbb{C} \backslash \mathbb{Z}$ definim:

$$z^\alpha = e^{\alpha \ln z} = \{e^{\alpha(\ln|z|+i(\arg z+2k\pi))} | k \in \mathbb{Z}\}.$$

Pentru $k \in \mathbb{Z}$ fixat, funcţia

$$f_k : \mathbb{C} \backslash T \to \mathbb{C}, f_k(z) = e^{\alpha(\ln|z|+i(\arg z+2k\pi))}$$

se numeşte *ramura continuă a puterii de ordin* α.

1.4 Integrala curbilinie în planul complex

1.4.1 Definiţii şi proprietăţi

Definiţia 1.44 Se numeşte *drum* o funcţie continuă $\gamma : [a, b] \to \mathbb{C}$. Este de clasă C^1 pe porţiuni dacă $(\exists) a = t_0 < t1 < ... < t_{n-1} < t_n = b$ a intervalului $[a, b]$ astfel încât $\gamma_{[t_i, t_{i+1}]} \in C^1$, $0 \leq i \leq n-1$.

Mulţimea $\gamma([a, b]) = \Gamma = \{\gamma(t) | t \in [a, b]\}$ se numeşte *curba imagine* a drumului γ; γ se mai numeşte *parametrizarea curbei* Γ.

Drumul γ(curba Γ) se numeşte *simplu(ă)* dacă $\gamma(t) \neq \gamma(t')$, $(\forall) t$, $t' \in (a, b)$ cu $t \neq t'$.

γ(curba Γ) se numeşte *simplă închisă* dacă este simplă şi $\gamma(a) = \gamma(b)$.

Dacă curba Γ este închisă, ea împarte planul $\mathbb{C}$ în două domenii, fixăm unul dintre acestea; orientarea pozitivă pe drumul γ(curba Γ) se consideră atunci când domeniul fixat rămâne în stânga în timpul deplasării.

Drumul $\gamma^- : [a,b] \to \mathbb{C}$ definit prin $\gamma^-(t) = \gamma(a+b-t)$ se numeşte *opusul drumului* γ. Are aceiaşi imagine cu γ, dar orientare inversă.

$\gamma_1 : [a,b] \to \mathbb{C}$, $\gamma_2 : [c,d] \to \mathbb{C}$ astfel încât $\gamma_1(b) = \gamma_2(c)$. Definim *recursiunea drumurilor*: $\gamma_1 \vee \gamma_2 = \gamma : [a, b+d-c] \to \mathbb{C}$ definim prin $\gamma(t) = \begin{cases} \gamma_1(t),\ t \in [a,b] \\ \gamma_2(t-b+c),\ t \in [b, b+d-c] \end{cases}$.

Dacă Γ_i este imaginea lui γ_i, $i = \overline{1,2}$, atunci $\Gamma_1 \cup \Gamma_2$ este imaginea drumului $\gamma_1 \vee \gamma_2$.

Definiţia 1.45 Fie $D \subset \mathbb{C}$ domeniu din $\mathbb{C}$, $f : D \to \mathbb{C}$, $\gamma : [a,b] \to \mathbb{C}$ un drum de clasă C^1 pe porţiuni astfel încât imaginea lui γ = curba Γ este inclusă în D. Definim integrala lui f de-a lungul drumului γ, numărul complex:

$$\int_\gamma f(z)dz = \int_a^b (f \circ \gamma)(t) \cdot \gamma'(t)dt.$$

Scriem de regulă :

$$\int_\Gamma f(z)dz; \ \int_{\Gamma^-} f(z)dz; \ \int_{\Gamma^+} f(z)dz.$$

Observaţia 1.46 Dacă

$$f(z) = u(x,y) + iv(x,y),$$

$$\gamma(t) = z(t) = x(t) + iy(t) \Rightarrow \gamma'(t) = z'(t) = x'(t) + iy'(t) \Rightarrow$$

$$\int_\gamma f(z)dz = \int_a^b [u(x(t),y(t)) + iv(x(t),y(t))] \cdot [x'(t) + iy'(t)]dt =$$

$$= \int_a^b [u(x(t),y(t)) \cdot x'(t) - v(x(t),y(t)) \cdot y'(t)]dt +$$

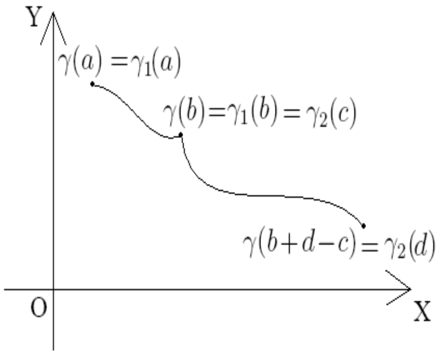

Figura 5.

$$+i\int_a^b [u(x(t),y(t))\cdot y'(t) + v(x(t),y(t))\cdot x'(t)]dt =$$

$$= \int_{\substack{(\Gamma)\\ \gamma}} udx - vdy + i\int_{\substack{(\Gamma)\\ \gamma}} udy - vdx \leftarrow \int_\gamma f(z)dz$$

Integrala $\int_\gamma f(z)dz$ se definite prin intermediul a două integrale curbilinii de speţa a doua pe domeniul γ.

Proprietăţi 1.47 (ale integralei (complexe) curbilinii în planul complex)

1. $\int_{\Gamma^+} f(z)dz = -\int_{\Gamma^-} f(z)dz$;
2. $\int_\Gamma (\lambda f + \mu g)dz = \lambda \int_\Gamma f dz + \mu \int_\Gamma g dz$;
3. $\int_{\Gamma_1 \vee \Gamma_2} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz$;
4. $\left|\int_\Gamma f(z)dz\right| \leq \left(\sup\limits_{z\in\Gamma} |f(z)|\right)\cdot L$

   unde $L = l(\gamma) = \int_a^b |z'(t)|dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}dt$;
5. $\int_{U_r(a)} \frac{dz}{z-a} = 2\pi i$.

Într-adevăr: $U_r(a) = \{z \in \mathbb{C} | \, |z-a| = r\}$ rezultă că pentru $z \in U_r(a)$ avem:

$$z = a + r \cdot e^{it}, t \in [0, 2\pi] \Rightarrow dz(t) = z'(t)dt = r \cdot i \cdot e^{it} dt$$

$$\int_{U_r(a)} \frac{dz}{z-a} = \int_0^{2\pi} \frac{r \cdot i \cdot e^{it}}{r \cdot e^{it}} dt = 2\pi i.$$

1.4.2 Teorema lui Cauchy

Un domeniu $D \subset \mathbb{C}$ se numeşte *simplu conex* dacă orice curbă închisă $\Gamma \subset D$ delimitează un domeniu Δ inclus în D.

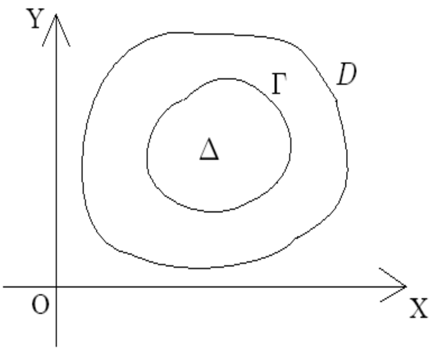

Figura 6.

Teorema 1.48 *Fie $D \subset \mathbb{C}$ un domeniu simplu conex şi $\Gamma \subset D$ o curba simplă, închisă, de clasă C^1 pe porţiuni şi $f : D \to \mathbb{C}$ o funcţie olomorfă. Atunci:*

$$\int_\Gamma f(z)dz = 0.$$

Demonstraţie. Fie Δ domeniul delimitat de curba Γ în D. Aplicând formula lui Green avem:

$$\int_\Gamma f(z)dz = \int_\Gamma udx - vdy + i\int_\Gamma vdx + udy =$$

$$\iint_\Delta \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\iint_\Delta \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy.$$

Cum f este olomorfă, u şi v satisfac condiţiile Cauchy-Riemann:

$$\int_\Gamma f(z)dz = 0 \Leftrightarrow \iint_\Delta 0dxdy + i\iint_\Delta 0dxdy = 0.$$

□

Observaţia 1.49 Rezultatul rămâne valabil dacă Γ este formată din mai multe curbe simple închise.

Observaţia 1.50 Fie $z_0, z_1 \in D$ şi γ_1, γ_2 doua drumuri simple, de clasă C^1 pe porţiuni ce leagă punctele z_0 şi z_1. Atunci: $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$. Acest număr depinde numai de capetele z_0 şi z_1 şi se notează $\int_{z_0}^{z_1} f(z)dz$.

Observaţia 1.51 Funcţia $F : D \to \mathbb{C}$, $F(w) = \int_{z_0}^{w} f(z)dz$, $z_0 \in D$ fixat este o primitivă a lui f.
F_1, F_2 primitive ale lui $f \Rightarrow F_1 - F_2 = const.$
Dacă F este primitivă a lui $f \Rightarrow \int_{z_1}^{z_2} f(z)dz = -F(z_1) + F(z_2) \to$ formula Leibniz-Newton.

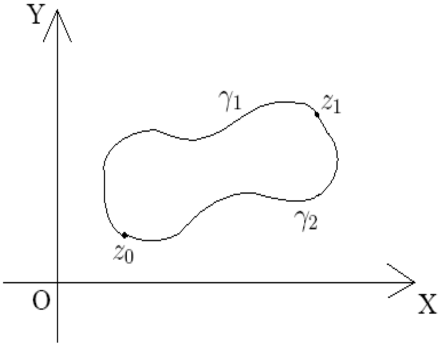

Figura 7.

1.4.3 Teorema lui Cauchy pentru domenii multiplu-conexe.

Definiţia 1.52 Un domeniu multiplu conex este un domeniu a cărui frontieră este formată din mai multe curbe disjuncte.

Un domeniu multiplu conex se poate transforma într-un domeniu simplu conex dacă se efectuează mai multe tăieturi.

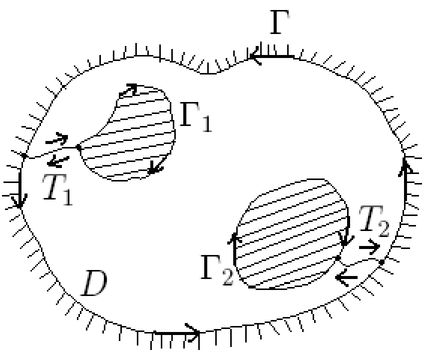

Figura 8.

Ariile haşurate le numim lagune. Tăietura T_i, $i = \overline{1,2}$ uneşte un punct de pe frontiera exterioară Γ şi unul de pe frontiera interioară $\Gamma_i,\ i=\overline{1,2}$.

Fără tăieturile $T_i,\ i=\overline{1,2}$, D este un multiplu conex cu frontiera $\Gamma\cup\Gamma_1\cup\Gamma_2$, trei curbe disjuncte între ele.

Cu tăieturile $T_i,\ i=\overline{1,2}$, D se transformă în domeniu simplu conex cu frontiera formată din reuniunea a cinci curbe $\Gamma\cup\Gamma_1\cup\Gamma_2\cup T_1\cup T_2$, care au legătură între ele. Domeniul $\widetilde{D}=D\backslash\{T_1\cup T_2\}$ devine simplu conex.

Generalizare: Dacă domeniul D este multiplu conex, se pot efectua tăieturile $T_1,T_2,T_3,...,T_p$ astfel încât domeniul $\widetilde{D}=D\backslash\{T_1\cup T_2\cup...\cup T_p\}$ să devină domeniu simplu conex.

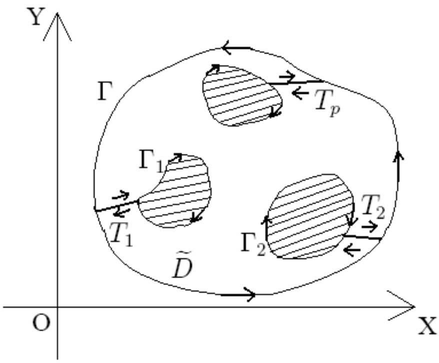

Figura 9.

Fie $\Gamma_1,\Gamma_2,\Gamma_3,...,\Gamma_p$ curbele închise, simple care închid lagunele. Curba de frontieră a domeniului simplu conex $\widetilde{D}=D\backslash\{T_1\cup T_2\cup...\cup T_p\}$ este: $\Gamma_0=\Gamma\cup T_1^+\cup\Gamma_1^-\cup T_1^-\cup...\cup T_p^+\cup\Gamma_p^-\cup T_p^-$.

Definiţia 1.53 Domeniul D se numeşte *multiplu conex* dacă frontiera lui este reuniunea mai multor curbe disjuncte între ele.

Observaţia 1.54 Un domeniu D multiplu conex se poate transforma într-un domeniu simplu conex $\widetilde{D}$ dacă se efectuează

$T_1, T_2, T_3, ..., T_p$ tăieturi (un segment care leagă un punct de pe Γ şi unul de pe $T_i,\ i = 1, 2, ..., p$). Atunci $\widetilde{D} = D \backslash \{T_1 \cup T_2 \cup ... \cup T_p\}$ şi $\partial\widetilde{D} = \Gamma \cup T_1^+ \cup \Gamma_1^- \cup T_1^- \cup ... \cup T_p^+ \cup \Gamma_p^- \cup T_p^-$ curbă închisă, simplă, C^1 pe porţiuni.

1.4.4 Teorema fundamentală a lui Cauchy pentru domenii multiplu conexe.

$D \subset \mathbb{C}$ domeniu multiplu conex cu frontiera

$$\underbrace{\Gamma \cup \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_p}_{\text{curbe simple de clasă } C^1 \text{ pe porţiuni}},$$

$f : D \to \mathbb{C}$ olomorfă. Atunci

$$\int_{\Gamma} f(z)dz = \sum_{k=1}^{p} \int_{\Gamma_k} f(z)dz.$$

Demonstraţie. Efectuând tăieturile $T_1, T_2, T_3, ..., T_p$,D multiplu conex se transformă în domeniul simplu conex $\widetilde{D}$ pe care vom aplica teorema fundamentala Cauchy pentru frontiera $\partial\widetilde{D}$. Avem: $0 = \int_{\partial\widetilde{D}} f(z)dz$ şi folosind proprietatea de aditivitate a integralei curbilinii complexe şi a integralei pe drum opus, obţinem:

$$0 = \int_{\Gamma} f(z)dz + \int_{T_1^+} f(z)dz + \int_{\Gamma_1^-} f(z)dz + \int_{T_1^-} f(z)dz + ...$$
$$+ \int_{T_p^+} f(z)dz + \int_{\Gamma_p^-} f(z)dz + \int_{T_p^-} f(z)dz,$$

$$\int_{\Gamma} f(z)dz = \sum_{k=1}^{p} \int_{\Gamma_k} f(z)dz.$$

□

1.5 Formula integrală a lui Cauchy.

Teorema 1.55 *Fie D un domeniu simplu conex, $f : D \to \mathbb{C}$ o funcţie olomorfă şi Γ o curbă simplă închisă, de clasă C^1 pe porţiuni care delimitează în D domeniul Δ. Atunci:*

1. $f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi,\ (\forall)z \in \Delta;$

2. *f este C^∞ derivabilă şi* $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi,\ (\forall)z \in \Delta,\ (\forall)n \geq 1.$

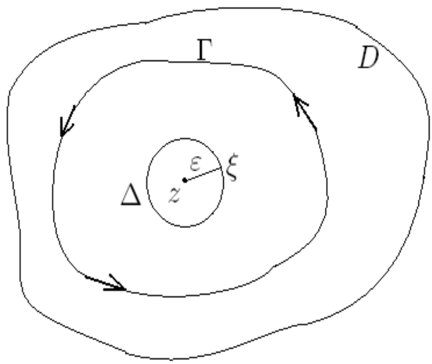

Figura 10.

Demonstraţie.

1.Fie $z \in \Delta$ şi fie discul centrat în z de rază ε: $D_\varepsilon(z) \subset \Delta$; fie $U_\varepsilon(z)$ frontiera discului.
Domeniul $\Delta \backslash D_\varepsilon(z)$ este multiplu conex şi funcţia $\varepsilon \mapsto \frac{f(\xi)}{\xi - z}$ este olomorfă pe $\Delta \backslash D_\varepsilon(z)$ şi conform teoremei fundamentale a lui Cauchy pe domenii multiplu conexe avem:

$$\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi = \int_{U_\varepsilon(z)} \frac{f(\xi)}{\xi - z}d\xi. \qquad (1.7)$$

Pe de altă parte avem:

$$\int_{U_\varepsilon(z)} \frac{f(\xi)}{\xi - z} d\xi = \int_{U_\varepsilon(z)} \frac{f(\xi) - f(z)}{\xi - z} d\xi + f(z) \int_{U_\varepsilon(z)} \frac{d\xi}{\xi - z} =$$

$$= \int_{U_\varepsilon(z)} \frac{f(\xi) - f(z)}{\xi - z} d\xi + 2\pi i f(z) \tag{1.8}$$

$\int_{U_\varepsilon(z)} \frac{f(\xi)-f(z)}{\xi-z} d\xi \leq \sup\limits_{\xi \in U_\varepsilon(z)} \frac{|f(\xi)-f(z)|}{|\xi-z|}.\ L(U_\varepsilon(z)) = \sup\limits_{\xi \in U_\varepsilon(z)} \frac{|f(\xi)-f(z)|}{\varepsilon}.$
$2\pi\varepsilon = 2\pi \sup\limits_{\xi \in U_\varepsilon(z)} |f(\xi) - f(z)|$

$$\tag{1.9}$$

f continuă $\Rightarrow \lim\limits_{\xi \to z} f(\xi) = f(z)$. Când $\varepsilon \to 0$ avem $\xi \to z \Rightarrow$

$$\lim_{\varepsilon \to 0} \sup_{\xi \in U_\varepsilon(z)} |f(\xi) - f(z)| = 0 \tag{1.10}$$

Trecând la limită în inegalitatea (1.9) folosind relaţia (1.10) obţinem:

$$\lim_{\varepsilon \to 0} \int_{U_\varepsilon(z)} \frac{f(\xi) - f(z)}{\xi - z} d\xi = 0 \tag{1.11}$$

şi trecând la limită în (1.9) folosind relaţia (1.8) obţinem:

$$\lim_{\varepsilon \to 0} \int_{U_\varepsilon(z)} \frac{f(\xi)}{\xi - z} d\xi = 2\pi i f(z) \tag{1.12}$$

Trecând la limită în relaţia (1.7) după $\varepsilon \to 0$ şi folosind relaţia (1.9) obţinem:

$$f(z) = \frac{1}{2\pi i} \underbrace{\int_\Gamma \frac{f(\xi)}{\xi - z} d\xi}_{\substack{\| \\ \text{integrala Cauchy}}} = 2\pi i f(z) \Leftrightarrow$$

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z} d\xi.$$

2. Fie $\gamma : [a, b] \to \mathbb{C}$ o parametrizare a lui Γ. Avem:

$$\int_\Gamma \frac{f(\xi)}{\xi - z} d\xi = \int_a^b \frac{(f \circ \gamma)(t)}{\gamma(t) - z} \cdot \gamma'(t) dt \tag{1.13}$$

Notăm aplicaţia:

$$f(z) = \int_\Gamma \frac{f(\xi)}{\xi - z} d\xi, z \in \Delta. \tag{1.14}$$

Aplicaţia $(z, t) \mapsto \frac{(f\circ\gamma)(t)}{\gamma(t)-z} \cdot \gamma'(t)$ este derivabilă în raport cu z. Privim integrala din partea dreaptă a relaţiei (1.13) ca o integrală cu parametrul sau $F(z)$ e dată prin intermediul unei integrale cu parametru $\overset{(1.14)}{\Rightarrow}$ F este derivabilă în raport cu z şi după regula de derivare în raport cu un parametru sub semnul integralei avem:

$$2\pi i f'(z) = \int_a^b \frac{\partial}{\partial z}\left[\frac{(f \circ \gamma)(t)}{\gamma(t) - z} \cdot \gamma'(t)\right] dt =$$

$$= \int_a^b \frac{(f \circ \gamma)(t)}{[\gamma(t) - z]^2} \cdot \gamma'(t) dt = \int_\Gamma \frac{f(\xi)}{(\xi - z)^2} d\xi$$

$$\Leftrightarrow f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^2} d\xi$$

Deci: $f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi-z)^2} d\xi$ şi observăm că derivata comută cu integrala.

Din acest motiv, derivăm de $(n - 1)$ ori şi ţinem cont că (inducţie):

$$\left[\frac{1}{(\xi - z)^2}\right]_z^{(n-1)} = \frac{n!}{(\xi - z)^{n+1}},\ (\forall) n \geq 2,\ n \in \mathbb{N}.$$

Deci:

$$f^{(n)}(z) = \frac{1}{2\pi i}\left[\int_\Gamma \frac{f(\xi)}{(\xi - z)^2} d\xi\right]^{(n-1)} =$$

$$= \frac{1}{2\pi i}\int_\Gamma \left[\frac{f(\xi)}{(\xi - z)^{n+1}}\right]^{(n-1)} d\xi =$$

$$= \frac{1}{2\pi i}\int_\Gamma f(\xi)\cdot\left[\frac{1}{(\xi - z)^{n+1}}\right]^{(n-1)} d\xi =$$

$$= \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Q.E.D. (ceea ce trebuia demonstrat) formula generală. □

În rezumat:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z} d\xi =$$

$$\left.\begin{array}{r} = \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)(t)}{\gamma(t)-z}\cdot\gamma'(t)dt \to \text{integrala în raport cu } z \\ (\xi, z) \mapsto \frac{f(\xi)}{\xi - z} \text{ derivat în raport cu } z \\ (t, z) \mapsto \frac{(f\circ\gamma)(t)}{\gamma(t)-z}\cdot\gamma'(t) \text{ derivat în raport cu } z \end{array}\right\} \Rightarrow$$

$\Rightarrow f$ este derivabilă şi

$$f'(z) = \frac{1}{2\pi i}\int_a^b \frac{\partial}{\partial z}\left[\frac{(f\circ\gamma)(t)}{\gamma(t) - z}\cdot\gamma'(t)\right] dt =$$

$$= \frac{1}{2\pi i}\int_a^b \frac{(f\circ\gamma)(t)}{[\gamma(t) - z]^2}\cdot\gamma'(t)dt =$$

$$= \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^2} d\xi \Rightarrow$$

$\Rightarrow$ putem comuta derivata cu integrala

$$\Rightarrow f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi.$$

Observaţia 1.56 Dacă D nu e simplu conex, ci multiplu conex $\Rightarrow$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{\xi - z}d\xi - \frac{1}{2\pi i}\sum_{k=1}^{p}\int_{\Gamma_k}\frac{f(\xi)}{\xi - z}d\xi,\ (\forall)z \in D.$$

1.6 Serii Taylor şi serii Laurent

1.6.1 Serii de puteri. Raza şi domeniul de convergenţă

Definiţia 1.57 Se numeşte serie de puteri o serie de funcţii de forma

$$\sum_{n=0}^{\infty} a_n z^n.$$

Numărul $\rho := \sup\{r > 0 | \sum_{n=0}^{\infty} |a_n| r^n \text{ convergent}\}$ se numeşte *raza* de convergenţă a seriei, iar discul $D_\rho(z_0)$ se numeşte *domeniul* de convergenţă al seriei.

Teorema 1.58 *(Teorema Cauchy-Hadamard)*

Dacă notăm $\omega = \lim_{n\to\infty} \sqrt[n]{|a_n|}$, *atunci raza de convegenţă este:*

$$\begin{cases} \frac{1}{\omega},\ \omega \in (0, \infty) \\ +\infty,\ \omega = 0 \\ 0,\ \omega = \infty. \end{cases}$$

Teorema 1.59 *(Teorema lui Abel)*

Dacă ρ *este raza de convergenţă a seriei de puteri* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *avem:*

1. $(\forall)z \in D_\rho(z_0)$ *seria* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *este absolut convergentă;*

2. $(\forall)z \notin D_\rho(z_0)$ *seria* $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ *este divergentă;*

3. $(\forall)0 < r < \rho$ *şi* $(\forall)z \in D_\rho(z_0)$ *seria* $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ *este uniformă şi absolut convergentă;*

1.6.2 Seria Laurent. Seria Taylor

Definiţia 1.60 Fie $f : D \to \mathbb{C}$, $z_0 \in D$ astfel încât există $f^{(n)}(z_0)$, $(\forall)n \in \mathbb{N}$.

1. Seria $\sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0)$ se numeşte seria Taylor asociată lui f în z_0.

2. Dacă există $r > 0$ astfel încât $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)$, $(\forall)z \in D_r(z_0)$ spunem că f se dezvoltă în serie Taylor în jurul lui z_0.

Teorema 1.61 *(Seria Taylor) Fie* $f : D \to \mathbb{C}$ *olomorfă şi* $z_0 \in D$. *Notăm* $r < dist(z_0, \partial D)$. *Atunci:*

$$f(z) = \sum_{n\geq 0} \frac{f^{(n)}(z_0)}{n!} \cdot (z-z_0), (\forall)z \in D_r(z_0).$$

Demonstraţie. Conform formulei integrale a lui Cauchy avem:

$$f(z) = \frac{1}{2\pi i} \int_{U_r(z_0)} \frac{f(\xi)}{\xi - z} d\xi,\ (\forall)z \in D_r(z_0).$$

Pentru $\xi \in U_r(z_0)$ şi $z \in D_r(z_0)$ avem:

$$|\xi - z_0| = r,\ |z - z_0| < r \Rightarrow \left| \frac{z-z_0}{\xi - z_0} \right| = \rho_z < 1.$$

$$\frac{1}{\xi - z} = \frac{1}{\xi - z_0 - (z - z_0)} = \frac{1}{\xi - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{\xi - z_0}} = \text{ seria geometrică}$$

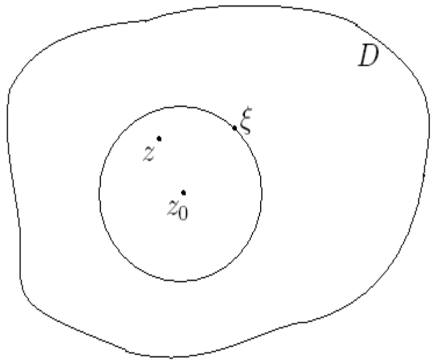

Figura 11.

$$= \frac{1}{\xi - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\xi - z_0} \right)^n, (\forall) z \in D_r(z_0)$$

Seria geometrică este uniform convergentă în ξ pe $U_r(z_0)$ şi deci integrala pe curbă permută cu suma infinită. Deci:

$$f(z) = \int_{U_r(z_0)} \frac{f(\xi)}{\xi - z_0} \left[\sum_{n=0}^{\infty} \frac{(z - z_0)^n}{(\xi - z_0)^n} \right] d\xi =$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \left\{ \int_{U_r(z_0)} \frac{f(\xi) d\xi}{(\xi - z_0)^{n+1}} \right\} \cdot (z - z_0)^n =$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{U_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi \right] \cdot (z - z_0)^n =$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} \cdot (z - z_0)^n, \ (\forall) z \in D_r(z_0).$$

□

Observaţia 1.62 Avem:

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_{U_r(z_0)} \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi, \ (\forall) n \in \mathbb{N}.$$

Teorema 1.63 *(Seria Laurent) Fie $W_{r,\rho}(z_0) = \{z \in \mathbb{C} | r < |z - z_0| < \rho\}$ coroana circulară de rază interioară r şi rază exterioară ρ şi funcţia $f : W_{r,\rho}(z_0) \to \mathbb{C}$ olomorfă. Atunci:*

$$f(z) = \sum_{n\in\mathbb{Z}} a_n \cdot (z - z_0)^n, (\forall) z \in W_{r,\rho}(z_0).$$

Se spune că f se dezvoltă în serie Laurent în coroana $W_{r,\rho}(z_0)$.
$a_n = \frac{1}{2\pi i} \int_\Gamma \frac{f(\xi)}{(\xi - z_0)^{n+1}} d\xi$, $(\forall) n \in \mathbb{Z}$, *unde:*
$\Gamma =$ *curbă închisă, simplă, de clasă C^1 pe porţiuni, ce înconjoară z_0 în coroană.*

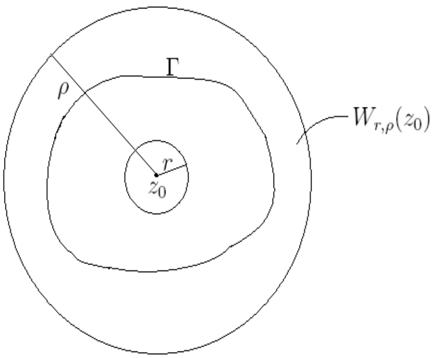

Figura 13.

Demonstraţie. Fie r' şi ρ' astfel încât $\Gamma \subset W_{r',\rho'}(z_0)$ şi fie $z \in W_{r',\rho'}(z_0)$ în care vrem să verificăm formula din teoremă.

Aplicăm formula integrală a lui Cauchy pentru domeniul dublu conex $W_{r',\rho'}(z_0)$ şi funcţia f. Avem:

$$f(z) = \frac{1}{2\pi i} \int_{U_{\rho'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi - \frac{1}{2\pi i} \int_{U_{r'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi \qquad (1.15)$$

Pentru $\xi \in U_{\rho'}(z_0)$ avem:

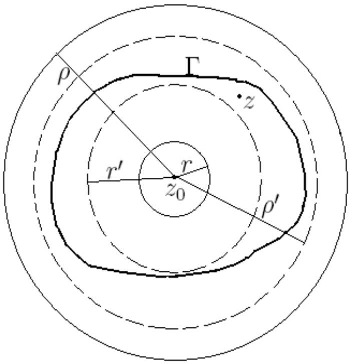

Figura 15.

$\frac{1}{1-\xi} = \frac{1}{\xi-z_0-(z-z_0)} = \frac{1}{\xi-z_0} \cdot \frac{1}{1-\frac{z-z_0}{\xi-z_0}} = \frac{1}{\xi-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{\xi-z_0}\right)^n$, serie care converge uniform în raport cu ξ pe $U_{\rho'}(z_0)$ şi deci:

$$\frac{1}{2\pi i}\int_{U_{\rho'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\int_{U_{\rho'}(z_0)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\right] \cdot (z-z_0)^n \tag{1.16}$$

Aplicând teorema fundamentală a lui Cauchy pentru domenii multiplu conexe (domeniul mărginit de curbele Γ şi $U_{\rho'}(z_0)$) funcţiei $\xi \mapsto \frac{f(\xi)}{(\xi-z_0)^{n+1}}$ găsim:

$$\int_{U_{\rho'}(z_0)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \tag{1.17}$$

şi înlocuind (1.17) în (1.15) obţinem:

$$\frac{1}{2\pi i}\int_{U_{\rho'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i}\int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi\right] \cdot (z - z_0)^n \tag{1.18}$$

Pentru $\xi \in U_{r'}(z_0)$ avem:

$$\frac{1}{1-\xi} = \frac{1}{\xi - z_0 - (z - z_0)} = -\frac{1}{z - z_0} \cdot \frac{1}{1 - \frac{\xi - z_0}{z - z_0}} =$$

$$= -\frac{1}{z - z_0} \sum_{n=0}^{\infty} \left(\frac{\xi - z_0}{z - z_0} \right)^n,$$

serie care converge uniform în raport cu ξ pe $U_{r'}(z_0)$ de unde:

$$-\frac{1}{2\pi i} \int_{U_{r'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi =$$

$$= \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_{U_{r'}(z_0)} \frac{f(\xi)}{(\xi - z)^{-n}} d\xi \right] \cdot (z - z_0)^{-n-1} =$$

$$= \sum_{n \leq -1} \left[\frac{1}{2\pi i} \int_{U_{r'}(z_0)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right] \cdot (z - z_0)^n. \tag{1.19}$$

Cu Teorema fundamentală Cauchy pentru domenii dublu conexe avem:

$$\int_{U_{r'}(z_0)} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi = \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \tag{1.20}$$

şi înlocuind (1.20) în (1.19) găsim:

$$-\frac{1}{2\pi i} \int_{U_{r'}(z_0)} \frac{f(\xi)}{\xi - z} d\xi = \sum_{n \leq -1} \left[\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi \right] \cdot (z - z_0)^n \tag{1.21}$$

Din (1.18) şi (1.21), notând $a_n = \int_{\Gamma} \frac{f(\xi)}{(\xi - z)^{n+1}} d\xi$, $(\forall) n \in \mathbb{Z}$, găsim:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n, (\forall) z \in W_{r,\rho}(z_0).$$

□

Exemplul 1.64 Verificaţi următoarele dezvoltări în serie Taylor în jurul lui 0:

1. $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \rho = 1;$
2. $\text{sh } z = \frac{e^z - e^{-z}}{2} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{z^n}{n!} [1 - (-1)^n] = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}, \rho = \infty;$
3. $\text{ch } z = \frac{e^z + e^{-z}}{2}, \rho = \infty;$
4. $\cos z = \frac{e^{iz} + e^{-iz}}{2}, \rho = \infty;$
5. $\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \rho = \infty;$
6. $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z| < 1.$

Aplicaţia 1.65 Să se dezvolte în serie Taylor în jurul lui $z_0 = 1$ funcţia

$$f(z) = \frac{z^2 - 1}{z^2 + 1}.$$

Soluţie.

$$f(z) = 1 - \frac{2}{z^2 + 1} = 1 - \frac{2}{(z + i)(z - i)} =$$

$$= 1 - \frac{2}{2i} \left(\frac{1}{z - i} - \frac{1}{z + i} \right) = 1 + i \left(\frac{1}{z - i} - \frac{1}{z + i} \right).$$

$$\frac{1}{z - i} = \frac{1}{z - 1 + (1 - i)} = \frac{1}{1 - i} \cdot \frac{1}{1 + \frac{z-1}{1-i}} =$$

$$= \frac{1}{1 - i} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(1 - i)^n} \cdot (z - 1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (z - 1)^n}{(1 - i)^{n+1}}$$

pentru $|z-1| < |1-i| = \sqrt{2}$. Am folosit seria geometrică

$$\begin{array}{l}\sum_{n=1}^{\infty} z^n = \frac{1}{1-z},\ |z| < 1\\ \sum_{n=1}^{\infty}(-1)^n \cdot z^n = \frac{1}{1+z},\ |z| < 1.\end{array}$$

$$\frac{1}{z-i} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1-i)^{n+1}} \cdot (z-1)^n,\ \text{pentru } |z-1| < \sqrt{2}.$$

$$\frac{1}{z+i} = \frac{1}{z-1+i+1} = \frac{1}{1+i} \cdot \frac{1}{1+\frac{z-1}{1+i}} =$$

$$= \frac{1}{1+i} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^n} \cdot (z-1)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} \cdot (z-1)^n$$

pentru $|z-1| < |1+i| = \sqrt{2}$.

Deci, pentru $|z-1| < \sqrt{2}$ avem:

$$f(z) = 1+i\left[\sum_{n=0}^{\infty} \frac{(-1)^n}{(1-i)^{n+1}} \cdot (z-1)^n - \sum_{n=0}^{\infty} \frac{(-1)^n}{(1+i)^{n+1}} \cdot (z-1)^n\right] =$$

$$= 1+i\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right) \cdot (z-1)^n =$$

$$= 1+i\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \cdot \left[(1+i)^{n+1} - (1-i)^{n+1}\right] \cdot (z-1)^n$$

$$\left\{\begin{array}{l} 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)\\ 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\end{array}\right| \Rightarrow (1+i)^{n+1} - (1-i)^{n+1} =$$

$$= \sqrt{2}\left(\cos\frac{(n+1)\pi}{4} + i\sin\frac{(n+1)\pi}{4} - \cos\frac{(n+1)\pi}{4} + \right.$$

$$\left. + i\sin\frac{(n+1)\pi}{4}\right) = 2i\sqrt{2}^{n+1}\sin\frac{(n+1)\pi}{4} \Rightarrow$$

$$\Rightarrow f(z) = 1 - \sum_{n=0}^{\infty} \frac{1}{2^{\frac{n+1}{2}}} \cdot \sin \frac{(n+1)\pi}{4} \cdot (z-1)^n$$

pentru $|z-1| < \sqrt{2}$.

Aplicaţia 1.66 Dezvoltaţi în serie de puteri ale lui $z+i$ funcţia

$$f(z) = \frac{z+1}{(z-1)(z+i)}.$$

Soluţie. Descompunem funcţia f în fracţii simple:

$$f(z) = \frac{z+1}{(z-1)(z+i)} =$$

$$= \frac{a}{z-1} + \frac{b}{z+i} \quad \begin{array}{l} \left| \cdot \begin{array}{c} (z-1) \\ z=1 \end{array} \right\rangle \Rightarrow a = \frac{2}{1+i} \\ \left| \cdot \begin{array}{c} (z+i) \\ z=-i \end{array} \right\rangle \Rightarrow b = \frac{z+1}{z-1}\Big|_{z=i} = \frac{1-i}{-1-i} = \frac{i-1}{1+i} \end{array}$$

$$f(z) = \frac{2}{1+i} \cdot \frac{1}{z-1} + \frac{i-1}{1+i} \cdot \frac{1}{z+i} =$$

$$= \frac{i-1}{i+1} \cdot \frac{1}{z+i} + \frac{2}{1+i} \cdot \frac{1}{z+i-1-i} =$$

$$= \frac{i-1}{i+1} \cdot \frac{1}{z+i} - \frac{2}{1+i} \cdot \frac{1}{(1+i)-(z+i)} =$$

$$= \frac{i-1}{i+1} \cdot \frac{1}{z+i} - \frac{2}{(1+i)^2} \cdot \frac{1}{1-\frac{z+i}{1+i}} =$$

$$= \frac{(i+1)(i-1)}{-2} \cdot \frac{1}{z+i} - \frac{2}{(1+i)^2} \cdot \sum_{n=0}^{\infty} \frac{(z+i)^n}{(1+i)^n}$$

pentru $|z+i| < |1+i| = \sqrt{2}$.

$$f(z) = \frac{i}{z+i} - 2\sum_{n=0}^{\infty} \frac{(z+i)^n}{(1+i)^{n+2}}, \text{ pentru } |z+i| < \sqrt{2}$$

(dezvoltarea în serie Laurent a lui f în jurul lui $z_0 = -i$).

Aplicaţia 1.67 Să se dezvolte în serie Laurent în jurul lui $z_0 = 0$ funcţia

$$f(z) = \frac{1}{z(z^2+1)}.$$

Soluţie.

$$f(z) = \frac{1}{z} - \frac{z}{z^2+1} = \frac{1}{z} - z \cdot \frac{1}{1+z^2} =$$

$$= \frac{1}{z} - z\sum_{n=0}^{\infty}(-1)^n \cdot z^{2n}, \text{ pentru } |z| < 1$$

$$f(z) = \frac{1}{z} - \sum_{n=0}^{\infty}(-1)^n \cdot z^{2n+1}, \ |z| < 1$$

1.6.3 Singularităţile izolate ale unei funcţii de variabilă complexă

Definiţia 1.68 $f : D \to \mathbb{C}$, $z_0 \in \overline{D}$ se numeşte *punct ordinar* pentru f dacă $(\exists)D_r(z_0)$ astfel încât f este derivabilă pe $D \cap D_r(z_0)$. În caz contrar, z_0 se numeşte *punct singular* al funcţiei f. (Punct singular: $(\forall)D_r(z_0)$ astfel încât f nu este derivabilă în toate punctele din $D_r(z_0)$.)

Definiţia 1.69 $f : D \to \mathbb{C}$, z_0 *punct singular* pentru f şi $(\exists)r > 0$ astfel încât f să se dezvolte în serie Laurent pe $W_{r,0}(z_0)$. Atunci z_0 se numeşte *punct singular izolat* pentru f.

Definiţia 1.70 Fie z_0 punct singular izolat pentru f şi fie dezvoltarea în serie Laurent a lui f pe $W_{r,0}(z_0)$:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(z-z_0)^n,\ (\forall)z \in W_{r,0}(z_0)$$

unde $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(\xi)}{(\xi-z_0)^n}d\xi$,unde Γ este curbă închisă, netedă pe porţiuni, simplă, ce înconjoară z_0 în $W_{r,0}(z_0)$.
Numim *reziduul* lui f în z_0 şi se notează $\text{Rez}[f, z_0]$ coeficientul a_{-1} din dezvoltarea în serie Laurent a lui f pe $W_{r,0}(z_0)$:

$$\text{Rez}[f, z_0] = a_{-1} = \frac{1}{2\pi i}\int_\Gamma f(\xi)d\xi.$$

1. $a_n = 0,\ (\forall)n \leq -1 \Rightarrow z_0$ se numeşte *punct singular aparent.*

2. $(\exists)k \in \mathbb{N}*$ astfel încât $a_n = 0,\ (\forall)n \leq -k-1$, z_0 se numeşte *pol de ordinul k.*

În acest caz $f(z) = \frac{g(z)}{(z-z_0)^k}$, g este olomorfă cu $g(z_0) \neq 0$ pe $D_r(z_0)\backslash\{z_0\} = W_{r,0}(z_0)$. z_0 se numeşte pol de ordinul k dacă $(\exists)\lim\limits_{z\to z_0}\left[(z-z_0)^k \cdot g(z)\right]$ finită.

1. Dacă $(\forall)k \in \mathbb{N}*, (\exists)k' \in \mathbb{N}*$ astfel încât $k' \geq k \Rightarrow a_{-k'} \neq 0$, atunci z_0 se numeşte punct singular esenţial pentru f.

Altfel spus:

a. f punct singular dacă în orice $D_r(z_0)$ f are şi puncte în care F este derivabilă cât şi punctele în care f nu e derivabilă.

b. f punct singular izolat dacă $(\exists)W_{r,0}(z_0)$ pe care f este derivabilă; f nu e derivabilă sau nici definită în z_0.

1.7 Teorema reziduurilor. Aplicaţii

1.7.1 Teorema reziduurilor

Proprietatea 1.71 Dacă z_0 este un pol de ordinul k pentru f atunci:

$$\text{Rez}\,[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} [(z-z_0)^k \cdot f(z)]^{(k-1)}$$

Demonstraţie. z_0 pol de ordinul k, atunci f are dezvoltarea în serie Laurent în jurul lui z_0 de forma:

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + ...$$

$$+\frac{a_{-1}}{z-z_0} + a_0 + a_1(z+z_0) + a_1(z+z_0)^2 + ..., \ (\forall) z \in W_{r,0}(z_0) \Rightarrow$$

$$\Rightarrow (z-z_0)^k \cdot f(z) = a_{-k} + a_{-k+1}(z-z_0) + ...$$

$$+a_{-1}(z+z_0)^{k-1} + a_0(z+z_0)^k + a_1(z+z_0)^{k+1} + ... \Rightarrow$$

$$\Rightarrow [(z-z_0)^k \cdot f(z)]^{(k-1)} = 0 + 0 + 0 + ...$$

$$+(k-1)! a_{-1} + k! \cdot a_0 \cdot (z+z_0) + \frac{(k+1)!}{2!} \cdot a_1 \cdot (z+z_0)^2 + ... \Rightarrow$$

$$\Rightarrow \lim_{z \to z_0} [(z-z_0)^k \cdot f(z)]^{(k-1)} = (k-1)! \cdot a_{-1} \Rightarrow$$

$$\text{Rez}\,[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} [(z-z_0)^k \cdot f(z)]^{(k-1)} = a_{-1}$$

$k = 1 \Rightarrow z_0$ pol de ordinul unu,

$$f(z) = \frac{g(z)}{h(z)} \Rightarrow \text{Rez}\,[f, z_0] = \frac{g(z_0)}{h'(z_0)}.$$

Dacă $z = \infty$, atunci tipul punctului ∞ pentru funcţia $f(z)$ (este) se defineşte ca fiind tipul punctului 0 pentru funcţia $f\left(\frac{1}{\xi}\right)$ şi:

$\text{Rez}\,[f, \infty] = \text{Rez}\,\left[-\frac{1}{\xi^2} \cdot f\left(\frac{1}{\xi}\right), 0\right] = a_{-1}$ $\leftarrow$coeficientul lui $\frac{1}{z}$ din dezvoltarea în serie Laurent a lui f în jurul lui $z = \infty$ (se dezvoltă $f\left(\frac{1}{\xi}\right)$ în jurul lui $\xi = 0$, apoi se substituie ξ cu $\frac{1}{z}$). $\square$

Teorema 1.72 *(teorema reziduurilor) Fie $D \subset \mathbb{C}$ domeniu în $\mathbb{C}$, $f : D \backslash \{z_1, z_2, ..., z_n\} \to \mathbb{C}$ o funcţie olomorfă şi $\Gamma \subset D$ o curbă simplă, închisă şi netedă pe porţiuni, orientată în sens pozitiv şi care cuprinde în interiorul domeniului delimitat de ea punctele singular izolate $z_1, z_2, ..., z_n$. Atunci:*

$$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^{n} \text{Rez}\,[f, z_0].$$

Demonstraţie. Pentru fiecare singulatitate z_k considerăm coroana $W_{r_k,0}(z_k)$ în care f este derivabilă şi Γ_k curba ce înconjoară pe z_k în coroană.

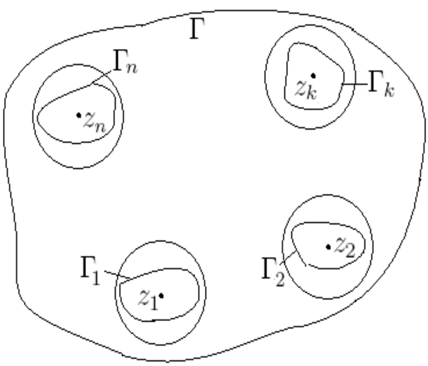

Figura 14.

Conform teoremei fundamentale a lui Cauchy pentru domenii multiplu conexe avem:

$\int_\Gamma f(z)dz = \sum_{k=1}^n \int_{\Gamma_k} f(z)dz$. Cu definiţia reziduului lui f în punctul z_k avem: $\text{Rez}[f, z_k] = \frac{1}{2\pi i}\int_{\Gamma_k} f(z)dz$, deci: $\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^n \text{Rez}[f, z_k]$. □

Consecinţa 1.73 $f : \mathbb{C}\backslash\{a_1, a_2, ..., a_n\} \to \mathbb{C}$, unde $a_1, a_2, ..., a_n$ sunt puncte singular izolate, iar f este olomorfă $\Rightarrow \sum_{k=1}^n \text{Rez}[f, a_k] + \text{Rez}[f, \infty] = 0$.

Teorema 1.74 *(teorema semireziduurilor) Fie $D \subset \mathbb{C}$, $f : D \to \mathbb{C}$ olomorfă, $\Gamma \in \bar{D}$ curbă închisă, simplă, netedă pe porţiuni ce cuprinde în interiorul delimitat punctele singular izolate $z_1, z_2, ..., z_n$ si pe care sunt doar puncte singular izolate de tip pol simplu $a_1, a_2, ..., a_n$. Atunci:*
$\int_\Gamma f(z)dz = 2\pi i \sum_{k=1}^n Rez[f, z_i] + \sum_{k=1}^m \delta_k i Rez[f, a_k]$, unde δ_k este unghiul sub care se vede curba Γ din punctul a_k.

1.7.2 Calculul unor integrale reale cu ajutorul teoremei reziduurilor

Lema 1.75 Fie arcul de cerc $\Gamma_r : z = r \cdot e^{i\theta}, \theta \in [\theta_1, \theta_2]$ şi f o funcţie continuă. Dacă: $\lim\limits_{\substack{r\to\infty \\ (r\to 0)}} \sup\limits_{z\in\Gamma_r} |z \cdot f(z)| = 0$, atunci:

$$\lim_{\substack{r\to\infty \\ (r\to 0)}} \int_{\Gamma_r} f(z)dz = 0.$$

Demonstraţie.

$$\left|\int_{\Gamma_r} f(z)dz\right| = \left|\int_{\theta_1}^{\theta_2} f(r \cdot e^{i\theta}) \cdot r \cdot i \cdot e^{i\theta} d\theta\right| \leq$$

$$\leq \int_{\theta_1}^{\theta_2} f(r \cdot e^{i\theta}) \cdot r d\theta \leq r(\sup_{\theta \in [\theta_1, \theta_2]} \left| f(r \cdot e^{i\theta}) \right|) \cdot (\theta_2 - \theta_1) \overset{r=|z|_{\Gamma_r}}{=}$$

$$= (\sup_{z \in \Gamma_r} |zf(z)|) \cdot (\theta_2 - \theta_1) \xrightarrow[]{\substack{r \to \infty \\ (r \to 0)}} 0$$

$$\lim_{\substack{r \to \infty \\ (r \to 0)}} \int_{\Gamma_r} f(z) dz = 0.$$

□

Lema 1.76 Fie semicercul $\Gamma_r : z = r \cdot e^{i\theta}, \theta \in [0, 2\pi]$ şi f o funcţie continuă. Dacă: $\lim\limits_{\substack{r \to \infty \\ (r \to 0)}} \sup\limits_{z \in \Gamma_r} |f(z)| = 0$, atunci:

$$\lim_{\substack{r \to \infty \\ (r \to 0)}} \left(\int_{\Gamma_r} f(z) \cdot e^{\lambda z i} dz \right) = 0, (\forall) \lambda > 0.$$

Teorema 1.77 *Fie polinoamele $P, Q \in \mathbb{R}[X], \lambda > 0$.*

1. *Dacă $Q(x) \neq 0, (\forall) x \in \mathbb{R}$ şi grad $P + 2 \leq$ grad Q, atunci:*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_j \text{Rez}[f, z_j],$$

   unde $f(z) = \frac{P(z)}{Q(z)},\ Q(z_j) = 0,\ Im z_j > 0$.

2. *Dacă $Q(x) \neq 0, (\forall) x \in \mathbb{R}$ şi grad $P + 1 \leq$ grad Q, atunci:*

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x} dx = 2\pi i \sum_j \mathrm{Rez}[f, z_j],$$

unde $f(z) = \frac{P(z)}{Q(z)} \cdot e^{i\lambda z}$, cu $Q(z_j) = 0$ *şi* $Im z_j > 0$.

Demonstraţie.

1. Fie $r > 0$ suficient de mare încât toate punctele singular izolate ale lui f cu partea imaginară > 0 să fie situate în domeniul delimitat de semicercul superior $\Gamma_r = \{z \in U_r(0) | Im z > 0\}$ şi fie segmentul $T = [-r, r]$.

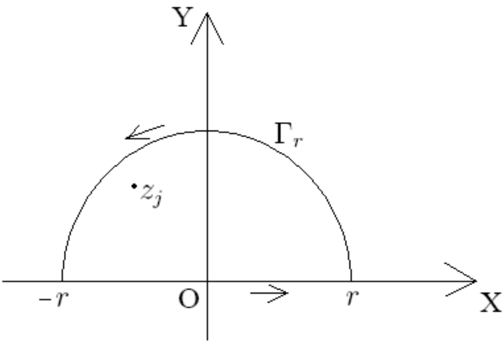

Figura 15.

Aplicând teorema reziduurilor funcţei f pe curba închisă $\Gamma_r \vee T$ avem:

$$\int_{\Gamma_r} f(z)dz + \int_T f(z)dz = 2\pi i \sum_j \mathrm{Rez}[f, z_j],\ Im z_j > 0. \tag{1.22}$$

Deoarece: $grad\ P+2 \leq grad\ Q \Rightarrow \lim\limits_{\substack{r \to \infty \\ (r \to 0)}} \sup\limits_{z \in \Gamma_r} |z \cdot f(z)| =$

0 şi din lema (1.75) avem:

$$\lim_{r\to\infty}\int_{\Gamma_r} f(z)dz = 0 \tag{1.23}$$

$$\lim_{r\to\infty}\int_T f(z)dz = \lim_{r\to\infty}\int_{-r}^{r}\frac{P(x)}{Q(x)}dx = \int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx. \tag{1.24}$$

În relaţia (1.22) trecem la limită după $r \to \infty$ şi cu relaţiile (1.23), (1.24) găsim:

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}dx = 2\pi i\sum_j \text{Rez}[f, z_k].$$

2. Analog pentru $f(z) = \frac{P(x)}{Q(x)}\cdot e^{i\lambda z}$.

   Cu teorema reziduurilor avem:

$$\int_{\Gamma_r} f(z)dz + \int_{-r}^{r}\frac{P(x)}{Q(x)}\cdot e^{i\lambda x}dx = 2\pi i\sum_j \text{Rez}[f, z_j] \tag{1.25}$$

   Avem: $grad\ P + 1 \leq grad\ Q \Rightarrow \lim\limits_{r\to\infty}\sup\limits_{z\in\Gamma_r}\left|\frac{P(x)}{Q(x)}\right| = 0 \Rightarrow$ conform lemei (1.76) avem:

$$\lim_{r\to\infty}\int_{\Gamma_r} f(z)dz = \lim_{r\to\infty}\left(\int_{\Gamma_r}\frac{P(z)}{Q(z)}\cdot e^{\lambda i z}dz\right) = 0. \tag{1.26}$$

   Trecând la limită după $r \to \infty$ în (1.25) şi ţinând cont de (1.26) $\Rightarrow$

$$\int_{-\infty}^{\infty}\frac{P(x)}{Q(x)}\cdot e^{i\lambda x}dx = 2\pi i\sum_j \text{Rez}[f, z_j].$$

□

Observaţia 1.78

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} dx = 2\pi i \sum_j \text{Rez}[f, z_j] + \pi i \sum_j \text{Rez}[f, a_j],$$

unde a_j sunt poli simpli;

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x} dx = 2\pi i \sum_j \text{Rez}[f, z_j] + \pi i \sum_j \text{Rez}[f, a_j];$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot \cos \lambda x dx = Re(\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x} dx);$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot \sin \lambda x dx = Im(\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cdot e^{i\lambda x} dx).$$

Observaţia 1.79 Integralele de forma $I = \int_0^{2\pi} R(\sin\theta, \cos\theta) \cdot e^{im\theta} d\theta,\ m \in \mathbb{N},\ R(x, y) = \frac{P(x,y)}{Q(x,y)}$, se calculează astfel:
se face schimbarea de variabilă:
$z = e^{i\theta}, \theta \in [0, 2\pi] \Rightarrow \begin{cases} \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z^2-1}{2iz} \\ \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z^2+1}{2z} \end{cases}$ şi $dz = i \cdot e^{i\theta} d\theta \Rightarrow d\theta = \frac{i}{iz} dz$.
Atunci: $I = \int_{\substack{|z| = 1 \\ U_1(0)}} R\left(\frac{z^2-1}{2iz}, \frac{z^2+1}{2z}\right) \cdot z^{m-1} dz$ care se calculează cu teorema reziduurilor.

Exerciţiul 1.80

a. Calculaţi reziduurile (inclusiv în ∞) pentru:

$$f(z) = \frac{z^2}{(z^2+1)^2},$$

$$f(z) = \frac{1-\cos z}{z^2},$$

$$f(z) = z \cdot e^{\frac{1}{z-1}}.$$

b. Calculaţi

$$\int_\Gamma \frac{1}{z} \sin \frac{1}{(z-1)^2} dz,$$

unde Γ este triunghiul de vârfuri $0,\ 2-2i,\ 2+2i$.

c.

$$I = \int_{-\infty}^{\infty} \frac{x^2}{1+x^4} dx;$$

$$I = \int_{-\infty}^{\infty} \frac{x\cos x}{x^2-2x+10} dx;$$

$$I = \int_0^{2\pi} \frac{\sin\theta \sin 2\theta}{5-4\sin\theta} d\theta = Im \int_0^{2\pi} \frac{\sin\theta \cdot (e^{i\theta})^2}{5-4\sin\theta} d\theta.$$

Aplicaţia 1.81 la teorema reziduurilor

$$\int_\Gamma \frac{e^{\frac{1}{z+1}}}{z(z+2)} dz, \Gamma : \frac{x^2}{2} + y^2 = 1$$

$$\int_\Gamma \frac{e^{\frac{1}{z+1}}}{z(z+2)} dz = 2\pi i \left[\mathrm{Rez}\,(f,-1) + \mathrm{Rez}\,(f,0) \right]$$

$$f(z) = \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right) \cdot e^{\frac{1}{z+1}} =$$

$$= \frac{1}{2}\left(\frac{1}{1+z-1}\cdot e^{\frac{1}{z+1}} - \frac{1}{1+(z+1)}\cdot e^{\frac{1}{z+1}}\right) =$$

$$= \frac{1}{2}\left(\frac{1}{-1+(z+1)}\cdot e^{\frac{1}{z+1}} - \frac{1}{1+(z+1)}\cdot e^{\frac{1}{z+1}}\right) =$$

$$= -\frac{1}{2}\left(\frac{1}{1-(z+1)} + \frac{1}{1+(z+1)}\right)\cdot e^{\frac{1}{z+1}} =$$

$$= -\frac{1}{2}\left[\sum_{n\geq 0}(z+1)^n + \sum_{n\geq 0}(-1)^n\cdot(z+1)^n\right]\cdot\sum_{m\geq 0}\frac{1}{m!(z+1)^m} =$$

$$= -\frac{1}{2}\sum_{n\geq 0}\left[1+(-1)^n\right]\cdot(z+1)^n\cdot\sum_{m\geq 0}\frac{1}{m!(z+1)^m} =$$

$$= -\frac{1}{2}\sum_{n,m\geq 0}\frac{1+(-1)^n}{m!}\cdot(z+1)^{n-m} \Rightarrow$$

$$-\frac{1}{2}\sum_{\substack{n,m\geq 0\\ n-m=p}}\frac{1+(-1)^n}{m!}\cdot(z+1)^{n-m} =$$

$$= -\frac{1}{2}\sum_{\substack{m\geq 0\\ p\in\mathbb{Z}}}\frac{1+(-1)^{m+p}}{m!}\cdot(z+1)^p \Rightarrow \mathrm{Rez}\,(f,-1) =$$

$$= -\frac{1}{2}\sum_{\substack{m\geq 0\\ p=-1}}\frac{1+(-1)^{m-1}}{m!} = -\frac{1}{2}(e-e^{-1}) = -\mathrm{sh}1$$

$$\mathrm{Rez}\,(f,-1) = c_{-1} = -\frac{1}{2}\sum_{\substack{n,m\geq 0\\ n-m=-1}}\frac{1+(-1)^n}{m!} =$$

$$= -\frac{1}{2} \sum_{\substack{n \geq 0 \\ m = m+1}} \frac{1 + (-1)^n}{(n+1)!} = -\frac{1}{2}(e - 1 - e^{-1} + 1) = -\mathrm{sh}1$$

$$\mathrm{Rez}\,(f, 0) = \lim_{z \to 0} z \cdot f(z) = \lim_{z \to 0} \frac{e^{\frac{1}{z+1}}}{z+2} = \frac{e}{2}$$

$$\int_\Gamma f(z)dz = \left(-\frac{e}{2} + \frac{e^{-1}}{2} + \frac{e}{2}\right) \cdot 2\pi i = 2\pi i \mathrm{ch}1.$$

Aplicaţia 1.82

$$\int_{|z|=3} \frac{e^{\frac{1}{z-1}}}{z(z-2)^2} dz = 2\pi i \left[\mathrm{Rez}\,(f, 1) + \mathrm{Rez}\,(f, 2) + \mathrm{Rez}\,(f, 0)\right]$$

$$\frac{1}{z(z-2)^2} = \frac{a}{z} + \frac{b}{z-2} + \frac{c}{(z-2)^2} \Rightarrow a = \frac{1}{4}; c = \frac{1}{2}$$

$$\frac{1}{4z} + \frac{{}^{2z/}1}{2(z-2)^2} + \frac{b}{z-2} = \frac{(z-2)^2 + 2z + 4(z^2 - 2z)b}{4z(z-2)^2} = \frac{4}{4z(z-2)^2}$$

$$\Rightarrow z^2 + 4bz^2 = 0 \Rightarrow b = -\frac{1}{4}$$

$$\frac{1}{z(z-2)^2} = \frac{1}{4z} - \frac{1}{4(z-2)} + \frac{1}{2(z-2)^2}$$

$$\Rightarrow f(z) = \frac{1}{4} \cdot \frac{1}{1 + (z-1)} \cdot e^{\frac{1}{z-1}} + \frac{1}{4} \cdot \frac{1}{1 - (z-1)} \cdot e^{\frac{1}{z-1}} +$$

$$+ \frac{1}{2} \cdot \frac{1}{\left[1 - (z-1)\right]^2} \cdot e^{\frac{1}{z-1}} =$$

$$= \frac{1}{4} \sum_{k \geq 0} (z-1)^k \cdot (-1)^k \sum_{m \geq 0} \frac{1}{m!(z-1)^m} +$$

$$+\frac{1}{4}\sum_{k\geq 0}(z-1)^k\cdot\sum_{m\geq 0}\frac{1}{m!(z-1)^m}+$$

$$+\frac{1}{2}\sum_{k\geq 0}(k+1)\cdot(z-1)^k\cdot\sum_{m\geq 0}\frac{1}{m!(z-1)^m}=$$

$$=\sum_{k\geq 0}\frac{1+(-1)^k+2(k+1)}{4}\cdot\sum_{k\geq 0}(z-1)^k\cdot(-1)^k=$$

$$=\frac{1}{4}\sum_{k,m\geq 0}\frac{1+(-1)^k+2(k+1)}{m!}\cdot(z-1)^{k-m}=$$

$$=\frac{1}{4}\sum_{\substack{m,k\geq 0\\ k-m=p}}\frac{1+(-1)^k+2(k+1)}{m!}\cdot(z-1)^{k-m}=$$

$$=\frac{1}{4}\sum_{\substack{m,k\geq 0\\ p\in\mathbb{Z}}}\frac{1+(-1)^{m+p}+2(m+p+1)}{m!}\cdot(z-1)^p$$

$$\left.\begin{array}{l}\mathrm{Rez}\,(f,1)=c_{-1}=\frac{1}{4}\sum_{m\geq 0}\frac{1+(-1)^{m-1}+2m}{m!}=\\ \qquad\qquad=\frac{1}{4}(e-e^{-1}+2e)=\frac{1}{4}(3e-e^{-1})\\ \mathrm{Rez}\,(f,0)=\frac{e^{-1}}{4}\\ \mathrm{Rez}\,(f,2)=\lim\limits_{z\to 2}\left(\frac{e^{\frac{1}{z-1}}}{z}\right)'=\left.\frac{-\frac{z}{(z-1)^2}\cdot e^{\frac{1}{z-1}}-e^{\frac{1}{z-1}}}{z^2}\right|_{z=2}=-\frac{3}{4}e\end{array}\right\}\Rightarrow$$

$$\Rightarrow\int_{|z|=3}f(z)dz=0$$

$$\int_{|z|=3}f(z)dz=-2\pi i\mathrm{Rez}\,(f,\infty)=2\pi i\mathrm{Rez}\left(\frac{1}{\xi^2}f\left(\frac{1}{\xi}\right),0\right)=$$

$$= 2\pi i \text{Rez}\left(\frac{1}{\xi^2}\cdot\frac{e^{\frac{\xi}{1-\xi}}}{\frac{1}{\xi}\cdot\frac{(1-2\xi)^2}{\xi^2}}, 0\right) = 2\pi i \text{Rez}\left(\xi\cdot e^{\frac{\xi}{1-\xi}}, 0\right) = 0 \leftarrow$$

$\xi = 0$ punct ordinar.

Aplicaţia 1.83

$$\int_\Gamma \frac{1}{z}\cdot\sin\frac{1}{(z-1)^2}dz,\ \Gamma : \Delta \text{ de vârfuri } 0, 2-2i, 2+2i.$$

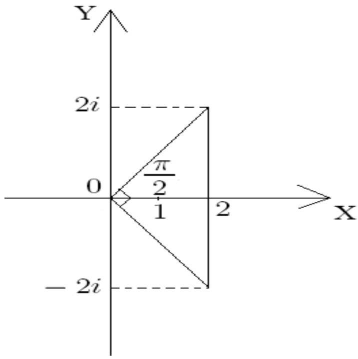

Figura 16.

$$\int_\Gamma \frac{1}{z}\cdot\sin\frac{1}{(z-1)^2}dz = 2\pi i \text{Rez}\,(f,1) + \frac{\pi i}{2}\text{Rez}\,(f,0)\,.$$

$$f(z) = \frac{1}{1+(z-1)}\cdot\sin\frac{1}{(z-1)^2} =$$

$$= \sum_{n\geq 0}(-1)^n\cdot(z-1)^n\cdot\sum_{m\geq 0}(-1)^m\cdot\frac{1}{(2m+1)!}\cdot\frac{1}{(z-1)^{2(2m+1)}} =$$

$$= \sum_{n,m\geq 0}\frac{(-1)^{n+m}}{(2m+1)!}\cdot(z-1)^{n-4m-2} = \sum_{\substack{m\geq 0\\ p\in\mathbb{Z}\\ n-4m-2=p}}$$

$$\frac{(-1)^{5m+2+p}}{(2m+1)!}\cdot(z-1)^p=\sum_{p\in\mathbb{Z}}(-1)^p\left[\sum_{m\geq 0}\frac{(-1)^m}{(2m+1)!}\right]\cdot(z-1)^p=$$

$$=\sum_{p\in\mathbb{Z}}(-1)^p\cdot(z-1)^p\cdot\underbrace{\sum_{m\geq 0}\frac{(-1)^m}{(2m+1)!}}_{\sin 1}=\sin 1\cdot\sum_{p\in\mathbb{Z}}(-1)^p\cdot(z-1)^p$$

$$\Rightarrow \operatorname{Rez}(f,1)=-\sin 1$$

$$\operatorname{Rez}(f,0)=\sin 1\Rightarrow\int_\Gamma f(z)dz-2\pi i\sin 1+\frac{\pi i}{2}\sin 1=-\frac{3\pi i}{2}\sin 1.$$

Aplicaţia 1.84

$$\int_{U_3(0)}\frac{z^n\cdot e^{\frac{1}{z}}}{z^2-1}dz=2\pi i\left[\operatorname{Rez}(f,1)+\operatorname{Rez}(f,-1)+\operatorname{Rez}(f,0)\right].$$

$$f(z)=-z^n\cdot\frac{1}{1-z^2}\cdot e^{\frac{1}{z}}=-z^n\left(\sum_{j\geq 0}\frac{1}{j!z^j}\right)\left(\sum_{k\geq 0}z^{2k}\right)=$$

$$=-\sum_{\substack{j,k\geq 0\\ 2k-j+n=p\in\mathbb{Z}}}\frac{z^{2k-j+n}}{j!}=-\sum_{\substack{k\geq 0\\ p\in\mathbb{Z}}}\frac{z^p}{(2k+n-p)!}$$

$$\Rightarrow \operatorname{Rez}(f,0)=c_{-1}=-\sum_{k\geq 0}\frac{1}{(2k+n+1)!}$$

$n=2p\Rightarrow$

$$\operatorname{Rez}(f,0)=-\sum_{k\geq 0}\frac{1}{[2(k+p)+1]!}=$$

$$= -1 - \sum_{k\geq 0} \frac{1}{(2k+1)!} + \frac{1}{1!} + \frac{1}{3!} + ... + \frac{1}{[2(p-1)+1]!} =$$

$$= -\mathrm{sh}1 + \sum_{k=0}^{p-1} \frac{1}{(2k-1)!}$$

$n = 2p+1 \Rightarrow$

$$\mathrm{Rez}\,(f,0) = -\sum_{k\geq 0} \frac{1}{2(k+p+1)!} = \sum_{k\geq 0} \frac{1}{(2k)!} + \sum_{k=0}^{p} \frac{1}{(2k)!} =$$

$$= -\mathrm{ch}1 + \sum_{k=0}^{p} \frac{1}{(2k)!}$$

$$\mathrm{Rez}\,(f,0) = \begin{cases} -\mathrm{sh}1 + \sum_{k=0}^{p-1} \frac{1}{(2k-1)!}, n = 2p \\ -\mathrm{ch}1 + \sum_{k=0}^{p} \frac{1}{(2k)!}, n = 2p+1 \end{cases}$$

$$\mathrm{Rez}\,(f,1) = \left. \frac{z^n \cdot e^{\frac{1}{z}}}{z+1} \right|_{z=1} = \frac{e}{2}$$

$$\mathrm{Rez}\,(f,-1) = \left. \frac{z^n \cdot e^{\frac{1}{z}}}{z-1} \right|_{z=-1} = (-1)^n \cdot \frac{e^{-1}}{-2} = (-1)^{n+1} \cdot \frac{e^{-1}}{2}$$

Pentru $n = 2p$ avem

$$\int_{U_3(0)} \frac{z^n \cdot e^{\frac{1}{z}}}{z-1} dz = 2\pi i \left[-\frac{e}{2} + \frac{e^{-1}}{2} + \sum_{k=0}^{p-1} \frac{1}{(2k+1)!} + \frac{e}{2} - \frac{e^{-1}}{2} \right] =$$

$$= 2\pi i \sum_{k=0}^{p-1} \frac{1}{(2k+1)!}$$

Pentru $n = 2p + 1$ avem

$$\int_{U_3(0)} \frac{z^n \cdot e^{\frac{1}{z}}}{z-1} dz = 2\pi i \left[-\frac{e}{2} - \frac{e^{-1}}{2} + \sum_{k=0}^{p} \frac{1}{(2k)!} + \frac{e}{2} + \frac{e^{-1}}{2} \right] =$$

$$= 2\pi i \sum_{k=0}^{p-1} \frac{1}{(2k)!}$$

Aplicaţia 1.85

$$\int_{U_r(0)} \frac{1}{(1+e^z)^2} dz, \ (2n-1)\pi < r < (2n+1)\pi, \ n \in \mathbb{N}^* \text{ dat.}$$

$f(z) = \frac{1}{(1+e^z)^2}$; Ecuaţia $1 + e^z = 0$ are soluţii în mulţimea $Ln(-1) \Rightarrow$

$$z_k = \ln|-1| + i(\arg(-1) + 2k\pi) = (2k+1)\pi i, \ k \in \mathbb{Z}.$$

$(1+e^z)'_{|z=z_k} = e^{z_k} = -1 \neq 0 \Rightarrow z_k \rightarrow$ soluţie simplă pentru ecuaţia $1 + e^z = 0 \Rightarrow z_k$ pol dublu pentru ecuaţia $(1+e^z)^2 = 0 \Rightarrow z_k \rightarrow$pol dublu pentru f:

$$\text{Rez}\,(f, z_k) = \frac{1}{1!} \lim_{z \to z_k} \left[\frac{(z-z_k)^2}{(1+e^z)^2} \right]' = \lim_{z \to z_k} \left[\frac{(z-z_k)^2}{(1+e^z)^2} \right]'$$

$$= \lim_{z \to z_k} 2 \frac{z-z_k}{1+e^z} \cdot \frac{1+e^z-(z-z_k) \cdot e^z}{(1+e^z)^2} =$$

$$= 2 \lim_{z \to z_k} \frac{z-z_k}{1+e^z} \cdot \lim_{z \to z_k} \frac{1+e^z-(z-z_k) \cdot e^z}{(1+e^z)^2} \overset{L.H.}{=}$$

$$2 \lim_{z \to z_k} \frac{1}{e^z} \cdot \lim_{z \to z_k} \frac{e^z - e^z - (z-z_k) \cdot e^z}{2e^z(1+e^z)} = (-1) \cdot (-1) \cdot (-1) = -1.$$

$$\left.\begin{array}{l} z_k \in \Delta_r(0) \Leftrightarrow |z_k| < r \\ (2n-1)\pi < r < (2n-1)\pi \end{array}\right\} \Rightarrow$$

$$z_k \in \Delta_r(0) \Leftrightarrow |z_k| \leq (2n-1)\pi \Leftrightarrow$$

$$\Leftrightarrow |2k+1| \leq |2n-1| \Leftrightarrow |k| \leq n-1 \Rightarrow$$

$$\Rightarrow \int_{U_r(0)} \frac{1}{(1+e^z)^2} dz = 2\pi i \sum_{k=-(n-1)}^{n-1} \operatorname{Rez}(f, z_k) =$$

$$= 2\pi i \cdot (-1) \cdot [n-1-(-(n-1))+1] =$$

$$= -2\pi i(2n-1) = 2(1-2n)\pi i.$$

Aplicaţia 1.86

$$\int_\Gamma \frac{z^3 e^{\frac{1}{z}}}{1+z} dz =, \Gamma : x^2 + y^2 + 2x + 2y - 2 = 0 \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2 + (y+1)^2 = 4 \Rightarrow |z + (1+i)| = 2$$

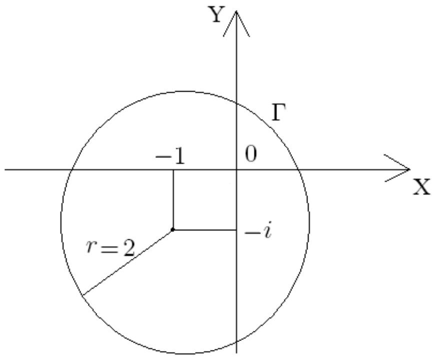

Figura 17.

$$\int_\Gamma \frac{z^3 e^{\frac{1}{z}}}{z+1} dz = 2\pi i \left[\operatorname{Rez}(f, 0) + \operatorname{Rez}(f, -1)\right]$$

$$f(z) = z^3 \left(\sum_{j \geq 0} (-1)^j z^j \right) \left(\sum_{k \geq 0} \frac{1}{k! z^k} \right) = \sum_{j,k \geq 0} \frac{(-1)^j}{k!} \cdot z^{j-k+3} =$$

$$= \sum_{\substack{j, k \geq 0 \\ j - k + 3 = p \in \mathbb{Z}}} \frac{(-1)^j}{k!} \cdot z^{j-k+3} =$$

$$= \sum_{\substack{j \geq 0 \\ p \in \mathbb{Z} \\ k = j + 3 - p}} \frac{(-1)^j}{(j+3-p)!} \cdot z^p$$

$$\Rightarrow f(z) = \sum_{\substack{j \geq 0 \\ p \in \mathbb{Z}}} \frac{(-1)^j}{(j+3-p)!} \cdot z^p$$

$$\Rightarrow \operatorname{Rez}(f, 0) = c_{-1} = \sum_{j \geq 0} \frac{(-1)^j}{(j+4)!} =$$

$$= \sum_{j \geq 4} \frac{(-1)^j}{j!} = \sum_{j \geq 0} \frac{(-1)^j}{j!} - \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right) =$$

$$= e^{-1} - \left(\frac{1}{2} - \frac{1}{6} \right) = e^{-1} - \frac{1}{3}$$

$$\operatorname{Rez}(f, -1) = -e^{-1} \Rightarrow$$

$$\Rightarrow \int_\Gamma \frac{z^3 \cdot e^{\frac{1}{z}}}{1+z} dz = 2\pi i \left(e^{-1} - \frac{1}{3} - e^{-1} \right) = -\frac{2\pi i}{3}$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx = \frac{2\pi i}{1 - e^{2\pi\lambda i}} \sum_j \operatorname{Rez}(f, z_j)$$

Avem identitatea următoare:

$$f(z) = \frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z| + i\arg z)},$$

$Q \neq 0$ pe $\mathbb{R}_+$, $Q(z_j) = 0$, $\lambda \in (-1,1)\backslash\{0\}$, gr $P+1+\lambda <$ gr Q.

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx =$$

$$= \frac{2\pi i}{1 - e^{2\pi\lambda i}} \left[\sum_j \mathrm{Rez}\,(f, z_j) + e^{2\pi\lambda i} \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx\right]$$

$$f(z) = \frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z| + i\arg z)} \cdot \lambda(\ln|z| + i\arg z),$$

gr$P + 1 + \lambda <$ gr Q, $Q(x) \neq 0$, $(\forall)x \in [0,\infty)$, $Q(z_j) = 0$, $\lambda \in (-1,1)\backslash\{0\}$.

Aplicaţia 1.87 i)

$$I = \int_0^\infty \frac{x^\alpha}{x^2 + a^2} dx, a > 0, \alpha \in (-1,1)\backslash\{0\}.$$

Soluţie:

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx = \frac{2\pi i}{1 - e^{2\pi\lambda i}} \sum_j \mathrm{Rez}\,(f, z_j);$$

$$f(z) = \frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z| + i\arg z)}$$

$$\begin{cases} P(x) = 1 \\ Q(x) = x^2 + a^2 \neq 0 \text{ pe } [0,\infty) \end{cases}$$

$\lambda = \alpha, Q(z) = 0 \Leftrightarrow z^2 + a^2 = 0 \Rightarrow z_{1,2} = \pm ia \rightarrow$ poli simpli pentru f, $|z_{1,2}| = a; \arg z_1 = \frac{\pi}{2}; \arg z_2 = \frac{3\pi}{2}$

$$f(z) = \frac{e^{\lambda}(\ln|z| + i\arg z)}{z^2 + a^2}$$

$$I = \int_0^{\infty} \frac{x^{\alpha}}{x^2 + a^2} dx =$$

$$= \frac{2\pi i}{1 - e^{2\pi\alpha i}} \left[\text{Rez} \left(\frac{e^{\alpha(\ln|z| + i\arg z)}}{z^2 + a^2}, ia \right) + \right.$$

$$\left. + \text{Rez} \left(\frac{e^{\alpha(\ln|z| + i\arg z)}}{z^2 + a^2}, -ia \right) \right] =$$

$$= \frac{2\pi i}{1 - e^{2\pi\alpha i}} \cdot \left[\frac{e^{\alpha(\ln a + i\frac{\pi}{2}}}{2ia} - \frac{e^{\alpha(\ln a + i\frac{3\pi}{2}}}{2ia} \right] =$$

$$= \frac{2\pi i}{1 - e^{2\pi i\alpha i}} \cdot \frac{1}{2ia} \cdot e^{\alpha\ln a} \left(e^{i\frac{\alpha\pi}{2}} - e^{i\frac{3\alpha\pi}{2}} \right) =$$

$$= \frac{\pi a^{\alpha}}{a(1 - e^{2\pi\alpha i})} \cdot e^{ia\pi} \left(\underbrace{e^{-i\frac{\alpha\pi}{2}} - e^{i\frac{\alpha\pi}{2}}}_{-2i\sin\frac{\alpha\pi}{2}} \right) =$$

$$= \left(\frac{-2\pi i}{a} \right) \cdot \frac{a^{\alpha}}{\underbrace{e^{-i\alpha\pi} - e^{i\alpha\pi}}_{(-2i)\sin\alpha\pi}} \cdot \sin\frac{\alpha\pi}{2} =$$

$$= \frac{\pi}{a} \cdot \frac{\sin\frac{\pi\alpha}{2} \cdot a^{\alpha}}{2\sin\frac{\pi\alpha}{2} \cdot \cos\frac{\pi\alpha}{2}} \Rightarrow$$

$$\Rightarrow \int_0^{\infty} \frac{x^{\alpha}}{x^2 + a^2} dx = \frac{\pi a^{\alpha}}{2a\cos\frac{\pi\alpha}{2}} = \frac{\pi a^{\alpha-1}}{2\cos\frac{\pi\alpha}{2}}.$$

ii)

$$\int_{U_1(0)} \frac{1}{z} \sin\frac{1}{z-1} dz = 2\pi i \text{Rez}\,(f, 0) + \pi i \text{Rez}\,(f, 1) = -3\pi i \sin 1$$

$$f(z) = \frac{1}{1+(z-1)} \cdot \sin\frac{1}{z-1} =$$

$$= \left[\sum_{n\geq 0}(-1)^n \cdot (z-1)^n\right] \cdot \left[\sum_{m\geq 0} \frac{(-1)^m}{(2m+1)! \cdot (z-1)^{2m+1}}\right] =$$

$$= \sum_{m,n\geq 0} \frac{(-1)^{m+n}}{(2m+1)!} \cdot (z-1)^{n-2m-1} =$$

$$= \sum_{p\in\mathbb{Z}} \left(\sum_{m\geq 0} \frac{(-1)^{3m}}{(2m+1)!}\right) \cdot (-1)^p \cdot (z-1)^p =$$

$$= \sum_{p\in\mathbb{Z}} \sin 1 \cdot (-1)^p \cdot (z-1)^p \Rightarrow$$

$$c_{-1} = \mathrm{Rez}\,(f,1) = -\sin 1$$

$$\mathrm{Rez}\,(f,0) = \sin\frac{1}{z-1}\bigg|_{z=0} = -\sin 1$$

iii)

$$\int_{x^2+\frac{y^2}{2}=1} \frac{1}{z+1}\sin\frac{1}{z}dz = ?.$$

Avem identitatea următoare:

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx =$$

$$= \frac{2\pi i}{1-e^{2\pi\lambda i}}\left[\sum_j \mathrm{Rez}\,(f,z_j) + e^{2\pi\lambda i}\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx\right]$$

$$f(z) = \frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z|+i\arg z)} \cdot \lambda(\ln|z| + i\arg z).$$

Aplicaţia 1.88

$$I = \int_0^\infty \frac{1}{x^2+1} \cdot x^{\frac{1}{2}} \ln x dx = ?$$

$$\lambda = \frac{1}{2}; z_{1,2} = \pm i \Rightarrow$$

$$|z_{1,2}| = 1, \arg z_1 = \frac{\pi}{2}, \arg z_2 = \frac{3\pi}{2}, f(z) = \frac{1}{z^2+1} \cdot e^{\frac{1}{2}(\ln|z| + i\arg z)}$$

$$I = \int_0^\infty \frac{1}{x^2+1} \cdot x^{\frac{1}{2}} \ln x dx =$$

$$= \frac{2\pi i}{2} \left[\text{Rez} \left(\frac{e^{\frac{1}{2}(\ln|z|+i\arg z)}}{z^2+1} \cdot (\ln|z| + i\arg z), i \right) + \right.$$

$$\left. + \text{Rez} \left(\frac{e^{\frac{1}{2}(\ln|z|+i\arg z)}}{z^2+1} \cdot (\ln|z| + i\arg z), -i \right) + e^{\frac{\overbrace{2\pi i}^{=-1}}{2}} \cdot \frac{\pi}{2\cos\frac{\pi}{4}} \right] =$$

$$= \pi i \left[\frac{1}{2i} \cdot e^{\frac{1}{2}\left(\ln|1| + i\frac{\pi}{2}\right)} \cdot \left(\ln|1| + i\frac{\pi}{2} \right) - \right.$$

$$\left. \frac{1}{2i} \cdot e^{\frac{1}{2}\left(\ln|1| + i\frac{3\pi}{2}\right)} \cdot \left(\ln|1| + i\frac{3\pi}{2} \right) + \frac{\pi i \cdot (-1)}{2 \cdot \frac{\sqrt{2}}{2}} \right] =$$

$$= \pi i \left[\frac{e^{i\frac{\pi}{4}}}{2} \cdot i\frac{\pi}{2} - \frac{e^{i\frac{3\pi}{4}}}{2} \cdot i\frac{3\pi}{2} - \frac{\pi\sqrt{2}}{2} \right] = \frac{\pi^2 i}{4} \left(e^{i\frac{\pi}{4}} - 3e^{i\frac{3\pi}{4}} - 2\sqrt{2} \right) =$$

$$= \frac{\pi^2 i}{4} \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} + 3\frac{\sqrt{2}}{2} - 3i\frac{\sqrt{2}}{2} - 2\sqrt{2} \right) =$$

$$= \frac{\pi^2 i}{4}\left(2\sqrt{2} - i\frac{2\sqrt{2}}{2} - 2\sqrt{2}\right) =$$

$$= -\pi^2 i^2 \frac{\sqrt{2}}{4} = \frac{\pi^2}{2\sqrt{2}} \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{1}{x^2+1} \cdot x^{\frac{1}{2}} \ln x dx = \frac{\pi^2}{2\sqrt{2}}.$$

Aplicaţia 1.89

$$\int_0^{2\pi} \frac{\sin\theta \sin n\theta}{5 - 4\sin\theta} d\theta, n \in \mathbb{N}^*$$

$$I = \int_0^{2\pi} \frac{\sin\theta}{5-4\sin\theta} \cdot Im(e^{in\theta}) d\theta = Im \int_0^{2\pi} \frac{\sin\theta}{5-4\sin\theta} \cdot (e^{in\theta}) d\theta$$

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta} d\theta \Rightarrow d\theta = \frac{1}{iz} dz; \sin\theta = \frac{z^2-1}{2iz} \Rightarrow$$

$$I = Im\left(\int_{|z|=1} \frac{\frac{(z^2-1)}{2iz}}{5 - \frac{4z^2-4}{2iz}} \cdot z^n \cdot \frac{1}{iz} dz\right) =$$

$$= Im\left(\frac{1}{i}\int_{|z|=1} \frac{(z^2-1)\cdot z^{n-1}}{-2(2z^2 - 5iz - 2} dz\right) =$$

$$= Im\left(\frac{1}{i} \cdot \frac{2\pi i}{-2} \text{Rez}\left(\frac{(z^2-1)\cdot z^{n-1}}{-2(2z^2-5iz-2}, \frac{i}{2}\right)\right) =$$

$$= -\pi Im\left(\left.\frac{(z^2-1)\cdot z^{n-1}}{4z-5i}\right|_{z=\frac{i}{2}}\right) =$$

$$= -\pi Im \frac{\frac{3}{2} \cdot \frac{i^{n-1}}{2^{n-1}}}{3i} = \frac{\pi}{2^n} Im(i^n) = \frac{\pi}{2^n} Im \begin{cases} \pm 1, n = 4k, 4k+2 \\ i, n = 4k+1 \\ -i, n = 4k+3 \end{cases} =$$

$$= \begin{cases} 0, n = 4k, 4k+2 \\ \frac{\pi}{2^n}, n = 4k+1 \\ -\frac{\pi}{2^n}, n = 4k+3 \end{cases}$$

$$\int_0^{2\pi} \frac{\sin\theta \sin n\theta}{5 - 4\sin\theta} d\theta = Im \left(\int_0^{2\pi} \frac{\sin\theta \left(e^{i\theta}\right)^2}{5 - 4\sin\theta} d\theta \right) =$$

$$= Im \left(\frac{1}{i} \int_{|z|=1} \frac{\frac{z^2-1}{2iz} \cdot z^2}{iz/5 - \frac{2(z^2-1)}{iz}} \cdot \frac{1}{z} dz \right) =$$

$$= -Im \left(\frac{1}{2i} \int_{|z|=1} \frac{z(z^2-1)}{2z^2 - 5iz - 2} dz \right) =$$

$$= -Im \left(\frac{1}{2i} \cdot 2\pi i \left. \frac{z(z^2-1)}{4z - 5i} \right|_{z=\frac{i}{2}} \right) =$$

$$= -\pi Im \left(\frac{\frac{i}{2} \cdot \left(\frac{-3}{4}\right)}{-3i} \right) = -\frac{\pi}{8} \cdot \frac{5}{3} = -\frac{5\pi}{24}.$$

Aplicaţia 1.90 Calculaţi integralele următoare:
i)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2} dx;$$

ii)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)} dx, a > 0, b > 0;$$

iii)

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^2(x^2+b^2)} dx, a > 0, b > 0.$$

Aplicaţia 1.91

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + 2x + 5} dx;$$

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4x + 20} dx;$$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 9)(x^2 + 2x + 2)} dx.$$

$$\int_{U_2(0)} \frac{1}{(z-1)^2} \cdot e^{\frac{1}{z}} dz;$$

$$\int_{0}^{2\pi} \frac{\cos 2\theta}{2 - \cos\theta} d\theta$$

Seria Taylor pentru $f(z) = \frac{1}{(z^2-1)^2}$ în jurul lui $z_0 = 1, z_0 = 0$.
Seria Laurent, precizând domeniul în care are sens pentru $f(z) = \frac{z}{z+3} \cdot e^{\frac{1}{z}}$.

Aplicaţia 1.92

$$\int_{\gamma} \frac{dz}{(z-1)^2 \cdot (z^2 + 1)}, \gamma : x^2 + y^2 = 2x + 2y.$$

$(x-1)^2 + (y-1)^2 = 2 \Leftrightarrow |z - (1+i)| = \sqrt{2}$
$z_1 = 1 \rightarrow$ pol dublu pe axa reală, în interiorul lui γ.
$z_{2,3} = \pm i \rightarrow$ poli simpli:

$$\left.\begin{array}{l} |z_2 - (1+i)| = |i - 1 + i| = 1 < \sqrt{2} \\ |z_3 - (1+i)| = |-1 - 2i| = \sqrt{5} > \sqrt{2} \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} z_2 \in Int\gamma \\ z_3 \notin Int\gamma \end{array}\right.$$

$$\int_\gamma \frac{dz}{(z-1)^2 \cdot (z^2+1)} = 2\pi i \left[\mathrm{Rez}\,(f,1) + \mathrm{Rez}\,(f,i)\right] =$$

$$= 2\pi i \left(-\frac{1}{2} + \frac{1}{4}\right) = \frac{-2\pi i}{4} = -\frac{\pi i}{2}$$

$$\mathrm{Rez}\,(f,1) = \frac{1}{1!} \lim_{z\to 1} \left[(z-1)^2 \cdot \frac{1}{(z-1)^2 \cdot (z^2+1)}\right]' =$$

$$= \left.\frac{-2z}{(z^2+1)^2}\right|_{z=1} = \frac{-2}{4} = -\frac{1}{2}$$

$$\mathrm{Rez}\,(f,i) = \left.\frac{1}{(z-1)^2 \cdot 2z + 2(z-1)\cdot(z^2+1)}\right|_{z=i} =$$

$$= \frac{1}{2i(1-i)^2} = \frac{1}{4}.$$

Aplicaţia 1.93 (temă)

$$\int_0^\infty \frac{dx}{1+x^4}.$$

Aplicaţia 1.94

$$\int_{-\infty}^\infty \frac{x\cdot\cos x}{x^2-2x+10}dx;$$

$$\int_0^\infty \frac{\sin x}{x}dx;$$

$$\int_{-\infty}^\infty \frac{x\cdot\cos x}{x^2-5x+6}dx;$$

$$\int_{-\infty}^\infty \frac{\sin x}{(x-1)\cdot(x^2+4)}dx.$$

$$\int_{-\infty}^{\infty} \frac{x}{x^2-5x+6} \cdot \cos x dx = Re\left(\int_{-\infty}^{\infty} \frac{x}{x^2-5x+6} \cdot e^{ix} dx\right) =$$

$$= Re\left[\pi i \left(\text{Rez}\left(\frac{z \cdot e^{iz}}{z^2-5z+6}, 2\right) + \text{Rez}\left(\frac{z \cdot e^{iz}}{z^2-5z+6}, 3\right)\right)\right] =$$

$$= Re\left[\pi i \left(\left.\frac{z \cdot e^{iz}}{z-3}\right|_{z=2} + \left.\frac{z \cdot e^{iz}}{z-2}\right|_{z=3}\right)\right] = Re\left[\pi i\left(-2e^{2i} + 3e^{3i}\right)\right] =$$

$$= Re\left[\pi i(-2\cos 2 - 2i \sin 2 + 3\cos 3 + 3i \sin 3)\right] =$$

$$= Re\left[2\pi \sin 2 - 3\pi \sin 3 + i(3\pi \cos 3 - 2\pi \cos 2)\right] =$$

$$= \pi(2\sin 2 - 3\sin 3).$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{(x-1)\cdot(x^2+4)} dx = Im\left(\int_{-\infty}^{\infty} \frac{e^{ix}}{(x-1)\cdot(x^2+4)} dx\right) =$$

$$= Im\left(2\pi i \text{Rez}\left(\frac{e^{iz}}{(z-1)(z^2+4)}, 2i\right) +\right.$$

$$\left. + \pi i \text{Rez}\left(\frac{e^{iz}}{(z-1)(z^2+4)}, 1\right)\right) =$$

$$= Im\left[2\pi i \cdot \left.\frac{e^{iz}}{z^2+4+2z(z-1)}\right|_{z=2i} + \pi i \cdot \left.\frac{e^{iz}}{z^2+4}\right|_{z=1}\right] =$$

$$= Im\left[2\pi i \frac{e^{-2}}{4i(4i-1} + \pi i \frac{e^{i}}{5}\right] =$$

$$= \pi \cdot Im\left[\frac{e^{-2}}{2(2i-1)} + \frac{i}{5}\cos 1 - \frac{\sin 1}{5}\right] =$$

$$= \pi \cdot Im\left[\frac{e^{-2}}{2 \cdot 5}(-1-2i) - \frac{\sin 1}{5} + \frac{i \cos 1}{5}\right] =$$

$$= \pi \cdot Im \left(\frac{-e^{-2}}{2 \cdot 5} - \frac{\sin 1}{5} + \frac{i}{5}(\cos 1 - e^{-2}) \right) = \frac{\pi}{5}(\cos 1 - e^{-2}).$$

$$\int_{U_2(0)} \frac{1}{z} \mathrm{ch} \frac{1}{z-1} dz;$$

$$f(z) = +\frac{1}{1+(z-1)} \mathrm{ch} \frac{1}{z-1} =$$

$$= \sum_{n \geq 0} (-1)^n (z-1)^n \sum_{m \geq 0} \frac{1}{(2m)!(z-1)^{2m}} =$$

$$= \sum_{m,n \geq 0} \frac{(-1)^n}{(2m)!} \cdot (z-1)^{n-2m} \overset{n-2m=p}{=}$$

$$= \sum_{p \in \mathbb{Z}} \left(\sum_{m \geq 0} \frac{(-1)^{2m}}{(2m)!} \right) \cdot (-1)^p \cdot (z-1)^p \Rightarrow c_{-1} = -\mathrm{ch}$$

(temă) $\int_{U_2(0)} \frac{1}{z+1} \mathrm{sh} \frac{1}{z} dz$ sau $\int_{U_2(0)} \frac{1}{z-1} \sin \frac{1}{z} dz$.

Aplicaţia 1.95 Calculaţi:

$$\int_0^{\infty} \frac{P(x)x^{\lambda}}{Q(x)} dx, \lambda \in (-1,1) \backslash \{0\}; fie functia f(z) = \frac{P(z)}{Q(z)} \cdot z^{\lambda}$$

$z \in [\varepsilon, r] \Rightarrow$

$$f(z) = \frac{P(x)}{Q(x)} \cdot x^{\lambda}$$

$z \in [r, \varepsilon] \Rightarrow$

$$f(z) = \frac{P(x)}{Q(x)} \cdot (x \cdot e^{2\pi i})^{\lambda} = \frac{P(x)}{Q(x)} \cdot e^{\lambda \ln(x \cdot e^{2\pi i})} = \frac{P(x)}{Q(x)} \cdot e^{\lambda(\ln x + 2\pi i)}$$

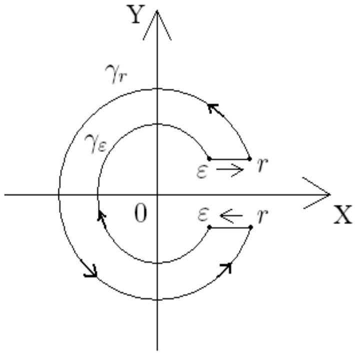

Figura 18.

$\gamma = [\varepsilon, r] \vee \gamma_r \vee [r, \varepsilon] \vee \gamma_\varepsilon^-$

$$\int_\gamma f(z)dz = 2\pi i \sum_{z \in \mathbb{C}*} \mathrm{Rez}\,(f, z) \Rightarrow$$

$$\int_\varepsilon^r \frac{P(x)}{Q(x)} \cdot x^\lambda dx + \underset{r \to \infty}{\int_{\gamma_r}} f(z)dz - \int_\varepsilon^r \frac{P(x)}{Q(x)} \cdot e^{\lambda \ln x} \cdot e^{2\lambda\pi i} dx -$$

$$-\underset{\varepsilon \to 0}{\int_{\gamma_\varepsilon}} f(z)dz = 2\pi i \sum_{z \in \mathbb{C}*} \mathrm{Rez}\,(f, z) \Rightarrow$$

$$(1 - e^{2\lambda\pi i}) \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx = 2\pi i \sum_{z \in \mathbb{C}*} \mathrm{Rez}\,(f, z) \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx = \frac{2\pi i}{1 - e^{2\pi\lambda i}} \sum_{z \in \mathbb{C}*} \mathrm{Rez}\left(\frac{P(x)}{Q(x)} \cdot e^{\lambda(\ln|z| + i \arg z)}, z\right).$$

Aplicaţia 1.96 Calculaţi:

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx.$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx - \int_0^\infty \frac{P(x)}{Q(x)} \cdot e^{\lambda \ln x \cdot e^{2\pi i}} \ln x \cdot e^{2\pi i} dx$$

$$= 2\pi i \sum_{z \in \mathbb{C}*} \operatorname{Rez}\left(\frac{P(x)}{Q(x)} \cdot e^{\lambda(\ln|z| + i \arg z)} \cdot (\ln|z| + i \arg z)\right) \Rightarrow$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx - \int_0^\infty e^{2\pi\lambda i} \frac{P(x)}{Q(x)} \cdot x^\lambda (\ln x + 2\pi i) dx =$$

$$= \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx - e^{2\pi\lambda i} \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx -$$

$$-2\pi i e^{2\pi\lambda i} \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx \Rightarrow$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx =$$

$$= \frac{2\pi i}{1 - e^{2\pi\lambda i}} \left(\sum_{z \in \mathbb{C}*} \operatorname{Rez}\left(\frac{P(x)}{Q(x)} \cdot e^{\lambda(\ln|z| + i \arg z)} \cdot (\ln|z| + i \arg z), z\right) + \right.$$

$$\left. + e^{2\pi\lambda i} \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx \right).$$

Aplicaţia 1.97 Calculaţi:

$$\int_0^\infty \frac{P(x)}{Q(x)} dx,\ \mathrm{gr}P + 2 \leq \mathrm{gr}Q,\ Q \neq 0.$$

Fie

$$f(z) = \frac{P(z)}{Q(z)} \ln z;$$

$$\lim_{\substack{z \to \infty \\ (0)}} z \cdot f(z) = \lim_{\substack{z \to \infty \\ (0)}} \frac{zP(z)\ln z}{Q(z)} = 0 \Rightarrow$$

$$\int_{\gamma_r} f(z)dz \xrightarrow[\substack{z \to \infty \\ (0)}]{} 0$$

$$\lim_{\substack{z \to \infty \\ (0)}} z \cdot f(z) = 0 \Rightarrow \int_{\gamma_r} f(z) \cdot e^{i\alpha z} dz \xrightarrow[\substack{z \to \infty \\ (0)}]{} 0(\alpha > 0)$$

$$\lim_{z\to\infty} \frac{f(z)}{z} = 0 \Rightarrow \int_{\gamma_r} f(z) \cdot e^{iz^2} dz \xrightarrow[\substack{z \to \infty \\ (0)}]{} 0,$$

$\gamma_r(\theta) = re^{i\theta}, \theta \in [0, 2\pi$

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx - \int_0^\infty \frac{P(x)}{Q(x)}(\ln x + 2\pi i)dx =$$

$$= 2\pi i \sum_{z\in\mathbb{C}*} \operatorname{Rez}\left(\frac{P(z)}{Q(z)}(\ln|z| + i\arg z), z\right) \Rightarrow$$

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = -\sum_{z\in\mathbb{C}*} \operatorname{Rez}\left(\frac{P(z)}{Q(z)}(\ln|z| + i\arg z), z\right)$$

Aplicaţia 1.98

$\displaystyle\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx,$ gr$P$+2 $\leq$ gr$Q, Q \neq 0$ pe $\mathbb{R}.f(z) = \dfrac{P(z)}{Q(z)} \ln^2 z.$

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln^2 x dx - \int_0^\infty \frac{P(x)}{Q(x)}(\ln x + 2\pi i)^2 dx =$$

$$= 2\pi i \sum_{z\in\mathbb{C}*} \operatorname{Rez}(f,z) \Rightarrow$$

$$-4\pi i \int_0^\infty \frac{P(x)}{Q(x)} \ln x dx + 4\pi i \int_0^\infty \frac{P(x)}{Q(x)} dx =$$

$$= 2\pi i Re \left(\sum_{z\in\mathbb{C}*} \operatorname{Rez}(f,z) \right) - 2\pi Im \left(\sum_{z\in\mathbb{C}*} \operatorname{Rez}(f,z) \right) \Rightarrow$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx = -\frac{1}{2} Re \left(\sum_{z\in\mathbb{C}*} \operatorname{Rez}(f,z) \right);$$

$$\int_0^\infty \frac{P(x)}{Q(x)} dx = -\frac{1}{2\pi} Im \left(\sum_{z\in\mathbb{C}*} \operatorname{Rez}(f,z) \right);$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \ln x dx = -\frac{1}{2} Re \left(\sum_{z\in\mathbb{C}*} \operatorname{Rez}\left(\frac{P(z)}{Q(z)} (\ln|z| + i \arg z)^2, z \right) \right).$$

Aplicaţia 1.99

$$\int_0^\infty \frac{x}{1+x^4} \cdot x^{\frac{1}{3}} dx$$

$f(z) = \frac{z}{1+z^4} \cdot z^{\frac{1}{3}}$

$$\int_0^\infty \frac{x}{1+x^4} \cdot x^{\frac{1}{3}} dx + \int_{\gamma_r} f(z) dz -$$

$$- \int_\varepsilon^r \frac{x}{1+x^4} \cdot e^{\frac{1}{3}(\ln x + 2\pi i)} dx - \int_{\gamma_\varepsilon} f(z) dz =$$

$$= 2\pi i \sum_{k=0}^{3} \operatorname{Rez}\left(\frac{z}{1+z^4} \cdot e^{\frac{1}{3}(\ln|z| + i \arg z)}, z_k \right) \Rightarrow$$

$$\int_0^\infty \frac{x}{1+x^4}\cdot x^{\frac{1}{3}}dx - e^{\frac{2\pi i}{3}}\int_0^\infty \frac{x}{1+x^4}\cdot x^{\frac{1}{3}}dx = 2\pi i\sum_k \operatorname{Rez}(f,z_k) \Rightarrow$$

$$\int_0^\infty \frac{x^{\frac{4}{3}}}{1+x^4}dx = \frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\sum_{k=0}^{3}\operatorname{Rez}(f,z_k)$$

$f(z) = \frac{z\cdot e^{\frac{1}{3}(\ln|z|+i\arg z)}}{1+z^4}$;
$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = e^{i\frac{\pi}{4}}; z_1 = e^{i\frac{3\pi}{4}}; z_2 = e^{i\frac{5\pi}{4}}; z_3 = e^{i\frac{7\pi}{4}}$;

$$\operatorname{Rez}(f,z_0) = \frac{1}{4e^{i\frac{\pi}{2}}}\cdot e^{\frac{1}{3}\left(i\frac{\pi}{4}\right)} = \frac{e^{\frac{\pi i}{12}}}{4i};$$

$$\operatorname{Rez}(f,z_1) = \frac{1}{4e^{i\frac{3\pi}{2}}}\cdot e^{i\frac{\pi}{4}} = -\frac{e^{\frac{\pi i}{4}}}{4i};$$

$$\operatorname{Rez}(f,z_2) = \frac{1}{4e^{i\frac{5\pi}{2}}}\cdot e^{i\frac{5\pi}{12}} = \frac{e^{\frac{5\pi i}{12}}}{12};$$

$$\operatorname{Rez}(f,z_3) = \frac{1}{4e^{i\frac{7\pi}{2}}}\cdot e^{i\frac{7\pi}{12}} = -\frac{e^{\frac{7\pi i}{12}}}{4i};$$

$$\int_0^\infty \frac{x^{\frac{4}{3}}}{1+x^4}dx =$$

$$= \frac{2\pi i}{{}^2/1+\underbrace{\cos\frac{\pi}{3}}_{\frac{1}{2}} - i\underbrace{\sin\frac{\pi}{3}}_{\frac{\sqrt{3}}{2}}}\cdot\frac{1}{4i}\left(e^{\frac{\pi i}{12}} - e^{\frac{\pi i}{4}} + e^{\frac{5\pi i}{12}} - e^{\frac{7\pi i}{12}}\right) =$$

$$\frac{\pi}{\sqrt{3}}\cdot\frac{\left(e^{\frac{\pi i}{12}} - e^{\frac{\pi i}{4}} + e^{\frac{5\pi i}{12}} - e^{\frac{7\pi i}{12}}\right)}{\sqrt{3}-i}.$$

Aplicaţia 1.100 Folosind identitatea următoare

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\alpha dx = \frac{2\pi i}{1-e^{2\pi i\alpha}} \sum_{z\in\mathbb{C}*} \text{Rez}\left(\frac{P(z)}{Q(z)} \cdot e^{\alpha(\ln|z|+i\arg z)}, z\right)$$

calculaţi direct următoarea integrală:

$$\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2}dx;$$

$$\int_\varepsilon^r \frac{\sqrt{x}\ln x}{1+x^2}dx + \int_{\gamma_r} f(z)dz -$$

$$-\int_\varepsilon^r \frac{e^{\frac{1}{2}(\ln x+2\pi i)}(\ln x+2\pi i)}{1+x^2}dx - \int_{\gamma_\varepsilon} f(z)dz =$$

$$= 2\pi i\left[\text{Rez}\left(\frac{e^{\frac{1}{2}(\ln x+2\pi i)}(\ln x+2\pi i)}{1+x^2}, i\right) + \right.$$

$$\left. + \text{Rez}\left(\frac{e^{\frac{1}{2}(\ln x+2\pi i)}(\ln x+2\pi i)}{1+x^2}, -i\right)\right]$$

$$\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2}dx - e^{\pi i}\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2}dx - 2\pi i e^{\pi i}\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx =$$

$$= 2\pi i\left[\text{Rez}\left(\frac{\sqrt{z}\ln z}{1+z^2}, \pm i\right)\right]$$

$$\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2}dx = \frac{2\pi i}{1-e^{\pi i}}\left[\text{Rez}\left(\frac{\sqrt{z}\ln z}{1+z^2}, \pm i\right) + e^{\pi i}\int_0^\infty \frac{\sqrt{x}}{1+x^2}dx\right]$$

$$\text{Rez}\left(\frac{e^{\frac{1}{2}(\ln|z|+i\arg z)}(\ln|z|+i\arg z)}{z^2+1}, i\right) =$$

$$= e^{\frac{1}{2}(\ln|i|+i\underbrace{\arg i}_{\frac{\pi}{2}})} \cdot \frac{(\ln|i|+i\arg i)}{2i} =$$

$$= e^{\frac{\pi i}{4}} \cdot \frac{i\pi}{2 \cdot 2 \cdot i} - \frac{\pi}{4} e^{\frac{\pi i}{4}}$$

$$\operatorname{Rez}(f, -i) = e^{\frac{3\pi i}{4}} \cdot \frac{i3\pi}{-4i} = \frac{-3\pi}{4} e^{\frac{3\pi}{4} i}$$

$$\int_0^\infty \frac{\sqrt{x}}{1+x^2} dx =$$

$$= \frac{2\pi i}{1 - e^{\pi i}} \left[\operatorname{Rez}\left(\frac{e^{\frac{1}{2}(\ln|z| + i\arg z)}}{z^2+1}, i \right) + \operatorname{Rez}\left(\frac{e^{\frac{1}{2}(\ln|z| + i\arg z)}}{z^2+1}, -i \right) \right] =$$

$$= \frac{2\pi i}{1+1} \left[\frac{e^{\frac{\pi i}{4}}}{2i} - \frac{e^{\frac{3\pi i}{4}}}{2i} \right] = \frac{\pi}{2} \left(e^{\frac{\pi i}{4}} - e^{\frac{3\pi i}{4}} \right)$$

$$\int_0^\infty \frac{\sqrt{x}\ln x}{1+x^2} dx = \pi i \left[\frac{\pi}{4} e^{\frac{\pi i}{4}} - \frac{3\pi}{4} e^{\frac{3\pi i}{4}} - \frac{\pi}{2} e^{\frac{\pi i}{4}} + \frac{\pi}{2} e^{\frac{3\pi i}{4}} \right] =$$

$$= \pi i \left(-\frac{\pi}{4} e^{\frac{\pi i}{4}} - \frac{\pi}{4} e^{\frac{3\pi i}{4}} \right) =$$

$$= -\frac{\pi^2}{4} i \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4} - \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} \right) = -\frac{\pi^2}{4} \cdot 2i^2 \sin\frac{\pi}{4} =$$

$$2$$

$$= \frac{\pi^2}{2} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}\pi^2}{4}$$

Aplicaţia 1.101

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx = \frac{2\pi i}{1 - e^{2\pi\lambda i}} \sum_{z \in \mathbb{C}*} \operatorname{Rez}\left(\frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z| + i\arg z)}, z \right)$$

$Q \neq 0$ pe $\mathbb{R}$;
$\lambda \in (-1, 1) \backslash \{0\}$;
grad $P + \lambda + 1 <$ grad Q.

Aplicaţia 1.102 Folosind

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda \ln x dx =$$

$$= \frac{2\pi i}{1-e^{2\pi\lambda i}} \sum_{z\in\mathbb{C}*} \text{Rez}\left(\frac{P(z)}{Q(z)} \cdot e^{\lambda(\ln|z|+i\arg z)} \cdot (\ln|z| + i\arg z), z\right) +$$

$$+e^{2\pi i} \int_0^\infty \frac{P(x)}{Q(x)} \cdot x^\lambda dx,$$

calculaţi:

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{1+x+x^2} dx.$$

Luăm

$$f(z) = \frac{z^{\frac{1}{3}}\ln z}{1+z+z^2},$$

$$\text{gr}P + \frac{1}{3} + 1 = \frac{4}{3} < \text{gr } Q = 2, Q \neq 0 \text{ pe } \mathbb{R} \Rightarrow Q(z) = 0 \Rightarrow$$

$$z_{-1,2} = -\frac{1}{2} \mp i\frac{\sqrt{3}}{2}, \lambda = \frac{1}{3} \quad \begin{array}{l} z_1 = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{i\frac{4\pi}{3}} \\ z_2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{i\frac{2\pi}{3}} \end{array}$$

$$\int_0^\infty \frac{\sqrt[3]{x}\ln x}{1+x+x^2} dx = \frac{2\pi i}{1-e^{\frac{2\pi i}{3}}} \cdot$$

$$\cdot \left[\text{Rez}\left(\frac{1}{1+z+z^2} \cdot e^{\frac{1}{3}(\ln|z|+i\arg z)} \cdot (\ln|z| + i\arg z), e^{i\frac{2\pi}{3}}\right) + \right.$$

$$+ \text{Rez}\left(\frac{1}{1+z+z^2} \cdot e^{\frac{1}{3}(\ln|z|+i\arg z)} \cdot (\ln|z| + i\arg z), e^{i\frac{4\pi}{3}}\right) +$$

$$\left. +e^{\frac{2\pi i}{3}} \int_0^\infty \frac{x^{\frac{1}{3}}\ln x}{1+x+x^2} dx\right]$$

$$\operatorname{Rez}\left(\frac{e^{\frac{1}{3}(\ln|z|+i\arg z)}\cdot(\ln|z|+i\arg z)}{1+z+z^2}, e^{i\frac{2\pi}{3}}\right)$$

$$=\frac{e^{\frac{1}{3}\cdot i\frac{2\pi}{3}}\cdot i\frac{2\pi}{3}}{2e^{i\frac{2\pi}{3}}+1}=\frac{2\pi i}{3}\cdot\frac{e^{i\frac{2\pi}{9}}}{2e^{i\frac{2\pi}{3}}+1}$$

$$\operatorname{Rez}\left(\frac{e^{\frac{1}{3}(\ln|z|+i\arg z)}\cdot(\ln|z|+i\arg z)}{1+z+z^2}, e^{i\frac{4\pi}{3}}\right)=\frac{e^{i\frac{4\pi}{9}}\cdot\frac{4\pi i}{3}}{2e^{i\frac{4\pi}{3}}+1}$$

$$\int_0^\infty\frac{x^{\frac{1}{3}}\ln x}{1+x+x^2}dx=$$

$$=\frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\left[\operatorname{Rez}\left(\frac{e^{\frac{1}{3}(\ln|z|+i\arg z)}\cdot(\ln|z|+i\arg z)}{1+z+z^2}, e^{i\frac{2\pi}{3}}\right)+\right.$$

$$\left.+\operatorname{Rez}\left(\frac{e^{\frac{1}{3}(\ln|z|+i\arg z)}\cdot(\ln|z|+i\arg z)}{1+z+z^2}, e^{i\frac{4\pi}{3}}\right)\right]=$$

$$=\frac{2\pi i}{1-e^{\frac{2\pi i}{3}}}\left[\frac{2\pi i}{3}\cdot\frac{e^{i\frac{2\pi}{9}}}{2e^{i\frac{2\pi}{3}}+1}+\frac{e^{i\frac{4\pi}{9}}\cdot\frac{4\pi i}{3}}{2e^{i\frac{4\pi}{3}}+1}\right]$$

$$e^{\frac{2\pi i}{3}}=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$

$$e^{\frac{4\pi i}{3}}=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}.$$

$$\int_0^\infty\frac{x^{\frac{1}{3}}\ln x}{1+x+x^2}dx=$$

$$=\frac{2\pi i}{\frac{3}{2}-i\frac{\sqrt{3}}{2}}\left[\frac{e^{i\frac{2\pi}{9}}}{i\sqrt{3}}+\frac{e^{i\frac{4\pi}{9}}}{-i\sqrt{3}}\right]=$$

$$= \frac{4\pi}{3(\sqrt{3}-i)}\left(e^{i\frac{2\pi}{9}} - e^{i\frac{4\pi}{9}}\right) = \frac{2\pi e^{-i\frac{11\pi}{6}}}{3}\left(e^{i\frac{2\pi}{9}} - e^{i\frac{4\pi}{9}}\right) =$$

$$= \frac{4\pi(\sqrt{3}+i)}{12}\left(e^{i\frac{2\pi}{9}} - e^{i\frac{4\pi}{9}}\right) = \frac{2\pi e^{i\frac{\pi}{6}}}{3}\left(e^{i\frac{2\pi}{9}} - e^{i\frac{4\pi}{9}}\right) =$$

$$= \frac{2\pi}{3}\left(e^{\frac{7\pi i}{18}} - e^{\frac{11\pi i}{18}}\right).$$

Tema 1.103

$$\int_{-\infty}^{\infty} \frac{x\cos x}{x^2-2x+10}dx,$$

$$\int_0^{2\pi} \frac{\sin\theta \cdot \sin n\theta}{5-4\sin\theta}d\theta, n \in \mathbb{N}*,$$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^6},$$

$$\int_0^{\infty} \frac{x^{\frac{4}{3}}}{1+x^4}dx,$$

$$\int_0^{\infty} \frac{\sqrt{x}}{1+x^2}\ln x dx.$$

Lema 1.104 f continuă în sectorul închis $S_0[\theta_1, \theta_2]$, iar γ_r drumul din acest sector definit de $\gamma_r(t) = r \cdot e^{i[\theta_1 + t(\theta_2-\theta_1)]}, t \in [0, 1]$.

$$\lim_{z\to\infty} z \cdot f(z) = 0 \Rightarrow \lim_{r\to\infty} \int_{\gamma_r} f = 0$$

$$\lim_{z\to 0} z \cdot f(z) = 0 \Rightarrow \lim_{r\to 0} \int_{\gamma_r} f = 0.$$

Demonstraţie.

$$\int_{\gamma_r} f = \int_0^1 f(\gamma_r(t)) \cdot \gamma_r'(t)dt =$$

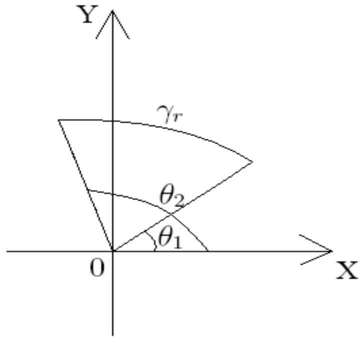

Figura 19.

$$= \int_0^1 ir(\theta_2 - \theta_1) \cdot e^{i[\theta_1 + t(\theta_2 - \theta_1)]} \cdot f(\gamma_r(t))dt \Rightarrow$$

$$\left| \int_{\gamma_r} f \right| \le \int_0^1 r(\theta_2 - \theta_1) \cdot |f(\gamma_r(t))| \, dt =$$

$$= r(\theta_1 - \theta_2) \cdot \int_0^1 |f(\gamma_r(t))| \, dt \le M(r) \cdot r(\theta_1 - \theta_2)$$

$$M(r) := \sup\{|f(\gamma_r(t))| | t \in [0, 1]\}.$$

Cum f este continuă pe $S_0[\theta_1, \theta_2] \Rightarrow |f|$ este continuă $\Rightarrow t \mapsto |f(\gamma_r(t))|, t \in [0, 1]$ din Th. W. este mărginită $\Rightarrow M(r) < +\infty$.

Deci $\left| \int_{\gamma_r} f \right| \le r \cdot M(r) \cdot (\theta_2 - \theta_1)$. Dacă $\lim\limits_{z \to \infty} z \cdot f(z) = 0 \Rightarrow \lim\limits_{r \to \infty} r \cdot M(r) = 0 \Rightarrow \lim\limits_{r \to \infty} \int_{\gamma_r} f = 0$.

Dacă $\lim\limits_{z \to 0} z \cdot f(z) = 0 \Rightarrow \lim\limits_{r \to 0} r \cdot f(r) = 0 \Rightarrow \lim\limits_{r \to 0} \int_{\gamma_r} f = 0$. $\square$

Aplicaţia 1.105 Calculaţi integrala

$$\int_0^\infty R(x) \log x dx,$$

cu condiţia $n \le m - 2, R = \frac{P}{Q}$ funcţie raţională cu $Q \neq 0$ pe $\mathbb{R}$ $(n = \mathrm{gr}P, m = \mathrm{gr}Q)$.

Demonstraţie. Condiţia $n \leq m-2$ asigură convergenţa integralei.

Alegem funcţia $g(z) = \frac{P(z)}{Q(z)} \cdot (\log z)^2$ şi drumul de mai jos:

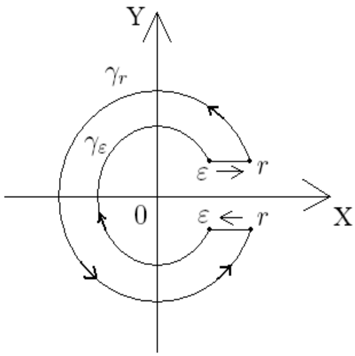

Figura 20.

$$\gamma := [\varepsilon, r] \vee \gamma_r \vee [r, \varepsilon] \vee \gamma_\varepsilon^- \Rightarrow$$

conform teoremei reziduurilor

$$\int_\gamma g(z)dz = 2\pi i \sum_{z \in \mathbb{C}*} \text{Rez}\,(g, z) \Leftrightarrow$$

$$\Leftrightarrow \int_\varepsilon^r g(x)dx + \int_{\gamma_r} g(z)dz + \int_r^\varepsilon g(z)dz - \int_{\gamma_\varepsilon} g(z)dz =$$

$$= 2\pi i \sum_{z \in \mathbb{C}*} \text{Rez}\,(g, z)$$

$$\lim_{z \to \infty} z \cdot g(z) = \lim_{z \to \infty} z \cdot \frac{P(z)}{Q(z)} \cdot (\log z)^2 =$$

$$= \lim_{z \to \infty} z^2 \cdot \frac{P(z)}{Q(z)} \cdot \frac{(\log z)^2}{z} = \lim_{z \to \infty} \frac{(\log z)^2}{z} =$$

$$= \lim_{z\to\infty} \frac{2}{z} \cdot c \cdot \log z = 2c^2 \cdot \lim_{z\to\infty} \frac{1}{z} = 0$$

$$grz^2 \cdot P(z) \leq grQ(z) \Rightarrow \lim_{z\to\infty} z^2 \cdot \frac{P(z)}{Q(z)} =$$

$$= \begin{cases} 0, n < m-2, \\ \alpha \neq \infty, n = m-2. \end{cases}$$

Deci:

$$\lim_{z\to\infty} z \cdot g(z) = 0 \Rightarrow \lim_{r\to\infty} \int_{\gamma_r} g(z)dz = 0$$

$$\lim_{z\to 0} z \cdot g(z) = \lim_{z\to 0} z \cdot \frac{P(z)}{Q(z)} \cdot (\log z)^2 = \lim_{z\to 0} z \cdot (\log z)^2 \cdot \lim_{z\to 0} \frac{P(z)}{Q(z)} \Rightarrow$$

$$\Rightarrow \lim_{z\to 0} z \cdot (\log z)^2 = \lim_{z\to 0} \frac{(\log z)^2}{\frac{1}{z}} = \lim_{z\to 0} \frac{-\log z}{\frac{1}{z}} = \lim_{z\to 0} \frac{\frac{1}{z}}{\frac{1}{z^2}} = 0$$

$$\int_{\gamma_\varepsilon} g(z)dz \xrightarrow[\varepsilon\to 0]{} 0.$$

Când argumentul lui z este $2\pi \Rightarrow \log z = \log|z| \cdot e^{2\pi i} = \log|z| + 2\pi i$

$$\log z = \log|z| \cdot e^{2\pi i} = \log|z| + 2\pi i \Rightarrow$$

$$\Rightarrow (\log z)^2 = \log^2|z| - 4\pi^2 + 4\pi \cdot \log|z| \cdot i.$$

În relaţia:

$$\int_\varepsilon^r g(z)dz + \int_{\gamma_r} g(z)dz - \int_{\gamma_\varepsilon} g(z)dz +$$

$$+ \int_r^\varepsilon \frac{P(z)}{Q(z)} \cdot \left[\log^2|z| - 4\pi^2 + i \cdot 4\pi \log|z|\right] dz =$$

$$= 2\pi i \sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)$$

facem

$$\begin{cases} \varepsilon \to 0 \\ r \to \infty \end{cases} \Rightarrow$$

$$\int_0^\infty g(x)dx + 0 - 0 + \int_\infty^0 g(x)dx -$$

$$-4\pi^2 \int_\infty^0 \frac{P(x)}{Q(x)}dx + i \cdot 4\pi \int_\infty^0 \frac{P(x)}{Q(x)} \cdot \log x dx =$$

$$= 2\pi i \sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z) \Rightarrow$$

$$\Rightarrow \int_0^\infty \frac{P(x)}{Q(x)}dx = \frac{-1}{2\pi} \cdot Im\left(\sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)\right) =$$

$$= 2\pi i \cdot Re\left(\sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)\right) - 2\pi Im\left(\sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)\right)$$

$$\int_0^\infty \frac{P(x)}{Q(x)}dx = -\frac{1}{2\pi} Im\left(\sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)\right)$$

$$\int_0^\infty \frac{P(x)}{Q(x)} \cdot \log dx = -\frac{1}{2} Re\left(\sum_{z\in\mathbb{C}*} \text{Rez}\,(g,z)\right).$$

□

Aplicaţia 1.106 Să se calculeze integrala

$$\int_0^\infty \frac{\ln x}{\sqrt{x}\cdot(x+1)^2}dx.$$

$x = y^2 \Rightarrow dx = 2ydy$

$$\int_0^\infty \frac{\ln x}{\sqrt{x}\cdot(x+1)^2}dx =$$

$$= 2\int_0^\infty \frac{\ln y^2}{y\cdot(1+y^2)^2}\cdot ydy = 4\int_0^\infty \frac{\ln y}{(1+y^2)^2}dy =$$

$$= 4\cdot\left(\frac{-1}{2}\right) Re\left[\text{Rez}\,(g,i) + \text{Rez}\,(g,)-i\right], g(z) = \frac{1}{(1+z^2)^2}\cdot(\ln z)^2$$

$$\text{Rez}\,(g,i) = \lim_{z\to i}\left[\frac{(\ln z)^2}{(z+i)^2}\right]' =$$

$$= \lim_{z\to i}\frac{2\frac{\ln z}{z}\cdot(z+i)^2 - 2\ln^2 z\cdot(z+i)}{(z+i)^4} =$$

$$= \lim_{z\to i}\frac{2\frac{\ln z}{z}\cdot(z+i) - 2\ln^2(z)}{(z+i)^3} = \frac{2\cdot\frac{\pi i}{2i}\cdot 2i + 2\cdot\frac{\pi^2}{42}}{-8i} = -\frac{\pi}{4} + \frac{\pi^2}{16}i$$

$$\text{Rez}\,(g,-i) = \lim_{z\to -i}\frac{2\cdot\frac{\ln z}{z}\cdot(z-i) - 2\ln^2(z)}{(z-i)^3} =$$

$$= \frac{2\cdot\frac{\frac{3\pi i}{2}}{-i}\cdot(-2i) + 2\cdot\frac{9\pi}{42}}{8i} = \frac{3\pi}{4} - \frac{9\pi^2}{16}i$$

$$\int_0^\infty \frac{\ln x}{\sqrt{x}\cdot(x+1)^2}dx = -2Re\left[\frac{3\pi}{4} - \frac{2\pi}{8} - \frac{9\pi^2}{16}i + \frac{\pi^2}{16}i\right] =$$

$$= -2Re\left(\frac{\pi}{2} - \frac{\pi^2}{2}i\right) \Rightarrow$$

$$\int_0^\infty \frac{\ln x}{\sqrt{x}\cdot(x+1)^2}dx = -\pi.$$

Aplicaţia 1.107 Să se calculeze integralele:

$$I_1 = \int_0^\infty \sin x^2 dx;$$

$$I_2 = \int_0^\infty \cos x^2 dx.$$

Soluţie:

$$I_2 + iI_1 = \int_0^\infty e^{ix^2} dx.$$

Considerăm drumul: $\lambda = [0, R] \cdot \lambda_1 \cdot [R \cdot e^{i\frac{\pi}{4}}, 0]$, unde $\lambda_1(t) = R \cdot e^{i\frac{\pi}{4}t}, 0 \leq t \leq 1$.
λ este drum neted, închis, $z \mapsto e^{iz^2}$ olomorfă pe $\mathbb{C} \Rightarrow$ conform teoremei fundamentale a lui Cauchy:

$$\int_\lambda e^{iz^2} dz = 0.$$

Altfel:

$$\int_0^R e^{ix^2} dx + \int_{\lambda_1} e^{iz^2} dz + \int_{[R\cdot e^{i\frac{\pi}{4}}, 0]} e^{iz^2} dz = 0.$$

Fie

$$z \in \lambda_1 \Rightarrow z = R \cdot e^{i\frac{\pi}{4}t} = R(\cos \tfrac{\pi}{4}t + i \sin \tfrac{\pi}{4}t), 0 \leq t \leq 1$$
$$|e^{iz^2}| = |\cos z^2 + i \sin z^2| = e^{-R^2 \sin \frac{\pi t}{2}}, 0 \leq t \leq 1$$
$$z \in \lambda_1 \Rightarrow \left|\frac{e^{iz^2}}{z}\right| \to 0 \text{ pentru } R \to \infty$$
$$\left|\frac{e^{iz^2}}{z}\right| = \frac{e^{-R \sin \frac{\pi t}{4}}}{R} \xrightarrow[R\to\infty]{} 0$$

Deci:

$$\int_{\lambda_1} \frac{e^{iz^2}}{z^2} dz \to 0 \text{ pentru } R \to \infty.$$

Integrând prin părţi:

$$\int_{\lambda_1} e^{iz^2}dz = \int_{\lambda_1} \frac{2iz \cdot e^{iz^2}}{2iz}dz = \int_{\lambda_1} \frac{\left(e^{iz^2}\right)'}{2iz}dz =$$

$$= \underbrace{\left(\frac{e^{iz^2}}{2iz}\right)_{|\lambda_1}}_{\xrightarrow[R\to\infty]{}0} + \frac{1}{2\pi}\underbrace{\int_{\lambda_1} \frac{e^{iz^2}}{z^2}dz}_{\xrightarrow[R\to\infty]{}0} \Rightarrow (R \to \infty \Rightarrow \int_{\lambda_1} e^{iz^2}dz \to 0)$$

Pentru $z \in [R \cdot e^{i\frac{\pi}{4}}, 0] \Rightarrow z = r \cdot e^{\frac{i\pi}{4}}, r \in [R, 0] \Rightarrow dz = e^{i\frac{\pi}{4}}dr$.

$$\int_{[R\cdot e^{\frac{i\pi}{4}},0]} e^{iz^2}dz =$$

$$= \int_R^0 e^{ir^2\cdot \underbrace{e^{\frac{i\pi}{2}}}_{=i}} \cdot e^{\frac{i\pi}{4}}dr = e^{\frac{i\pi}{4}}\int_R^0 e^{-r^2}dr = -e^{\frac{i\pi}{4}}\int_0^R e^{-r^2}dr.$$

Facem $R \to \infty \Rightarrow$

$$\int_{[R\cdot e^{\frac{i\pi}{4}},0]} e^{iz^2}dz \xrightarrow[R\to\infty]{} -e^{\frac{i\pi}{4}}\int_0^\infty e^{-r^2}dr = -e^{\frac{i\pi}{4}} \cdot \frac{\sqrt{\pi}}{2}.$$

Deci:

$$\int_0^R e^{ix^2}dx + \int_{\lambda_1} e^{iz^2}dz + \int_{[R\cdot e^{\frac{i\pi}{4}},0]} e^{iz^2}dz = 0.$$

Trecând la limită: $R \to \infty \Rightarrow$

$$\int_0^\infty e^{ix^2}dx + 0 - e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{\pi}}{2} = 0 \Leftrightarrow$$

$$\Leftrightarrow \int_0^\infty e^{ix^2}dx = \frac{\sqrt{\pi}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) \Rightarrow$$

$$\begin{cases} \int_0^\infty \cos x^2 dx = \frac{\sqrt{2\pi}}{4}; \\ \int_0^\infty \sin x^2 dx = \frac{\sqrt{2\pi}}{4}. \end{cases}$$

Capitolul 2

Funcţii speciale şi transformări integrale

2.1 Funcţiile euleriene Γ şi B

Teorema 2.1 *Fie domeniul* $D_0 = \{z \in \mathbb{C} \backslash Rez > 0\}$ *şi* $\Gamma : D_0 \to \mathbb{C}$, $\Gamma(z) = \int_0^\infty t^{z-1} \cdot e^{-t} dt$.

Atunci:

i) Γ *este bine definită (integrala este conjugată) si este olomorfă;*

ii) $\Gamma(z+1) = z\Gamma(z)$, $(\forall)\, z \in D_0$ *şi* $\Gamma(n+1) = n!$, $(\forall)\, n \in \mathbb{N}^*$.

Demonstraţie.

$$\Gamma(z+1) = \int_0^\infty t^z \cdot e^{-t} dt = -\int_0^\infty t^z \cdot \left(e^{-t}\right)' dt =$$

$$= -\frac{t^z}{e^t}\Bigg|_0^\infty + z\int_0^\infty t^{z-1} \cdot e^{-t} dt = z\Gamma(z);\ \lim_{x\to\infty} \frac{t^z}{e^t} = 0.$$

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = ... = n(n-1)...1\Gamma(1) = n!.$$

□

Teorema 2.2 *Fie* $D_0 = \{z \in \mathbb{C} \backslash Rez > 0\}$ *şi aplicaţia* $B : D_0 \times D_0 \to \mathbb{C}$ *definită prin:*

$$B(z, z') = \int_0^1 t^{z-1} \cdot (1-t)^{z'-1} dt.$$

Atunci:
i)$B(z, z') = \frac{\Gamma(z) \cdot \Gamma(z')}{\Gamma(z+z')}$, *deci* $B(\cdot, \cdot)$ *este corect definită şi olomorfă.*
ii) $B(z, z-1) = \frac{\pi}{\sin \pi z}$.

Demonstraţie.
i)Avem $\Gamma(z) = \int_0^\infty t^{z-1} \cdot e^{-t} dt$ şi facem schimbarea de variabilă:
$t = u^2 \Rightarrow dt = 2udu \Rightarrow$

$$\Gamma(z) = \int_0^\infty t^{z-1} \cdot e^{-t} dt = 2 \int_0^\infty \left(u^2\right)^{z-1} \cdot e^{-u^2} \cdot udu$$

Facem schimbarea de variabilă: $t = v^2 \Rightarrow$

$$\Gamma(z') = \int_0^\infty t^{z'-1} \cdot e^{-t} dt = 2 \int_0^\infty \left(v^2\right)^{z'-1} \cdot e^{-v^2} \cdot vdv$$

De unde:

$$u, v \geq 0 \Rightarrow \theta \in \left[0, \frac{\pi}{2}\right].$$

$$\Gamma(z) \Gamma(z') = 4 \int_0^\infty \int_0^\infty \left(u^2\right)^{z-\frac{1}{2}} \cdot \left(v^2\right)^{z'-\frac{1}{2}} \cdot e^{-\left(u^2+v^2\right)} \cdot dudv =$$

Facem schimbarea de variabilă:

$$\begin{cases} u = r\cos\theta & r \in [0, \infty] \\ v = r\sin\theta & \theta \in \left[0, \frac{\pi}{2}\right] \end{cases}$$

$$dudv = rdrd\theta \quad r = \begin{cases} iacobianul \\ transformarii \end{cases}$$

$$= 4\int_0^{\infty}\int_0^{\frac{\pi}{2}} \left(r^2\cos^2\theta\right)^{z-\frac{1}{2}} \cdot \left(r^2\sin^2\theta\right)^{z'-\frac{1}{2}} \cdot e^{-r^2} rdrd\theta =$$

$$= \left[\int_0^{\infty} \left(r^2\right)^{z+z'-1} \cdot e^{-r^2} \cdot 2rdr\right] \cdot$$

$$\cdot \left[\int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)^{z-1} \cdot \left(\sin^2\theta\right)^{z'-1} \cdot 2\cos\theta\sin\theta d\theta\right] =$$

$$= \left[\int_0^{\infty} \left(z^2\right)^{z+z'-1} \cdot e^{-r^2} \cdot \left(r^2\right)' dr\right] \cdot$$

$$\cdot \left[\int_0^{\frac{\pi}{2}} \left(\cos^2\theta\right)^{z-1} \cdot \left(1-\cos^2\theta\right)^{z'-1} \cdot \left(-\cos^2\theta\right)' d\theta\right] =$$

cu schimbarea de variabilă: $r^2 = t$ şi $\cos\theta = t$

$$= \left(\int_0^{\infty} t^{z+z'-1} \cdot e^{-t} dt\right)\left(-\int_1^0 t^{z-1} \cdot (1-t)^{z'-1} dt\right) =$$

$$= \Gamma(z+z') \cdot \beta(z+z') \Rightarrow$$

$$\Rightarrow B(z,z') = \frac{\Gamma(z)\cdot\Gamma(z')}{\Gamma(z+z')} = \int_0^1 t^{z-1} \cdot (1-t)^{z'-1} dt.$$

ii) Presupunem: $0<\text{Re}z<1$, atunci:

$$\beta(z,1-z) = \int_0^1 t^{z-1} \cdot (1-t)^{-z} dt =$$

$$= \int_0^1 \frac{1}{t} \cdot \left(\frac{t}{1-t}\right)^z dt = \frac{t}{1-t} = u \Rightarrow$$

Facem schimbarea de variabilă:

$$t = \frac{u}{1+u} \Rightarrow dt = \frac{du}{(1+u)^2}$$

$$= \int_0^\infty \frac{1+u}{u} \cdot u^z \cdot \frac{du}{(1+u)^2} = \int_0^\infty \frac{u^{z-1}}{1+u} du =$$

$$= \lim_{\substack{r \to \infty \\ \varepsilon \to 0}} \int_\varepsilon^r \frac{u^{z-1}}{1+u} du. \tag{2.1}$$

Fie funcţia:

$$f(w) = \frac{w^{z-1}}{1+w} = \frac{e^{(z-1)[\ln|w| + i(\arg w)]}}{1+w}.$$

Considerăm domeniul următor, avem: $w = u \in \mathbb{R}$

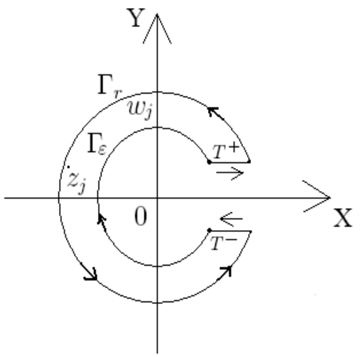

Figura 21.

1. Pe T^{+} avem: $|w| = u, \arg w = 0$

$$f|_{T^{+}}(w) = \frac{e^{(z-1)\ln u}}{1+u} = \frac{u^{z-1}}{1+u}$$

2. Pe T^{-} avem: $|w| = u$ şi $\arg w = 2\pi$

$$f\,|_{T^{-}}\,(w) = \frac{e^{(z-1)[\ln u + 2\pi i]}}{1+u} = \frac{u^{(z-1)}}{1+u} \cdot e^{2\pi i(z-1)} = \frac{u^{z-1}}{1+u} \cdot e^{2\pi i z}$$

pentru că: $e^{-\pi i} = \cos\pi - i\sin\pi = -1$, $e^{-2\pi i} = \cos 2\pi - i\sin 2\pi = 1$.

1. $\lim\limits_{\varepsilon\to 0} \sup\limits_{w\in\Gamma_\varepsilon} |w \cdot f(w)| = 0$ şi $\lim\limits_{r\to\infty} \sup\limits_{w\in\Gamma_r} |w \cdot f(w)| = 0$

$$\overset{lema}{\underset{Jordan}{\Rightarrow}} \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} f(w)\,dw = 0 \text{ şi } \lim_{r\to\infty} \int_{\Gamma_r} f(w)\,dw = 0. \qquad (2.2)$$

Aplicăm teorema reziduurilor funcţiei $f(w)$ pe domeniul D şi pe frontiera $D = \Gamma_r \vee T_{-} \vee \Gamma_\varepsilon^{-} \vee T_{+}$ curbă inchisă, simplă, netedă pe porţiuni:

$$\int_{\Gamma_r \vee T_{-} \vee \Gamma_\varepsilon^{-} \vee T_{+}} f(w)\,dw =$$

$$= \int_{\Gamma_r} f(w)\,dw - \int_\varepsilon^r \frac{u^{z-1}}{1+u} \cdot e^{2\pi i z} du - \int_{\Gamma_\varepsilon} f(w)\,dw + \int_\varepsilon^r \frac{u^{z-1}}{1+u} du =$$

$$= 2\pi i \sum_j Rez\,[f, w_j]$$

Trecem la limită după $r \to \infty$, $\varepsilon \to 0$ şi folosind (2.2) avem:

$$\left(1-e^{2\pi iz}\right)\lim_{\substack{r\to\infty\\ \varepsilon\to 0}}\int_{\varepsilon}^{r}\frac{u^{z-1}}{1+u}du=2\pi i\sum_{j}Rez\left[f,w_j\right]$$

Funcţia f are $w_0=-1$ pol de ordinul unu, cu relaţia (2.1) avem:

$$B\left(z,1-z\right)=\frac{2\pi i}{1-e^{2\pi iz}}\cdot\lim_{w\to-1}\left(w+1\right)\cdot\frac{w^{z-1}}{1+w}=$$

$$=\frac{2\pi i}{1-e^{2\pi iz}}\cdot\lim_{w\to-1}e^{(z-1)[\ln|w|+i\arg w]}=$$

$$=\frac{2\pi i}{1-e^{2\pi iz}}\cdot e^{(z-1)[\ln|-1|+i\arg(-1)]}=\frac{2\pi i}{1-e^{2\pi iz}}\cdot e^{i(z-1)\cdot\pi}=$$

$$=\frac{2\pi ie^{iz\pi}}{e^{2\pi iz}-1}=\frac{\pi}{\frac{e^{i\pi z}-e^{-i\pi z}}{2i}}=\frac{\pi}{\sin\pi z}.$$

□

Aplicaţia 2.3 $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$, $B\left(z,z'\right)=\frac{\Gamma(z)\cdot\Gamma(z')}{\Gamma(z+z')}$ în care facem

$$z=z'=\frac{1}{2}\Rightarrow\pi=\frac{\pi}{\sin\frac{\pi}{2}}=B\left(\frac{1}{2},\frac{1}{2}\right)=\Gamma^2\left(\frac{1}{2}\right)\Rightarrow$$

$$\Rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Aplicaţia 2.4

$$\int_{-\infty}^{\infty} e^{-x^2} dx = 2\int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-t} \cdot t^{-\frac{1}{2}} dt = \begin{array}{l} x^{2=t} \Rightarrow x = t^{\frac{1}{2}} \\ dx = \frac{1}{2} t^{-\frac{1}{2}} dt \end{array}$$

$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \Rightarrow \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{2}}{2}.$$

Aplicaţia 2.5

$$\int_0^{\infty} \frac{dx}{1+x^a}, a > 0.$$

Facem schimbarea de variabilă:

$$\frac{1}{1+x^a} = t \Leftrightarrow 1 + x^a = \frac{1}{t} \Rightarrow x^a = \frac{1}{t} - 1 \Rightarrow z = \left(\frac{1}{t} - 1\right)^{\frac{1}{a}} \Rightarrow$$

$$dx = \frac{1}{a}\left(\frac{1}{t} - 1\right)^{\frac{1}{a}-1} \cdot \left(\frac{-1}{t^2}\right) dt$$

$$\int_0^{\infty} \frac{dx}{1+x^a} = \frac{1}{a}\int_1^0 t \cdot \left(\frac{-1}{t^2}\right) \cdot \frac{(1-t)^{\frac{1}{a}-1}}{t^{\frac{1}{a}-1}} dt =$$

$$= \frac{1}{a}\int_0^1 t^{-\frac{1}{a}} \cdot (1-t)^{\frac{1}{a}-1} dt =$$

$$= \frac{1}{a}\int_0^1 t^{\left(1-\frac{1}{a}\right)-1} \cdot (1-t)^{\frac{1}{a}-1} dt = \frac{1}{a}\int_0^1 t^{\left(1-\frac{1}{a}\right)-1} \cdot (1-t)^{\frac{1}{a}-1} dt =$$

$$= \frac{1}{a} B\left(1 - \frac{1}{a}, \frac{1}{a}\right) = \frac{\pi}{a \sin \pi \left(1 - \frac{1}{a}\right)} = \frac{\pi}{a \sin \frac{\pi}{a}}.$$

Aplicaţia 2.6

$$\int_0^\infty \frac{z^{a-1}}{1+x}dx.$$

Facem schimbarea de variabilă:

$$x = \frac{t}{1-t} \Rightarrow t = \frac{x}{1+x} \Rightarrow \begin{array}{l} x=0 \Rightarrow t=0 \\ x \to \infty \Rightarrow t \to 1 \end{array} \; dx = \frac{dt}{(1-t)^2} \Rightarrow$$

$$\int_0^\infty \frac{z^{a-1}}{1+x}dx = \int_0^1 \frac{t^{a-1}}{(1-t)^{a-1}} \cdot (1-t) \frac{dt}{(1-t)^2} =$$

$$= \int_0^1 t^{a-1} \cdot (1-t)^{-a} dt =$$

$$= \int_0^1 t^{a-1} \cdot (1-t)^{(1-a)-1} dt = B(a, a-1) = \frac{\pi}{\sin \pi a}.$$

2.2 Polinoame ortogonale

Fie $C^0([a,b])$ spaţiul funcţiilor continue pe [a,b] şi funcţia pozitivă $\rho : [a,b] \to R_+$. Definim produsul scalar al funcţiilor f, g din spaţiul $C^0([a,b])$ cu ponderea ρ astfel:

$$< f, g >_\rho = \int_a^b f(x) \cdot g(x) \cdot \rho(x) \, dx.$$

În spaţiul $C^0([a,b])$ şirul format cu funcţiile: $1, x, x^2, \ldots, x^n$... formează un sistem de funcţii liniar independente şi utilizând procedeul Gram-Schmidt de ortogonalizare se poate transforma într-un şir ortogonal: $Q_0, Q_1, \ldots, Q_n, \ldots$ Şirul obţinut se numeşte *şir de polinoame ortogonale.*

În practică se utilizează trei tipuri de polinoame ortogonale depinzând de ponderea şi natura intervalului:

1. Polinoamele lui Jacobi, notate: $\left(j_n^{(p,q)}(x)\right)_{n\geq 0}$ ortogonale pe (-1,1) cu ponderea $\rho_{(x)} = (1-x)^p \cdot (1+x)^q$ cu $p, q \in \mathbb{R}$.

Cazuri particulare:

a) Pentru p=q=0 obţinem polinoamele **Lengendre** definite de relaţia:

$$P_n = \frac{(-1)^n}{n!} \cdot \frac{d^n}{dx^n}\left[x^n (1-x)^n\right].$$

Ponderea este $\rho(x) = 1$.

$$\int_{-1}^{1} P_n(x) \cdot P_m(x)dx = \begin{cases} 0, n \neq m, \\ \frac{2}{2n+1}, n = m. \end{cases}$$

Funcţia de recurenţă:

$$(n+1)\,P_{n+1}(x) - (2n+1)\,P_n(x) \cdot x + n \cdot P_{n-1}(x) = 0.$$

b) Pentru $p = q = -\frac{1}{2}$ obţinem polinoamele **Cebîşev**:

Ele verifică relaţia de recurenţa:

$$T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$$

Ponderea este :$\rho(x) = \frac{1}{\sqrt{1-x^2}}$.

Polinoamele Cebîşev sunt de forma:

$$T_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \cdot C_n^{2k} \cdot \left(1-x^2\right)^k \cdot x^{n-2k}.$$

c) Pentru $p = q = \lambda - \frac{1}{2}$ se obţin polinoamele **Gegenbauer**. Ponderea este:

$$\rho_{(x)} = \left(1 - x^2\right)^{\lambda - \frac{1}{2}}.$$

2. Polinoamele lui Laguerre notate $(L_n(x))_{n\geq 0}$ care sunt ortogonale pe $(0, \infty)$ cu ponderea $\rho_{(x)} = x^2 \cdot e^{-x}$, $\lambda \in \mathbb{R}$.

Polinoamele Laguerre verifică ecuaţia diferenţială:

$$xL_n^{''}(x) + (1 - x) L_n^{'}(x) + nL_n(x) = 0.$$

Sunt definite de relaţia:

$$L_n(x) = e^x \cdot \frac{d^n}{dx^n}\left[x^n \cdot e^{-x}\right].$$

Polinoamele Laguerre au forma:

$$L_n(x) = (-1)^n \left[x^n - \frac{n^2}{1!} \cdot x^{n-1} + \right.$$

$$\left. + \frac{n^2 (n-1)^2}{2!} \cdot x^{n-2} - ... + (-1)^n \cdot n!\right].$$

3. Polinoamele Hermite notate $(H_n(x))_{n\geq 0}$ care sunt ortogonale pe $(-\infty, \infty)$ cu ponderea

$$\rho_{(x)} = e^{-x^2}.$$

Polinoamele Hermite verifică relaţia de recurenţă:

$$H_{n+1}(x) - 2x \cdot H_n(x) + 2nH_{n-1}(x) = 0$$

unde:

$$H_n(x) = (-1)^n e^{-x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right).$$

2.3 Funcţii Bessel

Definiţia 2.7 Se numesc *funcţii Bessel* sau *funcţii cilindrice* soluţiile ecuaţiei

$$x^2y'' + xy' + \left(x^2 - \nu^2\right) y = 0, \tag{2.3}$$

unde ν este un parametru real sau complex.

Pentru rezolvarea ecuaţiei (2.3) se caută soluţiile de forma:

$$y\left(x\right) = x^r \sum_{i=0}^{\infty} a_i x^i. \tag{2.4}$$

Calculăm derivatele lui $y(x)$. Avem

$$y'\left(x\right) = rx^{r-1} \sum_{i=0}^{\infty} a_i x^i + x^r \sum_{i=0}^{\infty} i a_i x^{i-1}$$

$$y''\left(x\right) = r\left(r-1\right) x^{r-2} \sum_{i=0}^{\infty} a_i x^i +$$

$$+2rx^{r-1} \sum_{i=0}^{\infty} i a_i x^{i-1} + x^r \sum_{i=0}^{\infty} i\left(i-1\right) a_i x^{i-2}.$$

Şi înlocuind în (2.3) avem:

$$r\left(r-1\right) x^r \sum_{i=0}^{\infty} a_i x^i + 2rx^{r+1} \sum_{i=0}^{\infty} i a_i x^{i-1} +$$

$$+x^{r+2} \sum_{i=1}^{\infty} i\left(i-1\right) a_i x^{i-2} + rx^r \sum_{i=0}^{\infty} a_i x^i + x^{r+1} \sum_{i=0}^{\infty} i a_i x^{i-1} +$$

$$+x^{r+2} \sum_{i=0}^{\infty} a_i x^i - \nu^2 x^r \sum_{i=0}^{\infty} a_i x^i = 0 / : x^r$$

$$\sum_{i=0}^{\infty} r\left(r-1\right) a_i x^i + \sum_{i=0}^{\infty} 2ria_i x^{ri} + \sum_{i=1}^{\infty} i\left(i-1\right) a_i x^i +$$

$$+r\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} ia_i x^i + \sum_{i=0}^{\infty} a_i x^{i+2} - \nu^2 \sum_{i=0}^{\infty} a_i x^i = 0 \Leftrightarrow$$

$$\sum_{i=0}^{\infty} \left[r\left(r-1\right) + 2ir + i\left(i-1\right) + r + i - \nu^2\right] a_i x^i = -\sum_{i=0}^{\infty} a_i x^{i+2}$$

Indentificăm coeficenţii din relaţia:

$$\sum_{i=0}^{\infty} \left[\left(r+i\right)^2 - \nu^2\right] a_i x^i = -\sum_{i=0}^{\infty} a_i x^{i+2}$$

şi obţinem un sistem cu un număr infinit de ecuaţii şi necunoscute

$$\begin{array}{l}
\left(r^2-\nu^2\right) a_0 = 0 \\
\left[\left(r+1\right)^2 - \nu^2\right] a_1 = 0 \\
\left[\left(r+2\right)^2 - \nu^2\right] a_2 = 0 \\
\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
\left[\left(r+n\right)^2 - \nu^2\right] a_n + a_{n-2=0}
\end{array}$$

Punând condiţia ca $a_0 \neq 0$ deducem din prima ecuaţie: $r = \pm\nu$.
Pentru $r = \nu$ din a doua ecuaţie deducem $a_1 = 0$, deoarece $2\nu + 1 \neq 0$.
De aici rezultă că toti coeficenţii de indice impar sunt nuli.
Pentru coeficenţii de indice par avem relaţia:

$$\left[\left(\nu+2n\right)^2 - \nu^2\right] a_{2n} + a_{2n-2} = 0 \Leftrightarrow a_{2k} = -\frac{a_{2k-2}}{4k\left(k+\nu\right)},$$

$a_{2k} = -\frac{a_{2k-2}}{4k(k+\nu)}$. Aplicăm succesiv ultima formulă: $k = 1, 2, ...$

$$a_{2k} = (-1)^2 \cdot \frac{a_{2k-4}}{2^2 \cdot 2^2 k (\nu + k)(k-1)(\nu + k - 1)} =$$

$$= \ldots =$$

$$= (-1)^k \cdot \frac{a_0}{(2^2)^k k! \cdot (\nu + k)(\nu + k - 1) \cdot \ldots \cdot (\nu + 1)} =$$

$$= (-1)^k \cdot \frac{a_0}{2^{2k} \cdot k! (\nu + 1)(\nu + 2) \ldots (\nu + k)},$$

unde a_0 are o valoare nedeterminată.
Alegând $a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}$

$$\Rightarrow a_{2k} = (-1)^k \frac{1}{2^{2k+\nu} k! \Gamma(\nu + k + 1)} =$$

$$= \frac{(-1)^k}{2^{2k+\nu} \cdot \Gamma(k) \cdot \Gamma(\nu + k + 1)}$$

deci:

$$\left.\begin{array}{c} a_{2k+1} = 0, \ a_{2k} = \frac{(-1)^k}{2^{2k+\nu} \cdot \Gamma(k) \cdot \Gamma(k+\nu+1)} \\ r = \nu \end{array}\right\}$$

şi înlocuind în (2.4) găsim funcţia:

$$j(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{\Gamma(k+1)\Gamma(k+\nu+1)} \cdot \left(\frac{x}{2}\right)^{2k+\nu}$$

notată:

$$j_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\nu+1)} \cdot \left(\frac{x}{2}\right)^{2k+\nu}$$

şi numită funcţie *Bessel de speţa I-a*.

Pentru $r = -\nu$ analog găsim:

$$j_{-\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k-\nu+1)} \cdot \left(\frac{x}{2}\right)^{2k-\nu}.$$

Deci, funcţiile Bessel de speţa I-a sunt de forma:

$$j_{\pm\nu}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k\pm\nu+1)} \cdot \left(\frac{x}{2}\right)^{2k\pm\nu} \leftarrow$$

cu d'Alembert, seriile sunt convergente, $(\forall)\ x \in \mathbb{R}$.

Teorema 2.8 *Dacă $\nu \notin \mathbb{Z}$ atunci funcţiile Bessel de speţa I-a, $j_{\pm\nu}$ sunt liniar independente şi atunci orice funcţie Bessel se obţine prin particularizarea constantelor C_1 şi C_2:*

$$j(x) = C_1 j_\nu(x) + C_2 j_{-\nu}(x).$$

Observaţia 2.9

$$j_\nu(x) \xrightarrow[x\to 0]{} 0 \text{ şi } j_{-\nu}(x) \xrightarrow[x\to 0]{} +\infty$$

rezultă că funcţiile nu sunt liniar dependente.

Teorema 2.10

$$j_{-n}(x) = (-1)^n j_n(x),\ n \in \mathbb{Z}.$$

Teorema 2.11 *Funcţiile Bessel verifică următoarele relaţii de recurenţă:*

$$\frac{d}{dz}\left[z^\nu j_\nu(z)\right] = z^\nu \cdot j_{\nu-1}(z);\ \frac{d}{dz}\left[z^{-\nu} j_\nu(z)\right] = -z^{-\nu} \cdot j_{\nu+1}(z).$$

Demonstraţie.

$$\frac{d}{dz}\left[z^{\nu}\cdot y_{\nu}\left(z\right)\right]=$$

$$=\frac{d}{dz}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}z^{2k+2\nu}}{2^{2k+\nu}\cdot\Gamma\left(k+1\right)\cdot\Gamma\left(k+\nu+1\right)}=$$

$$=\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\cdot\left(2k+2\nu\right)z^{2k+2\nu-1}}{2^{2k+\nu}\cdot\Gamma\left(k+1\right)\cdot\Gamma\left(k+\nu+1\right)}=$$

$$=\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}\cdot\left(k+\nu\right)\cdot z^{\nu}\cdot z^{2k+\nu-1}}{2^{2k+\nu-1}\cdot\Gamma\left(k+1\right)\cdot\left(k+\nu\right)\Gamma\left(k+\nu+1\right)}=$$

$$=z^{\nu}\cdot\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}}{\Gamma\left(k+1\right)\cdot\Gamma\left(k+\left(\nu-1\right)+1\right)}\cdot\left(\frac{z}{2}\right)^{2k+\left(\nu-1\right)}=$$

$$=z^{\nu}\cdot y_{\nu-1}\left(z\right);$$

$$\frac{d}{dz}\left[z^{-\nu}\cdot j_{\nu}\left(z\right)\right]=$$

$$=\frac{d}{dz}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k}z^{2k}}{2^{2k+\nu}\cdot\Gamma\left(k+1\right)\cdot\Gamma\left(k+\nu+1\right)}$$

$$==\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k}\left(2k\right)z^{2k-1}}{2^{2k+\nu}\cdot k\Gamma\left(k\right)\cdot\Gamma\left(k+\nu+1\right)}=$$

$$=z^{-\nu}\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k}}{\Gamma\left(k\right)\cdot\Gamma\left(k+\nu+1\right)}\cdot\left(\frac{z}{2}\right)^{2k+\nu-1}=$$

$$=\mathrm{z}^{-\nu}\sum_{k=0}^{\infty}\frac{\left(-1\right)^{k+1}}{\Gamma\left(k+1\right)\cdot\Gamma\left(k+\nu+2\right)}\cdot\left(\frac{z}{2}\right)^{2k+\nu+1}=$$

$$= -z^{-\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1) \cdot \Gamma(k+\nu+2)} \cdot \left(\frac{z}{2}\right)^{2k+(\nu+1)} =$$

$$= -z^{-\nu} \cdot j_{\nu+1}(z).$$

Expresiile funcţiilor Bessel pentru citirea valorilor particulare ale indiciilor:

1. Pentru $\nu = \frac{1}{2}$ avem

$$j_{\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(k+\frac{3}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2k+\frac{1}{2}} =$$

$$= \left(\frac{2}{z}\right)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma\left(\frac{2k+3}{2}\right)} \cdot \frac{z^{2k+1}}{2^{2k+1}} =$$

$$= \sqrt{\frac{2}{z}} \sum_{k=0}^{\infty} \left(\frac{(-1)^k \cdot z^{2k+1}}{2 \cdot 4 \cdot \ldots \cdot (2k) \cdot 2^{k+1}} \cdot \right.$$

$$\left. \cdot \frac{1}{\frac{2k+1}{2} \cdot \frac{2k-1}{2} \cdot \frac{2k-3}{2} \cdot \ldots \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} \right) =$$

$$= \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} \cdot z^{2k+1} \Rightarrow$$

$$j_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cdot \sin z.$$

2. Pentru $\nu = -\frac{1}{2}$ avem

$$j_{-\frac{1}{2}}(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma\left(k+\frac{1}{2}\right)} \cdot \left(\frac{z}{2}\right)^{2k-\frac{1}{2}} =$$

$$= \sqrt{\frac{2}{z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \frac{2k-1}{2} \cdot \frac{2k-3}{2} \cdot \ldots \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}} \cdot \frac{z^{2k}}{2^{2k}} =$$

$$= \sqrt{\frac{2}{z}} \cdot \sum_{k=0}^{\infty} \left(\frac{(-1)^k}{2 \cdot 4 \cdot \ldots \cdot (2k)} \cdot \right.$$

$$\left. \cdot \frac{1}{(2k-1) \cdot (2k-3) \cdot \ldots \cdot 3 \cdot 1 \cdot \sqrt{\pi}} \cdot z^{2k} \right) =$$

$$= \sqrt{\frac{2}{\pi z}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} \cdot z^{2k} = \sqrt{\frac{2}{\pi z}} \cdot \cos z \Rightarrow$$

$$j_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \cdot \cos z.$$

3. Pentru
$$\frac{d}{dz} \left[z^{-\nu} \cdot j_\nu(z) \right] = -z^{-\nu} \cdot j_{\nu+1}(z)$$
facem $\nu = \frac{1}{2}$ şi rezultă

$$\frac{d}{dz} \left[\frac{1}{\sqrt{z}} \cdot j_{\frac{1}{2}}(z) \right] = -\frac{1}{\sqrt{z}} \cdot j_{\frac{3}{2}}(z) \Leftrightarrow$$

$$j_{\frac{3}{2}}(z) = -\sqrt{z} \cdot \frac{d}{dz} \left[\sqrt{\frac{2}{\pi}} \cdot \frac{\sin z}{z} \right] = -\sqrt{\frac{2z}{\pi}} \cdot \frac{z \cos z - \sin z}{z^2}$$

$$\Rightarrow j_{\frac{3}{2}}(z) = -\sqrt{\frac{2}{\pi z}} \left(\cos z - \frac{\sin z}{z} \right) =$$

$$= \sqrt{\frac{2}{\pi z}} \left(\frac{\sin z}{z} - \cos z \right).$$

În relaţia de recurenţa $\frac{d}{dz}\left[z^{\nu}\cdot j_{\nu}\left(z\right)\right]=z^{\nu}\cdot j_{\nu-1}\left(z\right)$ facem $\nu=-\frac{1}{2}$ şi rezultă

$$\frac{d}{dz}\left[\frac{1}{\sqrt{z}}\cdot\sqrt{\frac{2}{\pi z}}\cos z\right]=\frac{1}{\sqrt{z}}\cdot j_{-\frac{2}{3}}\left(z\right)\Rightarrow$$

$$j_{-\frac{2}{3}}\left(z\right)=\sqrt{z}\cdot\sqrt{\frac{2}{\pi}}\frac{d}{dz}\left(\frac{\cos z}{z}\right)=$$

$$=\sqrt{\frac{2z}{\pi}}\cdot\frac{-z\sin z-\cos}{z^{2}}=-\sqrt{\frac{2}{\pi z}}\left(\sin z+\frac{\cos z}{z}\right).$$

Analog: $j_{\frac{5}{2}}\left(z\right)$ şi $j_{-\frac{5}{2}}\left(z\right)$. $\square$

Aplicaţia 2.12 Să se găsească soluţia generală a ecuaţiei:

$$z^{2}y''-2zy'+4\left(z^{4}-1\right)y=0$$

Facem schimbarea de variabilă: $\begin{cases} z=kx^{u} \\ y=x^{\lambda}u\left(x\right) \end{cases}$ şi de funcţie. Avem:

1. Determinăm $y'\left(z\right)$

$$y'\left(z\right)=\frac{dy}{dz}=\frac{dy}{dx}\cdot\frac{1}{\frac{dz}{dx}}=$$

$$=\left[\lambda x^{\lambda-1}u\left(x\right)+x^{\lambda}\cdot\frac{du}{dx}\right]\cdot\frac{1}{k\mu x^{\mu-1}}=$$

$$=\frac{\lambda}{k\mu}x^{\lambda-\mu}\cdot u\left(x\right)+\frac{x^{\lambda-\mu+1}}{k\mu}\cdot\frac{du}{dx}.$$

2. Determinăm $y''(z)$

$$y''(z) = \frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) =$$

$$= \frac{d}{dx}\left[\frac{dy}{dz}\right] \cdot \frac{1}{\frac{dz}{dx}} = \frac{1}{k\mu x^{\mu-1}} \cdot$$

$$\cdot \frac{d}{dx}\left[\frac{\lambda}{k\mu} x^{\lambda-\mu} \cdot u(x) + \frac{x^{\lambda-\mu+1}}{k\mu} \cdot \frac{du}{dx}\right] =$$

$$= \frac{1}{k\mu x^{\mu-1}} \cdot \left[\frac{\lambda(\lambda-\mu) x^{\lambda-\mu}}{k\mu} \cdot u(x) + \right.$$

$$\left. + \frac{2\lambda-\mu+1}{k\mu} \cdot x^{\lambda-\mu} \cdot \frac{du}{dx} + \frac{x^{\lambda-\mu+1}}{k\mu} \cdot \frac{d^2u}{dx^2}\right].$$

Ecuaţia devine:

$$\frac{k^2x^{2\mu}}{k\mu x^{\mu-1}}\left[\frac{\lambda}{k\mu} \cdot (\lambda-\mu) \cdot x^{\lambda-\mu-1} \cdot u(x) + \frac{2\lambda-\mu+1}{k\mu} \cdot x^{\lambda-\mu} \cdot \frac{du}{dx} + \right.$$

$$\left. + \frac{x^{\lambda-\mu+1}}{k\mu} \cdot \frac{d^2u}{dx^2}\right] - 2kx^{\mu}\left[\frac{\lambda}{k\mu} \cdot x^{\lambda-\mu} u(x) + \frac{x^{\lambda-\mu+1}}{k\mu} \cdot \frac{du}{dx}\right] +$$

$$+4\left(k^4x^{4\mu}-1\right) \cdot x^{\lambda} u(x) = 0 \Leftrightarrow$$

$$\frac{\lambda}{\mu^2} \cdot (\lambda-\mu) x^{\lambda} u(x) + \frac{2\lambda-\mu+1}{\mu^2} \cdot x^{\lambda+1} \cdot \frac{du}{dx} + \frac{1}{\mu^2} \cdot x^{\lambda+2} \cdot \frac{d^2u}{dx^2} -$$

$$-2\frac{\lambda}{\mu} \cdot x^{\lambda} u(x) - \frac{2}{\mu} \cdot x^{\lambda+1}\frac{du}{dx} + 4\left(k^4x^{4\mu}-1\right) x^{\lambda} u(x) = 0.$$

$$\frac{1}{\mu^2} \cdot x^{\lambda+2}\frac{d^2u}{dx^2} + \frac{2\lambda-3\mu+1}{\mu^2} \cdot x^{\lambda+1} \cdot \frac{du}{dx} +$$

$$+\left(\frac{\lambda^2}{\mu^2}-3\frac{\lambda\mu}{\mu^2}+4k^4x^{4\mu}-4\right)x^\lambda u\left(x\right)=0/:\frac{x^\lambda}{\mu^2}\Rightarrow$$

$$x^2\frac{d^2u}{dx^2}+\left(2\lambda-3\mu+1\right)x\frac{du}{dx}+$$
$$+\left(\lambda^2-4\mu^2-3\lambda\mu+4k^4\mu^2x^{4\mu}\right)\cdot u\left(x\right)=0.$$

Punem:

$$\left\{\begin{array}{c} 2\lambda-3\mu+1=1 \\ 4\mu=2 \\ 4k^4\mu^2=1 \end{array}\right. \Rightarrow \lambda=\frac{3}{4},\mu=\frac{1}{2},\ k=1.$$

Considerăm: $-\nu^2=\lambda^2-4\mu^2-3\lambda\mu=\frac{9}{16}-4\cdot\frac{1}{4}-\frac{9}{4}\cdot\frac{1}{2}=\frac{9}{16}-{}^{16/}1-{}^{2/}\frac{9}{8}=$

$$=\frac{-7-18}{16}=\frac{-25}{16}\Rightarrow\ \nu^2=\left(\frac{5}{4}\right)^2\Rightarrow\nu=\pm\,\frac{5}{4}.$$

Deci, am obţinut ecuaţia Bessel $u(x)$:

$$x^2u''\left(x\right)+xu'\left(x\right)+\left(x^2-\frac{25}{16}\right)u\left(x\right)=0,$$

de unde:

$$\left.\begin{array}{l} u\left(x\right)=aj_{\frac{5}{4}}\left(x\right)+bj_{-\frac{5}{4}}\left(x\right) \\ \underbrace{\mathrm{z}=\mathrm{x}^{\frac{1}{2}}}_{z^2=x},\ \mathrm{y}\left(z\left(x\right)\right)=x^{\frac{3}{4}}u\left(x\right)=z^{\frac{3}{2}}u\left(x\right) \end{array}\right\}\Rightarrow$$

$$y\left(z\right)=az^{\frac{3}{4}}\cdot j_{\frac{5}{4}}\left(z^2\right)+bz^{\frac{3}{2}}\cdot j_{-\frac{5}{4}}\left(z^2\right).$$

2.4 Transformata Laplace

Definiţia 2.13 Fie $f : \mathbb{R} \to \mathbb{R}(\mathbb{C})$; dacă are sens integrala improprie cu parametrul $p \in \mathbb{C}$, $F(p) = \int_0^\infty e^{-pt} \cdot f(t)\, dt$ atunci F se numeşte *transformata Laplace* a lui f şi se notează prin: $L[f(t)](p)$.

Definiţia 2.14 Funcţia $f : \mathbb{R} \to \mathbb{R}(\mathbb{C})$ se numeşte *funcţie original Laplace* dacă îndeplineşte condiţiile:

i) $f(t) = 0$ pentru $t < 0$;

ii) f este continuă pe porţiuni

iii) $|f(t) \cdot e^{-s_0 t}| \leq M$, $M > 0$, $t > t_0$, cu s_0, t_0 şi $M \in \mathbb{R}_+$.

Observaţia 2.15

1. Transformata Laplace se numeşte *funcţia imagine.*

2. Condiţia iii) se numeşte condiţia de creştere exponenţială şi se scrie sub forma:

$$|f(t)| \leq M \cdot e^{s_0 t}, \ (\forall)\ \mathrm{t} > \mathrm{t}_0.$$

Considerăm:

$$\mathrm{Re\ p} = \tau > s_0 \qquad \left|e^{-pt}\right| = e^{-(\mathrm{Re\ p})t} < e^{-\tau t}, \quad \mathrm{t} > 0;$$

Avem:

$$\left|\int_0^\infty f(t) \cdot e^{-pt} dt\right| \leq \int_0^\infty |f(t)| \cdot \left|e^{-pt}\right| dt \leq \int_0^\infty M \cdot e^{s_0 t} \cdot e^{-\tau t} dt =$$

$$= M \int_0^\infty e^{-(\tau - s_0)t} dt =$$

$$= -\frac{M}{\tau - s_0} \cdot e^{-(\tau - s_0)t} \Big|_0^\infty = \frac{M}{s - \tau_0} \Rightarrow$$

conform criteriului comparaţiei pentru integrala improprie, avem că integrala

$$\int_0^\infty f(t) \cdot e^{-pt} dt$$

este absolut şi uniform convergentă $\Rightarrow$ F(p) este bine definită/şi olomorfă pe semiplanul $Rep > s_0$.

2.4.1 Proprietăţi ale transformatei Laplace

1) Transformata Laplace este liniară:

$$L[\lambda_1 f_1(t) + \lambda_2 f_2(t)](p) = \int_0^\infty [\lambda_1 f_1(t) + \lambda_2 f_2(t)] dt =$$

$$= \lambda_1 L[f_1(t)](p) + \lambda_2 L[f_2(t)].$$

2) **Teorema 2.16** *(a asemănării) Dacă f este o funcţie original şi $a > 0$ rezultă*

$$L[f(at)](p) = \int_0^\infty f(at) \cdot e^{-pt} dt =$$

$$= \frac{1}{a} \int_0^\infty f(u) \cdot e^{-\frac{p}{a}u} du = \frac{1}{a} L[f(t)] \left(\frac{p}{a}\right)$$

$$u = at \Rightarrow t = \frac{u}{a} \Rightarrow dt = \frac{1}{a} du.$$

3) **Teorema 2.17** *(a întârzierii) Dacă f este o funcţie original, $a > 0$;*

$$L[f(t-a)](p) = \int_0^\infty e^{-pt} \cdot f(t-a) dt =$$

$$= \int_0^\infty e^{-p(a+u)} \cdot f(u)\,du = \int_0^\infty e^{-pa} \cdot e^{-pu} f(u)\,du =$$

$u = t - a \Rightarrow dt = du;\ t = u + a$

$$= e^{-pa} \cdot \int_0^\infty e^{-pu} f(u)\,du = e^{-pa} \cdot L[f(t)](p).$$

4) **Teorema 2.18** *(a deplasării)*

$$L\left[e^{at} \cdot f(t)\right](p) = \int_0^\infty f(t) \cdot e^{-(p-a)t} dt = L[f(t)](p-a).$$

5) **Teorema 2.19** *(a derivării originalului) Dacă f este funcţie original,* $(\exists)$ $f'(t)$ *funcţie original:*

$$L[f'(t)](p) = pL[f(t)](p) - f(0+0).$$

$f =$ original şi pentru Re$p \geq +\tau > s_0$ avem: (şi pentru Re$p = \tau > s_0$)

$$\left|f(t) \cdot e^{-pt}\right| \leq |f(t)| \cdot \left|e^{-pt}\right| \leq M \cdot e^{s_0 t} \cdot e^{-\tau t} =$$

$$= M \cdot e^{-(\tau \pm s_0)t} \xrightarrow[t \to \infty]{} 0$$

$$L[f'(t)](p) = \int_0^\infty f'(t) \cdot e^{-pt} dt =$$

$$= f(t) \cdot e^{-pt} \Big|_0^\infty + p \int_0^\infty f(t) \cdot e^{-pt} dt =$$

$$= pL[f(t)](p) - f(0).$$

6) **Teorema 2.20** *(a derivării imaginii) Dacă f este funcţie original, atunci:*

$$L\left[t \cdot f(t)\right](p) = \int_0^\infty t f(t) \cdot e^{-pt} dt =$$

$$= -\int_0^\infty f(t) \cdot \left(e^{-pt}\right)'_p dt =$$

$$= -\left(\int_0^\infty f(t) \cdot e^{-pt} dt\right)'_p =$$

$$-\left(L\left[f(t)\right](p)\right)' = (-1)^1 \cdot \left(L\left[f(t)\right](p)\right)'$$

Altfel, cu derivarea integralei improprii cu parametru

$$F'(p) = \left(\int_0^\infty f(t) \cdot e^{-pt} dt\right)' = -\int_0^\infty t f(t) \cdot e^{-pt} dt =$$

$$= -L\left[t \cdot f(t)\right](p).$$

$$F^{(n)}(p) = (-1)^n \int_0^\infty t^n \cdot f(t) \cdot e^{-pt} dt \Rightarrow \int_0^\infty t^n \cdot f(t) \cdot e^{-pt} dt =$$

$$= (-1)^n F^{(n)}(p).$$

7) **Teorema 2.21** *(a integrării originalului)*

$$L\left[\int_0^t f(u)\, du\right](p) = \frac{1}{p} F(p).$$

Demonstraţie. Notăm: $f_1(t) = \int_0^t f(u)\, du \Rightarrow f_1'(t) = f(t) \Rightarrow$

$$F(p) = L[f(t)](p) = L\left[f_1^{'}(t)\right](p) =$$
$$= pL[f_1(t)](p) - f_1(0) =$$
$$= pL\left[\int_0^{\infty} f(u)\,du\right](p) \Rightarrow$$
$$L\left[\int_0^{t} f(u)\,du\right](p) = \frac{1}{p}F(p).$$

□

8) **Teorema 2.22** *(a integrării imaginii) Dacă f este o funcţie original rezultă*

$$L\left[\frac{f(t)}{t}\right](p) = \int_p^{\infty} L[f(t)](q)\,dq.$$

Demonstraţie.(Prima metodă:)

$$G(p) = \int_p^{\infty} F(q)\,dq = \lim_{z\to\infty}\int_p^{z} F(q)\,dq =$$
$$= \lim_{z\to\infty}[\Phi(z) - \Phi(p)] = -\Phi(p) \Rightarrow$$
$$G^{'}(p) = -F(p);$$

Fie g originalul funcţiei imagine G. Avem cu teorema (2.20):

$$G'(p) = -L[t\cdot g(t)](p) = L[-t\cdot g(t)](p).$$

Deci:

$$\left.\begin{array}{c} F(p) = L[f(t)](p) \\ G'(p) = L[-t\cdot g(t)](p) \\ G'(p) = -F(p) \end{array}\right\} \Rightarrow$$

$$\Rightarrow L\left[-f\left(t\right)\right]\left(p\right)=L\left[-t\cdot g\left(t\right)\right]\left(p\right)\Leftrightarrow$$

$$\Leftrightarrow \text{inversibilitatea lui Laplace}\ .$$

$$-f\left(t\right)=-t\cdot g\left(t\right)\Leftrightarrow g\left(t\right)=\frac{f\left(t\right)}{t}\Rightarrow$$

$$L\left[\frac{f\left(t\right)}{t}\right]\left(p\right)=L\left[g\left(t\right)\right]\left(p\right)=\int_{p}^{\infty}F\left(q\right)dq.$$

□

Demonstraţie.(A doua metodă:)

$$\int_{p}^{\infty}L\left[f\left(t\right)\right]\left(q\right)dq=\int_{p}^{\infty}\left(\int_{0}^{\infty}f\left(t\right)\cdot e^{-qt}dt\right)dq=$$

$$=\int_{0}^{\infty}f\left(t\right)\left(\int_{p}^{\infty}e^{-qt}dq\right)dt=\int_{0}^{\infty}f\left(t\right)\cdot\left(\left.\frac{e^{-qt}}{-t}\right|\begin{matrix}\infty\\ p\end{matrix}\right)dt=$$

$$=\int_{0}^{\infty}f\left(t\right)\cdot\frac{e^{-pt}}{t}dt=L\left[\frac{f\left(t\right)}{t}\right]\left(p\right).$$

□

Observaţia 2.23 Dacă

$$p=0\Rightarrow\int_{0}^{\infty}L\left[f\left(t\right)\right]\left(p\right)dp=L\left[\frac{f\left(t\right)}{t}\right]\left(0\right)=$$

$$=\int_{0}^{\infty}\frac{f\left(t\right)}{t}dt\Rightarrow\int_{0}^{\infty}\frac{f\left(t\right)}{t}dt=\int_{0}^{\infty}L\left[f\left(t\right)\right]\left(p\right)dp.$$

9) **Teorema 2.24** *Dacă*

$$F(p) = \int_0^\infty e^{-pt} \cdot f(t)\, dt = L[f(t)](p)$$

$$G(p) = \int_0^\infty e^{-pt} \cdot g(t)\, dt = L[g(t)](p),$$

atunci

$$L[(f * g)(t)](p) = F(p) \cdot G(p).$$

Demonstraţie.

$$(f * g)(t) = \int_0^t f(u) \cdot g(t-u)\, du$$

pentru $t - u < 0 \Leftrightarrow t < u$ avem $g(t-u) = 0$ pentru $u < t$.

$$L[(f * g)(t)](p) = \int_0^\infty e^{-pt} \cdot \left(\int_0^t f(u) \cdot g(t-u)\, du \right) dt =$$

$$= \int_0^\infty e^{-pt} \cdot \left(\int_0^\infty f(u) \cdot g(t-u)\, du \right) dt =$$

$$= \int_0^\infty \int_0^\infty f(u) \cdot e^{-pu} \cdot g(t-u) \cdot e^{-p(t-u)} du dt$$

cu $t - u = \xi$

$$= \int_0^\infty \int_0^\infty f(u) \cdot e^{-pu} \cdot g(\xi) \cdot e^{-p\xi} du d\xi \underset{Fubini}{=}$$

$$= \left(\int_0^\infty f(u) \cdot e^{-pu} du \right) \left(\int_0^\infty g(\xi) \cdot e^{-p\xi} d\xi \right) = F(p) \cdot G(p).$$

Sau:

$$L\left[(f*g)(t)\right](p)=\int_0^\infty\int_0^\infty f(u)\cdot g(t-u)\cdot e^{-pt}dudt=$$
$$=\int_0^\infty\int_0^\infty f(u)\cdot g(s)\cdot e^{-p(u+s)}duds$$

$$\begin{array}{ll} t-u=s\Rightarrow t=u+s\Rightarrow dt=ds & \\ t=o\Rightarrow s=-u & \text{pentru :}\quad u>t\Rightarrow \\ t=\infty\Rightarrow s=\infty & \end{array}$$

$$\Rightarrow g(t-u)=0.$$

Deci:

$$L\left[(f*g)(t)\right](p)=$$
$$=\int_0^\infty\left(\int_0^t f(u)\,g(t-u)\,du\right)e^{-pt}dt=$$
$$=\int_0^\infty\int_0^\infty f(u)\,g(t-u)\cdot e^{-pt}dudt=$$
$$=\int_0^\infty\int_{-s}^\infty f(u)\,g(s)\cdot e^{-p(u+s)}duds=$$
$$=\int_0^\infty\int_0^\infty f(u)\,g(s)\cdot e^{-p(u+s)}duds\overset{Fubini}{=}$$
$$=\left(\int_0^\infty f(u)\cdot e^{-pu}du\right)\left(\int_0^\infty g(s)\cdot e^{-ps}ds\right)=F(p)\cdot G(p)$$

Schimbarea de variabilă:

$$\begin{array}{l} t-u=s\Rightarrow t=u+s\Rightarrow dt=ds \\ t=0\Rightarrow s=-u \\ t=\infty\Rightarrow s=\infty. \end{array}$$

□

2.4.2 Inversa transformatei Laplace

Teorema 2.25 *(Mellin Fourier) Dacă*

$$F(t) = L[f(t)](p)$$

şi x este un punct de continuitate pentru f, atunci:

$$f(x) = \frac{1}{2\pi i}\int_{s-i\infty}^{s+i\infty} F(p) \cdot e^{pt} dp,$$

unde $s = s_0$.

Observaţia 2.26 Se calculează cu ajutorul reziduurilor şi se obţine: $f(x) = \sum_k Rez[f(p) \cdot e^{px}, pk]$, unde p_k sunt singularităţile lui $F(p)$ în semiplanul Re$p < s_0$.

Aplicaţia 2.27

$$L\left[e^{\lambda t}\right](p) = \frac{1}{p-\lambda}; \ \int_0^\infty e^{-(p-\lambda)t}dt = -\frac{e^{-(p-\lambda)t}}{p-\lambda}\Bigg| \begin{matrix}\infty \\ 0\end{matrix} = \frac{1}{p-\lambda}.$$

Aplicaţia 2.28

$$\underset{k>-1}{L}\left[t^k\right](p) = \int_0^\infty t^k \cdot e^{-pt}dt \underset{\substack{x=pt \\ dx=pdt \\ dt=\frac{dx}{p}}}{=} \int_0^\infty \left(\frac{x}{p}\right)^k \cdot e^{-x} \cdot \frac{dx}{p} =$$

$$= \frac{1}{p^{k+1}}\int_0^\infty x^k \cdot e^{-x}dx = \frac{1}{p^{k+1}} \cdot \Gamma(k+1) = \frac{\Gamma(k+1)}{p^{k+1}}.$$

Temă

1.

$$L[sh\ \alpha t](p) = \frac{\alpha}{p^2-\alpha^2};$$

2.
$$L\left[ch\ \alpha t\right](p) = \frac{p}{p^2 - \alpha^2};$$

3.
$$L\left[\cos\ \omega t\right](p) = \frac{p}{p^2 + \omega^2};$$

4.
$$L\left[\sin \omega t\right](p) = \frac{\omega}{p^2 + \omega^2};$$

5.
$$L\left[\mathrm{e}^{\alpha t} \cdot \cos\ \omega t\right](p) = \frac{p - \alpha}{(p-\alpha)^2 + \omega^2};$$

6.
$$L\left[\mathrm{e}^{\alpha t} \cdot \sin\ \omega t\right](p) = \frac{\omega}{(p-\alpha)^2 + \omega^2};$$

Aplicaţia 2.29

$$L\left[t^n \cdot \sin\ \omega t\right](p) = Im\left[L\left[t^n \cdot e^{i\omega t}\right](p)\right] =$$

$$= Im\left[(-1)^n \cdot \left(\frac{1}{p - i\omega}\right)^{(n)}\right] =$$

$$= Im\left[\frac{(-1)^n \cdot (-1)^n \cdot n!}{(p - i\omega)^{n+1}}\right] =$$

$$= Im\frac{n!\,(p + i\omega)^{n+1}}{(p^2 + \omega^2)^{n+1}} = \frac{n!}{(p^2 + \omega^2)^{n+1}} \cdot Im\left[(p + i\omega)^{n+1}\right].$$

Aplicaţia 2.30

$$L\left[t^n \cos\ \omega t\right](p) = n!\frac{Re\,(p + i\omega)^{n+1}}{(p^2 + \omega^2)^{n+1}};$$

Temă

1.
$$L\left[t \cdot e^{\alpha t} \cos\ \omega t\right](p);$$

2.
$$L\left[t \cdot e^{\alpha t} \sin\ \omega t\right](p);$$

Aplicaţia 2.31

$$L\left[t \cos \omega t \cdot e^{\alpha t}\right](p) = L\left[t \cos \omega t\right](p-\alpha) = \frac{(p-\alpha)^2 - \omega^2}{\left[(p-\alpha)^2 + \omega^2\right]^2};$$

Aplicaţia 2.32

$$L\left[t \cos \omega t\right](p) = \frac{Re\,(p+i\omega)^2}{p^2+\omega^2} = \frac{p^2-\omega^2}{\left(p^2+\omega^2\right)^2};$$

Aplicaţia 2.33

$$L\left[t^2 \cos \omega t\right](p) = \frac{2!}{\left(p^2+\omega^2\right)^3} \cdot \left[Re\,(p+i\omega)^3\right] =$$

$$= \frac{2}{\left(p^2+\omega^2\right)^3} \cdot \left(p^3 - 3p\omega^2\right) = 2p \cdot \frac{p^2-3\omega^2}{\left(p^2+\omega^2\right)^3};$$

Temă

$$F(p) = \frac{2p^2-6p+5}{(p-1)(p-2)(p-3)}; \quad f(t) = ?.$$

2.4.3 Metode variaţionale

1. Integrarea ecuaţiilor diferenţiale liniare cu coeficienţi constanţi.

$$a_0x^{(n)}(t)+a_1x^{(n-1)}(t)+\ldots+a_{n-1}x^1(t)+a_nx(t)=f(t) \quad (2.5)$$

$$x(0)=x_0,\ x^1(0)=x_1,\ldots,x^{(n-1)}(0)=x_{n-1}.$$

Aplicăm transformata Laplace şi obţinem:

$$a_0L\left[x^{(n)}(t)\right](p)+a_1L\left[x^{(n-1)}(t)\right](p)+\ldots+a_{n-1}L\left[x^1(t)\right](p)+$$
$$+a_nL\left[x(t)\right](p)=L\left[f(t)\right](p) \quad (2.6)$$

Notăm: $X(p)=L\left[x(t)\right](p)$ şi $F(p)=L\left[f(t)\right](p)$.

- $L\left[x'(t)\right](p)=pX(p)-x(0)=pX(p)-x_0$
- $L\left[x''(t)\right](p)=p\left(pX(p)-x_0\right)=p^2X(p)-x_0p-x_1=$
  $=p^2X(p)-\left(x_0p+x_1\right)$

  $\vdots$
- L $\left[x^{(n-1)}(t)\right](p)=p^{n-1}X(p)-$
  $-\left(p^{n-2}x_0+p^{n-3}x_1+...+x_{n-2}\right)$
- L $\left[x^{(n)}(t)\right](p)=p^nX(p)-$
  $-\left(p^{n-1}x_0+p^{n-2}x_1+...+px_{n-2}+x_{n-1}\right)$

Înlocuim în (2.6) şi obţinem:

$$\left(a_0p^n+a_1p^{n-1}+...+a_{n-1}p+a_n\right)X(p)=F(p)+G(p) \quad (2.7)$$

unde

$$G(p)=a_0\left(p^{n-1}x_0+\ldots+px_{n-1}+x_{n-1}\right)+$$

$$+a_1\left(p^{n-2}x_0+\ldots+px_{n-3}+x_{n-2}\right)+\ldots+a_{n-1}x_0=$$
$$=x_0\left(a_0p^{n-1}+a_1p^{n-2}+...+a_{n-2}p+a_{n-1}\right)+$$
$$+x_1\left(a_0p^{n-2}+a_1p^{n-3}+...+a_{n-2}\right)+...+$$
$$+x_{n-2}\left(a_0p+a_1\right)+x_{n-1}a_0.$$

Din (2.7) avem

$$X(p)=\frac{F(p)+G(p)}{\phi(p)},$$

unde: $\phi(p)=a_0p^n+a_1p^{n-1}+...+a_{n-1}p+a_n$.
Apoi:

$$x(t)=L^{-1}\left[X(p)\right](t)=L^{-1}\left[\frac{F(p)+G(p)}{\phi(p)}\right](t).$$

2. Rezolvarea ecuaţiei integrale de tipul:

$$Ax(t)+B\int_0^t x(T)\cdot K(t-T)\,dT=C\ f(t);$$

- A, B, C constante;
- f şi k funcţii cunoscute;
- $x()$ funcţia necunoscută.

Notăm: $X(p)=L[x(t)](p)$, $F(p)=L[f(t)](p)$, $K(p)=L[k(t)](p)$.

$$AX(p)+BX(p)\cdot K(p)=CF(p)\Rightarrow X(p)=\frac{CF(p)}{A+BK(p)}.$$

Am folosit:

$$L\left[\int_0^t x(T)\cdot k(t-T)\,dT\right](p)=L\left[x(t)*k(t)\right](p)=$$
$$=X(p)\cdot K(p).$$

Aplicaţia 2.34 (la transformata Laplace)

$$\begin{cases} x''' - 2x'' - x' + 2x = t \cdot sh\,(2t-1) \\ x\,(0) = x'\,(0) = x''\,(0) = 0. \end{cases}$$

Aplicăm în ecuaţie transformata Laplace:

$L\,[x\,(t)]\,(p) = X\,(p)$ (notaţie)

$$\begin{array}{l} L\,[x'\,(t)]\,(p) = pX\,(p) - x\,(0) = pX\,(p) \\ L\,[x'']\,(p) = L\left[(x'\,(t))'\right](p) = pL\,[x'\,(t)]\,(p) - x'\,(0) = p^2X\,(p) \\ L\,[x''']\,(p) = L\left[(x'')'\right](p) = pL\,[x''\,(t)]\,(p) - x''\,(0) = p^3X\,(p) \end{array}$$

Folosim formula:

$$L\,[f\,(at+b)]\,(p) =$$

$$= \frac{be^{p\frac{b}{a}}}{a}\left\{L\,[f\,(t)]\left(\frac{p}{a}\right) - \int_0^b f\,(t) \cdot e^{-\frac{pt}{a}} \cdot H\,(t)\,dt\right\}$$

$$L\,[sh\,(2t-1)]\,(p) =$$

$$= \frac{1}{2} \cdot e^{-\frac{p}{2}}\left\{L\,[sh\,(t)]\left(\frac{p}{2}\right) - \int_0^{-1} f\,(t) \cdot e^{-\frac{pt}{a}} \cdot H\,(t)\,dt\right\} =$$

$$= \frac{1}{2} \cdot e^{-\frac{p}{2}} \cdot \frac{1}{\left(\frac{p}{2}\right)^2 - 1} = \frac{2e^{-\frac{p}{2}}}{p^2-4}$$

$$L\,[t \cdot sh\,(2t-1)]\,(p) = -L'\,[sh\,(2t-1)]\,(p) = -2 \cdot \left(\frac{2e^{-\frac{p}{2}}}{p^2-4}\right)' =$$

$$= -2 \cdot \frac{-\frac{1}{2} \cdot e^{-\frac{p}{2}} \cdot (p^2-4) - 2p \cdot e^{-\frac{p}{2}}}{(p^2-4)^2} =$$

$$= \frac{e^{-\frac{p}{2}} \cdot (p^2-4) + 4p \cdot e^{-\frac{p}{2}}}{(p^2-4)^2} = \frac{e^{-\frac{p}{2}} \cdot (p^2+4p-4)}{(p^2-4)^2}$$

Ecuaţia nouă este:

$$p^3X(p) - 2p^2X(p) - pX(p) + 2X(p) = e^{-\frac{p}{2}} \cdot \frac{p^2+4p-4}{\left(p^2-4\right)^2}$$

$$\left(p^3 - 2p^2 - p + 2\right) \cdot X(p) = e^{-\frac{p}{2}} \cdot \frac{p^2+4p-4}{\left(p^2-4\right)^2} \Rightarrow$$

$$X(p) = e^{-\frac{p}{2}} \cdot \frac{p^2+4p-4}{(p-2)^3 \cdot (p+2)^2 \cdot (p-1)(p+1)} \cdot$$

$$\cdot \frac{p^2+4p-4}{(p-2)^3 \cdot (p+2)^2 \cdot (p-1)(p+1)} =$$

$$= \frac{a}{p-2} + \frac{b}{(p-2)^2} + \frac{c}{(p-2)^3} + \frac{d}{p+2} + \frac{e}{(p-2)^2} + \frac{f}{p-1} + \frac{g}{p+1}$$

$$/^{(p-1) \Rightarrow p=1}_{(p+1) \Rightarrow p=-1} \Rightarrow$$

$$g = \frac{1-4-4}{(-8)^3 \cdot (-2)} = \frac{-7}{2^{10}}; f = \frac{1+4-4}{(-1) \cdot 3^2 \cdot 2} = \frac{-1}{18};$$

$$e = \frac{p^2+4p-4}{(p-2)^3 \cdot (p^2-1)}/_{p=-2} = \frac{4-8-4}{(-4)^3 \cdot 3} = \frac{\cancel{8}}{3 \cdot 4 \cdot \cancel{16}_2} = \frac{1}{24};$$

$$c = \frac{p^2+4p-4}{(p+2)^2 \cdot (p^2-1)}/_{p=+2} = \frac{4+8-4}{\cancel{16}_2 \cdot 3} = \frac{1}{6}.$$

a, b, d se calculează prin metoda coeficienţiilor nedeterminaţi:

$$X(p) = a \cdot \frac{e^{-\frac{p}{2}}}{p-2} + b \cdot \frac{e^{-\frac{p}{2}}}{(p-2)^2} + \frac{1}{6} \cdot \frac{e^{-\frac{p}{2}}}{(p-2)^3} + d \cdot \frac{e^{-\frac{p}{2}}}{p+2} + \frac{1}{24} \cdot \frac{e^{-\frac{p}{2}}}{(p-2)^2} -$$

$$-\frac{1}{18}\cdot\frac{e^{-\frac{p}{2}}}{p-1}-\frac{7}{2^{10}}\cdot\frac{e^{-\frac{p}{2}}}{p+1}.$$

Aplicăm transformata Laplace inversă:

$$X(t)=a\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p-2}\right](t)+b\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p-2)^2}\right](t)+$$

$$+\frac{1}{6}\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p-2)^3}\right](t)+d\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p+2}\right](t)$$

$$+\frac{1}{24}\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p-2)^2}\right](t)-$$

$$-\frac{1}{18}\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p-1}\right](t)-\frac{7}{2^{10}}\cdot L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p+1}\right](t).$$

Aplicăm metoda Mellin-Fourier:

1. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p-2}\right](t)=Rez\left[\frac{e^{-\frac{p}{2}}\cdot e^{pt}}{p-2},2\right]=\lim\limits_{p\to 2}(p-2)\cdot\frac{e^{p\left(t-\frac{1}{2}\right)}}{p-2}=$
$=e^{2t-1};$

2. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p-2)^2}\right](t)=Rez\left[\frac{e^{-\frac{p}{2}}\cdot e^{pt}}{(p-2)^2},2\right]=\lim\limits_{p\to 2}\left[e^{p\left(t-\frac{1}{2}\right)}\right]'=$
$=\left(t-\frac{1}{2}\right)\cdot e^{p\left(t-\frac{1}{2}\right)}/_{p=2}=\frac{1}{2}(2t-1)\cdot e^{2t-1};$

3. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p-2)^3}\right](t)=\lim\limits_{p\to 2}\left[e^{p\left(t-\frac{1}{2}\right)}\right]''=\lim\limits_{p\to 2}\left(t-\frac{1}{2}\right)^2\cdot e^{p\left(t-\frac{1}{2}\right)}=$
$=\frac{1}{4}\cdot(2t-1)^2\cdot e^{2t-1};$

4. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p+2}\right](t)=\lim\limits_{p\to -2}e^{-\frac{p}{2}}\cdot e^{pt}=e^{-2t+1}=e^{-(2t-1)};$

5. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{(p+2)^2}\right](t)=\lim\limits_{p\to -2}\left[e^{p\left(t-\frac{1}{2}\right)}\right]'=\left(t-\frac{1}{2}\right)\cdot e^{-2t+1}=$
$=\frac{1}{2}(2t-1)\cdot e^{-(2t-1)};$

6. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p-1}\right](t)=\lim\limits_{p\to 1}(p-1)\cdot\frac{e^{p\left(t-\frac{1}{2}\right)}}{(p-1)}=e^{t-\frac{1}{2}}$;

7. $L^{-1}\left[\frac{e^{-\frac{p}{2}}}{p+1}\right](t)=\lim\limits_{p\to -1}(p+1)\cdot\frac{e^{-\frac{p}{2}}\cdot e^{pt}}{(p+1)}=e^{-t+\frac{1}{2}}=e^{-\left(t-\frac{1}{2}\right)}$.

$$x(t)=ae^{2t-1}+\frac{b}{2}(2t-1)\cdot e^{2t-1}+\frac{1}{24}(2t-1)^2\cdot e^{-(2t-1)}+$$
$$+d\cdot e^{-(2t-1)}+\frac{1}{48}\cdot e^{-(2t-1)}\cdot(2t-1)-\frac{1}{18}\cdot e^{t-\frac{1}{2}}-\frac{7}{2^{10}}\cdot e^{-\left(t-\frac{1}{2}\right)}.$$

Aplicaţia 2.35 (la transformata Laplace)

$$2.\begin{cases} x'''-2x''-x'+2x=e^{2t}\cdot\sin(3t-1)\\ x(0)=x'(0)=x''(0)=0.\end{cases}$$

Aplicăm transformata Laplace ecuaţie date:

$$L[x'''](p)-2L[x''](p)-L[x'](p)+2L[x](p)=$$
$$=L\left[e^{2t}\cdot\sin(3t-1)\right](p)$$

$L[x(t)](p)=X(p)$ (notaţie)

$$\begin{array}{l} L[x'](p)=pL[x](p)-x(0)=pX(p)\\ L[x''](p)=L\left[(x')'\right](p)=pL[x'](p)-x'(0)=p^2X(p)\\ L[x'''](p)=L\left[(x'')'\right](p)=pL[x''](p)-x''(0)=p^3X(p)\end{array}$$

$L\left[e^{2t}\cdot\sin(3t-1)\right](p)=L[\sin(3t-1)](p-2)$- teorema deplasării

$L[f(at+b)](p)=\frac{be^{p\frac{b}{a}}}{a}\left\{L[f(t)]\left(\frac{p}{a}\right)-\int_0^b f(t)\cdot e^{-\frac{pt}{a}}\cdot H(t)\,dt\right\}$

$\Downarrow$

$L[\sin(3t-1)](p)=\frac{1}{3}\cdot e^{-\frac{p}{3}}\cdot L[\sin t]\left(\frac{p}{3}\right)=$

$=\frac{1}{3}\cdot e^{-\frac{p}{3}}\cdot\frac{1}{\left(\frac{p}{3}\right)^2+1}=\frac{3e^{-\frac{p}{3}}}{p^2+9}$.

Deci:

$$L\left[e^{2t}\cdot\sin(3t-1)\right](p)=\frac{3e^{-\frac{p-2}{3}}}{(p-2)^2+9}.$$

Ecuaţia s-a transformat în:

$$\left(p^3-2p^2-p+2\right)\cdot X(p)=\frac{3\cdot e^{-\frac{p-2}{3}}}{(p-2)^2+9}\Rightarrow$$

$$X(p)=\frac{3\cdot e^{-\frac{p-2}{3}}}{(p-2)(p-1)(p+1)(p^2-4p+13)}$$

Descompunem în fracţii simple:

$$\frac{1}{(p-2)(p-1)(p+1)(p^2-4p+13)}=$$

$$=\frac{a}{p-1}+\frac{b}{p+1}+\frac{c}{p-2}+\frac{dp+e}{p^2-4p+13};/\begin{array}{c}\cdot(p-2)\Rightarrow p=2\\ \cdot(p-1)\Rightarrow p=1\\ \cdot(p+2)\Rightarrow p=-1\end{array}\Rightarrow$$

$$a=\frac{1}{(-2)\cdot 10}=-\frac{1}{20};b=\frac{1}{6\cdot 18};c=\frac{1}{3\cdot 9};$$

d şi e se găsesc prin metoda coeficienţiilor necunoscuţi.

Notăm $3d=\alpha,\ 3e=\beta\Rightarrow$

$$x(p)=\frac{-3}{20}\cdot\frac{e^{-\frac{p-2}{3}}}{p-1}+\frac{1}{36}\cdot\frac{e^{-\frac{p-2}{3}}}{p+1}+\frac{1}{9}\cdot\frac{e^{-\frac{p-2}{3}}}{p-2}+\frac{\alpha p}{p^2-4p+13}\cdot e^{-\frac{p-2}{3}}+$$

$$+\frac{\beta}{p^2-4p+13}\cdot e^{-\frac{p-2}{3}}.$$

Aplicăm Laplace inversă:

$$x(t) = \frac{-3}{20} \cdot L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-1}\right](t) + \frac{1}{36} \cdot L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p+1}\right](t) +$$

$$+\frac{1}{9} \cdot L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-2}\right](t) + \alpha \cdot L^{-1}\left[\frac{p-2}{(p-2)^2+9} \cdot e^{-\frac{p-2}{3}}\right](t) +$$

$$+(2\alpha+\beta) \cdot L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{(p-2)^2+9}\right](t).$$

Pentru a calcula $L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-1}\right](t)$ folosim formula Mellin-Fourier:

$$L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-1}\right](t) = Rez\left[\frac{e^{-\frac{p-2}{3}} \cdot e^{pt}}{p-1}, 1\right] =$$

$$= \lim_{p \to 1}(p-1) \cdot \frac{e^{-\frac{p-2}{3}} \cdot e^{pt}}{p-1} = e^{\frac{1}{3}} \cdot e^{t} = e^{t+\frac{1}{3}}$$

$$L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p+1}\right](t) = Rez\left[\frac{e^{-\frac{p-2}{3}} \cdot e^{pt}}{p+1}, -1\right] =$$

$$= \lim_{p \to -1}(p+1) \cdot \frac{e^{-\frac{p-2}{3}} \cdot e^{pt}}{p+1} = e^{-t} \cdot e^{1} = e^{-t+1}$$

Pentru a calcula $L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-2}\right](t)$ calculăm întâi:

$L[f(t)](p)$, unde $f(t) = \begin{cases} e^{2t}, \ t \geq \frac{1}{3} \\ 0, \ \text{t} < \frac{1}{3} \end{cases}$

$$L[f(t)](p) = \int_0^\infty f(t) \cdot e^{-pt} dt = \int_{\frac{1}{3}}^\infty e^{2t} \cdot e^{-pt} dt = \int_3^\infty e^{-(p-2)t} dt =$$

$$= -\frac{1}{p-2} \cdot e^{-(p-2)t} \Big|_{\frac{1}{3}}^{\infty} = \frac{1}{p-2} \cdot e^{-\frac{p-2}{3}}.$$

Deci

$$L\left[f\left(t\right)\right]\left(p\right) = \frac{e^{-\frac{p-2}{3}}}{p-2}$$

unde $f\left(t\right) = \begin{cases} e^{2t},\ t \geq \frac{1}{3} \\ 0,\ \text{t} <\frac{1}{3}. \end{cases}$

Aplicăm Laplace inversă:

$$L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{p-2}\right](t) = f\left(t\right) = \begin{cases} e^{2t},\ t \geq \frac{1}{3} \\ 0,\ \text{t} <\frac{1}{3} \end{cases}$$

$$L^{-1}\left[\frac{(p-2)\cdot e^{-\frac{p-2}{3}}}{(p-2)^2+9}\right](t) =?$$

$$\bullet L\left[e^{2t} \cdot \cos\left(3t-1\right)\right](p) = L\left[\cos\left(3t-1\right)\right](p-2)$$

$$\bullet L\left[\cos\left(3t-1\right)\right](p) =$$

$$= \frac{e^{-\frac{p}{3}}}{3} \cdot L\left[\cos t\right]\left(\frac{p}{3}\right) = \frac{e^{-\frac{p}{3}}}{3} \cdot \frac{\frac{p}{3}}{\left(\frac{p}{3}\right)^2+1} = \frac{e^{-\frac{p}{3}} \cdot p}{p^2+9} \Rightarrow$$

$$L\left[e^{2t} \cdot \cos\left(3t-1\right)\right](p) = \frac{(p-2)\cdot e^{-\frac{p-2}{3}}}{(p-2)^2+9} \Rightarrow$$

$$\Rightarrow L^{-1}\left[\frac{(p-2)\cdot e^{-\frac{p-2}{3}}}{(p-2)^2+9}\right](t) = e^{2t} \cdot \cos\left(3t-1\right)$$

$$L\left[e^{2t} \cdot \cos\left(3t-1\right)\right](p) = \frac{3e^{-\frac{p}{3}}}{(p-2)^2+9} \Rightarrow$$

$$L^{-1}\left[\frac{e^{-\frac{p-2}{3}}}{(p-2)^2+9}\right](t)=\frac{1}{3}\cdot e^{2t}\cdot\sin(3t-1).$$

Deci

$$x(t)=-\frac{3}{20}\cdot e^{t+\frac{1}{3}}+\frac{1}{36}\cdot e^{-t+1}+\frac{1}{9}\cdot f(t)+\alpha\cdot e^{2t}\cdot\cos(3t-1)+$$

$$+\frac{(2\alpha+\beta)}{3}\cdot e^{2t}\cdot\sin(3t-1)$$

unde $f(t)=\begin{cases} e^{2t},\ t\geq\frac{1}{3} \\ 0,\ \mathrm{t}<\frac{1}{3}\end{cases}$, iar $\alpha=3d$, $\beta=3e$ unde coeficientul d şi e se deduc prin metoda coeficientului nedeterminat.

Aplicaţia 2.36 (la transformata Laplace)

$$\begin{cases} x'''+y'=t\cdot e^t\cdot\sin 2t, & x(0)=x'(0)=x''(0)=0, \\ x''+y'''=1, & y(0)=y'(0)=y''(0)=0.\end{cases}$$

Aplicăm transformata Laplace sistemului de ecuaţii:

$$\begin{cases} L[x'''](p)+L[y'](p)=L[t\cdot e^t\cdot\sin 2t](p) \\ L[x''](p)+L[y'''](p)=L[1](p)\end{cases}$$

$$\begin{array}{l} \bullet L[x''](p)=p^2X(p);\quad L[x'''](p)=p^3X(p); \\ \bullet L[y'''](p)=p^3Y(p);\quad L[y'](p)=p\cdot Y(p). \end{array}$$

$\bullet L[t\cdot e^t\cdot\sin 2t](p)=L[t\cdot e^t\cdot\sin 2t](p-1)$

$\bullet L[t\cdot\sin 2t](p)=-L'[\sin 2t](p)=-\left(\frac{2}{p^2+4}\right)'=\frac{4p}{(p^2+4)^2}\Rightarrow$

$$\Rightarrow L[t\cdot e^t\cdot\sin 2t](p)=\frac{4(p-1)}{\left[(p-1)^2+4\right]^2}$$

$$L[1](p)=\frac{1}{p}$$

Avem:

$$\begin{cases} p^3X(p) + pY(p) = \frac{4(p-1)}{(p^2-2p+5)^2} \\ p^2X(p) + p^3Y(p) = \frac{1}{p} \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} p^3X(p) + pY(p) = \frac{4(p-1)}{(p^2-2p+5)^2} \\ X(p) + pY(p) = \frac{1}{p^3} \end{cases} \Rightarrow$$

$$\Rightarrow \left(p^3 - 1\right) \cdot X(p) = \frac{4(p-1)}{(p^2-2p+5)^2} - \frac{1}{p^3} \Rightarrow$$

$$X(p) = \frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)} - \frac{1}{p^3(p-1)(p^2+p+1)} =$$

$$= \frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)} - \frac{1}{p^3(p^3-1)} =$$

$$= \frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)} - \frac{1}{p^3-1} + \frac{1}{p^3} =$$

$$= \frac{1}{p^3} - \frac{1}{(p-1)(p^2+p+1)} + \frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)}.$$

Se aplică transformata Laplace inversă:

$$x(t) = 4L^{-1}\left[\frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)}\right](t) + L^{-1}\left[\frac{1}{p^3}\right](t) -$$

$$-L^{-1}\left[\frac{1}{(p-1)(p^2+p+1)}\right](t) =$$

$$= \frac{t^2}{2} + 4L^{-1}\left[\frac{4}{(p^2-2p+5)^2 \cdot (p^2+p+1)}\right](t) -$$

$$-L^{-1}\left[\frac{1}{(p-1)(p^2+p+1)}\right](t) \overset{f.Mellin-Fourier}{=}$$

$$= \frac{t^2}{2} + 4\left\{Rez\left[\frac{e^{pt}}{\left(p^2-2p+5\right)^2\cdot\left(p^2+p+1\right)}\right], 1+2i\right\} =$$

$$+Rez\left[\frac{e^{pt}}{\left(p^2-2p+5\right)^2\cdot\left(p^2+p+1\right)}, 1-2i\right] +$$

$$+Rez\left[\frac{e^{pt}}{\left(p^2-2p+5\right)^2\cdot\left(p^2+p+1\right)}, -\frac{1}{2}-\frac{i\sqrt{3}}{2}\right] +$$

$$+Rez\left[\frac{e^{pt}}{\left(p^2-2p+5\right)^2\cdot\left(p^2+p+1\right)}, -\frac{1}{2}+\frac{i\sqrt{3}}{2}\right] -$$

$$-\frac{e^t}{3} - Rez\left[\frac{e^{pt}}{(p-1)\left(p^2+p+1\right)}, -\frac{1}{2}\mp\frac{i\sqrt{3}}{2}\right] = \ldots\ldots$$

$$\bullet L\left[t^n\right](p) = \tfrac{n!}{p^{n+1}} \Rightarrow$$
$$\bullet L^{-1}\left[\tfrac{1}{p^{n+1}}\right](t) = \tfrac{t^n}{n!}$$

Scriem încă odată sistemul în necunoscutele $X(p)$ şi $Y(p)$:

$$\left\{\begin{array}{c} p^3X(p) + pY(p) = \frac{4(p-1)}{\left(p^2-2p+5\right)^2} \\ p^2X(p) + p^3Y(p) = \frac{1}{p} / \cdot p \end{array}\right. \Rightarrow$$

$$\left(p-p^4\right)\cdot Y(p) = \frac{4(p-1)}{\left(p^2-2p+5\right)^2} - 1 / : \left(-p^4+p\right) \Rightarrow$$

$$Y(p) = \frac{1}{p\left(p^3-1\right)} - \frac{4(p-1)}{p(p-1)\left(p^2+p+1\right)\left(p^2-2p+5\right)^2} =$$

$$= \frac{1}{p\left(p^3-1\right)} - \frac{4}{p\left(p^2+p+1\right)\left(p^2-2p+5\right)^2}.$$

Cu formulele Mellin-Fourier aflăm funcţia original:

$$y(t) = Rez\left[\frac{e^{pt}}{p(p^3-1)}, 0\right] + Rez\left[\frac{e^{pt}}{p(p^3-1)}, 1\right] +$$

$$+Rez\left[\frac{e^{pt}}{p(p^3-1)}, -\frac{1}{2} \mp i\frac{\sqrt{3}}{2}\right] -$$

$$-4Rez\left[\frac{e^{pt}}{p(p^2+p+1)(p^2-2p+5)^2}, 0\right] -$$

$$-4Rez\left[\frac{e^{pt}}{p(p^2+p+1)(p^2-2p+5)^2}, -\frac{1}{2} \mp i\frac{\sqrt{3}}{2}\right] -$$

$$-4Rez\left[\frac{e^{pt}}{p(p^2+p+1)(p^2-2p+5)^2}, 1 \mp 2i\right] = ...$$

Aplicaţia 2.37

$$\begin{cases} x'' + y' - x = t^3 \cdot e^t, \\ x' + y'' - x = sht, \end{cases} \qquad x(0) = x'(0) = y(0) = y'(0) = 0.$$

Aplicăm transformata Laplace în ecuaţiile sistemului:

$$L[x](p) = X(p); \quad L[x'](p) = pX(p) - x(0) = pX(p); \\ L[x''](p) = p^2X(p),$$

$$L[y](p) = Y(p); \quad L[y'](p) = pY(p) - y(0) = pY(p); \\ L[y''](p) = p^2Y(p).$$

Avem sistemul:

$$\begin{cases} p^2X(p) + pX(p) - X(p) = L[t^3 \cdot e^t](p) \\ pX(p) + p^2Y(p) - X(p) = L[sht](p). \end{cases}$$

$$\bullet L\left[t^3\cdot e^t\right](p)=L\left[t^3\right](p-1)=\frac{6}{(p-1)^4}.$$

Deci:

$$L\left[t^3\right](p)=\frac{3!}{p^{3+1}}=\frac{3!}{p^4}$$

$$\bullet L\left[sht\right](p)=\frac{1}{p^2-1}$$

$$\left\{\begin{array}{c} p^2X(p)+pY(p)-X(p)=\frac{6}{(p-1)^4}\\ pX(p)+p^2Y(p)-X(p)=\frac{1}{p^2-1}\end{array}\right.\quad\Leftrightarrow$$

$$\left\{\begin{array}{c} \left(p^2-1\right)\cdot X(p)+pY(p)=\frac{6}{(p-1)^4}/\cdot p\\ (p-1)\cdot X(p)+p^2Y(p)=\frac{1}{p^2-1}/\cdot(p+1)\end{array}\right.\quad\Rightarrow$$

$$\Rightarrow\left[p\left(p^2-1\right)-(p-1)\right]\cdot X(p)=\frac{6p}{(p-1)^4}-\frac{1}{p^2-1}\Leftrightarrow$$

$$X(p)=\frac{6p}{(p-1)^5\cdot\left(p^2+p-1\right)}-\frac{1}{(p-1)^2(p+1)\left(p^2+p-1\right).}$$

Cu formulele Mellin-Fourier aflăm funcţia original:

$$X(t)=6Rez\left[\frac{p\cdot e^{pt}}{(p-1)^5\cdot\left(p^2+p-1\right)},1\right]+$$

$$+\left[\frac{p\cdot e^{pt}}{(p-1)^5\cdot\left(p^2+p-1\right)},-\frac{1}{2}\mp\frac{\sqrt{5}}{2}\right]-$$

$$-Rez\left[\frac{p\cdot e^{pt}}{(p-1)^2(p+1)\left(p^2+p-1\right)},1\right]-$$

$$-Rez\left[\frac{p\cdot e^{pt}}{(p-1)^2(p+1)\left(p^2+p-1\right)},-1\right]-$$

$$-Rez\left[\frac{p\cdot e^{pt}}{(p-1)^2(p+1)(p^2+p-1)},-\frac{1}{2}\mp\frac{\sqrt{5}}{2}\right]$$

$$\Leftrightarrow\left[p\left(p^2-1\right)-(p-1)\right]\cdot X(p)=\frac{(p-1)^3/1}{p-1}-\frac{6}{(p-1)^4}=$$

$$=\frac{p^3-3p^2+3p-7}{(p-1)^4}\Rightarrow$$

$$Y(p)=\frac{p^3-3p^2+3p-7}{p(p-1)^4(p^2+p-1)}\Rightarrow$$

Cu formulele lui Mellin-Fourier obţinem:

$$Y(t)=Rez\left[\frac{(p^3-3p^2+3p-7)\cdot e^{pt}}{p(p-1)^4\cdot(p^2+p-1)},0\right]+$$

$$+Rez\left[\frac{(p^3-3p^2+3p-7)\cdot e^{pt}}{p(p-1)^4\cdot(p^2+p-1)},1\right]+$$

$$+Rez\left[\frac{(p^3-3p^2+3p-7)\cdot e^{pt}}{p(p-1)^4\cdot(p^2+p-1)},-\frac{1}{2}\mp\frac{\sqrt{5}}{2}\right]=...$$

Aplicaţia 2.38

$$x'''-2x''-x'=f(t)=\begin{cases}0, & \text{rest},\\ 1, & \text{t}\in[0,1],\end{cases}$$

$$x(0)=x'(0)=0,\ x''(0)=1.$$

$$L[x](p)=X(p)$$

$L[x'](p)=pX(p)-x(0)=pX(p)$
$L[x''](p)=L\left[(x')'\right](p)=pL[x'](p)-x'(0)=p^2X(p)$
$L[x'''](p)=L\left[(x'')'\right](p)=pL[x''](p)-x''(0)=p^3X(p)-1$

$$L\left[f\left(t\right)\right]\left(p\right) =$$

$$= \int_0^1 e^{-pt}dt = -\frac{1}{p} \cdot e^{-pt} \Big| \begin{matrix} 1 \\ 0 \end{matrix} = -\frac{1}{p}\left(e^{-p} - 1\right) = \frac{1 - e^{-p}}{p}.$$

Ecuaţia a devenit:

$$\begin{array}{l} p^3 X\left(p\right) - 2p^2 X\left(p\right) - pX\left(p\right) - 1 = \frac{1-e^{-p}}{p} \\ p\left(p^2 - 2p - 1\right) \cdot \frac{1-e^{-p}}{p} = 1 + \frac{1-e^{-p}}{p} \end{array}$$

$$L\left[f\left(t-1\right)\right]\left(p\right) = e^{-p} L\left[f\left(t\right)\right]\left(p\right) \Rightarrow$$

$$\Rightarrow L^{-1}\left[e^{-p} \cdot L\left[f\left(t\right)\right]\left(p\right)\right]\left(t\right) = f\left(t-1\right)$$

$$X\left(p\right) = \frac{1}{p\left(p^2 - 2p - 1\right)} + \frac{1 - e^{-p}}{p^2\left(p^2 - 2p - 1\right)}$$

$$\begin{cases} \frac{1}{p(p^2-2p-1)} = \frac{p-2}{p^2-2p-1} - \frac{1}{p} \\ \frac{1-e^{-p}}{p^2(p^2-2p-1)} = \frac{1}{p^2-2p-1} - \frac{1}{p^2} - \frac{2}{p} \end{cases}$$

$$X\left(p\right) = \frac{p-2}{p^2-2p-1} - \frac{1}{p} + \frac{1-e^{-p}}{p^2-2p-1} - \frac{1-e^{p}}{p^2} - \frac{2\left(1-e^{p}\right)}{p} =$$

$$= \frac{p-2}{p^2-2p-1} - \frac{1}{p} + \frac{1-e^{-p}}{p^2-2p-1} -$$

$$-\left(\frac{1-e^{p}}{p^2} - \frac{e^{-p}}{p}\right) - \frac{1}{p} - \frac{1-e^{-p}}{p} =$$

$$= \frac{p-1-e^{-p}}{p^2-2p-1} - \frac{2}{p} - \frac{1-e^{-p}}{p} - \left(\frac{1-e^{p}}{p^2} - \frac{e^{-p}}{p}\right).$$

Aplicăm transformata Laplace inversă: $\left(L^{-1}\left[\frac{1}{p}\right]\left(t\right) = 1\right)$

$$x\left(t\right) = L^{-1}\left[\frac{p-1-e^{-p}}{p^2-2p-1}\right]\left(t\right) - 2 - L^{-1}\left[\frac{1-e^{-p}}{p}\right]\left(t\right) -$$

$$-L^{-1}\left[\frac{1-e^{p}}{p^2}-\frac{e^{-p}}{p}\right](t)$$

$$L\left[f\left(t\right)\right]\left(p\right)=\frac{1-e^{-p}}{p}\Rightarrow$$

$$\Rightarrow L^{-1}\left[\frac{1-e^{-p}}{p}\right](t)=f\left(t\right)=\begin{cases}1, & \text{t}\in[0,1]\\ 0, & \text{rest.}\end{cases}$$

$$g\left(t\right)=\begin{cases}1, & \text{t}\in[0,1]\\ 0, & \text{rest}\end{cases}\Rightarrow$$

$$L\left[g\left(t\right)\right]\left(p\right)=\int_0^1 t\cdot e^{-pt}dt=-\frac{t}{p}\cdot e^{-pt}\Big|\begin{matrix}1\\0\end{matrix}+\frac{1}{p}\int_0^1\left(e^{-pt}\right)'dt=$$

$$=-\frac{e^{-p}}{p}-\frac{1}{p^2}\cdot e^{-pt}\Big|\begin{matrix}1\\0\end{matrix}=\frac{1-e^{-p}}{p^2}-\frac{e^{-p}}{p}\Rightarrow$$

$$L^{-1}\left[\frac{1-e^{-p}}{p^2}-\frac{e^{-p}}{p}\right](t)=g\left(t\right)=t\cdot f\left(t\right)=\begin{cases}1, & \text{t}\in[0,1]\\ 0, & \text{rest.}\end{cases}$$

$$\bullet L^{-1}\left[\frac{p-1-e^{-p}}{p^2-2p-1}\right](t)\underset{Fourier}{\overset{Mellin-}{=}}Rez\left[\frac{e^{pt}\cdot\left(p-1-e^{-p}\right)}{p^2-2p-1},1+\sqrt{2}\right]+$$

$$+Rez\left[\frac{e^{pt}\cdot\left(p-1-e^{-p}\right)}{p^2-2p-1},1-\sqrt{2}\right]=$$

$$=\frac{e^{pt}\cdot\left(p-1-e^{-p}\right)}{2\left(p-1\right)}\Big|_{p=1+\sqrt{2}}+\frac{e^{pt}\cdot\left(p-1-e^{-p}\right)}{2\left(p-1\right)}\Big|_{p=1-\sqrt{2}}=$$

$$=\frac{e^{\left(1+\sqrt{2}\right)t}}{2}-\frac{e^{\left(1+\sqrt{2}\right)(t-1)}}{2\sqrt{2}}+\frac{e^{\left(1-\sqrt{2}\right)t}}{2}+\frac{e^{\left(1-\sqrt{2}\right)(t-1)}}{2\sqrt{2}}=$$

$$=e^t\cdot ch\sqrt{2}t-\frac{e^{t-1}}{\sqrt{2}}\cdot sh\sqrt{2}\left(t-1\right)$$

Deci:

$$x(t) = e^t \cdot ch\sqrt{2}t - \frac{e^{t-1}}{\sqrt{2}} \cdot sh\sqrt{2}(t-1) - 2 - f(t) - t \cdot f(t),$$

unde

$$f(t) = \begin{cases} 1 \ , & \text{t} \in [0,1] \\ 0 \ , & \text{rest.} \end{cases}$$

2.5 Transformata Laplace discretă

Considerăm **seria Laurent**

$$\sum_{n\in\mathbb{Z}} x_n \cdot z^{-n} = \sum_{n\in\mathbb{Z}} \frac{x^n}{z^n},$$

unde $x^n \in \mathbb{C}$, x^n-un şir- o funcţie $x : \mathbb{Z} \to \mathbb{C}$, $\quad x(n) := x_n$.

Notăm $\omega_1 = \lim\limits_{n\to\infty} \sqrt[n]{|x_{-n}|}$ şi $\omega_2 = \lim\limits_{n\to\infty} \sqrt[n]{|x_n|}$.

Dacă $\omega_2 < \frac{1}{\omega_1}$ atunci seria $\sum_{n\in\mathbb{Z}} x_n \cdot z^{-n}$ este convergentă pe $W_{\omega_2, \frac{1}{\omega_1}}(0)$.

Notăm $x = (x_n)_{n\in\mathbb{Z}}$ şirul x; atunci x se numeşte *semnal discret* şi funcţia olomorfă:

$X \equiv L[x_n] : W_{\omega_2, \frac{1}{\omega_1}}(0) \to \mathbb{C}$, $X(z) = \sum_{n\in\mathbb{Z}} x_n \cdot z^{-n}$ se *numeşte transformată* $z \in W_{\omega_2, \frac{1}{\omega_1}}(0)$

$$z(x) = L[x_n](z)$$

Laplace discretă - sau transformata "z"- a lui x.

Proprietăţi 2.39 $(x * y)_n = (y * x)_n = \sum_{k=0}^{n} y_k x_{n-k}$

1. **Liniaritate:**

$$L[\alpha x_n + \beta y_n](z) = \alpha L[x_n](z) + \beta L[y_n](z), \ \ (\forall) \, z \in W_{r,\rho}(0),$$

$$r = \max\left\{\omega_{2,x}, \omega_{2,y}\right\}, \quad \rho = \min\left\{\frac{1}{\omega_{1,x}}, \frac{1}{\omega_{1,y}}\right\}.$$

2. **Convoluţia:**

$$L\left[(x * y)_n\right](z) = L\left[x_n\right](z) \cdot L\left[y_n\right](z), \ \ (\forall)\, z \in W_{r,\rho}(0).$$

3. **Schimbarea de variabilă pentru semnal:**

$$L\left[x_{n-m}\right](z) = z^{-m} L\left[x_n\right](z), \ \ (\forall)\, z \in W_{\omega_{2,x}, \frac{1}{\omega_{1,x}}}(0).$$

4. **Derivarea:**

$$L\left[n x_n\right](z) = -z\left(L\left[x_n\right](z)\right)', \ \ (\forall)\, z \in W_{\omega_{2,x}, \frac{1}{\omega_{1,x}}}(0).$$

5. **Teorema valorii iniţiale:**

$$\lim_{z \to \infty} L\left[x_n\right](z) = x_0.$$

6. **Teorema valorii finale:**

$$\lim_{n \to \infty} x_n = l \to convergent \Rightarrow \lim_{\substack{z \to 1 \\ z > 1}} \frac{z-1}{z} \cdot L\left[x_n\right](z) = l.$$

Aplicaţia 2.40 Să se calculeze transformata Laplace discretă pentru următoarele semnale:
i) Semnalul discret treapta-unitară:

$$\tau : \mathbb{Z} \to \mathbb{C}, \ \tau_n = \begin{cases} 1, & \text{pentru } n \in \mathrm{N}. \\ 0, & \text{pentru } n \notin \mathrm{N}. \end{cases}$$

$L\left[\tau_n\right](z) = \sum_{n \in \mathbb{Z}} \tau_n z^n = \sum_{n \geq 0} \frac{1}{z^n} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$ pentru $|z| > 1$.

ii) Semnalul impuls unitar la momentul k:

$$k \in \mathbb{Z},\ \ \delta_k : \mathbb{Z} \to \mathbb{C},\ \ \delta_k(n) = \begin{cases} 1, & \text{pentru } \ n = k, \\ 0, & \text{pentru } \ n \neq k, \end{cases}$$

$L[\delta_k(n)](z) = \frac{1}{z^k}$ pentru orice z dacă $k \leq 0$. Pentru orice $z \neq 0$ dacă $k > 0$.

iii)

$$x = (x_n)_{n \in \mathbb{Z}}\ ,\ \ x_n = \begin{cases} n, \text{ pentru } n \in \mathbb{N}, \\ 0, \text{ pentru } n \notin \mathbb{N}, \end{cases}$$

Folosim proprietatea de derivare (4):
$L[x_n](z) = L[n\tau_n](z) = -z(L[\tau_n](z))' = -z \cdot \left(\frac{z}{z-1}\right)' = \frac{z}{(z-1)^2}$, pentru $|z| > 1$.

iv)

$$(y_n)_{n \in \mathbb{Z}} = y,\ \ y_n = \begin{cases} n^2, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases}$$

Folosim proprietatea de derivare (4):
$L[y_n](z) = L[nx_n](z) = -z(L[x_n](z))' = -z \cdot \left(\frac{z}{(z-1)^2}\right)' = \frac{z(z+1)}{(z-1)^3}$, pentru $|z| > 1$.

v)

$$(x_n)_{n \in \mathbb{Z}} = x,\ \ x_n = \begin{cases} a^n, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases} \qquad \text{a} \in \mathbb{C}.$$

$$L[x_n](z) = \sum_{n \geq 0} \left(\frac{a}{z}\right)^n = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z - a},$$

$$|z| > |a| = L\left[a^n \cdot \tau_n\right](z) = \frac{z}{z-a}.$$

vi)

$$y = (y_n)_{n\in\mathbb{Z}}\,,\ \ y_n = \begin{cases} e^{an}, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases} \qquad a \in \mathbb{C}.$$

$$L\left[y_n\right](z) = L\left[(e^a)^n \cdot \tau_n\right](z) = \frac{z}{z-e^a},\ \ |z| > |e^a|\,.$$

vii)

$$x = (x_n)_{n\in\mathbb{Z}}\,,\ \ x_n = \begin{cases} \sin\omega n, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases} \qquad \omega \in \mathbb{R}.$$

$$L\left[x_n\right](z) = L\left[\frac{e^{i\omega n} - e^{-i\omega n}}{2i} \cdot \tau_n\right](z) =$$

$$= \frac{1}{2i}\left(L\left[e^{i\omega n} \cdot \tau_n\right](z) - L\left[e^{-i\omega n} \cdot \tau_n\right](z)\right) =$$

$$= \frac{1}{2i}\left(\frac{z}{z-e^{i\omega}} - \frac{z}{z-e^{-i\omega}}\right) = \frac{(2i)\,z \cdot \sin\omega}{2i\left(z^2 - z\left(e^{i\omega} + e^{-i\omega}\right) + 1\right)} =$$

$$= \frac{z \cdot \sin\omega}{z^2 - 2z \cdot \cos\omega + 1}.$$

viii)

$$y = (y_n)_{n\in\mathbb{Z}}\,,\ \ y_n = \begin{cases} \cos\omega n, & n \in \mathbb{N}, \\ 0, & n \notin \mathbb{N}, \end{cases} \qquad \omega \in \mathbb{R}.$$

$$L\left[y_n\right](z) = \frac{1}{2}\left(L\left[e^{i\omega n} \cdot \tau_n\right](z) + L\left[e^{-i\omega n} \cdot \tau_n\right](z)\right) =$$

$$= \frac{1}{2}\left(\frac{z}{z-e^{i\omega}} + \frac{z}{z-e^{-i\omega}}\right) = \frac{2z^2 - z\left(e^{i\omega}+e^{-i\omega}\right)}{2\left(z^2-2z\cos\omega+1\right)} =$$

$$= \frac{z\left(z-\cos\omega\right)}{z^2-2z\cdot\cos\omega+1}.$$

Transformata Laplace discretă inversă:
$x = L^{-1}\left[X\left(z\right)\right]$ definită prin:

$$x_n = L^{-1}\left[X\left(z\right)\right]\left(n\right) = \frac{1}{2\pi i}\int_{\Gamma} z^{n-1}\cdot X\left(z\right)dz, \quad \left(\forall\right) n\in\mathbb{Z},$$

Γ curbă inchisă, simplă, netedă ce înconjoară pe 0 în coroană.

Aplicaţia 2.41 Pentru $x = \left(x_n\right)_{n\in\mathbb{Z}}$ cu

$$x_n = \begin{cases} n, & n\in\mathbb{N}, \\ 0, & n\notin\mathbb{N}, \end{cases}$$

şi $y = \left(y_n\right)_{n\in\mathbb{Z}}$ cu

$$y_n = \begin{cases} \sin\omega n, & n\in\mathbb{N}, \\ 0, & n\notin\mathbb{N}, \end{cases}$$

calculaţi x*y.

Avem

$$\left(x*y\right)_n = \left(y*x\right)_n = \sum_{k=0}^{n} y_k x_{n-k} = \sum_{k=0}^{n}\left(n-k\right)\sin\omega k$$

$\leftarrow$ ocolim calcule!

$$L\left[\left(x*y\right)_n\right]\left(z\right) = L\left[x_n\right]\left(z\right)\cdot L\left[y_n\right]\left(z\right) =$$

$$= \frac{z}{\left(z-1\right)^2}\cdot\frac{z\cdot\sin\omega}{z^2-2z\cdot\cos\omega+1}, \quad \text{pentru } \left|z\right|>1$$

$$L\left[(x*y)_n\right](z)=\frac{z^2\cdot\sin\omega}{(z-1)^2\left(z-e^{i\omega}\right)\left(z-e^{-i\omega}\right)},$$

şi cu transformata Laplace discretă inversă avem:

$$(x*y)_n=\frac{1}{2\pi i}\int_{|z|=2}\frac{z^{n-1}\cdot z^2\cdot\sin\omega}{(z-1)^2\left(z-e^{i\omega}\right)\left(z-e^{-i\omega}\right)}dz=$$

$$=\frac{1}{2\pi i}\int_{|z|=2}\frac{z^{n-1}\cdot z^2\cdot\sin\omega\;dz}{(z-1)^2\left(z-e^{i\omega}\right)\left(z-e^{-i\omega}\right)}=$$

$$=Rez\,[f,1]+Rez\left[f,e^{i\omega}\right]+Rez\left[f,e^{-i\omega}\right]=$$

$$=\frac{n}{2}ctg\frac{\omega}{2}+\frac{\sin\omega\cdot\cos(n+1)\,\omega-\cos\omega\cdot\sin(n+1)\,\omega}{4\sin^2\frac{\omega}{2}}$$

$z_1=1$ un pol dublu, $z_{2,3}=e^{\pm i\omega}$ poli simpli pentru: $f(z)=\frac{z^{n+1}\cdot\sin\omega}{(z-1)^2(z-e^{i\omega})(z-e^{-i\omega})}$

Aplicaţia 2.42 Să se determine semnalul discret x astfel incât

$$x_{n+2}+x_{n+1}-2x_n=a_n-a_{n-1},\quad (\forall)\,n\in\mathbb{Z},$$

unde $a_n=\begin{cases} n, & \text{pentru } n\in\mathbb{N},\\ 0, & \text{pentru } n\notin\mathbb{N}.\end{cases}$

Aplicăm transformata Laplace discretă:

$$L\left[x_{n+2}\right](z)+L\left[x_{n+1}\right](z)-2L\left[x_n\right](z)=L\left[a_n\right](z)-L\left[a_{n-1}\right](z)$$

Aplicăm proprietatea 3 şi avem:

$$z^2L\left[x_n\right](z)+zL\left[x_n\right](z)-2L\left[x_n\right](z)=L\left[a_n\right](z)-z^{-1}L\left[a_n\right](z)\Leftrightarrow$$

$$\left(z^2+z-2\right)L\left[x_n\right](z)=\frac{z}{(z-1)^2}-\frac{1}{(z-1)^2}=\frac{1}{z-1}\Rightarrow$$

$$\Rightarrow (z-1)(z+2), \text{ pentru } |z| > 1.$$

$$L[x_n](z) = \frac{1}{(z-1)^2 (z+2)}, \quad |z| > 1 \Rightarrow$$

$$\Rightarrow x_n = \frac{1}{2\pi i} \int_{|z|=3} \frac{z^{n-1}}{(z-1)^2 (z+2)} dz$$

•$n \leq 0 ... z_1 = 0$ pol de ordinul 1-n,
•$n \geq 1 ... z_1 = 1$ pol de ordinul 2, $z_2 = -2$ pol simplu;
•$n \geq 0$

$$x_n = Rez[f, 0] + Rez[f, 1] + Rez[f, -2] = -Rez[f, \infty] = 0$$

$$Rez[f, \infty] = Rez\left[-\frac{1}{\xi^2} \cdot f\left(\frac{1}{\xi}\right), 0\right] =$$

$$= Rez\left[\frac{\xi^{2-n}}{(1-\xi)^2 \cdot (1+2\xi)}, 0\right] = 0$$

•$n \geq 1$

$$x_n = Rez[f, 1] + Rez[f, -2] = ... = \frac{3n - 4 + (-2)^{n-1}}{9}.$$

2.6 Transformata Fourier

2.6.1 Integrala Fourier. Transformata Fourier prin cosinus şi sinus.

Definiţia 2.43 O funcţie $f : [a, b] \to \mathbb{R}$ se numeşte *absolut integrabilă* pe $[a, b]$ dacă există $\int_b^a f(x)\, dx$ şi este finită.

Fie $f:(-\infty,\infty)\to\mathbb{R}\,(\mathbb{C})$ absolut integrabilă pe $(-\infty,+\infty)$ şi care admite o dezvoltare în serie Fourier pe $(-l,l)$, adică:

$$f\left(x\right)=\frac{a_0}{2l}+\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi x}{l}+b_k\sin\frac{k\pi x}{l}\right) \tag{2.8}$$

unde:

$$a_0=\frac{1}{l}\int_{-l}^{l}f\left(t\right)dt,$$
$$a_k=\frac{1}{l}\int_{-l}^{l}f\left(t\right)\cos\frac{k\pi x}{l}dt, \tag{2.9}$$
$$b_k=\frac{1}{l}\int_{-l}^{l}f\left(t\right)\sin\frac{k\pi x}{l}dt.$$

Introducând (2.9) în (2.8) obţinem:

$$f\left(x\right)=\frac{1}{2l}\int_{-l}^{l}f\left(t\right)dt+$$

$$+\frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l}f\left(t\right)\left[\cos\frac{k\pi t}{l}\cdot\cos\frac{k\pi x}{l}+\sin\frac{k\pi t}{l}\cdot\sin\frac{k\pi x}{l}\right]dt\right)=$$
$$=\frac{1}{2l}\int_{-l}^{l}f\left(t\right)dt+\frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l}f\left(t\right)\cdot\cos\frac{k\pi\left(x-t\right)}{l}dt\right)$$

Notăm $\alpha_k=\frac{k\pi}{l},\;\; k=1,2,.....\Rightarrow$

$$f\left(x\right)=\frac{1}{2l}\cdot\int_{-l}^{l}f\left(t\right)dt+\frac{1}{l}\sum_{k=1}^{\infty}\left(\int_{-l}^{l}f\left(t\right)dt\cdot\cos\alpha_k\left(x-t\right)dt\right) \tag{2.10}$$

$$\left|\frac{1}{2l}\cdot\int_{-l}^{l}f\left(t\right)dt\right|\leq\frac{1}{2l}\cdot\int_{-l}^{l}\left|f\left(t\right)\right|dt\leq$$

$$\leq \frac{1}{2l} \cdot \int_{-\infty}^{\infty} |f(t)|\, dt = \frac{M}{2l} \xrightarrow[l\to\infty]{} 0$$

$$\Rightarrow \lim_{l\to\infty} \left[\frac{1}{2l} \cdot \int_{-l}^{l} f(t)\, dt\right] = 0. \tag{2.11}$$

Definiţia 2.44 $f : [a, b] \to \mathbb{R}$ se numeşte *monotonă pe porţiuni* pe $[a, b]$ dacă f este continuă pe $[a, b]$ cu excepţia unui număr finit de puncte în care are limite laterale finite de subintervale pe care f este monotonă.

Se demonstrează că dacă f este monotonă pe porţiuni, marginită pe $(-\infty, +\infty)$ şi absolut integrabilă pe $(-\infty, +\infty)$ atunci trecând la limită după $l \to \infty$ în (2.10) şi utilizând (2.11) obţinem:
Integrala Fourier a lui f:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cos \alpha (x - t)\, dt \right) d\alpha. \tag{2.12}$$

Dezvoltând $\cos \alpha (x - t)$ formula (2.12) devine:

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot \cos \alpha t\ dt \right) \cos \alpha t\ d\alpha +$$

$$+ \frac{1}{\pi} \int_0^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot \sin \alpha t\ dt \right) \sin \alpha t\ d\alpha. \tag{2.13}$$

Dacă funcţia f este pară formula (2.13) devine:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cdot \cos \alpha t\ dt \right) \cos \alpha x\ d\alpha. \tag{2.14}$$

Dacă funcţia f este impară atunci formula (2.13) devine:

$$f(x) = \frac{2}{\pi} \int_0^{\infty} \left(\int_0^{\infty} f(t) \cdot \sin \alpha t\ dt \right) \sin \alpha x\ d\alpha. \tag{2.15}$$

Dacă notăm în (2.10): $F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t \, dt$ atunci (2.10) devine:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\alpha) \cos \alpha x \, d\alpha. \tag{2.16}$$

Definiţia 2.45 Funcţia

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \alpha t \, dt$$

se numeşte *transformata Fourier prin cosinus* şi se notează:

$$F_c(f)(\alpha).$$

Analog, dacă notăm

$$F(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin \alpha t \, dt, \tag{2.17}$$

atunci relaţia (2.15) devine

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(\alpha) \sin \alpha x \, d\alpha. \tag{2.18}$$

Definiţia 2.46 Formula (2.17) se numeşte *transformata Fourier prin sinus* şi se notează:

$$F_s(f)(\alpha).$$

Observaţia 2.47 Are sens problema de forma: Să se determine funcţia f ce satisface relaţia (2.16) sau (2.18), unde $F(\alpha)$ se presupune cunoscută. Aceste relaţii se numesc *ecuaţii integrale*, deoarece funcţia necunoscută figurează sub integrală.

2.6.2 Forma complexă a integralei Fourier

Avem:

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\alpha(x-t)\,dtd\alpha =$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\left[\int_0^{\infty}\cos\alpha(x-t)\,d\alpha\right] f(t)\,dt =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\cos\alpha(x-t)\,dtd\alpha.$$

Pe de altă parte din imparitatea funcţiei de α avem:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\sin\alpha(x-t)\,dtd\alpha = 0.$$

Într-adevăr:

$$\bullet f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cdot\cos\alpha(x-t)\,dtd\alpha =$$

$$= \frac{1}{\pi}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cdot\cos\alpha(x-t)\,dt\right)d\alpha =$$

$$= \frac{1}{\pi}\lim_{l\to\infty}\int_0^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cdot\cos\alpha(x-t)\,dt\right)d\alpha = \tag{2.19}$$

$$= \frac{1}{2\pi}\lim_{l\to\infty}\int_{-l}^{l}\left(\int_{-\infty}^{\infty} f(t)\cdot\cos\alpha(x-t)\,dt\right)d\alpha =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cdot\cos\alpha(x-t)\,dt\right)d\alpha.$$

$$\bullet \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cdot\sin\alpha(x-t)\,dt\right)d\alpha =$$

$$= \lim_{l \to \infty} \int_{-l}^{l} \left(\int_{-\infty}^{\infty} f(t) \cdot \sin \alpha (x-t)\, dt \right) d\alpha = 0. \qquad (2.20)$$

Înmulţim (2.20) cu $-\frac{i}{2\pi}$ şi adunăm la relaţia (2.19), găsind:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \left[\cos \alpha (x-t) - i \sin \alpha (x-t) \right] dt \right) d\alpha =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t) \cdot e^{-i\alpha(x-t)} dt \right] d\alpha =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(t) \cdot e^{i\alpha t} dt \right) \cdot e^{-i\alpha x} d\alpha.$$

Notăm:

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \cdot e^{i\alpha t} dt \quad (15) \qquad (2.21)$$

şi o numim *transformata Fourier* a lui $f(x)$- o notăm $F[f(t)](\alpha)$. Înlocuind (2.21) în (2.20) obţinem:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\alpha) \cdot e^{-i\alpha x} d\alpha \Rightarrow F^{-1}\left[F\left[f(x)\right](\alpha)\right](x)$$

pe care o numim *transformata Fourier inversă* a lui $F(\alpha)$.

2.6.3 Proprietaţi ale transformatei Fourier

Teorema 2.48 *(Liniaritatea)*

$$F(c_1 f_1 + c_2 f_2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[c_1 f_1(t) + c_2 f_2(t) \right] \cdot e^{+i\alpha t} dt = ...$$

$$... = c_1 F\left(f_1^{(t)}\right)(\alpha) + c_2 F\left(f_2^{(t)}\right)(\alpha).$$

Teorema 2.49 *(Translaţia)*

$$F\left[f\left(x+h\right)\right]\left(\alpha\right)=$$
$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(x+h\right)\cdot e^{i\alpha x}dx=$$

$$cu\left\{\begin{array}{l} x+h=y\Rightarrow dx=dy\\ x=y-h\end{array}\right.$$

$$=e^{-ih\alpha}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(y\right)\cdot e^{i\alpha y}dy=e^{-i\alpha h}F\left[f\left(x\right)\right]\left(\alpha\right).$$

Teorema 2.50

$$F\left[f\left(ax\right)\right]\left(\alpha\right)=\frac{1}{|a|}F\left[f\left(x\right)\right]\left(\frac{\alpha}{a}\right)=?$$

Fie

1. $a>0\Rightarrow$

$$F\left[f\left(ax\right)\right]\left(\alpha\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(ax\right)\cdot e^{i\alpha x}dx=$$

$$dar\left\{\begin{array}{l} ax=y\\ x=\frac{y}{a}\Rightarrow dx=dy/a\end{array}\right.$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(y\right)\cdot e^{i\alpha\frac{y}{a}}\frac{dy}{a}=\frac{1}{a}F\left[f\left(x\right)\right]\left(\frac{\alpha}{a}\right).$$

2. $a<0\Rightarrow$

$$F\left[f\left(ax\right)\right]\left(\alpha\right)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(ax\right)\cdot e^{i\alpha x}dx=$$

$$=+\frac{1}{\sqrt{2\pi}}\int_{+\infty}^{-\infty}f\left(y\right)\cdot e^{i\frac{\alpha}{a}y}\frac{dy}{a}=-\frac{1}{a}\cdot\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f\left(y\right)\cdot e^{i\frac{\alpha}{a}y}dy=$$

$$= \frac{1}{|a|} F\left[f\left(x\right)\right]\left(\frac{\alpha}{a}\right) \Rightarrow$$

$$F\left[f\left(ax\right)\right]\left(\alpha\right) = \frac{1}{|a|} F\left[f\left(x\right)\right]\left(\frac{\alpha}{a}\right), \quad a \in \mathbb{R}^{*}.$$

Teorema 2.51

$$F\left[e^{ixh} \cdot f\left(x\right)\right] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(x\right) \cdot e^{ixh} \cdot e^{i\alpha x} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(x\right) \cdot e^{i(\alpha+h)x} dx = F\left[f\left(x\right)\right]\left(\alpha + h\right).$$

Teorema 2.52 *Fie $k \in \mathbb{N}^{*}$ şi derivăm în raport cu α formula:*

$$F\left(\alpha\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(t\right) \cdot e^{i\alpha t} dt.$$

$$\frac{d}{d\alpha} F\left(\alpha\right) = \frac{1}{\sqrt{2\pi}} \frac{d}{d\alpha} \left(\int_{-\infty}^{\infty} f\left(t\right) \cdot e^{i\alpha t} dt \right) =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(t\right) \cdot \frac{d}{d\alpha} \left(e^{i\alpha t}\right) dt =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f\left(t\right) \cdot it \cdot e^{i\alpha t} dt = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t f\left(t\right) \cdot e^{i\alpha t} dt =$$

$$= iF\left[tf(t)\right](\alpha).$$

Proprietate: $\frac{d^k F}{d\alpha^k}\left(\alpha\right) = \frac{i^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k f\left(t\right) \cdot e^{i\alpha t} dt$ *şi determinăm pentru (k+1). Mai derivăm o dată în raport cu α şi formula se confirmă.*

Teorema 2.53 *Definim produsul de convoluţie pentru 2 funcţii absolut integrabile:*

$$\left(f * g\right)\left(x\right) = \int_{-\infty}^{\infty} f\left(x - y\right) \cdot g\left(y\right) dy = \left(g * f\right)\left(x\right).$$

şi îi aplicăm transformata Fourier:

$$F\left[(f*g)(x)\right](\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(f*g)(x)\cdot e^{i\alpha x}dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}f(x-y)\cdot g(y)\,dy\right]\cdot e^{i\alpha x}dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x-y)\cdot e^{i\alpha(x-y)}\cdot g(y)\cdot e^{i\alpha y}dxdy=$$

$$\overset{x-y=u}{=}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(u)\cdot e^{i\alpha u}\cdot g(y)\cdot e^{i\alpha y}dudy=$$

$$\underset{Fubini}{=}\sqrt{2\pi}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(u)\cdot e^{i\alpha u}du\right)\cdot\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(y)\,e^{i\alpha y}dy\right)=$$

$$=\sqrt{2\pi}F[f](\alpha)\cdot F[g](\alpha).$$

Deci:

$$F\left[(f*g)(x)\right](\alpha)=\sqrt{2\pi}F[f](\alpha)\cdot F[g](\alpha).$$

Teorema 2.54

$$F^{-1}\left[F(\alpha)\cdot G(\alpha)\right](x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\alpha)\cdot G(\alpha)\cdot e^{-i\alpha x}d\alpha=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\alpha)\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}g(v)\cdot e^{-i\alpha v}dv\right)\cdot e^{-i\alpha x}dx=$$

$$\underset{Fubini}{=}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}F(\alpha)\cdot e^{-i\alpha(x-v)}\cdot g(v)\;d\alpha dv=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(\alpha)\cdot e^{-i\alpha(x-v)}d\alpha\right]\cdot g(v)\,dv=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x-v)\cdot g(v)\;dv=\frac{1}{\sqrt{2\pi}}(f*g)(x)\Rightarrow$$

$$\Rightarrow F^{-1}\left[F(\alpha)\cdot G(\alpha)\right](x)=\frac{1}{\sqrt{2\pi}}(f*g)(x).$$

Aplicaţia 2.55 Funcţia lui Heaviside:

$$H(x)=\begin{cases}1, & x\geq 0,\\ 0, & x<0.\end{cases}$$

1.

$$F\left[H(x)\cdot e^{-ax}\right](\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}H(x)\cdot e^{-ax}\cdot e^{i\alpha x}dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{(i\alpha-a)x}dx=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{(i\alpha-a)x}}{i\alpha-a}\Bigg|\begin{matrix}\infty\\0\end{matrix}=$$

$$=\frac{1}{\sqrt{2\pi}(a-i\alpha)}.$$

2.

$$F\left[H(-x)\cdot e^{ax}\right](\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{ax}\cdot e^{i\alpha x}dx=$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}e^{(a+i\alpha)x}dx=\frac{1}{\sqrt{2\pi}}\cdot\frac{e^{(a+i\alpha)x}}{a+i\alpha}\Bigg|\begin{matrix}0\\\infty\end{matrix}=$$

$$=\frac{1}{\sqrt{2\pi}(a+i\alpha)}.$$

Aplicaţia 2.56

$$F\left[e^{-a|x|}\right](\alpha)=F\left[H(x)\cdot e^{-\alpha x}+H(-x)\cdot e^{ax}\right](\alpha)=$$

$$=\frac{1}{\sqrt{2\pi}}\left(\frac{1}{a-i\alpha}+\frac{1}{a+i\alpha}\right)=\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\alpha^2}.$$

Aplicaţia 2.57 Fie $f(x) = e^{-ax^2}$, cu $a > 0$. Calculaţi

$$F[f(x)](\alpha) = F(\alpha).$$

$$F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) \cdot e^{i\alpha x} dx =$$

$$\underset{f=par}{\overset{Euler}{=}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} \cdot \cos \alpha x \, dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax^2} \cdot \cos \alpha x \, dx$$

Deoarece, după derivarea sub integrală în raport cu α obţinem o integrală improprie uniform convergentă, avem:

$$F'(x) = -\sqrt{\frac{2}{\pi}} \int_{0}^{\infty} e^{-ax^2} \cdot x \cdot \sin \alpha x \, dx =$$

$$= \frac{1}{2a} \int_{0}^{\infty} \left(e^{-ax^2}\right)' \cdot \sin \alpha x \, dx = -\frac{\alpha}{a\sqrt{2\pi}} \cdot \int_{0}^{\infty} e^{-ax^2} \cdot \cos \alpha x \, dx =$$

$$= -\frac{\alpha}{a\sqrt{2\pi}} \cdot \sqrt{\frac{2}{\pi}} F(\alpha) = -\frac{\alpha}{2a} F(\alpha)$$

şi integrăm ecuaţia cu variabile separabile în necunoscuta $F(\alpha)$:

$$\frac{dF}{F} = -\frac{\alpha}{2a} d\alpha \Leftrightarrow \ln F(\alpha) = \underbrace{-\frac{1}{4a} \cdot \alpha^2}_{\ln e^{-\frac{\alpha^2}{4a}}} + \ln C \Rightarrow F(\alpha) = C \cdot e^{-\frac{\alpha^2}{4a}}.$$

Determinăm constanta C, făcând $\alpha = 0$:

$$F(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} dx = \frac{1}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} e^{-ax^2} dx =$$

$$= \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{a}} \int_{0}^{\infty} e^{-y^2} dy = \sqrt{\frac{2}{\pi}} \cdot \sqrt{\frac{\pi}{a}} = \sqrt{\frac{2}{a}} = C \Rightarrow$$

$$F(\alpha) = F\left[e^{-ax^2}\right](\alpha) = \sqrt{\frac{2}{a}} \cdot e^{-\frac{\alpha^2}{4a}}.$$

Observaţia 2.58 Putem lua

$$F[f(x)](\xi) = \hat{f}(\xi) = \int_{-\infty}^{+\infty} f(x) \cdot e^{ix\xi} dx$$

deci, fără $\frac{1}{\sqrt{2\pi}}$; $\hat{f}(\xi)$ este o altă notaţie pentru transformata Fourier.

Capitolul 3

Ecuaţiile fizicii matematice

3.1 Formulările problemelor la limită ale fizicii matematice

Exemplul 3.1 Vibraţiile coardei. Fie o coardă de lungime l, întinsă cu o forţă T_0 şi aflată în poziţie rectilinie de echilibru. La momentul $t = 0$, punctele coardei, depărtate din poziţiile lor de echilibru, capătă o anumită viteză.

Avem următoarele formulări ale micilor vibraţii transversale ale punctelor cordei pentru $t > 0$:

a) dacă extremităţile coardei sunt fixate rigid:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t), 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0 \text{ condiţia la limită} \\ u(x,0) = \phi(x) \quad \text{şi} \quad u_t(x,0) = \psi(x) \quad \text{condiţii iniţiale.} \end{cases} \tag{3.1}$$

unde $a^2 = \frac{T_0}{\rho}$; $g(t,x) = \frac{p(x,t)}{\rho}$; $\rho(x) = \rho \equiv$ constantă este densitatea; T_0 este tensiunea coardei; $u = u(x)$ este deplasarea; $p(x,t)$ este densitatea liniară continuă a forţelor externe; $\phi(x)$ şi $\psi(x)$ sunt funcţii date.

b) dacă extremităţile coardei sunt libere, adică ele se pot deplasa liber de-a lungul unor drepte paralele cu deplasarea u:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g(x,t), \quad 0 < x < l, \\ u(x,0) = \varphi(x), \quad 0 < x < l, \\ u_t(x,0) = \psi(x), \quad 0 < x < l, \\ \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0. \end{cases} \tag{3.2}$$

c) dacă extremităţile sunt fixate elastic, adică asupra fiecărei extremităţi se exercită din partea sprijinului, o reacţiune proporţională cu deplasarea şi de sens contrar:

-în acest caz forţele elastice care apar în punctul de încastrare,

$x = 0$, sunt date de $-ku\,(0,t)$, obţinem:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g\,(x,t)\,, \quad 0 < x < l, \\ \\ u\,(x,0) = \phi\,(x)\ \ ,\ \ u_t\,(x,0) = \psi\,(x)\,, \quad 0 < x < l, \\ \\ \frac{\partial u}{\partial x}\,(0,t) = hu\,(0,t) \quad \text{şi } \frac{\partial u}{\partial x}\,(l,t) + hu\,(l,t) = 0, \quad t > 0. \end{cases} \tag{3.3}$$

unde: $h = \frac{k}{T_0}$.

d) animate de o mişcare transversală care se desfaşoară conform unor legi date:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g\,(x,t)\,, \quad 0 < x < l, \ \ t > 0, \\ \\ u\,(x,0) = \phi\,(x)\ \ ,\ \ \frac{\partial u}{\partial t}\,(x,0) = \psi\,(x)\,, \quad 0 < x < l, \\ \\ u\,(0,t) = \mu_1\,(t)\ \ ,\ \ u\,(l,t) = \mu_2\,(t)\,. \end{cases} \tag{3.4}$$

unde μ_1, μ_2 sunt funcţiile care determină legea de mişcare a extremităţilor $\mu_1\,(0) = \theta\,(0)$ şi $\mu_2\,(0) = \theta\,(l)$.

Exemplul 3.2 Problema vibraţiilor barei omogene:

$$(\rho(x) = \rho \equiv \text{constant})\,.$$

a) când extremităţile sunt fixe:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g\,(x,t)\,, \quad 0 < x < l, \ \ t > 0, \\ \\ u\,(0,t) = u\,(l,t) = 0, \quad t > 0, \\ \\ u\,(x,0) = \phi\,(x)\ \ ,\ \ \frac{\partial u}{\partial t}\,(x,0) = \psi\,(x)\,, \quad 0 < x < l. \end{cases} \tag{3.5}$$

unde: $a^2 = \frac{E}{\rho}$, E constanta lui Young, $g\left(x,t\right) = \frac{p(x,t)}{\rho}$.

b) când extremităţile sunt libere:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + g\left(x,t\right), \\ \\ \frac{\partial u}{\partial x}\left(0,t\right) = \frac{\partial u}{\partial x}\left(l,t\right) = 0, \\ \\ u\left(x,0\right) = \phi\left(x\right) \ , \ \ \frac{\partial u}{\partial t}\left(x,0\right) = \psi\left(x\right). \end{cases} \tag{3.6}$$

Exemplul 3.3 Problema vibraţiilor unei membrane fixate.

$$\begin{cases} T\Delta u = -f\left(x\right), \quad x \in G, \\ u_{|L} = 0, \quad L = \partial G, \end{cases} \tag{3.7}$$

unde G reprezintă membrana, T coeficientul de proporţionalitate pozitiv numit tensiunea membranei, $f\left(x\right)$ densitatea într-un punct $x \in G$ a forţei perpendiculare pe planul membranei, $u = u\left(x\right)$ se numeşte deformarea membranei.

Exemplul 3.4 Ecuaţia de continuitate.
Fie mişcarea unui lichid (gaz) perfect-fluid fără vâscozitate. Avem: $\overrightarrow{v} = \left(v_1, v_2, v_3\right)$ viteza fluidului, $\rho\left(x,t\right)$ densitatea fluidului, $f\left(x,t\right)$ intensitatea surselor.

Fie Ω un domeniu în $\mathbb{R}^3$, mărginit şi $S = \partial\Omega$. Ecuaţia de continuitate a mişcării unui fluid perfect în Ω:

$$\frac{\partial \rho}{\partial t} + div\left(\rho\overrightarrow{v}\right) = f\left(x,t\right), \quad x \in \Omega. \tag{3.8}$$

Dacă avem - în absenţa surselor - curgerea irotaţională (potenţială) în jurul unui corp solid Ω de frontieră S, a unui fluid omogen, incompresibil, care are viteza v_0 la infinit, obţinem ($\rho \equiv$ constant, $f \equiv 0$):

$$\begin{cases} div\overrightarrow{v} = 0, & x \notin \Omega, \\ (\overrightarrow{v} \cdot \overrightarrow{n})_{|S} = 0, & \overrightarrow{n} \;=\; \text{normala la suprafa la S.} \end{cases}$$

Dacă $\overrightarrow{v} = grad\ u$, unde u este potenţialul vitezelor, obţinem următoarea problemă:

$$\begin{cases} \Delta u = 0, & x \notin \Omega, \\ \frac{\partial u}{\partial \overrightarrow{n}}\Big|_S = 0, & \lim\limits_{|x|\to\infty} grad\ u = \overrightarrow{v_0}. \end{cases} \tag{3.9}$$

Exemplul 3.5 Problema de propagare a căldurii într-un corp $\Omega \subset \mathbb{R}^3$.

$$\frac{\partial u}{\partial t} = a^2 \Delta u + f(x,t)\,,\ x \in \Omega,\ t > 0, \tag{3.10}$$

unde u este temperatura corpului, $a^2 = \frac{k}{c \cdot \rho}$, $f(x,t) = \frac{F(x,t)}{c \cdot \rho}$, unde: $F(x,t)$ este densitatea surselor de căldură; $\rho(x)$ este densitatea materialului şi $c(x)$ căldura sa specifică în punctul x la momentul t. În cazul (3.10) avem ρ şi c constante, precum şi coeficientul conductibilităţii termice $k(x,u) \equiv$ constant.
Pentru a descrie propagarea căldurii într-un corp Ω trebuie să se specifice temperatura iniţială $u(x,0) = u^0(x)$ precum şi regimul termic pe frontieră. Fie $\Gamma = \partial\Omega$.
a) Când frontiera Γ este menţinută la o temperatură dată problema este:

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \Delta u + f(x,t)\ \,, x \in \Omega,\ t > 0, \\ \\ u(x,0) = u^0(x)\,,\ \text{condiţie iniţială}, \\ \\ u|_\Gamma = \psi,\ \text{condiţie la limită}. \end{cases} \tag{3.11}$$

b) Dacă prin Γ trece un flux de căldură q, condiţia la limită se scrie:

$$\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{\Gamma} = h \quad , \text{unde } h = \frac{q}{k}.$$

În particular, dacă Ω este izolat termic pe frontiera Γ, atunci avem

$$\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{\Gamma} = 0.$$

c) Dacă temperatura mediului ambiant este dată, se presupune că schimbările de căldură au loc conform legii lui Newton, adică $q|_{\Gamma} = \alpha\,(u_1 - u)|_{\Gamma}$, unde q este fluxul termic, α este coeficientul de schimb la suprafaţă, iar u_1 este temperatura mediului ambiant. Atunci condiţia la limită se scrie:

$$k\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{\Gamma} = \alpha\,(u_1 - u)|_{\Gamma}\,.$$

Exemplul 3.6 Probleme de difuzie.
Aici este vorba despre ecuaţia de difuzie a unei substanţe într-un mediu mobil, care ocupă volumul Ω de frontieră Γ, când se cunoaşte densitatea surselor $F\,(x,t)$ şi când difuzia se face cu absorţie - viteza de absorţie fiind proporţională în fiecare punct $x \in \Omega$ cu densitatea $u\,(x,t)$ a substanţei care difuzează. Presupunem că ştim densitatea iniţială $u|_{t=0} = \phi\,(x)\,,\ x \in \Omega$.
Avem ecuaţia de difuzie:

$$\rho\,(x)\,\frac{\partial u}{\partial t} = div\,(D\,(x)\,grad\ u) - q\cdot u + F\,(x,t)\,,\quad t>0,\ \text{în } \Omega \tag{3.12}$$

unde: $\rho\,(x)$ porozitatea mediului, $D\,(x)$ coeficientul de difuzie, $-q\cdot u$ reprezintă pierderea în volum datorită absorţiei în mediul ambiant.
Condiţia iniţială:

$$u\,(x,0) = \phi\,(x)\,,\quad x \in \Omega. \tag{3.13}$$

Condiţia la limită în anumite cazuri:

a) frontiera Γ a domeniului este menţinută la o densitate dată $u|_{\Gamma} = u_0$.

b) frontiera Γ este impermeabilă: $\frac{\partial u}{\partial \overrightarrow{n}}\Big|_{\Gamma} = 0$.

c) frontiera Γ este semi-impermeabilă, difuzia prin suprafaţa de separaţie (Γ) efectuându-se după o lege similară legii lui Newton pentru schimbul de căldură prin convecţie:

$$D\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{\Gamma} = \alpha \left(u_1 - u\right)|_{\Gamma},$$

unde u_1 este temperatura mediului ambiant, α coeficientul de permeabilitate al frontierei Γ.

3.2 Clasificarea ecuaţiilor diferenţiale cvasiliniare de ordinul doi

Considerăm ecuaţia diferenţială cvasiliniară de ordinul doi

$$\sum_{i,j=1}^{n} a_{ij}\left(x\right) \frac{\partial^2 u}{\partial x_i \partial x_j}\left(x\right) + \Phi\left(x, u, \nabla u\right) = 0 \tag{3.14}$$

$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = grad\ u = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot \overrightarrow{e_i}$, unde $B = \{\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}\}$ este baza canonică în $\mathbb{R}^n$.
Fie x_0 un punct şi $y = y\left(x\right)$ o transformare nesingulară de forma:

$$\begin{aligned} &y_l = y_l\left(x_1, x_2, \ldots, x_n\right), 1 \leq l \leq n\ ;\ y \in C^2\left(\mathbb{R}^n\right) \\ &\tfrac{D(y_1,\ldots,y_n)}{D(x_1,\ldots,x_n)} \neq 0. \end{aligned} \tag{3.15}$$

Notăm $\widetilde{u}\ (y) = u\,(x\,(y)) \Rightarrow \widetilde{u}\ (y\,(x)) = u\,(x)$ şi cu formula de derivare a funcţiilor compuse avem:

$$\frac{\partial u}{\partial x_i} = \sum_{e=1}^{n} \frac{\partial \widetilde{u}}{\partial y_e} \cdot \frac{\partial y_e}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial u}{\partial x_j} \right) =$$

$$\sum_{k,e=1}^{n} \frac{\partial^2 \widetilde{u}}{\partial y_e \partial y_k} \cdot \frac{\partial y_e}{\partial x_i} \cdot \frac{\partial y_k}{\partial x_j} + \sum_{e=1}^{n} \frac{\partial \widetilde{u}}{\partial y_e} \cdot \frac{\partial^2 y_e}{\partial x_i \partial x_j} \tag{3.16}$$

Ecuaţia (3.14) devine:

$$\sum_{k,e=1}^{n} \frac{\partial^2 \widetilde{u}}{\partial y_e \partial y_k} \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_e}{\partial x_i} \cdot \frac{\partial y_k}{\partial x_j} +$$

$$+ \sum_{e=1}^{n} \frac{\partial \widetilde{u}}{\partial y_e} \sum_{i,j=1}^{n} \frac{\partial^2 y_e}{\partial x_i \partial x_j} + \Phi^* \left(y, \widetilde{u}\ , \nabla \widetilde{u}\ \right) = 0,$$

care se mai scrie:

$$\sum_{k,e=1}^{n} \widetilde{a_{ke}}\,(y) \frac{\partial^2 \widetilde{u}}{\partial y_e \partial y_k} + \widetilde{\Phi}\left(y, \widetilde{u}\ , \nabla \widetilde{u}\ \right) = 0 \tag{3.17}$$

unde:

$$\widetilde{a_{ke}}\,(y) = \sum_{i,j=1}^{n} a_{ij} \frac{\partial y_e}{\partial x_i} \cdot \frac{\partial y_k}{\partial x_j} \tag{3.18}$$

Fie $y_0 = y\,(x_0)$ şi $\alpha_{ei} = \frac{\partial y_e}{\partial x_i}\,(x_0)$; atunci (3.18) devine :

$$\widetilde{a}_{ke}\,(y_0) = \sum_{i,j=1}^{n} a_{ij}\,(x_0) \cdot \alpha_{ei} \cdot \alpha_{kj} \tag{3.19}$$

Formula de transformare a coeficienţilor a_{ij} în punctul x_0 coincide cu formula de transformare a coeficienţilor formei pătratice:

$$\sum_{i,j=1}^{n} a_{ij}\,(x_0) \cdot p_i \cdot p_j \tag{3.20}$$

la transformarea liniară nesingulară

$$p_i = \sum_{e=1}^{n} \alpha_{ei} \cdot q_e, \quad \det(\alpha_{ei}) \neq 0. \tag{3.21}$$

Aceasta transformă forma pătratică (3.20) în forma

$$\sum_{k,e=1}^{n} \widetilde{a}_{ek}(y_0) \cdot q_k \cdot q_e. \tag{3.22}$$

Se ştie de la algebră că există o transformare (3.21) prin care forma pătratică (3.20) este adusă la forma canonică:

$$\sum_{e=1}^{r} q_e^2 - \sum_{e=r+1}^{m} q_e^2, \; m \leq n.$$

Avem clasificarea următoare:

a) dacă $m = n$ şi ($r = n$ sau $r = 0$) spunem că (3.14) este de tip eliptic în x_0;

b) dacă $m = n$ şi $1 \leq r \leq n-1$ spunem că (3.14) este de tip hiperbolic în x_0;

c) dacă $m < n$ spunem că (3.14) este de tip parabolic în x_0.

Remarca 3.7 Dacă (3.14) are coeficienţii a_{ij} constanţi, atunci transformarea liniară

$$y_e = \sum_{i=1}^{n} \alpha_{ei} \cdot x_i, \; 1 \leq e \leq n$$

reduce ecuaţia (3.14) la forma canonică

$$\sum_{e=1}^{r} \frac{\partial^2 \widetilde{u}}{\partial y_e^2} - \sum_{e=r+1}^{m} \frac{\partial^2 \widetilde{u}}{\partial y_e^2} + \widetilde{\Phi}(y, \widetilde{u}, \nabla \widetilde{u}) = 0.$$

Aplicaţia 3.8 Să se aducă ecuaţia următoare la forma canonică:

$$4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y \partial z} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Soluţie: Forma pătratică asociată este:

$$\begin{gathered} 4p_1^2 - 4p_1p_2 - 2p_2p_3 \Leftrightarrow \\ 4p_1^2 + p_2^2 + p_3^2 - 4p_1p_2 - 2p_2p_3 - p_2^2 - p_3^2 \Leftrightarrow \\ 4p_1^2 - 4p_1p_2 + p_2^2 - p_2^2 - 2p_2p_3 - p_3^2 + p_3^2 \Leftrightarrow \\ (2p_1 - p_2)^2 - (p_2 + p_3)^2 + p_3^2 \end{gathered}$$

forma canonică a formei pătratice asociate. Facem schimbarea de variabilă:

$$\begin{cases} q_1 = 2p_1 - p_2 \\ q_2 = p_2 + p_3 \\ q_3 = p_3 \end{cases} \Rightarrow 2p_1 + q_3 = q_1 + q_2$$

$$\Rightarrow \begin{cases} p_1 = \frac{1}{2}(q_1 + q_2 - q_3) \\ p_2 = q_2 - q_3 \\ p_3 = q_3 \end{cases}$$

Cu această schimbare de variabilă forma canonică a formei pătratice este:

$$q_1^2 - q_2^2 + q_3^2.$$

$$\Leftrightarrow \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = B \cdot \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix},$$

unde

$$B=\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 1\end{pmatrix}.$$

Facem schimbare de variabilă:

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}=B^T\begin{pmatrix}x\\ y\\ z\end{pmatrix}\Leftrightarrow$$

$$\begin{pmatrix}\xi\\ \eta\\ \zeta\end{pmatrix}=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ \frac{1}{2} & 1 & 0\\ -\frac{1}{2} & -1 & 1\end{pmatrix}\cdot\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}\frac{1}{2}x\\ \frac{1}{2}x+y\\ -\frac{1}{2}x-y+z\end{pmatrix}\Rightarrow$$

$$\begin{cases}\xi=\frac{1}{2}x\\ \eta=\frac{1}{2}x+y\\ \zeta=-\frac{1}{2}x-y+z\end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}\,(\xi,\eta,\zeta)=u\,(x,y,z)\Rightarrow$$

$$\begin{cases}\dfrac{\partial u}{\partial x}=\dfrac{\partial \widetilde{u}}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial x}+\dfrac{\partial \widetilde{u}}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial x}+\dfrac{\partial \widetilde{u}}{\partial \zeta}\cdot\dfrac{\partial \zeta}{\partial x}=\frac{1}{2}\left(\dfrac{\partial \widetilde{u}}{\partial \xi}+\dfrac{\partial \widetilde{u}}{\partial \eta}-\dfrac{\partial \widetilde{u}}{\partial \zeta}\right)\\ \dfrac{\partial u}{\partial y}=\dfrac{\partial \widetilde{u}}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial y}+\dfrac{\partial \widetilde{u}}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial y}+\dfrac{\partial \widetilde{u}}{\partial \zeta}\cdot\dfrac{\partial \zeta}{\partial y}=\dfrac{\partial \widetilde{u}}{\partial \eta}-\dfrac{\partial \widetilde{u}}{\partial \zeta}\\ \dfrac{\partial u}{\partial z}=\dfrac{\partial \widetilde{u}}{\partial \xi}\cdot\dfrac{\partial \xi}{\partial z}+\dfrac{\partial \widetilde{u}}{\partial \eta}\cdot\dfrac{\partial \eta}{\partial z}+\dfrac{\partial \widetilde{u}}{\partial \zeta}\cdot\dfrac{\partial \zeta}{\partial z}=\dfrac{\partial \widetilde{u}}{\partial \zeta}\end{cases}\Rightarrow$$

$$\begin{cases}\frac{\partial}{\partial x}=\frac{1}{2}\left(\frac{\partial}{\partial \xi}+\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \zeta}\right)\\ \frac{\partial}{\partial y}=\frac{\partial}{\partial \eta}-\frac{\partial}{\partial \zeta}\\ \frac{\partial}{\partial z}=\frac{\partial}{\partial \zeta}\end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2}{\partial x^2} \cdot u = \left(\frac{\partial}{\partial x}\right)^2 \cdot u = \frac{1}{2^2}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right)^2 \cdot u =$$

$$= \frac{1}{4}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \frac{\partial^2 \widetilde{u}}{\partial \zeta^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \zeta} - 2\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta}\right)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \left(\frac{\partial}{\partial x} \cdot \frac{\partial}{\partial y}\right) \cdot u =$$
$$= \frac{1}{2}\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right) \cdot \left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right) \cdot \widetilde{u} =$$

$$= \frac{1}{2}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \zeta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} - \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} + \frac{\partial^2 \widetilde{u}}{\partial \zeta^2}\right) =$$

$$= \frac{1}{2}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \zeta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - 2\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} + \frac{\partial^2 \widetilde{u}}{\partial \zeta^2}\right)$$

$$\frac{\partial^2 u}{\partial y \partial z} = \left(\frac{\partial}{\partial y} \cdot \frac{\partial}{\partial z}\right) \cdot u = \frac{\partial}{\partial \zeta}\left(\frac{\partial}{\partial \eta} - \frac{\partial}{\partial \zeta}\right) \cdot \widetilde{u} = \frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} - \frac{\partial^2 \widetilde{u}}{\partial \zeta^2}.$$

Ecuaţia devine:

$$\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \frac{\partial^2 \widetilde{u}}{\partial \zeta^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \zeta} - 2\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} -$$

$$-2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \zeta} - 2\frac{\partial^2 \widetilde{u}}{\partial \eta^2} + 4\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} - 2\frac{\partial^2 \widetilde{u}}{\partial \zeta^2} -$$

$$-2\frac{\partial^2 \widetilde{u}}{\partial \eta \partial \zeta} + 2\frac{\partial^2 \widetilde{u}}{\partial \zeta^2} + \frac{\partial \widetilde{u}}{\partial \eta} - \frac{\partial \widetilde{u}}{\partial \zeta} + \frac{\partial \widetilde{u}}{\partial \zeta} = 0 \Leftrightarrow$$

$$\frac{\partial^2\widetilde{u}}{\partial\xi^2}-\frac{\partial^2\widetilde{u}}{\partial\eta^2}+\frac{\partial^2\widetilde{u}}{\partial\zeta^2}+\frac{\partial\widetilde{u}}{\partial\eta}=0.$$

este forma canonică a ecuaţiei cvasiliniare de gradul al doilea.

Aplicaţia 3.9 Să se aducă la forma canonică:

$$\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial z}-2\frac{\partial^2 u}{\partial x\partial t}+\frac{\partial^2 u}{\partial y^2}+2\frac{\partial^2 u}{\partial y\partial z}+2\frac{\partial^2 u}{\partial y\partial t}+3\frac{\partial^2 u}{\partial z^2}+3\frac{\partial^2 u}{\partial t^2}=0.$$

Soluţie:

$$\begin{gathered}
p_1^2+2p_1p_3-2p_1p_4+p_2^2+2p_2p_3+2p_2p_4+3p_3^2+3p_4^2=\\
=\left(p_1^2+p_3^2+p_4^2+2p_1p_3-2p_1p_4-2p_3p_4\right)+\\
+\left(p_2^2+p_3^2+p_4^2+2p_2p_3+2p_2p_4+2p_3p_4\right)+p_3^2+p_4^2=\\
=(p_1+p_3-p_4)^2+(p_2+p_3+p_4)^2+p_3^2+p_4^2
\end{gathered}$$

este forma pătratică asociată

$$\begin{cases} q_1=p_1+p_3-p_4\\ q_2=p_2+p_3+p_4\\ q_3=p_3\\ q_4=p_4\end{cases}\Rightarrow \begin{array}{c}\text{Cu această schimbare de variabilă}\\ \text{forma canonică a formei pătratice}\\ \text{este : } q_1^2+q_2^2+q_3^2+q_4^2=0\end{array}$$

$$\Rightarrow\begin{cases} p_1=q_1-q_3+q_4\\ p_2=q_2-q_3-q_4\\ p_3=q_3\\ p_4=q_4\end{cases}$$

$$B=\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} \xi \\ \eta \\ \varsigma \\ \sigma \end{pmatrix}=\begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\cdot\begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}=\begin{pmatrix} x \\ y \\ -x-y+z \\ x-y+t \end{pmatrix};$$

$$u\left(x,y,z,t\right)=\widetilde{u}\left(\xi,\eta,\varsigma,\sigma\right)$$

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial \widetilde{u}}{\partial \xi}-\frac{\partial \widetilde{u}}{\partial \varsigma}+\frac{\partial \widetilde{u}}{\partial \sigma} \\ \frac{\partial u}{\partial y}=\frac{\partial \widetilde{u}}{\partial \eta}-\frac{\partial \widetilde{u}}{\partial \varsigma}-\frac{\partial \widetilde{u}}{\partial \sigma} \\ \frac{\partial u}{\partial z}=\frac{\partial \widetilde{u}}{\partial \varsigma} \\ \frac{\partial u}{\partial t}=\frac{\partial \widetilde{u}}{\partial \sigma} \end{cases}$$

Ecuaţia devine:

$$\frac{\partial^2\widetilde{u}}{\partial\xi^2}+\frac{\partial^2\widetilde{u}}{\partial\varsigma^2}+\frac{\partial^2\widetilde{u}}{\partial\sigma^2}-2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\varsigma}-2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}+2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\sigma}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}+\frac{\partial^2\widetilde{u}}{\partial\varsigma^2}+$$

$$+\frac{\partial^2\widetilde{u}}{\partial\sigma^2}-2\frac{\partial^2\widetilde{u}}{\partial\eta\partial\varsigma}-2\frac{\partial^2\widetilde{u}}{\partial\eta\partial\sigma}+2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}+2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\varsigma}-2\frac{\partial^2\widetilde{u}}{\partial\varsigma^2}+$$

$$+2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}-2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\sigma}+2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}-2\frac{\partial^2\widetilde{u}}{\partial\sigma^2}+2\frac{\partial^2\widetilde{u}}{\partial\eta\partial\varsigma}-2\frac{\partial^2\widetilde{u}}{\partial\varsigma^2}-$$

$$-2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}+2\frac{\partial^2\widetilde{u}}{\partial\eta\partial\sigma}-2\frac{\partial^2\widetilde{u}}{\partial\varsigma\partial\sigma}-2\frac{\partial^2\widetilde{u}}{\partial\sigma^2}+3\frac{\partial^2\widetilde{u}}{\partial\varsigma^2}+3\frac{\partial^2\widetilde{u}}{\partial\sigma^2}=0$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \frac{\partial^2 \widetilde{u}}{\partial \varsigma^2} + \frac{\partial^2 \widetilde{u}}{\partial \sigma^2} = 0.$$

3.2.1 Forma canonică a ecuaţiilor diferenţiale cu două variabile independente

I. Ecuaţia:

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + 2b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = \Phi(x,y,u,\nabla u) \tag{3.23}$$

-unde $|a| + |b| + |c| \neq 0$, este:

1. de tip hiperbolic dacă $\delta = b^2 - 4ac > 0$;
2. de tip parabolic dacă $\delta = b^2 - 4ac = 0$;
3. de tip eliptic dacă $\delta = b^2 - 4ac < 0$.

Ecuaţia caracteristică a ecuaţiei (3.23) este:

$$a(x,y)(dx)^2 + 2b(x,y)\,dxdy + c(x,y)(dy)^2 = 0$$

şi se descompune în două ecuaţii:

$$\begin{cases} a \cdot dy - \left(b + \sqrt{b^2 - ac}\right) dx = 0 \\ a \cdot dy - \left(b - \sqrt{b^2 - ac}\right) dx = 0 \end{cases} \tag{3.24}$$

Ecuaţii de tip hiperbolic: $b^2 - ac > 0$.
Integralele prime: $\phi(x,y) = C_1$, $\Psi(x,y) = C_2$ ale ecuaţiilor (3.24) sunt reale şi distincte. Ele determină două familii distincte ale caracteristicilor reale ale ecuaţiei (3.23). Schimbarea de variabile $\xi(x,y) = \phi(x,y)$, $\eta(x,y) = \Psi(x,y)$ şi de funcţie:

$\widetilde{u}(\xi,\eta)=\widetilde{u}(\xi(x,y),\eta(x,y))=u(x,y)$ aduce ecuaţia (3.23) la forma canonică:

$$\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}=\widetilde{\Phi}(\xi,\eta,\widetilde{u},\nabla\widetilde{u}) \tag{3.25}$$

Ecuaţii de tip parabolic: $b^2-ac=0$.
Ecuaţiile din (3.24) coincid. Integrala primă $\phi(x,y)=C$ a ecuaţiei (3.24) determină o familie de caracteristici reale pentru (3.23). Cu schimbarea de variabile:

$$\begin{cases} \xi=\phi(x,y) \\ \eta=\Psi(x,y) \end{cases}$$

unde $\Psi(x,y)$ este o funcţie regulată oarecare, astfel aleasă încât transformarea să fie bijectivă pe domeniul considerat, ecuaţia (3.23) devine:

$$\frac{\partial^2\widetilde{u}}{\partial\eta^2}=\widetilde{\Phi}(\xi,\eta,\widetilde{u},\nabla\widetilde{u}) \tag{3.26}$$

Ecuaţii de tip eliptic: $b^2-ac<0$.
Fie $\phi(x,y)+i\cdot\Psi(x,y)=C$- integrală primă pentru (3.23), unde $\phi(x,y)$ şi $\Psi(x,y)$ sunt funcţii reale. Atunci, cu ajutorul schimbării de variabile: $\xi=\phi(x,y)$, η=$\Psi(x,y)$ şi schimbării de funcţie $\widetilde{u}(\xi,\eta)=u(x,y)$, ecuaţia (3.23) are forma canonică:

$$\frac{\partial^2\widetilde{u}}{\partial\xi^2}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}=\widetilde{\Phi}(\xi,\eta,\widetilde{u},\nabla\widetilde{u}) \tag{3.27}$$

Observaţia 3.10 Pentru uşurinţa calculelor, putem folosi următoarele formule de derivare pentru funcţii compuse:

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} \\ \\ \frac{\partial u}{\partial y}=\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} \end{cases}$$

respectiv:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot \left(\dfrac{\partial \xi}{\partial x}\right)^2 + 2 \cdot \dfrac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot \dfrac{\partial \xi}{\partial x} \cdot \dfrac{\partial \eta}{\partial x} + \dfrac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot \left(\dfrac{\partial \eta}{\partial x}\right)^2 + \\ \qquad + \dfrac{\partial \widetilde{u}}{\partial \xi} \cdot \dfrac{\partial^2 \xi}{\partial x^2} + \dfrac{\partial \widetilde{u}}{\partial \eta} \cdot \dfrac{\partial^2 \eta}{\partial x^2}, \\ \dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot \dfrac{\partial \xi}{\partial x} \dfrac{\partial \xi}{\partial y} + \dfrac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \left(\dfrac{\partial \xi}{\partial x} \cdot \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \xi}{\partial y} \cdot \dfrac{\partial \eta}{\partial x}\right) + \\ \qquad + \dfrac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot \dfrac{\partial \eta}{\partial x} \dfrac{\partial \eta}{\partial y} + \dfrac{\partial \widetilde{u}}{\partial \xi} \cdot \dfrac{\partial^2 \xi}{\partial x \partial y} + \dfrac{\partial \widetilde{u}}{\partial \eta} \cdot \dfrac{\partial^2 \eta}{\partial x \partial y}, \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 \widetilde{u}}{\partial \xi^2} \cdot \left(\dfrac{\partial \xi}{\partial y}\right)^2 + 2 \cdot \dfrac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} \cdot \dfrac{\partial \xi}{\partial y} \cdot \dfrac{\partial \eta}{\partial y} + \dfrac{\partial^2 \widetilde{u}}{\partial \eta^2} \cdot \left(\dfrac{\partial \eta}{\partial y}\right)^2 + \\ \qquad + \dfrac{\partial \widetilde{u}}{\partial \xi} \cdot \dfrac{\partial^2 \xi}{\partial y^2} + \dfrac{\partial \widetilde{u}}{\partial \eta} \cdot \dfrac{\partial^2 \eta}{\partial y^2}. \end{cases}$$

În următoarea parte vom demonstra cele afirmate în I.
II. Fie ecuaţia:

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + \Phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0 \qquad (3.28)$$

şi ecuaţia caracteristicilor ataşată

$$a\left(\frac{\partial \omega}{\partial x}\right)^2 + 2b\frac{\partial \omega}{\partial x} \cdot \frac{\partial \omega}{\partial y} + c\left(\frac{\partial \omega}{\partial y}\right)^2 = 0. \qquad (3.29)$$

Notăm:

$$\lambda_1 = \frac{b - \sqrt{d}}{a},\ \lambda_2 = \frac{b + \sqrt{d}}{a}, \qquad (3.30)$$

unde $d = b^2 - ac$.

Lema 3.11 Fie $\omega(x, y)$ de clasă C^1, astfel ca $\frac{\partial \omega}{\partial y} \neq 0$. Curba $\omega(x, y)$ este caracteristică a ecuaţiei (3.28) dacă şi numai dacă $\omega(x, y) = C$ este integrală primă pentru una din ecuaţiile:

$$\frac{dx}{dy} = \lambda_1(x, y),\ \frac{dx}{dy} = \lambda_2(x, y). \qquad (3.31)$$

Avem
$$\frac{\frac{\partial\omega}{\partial x}}{\frac{\partial\omega}{\partial y}}=-\lambda_1 \ \text{ sau } \ \frac{\frac{\partial\omega}{\partial x}}{\frac{\partial\omega}{\partial y}}=-\lambda_2. \tag{3.32}$$

În continuare, facem schimbarea de variabile:

$$\xi{=}\xi\left(x,y\right),\ \eta{=}\eta\left(x,y\right)\ \text{ cu } \xi{,}\eta\in C^2 \text{ şi } \frac{D\left(\xi,\eta\right)}{D\left(x,y\right)}\neq 0 \tag{3.33}$$

şi schimbarea de funcţie: $\widetilde{u}\left(\xi\left(x,y\right),\eta\left(x,y\right)\right)=u\left(x,y\right)$.
Avem derivatele parţiale de ordinul unu şi doi:

$$\begin{cases} \frac{\partial u}{\partial x}=\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial x} \\ \\ \frac{\partial u}{\partial y}=\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial y} \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)=$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\widetilde{u}}{\partial\xi}\right)\cdot\frac{\partial\xi}{\partial x}+\frac{\partial}{\partial x}\left(\frac{\partial\widetilde{u}}{\partial\eta}\right)\cdot\frac{\partial\eta}{\partial x}+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2}=$$
$$=\left(\frac{\partial^2\widetilde{u}}{\partial\xi^2}\cdot\frac{\partial\xi}{\partial x}+\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\xi}{\partial x}+\left(\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial x}+\right.$$
$$\left.+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\cdot\frac{\partial\eta}{\partial x}\right)\cdot\frac{\partial\eta}{\partial x}+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2}=$$
$$=\frac{\partial^2\widetilde{u}}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2+2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}\cdot\frac{\partial\xi}{\partial x}\cdot\frac{\partial\eta}{\partial x}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2+\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial^2\xi}{\partial x^2}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial^2\eta}{\partial x^2},$$

$$\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial\widetilde{u}}{\partial\xi}\cdot\frac{\partial\xi}{\partial y}+\frac{\partial\widetilde{u}}{\partial\eta}\cdot\frac{\partial\eta}{\partial y}\right)=$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right)\cdot\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right)\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x\partial y} =$$

$$= \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial x}\right)\cdot\frac{\partial \xi}{\partial y} +$$

$$+ \left(\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\right)\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x\partial y} =$$

$$= \frac{\partial^2 \widetilde{u}}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x}\cdot\frac{\partial \xi}{\partial y} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\left(\frac{\partial \xi}{\partial x}\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial x}\right) +$$

$$+ \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial x\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial x\partial y},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right) =$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right)\cdot\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right)\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2} =$$

$$= \left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\cdot\frac{\partial \eta}{\partial y}\right)\cdot\frac{\partial \xi}{\partial y} +$$

$$+ \left(\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial y}\right)\cdot\frac{\partial \eta}{\partial y} + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2} =$$

$$= \frac{\partial^2 \widetilde{u}}{\partial \xi^2}\left(\frac{\partial \xi}{\partial y}\right)^2 + 2\frac{\partial^2 \widetilde{u}}{\partial \xi\partial \eta}\cdot\frac{\partial \xi}{\partial y}\cdot\frac{\partial \eta}{\partial y} +$$

$$+ \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\left(\frac{\partial \eta}{\partial y}\right)^2 + \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial^2 \xi}{\partial y^2} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial^2 \eta}{\partial y^2}.$$

Cu aceste derivate parţiale obţinute ecuaţia (3.28) devine:

$$\left[a \cdot \left(\frac{\partial \xi}{\partial x}\right)^2 + 2b \cdot \frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial y} + c \cdot \left(\frac{\partial \xi}{\partial y}\right)^2\right] \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} +$$

$$+2\left[a \cdot \frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial x} + b\left(\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x}\right) + c \cdot \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y}\right] \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \tag{3.34}$$

$$+\left[a \cdot \left(\frac{\partial \eta}{\partial x}\right)^2 + 2b \cdot \frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial y} + c \cdot \left(\frac{\partial \eta}{\partial y}\right)^2\right] \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} +$$

$$+\widetilde{\Phi}\left(\xi, \eta, \widetilde{u}, \frac{\partial \widetilde{u}}{\partial \xi}, \frac{\partial \widetilde{u}}{\partial \eta}\right) = 0.$$

Notăm:

$$\begin{cases} \widetilde{a} = a\left(\frac{\partial \xi}{\partial x}\right)^2 + 2b\frac{\partial \xi}{\partial x} \cdot \frac{\partial \xi}{\partial y} + c\left(\frac{\partial \xi}{\partial y}\right)^2, \\ \widetilde{b} = a\frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial x} + b\left(\frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y}\right) + c\frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y}, \\ \widetilde{c} = a\left(\frac{\partial \eta}{\partial x}\right)^2 + 2b\frac{\partial \eta}{\partial x} \cdot \frac{\partial \eta}{\partial y} + c\left(\frac{\partial \eta}{\partial y}\right)^2. \end{cases} \tag{3.35}$$

Ecuaţia (3.34) devine:

$$8\widetilde{a}\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\widetilde{b}\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \widetilde{c}\frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \widetilde{\Phi}\left(\xi, \eta, \widetilde{u}, \frac{\partial \widetilde{u}}{\partial \xi}, \frac{\partial \widetilde{u}}{\partial \eta}\right) = 0. \tag{3.36}$$

Vom căuta $\xi(x, y)$ şi $\eta(x, y)$ astfel încât $\widetilde{a}$ şi $\widetilde{c}$ să fie nule, adică ξ şi η sunt soluţii ale ecuaţiei caracteristicilor (3.29) şi conform lemei sunt integrale prime pentru ecuaţiile (3.31), iar cu (3.32) avem:

$$9\frac{\partial \xi}{\partial x} + \lambda_1 \frac{\partial \xi}{\partial y} = 0 \text{ şi } \frac{\partial \eta}{\partial x} + \lambda_2 \frac{\partial \eta}{\partial y} = 0. \tag{3.37}$$

În funcţie de semnul lui $d = b^2 - ac$ avem următoarele trei situaţii:

Cazul (α): $d = b^2 - ac > 0$; atunci $\lambda_1 \neq \lambda_2$ şi avem două familii de caracteristici ale ecuaţiei (3.28): $\xi(x, y) = c_1$ şi $\eta(x, y) = c_2$, unde $\xi,\eta \in \mathrm{C}^1$, $\frac{\partial \xi}{\partial y} \neq 0$, $\frac{\partial \eta}{\partial y} \neq 0$.
Avem:

$$\frac{D(\xi, \eta)}{D(x, y)} = \frac{\partial \xi}{\partial x} \cdot \frac{\partial \eta}{\partial y} - \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial x} = (\lambda_2 - \lambda_1) \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} \neq 0,$$

deci $\xi(x, y) = c_1$ şi $\eta(x, y) = c_2$ reprezintă o schimbare de variabilă.
Cu (3.37) rezultă

$$\widetilde{b} = [a\lambda_1\lambda_2 - b(\lambda_1 + \lambda_2) + c] \cdot \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} =$$

$$= \left(a\frac{b^2 - d}{a^2} + c - b\frac{2b}{a} \right) \cdot \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} =$$

$$= (-2) \cdot \frac{d}{a} \cdot \frac{\partial \xi}{\partial y} \cdot \frac{\partial \eta}{\partial y} \neq 0.$$

Împărţim prin $2\widetilde{b}$ şi ecuaţia (3.36) dacă o împărţim prin $2\widetilde{b}$:

$$10\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \widetilde{\Phi}\left(\xi, \eta, \widetilde{u}, \frac{\partial \widetilde{u}}{\partial \xi}, \frac{\partial \widetilde{u}}{\partial \eta} \right) = 0. \qquad (3.38)$$

Continuăm cu schimbarea de variabile $\rho = \xi + \eta$, $\sigma = \xi - \eta$ şi schimbarea de funcţie $u_1(\rho, \sigma) = \widetilde{u}(\xi, \eta)$.
Avem:

$$\frac{\partial \widetilde{u}}{\partial \xi} = \frac{\partial u_1}{\partial \rho} \cdot \frac{\partial \rho}{\partial \xi} + \frac{\partial u_1}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial \xi} = \frac{\partial u_1}{\partial \rho} + \frac{\partial u_1}{\partial \sigma}$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} = \frac{\partial}{\partial \eta}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) = \frac{\partial}{\partial \rho}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) \cdot \frac{\partial \rho}{\partial \eta} + \frac{\partial}{\partial \sigma}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) \cdot \frac{\partial \sigma}{\partial \eta} =$$

$$= \frac{\partial}{\partial \rho}\left(\frac{\partial u_1}{\partial \rho} + \frac{\partial u_1}{\partial \sigma}\right) - \frac{\partial}{\partial \sigma}\left(\frac{\partial u_1}{\partial \rho} + \frac{\partial u_1}{\partial \sigma}\right) = \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2}$$

şi atunci ecuaţia (3.38) devine:

$$\frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2} + \Phi_1\left(\rho, \sigma, u_1, \frac{\partial u_1}{\partial \rho}, \frac{\partial u_1}{\partial \sigma}\right) = 0,$$

astfel, în acest caz ecuaţia (3.28) este de tip *eliptic*.

Cazul (β): $d = b^2 - ac = 0$; atunci $\lambda_1 = \lambda_2$ şi atunci avem o singură familie de caracteristici pentru ecuaţia (3.28): $\xi(x, y) = c,$ unde $\xi \in \mathrm{C}^1$ şi $\frac{\partial \xi}{\partial y} \neq 0$.
Alegând $\eta(x, y) = x$ avem $\frac{D(\xi,\eta)}{D(x,y)} = -\frac{\partial \xi}{\partial y} \neq 0$, deci $\xi(x, y)$ şi $\eta(x, y)$ reprezintă o schimbare de variabile. Din (3.35) avem, deoarece $\xi(x, y)$ este soluţie a ecuaţiei caracteristicilor, $\widetilde{a} = 0$.
Apoi, folosind $\eta(x, y) = x \Rightarrow \widetilde{b} = a\frac{\partial \xi}{\partial x} + b\frac{\partial \xi}{\partial y} = a\left[\frac{\partial \xi}{\partial x} + \lambda_1 \frac{\partial \xi}{\partial y}\right] = 0$, $\widetilde{c} = a$ şi atunci ecuaţia (3.36) devine:

$$\frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \Phi_1\left(\xi, \eta, \widetilde{u}, \frac{\partial \widetilde{u}}{\partial \xi}, \frac{\partial \widetilde{u}}{\partial \eta}\right) = 0,$$

şi astfel, în acest caz, ecuaţia (3.28) este de tip *parabolic*.

Aplicaţii la reducerea la forma canonică a ecuaţiilor cu derivate parţiale de ordin al doilea.

Aplicaţia 3.12 Să se reducă la forma canonică ecuaţia:

$$4y^2\frac{\partial^2 u}{\partial x^2} - e^{2x}\frac{\partial^2 u}{\partial y^2} = 0.$$

Soluţie:

$$\begin{cases} a = 4y^2 \\ b = 0 \\ c = -e^{2x} \end{cases} \Rightarrow \delta = b^2 - ac = 4y^2e^{2x} > 0 \text{ tip parabolic.}$$

$$4y^2\left(\frac{dy}{dx}\right)^2 - e^{2x} = 0 \Leftrightarrow \frac{dy}{dx} = \pm\sqrt{\frac{e^{2x}}{4y^2}} \Rightarrow$$

$$\begin{cases} \frac{dy}{dx} = \sqrt{\frac{e^{2x}}{4y^2}} \\ \frac{dy}{dx} = -\sqrt{\frac{e^{2x}}{4y^2}} \end{cases} \Leftrightarrow \begin{cases} \frac{dy}{dx} = \frac{e^x}{2y} \\ \frac{dy}{dx} = -\frac{e^x}{2y} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2ydy = e^x dx \\ 2ydy = -e^x dx \end{cases} \Leftrightarrow \begin{cases} e^x - y^2 = c_1 \\ e^x + y^2 = c_2 \end{cases} - \text{integralele prime.}$$

Facem schimbarea de variabile şi de funcţie:

$$\begin{cases} \xi = \xi(x,y) = e^x - y^2 \\ \eta = \eta(x,y) = e^x + y^2 \end{cases} \qquad \widetilde{u}(\xi,\eta) = u(x,y).$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = (-2y)\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) \end{cases} \Rightarrow$$

$$\begin{cases} \frac{\partial}{\partial x} = e^x\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right) \\ \frac{\partial}{\partial y} = (-2y)\left(\frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}\right) \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left[e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] =$$

$$= e^x \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) + e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= e^x\left[e^x\left(\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) + \frac{\partial}{\partial \eta}\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right)\right)\right] +$$

$$+e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= e^{2x}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) + e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right).$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left[(-2y)\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] =$$

$$= (-2y)\,\frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) - 2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= (-2y)\cdot(-2y)\left[\frac{\partial}{\partial \xi}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) - \frac{\partial}{\partial \eta}\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right)\right] -$$

$$-2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= 4y^2\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) - 2\left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{\partial \widetilde{u}}{\partial \eta}\right).$$

Ecuaţia devine:

$$4y^2 e^{2x}\left(\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) + 4y^2 e^x\left(\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) -$$

$$-4y^2e^{2x}\left(\frac{\partial^2\widetilde{u}}{\partial\xi^2}-2\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}\right)-2e^{2x}\left(\frac{\partial\widetilde{u}}{\partial\xi}-\frac{\partial\widetilde{u}}{\partial\eta}\right)=0\Leftrightarrow$$

$$8y^2e^{2x}\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\left(2y^2-e^x\right)e^x\frac{\partial\widetilde{u}}{\partial\eta}+\left(2y^2+e^x\right)e^x\frac{\partial\widetilde{u}}{\partial\xi}=0\,|\ :e^x\Leftrightarrow$$

$$8y^2e^x\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\left(2y^2+e^x\right)\frac{\partial\widetilde{u}}{\partial\xi}+\left(2y^2-e^x\right)\frac{\partial\widetilde{u}}{\partial\eta}=0.$$

$$\begin{cases}\xi=e^x-y^2\\ \eta=e^x+y^2\end{cases}\Rightarrow\begin{cases}e^x=\frac{\xi+\eta}{2}\\ y^2=\frac{\eta-\xi}{2}\end{cases}\Rightarrow\ \text{ecuaţia devine:}$$

$$2\left(\eta^2-\xi^2\right)\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\left(\eta-\xi+\frac{\xi+\eta}{2}\right)\cdot\frac{\partial\widetilde{u}}{\partial\xi}+$$

$$+\left(\eta-\xi-\frac{\xi+\eta}{2}\right)\cdot\frac{\partial\widetilde{u}}{\partial\eta}=0\Leftrightarrow$$

$$2\left(\eta^2-\xi^2\right)\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\frac{3\eta-\xi}{2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}+\frac{\eta-3\xi}{2}\cdot\frac{\partial\widetilde{u}}{\partial\eta}=0.$$

Facem schimbare de variabilă şi de funcţie:

$$\begin{cases}\rho=\xi+\eta\\ \sigma=\xi-\eta\end{cases}\qquad u_1(\rho,\sigma)=\widetilde{u}(\xi,\eta)\Rightarrow$$

$$\begin{cases}\frac{\partial\widetilde{u}}{\partial\xi}=\frac{\partial u_1}{\partial\rho}\cdot\frac{\partial\rho}{\partial\xi}+\frac{\partial u_1}{\partial\sigma}\cdot\frac{\partial\sigma}{\partial\xi}=\frac{\partial u_1}{\partial\rho}+\frac{\partial u_1}{\partial\sigma}\\ \frac{\partial\widetilde{u}}{\partial\eta}=\frac{\partial u_1}{\partial\sigma}\cdot\frac{\partial\sigma}{\partial\eta}+\frac{\partial u_1}{\partial\rho}\cdot\frac{\partial\rho}{\partial\eta}=\frac{\partial u_1}{\partial\rho}-\frac{\partial u_1}{\partial\sigma}\end{cases}\Rightarrow$$

$$\begin{cases}\frac{\partial}{\partial\xi}=\frac{\partial}{\partial\rho}+\frac{\partial}{\partial\sigma}\\ \frac{\partial}{\partial\eta}=\frac{\partial}{\partial\rho}-\frac{\partial u}{\partial\sigma}\end{cases}$$

$$\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}=\frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}\right)=\frac{\partial}{\partial\xi}\left(\frac{\partial u_1}{\partial\rho}-\frac{\partial u_1}{\partial\sigma}\right)=$$

$$=\frac{\partial}{\partial\rho}\left(\frac{\partial u_1}{\partial\rho}-\frac{\partial u_1}{\partial\sigma}\right)+\frac{\partial}{\partial\sigma}\left(\frac{\partial u_1}{\partial\rho}-\frac{\partial u_1}{\partial\sigma}\right)=$$

$$= \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \rho \partial \sigma} + \frac{\partial^2 u_1}{\partial \rho \partial \sigma} - \frac{\partial^2 u_1}{\partial \sigma^2} = \frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2}.$$

Ecuaţia devine:

$$(-2)\,\rho \cdot \sigma \left(\frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2} \right) + \left(\frac{3\eta - \xi}{2} + \frac{\eta - 3\xi}{2} \right) \frac{\partial u_1}{\partial \rho} +$$

$$+ \left(\frac{-\eta + 3\xi}{2} + \frac{3\eta - \xi}{2} \right) \frac{\partial u_1}{\partial \sigma} = 0$$

$$(-2)\,\rho \cdot \sigma \left(\frac{\partial^2 u_1}{\partial \rho^2} - \frac{\partial^2 u_1}{\partial \sigma^2} \right) - 2\sigma \cdot \frac{\partial u_1}{\partial \rho} + \rho \cdot \frac{\partial u_1}{\partial \sigma} = 0.$$

Aplicaţia 3.13 Să se rezolve ecuaţia:

$$4\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} - 6\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} - 4u = 2e^{x-y}.$$

Soluţie:

$$\begin{cases} a = 4 \\ b = -2 \quad \Rightarrow \delta = b^2 - ac = 4 - 4 = 0 \\ c = 1 \end{cases}$$

$\Rightarrow$ ecuaţie de tip parabolic. Ecuaţia caracteristicilor este:

$$4\left(\frac{\partial y}{\partial x}\right)^2 + 4\frac{\partial y}{\partial x} + 1 = 0 \Rightarrow \frac{\partial y}{\partial x} = \frac{-1}{2} \Leftrightarrow 2dy = -dx \Leftrightarrow$$

$$2y^2 + x = c \text{ integrala primă.}$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi = x + 2y \\ \eta = x \end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi,\eta) = u(x,y) \Rightarrow$$

$$\left.\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = 2\frac{\partial \widetilde{u}}{\partial \xi} \end{array}\right\} \Rightarrow \text{operatorii :}$$

$$\left\{\begin{array}{l} \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} \end{array}\right. \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l} \frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)^2 = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \\ \frac{\partial^2}{\partial x \partial y} = 2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\cdot\frac{\partial}{\partial \xi} = 2\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} \\ \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial \xi^2}. \end{array}\right.$$

Atunci ecuaţia devine:

$$4\left(\frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)\cdot\widetilde{u} - 8\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \xi \partial \eta}\right)\cdot\widetilde{u} + 4\frac{\partial^2 \widetilde{u}}{\partial \xi^2} -$$

$$-6\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\cdot\widetilde{u} + 6\frac{\partial \widetilde{u}}{\partial \xi} - 4\widetilde{u} = 2e^{\frac{3\eta-\xi}{2}} \Leftrightarrow$$

$$4\frac{\partial^2 \widetilde{u}}{\partial \eta^2} - 6\frac{\partial \widetilde{u}}{\partial \eta} - 4\widetilde{u} = 2e^{\frac{3\eta-\xi}{2}} \quad |:2 \quad \Leftrightarrow$$

$$2\frac{\partial^2 \widetilde{u}}{\partial \eta^2} - 3\frac{\partial \widetilde{u}}{\partial \eta} - 2\widetilde{u} = e^{\frac{3\eta-\xi}{2}}.$$

Pentru a simplifica ecuaţia facem schimbarea de funcţie:

$$\widetilde{u}(\xi,\eta) = v(\xi,\eta)\cdot e^{\alpha\xi+\beta\eta} \;\Rightarrow\; \text{ecuaţia devine:}$$

$$\left.\begin{array}{l}\frac{\partial \widetilde{u}}{\partial \eta} = \frac{\partial v}{\partial \eta} \cdot e^{\alpha\xi+\beta\eta} + \beta \cdot v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta} \\ \\ \frac{\partial^2 \widetilde{u}}{\partial \eta^2} = \frac{\partial^2 v}{\partial \eta^2} \cdot e^{\alpha\xi+\beta\eta} + 2\beta\frac{\partial v}{\partial \eta} \cdot e^{\alpha\xi+\beta\eta} + \beta^2 \cdot v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta}\end{array}\right\} \Rightarrow$$

$$2\frac{\partial^2 v}{\partial \eta^2} \cdot e^{\alpha\xi+\beta\eta} + 4\beta\frac{\partial v}{\partial \eta} \cdot e^{\alpha\xi+\beta\eta} + 2\beta^2 \cdot v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta} -$$

$$-3\frac{\partial v}{\partial \eta} \cdot e^{\alpha\xi+\beta\eta} - 3\beta \cdot v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta} - 2v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta} = e^{\frac{3\eta-\xi}{2}}$$

$$2\frac{\partial^2 v}{\partial \eta^2} \cdot e^{\alpha\xi+\beta\eta} + (4\beta-3) \cdot \frac{\partial v}{\partial \eta} \cdot e^{\alpha\xi+\beta\eta} +$$

$$+\left(2\beta^2 - 3\beta - 2\right) v\left(\xi,\eta\right) \cdot e^{\alpha\xi+\beta\eta} = e^{\frac{3\eta-\xi}{2}}$$

Pentru a simplifica ecuaţia impunem: $2\beta^2 - 3\beta - 2 = 0$

$$\Rightarrow \beta_{1,2} = \frac{3 \mp \sqrt{9+16}}{4} = \frac{3 \mp 5}{4} \Rightarrow \left\{\begin{array}{l}\beta_1 = \frac{-1}{2}, \\ \beta_2 = 2.\end{array}\right.$$

Alegem $\alpha = 0,\ \beta = \frac{-1}{2} \Rightarrow$ ecuaţia devine:

$$2\frac{\partial^2 v}{\partial \eta^2} \cdot e^{\frac{-\eta}{2}} - 5\frac{\partial v}{\partial \eta} \cdot e^{\frac{-\eta}{2}} = e^{\frac{3\eta-\xi}{2}} \Leftrightarrow$$

$$2\frac{\partial^2 v}{\partial \eta^2} - 5\frac{\partial v}{\partial \eta} = e^{2\eta-\frac{\xi}{2}} \Leftrightarrow$$

$$\frac{\partial}{\partial \eta}\left(2\frac{\partial v}{\partial \eta} - 5v\right) = e^{2\eta-\frac{\xi}{2}} \Rightarrow$$

$$\Rightarrow 2\frac{\partial v}{\partial \eta} - 5v = \frac{1}{2}e^{2\eta-\frac{\xi}{2}} + \phi_1\left(\xi\right)$$

- ecuaţie afină (ecuaţie liniară neomogenă). Îi asociem ecuaţia liniară omogenă.

$$2\frac{\partial\overline{v}}{\partial\eta}-5\overline{v}=0\Leftrightarrow 2\frac{\partial\overline{v}}{\overline{v}}=5\cdot\partial\eta\Leftrightarrow \ln\overline{v}=\frac{5}{2}\eta+\ln\phi_2\left(\xi\right)\Rightarrow$$

$$\Rightarrow\ln\overline{v}\left(\xi,\eta\right)=\frac{5}{2}\eta+\ln\phi_2\left(\xi\right)\Leftrightarrow\overline{v}\left(\xi,\eta\right)=e^{\frac{5\eta}{2}}\cdot\phi_2\left(\xi\right).$$

Căutăm (efectuând metoda variaţiei constantelor) soluţie de forma:

$$v\left(\xi,\eta\right)=e^{\frac{5\eta}{2}}\cdot\phi_2\left(\xi,\eta\right)$$

şi ecuaţia devine:

$$2\cdot\frac{5}{2}\cdot e^{\frac{5\eta}{2}}\cdot\phi_2\left(\xi,\eta\right)+2\cdot e^{\frac{5\eta}{2}}\cdot\frac{\partial\phi_2}{\partial\eta}-5\cdot e^{\frac{5\eta}{2}}\cdot\phi_2\left(\xi,\eta\right)=$$

$$=\frac{1}{2}\cdot e^{2\eta-\frac{\xi}{2}}+\phi_1\left(\xi\right)\Leftrightarrow$$

$$2\cdot\frac{\partial\phi_2}{\partial\eta}=\frac{1}{2}\cdot e^{\frac{-\xi+\eta}{2}}+\phi_1\left(\xi\right)\cdot e^{\frac{-5\eta}{2}}\Rightarrow$$

$$\phi_2\left(\xi,\eta\right)=\frac{-1}{2}\cdot e^{\frac{-\xi+\eta}{2}}-\frac{1}{5}\cdot\phi_1\left(\xi\right)\cdot e^{\frac{-5\eta}{2}}+\phi_3\left(\xi\right)\Rightarrow$$

$$v\left(\xi,\eta\right)=e^{\frac{5\eta}{2}}\left(\frac{-1}{2}\cdot e^{\frac{-\xi}{2}-\frac{\eta}{2}}-\frac{1}{5}\cdot\phi_1\left(\xi\right)\cdot e^{\frac{-5\eta}{2}}+\phi_3\left(\xi\right)\right)=$$

$$=\frac{-1}{2}\cdot e^{2\eta-\frac{\xi}{2}}-\frac{1}{5}\cdot\phi_1\left(\xi\right)+e^{\frac{5\eta}{2}}\cdot\phi_3\left(\xi\right)$$

$$\widetilde{u}\left(\xi,\eta\right)=v\left(\xi,\eta\right)\cdot e^{\frac{-\eta}{2}}=\frac{-1}{2}\cdot e^{\frac{3\eta}{2}-\frac{\xi}{2}}-\frac{1}{5}\cdot e^{\frac{-\eta}{2}}\phi_1\left(\xi\right)+e^{2\eta}\cdot\phi_3\left(\xi\right).$$

Notăm: $\Phi\left(\xi\right)=\phi_1\left(\xi\right)$ şi $\Psi\left(\xi\right)=\phi_3\left(\xi\right)\Rightarrow$

$$\widetilde{u}(\xi,\eta)=\frac{-1}{2}\cdot e^{\frac{3\eta}{2}-\frac{\xi}{2}}-\frac{1}{5}\cdot e^{\frac{-\eta}{2}}\cdot\Phi(\xi)+e^{2\eta}\cdot\Psi(\xi)$$

Revenim la notaţii:

$$\left\{\begin{array}{l} x=\eta \\ 2y+x=\xi \end{array}\right. \Rightarrow \frac{3\eta-\xi}{2}=\frac{3x-x-2y}{2}=x-y\Rightarrow$$

soluţia generală a ecuaţiei este:

$$u(x,y)=\frac{-1}{2}\cdot e^{x-y}-\frac{1}{5}\cdot e^{\frac{-x}{2}}\cdot\Phi(x+2y)+e^{2x}\cdot\Psi(x+2y).$$

Aplicaţia 3.14 Să se rezolve problema:

$$\left\{\begin{array}{l} 4x^2\frac{\partial^2 u}{\partial x^2}-y^2\frac{\partial^2 u}{\partial y^2}+2x\frac{\partial u}{\partial x}=0 \\ u(x,1)=f(x) \\ \frac{\partial u}{\partial y}(x,1)=g(x). \end{array}\right.$$

Soluţie:

$$\left.\begin{array}{l} a=4x^2 \\ b=0 \\ c=-y^2 \end{array}\right\}\Rightarrow\delta=b^2-ac=4x^2y^2>0$$

$\Rightarrow$ ecuaţie de tip hiperbolic. Ecuaţia caracteristicilor este:

$$4x^2\left(\frac{dy}{dx}\right)^2-y^2=0\Rightarrow\frac{dy}{dx}=\pm\frac{y}{2x}\Rightarrow\frac{dy}{dx}=\frac{y}{2x}\text{ şi }\frac{dy}{dx}=\frac{-y}{2x}\Rightarrow$$

integralele prime sunt:

$$\frac{dy}{dx} = \frac{y}{2x} \Rightarrow 2\ln y = \ln x + \ln C_0 \Leftrightarrow \ln y^2 = \ln C_0 \cdot x \Rightarrow$$

$$\Rightarrow y^2 = C_0 \cdot x \Rightarrow \frac{y^2}{x} = C_0 \Rightarrow$$

prima integrală primă este:

$$\frac{y^2}{x} = C_0 \text{ sau } \frac{x}{y^2} = C_1$$

$$\frac{dy}{y} = \frac{-dx}{2x} \Leftrightarrow 2\ln y = \ln\frac{1}{x} + \ln C_2 \Rightarrow xy^2 = C_2.$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi(x,y) = \frac{x}{y^2} \\ \eta(x,y) = xy^2 \end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi,\eta) = u(x,y) \Rightarrow$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{1}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + y^2 \frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{-2x}{y^3} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy \frac{\partial \widetilde{u}}{\partial \eta} \end{cases} \xrightarrow{operatorii}$$

$$\begin{cases} \frac{\partial}{\partial x} = \frac{1}{y^2} \cdot \frac{\partial}{\partial \xi} + y^2 \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = \frac{-2x}{y^3} \cdot \frac{\partial}{\partial \xi} + 2xy \frac{\partial}{\partial \eta}. \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + y^2 \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{1}{y^2} \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + y^2 \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{1}{y^2}\left(\frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + y^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta}\right) + y^2\left(\frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} + y^2 \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right)$$

$$= \frac{1}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} + y^4 \frac{\partial^2 \widetilde{u}}{\partial \eta^2}.$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-2x}{y^3} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \frac{-2x}{y^3} \cdot \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) + 2x \frac{\partial \widetilde{u}}{\partial \eta} =$$

$$= \frac{-2x}{y^3}\left(\frac{-2x}{y^3} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2xy \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta}\right) +$$

$$+ 2xy\left(\frac{-2x}{y^3} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} + 2xy \frac{\partial^2 \widetilde{u}}{\partial \eta^2}\right) +$$

$$+ \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2x \frac{\partial \widetilde{u}}{\partial \eta} =$$

$$= \frac{4x^2}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - \frac{8x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} + 4x^2 y^2 \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \frac{6x}{y^4} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2x \frac{\partial \widetilde{u}}{\partial \eta}.$$

Ecuaţia devine:

$$\frac{4x^2}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 8x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} + 4x^2 y^4 \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - \frac{4x^2}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 8x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \cdot \partial \eta} -$$

$$-4x^2 y^4 \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - \frac{6x}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \xi} - 2xy^2 \frac{\partial \widetilde{u}}{\partial \eta} + \frac{2x}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \xi} + 2xy^2 \frac{\partial \widetilde{u}}{\partial \eta} = 0$$

$$16x^2\frac{\partial^2\widetilde{u}}{\partial\xi\cdot\partial\eta}-4\frac{x}{y^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\,\Big|:4x^2\Rightarrow 4\frac{\partial^2\widetilde{u}}{\partial\xi\cdot\partial\eta}-\frac{1}{xy^2}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\Leftrightarrow$$

$$\frac{\partial^2\widetilde{u}}{\partial\xi\cdot\partial\eta}-\frac{1}{4\eta}\cdot\frac{\partial\widetilde{u}}{\partial\xi}=0\Leftrightarrow\frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\widetilde{u}\right)=0\Rightarrow$$

$$\frac{\partial\widetilde{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\widetilde{u}=\phi(\eta)\Leftrightarrow$$

- ecuaţie afină, căreia îi ataşăm ecuaţia liniară:

$$\frac{\partial\overline{u}}{\partial\eta}-\frac{1}{4\eta}\cdot\overline{u}=0-\text{ecuaţie cu variabile separabile}\Rightarrow$$

$$\frac{\partial\overline{u}}{\overline{u}}=\frac{1}{4\eta}\partial\eta\Rightarrow\ln\overline{u}=\frac{1}{4}\ln\eta+\ln\Phi(\xi)\Rightarrow$$

$$\overline{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi).$$

Căutăm soluţie de forma:

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi,\eta);$$

Introducând în ecuaţie avem:

$$\frac{1}{4\sqrt[4]{\eta^3}}\Phi(\xi,\eta)+\sqrt[4]{\eta}\cdot\frac{\partial\Phi}{\partial\eta}-\frac{1}{4\sqrt[4]{\eta^3}}\Phi(\xi,\eta)=\phi(\eta)\Rightarrow$$

$$\frac{\partial\Phi}{\partial\eta}=\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\Rightarrow\Phi(\xi,\eta)=\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta+\psi(\xi)\Rightarrow$$

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\Phi(\xi,\eta)=\sqrt[4]{\eta}\left(\psi(\xi)+\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta\right)=$$

$$=\sqrt[4]{\eta}\cdot\psi(\xi)+\phi_0(\eta),\ \phi_0(\eta)=\sqrt[4]{\eta}\cdot\int\frac{1}{\sqrt[4]{\eta}}\cdot\phi(\eta)\,d\eta$$

Deci:

$$\widetilde{u}(\xi,\eta)=\sqrt[4]{\eta}\cdot\psi(\xi)+\phi_0(\eta)\Rightarrow u(x,y)=$$

$$= \sqrt[4]{xy^2} \cdot \psi\left(\frac{x}{y^2}\right) + \phi_0\left(xy^2\right).$$

Condiţiile iniţiale sunt:

$$\begin{cases} u(x,1) = f(x) \\ \frac{\partial u}{\partial y}(x,1) = g(x). \end{cases}$$

Ele devin:

$$\begin{cases} \sqrt[4]{\eta} \cdot \psi(x) + \phi_0(x) = f(x) \\ \sqrt[4]{xy^2} \cdot \psi^I\left(\frac{x}{y^2}\right) \cdot \frac{-2x}{y^3} + \\ +\frac{1}{4} \cdot \frac{2xy}{\sqrt[4]{x^3y^6}} \cdot \psi\left(\frac{x}{y^2}\right) + 2xy\phi_0^I\left(xy^2\right)\big|_{y=1} = g(x) \end{cases} \Leftrightarrow$$

$$\begin{cases} \sqrt[4]{\eta} \cdot \psi(x) + \phi_0(x) = f(x) \\ \\ -2x\sqrt[4]{x} \cdot \psi^I(x) + \frac{\sqrt[4]{x}}{2} \cdot \psi(x) + 2x\phi_0^I(x) = g(x) \end{cases} \Rightarrow$$

$$\begin{cases} \phi_0(x) = f(x) - \sqrt[4]{\eta} \cdot \psi(x) \\ \\ \phi_0^I(x) = f^I(x) - \frac{1}{4\sqrt[4]{x^3}}\psi(x) - \sqrt[4]{x} \cdot \psi^I(x) \end{cases} \Rightarrow$$

$$-2x\sqrt[4]{x} \cdot \psi^I(x) + \frac{\sqrt[4]{x}}{2} \cdot \psi(x) +$$

$$+2x \cdot f^I(x) - \frac{\sqrt[4]{x}}{2} \cdot \psi(x) - 2x\sqrt[4]{x} \cdot \psi^I(x) = g(x) \Rightarrow$$

$$-4x\sqrt[4]{x} \cdot \psi^I(x) = g(x) - 2x \cdot f^I(x) \Rightarrow$$

$$\psi^I(x) = \frac{-1}{4x^{\frac{5}{4}}}g(x) + \frac{1}{2x^{\frac{1}{4}}}f^I(x) = \frac{x^{\frac{-5}{4}}}{4}\left(2x \cdot f^I(x) - g(x)\right) \Rightarrow$$

$$\begin{cases} \psi(x) = \int_{x_0}^{x} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + C \\ \phi_0(x) = f(x) - \sqrt[4]{\eta}\left[\int_{x_0}^{x} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + C\right]. \end{cases}$$

Soluţia ecuaţiei este:

$$u(x,y) = \sqrt[4]{xy^2}\left[\int_{x_0}^{\frac{x}{y^2}} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + C\right] + f\left(xy^2\right) -$$

$$- \sqrt[4]{xy^2}\left[\int_{x_0}^{xy^2} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + C\right] =$$

$$= \sqrt[4]{xy^2} \cdot \int_{x_0}^{\frac{x}{y^2}} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt +$$

$$+ \sqrt[4]{xy^2} \cdot \int_{xy^2}^{x_0} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + f\left(xy^2\right) \Rightarrow$$

$$u(x,y) = \sqrt[4]{xy^2} \cdot \int_{xy^2}^{\frac{x}{y^2}} \frac{t^{\frac{-5}{4}}}{4}\left(2t \cdot f^I(t) - g(t)\right) dt + f\left(xy^2\right) \Leftrightarrow$$

$$u(x,y) = \frac{\sqrt[4]{xy^2}}{4} \cdot \int_{xy^2}^{\frac{x}{y^2}} t^{\frac{-5}{4}} \cdot \left(2t \cdot f^I(t) - g(t)\right) dt + f\left(xy^2\right).$$

Aplicaţia 3.15 Să se aducă la forma canonică ecuaţia:

$$\frac{\partial^2 u}{\partial x^2} - 2\sin x \cdot \frac{\partial^2 u}{\partial x \partial y} + \left(2 - \cos^2 x\right)\frac{\partial^2 u}{\partial y^2} = 0.$$

Soluţie:

$$\begin{cases} a = 1 \\ b = -\sin x \\ c = 2 - \cos^2 x \end{cases}$$

şi

$$\delta = b^2 - ac = \sin^2 x - 2 + \cos^2 x = -1 < 0 \Rightarrow$$

ecuaţia este de tip eliptic.

Ecuaţia caracteristicilor este:

$$\left(\frac{dy}{dx}\right)^2 + 2\sin x \frac{dy}{dx} + \left(2-\cos^2 x\right) = 0 \Rightarrow$$

$$\frac{dy}{dx} = \frac{-2\sin x \mp 2i}{2} \Rightarrow \frac{dy}{dx} = -\sin x \mp i \Leftrightarrow$$

$$dy = (-\sin x \mp i)\, dx \Rightarrow \int dy = \int (-\sin x \mp i)\, dx \Leftrightarrow$$

$$\Leftrightarrow y = \cos x \pm ix + C \Leftrightarrow (y - \cos x) \mp ix = C \Rightarrow$$

$$\begin{cases} \xi = y - \cos x \\ \eta = x \end{cases} \quad \text{şi } \widetilde{u}(\xi,\eta) = u(x,y) \Rightarrow$$

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \sin x \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta}\cdot\frac{\partial \eta}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = \sin x \cdot \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi} \end{cases}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\sin x \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta}\right) =$$

$$= \sin x \cdot \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \eta}\right) + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} =$$

$$\sin x \left(\sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}\right) +$$

$$+ \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} =$$

$$= \sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) = \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \widetilde{u}}{\partial \xi}\right) = \frac{\partial^2 \widetilde{u}}{\partial \xi^2}$$

Ecuaţia devine:

$$\sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} -$$

$$-2 \sin^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2 \sin x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} +$$

$$+2 \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - \cos^2 x \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} = 0$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + \cos x \cdot \frac{\partial \widetilde{u}}{\partial \xi} = 0.$$

Aplicaţia 3.16 Să se aducă la forma canonică ecuaţia:

$$\begin{cases} x^2 \frac{\partial^2 u}{\partial x^2} - y^2 \frac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{y=1} = x^2, \ \ \frac{\partial u}{\partial y}\Big|_{y=1} = 2x. \end{cases}$$

Soluţie:

$$\begin{cases} a = x^2 \\ b = 0 \\ c = -y^2 \end{cases}$$

şi

$$\delta = b^2 - ac = x^2y^2 > 0 \Rightarrow$$

ecuaţia este de tip hiperbolic.

Ecuaţia caracteristicilor:

$$x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0 \Rightarrow \frac{dy}{dx} = \pm\frac{y}{x} \Leftrightarrow$$

$$\begin{cases} \frac{dy}{dx} = \frac{-y}{x} \\ \frac{dy}{dx} = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} \int \frac{dy}{y} = \int \frac{-dx}{x} \\ \int \frac{dy}{y} = \int \frac{dx}{x} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \ln y = \ln \frac{1}{x} + \ln C_1 \\ \ln y + \ln C_2 = \ln x \end{cases} \Leftrightarrow \begin{cases} \ln xy = \ln C_1 \\ \ln \frac{x}{y} = \ln C_2 \end{cases}$$

$$\Leftrightarrow \begin{cases} xy = C_1 \\ \frac{x}{y} = C_2 \end{cases} - \text{integrale prime.}$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi = \xi(x,y) = xy \\ \eta = \eta(x,y) = \frac{x}{y} \end{cases}$$

şi schimbarea de funcţie:

$$\widetilde{u}(\xi(x,y), \eta(x,y)) = u(x,y).$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = y \cdot \frac{\partial \widetilde{u}}{\partial \xi} + \frac{1}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = x \cdot \frac{\partial \widetilde{u}}{\partial \xi} - \frac{x}{y^2} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{\partial}{\partial x} = y \cdot \frac{\partial}{\partial \xi} + \frac{1}{y} \cdot \frac{\partial}{\partial \eta} \\ \\ \frac{\partial}{\partial y} = x \cdot \frac{\partial}{\partial \xi} - \frac{x}{y^2} \cdot \frac{\partial}{\partial \eta} \end{cases}$$

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{1}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \Big| \cdot x^2 \\ \\ \frac{\partial^2 u}{\partial x^2} = x^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2\frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{x^2}{y^4} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} + 2\frac{x}{y^3} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \Big| \cdot y^2 \end{cases} \Rightarrow$$

$$0 = x^2 \cdot \frac{\partial^2 u}{\partial x^2} - y^2 \cdot \frac{\partial^2 u}{\partial y^2} =$$

$$= x^2 y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - x^2 y^2 \cdot \frac{\partial^2 \widetilde{u}}{\partial \xi^2} +$$

$$+2x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{x^2}{y^2} \cdot \frac{\partial^2 \widetilde{u}}{\partial \eta^2} - 2\frac{x}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} \Leftrightarrow$$

$$4x^2 \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - 2\frac{x}{y} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \Big| : 4x^2 \Rightarrow$$

$$\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{1}{2\underbrace{xy}_{\xi}} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \Leftrightarrow \frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} - \frac{1}{2\xi} \cdot \frac{\partial \widetilde{u}}{\partial \eta} = 0 \Leftrightarrow$$

$$\frac{\partial}{\partial \eta} \left(\frac{\partial \widetilde{u}}{\partial \xi} - \frac{1}{2\xi} \cdot \widetilde{u} \right) = 0 \Rightarrow$$

$$\frac{\partial \widetilde{u}}{\partial \xi} - \frac{1}{2\xi} \cdot \widetilde{u} = \phi_0 (\xi) .$$

Rezolvăm ecuaţia omogenă ataşată:

$$\left. \begin{array}{l} \frac{\partial \overline{u}}{\partial \xi} = \frac{1}{2\xi} \overline{u} \Leftrightarrow \frac{\partial \overline{u}}{\overline{u}} = \frac{\partial \xi}{2\xi} \Leftrightarrow \overline{u} (\xi, \eta) = \phi_1 (\eta) \cdot \sqrt{\xi} \\ \ln \overline{u} = \int \frac{\partial \overline{u}}{\overline{u}} d\xi = \frac{1}{2} \int \frac{\partial \xi}{\xi} = \ln \sqrt{\xi} + \ln \phi_1 (\eta) \end{array} \right| \Rightarrow$$

căutăm $\widetilde{u}$ de forma:

$$\widetilde{u} (\xi, \eta) = \phi_1 (\xi, \eta) \cdot \sqrt{\xi}.$$

Înlocuim în ecuaţia neomogenă:

$$\frac{\partial \overline{u}}{\partial \xi} = \frac{\partial \phi_1}{\partial \xi} \cdot \sqrt{\xi} + \frac{1}{2\sqrt{\xi}} \cdot \phi_1 (\xi, \eta)$$

şi ecuaţia devine:

$$\frac{\partial \phi_1}{\partial \xi} \cdot \sqrt{\xi} + \frac{1}{2\sqrt{\xi}} \cdot \phi_1 (\xi, \eta) - \frac{1}{2\sqrt{\xi}} \cdot \phi_1 (\xi, \eta) = \phi_0 (\xi) \Rightarrow$$

$$\frac{\partial \phi_1}{\partial \xi} = \frac{1}{\sqrt{\xi}} \cdot \phi_0 (\xi) \Rightarrow \phi_1 (\xi, \eta) = \phi_2 (\xi) + \phi_3 (\eta),$$

unde am notat:

$$\phi_2 (\xi) = \int \phi_0 (\xi) \cdot \frac{1}{\sqrt{\xi}} d\xi.$$

$$\phi_1 (\xi, \eta) = \phi_2 (\xi) + \phi_3 (\eta) \Rightarrow \widetilde{u} (\xi, \eta) = \sqrt{\xi} \cdot \phi_2 (\xi) + \sqrt{\xi} \cdot \phi_3 (\eta).$$

Revenim la notaţiile:

$$\begin{cases} \xi = xy \\ \eta = \frac{x}{y} \end{cases} \Rightarrow$$

$$\begin{cases} u (x, y) = \sqrt{xy} \cdot \phi_2 (xy) + \sqrt{xy} \cdot \phi_3 \left(\frac{x}{y}\right) \\ \frac{\partial u}{\partial y} = \frac{1}{2}\sqrt{\frac{x}{y}} \cdot \phi_2 (xy) + x\sqrt{xy} \cdot \phi_2^I (xy) + \\ \qquad + \frac{1}{2}\sqrt{\frac{x}{y}} \cdot \phi_3 \left(\frac{x}{y}\right) - \frac{x\sqrt{xy}}{y^2} \cdot \phi_3^I \left(\frac{x}{y}\right). \end{cases}$$

Impunem lui u condiţiile iniţiale:

$$\begin{cases} u (x, 1) = \sqrt{x} \left[\phi_2 (x) + \phi_3 (x)\right] = x^2 \\ \frac{\partial u}{\partial y} (x, 1) = \frac{1}{2}\sqrt{x} \cdot \phi_2 (x) + x\sqrt{x} \cdot \phi_2^I (x) + \\ \qquad + \frac{1}{2}\sqrt{x} \cdot \phi_3 (x) - x\sqrt{x} \cdot \phi_3^I (x) = 2x \Big| : \frac{\sqrt{x}}{2} \end{cases} \Leftrightarrow$$

$$\begin{cases} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ \phi_2(x) + 2x \cdot \phi_2^I(x) + \phi_3(x) - 2x \cdot \phi_3^I(x) = 4\sqrt{x} \end{cases}$$

$$\Rightarrow \begin{cases} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ 2x\left(\phi_2^I(x) - \phi_3^I(x)\right) = 4\sqrt{x} - x\sqrt{x} \end{cases} \Leftrightarrow$$

$$\begin{cases} \phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \\ \phi_2^I(x) - \phi_3^I(x) = \frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2} \end{cases} \Rightarrow$$

$$\begin{cases} \phi_2^I(x) + \phi_3^I(x) = \frac{3}{2}x^{\frac{1}{2}} \\ \phi_2^I(x) - \phi_3^I(x) = \frac{2}{\sqrt{x}} - \frac{\sqrt{x}}{2} \end{cases} \Rightarrow$$

$$\phi_2^I(x) = \left(\frac{2}{\sqrt{x}} + \sqrt{x}\right) : 2 = \frac{1}{\sqrt{x}} + \frac{\sqrt{x}}{2} \Rightarrow$$

$$\phi_2(x) = 2\sqrt{x} + \frac{1}{2} \cdot \frac{2}{3}x^{\frac{3}{2}} + C_0 = 2\sqrt{x} + \frac{3}{2}x\sqrt{x} + C_0$$

$$2\phi_3^I(x) = 2\sqrt{x} - \frac{2}{\sqrt{x}} \Rightarrow$$

$$\phi_3^I(x) = \sqrt{x} - \frac{1}{\sqrt{x}} \Rightarrow \phi_3(x) = \frac{2}{3}x^{\frac{3}{2}} - 2\sqrt{x} + C_1.$$

Din egalitatea: $\phi_2(x) + \phi_3(x) = x^{\frac{3}{2}} \Rightarrow x^{\frac{3}{2}} + C_0 + C_1 = x^{\frac{3}{2}} \Rightarrow C_0 + C_1 = 0.$
Deci:

$$u(x,y) = \sqrt{xy}\left[\phi_2(xy) + \phi_3\left(\frac{x}{y}\right)\right] =$$

$$= \sqrt{xy}\left[2\sqrt{xy} + \frac{1}{3}xy\sqrt{xy} + C_0 + \frac{2}{3}\left(\frac{x}{y}\right)^{\frac{3}{2}} - 2\sqrt{\frac{x}{y}} + C_1\right] =$$

$$= 2xy + \frac{x^2y^2}{3} + \frac{2}{3} \cdot \frac{x^2}{y} - 2x$$

$$u(x,y) = \frac{x^2y^2}{3} + \frac{2}{3} \cdot \frac{x^2}{y} + 2xy - 2x.$$

Aplicaţia 3.17 Să se aducă la forma canonică ecuaţia:

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0 \\ u\big|_{x=0} = y^2, \quad \frac{\partial u}{\partial x}\Big|_{x=0} = 1. \end{cases}$$

Soluţie:

$$\begin{cases} a = 1 \\ b = 1 \\ c = -3 \end{cases}$$

şi

$$\delta = b^2 - ac = 4 > 0 \Rightarrow$$

ecuaţia este de tip hiperbolic.

Ecuaţia caracteristicilor este:

$$\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} - 3 = 0 \Rightarrow \begin{cases} \frac{dy}{dx} = 3 \\ \frac{dy}{dx} = -1 \end{cases} \Rightarrow \begin{cases} 3x - y = C_1 \\ x + y = C_2 \end{cases} \Rightarrow$$

facem schimbarea de variabilă

$$\begin{cases} \xi = 3x - y \\ \eta = x + y \end{cases}$$

şi de funcţie

$$u(x, y) = \widetilde{u}(\xi(x, y), \eta(x, y)) \Rightarrow$$

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = 3\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \\ \frac{\partial u}{\partial y} = \frac{\partial \widetilde{u}}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial \widetilde{u}}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = -\frac{\partial \widetilde{u}}{\partial \xi} + \frac{\partial \widetilde{u}}{\partial \eta} \end{cases} \Rightarrow \begin{cases} \frac{\partial}{\partial x} = 3\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial y} = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \end{cases} \Rightarrow$$

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 6\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial x \partial y} = -3\frac{\partial^2 \widetilde{u}}{\partial \xi^2} + 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 \widetilde{u}}{\partial \xi^2} - 2\frac{\partial^2 \widetilde{u}}{\partial \xi \partial \eta} + \frac{\partial^2 \widetilde{u}}{\partial \eta^2} \end{cases}$$

Ecuaţia devine:

$$9\frac{\partial^2\widetilde{u}}{\partial\xi^2}+6\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+\frac{\partial^2\widetilde{u}}{\partial\eta^2}-6\frac{\partial^2\widetilde{u}}{\partial\xi^2}+4\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}+2\frac{\partial^2\widetilde{u}}{\partial\eta^2}-$$

$$-3\frac{\partial^2\widetilde{u}}{\partial\xi^2}+6\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}-3\frac{\partial^2\widetilde{u}}{\partial\eta^2}=0\Leftrightarrow$$

$$16\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}=0\Rightarrow\frac{\partial^2\widetilde{u}}{\partial\xi\partial\eta}=0\Rightarrow\frac{\partial}{\partial\xi}\left(\frac{\partial\widetilde{u}}{\partial\eta}\right)=0\Rightarrow\frac{\partial\widetilde{u}}{\partial\eta}=\phi_0(\eta)\Rightarrow$$

$$\widetilde{u}(\xi,\eta)=\int\phi_0(\eta)\,d\eta=\phi_1(\eta)+\phi_2(\xi)$$

$$\Leftrightarrow\begin{cases}u(x,y)=\phi_2(3x-y)+\phi_1(x+y)\\ \frac{\partial u}{\partial x}(x,y)=3\phi_2^I(3x-y)+\phi_1^I(x+y).\end{cases}$$

Condiţiile iniţiale devin:

$$\begin{cases}\phi_2(-y)+\phi_1(y)=y^2\\ 3\phi_2^I(-y)+\phi_1^I(y)=1\end{cases}\xrightarrow{\text{derivăm prima ecuaţie}}$$

$$\begin{cases}-\phi_2^I(-y)+\phi_1^I(y)=2y|\cdot 3\\ 3\phi_2^I(-y)+\phi_1^I(y)=1\end{cases}\Rightarrow$$

$$4\phi_1^I(y)=1+6y\Rightarrow\phi_1^I(y)=\frac{1}{4}+\frac{3}{2}y\Rightarrow\phi_1(y)=\frac{y}{4}+\frac{3}{4}y^2+C_0$$

$$4\phi_2^I(-y)=1-2y\Rightarrow\phi_2^I(-y)=\frac{1}{4}-\frac{y}{2}\xrightarrow{\text{facem } y\to-y}$$

$$\phi_2^I(y)=\frac{1}{4}+\frac{y}{2}\Rightarrow\phi_2(y)=\frac{y}{4}+\frac{y^2}{4}+C_1.$$

Ecuaţia $\phi_2(-y)+\phi_1(y)=y^2$ devine:

$$-\frac{y}{4}+\frac{(-y)^2}{4}+C_1+\frac{y}{4}+\frac{3y^2}{4}+C_0=y^2\Leftrightarrow$$

$$y^2+C_1+C_0=y^2\Rightarrow C_1+C_0=0\Rightarrow$$

$$u(x,y) = \frac{1}{4}(3x-y) + \frac{(3x-y)^2}{4} + C_1 + \frac{x+y}{4} + \frac{(x+y)^2}{2} + C_0 =$$

$$= x + \frac{1}{4}\left[(3x-y)^2 + (x+y)^2\right] =$$

$$= x + \frac{1}{4}\left(9x^2 - 6xy + y^2 + 3x^2 + 6xy + 3y^2\right) =$$

$$= x + \frac{1}{4}\left(12x^2 + 4y^2\right) = 3x^2 + y^2 + x$$

$$u(x,y) = 3x^2 + y^2 + x.$$

Observaţia 3.18 Avem următoarele probleme pentru ecuaţiile fizicii matematice:

I. *Problema Cauchy* (pentru ecuaţii de tip hiperbolic şi parobolic) în care avem numai condiţii iniţiale.
a) Problema Cauchy pentru ecuaţia oscilaţiilor:

$$\begin{cases} \rho \cdot \frac{\partial^2 u}{\partial t^2} = div\,(p\nabla u) - qu + F(x,t)\,, \quad x \in \mathbb{R}^n,\ t > 0 \\ u\big|_{t=0} = u_0(x)\,, \quad \frac{\partial u}{\partial t}\big|_{t=0} = u_1(x)\,, \quad x \in \mathbb{R}^n. \end{cases}$$

b) Problema lui Cauchy pentru ecuaţia difuziei:

$$\begin{cases} \rho \cdot \frac{\partial u}{\partial t} = div\,(p\nabla u) - qu + F(x,t)\,, \quad x \in \mathbb{R}^n,\ t > 0 \\ u\big|_{t=0} = u_0(x)\,, \quad x \in \mathbb{R}^n. \end{cases}$$

II. *Problema la limită* (pentru ecuaţii de tip eliptic) în care avem numai condiţii la limită.
Problema la limită pentru ecuaţia proceselor staţionare:

$$\begin{cases} -div\,(p\nabla u) + qu = F(x) \text{ în } \Omega \\ \alpha u + \beta \cdot \frac{\partial u}{\partial \vec{n}}\big|_{\Sigma} = v \text{ pe } \Sigma \end{cases}$$

unde

$$\Sigma=\partial\Omega,\ F\in C\left(\Omega\right),\ \alpha,\beta,v\in C\left(\Sigma\right),\ \alpha,\beta\geq 0,\ \alpha+\beta>0.$$

III. *Problema mixtă* (pentru ecuaţii de tip hiperbolic şi parabolic) în care avem atât condiţii iniţiale, cât şi pe frontieră.
a) Problema mixtă pentru ecuaţia oscilaţiilor:

$$\begin{cases} \rho\cdot\frac{\partial^2 u}{\partial t^2}=div\left(p\nabla u\right)-qu+F\left(x,t\right)\ \text{în};\mathrm{Q}_T=\Omega\times\left(0,T\right)\\ u\big|_{t=0}=u_0\left(x\right),\ \ \frac{\partial u}{\partial t}\Big|_{t=0}=u_1\left(x\right)\ \text{în}\ \Omega\\ \alpha u+\beta\cdot\frac{\partial u}{\partial\overrightarrow{n}}\Big|_{\Sigma}=v\ \text{pe}\ \Sigma\times\left[0,T\right],\\ \ \text{cu corelarea}:\alpha u_0+\beta\cdot\frac{\partial u_0}{\partial\overrightarrow{n}}\Big|_{\Sigma}=v\big|_{t=0}. \end{cases}$$

b) Problema mixtă pentru ecuaţia difuziei:

$$\begin{cases} \rho\cdot\frac{\partial u}{\partial t}=div\left(p\nabla u\right)-qu+F\left(x,t\right)\ \text{în};\mathrm{Q}_T\\ u\big|_{t=0}=u_0\left(x\right)\ \text{în}\ \Omega\\ \alpha u+\beta\cdot\frac{\partial u}{\partial\overrightarrow{n}}\Big|_{\Sigma}=v\ \text{pe}\ \Sigma\times\left[0,T\right]. \end{cases}$$

3.3 Metoda separării variabilelor pentru problema mixtă

3.3.1 Metoda separării variabilelor în cazul general

Considerăm Ω un domeniu din $\mathbb{R}^n$ şi domeniul cilindric $Q_T=\Omega\times\left(0,T\right),\ \Sigma$ frontiera lui Ω. Fie operatorii:

$$L_1\left(t,\frac{\partial}{\partial t}\right)=\sum_{k=0}^{q}b_k\left(t\right)\frac{\partial^k}{\partial t^k}\ \text{şi}\ L_2\left(x,\nabla\right)=\sum_{\substack{\alpha\in\mathbb{N}^n\\ [\alpha]\ \leq m}}a_\alpha\left(x\right)\cdot\frac{\partial^\alpha}{\partial x^\alpha},$$

unde

$$\alpha = (\alpha_1, ..., \alpha_n), \ \ [\alpha] = \alpha_1 + \alpha_2 + ... + \alpha_n$$

$$\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{[\alpha]}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}}, \ \rho(x) \text{ este o funcţie.}$$

Considerăm problema mixtă:
Să se găsească $u(x,t)$ astfel încât:

$$\begin{cases} \frac{1}{\rho(x)} \cdot L_1 \left(t, \frac{\partial}{\partial t}\right) \cdot u(x,t) = L_2(x, \nabla) \cdot u(x,t) + F(x,t) \text{ în } Q_T \\ \frac{\partial^j u}{\partial t^j} \left.\right|_{t=0} = u_j \text{ în } \Omega, \ 0 \le j \le q-1 \text{ condiţiile iniţiale} \\ Au = \phi \text{ pe } \Sigma \times [0,T] \quad \text{condiţia la limită} \end{cases} \tag{3.39}$$

unde $F(x,t)$; $\mathrm{u}_j(x)$; $0 \le j \le q-1$; $\phi(x,t)$ sunt date.

Metoda separării variabilelor rezolvă problema (3.39) în patru etape.
Etapa I.
Se determină o funcţie $\widetilde{w}(x,t)$ astfel încât $A\widetilde{w} = \phi$ pe $\Sigma \times [o,T]$.
Se caută $u(x,t)$ sub forma: $u(x,t) = u^*(x,t) + \widetilde{w}(x,t)$, unde $u^*(x,t)$ satisface problema mixtă:

$$\begin{cases} \frac{1}{\rho(x)} \cdot L_1 \left(t, \frac{\partial}{\partial t}\right) \cdot u^*(x,t) = \\ \qquad = L_2(x, \nabla) \cdot u^*(x,t) + F^*(x,t) \text{ în } Q_T \\ \left.\frac{\partial^j u^*}{\partial t^j}\right|_{t=0} = u_j^*, \ 0 \le j \le q-1, \text{ în } \Omega \\ Au^* = 0 \text{ pe } \Sigma \times [0,T] \end{cases} \tag{3.40}$$

unde

$$\begin{cases} F^*(x,t) = F(x,t) - \frac{1}{\rho(x)} \cdot L_1 \left(t, \frac{\partial}{\partial t}\right) \cdot \widetilde{w}(t,x) + \\ \qquad + L_2(x, \nabla) \cdot \widetilde{w}(x,t) \\ u_j^* = u_j - \left.\frac{\partial^j \widetilde{w}}{\partial t^j}\right|_{t=0}, \ 0 \le \mathrm{j} \le q-1. \end{cases}$$

Etapa a-II-a.
Se scrie $u^*(x,t) = u_p(x,t) + u_h(x,t)$ astfel încât:

$$\begin{cases} \frac{1}{\rho(x)} L_1\left(t, \frac{\partial}{\partial t}\right) \cdot u_p(x,t) = \\ \qquad\qquad = L_2(x,\nabla) \cdot u_p(x,t) + F^*(x,t) \text{ în } Q_T \\ \left.\frac{\partial^j u_p}{\partial t^j}\right|_{t=0} = 0,\ 0 \le j \le q-1, \text{ în } \Omega \\ \left.Au_p\right|_{\Sigma\times[0,T]} = 0 \end{cases} \tag{3.41}$$

resprectiv

$$\begin{cases} \frac{1}{\rho(x)} L_1\left(t, \frac{\partial}{\partial t}\right) \cdot u_h(x,t) = L_2(x,\nabla) \cdot u_h(x,t) \text{ în } Q_T \\ \left.\frac{\partial^j u_h}{\partial t^j}\right|_{t=0} = u_j^*,\ 0 \le j \le q-1, \text{ în } \Omega \\ Au_h = 0 \text{ pe } \Sigma\times[0,T] \end{cases} \tag{3.42}$$

Etapa a-III-a.
Se scrie $\widetilde{u_p}(x,t) = \int_0^t \widetilde{u}(x,t-s,s)\,ds$, unde $\widetilde{u}(\cdot,\cdot,s)$ este soluţia problemei:

$$\begin{cases} \frac{1}{\rho(x)} L_1\left(t, \frac{\partial}{\partial t}\right) \cdot \widetilde{u}(x,t,s) = L_2(x,\nabla) \cdot \widetilde{u}(x,t,s),\ (\forall)(x,t) \in Q_T \\ \frac{\partial^j \widetilde{u}}{\partial t^j}(x,0,s) = 0,\ x \in \Omega,\ 0 \le j \le q-2; \\ \frac{\partial^{q-1} \widetilde{u}}{\partial t^{q-1}}(x,0,s) = \frac{\rho(x)}{b_q(s)} \cdot F^*(x,s),\ (\forall)\, x \in \Omega \\ A\widetilde{u}(x,t,s) = 0,\ (\forall)(x,t) \in \Sigma\times[0,T]. \end{cases}$$

Această metodă poartă numele de principiul lui Duhamel.
Etapa a-IV-a.
Determinăm u_h (şi analog $\widetilde{u}(\cdot,\cdot,s)$) prin metoda separării variabilelor.
Considerăm $u_h(x,t) = v(x) \cdot f(t)$. Din prima ecuaţie a relaţiei (3.42) deducem:

$$\frac{v(x)}{\rho(x)} \cdot L_1\left(t, \frac{\partial}{\partial t}\right) \cdot f(t) = f(t) \cdot L_2(x,\nabla) \cdot v(x)$$

echivalent cu:

$$\frac{L_1\left(t, \frac{\partial}{\partial t}\right) \cdot f(t)}{f(t)} = \rho(x) \frac{L_2(x, \nabla) \cdot v(x)}{v(x)} = \lambda\,(\text{constant}) \tag{3.43}$$

Din a treia ecuaţie a relaţiei (3.42) şi din (3.43) obţinem o problemă limită (a valorilor proprii pentru operatorul $\rho(x) \cdot L_2(x, \nabla)$) de soluţie (λ, v) - pereche proprie a operatorului $\rho(x) \cdot L_2(x, \nabla)$:

$$\begin{cases} \rho(x) \cdot L_2(x, \nabla) \cdot v(x) = \lambda \cdot v(x) \\ Av|_{\Sigma} = 0. \end{cases}$$

Fie

$$\begin{cases} \lambda_0, \lambda_1, ... \text{ valorile proprii} \\ \text{(cu repetiţie pentru cele de multiplicitate } \geq 1) \\ v_0, v_1, ... \text{ vectorii proprii (presupuşi ortonormaţi).} \end{cases}$$

Din prima relaţie a lui (3.4.5.) găsim:

$$L_1\left(t, \frac{\partial}{\partial t}\right) \cdot f(t) - \lambda_k \cdot f(t) = 0,$$

ecuaţie cu soluţia generală de forma:

$$f_k(t) = a_{k1} \cdot f_{k1}(t) + ... + a_{kq} \cdot f_{kq}(t),$$

cu $f_{k1}(t), \cdots, a_{k1} \cdot f_{kq}(t)$ soluţii fundamentale.
Avem: $u_k(x,t) = \sum_k v_k(x) \cdot f_k(t)$ care verifică prima şi a treia ecuaţie a relaţiei (3.42).
Pentru a verifica a doua ecuaţie a relaţiei (3.42) impunem:

$$\begin{cases} \sum_k v_k(x) \cdot f_k(0) = u_0^*(x) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \sum_k v_k(x) \cdot f_k^{(q-1)}(0) = u_{q-1}^*(x) \end{cases} \tag{3.44}$$

şi cum $\langle v_k, v_e \rangle = \delta_{ke}$ avem:

$$\begin{cases} f_k(0) = \langle u_0^*, v_k \rangle \\ \dots\dots\dots\dots\dots\dots \\ f_k^{(q-1)}(0) = \left\langle u_{q-1}^*, v_k \right\rangle \end{cases} \quad (\forall)\, k$$

echivalent cu familia de sisteme:

$$\begin{cases} a_{k1} f_{k1}(0) + \cdots + a_{kq} f_{kq}(0) = \langle u_0^*, v_k \rangle \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{k1} f_{k1}^{(q-1)}(0) + \cdots + a_{kq} f_{kq}^{(q-1)}(0) = \left\langle u_{q-1}^*, v_k \right\rangle \end{cases} \quad (\forall)\, k.$$

Aceste sisteme sunt compatibil determinate deoarece determinanţii matricilor coeficienţilor necunoscutelor sunt wronskienii:

$$W[f_{k1}, \cdot \cdot \cdot, f_{kq}](0) \neq 0, \ \ (\forall)\, k.$$

În continuare prezentăm mai multe aplicaţii la această metodă.

Aplicaţia 3.19 Metoda separării variabilelor

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4t(\sin x - x)\,,\ 0 < \mathrm{x} < \frac{\pi}{2} \\ u(0,t) = 3,\ \frac{\partial u}{\partial x}\left(\frac{\pi}{2}, t\right) = t^2 + t \\ u(x,0) = 3,\ \frac{\partial u}{\partial t}(x,0) = x + \sin x. \end{cases}$$

Soluţie: Funcţia $w(x,t) = 3 + x(t^2 + t)$ satisface condiţiile la limită ale problemei mixte.
Căutăm soluţie de forma: $u(x,t) = u^*(x,t) + 3 + x(t^2 + t)$.
În acest caz problema mixtă devine:

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} - 2\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2} + 4t \sin x \\ u^*(0,t) = \frac{\partial u^*}{\partial x}\left(\frac{\pi}{2}, t\right) = 0 \\ u^*(x,0) = 0 \\ \frac{\partial u^*}{\partial t}(x,0) = \sin x \end{cases}$$

Căutăm soluţie de forma: $u^*(x,t) = u_p(x,t) + u_h(x,t)$, unde:

$$\begin{cases} \frac{\partial^2 u_p}{\partial t^2} - 2\frac{\partial u_p}{\partial t} = \frac{\partial^2 u_p}{\partial x^2} + 4t\sin x,\ 0 < x < \frac{\pi}{2} \\ u_p(0,t) = \frac{\partial u_p}{\partial x}\left(\frac{\pi}{2}, t\right) = 0 \\ u_p(x,0) = \frac{\partial u_p}{\partial t}(x,0) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} - 2\frac{\partial u_h}{\partial t} = \frac{\partial^2 u_h}{\partial x^2} + 4t\sin x,\ 0 < x < \frac{\pi}{2} \\ u_h(0,t) = \frac{\partial u_h}{\partial x}\left(\frac{\pi}{2}, t\right) = 0 \\ u_h(x,0) = 0 \\ \frac{\partial u_h}{\partial t}(x,0) = \sin x. \end{cases}$$

Pentru aflarea soluţiei u_p aplicăm principiul lui Duhamel: $u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds$, unde $\widetilde{u}(x,t,s)$ satisface problema mixtă:

$$\begin{cases} \frac{\partial^2 \widetilde{u}}{\partial t^2}(x,t,s) - 2\frac{\partial \widetilde{u}}{\partial t}(x,t,s) = \frac{\partial^2 \widetilde{u}}{\partial x^2}(x,t,s) \\ \widetilde{u}(0,t,s) = \frac{\partial \widetilde{u}}{\partial x}\left(\frac{\pi}{2}, t, s\right) = 0 \\ \widetilde{u}(x,0,s) = 0,\ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = 4s \cdot \sin x. \end{cases}$$

Cum problemele mixte pe care le satisface $u_h(x,t)$ şi $\widetilde{u}(x,t,s)$ sunt similare, vom rezolva problema lui u_h aplicând metoda separării variabilelor.
Căutăm soluţie u_h de forma:

$$u_h(x,t) = f(t) \cdot v(x) \Rightarrow \quad \text{ecuaţia devine :}$$

$$v(x) \cdot f^{II}(t) - 2v(x) \cdot f^{I}(t) = v^{II}(x) \cdot f(t) \Leftrightarrow$$

$$\frac{f^{II}(t) - 2f^{I}(t)}{f(t)} = \frac{v^{II}(x)}{v(x)} = \lambda \Rightarrow$$

$$\begin{cases} v^{II}(x) - \lambda \cdot v(x) = 0 \\ v(0) = v^{I}\left(\frac{\pi}{2}\right) = 0 \end{cases} \quad \text{şi } f^{II}(t) - 2f^{I}(t) - \lambda \cdot f(t) = 0.$$

Ecuaţia caracteristică asociată ecuaţiei lui v este:

$$r^2 - \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{\lambda} \Rightarrow \text{ soluţia este:}$$

$$v(x) = ae^{\sqrt{\lambda}x} + be^{-\sqrt{\lambda}x}.$$

$$v \not\equiv 0 \Leftrightarrow \lambda \neq 0;\ v(0) = v^I\left(\frac{\pi}{2}\right) = 0 \Rightarrow \begin{cases} a + b = 0 \\ ae^{\sqrt{\lambda}\frac{\pi}{2}} - be^{-\sqrt{\lambda}\frac{\pi}{2}} = 0. \end{cases}$$

Sistemul în a şi b are soluţie neidentic nulă în

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\frac{\pi}{2}} & -e^{-\sqrt{\lambda}\frac{\pi}{2}} \end{vmatrix} = 0 \Leftrightarrow e^{\sqrt{\lambda}\frac{\pi}{2}} = -e^{-\sqrt{\lambda}\frac{\pi}{2}} \Leftrightarrow$$

$$e^{\sqrt{\lambda}\pi} = -1 \Rightarrow \sqrt{\lambda_k}\pi = \mathrm{Ln}(-1) = \mathrm{Ln}\, e^{(2k+1)\pi i},\ \mathrm{k} \in \mathbb{Z} \Rightarrow$$

$$\sqrt{\lambda_k}\pi = (2k+1)\pi i,\ \mathrm{k} \in \mathbb{Z} \Leftrightarrow$$

$$\lambda_k = -(2k+1)^2,\ \mathrm{k} \in \mathbb{N}.$$

Din

$$a_k + b_k = 0 \Rightarrow b_k = -a_k \Rightarrow v_k(x) = a_k e^{(2k+1)xi} - a_k e^{-(2k+1)xi} =$$

$$= 2ia_k \frac{e^{(2k+1)xi} - e^{-(2k+1)xi}}{2i} = \gamma_k \sin(2k+1)x,\ \mathrm{k} \geq 0.$$

$$f_k^{II}(t) - 2f_k^I(t) + (2k+1)^2 f_k(t) = 0,\ \mathrm{k} \geq 0.$$

Pentru

$$k = 0 \Rightarrow f_0^{II}(t) - 2f_0^I(t) + f(t) = 0,\ r^2 - 2r + 1 = 0 \Rightarrow r_{1,2} = 1 \Rightarrow$$

$$f_0(t) = (\alpha_0 + \beta_0 t)e^t.$$

Pentru $k \geq 1 \Rightarrow$

$$r^2 - 2r + (2k+1)^2 = 0 \Rightarrow r_{1,2} = 1 \mp 2i\sqrt{k^2+k} \Rightarrow$$

$$f_k(t) = e^t \left(\alpha_k \cos 2\sqrt{k^2+k} \cdot t + \beta_k \sin 2\sqrt{k^2+k} \cdot t \right).$$

Am obţinut şirul de soluţii:

$$u_h^k = v_k(x) \cdot f_k(t) =$$

$$= \begin{cases} (\alpha_0 + \beta_0 t)\gamma_0 e^t \sin x = (c_0 + d_0 t) e^t \sin x \\ e^t \left(\alpha_k \cos 2\sqrt{k^2+k} \cdot t + \right. \\ \left. + \beta_k \sin 2\sqrt{k^2+k} \cdot t \right) \gamma_k \sin(2k+1)x = \\ = e^t \left(c_k \cos 2\sqrt{k^2+k} \cdot t + d_k \sin 2\sqrt{k^2+k} \cdot t \right) \cdot \\ \cdot \sin(2k+1)x, \ k \geq 1. \end{cases}$$

Căutăm soluţie u_h de forma:

$$u_h(x,t) = \sum_{k=0}^{\infty} u_h^k(x,t) = (c_0 + d_0 t) e^t \sin x +$$

$$+ \sum_{k \geq 1} e^t \left(c_k \cos 2\sqrt{k^2+k} \cdot t + d_k \sin 2\sqrt{k^2+k} \cdot t \right) \sin(2k+1)x.$$

Determinăm coeficienţii din condiţiile iniţiale:

$$u_h(x,0) = 0 \Rightarrow c_0 \sin x + \sum_{k=1}^{\infty} c_k \sin(2k+1)x = 0$$

$\{\sin(2k+1)x\}_{0<x<\frac{\pi}{2}} =$ sistem de vectori ortonormat complet

$$\int_0^{\frac{\pi}{2}} \sin(2k+1)x \cdot \sin(2m+1)x dx = \begin{cases} 0, \ k \neq m \\ \frac{\pi}{4}, \ k = m \end{cases}$$

$$\Rightarrow c_k = 0, \ k \geq 1 \Rightarrow$$

$$u_h(x,t) = d_0 t e^t \sin x + \sum_{k \geq 1} d_k e^t \sin 2\sqrt{k^2+k} \cdot t \cdot \sin(2k+1)x \Rightarrow$$

$$\frac{\partial u_h}{\partial t} = d_0\left(1+t\right)e^t \sin x +$$
$$+\sum_{k\geq 1} d_k \left(\sin 2\sqrt{k^2+k}\cdot t + 2\sqrt{k^2+k}\cos 2\sqrt{k^2+k}\cdot t\right)e^t\cdot$$
$$\cdot \sin\left(2k+1\right)x \Rightarrow$$
$$\frac{\partial u_h}{\partial t}\left(x,0\right) = d_0 \sin x + \sum_{k\geq 1} 2d_k\sqrt{k^2+k}\cdot \sin\left(2k+1\right)x = \sin x \Rightarrow$$
$$u_h\left(x,t\right) = t\cdot e^t \sin x.$$

Cum u_h şi $\widetilde{u}$ satisfac probleme similare căutăm $\widetilde{u}$ de forma:

$$\widetilde{u}\left(x,t,s\right) = \sum_{k=0}^{\infty} f_k\left(t,s\right)\cdot v_k\left(x\right) = \sum_{k=0}^{\infty} f_k\left(t,s\right)\cdot \sin\left(2k+1\right)x.$$

$\widetilde{u}$ verifică ecuaţia:

$$\sum_{k=0}^{\infty}\frac{\partial^2 f_k}{\partial t^2}\left(t,s\right)\cdot\sin\left(2k+1\right)x - 2\sum_{k=0}^{\infty}\frac{\partial f_k}{\partial t}\left(t,s\right)\cdot\sin\left(2k+1\right)x +$$
$$+\sum_{k=0}^{\infty}\left(2k+1\right)^2\cdot f_k\left(t,s\right)\cdot\sin\left(2k+1\right)x = 0$$

$$\Rightarrow \left\{\begin{array}{l} \frac{\partial^2 f_k}{\partial t^2}\left(t,s\right) - 2\frac{\partial f_k}{\partial t}\left(t,s\right) + \left(2k+1\right)^2\cdot f_k\left(t,s\right) = 0,\ \mathrm{k}\geq 0 \\ f_k\left(0,s\right) = 0,\ \sum_{k=0}^{\infty}\frac{\partial f_k}{\partial t}\left(0,s\right)\cdot\sin\left(2k+1\right)x = 4s\cdot\sin x \end{array}\right. \Rightarrow$$

$$\left\{\begin{array}{l} \frac{\partial f_k}{\partial t}\left(0,s\right) = 4s,\ k=0 \\ \frac{\partial f_k}{\partial t}\left(0,s\right) = 0,\ k\geq 1 \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} \frac{\partial^2 f_0}{\partial t^2}\left(t,s\right) - 2\frac{\partial f_0}{\partial t}\left(t,s\right) + f_0\left(t,s\right) = 0 \\ f_0\left(0,s\right) = 0,\ \frac{\partial f_0}{\partial t}\left(0,s\right) = 4s \end{array}\right.$$

şi

$$\left\{\begin{array}{l} \frac{\partial^2 f_k}{\partial t^2} - 2\frac{\partial f_k}{\partial t} + \left(2k+1\right)^2\cdot f_k = 0,\ \mathrm{k}\geq 1 \\ f_k\left(0,s\right) = \frac{\partial f_k}{\partial t}\left(0,s\right) = 0,\ f_k\left(t,s\right) = 0,\ k\geq 1. \end{array}\right.$$

Căutăm $f_0(t,s) = \alpha(t) \cdot \beta(s) \Rightarrow$

$$\begin{cases} \alpha^{II}(t) \cdot \beta(s) - 2\alpha^{I}(t) \cdot \beta(s) + \alpha(t) \cdot \beta(s) = 0 \Big| : \beta(s) \not\equiv 0 \\ \alpha(0) = 0 \\ \alpha^{I}(0) \cdot \beta(s) = 4s \Rightarrow \beta(s) = 4s, \ \alpha^{I}(0) = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha^{II}(t) - 2\alpha^{I}(t) + \alpha(t) = 0 \\ \alpha(0) = 0 \\ \alpha^{I}(0) = 1 \end{cases} \Rightarrow$$

$$\begin{cases} \alpha(t) = (a_0 + b_0 t)\, e^t \\ \left.\begin{matrix} \alpha(0) = 0 \\ \alpha^{I}(0) = 1 \end{matrix}\right\} \Rightarrow \begin{cases} a_0 = 0 \Rightarrow \alpha(t) = b_0 \cdot t \cdot e^t \\ \alpha^{I}(t) = b_0(1+t)\, e^t \\ \alpha^{I}(0) = 1 \Rightarrow b_0 = 1 \Rightarrow \alpha(t) = t \cdot e^t. \end{cases} \end{cases}$$

Deci: $f_0(t,s) = 4t \cdot e^t \cdot s \Rightarrow$

$$\widetilde{u}(x,t,s) = f_0(t,s) \cdot v_0(x) = 4t \cdot e^t \cdot s \cdot \sin x \Rightarrow$$

$$\widetilde{u}(x,t,s) = 4s \cdot t \cdot e^t \sin x$$
$$\Downarrow$$

$$u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds = \int_0^t 4s(t-s)\, e^t \sin x ds =$$

$$= 4e^t \sin x \cdot \int_0^t \left(st - s^2\right) e^{-s} ds =$$

$$= -4e^t \sin x \cdot \int_0^t \left(st - s^2\right) \left(e^{-s}\right)^I ds =$$

$$= -4e^t \sin x \cdot \left[\left(st - s^2\right) e^{-s}\Big|_0^t - \int_0^t (t-2s)\, e^{-s} ds \right] =$$

$$= -4e^t \sin x \cdot \int_0^t (t-2s) \left(e^{-s}\right)^I ds =$$

$$= -4e^t \sin x \cdot \left[(t-2s)\, e^{-s} \Big|_0^t + 2\int_0^t e^{-s} ds \right] =$$

$$= -4e^t \sin x \cdot \left[\left(-te^{-t} - t - 2e^{-s} \right) \Big|_0^t \right] =$$

$$= 4e^t \sin x \left(te^{-t} + t + 2e^{-t} - 2 \right) = \left(4t + 4te^t - 8e^t + 8 \right) \sin x \Rightarrow$$

$$u_p(x,t) = \left(4te^t + 4t - 8e^t + 8 \right) \sin x$$

$$u_h(x,t) = te^t \sin x$$

$$\Downarrow$$

$$u^*(x,t) = \left(5te^t + 4t - 8e^t + 8 \right) \sin x$$

$$\Downarrow$$

$$u(x,t) = 3 + x\left(t^2 + t \right) + \left(5te^t + 4t - 8e^t + 8 \right) \sin x.$$

Aplicaţia 3.20

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 3\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} - 3x - 2t,\ 0 < x < \pi \\ u(0,t) = 0,\ \mathrm{u}(\pi,t) = \pi t \\ u(x,0) = e^{-x}(\sin x + \sin 3x) \\ \frac{\partial u}{\partial t}(x,0) = x. \end{cases}$$

Soluţie:
$w(x,t) = \frac{x}{\pi}(\pi t - 0) = \frac{\pi x t}{\pi} = xt$ verifică condiţiile la limită.
Căutăm u de forma:

$$u(x,t) = u^*(x,t) + xt,\ \text{unde :}$$
$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} - 3\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2} + 2\frac{\partial u^*}{\partial x},\ 0 < x < \pi \\ u^*(0,t) = 0,\ u^*(\pi,t) = 0 \\ u^*(x,0) = e^{-x}(\sin x + \sin 3x) \\ \frac{\partial u^*}{\partial t}(x,0) = 0. \end{cases}$$

Deoarece ecuaţia este omogenă se aplică direct metoda separării variabilelor.

Căutăm $u^*(x,t) = \alpha(x) \cdot f(t)$.

$$\begin{cases} \alpha(x) \cdot f''(t) - 3\alpha(x) \cdot f'(t) = \alpha''(x) \cdot f(t) + 2\alpha'(x) \cdot f(t) \\ \alpha(0) = \alpha(\pi) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \frac{\alpha''(x)+2\alpha'(x)}{\alpha(x)} = \frac{f''(t)-3f'(t)}{f(t)} \\ \alpha(0) = \alpha(\pi) = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \alpha''(x) + 2\alpha'(x) - \lambda\alpha(x) = 0 \\ f''(t) - 3f'(t) - \lambda f(t) = 0 \\ \alpha(0) = \alpha(\pi) = 0 \end{cases}$$

$$r^2 + 2r - \lambda = 0 \Rightarrow r_{1,2} = -1 \mp \sqrt{1+\lambda} \Rightarrow$$

$$\alpha(x) = \left(C_1 e^{\sqrt{1+\lambda}\cdot x} + C_2 e^{-\sqrt{1+\lambda}\cdot x}\right) e^{-x}$$

$$\alpha(0) = \alpha(\pi) = 0 \Rightarrow \begin{cases} C_1 + C_2 = 0, \quad C_1, C_2 \not\equiv 0 \\ C_1 e^{\sqrt{1+\lambda}\cdot\pi} + C_2 e^{-\sqrt{1+\lambda}\cdot\pi} = 0 \end{cases} \Rightarrow$$

$$\begin{vmatrix} 1 & 1 \\ e^{\sqrt{1+\lambda}\cdot\pi} & e^{-\sqrt{1+\lambda}\cdot\pi} \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow e^{\sqrt{1+\lambda}\cdot\pi} = e^{-\sqrt{1+\lambda}\cdot\pi} \Leftrightarrow e^{2\sqrt{1+\lambda}\cdot\pi} = 1 \Rightarrow$$

$$2\sqrt{1+\lambda}\cdot\pi = \operatorname{Ln} 1 = 2k\pi i,\ k \in \mathbb{N} \Rightarrow \lambda_k = -1 - k^2,\ k \geq 0$$

$$k = 0 \Rightarrow r_1 = r_2 = -1 \Rightarrow \alpha_0(x) = (a_0 + b_0 x) e^{-x}.$$

Din

$$\begin{array}{l} \alpha_0(0) = 0 \Rightarrow a_0 = 0 \Rightarrow \alpha_0(x) = b_0 x e^{-x} \\ \alpha_0(\pi) = 0 \Rightarrow b_0 \pi e^{-\pi} = 0 \Rightarrow b_0 = 0 \Rightarrow \alpha_0(x) = 0. \end{array}$$

$k \geq 1 \Rightarrow$

$$\left.\begin{array}{l} \alpha_k(x) = \left(a_k e^{kxi} + b_k e^{-kxi}\right) e^{-x} \\ \alpha_k(0) = 0 \end{array}\right\} \Rightarrow b_k = -a_k \Rightarrow$$

$$\alpha_k\left(x\right)=\underbrace{2ia_k}_{\gamma_k}e^{-x}\cdot\frac{e^{kxi}-e^{-kxi}}{2i}=\gamma_k e^{-x}\sin kx.$$

$$\alpha_k\left(x\right)=\gamma_k e^{-x}\sin kx,\ k\geq 1.$$

Pentru $k\geq 1\Rightarrow$

$$\lambda_k=-1-k^2\Rightarrow f_k''\left(t\right)-3f_k'\left(t\right)+\left(k^2+1\right)f_k\left(t\right)=0$$

$$\mathrm{r}^2-3r+k^2+1=0\Rightarrow r_{1,2}=\frac{3}{2}\mp\frac{\sqrt{5-4k^2}}{2},\ k\geq 1.$$

Pentru $k=1\Rightarrow$

$$f_1\left(t\right)=\alpha_1 e^{2t}+\alpha_2 e^{t}.$$

Pentru $k>1\Rightarrow$

$$f_k\left(t\right)=\left(\alpha_k\cos\frac{\sqrt{4k^2-5}}{2}t+\beta_k\sin\frac{\sqrt{4k^2-5}}{2}t\right)e^{\frac{3t}{2}}.$$

Avem şirul de soluţii particulare:

$$\begin{cases} u_1^*\left(x,t\right)=\left(c_1e^{2t}+d_1e^{t}\right)e^{-x}\sin x\\ u_k^*\left(x,t\right)=e^{\frac{3t}{2}}\left(c_k\cos\frac{\sqrt{4k^2-5}}{2}t+d_k\sin\frac{\sqrt{4k^2-5}}{2}t\right)e^{-x}\sin kx,\ k\geq 2.\end{cases}$$

Se caută soluţie de forma:

$$u^*\left(x,t\right)=\left(c_1e^{2t}+c_2e^{t}\right)e^{-x}\sin x+$$

$$+\sum_{k=2}^{\infty}e^{\frac{3t}{2}}\left(c_k\cos\frac{\sqrt{4k^2-5}}{2}t+d_k\sin\frac{\sqrt{4k^2-5}}{2}t\right)e^{-x}\sin kx.$$

Condiţiile iniţiale:

$$u^*\left(x,0\right)=\left(c_1+d_1\right)e^{-x}\sin x+\sum_{k=2}^{\infty}c_k e^{-x}\sin kx=$$

$$= e^{-x}\sin x + e^{-x}\sin 3x \Rightarrow$$

$$\begin{cases} c_1 + d_1 = 1 \\ c_3 = 1 \\ c_k = 0,\ k \neq 3,\ k \geq 2 \end{cases}$$

$$\Rightarrow u^*(x,0) = \left(c_1 e^{2t} + d_1 e^t\right) e^{-x}\sin x +$$

$$+ e^{\frac{3t}{2}} \left(c_3 \cos\frac{\sqrt{31}}{2}t + d_3 \sin\frac{\sqrt{31}}{2}t\right) e^{-x}\sin 3x +$$

$$+ \sum_{\substack{k \geq 2 \\ k \neq 3}}^{\infty} d_k e^{\frac{3t}{2}} \sin\frac{\sqrt{4k^2-5}}{2}t \cdot e^{-x}\sin kx.$$

$$\frac{\partial u^*}{\partial t}(x,t) = \left(2c_1 e^{2t} + d_1 e^t\right) e^{-x}\sin x +$$

$$+ e^{\frac{3t}{2}} \left(\frac{3}{2}\cos\frac{\sqrt{31}}{2}t + \frac{3}{2}d_3 \sin\frac{\sqrt{31}}{2}t -\right.$$

$$\left. -\frac{\sqrt{31}}{2}\sin\frac{\sqrt{31}}{2}t + d_3\frac{\sqrt{31}}{2}\cos\frac{\sqrt{31}}{2}t\right) e^{-x}\sin 3x +$$

$$+ \sum_{\substack{k \geq 2 \\ k \neq 3}}^{\infty} d_k \left(\frac{3}{2}\sin\frac{\sqrt{4k^2-5}}{2}t +\right.$$

$$\left. + \frac{\sqrt{4k^2-5}}{2}\cos\frac{\sqrt{4k^2-5}}{2}t\right) e^{-x}\sin kx.$$

$$\frac{\partial u^*}{\partial t}(x,0)=0 \Rightarrow (2c_1+d_1)\,e^{-x}\sin x+\left(\frac{3}{2}+d_3\frac{\sqrt{31}}{2}\right)e^{-x}\sin 3x+$$

$$+\sum_{\substack{k\geq 2\\ k\neq 3}}^{\infty} d_k\frac{\sqrt{4k^2-5}}{2}e^{-x}\sin kx=0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} 2c_1+d_1=0\\ \frac{3}{2}+d_3\frac{\sqrt{31}}{2}=0\\ d_k=0\end{cases}$$

$$\begin{cases} c_1+d_1=1\\ 2c_1+d_1=0\end{cases} \Rightarrow$$

$$\begin{cases} c_1=-1\\ d_1=2\end{cases} \qquad \begin{cases} c_3=1\\ d_3=-\frac{3}{\sqrt{31}}\end{cases} \qquad c_k=d_k=0,\ k\geq 2,\ k\neq 3$$

$$u^*(x,t)=\left(2e^t-e^{2t}\right)e^{-x}\sin x+$$

$$+e^{\frac{3t}{2}}\left(\cos\frac{\sqrt{31}}{2}t-\frac{3}{\sqrt{31}}\sin\frac{\sqrt{31}}{2}t\right)e^{-x}\sin 3x.$$

Soluţia problemei este:

$$u(x,t)=xt+\left(2e^t-e^{2t}\right)e^{-x}\sin x+$$

$$+e^{\frac{3t}{2}}\left(\cos\frac{\sqrt{31}}{2}t-\frac{3}{\sqrt{31}}\sin\frac{\sqrt{31}}{2}t\right)e^{-x}\sin 3x$$

$$u^*(x,t)=\gamma_1 f_1(t)\,e^{-x}\sin x+\sum_{k=2}^{\infty}\gamma_k f_k(t)\,e^{-x}\sin kx \Rightarrow$$

$$\Rightarrow \begin{cases} u^*(x,0) = e^{-x}\sin x + e^{-x}\sin 3x \\ \frac{\partial u^*}{\partial t}(x,0) = 0 \end{cases} \Rightarrow$$

$$\begin{cases} \gamma_1 f_1(0) = 1 \\ \gamma_1 f_1'(0) = 0 \end{cases}, \begin{cases} \gamma_3 f_3(0) = 1 \\ \gamma_3 f_3'(0) = 0 \end{cases}, \begin{cases} \gamma_k f_k(0) = 0 \\ \gamma_k f_k'(0) = 0 \end{cases} \Rightarrow$$

$$\Rightarrow f_k(t) = 0,\ k \neq 1,\ k \neq 3.$$

$$\begin{cases} \alpha_1 + \beta_1 = \frac{1}{\gamma_1} \\ 2\alpha_1 + \beta_1 = 0 \end{cases} \Rightarrow \alpha_1, \beta_1 = \cdots$$

$\alpha_3 = \frac{1}{\gamma_3}$

$$\begin{cases} f_3(t) = e^{\frac{3t}{2}}\left(\alpha_3 \cos\frac{\sqrt{31}}{2}t + \beta_3 \sin\frac{\sqrt{31}}{2}t\right),\ f_3(0) = \alpha_3 = \frac{1}{\gamma_3} \\ f_3^I(t) = e^{\frac{3t}{2}}\left(-\alpha_3\frac{\sqrt{31}}{2}\sin\frac{\sqrt{31}}{2}t + \right. \\ \qquad \left. +\beta_3\frac{\sqrt{31}}{2}\cos\frac{\sqrt{31}}{2}t + \frac{3}{2}\alpha_3\cos\frac{\sqrt{31}}{2}t + \frac{3}{2}\beta_3\sin\frac{\sqrt{31}}{2}t\right) \end{cases}$$

$$\Rightarrow \frac{\sqrt{31}}{2}\beta_3 + \frac{3}{2}\alpha_3 = 0 \Rightarrow \beta_3 = \frac{-3\alpha_3}{\sqrt{31}} = \frac{-3}{\sqrt{31}}\alpha_3.$$

Aplicaţia 3.21 Să se rezolve problema:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 7\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} - 2t - 7x + e^{-x}\sin 3x,\ 0 < x < \pi \\ u(0,t) = 0,\ u(\pi,t) = \pi t \\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = x. \end{cases}$$

Soluţie:
$w(x,t) = \frac{x}{\pi}(\pi t - 0) = xt$ verifică condiţiile la limită.
Căutăm soluţie de forma: $u(x,t) = u^*(x,t) + xt$. În acest caz u^* satisface problema:

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} - 7\frac{\partial u^*}{\partial t} = \frac{\partial^2 u^*}{\partial x^2} + 2\frac{\partial u^*}{\partial x} + e^{-x}\sin 3x,\ 0 < x < \pi \\ u^*(0,t) = u^*(\pi,t) = 0 \\ u^*(x,0) = 0,\ \frac{\partial u^*}{\partial t}(x,0) = 0. \end{cases}$$

Pentru rezolvarea acestei probleme aplicăm pricipiul lui Duhamel:

$$u^* (x,t) = \int_0^t \widetilde{u}\,(x, t-s, s)\, ds,$$

unde

$$\begin{cases} \frac{\partial^2 \widetilde{u}}{\partial t^2}(x,t,s) - 7\frac{\partial \widetilde{u}}{\partial t}(x,t,s) = \frac{\partial^2 \widetilde{u}}{\partial x^2}(x,t,s) + 2\frac{\partial \widetilde{u}}{\partial x}(x,t,s), \\ \qquad 0 < x < \pi \\ \widetilde{u}\,(0,t,s) = \widetilde{u}\,(\pi,t,s) = 0, \\ \widetilde{u}\,(x,0,s) = 0,\ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = e^{-x}\sin 3x. \end{cases}$$

Deoarece datele problemei nu depind de s avem: $\widetilde{u}\,(x,t,s) = \widetilde{u}\,(x,t)$.
Pentru a afla $\widetilde{u}\,(x,t)$ aplicăm metoda separării variabilelor.
Căutăm $\widetilde{u}\,(x,t) = f(t) \cdot v(x)$.

$$\begin{cases} f^{II}(t) \cdot v(x) - 7f^{I}(t) \cdot v(x) = f(t) \cdot v^{II}(x) + 2f(t) \cdot v^{I}(x) \\ v(0) = v(\pi) = 0 \end{cases} \Leftrightarrow$$

$$\frac{f^{II}(t) - 7f^{I}(t)}{f(t)} = \frac{v^{II}(x) + 2v^{I}(x)}{v(x)} = \lambda,\ v(0) = v(\pi) = 0.$$

$$\begin{cases} v^{II}(x) + 2v^{I}(x) - \lambda v(x) = 0 \\ v(0) = v(\pi) = 0 \end{cases} \qquad f^{II}(t) - 7f^{I}(t) - \lambda f(t) = 0$$

$$r^2 + 2r - \lambda = 0 \Rightarrow r_{1,2} = -1 \mp \sqrt{1+\lambda} \Rightarrow$$

$$v(x) = \left(c_1 e^{-\sqrt{1+\lambda}x} + c_2 e^{\sqrt{1+\lambda}x} \right) e^{-x}$$

$$v(0) = v(\pi) = 0 \Leftrightarrow \begin{cases} c_1 + c_2 = 0 \\ c_1 e^{-\sqrt{1+\lambda}x} + c_2 e^{\sqrt{1+\lambda}x} = 0 \end{cases}$$

$$\Leftrightarrow \begin{vmatrix} 1 & 1 \\ e^{-\sqrt{1+\lambda}\pi} & e^{\sqrt{1+\lambda}\pi} \end{vmatrix} = 0 \Leftrightarrow$$

$$\Leftrightarrow e^{\sqrt{1+\lambda}\pi} = e^{-\sqrt{1+\lambda}\pi} \Leftrightarrow$$

$$\Leftrightarrow e^{2\pi\sqrt{1+\lambda}} = 1 \Rightarrow 2\pi\sqrt{1+\lambda_k} = \mathrm{Ln}1 = 2k\pi i,\ k \in \mathbb{Z} \Rightarrow$$

$$1 + \lambda_k = -k^2,\ k \in \mathbb{N}.$$

$$k = 0 \Rightarrow \lambda_0 = -1 \Rightarrow r_1 = r_2 = -1 \Rightarrow v_0(x) = (a_1 + b_1 x) e^{-x}$$

$$\left\{ \begin{array}{l} v_0(0) = 0 \Rightarrow a_1 = 0 \\ v_0(\pi) = 0 \Rightarrow b_1 = 0 \end{array} \right. \Rightarrow v_0 \equiv 0.$$

$k \geq 1 \Rightarrow$

$$\left. \begin{array}{l} v_k(x) = e^{-x}\left(a_k e^{kxi} + b_k e^{-kxi}\right) \\ \gamma_k(0) = 0 \Rightarrow a_k + b_k = 0 \Rightarrow b_k = -a_k \end{array} \right\} \Rightarrow$$

$$v_k(x) = \frac{2a_k}{i} e^{-x} \frac{e^{kxi} - e^{-kxi}}{2i} = \gamma_k \sin kx \cdot e^{-x}.$$

Deci: $\lambda_k = -1 - k^2,\ k \geq 0,\ v_0(x) = (a_0 + b_0 x) e^{-x},\ k \geq 1,$ $v_k(x) = \gamma_k e^{-x} \cdot \sin kx.$

$$f_k^{II}(t) - 7 f_k^{I}(t) + \left(k^2 + 1\right) f(t) = 0$$

$$r^2 - 7r + \left(k^2 + 1\right) = 0 \Rightarrow r_{1,2} = \frac{7}{2} \mp \frac{\sqrt{45 - 4k^2}}{2},\ k \geq 0.$$

$$f_k(t) = e^{\frac{7t}{2}} \left(\alpha_k e^{\frac{\sqrt{45-4k^2}}{2} t} + \beta_k e^{\frac{-\sqrt{45-4k^2}}{2} t} \right)$$

$$\left\{ \begin{array}{l} f_0(t) = e^{\frac{7t}{2}} \left(\alpha_0 e^{\frac{\sqrt{45}}{2} t} + \beta_0 e^{\frac{-\sqrt{45}}{2} t} \right) \\ f_1(t) = e^{\frac{7t}{2}} \left(\alpha_1 e^{\frac{\sqrt{41}}{2} t} + \beta_1 e^{\frac{-\sqrt{41}}{2} t} \right) \\ f_2(t) = e^{\frac{7t}{2}} \left(\alpha_2 e^{\frac{\sqrt{29}}{2} t} + \beta_2 e^{\frac{-\sqrt{29}}{2} t} \right) \\ f_3(t) = e^{\frac{7t}{2}} \left(\alpha_3 e^{\frac{3}{2} t} + \beta_3 e^{\frac{-3}{2} t} \right) = \alpha_3 e^{5t} + \beta_3 e^{2t} \\ f_k(t) = e^{\frac{7t}{2}} \left(\alpha_k \cos \frac{\sqrt{4k^2-45}}{2} t + \beta_k \sin \frac{\sqrt{4k^2-45}}{2} t \right),\ k \geq 4. \end{array} \right.$$

Căutăm

$$\widetilde{u}(x,t)=\sum_{k=0}^{\infty} f_k(t)\cdot v_k(x)=$$

$$=\underbrace{(a_1+b_1x)}_{0}e^{-x}\cdot e^{\frac{7t}{2}}\left(\alpha_0 e^{\frac{\sqrt{45}}{2}t}+\beta_0 e^{\frac{-\sqrt{45}}{2}t}\right)+$$

$$+e^{-x}\sin x\left(\alpha_1 e^{\frac{\sqrt{41}}{2}t}+\beta_1 e^{\frac{-\sqrt{41}}{2}t}\right)e^{\frac{7t}{2}}+$$

$$+e^{-x}\sin 2x\left(\alpha_2 e^{\frac{\sqrt{29}}{2}t}+\beta_2 e^{\frac{-\sqrt{29}}{2}t}\right)e^{\frac{7t}{2}}+$$

$$+e^{-x}\sin 3x\left(\alpha_3 e^{5t}+\beta_3 e^{2t}\right)+$$

$$+\sum_{k=4}^{\infty} e^{-x}\sin kx\left(\alpha_k\cos\frac{\sqrt{4k^2-45}}{2}t+\beta_k\sin\frac{\sqrt{4k^2-45}}{2}t\right)e^{\frac{7t}{2}}.$$

$$\widetilde{u}(x,0)=0\Rightarrow\begin{cases} a_1+b_1=0\\ a_2+b_2=0\\ a_3+b_3=0\\ a_k+b_k=0,\ k\geq 4.\end{cases}$$

$$\widetilde{u}(x,t)=\sum_{k=1}^{\infty} f_k(t)\cdot v_k(x)=\sum_{k=1}^{\infty} f_k(t)\cdot\gamma_k e^{-x}\sin kx.$$

$$\widetilde{u}(x,0)=0\Rightarrow f_k(0)=0,\ k\geq 1$$

$$\frac{\partial\widetilde{u}}{\partial t}(x,t)=\sum_{k=1}^{\infty}\gamma_k\cdot f_k^I(t)\cdot e^{-x}\sin kx$$

$$\frac{\partial\widetilde{u}}{\partial t}(x,0)=\sum_{k=1}^{\infty}\gamma_k\cdot f_k^I(0)\cdot e^{-x}\sin kx=e^{-x}\sin 3x\Rightarrow$$

$$f_k^I(0)=0,\ k\neq 3,\ f_3^I(0)=\frac{1}{\lambda_3}.$$

Deci:

$$\begin{cases} f_k(0) = 0 \\ f_k^I(0) = 0 \end{cases}, \ k \neq 3 \Rightarrow f_k(t) = 0$$

$$\begin{cases} f_3(0) = 0 \\ f_3^I(0) = \frac{1}{\gamma_3} \end{cases} \Leftrightarrow \begin{cases} \alpha_3 + \beta_3 = 0 \\ 5\alpha_3 + 2\beta_3 = \frac{1}{\gamma_3} \end{cases} \Rightarrow \begin{cases} \alpha_3 = \frac{1}{3\gamma_3} \\ \beta_3 = \frac{-1}{3\gamma_3}. \end{cases}$$

$$\underbrace{f_3(t)}_{\Downarrow} = \frac{1}{3\gamma_3}\left(e^{5t} - e^{2t}\right)$$

$$\widetilde{u}(x,t) = f_3(t) \cdot v_3(x) = \gamma_3 \cdot e^{-x} \sin 3x \cdot \frac{1}{3\gamma_3}\left(e^{5t} - e^{2t}\right) \Leftrightarrow$$

$$\underbrace{\widetilde{u}(x,t)}_{\Downarrow} = \frac{1}{3} \cdot e^{-x}\left(e^{5t} - e^{2t}\right)\sin 3x$$

$$u^*(x,t) = \int_0^t \widetilde{u}(x, t-s)\, ds =$$

$$= \int_0^t \frac{1}{3} \cdot e^{-x} \sin 3x \cdot \left(e^{5(t-s)} - e^{2(t-s)}\right) ds =$$

$$= \frac{e^{-x}\sin 3x}{3}\left(e^{5t}\int_0^t e^{-5s} ds - e^{2t}\int_0^t e^{-2s} ds\right) =$$

$$= \frac{e^{-x}\sin 3x}{3}\left(\frac{-e^{5t}}{5} \cdot e^{-5s}\Big|_0^t + \frac{e^{2t}}{2} \cdot e^{-2s}\Big|_0^t\right) =$$

$$= \frac{e^{-x}\sin 3x}{3}\left(\frac{-1}{5} + \frac{e^{5t}}{5} + \frac{1}{2} - \frac{e^{2t}}{2}\right) =$$

$$= \frac{e^{-x}\sin 3x}{10} + \frac{e^{5t}}{15}e^{-x}\sin 3x - \frac{e^{2t}}{6}e^{-x}\sin 3x$$

Soluţia problemei este:

$$u(x,t) = xt + e^{-x\sin 3x}\left(\frac{1}{10} + \frac{e^{5t}}{15} - \frac{e^{2t}}{6}\right).$$

Aplicaţia 3.22

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4x + 8e^t \cos x, \ x \in \left(0, \frac{\pi}{2}\right), \ t > 0 \\ u(x,0) = \cos x \\ \frac{\partial u}{\partial t}(x,0) = 2x \\ \frac{\partial u}{\partial x}(0,t) = 2t, \ u\left(\frac{\pi}{2}, t\right) = \pi t \end{cases}$$

Soluţie:
$w(x,t) = 2xt$ verifică condiţiile limită.

$$u(x,t) = u^*(x,t) + 2xt \Rightarrow$$

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} + 2\frac{\partial u^*}{\partial t} + 4x = \frac{\partial^2 u^*}{\partial x^2} + 4x + 8e^t \cos x \\ u^*(x,0) = \cos x \\ \frac{\partial u^*}{\partial t}(x,0) = 0 \\ \frac{\partial u^*}{\partial x}(0,t) = 0, \ u^*\left(\frac{\pi}{2}, t\right) = 0 \end{cases}$$

$$u^*(x,t) = u_h(x,t) + u_p(x,t) \Rightarrow$$

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} + 2\frac{\partial u_h}{\partial t} = \frac{\partial^2 u_h}{\partial x^2} \\ u_h(x,0) = \cos x \\ \frac{\partial u_h}{\partial t}(x,0) = 0 \\ \frac{\partial u_h}{\partial x}(0,t) = u_h\left(\frac{\pi}{2}, t\right) = 0 \end{cases} \quad \text{şi} \quad \begin{cases} \frac{\partial^2 u_p}{\partial t^2} + 2\frac{\partial u_p}{\partial t} = \frac{\partial^2 u_p}{\partial x^2} + 8e^t \cos x \\ u_p(x,0) = \frac{\partial u_p}{\partial t}(x,0) = 0 \\ \frac{\partial u_p}{\partial x}(0,t) = u_p\left(\frac{\pi}{2}, t\right) = 0 \end{cases}$$

$$u_h(x,t) = f(t) \cdot v(x) \Rightarrow$$

$$\begin{cases} f^{II}(t) \cdot v(x) + 2f^{I}(t) \cdot v(x) = f(t) \cdot v^{II}(x) \\ v^{I}(0) = v\left(\frac{\pi}{2}\right) = 0 \end{cases} \Leftrightarrow$$

$$\left.\begin{array}{l} \frac{f^{II}(t) + 2f^{I}(t)}{f(t)} = \frac{v^{II}(x)}{v(x)} = \lambda \\ v^{I}(0) = v\left(\frac{\pi}{2}\right) = 0 \end{array}\right\} \Rightarrow$$

$$* \begin{cases} v^{II}(x) - \lambda v(x) = 0 \\ v^{I}(0) = v\left(\frac{\pi}{2}\right) = 0 \end{cases} \qquad r^2 - \lambda = 0 \Rightarrow r_{1,2} = \pm\sqrt{\lambda} \Rightarrow$$

$$\Rightarrow v(x) = a \cdot e^{\sqrt{\lambda}x} + b \cdot e^{-\sqrt{\lambda}x}$$

$$\begin{cases} v^{I}(0)=0 \\ v\left(\frac{\pi}{2}\right)=0 \end{cases} \Leftrightarrow \begin{cases} a-b=0 \\ a\cdot e^{\sqrt{\lambda}\frac{\pi}{2}}+b\cdot e^{-\sqrt{\lambda}\frac{\pi}{2}}=0 \end{cases}, \quad a,b\neq 0 \Leftrightarrow$$

$$\Leftrightarrow \begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\frac{\pi}{2}} & e^{-\sqrt{\lambda}\frac{\pi}{2}} \end{vmatrix}=0 \Leftrightarrow$$

$$e^{\sqrt{\lambda}\frac{\pi}{2}}=-e^{-\sqrt{\lambda}\frac{\pi}{2}} \Rightarrow e^{\sqrt{\lambda}\pi}=-1 \Rightarrow \sqrt{\lambda_k}=(2k+1)\,i \Rightarrow$$

$$\boxed{\lambda_k=-(2k+1)^2},\ k\geq 0.$$

$$\left.\begin{array}{l} v_k(x)=a_k e^{(2k+1)xi}+b_k e^{-(2k+1)xi} \\ a_k=b_k \end{array}\right\} \Rightarrow$$

$$v_k(x)=\overbrace{(2a_k)}^{\gamma_k}\frac{e^{(2k+1)xi}+e^{-(2k+1)xi}}{2}=\gamma_k\cdot\cos(2k+1)\,x,\ k\geq 0.$$

$$*\boxed{v_k(x)=\gamma_k\cdot\cos(2k+1)\,x}$$

$$**\, f_k^{II}(t)+2f_k^{I}(t)+(2k+1)^2 f_k(t)=0,\ k\geq 0.$$

$$k=0 \Rightarrow f_0^{II}(t)+2f_0^{I}(t)+f_0(t)=0$$

$$r^2+2r+1=0 \Rightarrow r_{1,2}=-1 \Rightarrow \boxed{f_0(t)=(a_0+b_0t)\,e^{-t}}.$$

$$k\geq 1 \Rightarrow r^2+2r+(2k+1)^2=0 \Rightarrow$$

$$r_{1,2}=\frac{-2\mp 2\sqrt{1-4k^2-4k-1}}{2}=-1\mp 2\sqrt{k^2+k}i \Rightarrow$$

$$*\boxed{f_k(t)=e^{-t}\left(a_k\cos 2\sqrt{k^2+k}\cdot t+b_k\sin 2\sqrt{k^2+k}\cdot t\right)}$$

$$\begin{cases} u_h^k(x,t)=e^{-t}\left(\alpha_k\cos 2\sqrt{k^2+k}\cdot t+\beta_k\sin 2\sqrt{k^2+k}\cdot t\right)\cdot \\ \qquad\qquad \cdot\cos(2k+1)\,x,\ k\geq 0 \\ u_h^0(x,t)=(\alpha_0+\beta_0t)\,e^{-t}\cos x. \end{cases}$$

$$u_k(x,t)=(\alpha_0+\beta_0t)\,e^{-t}\cos x+$$

$$+\sum_{k\geq 1} e^{-t}\left(\alpha_k \cos 2\sqrt{k^2+k}\cdot t+\beta_k \sin 2\sqrt{k^2+k}\cdot t\right)\cdot\cos\left(2k+1\right)x.$$

$$\begin{cases} u_h\left(x,0\right)=\cos x \\ \frac{\partial u_h}{\partial t}\left(x,0\right)=0 \end{cases} \Rightarrow$$

$$\Rightarrow \alpha_0\cos x+\sum_{k\geq 1} e^{-t}\alpha_k\cos\left(2k+1\right)x=\cos x\Rightarrow \boxed{\alpha_0=1}.$$

$$\boxed{u_h\left(x,t\right)=\sum_{k\geq 0}\gamma_k f_k\left(t\right)\cdot\cos\left(2k+1\right)x}$$

$$\begin{cases} u_h\left(x,0\right)=\sum_{k\geq 0}\gamma_k f_k\left(0\right)\cdot\cos\left(2k+1\right)x=\cos x \\ \frac{\partial u}{\partial t}\left(x,0\right)=\sum_{k\geq 0}\gamma_k f_k^I\left(0\right)\cdot\cos\left(2k+1\right)x=0 \end{cases} \Rightarrow$$

$$\begin{cases} f_k\left(0\right)=f_k^I\left(0\right)=0,\ \ (\forall)\,k\geq 1\Rightarrow f_k\left(t\right)=0,\ \ (\forall)\,k\geq 1 \\ \left.\begin{matrix}\gamma_0 f_0\left(0\right)=1 \\ \gamma_0 f_0^I\left(0\right)=0\end{matrix}\right\}\Leftrightarrow a_0=\frac{1}{\gamma_0} \end{cases}$$

$$f_0\left(t\right)=\frac{e^{-t}}{\gamma_0}+b_0 t\cdot e^{-t}\Rightarrow f_0^I\left(t\right)=-e^{-t}+\left(b_0-b_0 t\right)\cdot e^{-t}\Rightarrow$$

$$f_0^I\left(t\right)=0\Leftrightarrow -1+b_0=0\Rightarrow b_0=1\Rightarrow \boxed{f_0\left(t\right)=\left(t+1\right)e^{-t}}.$$

$$\begin{cases} \gamma_0 f_0\left(0\right)=1 \\ f_0^I\left(0\right)=0 \end{cases} \Leftrightarrow \boxed{a_0=\frac{1}{\gamma_0}}.$$

$$f_0^I\left(t\right)=b_0\cdot e^{-t}-\left(a_0+b_0 t\right)e^{-t}=\left(b_0-a_0-b_0 t\right)e^{-t}$$

$$f_0^I\left(t\right)=0\Rightarrow a_0=b_0=\frac{1}{\gamma_0}\Rightarrow \boxed{f_0\left(t\right)=\frac{1}{\gamma_0}\left(1+t\right)e^{-t}}.$$

$$*\boxed{u_h\left(x,t\right)=\left(1+t\right)e^{-t}\cos x}$$

$$*u_p\left(x,t\right)=\int_0^t \widetilde{u}\left(x,t-s,s\right)ds$$

$$\begin{cases} \frac{\partial^2 \widetilde{u}}{\partial t^2} + 2\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial^2 \widetilde{u}}{\partial x^2} \\ \widetilde{u}(x,0,s) = 0, \quad \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = 8e^s \cos x \\ \frac{\partial \widetilde{u}}{\partial x}(0,t,s) = 0, \quad \widetilde{u}\left(\frac{\pi}{2},t,s\right) = 0 \end{cases}$$

$\widetilde{u}(x,t,s)$ verifică o problemă similară cu $u_h \Rightarrow$ o căutăm de forma:

$$\widetilde{u}(x,t,s) = \sum_{k\geq 0} f_k(t,s) \cdot \cos(2k+1)x.$$

$$\begin{cases} f_k(0,s) = 0 \\ \frac{\partial f_k^I}{\partial t}(0,s) = 0 \end{cases} \Rightarrow f_k(t,s) \equiv 0.$$

$$\begin{cases} f_0(0,s) = 0 \\ \frac{\partial f_0}{\partial t}(0,s) = 8e^s \end{cases}, \ \widetilde{u}(x,t,s) = f_0(t,s) \cdot \cos x \Rightarrow$$

Ecuaţia devine:

$$\begin{cases} \frac{\partial^2 f_0}{\partial t^2} + 2\frac{\partial f_0}{\partial t} + f_0 = 0 \\ f_0(0,s) = 0, \ \frac{\partial f_0}{\partial t}(0,s) = 8e^s \Rightarrow \\ \alpha^I(0) \cdot \beta(s) = 8e^s \Rightarrow \ \begin{matrix} \boxed{\beta(s) = 8e^s} \\ \alpha(t) = t \cdot e^{-t} \end{matrix} \end{cases} \Bigg\}$$

$$f_0(t,s) = \alpha(t) \cdot \beta(s) \Rightarrow \alpha^{II}(t) + 2\alpha^I(t) + 1 = 0 \Rightarrow$$

$$\begin{cases} \alpha(t) = (c_0 + c_1 t)e^{-t} \\ \alpha(0) = 0, \ \alpha^I(0) = 1 \Rightarrow c_0 = 0 \Rightarrow \underbrace{\alpha^I(t) = (c_1 - c_1 t)e^{-t}} \\ \qquad\qquad\qquad\qquad\qquad\qquad \Downarrow \\ \qquad\qquad\qquad\qquad\qquad\qquad \alpha^I(0) = 1 \Leftrightarrow c_1 = 1 \end{cases}$$

$$*f_0(t,s) = 8t \cdot e^{s-t}.$$

$$\widetilde{u}(x,t,s) = 8t \cdot e^{s-t} \cdot \cos x \Rightarrow$$

$$u_p(x,t) = 8\int_0^t \widetilde{u}(x,t-s,s)\,ds = 8\int_0^t (t-s) \cdot e^{s-(t-s)} \cdot \cos x dx =$$

$$= 8e^{-t}\cos x\int_0^t (t-s)\cdot e^{2s}dx = 4e^{-t}\cos x\int_0^t (t-s)\cdot\left(e^{2s}\right)^I dx =$$

$$= 4e^{-t}\cos x\left[(t-s)\cdot e^{2s}\Big|_0^t + \int_0^t e^{2s}ds\right] =$$

$$= 4e^{-t}\cos x\left[t+\frac{e^{2t}}{2}-\frac{1}{2}\right] =$$

$$= 2e^{-t}\cos x\left[e^{2t}-2t-1\right] \Rightarrow$$

$$*\boxed{u_p(x,t) = 4t\cdot e^{-t}\cos x + 2e^t\cos x - 2e^{-t}\cos x}$$

$$u(x,t) = (t-1)e^{-t}\cos x - 4t\cdot e^{-t}\cos x + 2e^t\cos x + 2xt =$$

$$= -(1+3t)e^{-t}\cos x + 2e^t\cos x + 2xt.$$

În cele ce urmează aplicăm metoda separării variabilelor pentru ecuaţia coardei vibrante şi ecuaţia căldurii, aflând soluţia particulară u_p folosind dezvoltarea în serie Fourier de sinusuri. Aplicaţiile se rezolvă, însă, şi cu principiul lui Duhamel.

Aplicaţia 3.23 i) Ecuaţia coardei vibrante

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0<x<l \\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x) \text{ - condiţiile iniţiale} \\ u(0,t) = \phi_1(t),\ u(l,t) = \phi_2(t) \text{ - condiţiile la limită.} \end{cases}$$

Funcţia $w(x,t) = \phi_1(t) + \frac{x}{l}(\phi_2(t) - \phi_1(t))$ satisface condiţiile la limită.
Căutăm soluţie de forma: $u(x,t) = u^*(x,t) + w(x,t)$ unde u^* satisface problema Cauchy-Dirichlet:

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} = a^2\frac{\partial^2 u^*}{\partial x^2} + g(x,t),\ 0<x<l \\ u^*(x,0) = \alpha_0(x),\ \frac{\partial u^*}{\partial t}(x,0) = \alpha_1(x) \\ u^*(0,t) = u^*(l,t) = 0. \end{cases}$$

Căutăm soluţie de forma: $u^*(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$I. \begin{cases} \frac{\partial^2 u_h}{\partial t^2} = a^2 \frac{\partial^2 u_h}{\partial x^2},\ 0 < x < l \\ u_h(x,0) = \alpha_0(x),\ \frac{\partial u_h}{\partial t}(x,0) = \alpha_1(x) \\ u_h(0,t) = u_h(l,t) = 0 \end{cases}$$

$$II. \begin{cases} \frac{\partial^2 u_p}{\partial t^2} = a^2 \frac{\partial^2 u_p}{\partial x^2} + g(x,t),\ 0 < x < l \\ u_p(x,0) = \frac{\partial u_p}{\partial t}(x,0) = 0 \\ u_p(0,t) = u_p(l,t) = 0. \end{cases}$$

$$u_h(x,t) = \sum_{k=1}^{\infty} \left(a_k \cos \frac{k\pi a t}{l} + b_k \sin \frac{k\pi a t}{l} \right) \sin \frac{k\pi x}{l},$$

unde

$$a_k = \frac{2}{l} \int_0^l \alpha_0(x) \sin \frac{k\pi x}{l} dx;$$

$$b_k = \frac{2}{k\pi a} \int_0^l \alpha_1(x) \sin \frac{k\pi x}{l} dx,\ k \geq 1.$$

Pentru problema II. cu ecuaţia neomogenă se caută soluţie de forma:

$$u_p(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin \frac{k\pi x}{l}$$

care introdusă în ecuţia din II. ne dă:

$$\left. \begin{array}{l} \sum_{k=1}^{\infty} \left[T_k^{II}(t) + \left(\frac{k\pi a}{l} \right)^2 \cdot T_k(t) \right] \cdot \sin \frac{k\pi x}{l} = g(x,t) \\ \text{Dezvoltăm pe g în serie Fourier de sinusuri:} \\ g(x,t) = \sum_{k=1}^{\infty} g_k(t) \cdot \sin \frac{k\pi x}{l} \end{array} \right\} \Rightarrow$$

Ţinând cont că sistemul $\left\{ \frac{2}{l} \cdot \sin \frac{k\pi x}{l} \right\}_{k \geq 1}$ este ortonormat $\Rightarrow$

$$\begin{cases} T_k^{II}(t) + \left(\frac{k\pi a}{l} \right)^2 \cdot T_k(t) = g_k(t), \\ T_k(0) = T_k^I(0) = 0 \end{cases}$$

unde $g_k(t) = \frac{2}{l}\int_0^l g(\xi,t)\cdot\sin\frac{k\pi\xi}{l}d\xi$. Aplicând metoda variaţiei constantelor rezultă:

$$T_k(t) = \frac{2}{k\pi a}\int_0^t \left(\int_0^l g(\xi,\tau)\cdot\sin\frac{k\pi\xi}{l}d\xi\right)\cdot\sin\frac{k\pi a(t-\tau)}{l}d\tau$$

$$T_k(t) = \frac{2}{k\pi a}\int_0^t\int_0^l g(\xi,\tau)\cdot\sin\frac{k\pi\xi}{l}\cdot\sin\frac{k\pi a(t-\tau)}{l}d\xi d\tau.$$

ii) Ecuaţia căldurii

Analog la ecuaţia căldurii căreia i se asociază problema mixtă.

$$\begin{cases} \frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2} + f(x,t),\ 0<x<l \\ u(x,0) = u_0(x) \\ u(0,t) = u(l,t) = 0 \end{cases}$$

Dacă avem condiţiile la limită nenule procedăm ca la ecuaţia undelor. Se caută soluţie de forma: $u(x,t) = u_h(x,t) + u_p(x,t)$, unde:

$$\begin{cases} \frac{\partial u_h}{\partial t} = a^2\frac{\partial^2 u_h}{\partial x^2},\ 0<x<l \\ u_h(x,0) = u_0(x) \\ \underbrace{u_h(0,t) = u_h(l,t) = 0}_{\Downarrow} \end{cases} \qquad \begin{cases} \frac{\partial u_p}{\partial t} = a^2\frac{\partial^2 u_p}{\partial x^2} + f(x,t), \\ \qquad 0<x<l \\ u_p(x,0) = 0 \\ u_p(0,t) = u_p(l,t) = 0 \end{cases}$$

$$\begin{cases} u_h(x,t) = \sum_{k=1}^{\infty} a_k\cdot e^{-\left(\frac{k\pi a}{l}\right)^2 t}\cdot\sin\frac{k\pi x}{l} \\ a_k = \frac{2}{l}\int_0^l u_o(x)\cdot\sin\frac{k\pi x}{l}dx. \end{cases}$$

Pentru u_p se procedează analog ca la ecuaţia undelor:

$$u_p(x,t) = \sum_{k=1}^{\infty} T_k(t)\cdot\sin\frac{k\pi x}{l},$$

unde:

$$T_k(t) = \frac{2}{k\pi a}\int_0^t\left(\int_0^l f(\xi,\tau)\cdot\sin\frac{k\pi\xi}{l}d\xi\right)\cdot e^{-\left(\frac{k\pi a}{l}\right)^2(t-\tau)}d\tau$$

şi

$$\sum_{k=1}^{\infty} T_k(t) \cdot \sin \frac{k\pi x}{l},$$

unde:

$$T_k(t) = \frac{2}{k\pi a} \int_0^t \int_0^l f(\xi, \tau) \cdot \sin \frac{k\pi \xi}{l} \cdot e^{-\left(\frac{k\pi a}{l}\right)^2 (t-\tau)} d\xi d\tau.$$

Problema mixtă pentru operatorul undelor şi operatorul căldurii.

Aplicaţia 3.24 Să se rezolve problema Cauchy-Dirichlet:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + x \cdot e^{-t}, \ 0 < x < 1 \\ u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \\ u(0, t) = 0, \ \ u(1, t) = \sin t. \end{cases} \tag{1}$$

Se observă că funcţia $w(x,t) = x \cdot \sin t$ verifică condiţiile la limită.
Facem substituţia $u(x,t) = u^*(x,t) + x \cdot \sin t$ şi înlocuind în problema mixtă de mai sus obţinem următoarea problemă pentru u^*:

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} = 4\frac{\partial^2 u^*}{\partial x^2} + x\left(e^{-t} + \sin t\right), \ 0 < x < 1 \\ u^*(x, 0) = 0, \ \frac{\partial u^*}{\partial t}(x, 0) = -x \\ u^*(0, t) = 0, \ u^*(1, t) = 0. \end{cases} \tag{2}$$

Căutăm soluţie pentru (2) de forma: $u^*(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} = 4\frac{\partial^2 u_h}{\partial x^2}, \ 0 < x < l \\ u_h(x, 0) = 0, \ \frac{\partial u_h}{\partial t}(x, 0) = -x \\ u_h(0, t) = u_h(1, t) = 0 \end{cases} \tag{3}$$

$$\begin{cases} \frac{\partial^2 u_p}{\partial t^2} = 4\frac{\partial^2 u_p}{\partial x^2} + x\left(e^{-t} + \sin t\right),\ 0 < x < l \\ u_p\left(x,0\right) = \frac{\partial u_p}{\partial t}\left(x,0\right) = 0 \\ u_p\left(0,t\right) = u_p\left(1,t\right) = 0. \end{cases} \qquad (4)$$

Problema (3) are ecuaţia omogenă cu soluţia: $(a = 2,\ l = 1)$

$$u_h\left(x,t\right) = \sum_{k=1}^{\infty}\left(a_k \cos 2k\pi t + b_k \sin 2k\pi t\right) \cdot \sin k\pi x.$$

Coeficienţii a_k şi b_k îi găsim din condiţiile iniţiale:

$$u_h\left(x,0\right) = \sum_{k=1}^{\infty} a_k \sin k\pi x = 0 \Rightarrow a_k = 0,\ \ (\forall)\, k \geq 1.$$

$$u_h\left(x,t\right) = \sum_{k=1}^{\infty} b_k \sin 2k\pi t \cdot \sin k\pi x \Rightarrow \frac{\partial u_h}{\partial t}\left(x,0\right) =$$

$$\sum_{k=1}^{\infty} 2k\pi b_k \cdot \sin k\pi x = 2k\pi b_k =$$

$$= -2\int_0^1 x \cdot \sin k\pi x dx \Leftrightarrow b_k = -\frac{1}{k\pi}\int_0^1 x \cdot \sin k\pi x dx =$$

$$\frac{1}{\left(k\pi\right)^2}\int_0^1 x \cdot \left(\cos k\pi x\right)^I dx =$$

$$= \frac{x}{\left(k\pi\right)^2} \cos k\pi x\Big|_0^1 - \frac{1}{\left(k\pi\right)^2}\int_0^1 \underbrace{\cos k\pi x}_{0} dx = \frac{\cos k\pi}{\left(k\pi\right)^2} = \frac{\left(-1\right)^k}{k^2\pi^2}.$$

Deci:

$$\boxed{u_h\left(x,t\right) = \sum_{k=1}^{\infty} \frac{\left(-1\right)^k}{\left(k\pi\right)^2} \sin 2k\pi t \cdot \sin k\pi x},\ 0 < x < 1.$$

Problema (4) are ecuaţia neomogenă cu $f(x,t) = x\left(e^{-t} + \sin t\right)$. Pentru (4) căutăm soluţie de forma:

$$u_p(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin k\pi x,$$

unde:

$$T_k(t) = \frac{2}{2k\pi} \int_0^t \left(\int_0^1 f(\xi,\tau) \cdot \sin k\pi\xi d\xi \right) \sin 2k\pi (t-\tau)\, d\tau =$$

$$= \frac{2}{2k\pi} \int_0^t \underbrace{\left(\int_0^1 \xi \cdot \sin k\pi\xi d\xi \right)}_{\frac{(-1)^{k-1}}{k\pi}} \cdot \left(e^{-\tau} + \sin\tau\right) \sin 2k\pi (t-\tau)\, d\tau =$$

$$= \frac{2(-1)^{k-1}}{2(k\pi)^2} \cdot \left\{ \int_0^t e^{-\tau} \sin 2k\pi (t-\tau)\, d\tau + \right.$$

$$\left. + \int_0^t \sin\tau \cdot \sin 2k\pi (t-\tau)\, d\tau \right\}.$$

$$I_1 = \int_0^t e^{-\tau} \sin 2k\pi (t-\tau)\, d\tau = -\int_0^t \left(e^{-\tau}\right)^I \sin 2k\pi (t-\tau)\, d\tau =$$

$$= -\, e^{-\tau} \sin 2k\pi (t-\tau)\Big|_0^t - 2k\pi \int_0^t e^{-\tau} \cos 2k\pi (t-\tau)\, d\tau$$

$$= \sin 2k\pi t + 2k\pi \int_0^t \left(e^{-\tau}\right)^I \cos 2k\pi (t-\tau)\, d\tau =$$

$$= \sin 2k\pi t + 2k\pi\, e^{-\tau} \cos 2k\pi (t-\tau)\Big|_0^t -$$

$$-(2k\pi)^2 \int_0^t e^{-\tau} \sin 2k\pi (t-\tau)\, d\tau =$$

$$= \sin 2k\pi t + 2k\pi e^{-t} - 2k\pi \cos 2k\pi t - (2k\pi)^2 I_1 \Rightarrow$$

$$I_1 = \frac{1}{1+4k^2\pi^2} \left\{ \sin 2k\pi t + 2k\pi \left(e^{-t} - \cos 2k\pi t \right) \right\}.$$

$$I_2 = \int_0^t \sin \tau \cdot \sin 2k\pi (t-\tau)\, d\tau =$$

$$= \frac{1}{2} \int_0^t \{\cos [2k\pi (t-\tau) - \tau] - \cos [2k\pi (t-\tau) + \tau]\}\, d\tau =$$

$$= \frac{1}{2} \int_0^t \{\cos [2k\pi t - (2k\pi + 1)\tau] - \cos [2k\pi t - (2k\pi - 1)\tau]\}\, d\tau =$$

$$= \frac{-1}{2} \cdot \frac{1}{2k\pi + 1} \cdot \sin [2k\pi t - (2k\pi + 1)\tau] \Big|_0^t +$$

$$+ \frac{1}{2} \cdot \frac{1}{2k\pi - 1} \cdot \sin [2k\pi t - (2k\pi - 1)\tau] \Big|_0^t =$$

$$= \frac{-1}{2(2k\pi + 1)} (-\sin t - \sin 2k\pi t) + \frac{1}{2(2k\pi - 1)} (\sin t - \sin 2k\pi t) =$$

$$= \frac{1}{2} \cdot \sin t \cdot \left(\frac{1}{2k\pi + 1} + \frac{1}{2k\pi - 1} \right) +$$

$$+ \frac{\sin 2k\pi t}{2} \left(\frac{1}{2k\pi + 1} - \frac{1}{2k\pi - 1} \right) =$$

$$= \frac{2k\pi \cdot \sin t - \sin 2k\pi t}{4k^2\pi^2 - 1}.$$

$$T_k(t) = \frac{(-1)^{k-1}}{(k\pi)^2} \cdot \left\{ \frac{1}{4k^2\pi^2 + 1} \left[\sin 2k\pi t + 2k\pi \left(e^{-t} - \cos 2k\pi t \right) \right] + \right.$$

$$\left. + \frac{2k\pi \sin t - \sin 2k\pi t}{4k^2\pi^2 - 1} \right\},\ k \geq 1. \ (5)$$

Deci:

$$u^*(x,t) = u_h(x,t) + u_p(x,t) =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\pi^2} \cdot \sin 2k\pi t \cdot \sin k\pi x + \sum_{k=1}^{\infty} T_k(t) \cdot \sin k\pi x.$$

Soluţia problemei iniţiale este:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\pi^2} \cdot \sin 2k\pi t \cdot \sin k\pi x + \sum_{k=1}^{\infty} T_k(t) \cdot \sin k\pi x + x \sin t,$$

unde $T_k(t)$ este dată de formula (5).

Metoda separării variabilelor

Aplicaţia 3.25

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + x \cdot e^{-t},\ 0 < x < 1 \\ u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0 \\ u(0,t) = 0, \quad u(1,t) = \sin t. \end{cases}$$

Soluţie: $u(x,t) = u^*(x,t) + x \sin t$, unde:

$$\begin{cases} \frac{\partial^2 u^*}{\partial t^2} = 4\frac{\partial^2 u^*}{\partial x^2} + x(e^{-t} + \sin t),\ 0 < x < 1 \\ u^*(x,0) = 0, \frac{\partial u^*}{\partial t}(x,0) = -x \\ u^*(0,t) = u^*(1,t) = 0. \end{cases}$$

Căutăm u^* de forma: $u^*(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} = 4\frac{\partial^2 u_h}{\partial x^2},\ 0 < x < l \\ u_h(x,0) = 0,\ \frac{\partial u_h}{\partial t}(x,0) = -x \\ u_h(0,t) = u_h(1,t) = 0 \end{cases} \qquad (3)$$

$$\left\{\begin{array}{l} \frac{\partial^2 u_p}{\partial t^2} = 4\frac{\partial^2 u_p}{\partial x^2} + x\left(e^{-t} + \sin t\right),\ 0 < x < l \\ u_p\left(x,0\right) = \frac{\partial u_p}{\partial t}\left(x,0\right) = 0 \\ u_p\left(0,t\right) = u_p\left(1,t\right) = 0. \end{array}\right. \qquad (4)$$

Pentru aflarea lui u_h aplicăm metoda separării variabilelor:

$$u_h\left(x,t\right) = f\left(t\right)\cdot v\left(x\right) \Rightarrow$$

$$\left\{\begin{array}{l} f^{II}\left(t\right)\cdot v\left(x\right) = 4f\left(t\right)\cdot v^{II}\left(x\right) \\ v\left(0\right) = v\left(1\right) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \frac{v^{II}(x)}{v(x)} = \frac{f^{II}(t)}{4f(t)} = \lambda \\ v\left(0\right) = v\left(1\right) = 0. \end{array}\right.$$

$$\left\{\begin{array}{l} v^{II}\left(x\right) - \lambda v\left(x\right) = 0 \\ v\left(0\right) = v\left(1\right) = 0 \end{array}\right. \Rightarrow \left.\begin{array}{l} v\left(x\right) = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \\ v\left(0\right) = v\left(1\right) = 0 \end{array}\right\} \Rightarrow$$

$$\left\{\begin{array}{l} c_1 + c_2 = 0 \\ c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} = 0 \end{array}\right. \Leftrightarrow$$

sistemul are soluţie nenulă

$$\Leftrightarrow \left|\begin{array}{cc} 1 & 1 \\ e^{\sqrt{\lambda}} & e^{-\sqrt{\lambda}} \end{array}\right| = 0 \Leftrightarrow e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}} \Leftrightarrow e^{2\sqrt{\lambda}} = 1 \Rightarrow$$

$$2\sqrt{\lambda_k} \in \mathrm{Ln}1 = \left\{i\ \left(0 + 2k\pi\right)\middle|\ k \in \mathbb{Z}\right\} \Rightarrow \lambda_k = -k^2\pi^2,\ k \in \mathbb{N}^*.$$

$$v_k\left(x\right) = a_k e^{ik\pi x} + b_k e^{-ik\pi x};\ v_k\left(0\right) = v_k\left(1\right) = 0;\ v_k\left(0\right) = 0$$

$$\Leftrightarrow a_k + b_k = 0 \Rightarrow b_k = -a_k \Rightarrow$$

$$v_k\left(x\right) = 2ia_k\frac{e^{ik\pi x} - e^{-ik\pi x}}{2i} = \gamma_k\cdot\sin k\pi x.$$

$$\boxed{v_k\left(x\right) = \gamma_k\cdot\sin k\pi x},\ k \geq 1.$$

$$f_k^{II}\left(t\right) + 4k^2\pi^2\cdot f_k\left(t\right) = 0,\ \mathrm{k} \geq 1$$

Ecuaţia caracteristică:

$$r^2 + 4\pi^2 k^2 = 0 \Rightarrow r_{1,2} = \pm i2k\pi \Rightarrow$$

$$\boxed{f_k(t) = a_k \cos 2k\pi t + b_k \sin 2k\pi t},\ k \geq 1.$$

$$u_h^k(x,t) = f_k(t) \cdot v_k(x) = \gamma_k f_k(t) \cdot \sin k\pi x,\ k \geq 1 \Rightarrow$$

căutăm u_h de forma:

$$\boxed{\mathrm{u}_h(x,t) = \sum_{k=1}^{\infty} \gamma_k f_k(t) \cdot \sin k\pi x}.$$

Folosim condiţiile iniţiale pentru a determina u_h.

$$u_h(x,0) = 0 \text{ şi } \frac{\partial u_h}{\partial t}(x,0) = -x \Rightarrow$$

$$u_h(x,0) = 0 \Leftrightarrow \sum_{k=1}^{\infty} \gamma_k f_k(0) \cdot \sin k\pi x = 0 \Rightarrow f_k(0) = 0,\ (\forall)\, k \geq 1.$$

$$\frac{\partial u_h}{\partial t}(x,0) = -x \Leftrightarrow \sum_{k=1}^{\infty} \gamma_k f_k^I(0) \cdot \sin k\pi x = -x \Rightarrow \gamma_k f_k^I(0) =$$

$$= -2\int_0^1 x \sin k\pi x dx = \frac{2(-1)^k}{k\pi}.$$

Deci:

$$\begin{cases} f_k(0) = 0 \\ \gamma_k f_k^I(0) = \frac{2(-1)^k}{k\pi}, \end{cases}$$

$$f_k(t) = \underbrace{a_k \cos 2k\pi t + b_k \sin 2k\pi t}_{\Downarrow}$$

$$\begin{cases} a_k = 0 \\ \gamma_k \cdot b_k \cdot 2k\pi = \frac{2(-1)^k}{k\pi} \end{cases}$$

$$\Rightarrow \boxed{a_k = 0,\ b_k = \frac{1}{\gamma_k} \cdot \frac{(-1)^k}{k^2\pi^2}}$$

$$\Rightarrow \boxed{u_h(x,t) = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2\pi^2} \cdot \sin 2k\pi t \cdot \sin k\pi x,\ 0 < x < 1}.$$

Pentru aflarea lui u_p aplicăm principiul lui Duhamel:

$$u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds,$$

unde:

$$\begin{cases} \frac{\partial^2 \widetilde{u}}{\partial t^2}(x,t,s) = 4\frac{\partial^2 \widetilde{u}}{\partial x^2}(x,t,s),\ 0 < x < 1 \\ \widetilde{u}(x,0,s) = 0,\ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = x\left(e^{-s} + \sin s\right) \\ \widetilde{u}(0,t,s) = \widetilde{u}(1,t,s) = 0. \end{cases}$$

Aplicând tot metoda separării variabilelor $\Rightarrow$

$$\boxed{\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} f_k(t,s) \cdot \sin k\pi x}.$$

$$\boxed{f_k(t,s) = \alpha_k(t) \cdot \beta_k(s)}$$

şi din:

$$\widetilde{u}(x,0,s) = 0,\ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = x\left(e^{-s} + \sin s\right) \Rightarrow$$

$$f_k(0,s) = \alpha_k(0) \cdot \beta_k(s) = 0 \Rightarrow \alpha_k(0) = 0$$

şi

$$\sum_{k=1}^{\infty} \alpha_k(0) \cdot \beta_k(s) \cdot \sin k\pi x = x\left(e^{-s} + \sin s\right)$$

$$\Rightarrow \beta(s) = e^{-s} + \sin s \text{ şi } \alpha_k^I(0) = \frac{2}{k\pi} \cdot (-1)^{k-1},\ k \geq 1.$$

Introducem $\widetilde{u}$ în ecuaţia din sistemul de mai sus:

$$\sum_{k=1}^{\infty} \alpha_k^{II}(t) \cdot \beta(s) \cdot \sin k\pi x = -4k^2\pi^2 \sum_{k=1}^{\infty} \alpha_k(0) \cdot \beta_k(s) \cdot \sin k\pi x \Leftrightarrow$$

$$\begin{cases} \alpha_k^{II}(t) + 4k^2\pi^2\alpha_k(t) = 0 \\ \alpha_k(0) = 0, \ \alpha_k^{I}(t) = \frac{2}{k\pi}(-1)^{k-1} \end{cases} \Rightarrow$$

$$\begin{cases} \alpha_k(t) = a_k \cos 2k\pi t + b_k \sin 2k\pi t \\ \alpha_k(0) = 0 \Rightarrow a_k = 0; \ 2k\pi b_k = \frac{2}{k\pi}(-1)^{k-1} \Rightarrow b_k = \frac{(-1)^{k-1}}{k^2\pi^2}. \end{cases}$$

$$\boxed{\alpha_k(t) = \frac{(-1)^{k-1}}{k^2\pi^2} \cdot \sin 2k\pi t}$$

$$f_k(t,s) = \frac{(-1)^{k-1}}{k^2\pi^2} \cdot \sin 2k\pi t \cdot \left(e^{-s} + \sin s\right) \Rightarrow$$

$$\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2\pi^2} \cdot \sin 2k\pi t \cdot \left(e^{-s} + \sin s\right) \cdot \sin k\pi x \Rightarrow$$

$$u_p = \int_0^t \widetilde{u}(x, t-s, s)\, ds =$$

$$= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2\pi^2} \cdot \left[\int_0^t \sin 2k\pi(t-s) \cdot \left(e^{-s} + \sin s\right) ds\right] \sin k\pi x.$$

Observăm că

$$\frac{(-1)^{k-1}}{k^2\pi^2} \cdot \int_0^t \sin 2k\pi(t-s) \cdot \left(e^{-s} + \sin s\right) ds = T_k(t) \Rightarrow$$

$$\frac{(-1)^{k-1}}{k^2\pi^2} \cdot \int_0^t \sin 2k\pi(t-s) \cdot \left(e^{-s} + \sin s\right) ds =$$

$$= \frac{(-1)^{k-1}}{(k\pi)^2} \cdot \left\{ \frac{1}{4k^2\pi^2 + 1} \left[\sin 2k\pi t + \right.\right.$$

$$\left. + 2k\pi\left(e^{-t} - \cos 2k\pi t\right)\right] + \frac{2k\pi\sin t - \sin 2k\pi t}{4k^2\pi^2 - 1}\Bigg\},\ \text{k} \geq 1 \Rightarrow$$

$$u(x,t) = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^2\pi^2}\sin 2k\pi t \cdot \sin k\pi x + \sum_{k=1}^{\infty}T_k(t)\sin k\pi x + x\sin t.$$

Aplicaţia 3.26

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} + t^2 + t + 1,\ 0 < x < 1 \\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = \frac{1}{1+x} \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Căutăm soluţie de forma: $u(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$I.\begin{cases} \frac{\partial^2 u_h}{\partial t^2} = 9\frac{\partial^2 u_h}{\partial x^2},\ 0 < x < l \\ u_h(x,0) = 0,\ \frac{\partial u_h}{\partial t}(x,0) = \frac{1}{1+x} \\ u_h(0,t) = u_h(1,t) = 0 \end{cases}$$

$$II.\begin{cases} \frac{\partial^2 u_p}{\partial t^2} = 9\frac{\partial^2 u_p}{\partial x^2} + \overbrace{t^2 + t + 1}^{f(x,t)},\ 0 < x < l \\ u_p(x,0) = \frac{\partial u_p}{\partial t}(x,0) = 0 \\ u_p(0,t) = u_p(1,t) = 0. \end{cases}$$

$$a = 3,\ l = 1 \Rightarrow \mathrm{u}_h(x,t) = \sum_{k=1}^{\infty}(a_k\cos 3k\pi t + b_k\sin 3k\pi t)\sin k\pi x.$$

$$u_h(x,0) = \sum_{k=1}^{\infty}a_k\sin k\pi x = 0 \Rightarrow a_k = 0,\ \ (\forall)\,k \geq 1.$$

$$u_h(x,t) = \sum_{k=1}^{\infty}b_k\sin 3k\pi t \cdot \sin k\pi x;$$

$$\frac{\partial u_h}{\partial t}(x,0) = \sum_{k=1}^{\infty}3k\pi b_k\sin k\pi x = \frac{1}{1+x} \Rightarrow$$

$$b_k = \frac{2}{3k\pi}\int_0^1 \frac{\sin k\pi x}{1+x}dx,\ k \geq 1.$$

$$u_h(x,t) = \sum_{k=1}^{\infty}\frac{2}{3k\pi}\left(\int_0^1 \frac{\sin k\pi\xi}{1+\xi}d\xi\right)\cdot \sin 3k\pi t \cdot \sin k\pi x.$$

$$\cdot u_p(x,t) = \sum_{k=1}^{\infty} T_k(t)\cdot \sin k\pi x,$$

unde:

$$T_k(t) = \frac{2}{3k\pi}\int_0^t \left(\int_0^1 f(\xi,\tau)\cdot \sin k\pi\xi d\xi\right).$$

$$\cdot\ \mathrm{sin3k}\pi\left(t-\tau\right),\ \text{unde}: f(\xi,\tau) = \tau^2+\tau+1.$$

$$T_k(t) = \frac{2}{3k\pi}\int_0^t \left(\tau^2+\tau+1\right)\sin 3k\pi(t-\tau)\underbrace{\left(\int_0^1 \sin k\pi\xi d\xi\right)}_{\frac{-\cos k\pi\xi}{k\pi}\Big|_0^1 = \frac{1-(-1)^k}{k\pi}} d\tau =$$

$$= \frac{2}{3(k\pi)^2}\left[1-(-1)^k\right]\cdot\int_0^t \left(\tau^2+\tau+1\right)\sin 3k\pi(t-\tau)\,d\tau =$$

$$= \frac{2}{k\pi}\left[1-(-1)^k\right]\cdot\frac{1}{3k\pi}\cdot\int_0^t \left(\tau^2+\tau+1\right)\cdot\left[\cos 3k\pi(t-\tau)\right]^I d\tau =$$

$$= \frac{2}{3^2(k\pi)^3}\left[1-(-1)^k\right]\cdot\Big[\left(\tau^2+\tau+1\right)\cos 3k\pi(t-\tau)\Big|_0^t -$$

$$-\int_0^t (2\tau+1)\cdot\cos 3k\pi(t-\tau)\,d\tau\Big] =$$

$$= \frac{2}{3^2}\cdot\frac{1-(-1)^k}{k^3\pi^3}\Big[t^2+t+1-\cos 3k\pi t+\frac{1}{3k\pi}\cdot\int_0^t (2\tau+1)\cdot$$

$$\cdot\left[\sin 2k\pi(t-\tau)\right]^I d\tau\Big] =$$

$$= \frac{2}{3^2} \cdot \frac{1-(-1)^k}{k^3\pi^3} \cdot \left[t^2 + t + 1 - \cos 3k\pi t + \frac{1}{3k\pi}(2\tau+1) \cdot \right.$$

$$\left. \cdot \sin 3k\pi\,(t-\tau)\big|_0^t - \frac{2}{3k\pi}\int_0^t \sin 3k\pi\,(t-\tau)\,d\tau\right] =$$

$$= \frac{2}{3^2} \cdot \frac{1-(-1)^k}{k^3\pi^3} \cdot \left[t^2 + t + 1 - \cos 3k\pi t - \frac{1}{3k\pi}\sin 3k\pi t - \right.$$

$$\left. - \frac{2}{(3k\pi)^2}\int_0^t \left[\cos 3k\pi\,(t-\tau)\right]^I d\tau\right] =$$

$$= \frac{2}{3^2} \cdot \frac{1-(-1)^k}{k^3\pi^3} \cdot \left[t^2 + t + 1 - \cos 3k\pi t - \frac{\sin 3k\pi t}{3k\pi} - \right.$$

$$\left. - \frac{2}{(3k\pi)^2} + \frac{2}{(3k\pi)^2}\cos 3k\pi t\right].$$

Deci:

$$T_k(t) = \frac{2}{3^2} \cdot \frac{1-(-1)^k}{k^3\pi^3} \cdot$$

$$\cdot \left[t^2 + t + 1 - \cos 3k\pi t - \frac{\sin 3k\pi t}{3k\pi} - \frac{2}{(3k\pi)^2} + \frac{2}{(3k\pi)^2}\cos 3k\pi t\right] =$$

$$= \frac{2}{3^2} \cdot \frac{1-(-1)^k}{k^3\pi^3} \cdot \left[t^2 + t + 1 + \right.$$

$$\left. + \left(1 - \frac{2}{(3k\pi)^2}\right)(1 - \cos 3k\pi t) - \frac{\sin 3k\pi t}{3k\pi}\right]. \quad \text{III.}$$

$$u(x,t) = \sum_{k=1}^{\infty} \frac{2}{3k\pi}\left(\int_0^1 \frac{\sin k\pi\xi}{1+\xi}d\xi\right)\sin 3k\pi t \cdot \sin k\pi x +$$

$$+ \frac{2}{3^2}\sum_{k=1}^{\infty}\frac{1-(-1)^k}{k^3\pi^3} \cdot \left[t^2 + t + 1 + \right.$$

$$+\left(1-\frac{2}{(3k\pi)^2}\right)(1-\cos 3k\pi t)-\frac{\sin 3k\pi t}{3k\pi}\Bigg]\cdot \sin k\pi x.$$

Altfel: pentru aflarea lui u_p aplicăm principiul lui Duhamel:

$$u_p\left(x,t\right)=\int_0^t \widetilde{u}\left(x,t-s,s\right)ds,\ \text{unde :}$$
$$\begin{cases} \frac{\partial^2\widetilde{u}}{\partial t^2}\left(x,t,s\right)=9\frac{\partial^2\widetilde{u}}{\partial x^2}\left(x,t,s\right),\ 0<x<1 \\ \widetilde{u}\left(x,0,s\right)=0,\ \frac{\partial\widetilde{u}}{\partial t}\left(x,0,s\right)=s^2+s+1 \\ \widetilde{u}\left(0,t,s\right)=\widetilde{u}\left(1,t,s\right)=0. \end{cases}$$

Problema lui $\widetilde{u}$ este similară problemei lui $u_h \Rightarrow$

$$\widetilde{u}\left(x,t,s\right)=\sum_{k=1}^{\infty} f_k\left(t,s\right)\cdot \mathrm{sink\pi x},\quad f_k\left(t,s\right)=\alpha_k\left(t\right)\cdot\beta_k\left(s\right).$$

Din condiţiile iniţiale $\Rightarrow \beta\left(s\right)=s^2+s+1$.

$$\alpha_k\left(0\right)=0;$$

$$\alpha_k^I\left(0\right)=2\int_0^1 sink\pi xdx=-\frac{2}{k\pi}\left[\left(-1\right)^k-1\right]=\frac{2}{k\pi}\left[1-\left(-1\right)^k\right],$$

deoarece:

$$\frac{\partial\widetilde{u}}{\partial t}\left(x,0,s\right)=\sum_{k=1}^{\infty}\alpha_k^I\left(t\right)\left(s^2+s+1\right)\cdot\sin k\pi x=s^2+s+1\Rightarrow$$

$$\alpha_k^I\left(0\right)=2\int_0^1 sink\pi xdx=\frac{2}{k\pi}\left[1-\left(-1\right)^k\right].$$

Înlocuind

$$\widetilde{u}\left(x,t,s\right)=\sum_{k=1}^{\infty}\alpha_k\left(t\right)\cdot\left(s^2+s+1\right)\sin k\pi x$$

în ecuaţia din sistem $\Rightarrow$

$$\left\{\begin{array}{l} \alpha_k^{II}(t) + 9k^2\pi^2\alpha_k(t) = 0 \\ \alpha_k(0) = 0,\ \alpha_k^I(0) = \frac{2}{k\pi}\left[1 - (-1)^k\right] \end{array}\right. \Rightarrow$$

$$\left\{\begin{array}{l} \alpha_k(t) = a_k \cos 3k\pi t + b_k \sin 3k\pi t \\ \alpha_k(0) = 0 \Rightarrow a_k = 0 \\ \alpha_k^I(0) = \frac{2}{k\pi}\left[1 - (-1)^k\right] \Rightarrow 3k\pi b_k = \frac{2}{k\pi}\left[1 - (-1)^k\right] \end{array}\right.$$

$$\Rightarrow \boxed{b_k = \frac{2}{3(k\pi)^2}\left[1 - (-1)^k\right]} \Rightarrow$$

$$\boxed{\alpha_k(t) = \frac{2}{3(k\pi)^2}\left[1 - (-1)^k\right] \cdot \sin 3k\pi t} \Rightarrow$$

$$\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} \frac{2}{3(k\pi)^2}\left[1 - (-1)^k\right] \cdot \left(s^2 + s + 1\right) \cdot \sin 3k\pi t \cdot \sin k\pi x$$

$$\Rightarrow u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds =$$

$$= \sum_{k=1}^{\infty} \underbrace{\frac{2}{3(k\pi)^2}\left[1 - (-1)^k\right] \int_0^t \left(s^2 + s + 1\right) \cdot \sin 3k\pi(t-s)\, ds}_{T_k(t)} \cdot \sin k\pi x.$$

Aplicaţia 3.27

$$\left\{\begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \mathrm{sh} t,\ 0 < x < 1 \\ u(x,0) = 0 \\ u(0,t) = -t,\ u(1,t) = t. \end{array}\right.$$

Funcţia $w(x,t) = -t + x(t - (-t)) = 2xt - t = (2x-1)t$ satisface condiţiile la limită.

Facem substituţia $u\left(x,t\right)=u^{*}\left(x,t\right)+\left(2x-1\right)t$ care înlocuită în problema Cauchy-Dirichlet ne dă următoarea problemă:

$$\begin{cases} \frac{\partial u^{*}}{\partial t}=\frac{\partial^{2}u^{*}}{\partial x^{2}}-2x+1+\mathrm{sh}t,\ 0<x<1\\ u^{*}\left(x,0\right)=0\\ u^{*}\left(0,t\right)=u^{*}\left(1,t\right)=0. \end{cases}$$

Aceasta este o problemă Cauchy-Dirichlet cu ecuaţie neomogenă şi condiţia iniţială plus condiţiile la limită nule. Avem cazul general:

$$\begin{cases} \frac{\partial u}{\partial t}=a^{2}\frac{\partial^{2}u}{\partial x^{2}}+f\left(x,t\right),\ 0<x<l\\ u\left(x,0\right)=0\\ u\left(0,t\right)=u\left(l,t\right)=0. \end{cases}$$

$$u\left(x,t\right)=\sum_{k=1}^{\infty}T_{k}\left(t\right)\cdot\sin\frac{k\pi x}{l},$$

unde:

$$T_{k}\left(t\right)=\frac{2}{k\pi l}\int_{0}^{t}\left(\int_{0}^{l}f\left(\xi,\tau\right)\sin\frac{k\pi\xi}{l}d\xi\right)\cdot e^{-\left(\frac{k\pi a}{l}\right)^{2}\cdot(t-\tau)}d\tau.$$

Deci, în cazul nostru $a=1,\ \mathrm{l}=1$ şi rezultă:

$$u^{*}\left(x,t\right)=\sum_{k=1}^{\infty}T_{k}\left(t\right)\cdot\sin k\pi x;$$

$$T_{k}\left(t\right)=2\int_{0}^{t}\left(\int_{0}^{1}\left(\mathrm{sh}\tau-2\xi+1\right)\cdot\sin k\pi\xi d\xi\right)\cdot e^{-(k\pi)^{2}\cdot(t-\tau)}d\tau.$$

Calculăm:

$$\int_{0}^{1}\left(\mathrm{sh}\tau-2\xi+1\right)\cdot\sin k\pi\xi d\xi=$$

$$=\frac{-1}{k\pi}\int_{0}^{1}\left(\mathrm{sh}\tau-2\xi+1\right)\cdot\left(\cos k\pi\xi\right)^{I}d\xi=$$

$$= \frac{-1}{k\pi}\left(\mathrm{sh}\tau - 2\xi + 1\right)\cos k\pi\xi\Big|_0^1 - \frac{2}{k\pi}\underbrace{\int_0^1 \cos k\pi\xi d\xi}_{0} =$$

$$= \frac{(-1)^{k-1}}{k\pi}\left(\mathrm{sh}\tau - 1\right) + \frac{1}{k\pi}\left(\mathrm{sh}\tau + 1\right)\cdot \int_0^1 \left(\mathrm{sh}\tau - 2\xi + 1\right)\cdot \sin k\pi\xi d\xi =$$

$$= \begin{cases} \frac{1}{n\pi} & ,k = 2n, \\ \frac{2\mathrm{sh}\tau}{(2n-1)\pi} & ,k = 2n-1, \end{cases} \quad n \geq 1.$$

Calculăm:

$$T_{2n}\left(t\right) = 2\int_0^t \frac{1}{n\pi}\mathrm{e}^{-(2n\pi)^2(t-s)}ds = \frac{2}{n\pi}\int_0^t \mathrm{e}^{-(2n\pi)^2(t-s)}ds = ...$$

$$T_{2n-1}\left(t\right) = \frac{4}{\left(2n-1\right)\pi}\int_0^t \mathrm{sh}\tau \cdot e^{-\left[(2n-1)\pi\right]^2(t-\tau)}d\tau \Rightarrow$$

$$u^*\left(x,t\right) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left\{\int_0^t e^{-(2n\pi)^2(t-\tau)}d\tau\right\}\cdot \sin 2n\pi x +$$

$$+ \sum_{n=1}^{\infty}\frac{4}{\left(2n-1\right)\pi}\left\{\int_0^t \mathrm{sh}\tau \cdot e^{-\left[(2n-1)\pi\right]^2(t-\tau)}d\tau\right\}\cdot \sin\left(2n-1\right)\pi x.$$

$$* \int_0^t e^{-(2n\pi)^2(t-\tau)}d\tau = \frac{1}{\left(2n\pi\right)^2}\cdot e^{-(2n\pi)^2(t-\tau)}\Big|_0^t = \frac{1 - e^{-4n^2\pi^2 t}}{4n^2\pi^2}.$$

$$** I = \int_0^t \mathrm{sh}\tau \cdot e^{-\left[(2n-1)\pi\right]^2(t-\tau)}d\tau =$$

$$= \frac{1}{\left[\left(2n-1\right)\pi\right]^2}\cdot \mathrm{sh}\tau \cdot e^{-\left[(2n-1)\pi\right]^2(t-\tau)}\Big|_0^t -$$

$$- \frac{1}{\left[\left(2n-1\right)\pi\right]^2}\cdot \int_0^t \mathrm{ch}\tau \cdot e^{-\left[(2n-1)\pi\right]^2(t-\tau)}d\tau =$$

$$= \frac{\mathrm{sh}t}{[(2n-1)\pi]^2} - \frac{1}{[(2n-1)\pi]^4} \cdot \mathrm{ch}\tau \cdot e^{-[(2n-1)\pi]^2(t-\tau)}\Big|_0^t +$$

$$+\frac{1}{[(2n-1)\pi]^4} \cdot I \Rightarrow$$

$$I \cdot \left(1 + \frac{1}{[(2n-1)\pi]^4}\right) =$$

$$= \frac{\mathrm{sh}t}{[(2n-1)\pi]^2} - \frac{\mathrm{ch}t}{[(2n-1)\pi]^4} + \frac{e^{-[(2n-1)\pi]^2 t}}{[(2n-1)\pi]^4} \Rightarrow$$

$$I = \frac{1}{[(2n-1)\pi]^4 + 1} \left\{ \mathrm{sh}t\,[(2n-1)\pi]^2 + e^{-[(2n-1)\pi]^2 t} - \mathrm{ch}t \right\} \Rightarrow$$

$$u^*(x,t) = \sum_{n=1}^{\infty} \frac{1}{2n^3\pi^3} \cdot \left(1 - e^{-4\pi^2 n^2 t}\right) \cdot \sin 2n\pi x +$$

$$+\sum_{n=1}^{\infty} \frac{4\left[((2n-1)\pi)^2 \cdot \mathrm{sh}t - \mathrm{ch}t + e^{-[(2n-1)\pi]^2 t}\right]}{(2n-1)\pi\left[1 + ((2n-1)\pi)^4\right]} \cdot \sin(2n-1)\pi x.$$

Altfel, calculăm u^* cu principiul lui Duhamel:

$$u^*(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds,$$

unde:

$$\begin{cases} \frac{\partial \widetilde{u}}{\partial t}(x,t,s) = \frac{\partial^2 \widetilde{u}}{\partial x^2}(x,t,s), \quad 0 < x < 1 \\ \widetilde{u}(x,0,s) = -2x + 1 + \mathrm{sh}s \\ \widetilde{u}(0,t,s) = \widetilde{u}(1,t,s) = 0. \end{cases}$$

Aplicăm metoda separării variabilelor: $\widetilde{u}(x,t,s) = f_k(t,s) \cdot v(x) \Rightarrow$

$$\begin{cases} \frac{\partial f}{\partial t}(t,s)\cdot v(x) = f(t,s)\cdot v^{II}(x) \\ v(0) = v(1) = 0 \end{cases} \Rightarrow \frac{v^{II}(x)}{v(x)} = \lambda \Rightarrow$$

$$\begin{cases} v^{II}(x) = \lambda v(x) \\ v(0) = v(1) = 0 \end{cases} \Leftrightarrow$$

$$\begin{cases} \boxed{v_k(x) = \sin k\pi x}, k \geq 1 \\ \lambda_k = -(k\pi)^2 \qquad , k \geq 1. \end{cases}$$

Deci:

$$\widetilde{u_k}(x,t,s) = f_k(t,s)\cdot \sin k\pi x \Rightarrow$$

$$\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} f_k(t,s)\cdot \sin k\pi x$$

$$\widetilde{u}(x,0,s) = \sum_{k=1}^{\infty} f_k(0,s)\cdot \sin k\pi x = -2x+1+\mathrm{sh}s \Rightarrow$$

$$\cdot f_k(0,s) = 2\int_0^1 (-2x+1+\mathrm{sh}s)\cdot \sin k\pi x ds =$$

$$= \begin{cases} \frac{2}{n\pi} & ,k = 2n, \\ \frac{2\cdot 2}{(2n-1)\pi}\cdot \mathrm{sh}s & ,k = 2n-1, \end{cases} \quad n \geq 1.$$

Dar:

$$f_k(t,s) = \alpha_k(t)\cdot \beta_k(s) \Rightarrow$$

$$\Rightarrow \boxed{\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} \alpha_k(t)\cdot \beta_k(s)\cdot \sin k\pi x}$$

$$f_{2n}(0,s) = \alpha_{2n}(0)\cdot\beta_{2n}(s) = \frac{2}{n\pi} \Rightarrow \alpha_{2n}(0) = \frac{2}{n\pi}, \ \boxed{\beta_{2n}(s) = 1}.$$

$$f_{2n-1}(0,s) = \alpha_{2n-1}(0)\cdot \beta_{2n-1}(s) = \frac{2\cdot 2}{(2n-1)\pi}\cdot \mathrm{sh}s \Rightarrow$$

$$\alpha_{2n}(0) = \frac{2 \cdot 2}{(2n-1)\pi}, \ \boxed{\beta_{2n-1}(s) = \text{sh}s}.$$

Ecuaţia lui α_k este:

$$\alpha_k^I(t) + (k\pi)^2 \alpha_k(t) = 0 \Rightarrow \boxed{\alpha_k(t) = c_k \cdot e^{-(k\pi)^2 t}}$$

$$\cdot\alpha_{2n}(0) = \frac{2}{n\pi} \Rightarrow \boxed{c_{2n} = \frac{2}{n\pi}}; \quad \alpha_{2n-1}(0) = \boxed{\frac{2 \cdot 2}{(2n-1)\pi} = c_{2n-1}}.$$

$$\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} \frac{4}{(2n-1)\pi} \cdot \text{sh}s \cdot e^{-[(2n-1)\pi]^2 t} \cdot \sin(2n-1)\pi x +$$

$$+ \sum_{k=1}^{\infty} \frac{2}{n\pi} \cdot e^{-(2n\pi)^2 t} \cdot \sin 2n\pi x.$$

$$\Rightarrow u^*(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds =$$

$$= \sum_{n=1}^{\infty} \frac{2 \cdot 2}{(2n-1)\pi} \left\{ \int_0^t \text{sh}s \cdot e^{-[(2n-1)\pi]^2 (t-s)} ds \right\} \cdot \sin(2n-1)\pi x +$$

$$+ \sum_{n=1}^{\infty} \frac{2}{n\pi} \left\{ \int_0^t e^{-(2n\pi)^2 (t-s)} ds \right\} \cdot \sin 2n\pi x = \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots\ldots}_{\substack{\text{se calculează prima} \\ \text{integrală prin părţi}}}$$

şi cu principiul lui Duhamel am ajuns la aceeaşi formulă pentru u^*.

Aplicaţia 3.28

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} + e^{-2t} \cdot \sin \pi x, \ 0 < x < 1 \\ u(x,0) = \sin 2\pi x + 3\sin 3\pi x \\ \frac{\partial u}{\partial t}(x,0) = 2\sin 2\pi x + \sin 3\pi x \\ u(0,t) = u(1,t) = 0. \end{cases}$$

Căutăm soluţie de forma: $u(x,t) = u_h(x,t) + u_p(x,t)$ unde:

$$\begin{cases} \frac{\partial^2 u_h}{\partial t^2} = 9\frac{\partial^2 u_h}{\partial x^2},\ 0 < x < 1 \\ u_h(x,0) = \sin 2\pi x + 3\sin 3\pi x \\ \frac{\partial u_h}{\partial t}(x,0) = 2\sin 2\pi x + \sin 3\pi x \\ u_h(0,t) = u_h(1,t) = 0 \end{cases}$$

$$\begin{cases} \frac{\partial^2 u_p}{\partial t^2} = 9\frac{\partial^2 u_p}{\partial x^2} + e^{-2t}\cdot\sin\pi x \\ u_p(x,0) = \frac{\partial u_p}{\partial t}(x,0) = 0 \\ u_p(0,t) = u_p(1,t) = 0. \end{cases}$$

$$u_h(x,t) = \sum_{k=1}^{\infty}(a_k\cos 3k\pi t + b_k\sin 3k\pi t)\cdot\sin k\pi x.$$

$$\begin{cases} u_h(x,0) = \sum_{k=1}^{\infty} a_k\sin k\pi x = \sin 2\pi x + 3\sin 3\pi x \\ \frac{\partial u}{\partial t}(x,0) = \sum_{k=1}^{\infty} 3k\pi b_k\sin k\pi x = 2\sin 2\pi x + \sin 3\pi x \end{cases} \Rightarrow$$

$$\begin{cases} a_2 = 1, \quad a_3 = 3 \\ 3\cdot 2\pi b_2 = 2,\ 9\pi b_3 = 1 \end{cases} \Rightarrow$$

$$\begin{cases} a_2 = 1, \quad a_3 = 3 \\ b_2 = \frac{1}{3\pi},\ b_3 = \frac{1}{9\pi} \end{cases} \Rightarrow \begin{cases} a_k = 0, \quad (\forall)\, k \neq \{2,3\} \\ b_k = 0, \quad (\forall)\, k \neq \{2,3\} \end{cases} \Rightarrow$$

$$\cdot u_h(x,t) = \left(\cos 6\pi t + \frac{1}{3\pi}\sin 6\pi t\right)\cdot\sin 2\pi x +$$
$$+\left(3\cos 9\pi t + \frac{1}{9\pi}\sin 9\pi t\right)\cdot\sin 3\pi x$$

$$\cdot u_h(x,t) = \left(\cos 6\pi t + \frac{1}{3\pi}\sin 6\pi t\right)\sin 2\pi x +$$
$$+\left(3\cos 9\pi t + \frac{1}{9\pi}\sin 9\pi t\right)\sin 3\pi x.$$

$$\cdot u_p(x,t) = \sum_{k=1}^{\infty} T_k(t)\sin k\pi x,$$

unde:

$$T_k(t) = \frac{2}{3k\pi}\int_0^t \left(\int_0^1 f(\xi,\tau)\sin k\pi\xi d\xi\right)\sin 3k\pi(t-\tau)\,d\tau,$$

unde:

$$\int_0^1 \sin\pi\xi\cdot\sin k\pi\xi d\xi = \begin{cases} 0 & ,k\geq 2 \\ \frac{1}{2} & ,k=1 \end{cases}$$

$$T_k(t) \equiv 0,\ \ (\forall)\,k\geq 2.$$

$$T_1(t) = \frac{1}{3\pi}\int_0^t e^{-2\tau}\sin 3\pi(t-\tau)\,d\tau =$$

$$= \frac{-1}{3\pi}\int_0^t \left(e^{-2\tau}\right)^I \sin 3\pi(t-\tau)\,d\tau =$$

$$= \frac{-1}{3\pi}\cdot e^{-2\tau}\cdot\sin 3\pi(t-\tau)\Big|_0^t - \frac{3\pi}{3\pi}\int_0^t e^{-2\tau}\cos 3\pi(t-\tau)\,\tau =$$

$$= \frac{1}{\pi}\sin 3\pi t + \frac{3\pi}{\pi}\int_0^t \left(e^{-2\tau}\right)\cos 3\pi(t-\tau)\,d\tau =$$

$$= \frac{1}{\pi}\sin 3\pi t + \frac{3\pi}{\pi}\cdot e^{-2\tau}\cdot\cos 3\pi(t-\tau)\Big|_0^t -$$

$$-\frac{9\pi^2}{\pi}\int_0^t e^{-2\tau}\sin 3\pi(t-\tau)\,d\tau =$$

$$= \frac{2}{\pi}\sin 3\pi t + \frac{3\pi}{3\pi}e^{-2t} - \frac{3\pi}{3\pi}\cos 3\pi t - \frac{9\pi^2}{\pi}\cdot T_1(t) \Rightarrow$$

$$(4+9\pi)\cdot T_1(t) = \frac{2}{\pi}\sin 3\pi t + \frac{3\pi}{3\pi}\left(e^{-t} - \cos 3\pi t\right) \Rightarrow$$

$$T_1(t) = \frac{1}{4+9\pi}\left[\frac{2}{3\pi}\sin 3\pi t + \frac{3\pi}{3\pi}\left(e^{-2t} - \cos 3\pi t\right)\right].$$

$$u_p(x,t) = T_1(t) \cdot \sin \pi x =$$
$$= \frac{1}{4+9\pi^2}\left[\frac{2}{3\pi}\sin 3\pi t + \frac{3\pi}{3\pi}\left(e^{-2t} - \cos 3\pi t\right)\right] \cdot \sin \pi x =$$
$$= \frac{1}{4+9\pi^2}\left[\frac{2}{3\pi}\sin 3\pi t + e^{-2t} - \cos 3\pi t\right] \cdot \sin \pi x.$$

Deci:

$$u(x,t) = \left(\cos 6\pi t + \frac{1}{3\pi}\sin 6\pi t\right) \cdot \sin 2\pi x +$$
$$+\left(3\cos 9\pi t + \frac{1}{9\pi}\sin 9\pi t\right)\sin 3\pi x +$$
$$+\frac{1}{4+9\pi}\left(\frac{2}{3\pi}\sin 3\pi t + e^{-2t} - \cos 3\pi t\right)\sin \pi x.$$

Altfel, aflăm u_p cu principiul lui Duhamel:

$$\begin{cases} \frac{\partial^2 \widetilde{u}}{\partial t^2}(x,t,s) = 9\frac{\partial^2 \widetilde{u}}{\partial x^2}(x,t,s), \ 0 < x < 1 \\ \widetilde{u}(x,0,s) = 0, \ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = e^{-2s}\sin \pi x \\ \widetilde{u}(0,t,s) = \widetilde{u}(1,t,s) = 0. \end{cases}$$

$$\widetilde{u}(x,t,s) = \sum_{k=1}^{\infty} f_k(t,s) \cdot \sin k\pi x.$$

Căutăm

$$f_k(t,s) = \alpha_k(t) \cdot \beta_k(s).$$

Din condiţiile iniţiale:

$$\begin{cases} \widetilde{u}(x,0,s) = 0 \Rightarrow \alpha_k(0) = 0 \\ \frac{\partial \widetilde{u}}{\partial t}(x,0,s) = \sum_{k=1}^{\infty} \alpha_k^I(0) \cdot \beta_k(s) \cdot \sin k\pi dx = e^{-2s} \cdot \sin \pi x \end{cases} \Rightarrow$$

$$\boxed{\beta_k(s) = e^{-2s}}$$

$$\alpha_1^I(0) = 2\int_0^1 \sin^2 \pi x dx = 1. \quad \alpha_k^I(0) = 0, \ \ (\forall)\, k \geq 2.$$

Ecuaţia lui $\alpha_1(t)$ o obţinem înlocuind

$$\widetilde{u}(x,t,s) = \alpha_1(t) \cdot e^{-2s} \cdot \sin \pi x$$

în ecuaţia sistemului:

$$\begin{cases} \alpha_1^{II}(t) + 9\pi^2 \alpha_1(t) = 0 \\ \alpha_1(0) = 0, \ \alpha_1^I(0) = 1 \end{cases} \Rightarrow \begin{cases} \alpha_1(t) = a_1 \cos 3\pi t + b_1 \sin 3\pi t \\ \alpha_1(0) = 0 \Rightarrow a_1 = 0 \\ \alpha_1^I(t) = 3\pi \cdot b_1 \cos 3\pi t \\ \alpha_1^I(0) = 1 \Rightarrow b_1 = \frac{1}{3\pi}. \end{cases}$$

$$\Rightarrow \boxed{\alpha_1(t) = \frac{1}{3\pi} \cdot \sin 3\pi t}$$

$$\widetilde{u}(x,t,s) = \frac{1}{3\pi} \sin 3\pi t \cdot e^{-2s} \cdot \sin \pi x \Rightarrow$$

$$u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds =$$

$$= \frac{1}{3\pi} \sin \pi x \int_0^t e^{-2s} \cdot \sin 3\pi (t-s)\, ds.$$

$$I = \int_0^t e^{-2s} \cdot \sin 3\pi (t-s)\, ds =$$

$$= \frac{1}{3\pi} \cdot e^{-2s} \cdot \cos 3\pi (t-s) \Big|_0^t + \frac{2}{3\pi} \int_0^t e^{-2s} \cdot \cos 3\pi (t-s)\, ds =$$

$$= \frac{e^{-2t}}{3\pi} - \frac{\cos 3\pi t}{3\pi} - \frac{2}{(3\pi)^2} \int_0^t e^{-2s} \left[\sin 3\pi (t-s)\right]^I ds =$$

$$= \frac{e^{-2t}}{3\pi} - \frac{\cos 3\pi t}{3\pi} - \frac{2}{9\pi^2} \cdot e^{-2s} \cdot \sin 3\pi (t-s) \Big|_0^t -$$

$$-\frac{4}{9\pi^2}\int_0^t e^{-2s}\cdot\sin 3\pi\,(t-s)\,ds \Rightarrow$$

$$I\left(1+\frac{4}{9\pi^2}\right)=\frac{e^{-2t}}{3\pi}-\frac{\cos 3\pi t}{3\pi}+\frac{2}{9\pi^2}\sin 3\pi t \Rightarrow$$

$$I=\int_0^t e^{-2s}\cdot\sin 3\pi\,(t-s)\,ds=$$

$$=\frac{1}{9\pi^2+4}\left\{3\pi e^{-2t}-3\pi\cos 3\pi t+2\sin 3\pi t\right\}.$$

$$\cdot u_p\,(x,t)=\frac{\sin\pi x}{4+9\pi^2}\left[e^{-2t}-\cos 3\pi t+\frac{2}{3\pi}\sin 3\pi t\right].$$

Aplicaţia 3.29

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}=\frac{1}{4}\frac{\partial^2 u}{\partial x^2}.\ 0<x<1 \\ u\,(x,0)=x\,(1-x)\,,\ \frac{\partial u}{\partial t}\,(x,0)=1 \\ u\,(0,t)=u\,(1,t)-0. \end{cases}$$

Avem problema omegenă. Deci: $\left(a=\frac{1}{2},\ l=1\right)$

$$u\,(x,t)=\sum_{k=1}^{\infty}\left(a_k\cos\frac{k\pi t}{2}+b_k\sin\frac{k\pi t}{2}\right)\cdot\sin k\pi x.$$

$$\begin{cases} u\,(x,0)=\sum_{k=1}^{\infty} a_k\cdot\sin k\pi x=x-x^2 \\ \frac{\partial u}{\partial t}\,(x,0)=\sum_{k=1}^{\infty}\frac{k\pi}{2}b_k\cdot\sin k\pi x=1 \end{cases}\Rightarrow$$

$$\begin{cases} a_k=2\int_0^1\left(x-x^2\right)\sin k\pi x dx, \\ b_k=\frac{2}{k\pi}\cdot 2\int_0^1\sin k\pi x dx, \end{cases}\qquad k\geq 1.$$

$$*a_k=\frac{-2}{k\pi}\int_0^1\left(x-x^2\right)\left(\cos k\pi x\right)^I dx=$$

$$=\frac{-2}{k\pi}\cdot\left(x-x^2\right)\cos k\pi x\Big|_0^1+\frac{2}{k\pi}\int_0^1\left(1-2x\right)\cos k\pi x dx=$$

$$= \frac{2}{(k\pi)^2} \int_0^1 (1-2x)(\sin k\pi x)^I \, dx = \frac{4}{(k\pi)^2} \int_0^1 dink\pi x dx =$$

$$= \frac{-4}{(k\pi)^3} \cdot \cos k\pi x|_0^1 =$$

$$= \frac{4\left[1-(-1)^k\right]}{(k\pi)^3} = \begin{cases} 0, & k = 2n, \\ \frac{8}{\pi^3(2n-1)^3}, & k = 2n-1, \end{cases} \quad n \geq 1.$$

$$**\, b_k = \frac{-4}{(k\pi)^2} \cdot \cos k\pi x|_0^1 = \frac{4\left[1-(-1)^k\right]}{(k\pi)^3} =$$

$$= \begin{cases} 0, & k = 2n, \\ \frac{8}{\pi^3(2n-1)^3}, & k = 2n-1, \end{cases} \quad n \geq 1.$$

$$u(x,t) = \sum_{k=1}^{\infty} \left[\frac{8}{\pi^3 (2n-1)^3} \cdot \cos \frac{(2n-1)\pi t}{2} + \right.$$

$$\left. + \frac{8}{\pi^3 (2n-1)^3} \cdot \sin \frac{(2n-1)\pi t}{2} \right] \cdot \sin (2n-1)\pi x.$$

Remarca 3.30 Următoarele probleme mixte se rezolvă cu metoda separării variabilelor, fără principiul lui Duhamel.

Aplicaţia 3.31 Să se rezolve problema mixtă:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 3\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} - 3x - 2t, \ 0 < x < \pi \\ u(x,0) = e^{-x}(\sin x + \sin 3x), \ \frac{\partial u}{\partial t}(x,0) = x \\ u(0,t) = 0, \ u(\pi,t) = \pi t. \end{cases}$$

Soluţie:

Facem substituţia

$$u(x,t) = v(x,t) \cdot e^{\alpha x + \beta t} + xt \Rightarrow$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} \cdot e^{\alpha x+\beta t} + \beta v \cdot e^{\alpha x+\beta t} + x$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} \cdot e^{\alpha x+\beta t} + 2\beta \frac{\partial v}{\partial t} \cdot e^{\alpha x+\beta t} + \beta^2 v \cdot e^{\alpha x+\beta t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} \cdot e^{\alpha x+\beta t} + \alpha \cdot v \cdot e^{\alpha x+\beta t} + t$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \cdot e^{\alpha x+\beta t} + 2\alpha \frac{\partial v}{\partial x} e^{\alpha x+\beta t} + \alpha^2 v \cdot e^{\alpha x+\beta t}.$$

Ecuaţia devine:

$$\frac{\partial^2 v}{\partial t^2} \cdot e^{\alpha x+\beta t} + 2\beta \frac{\partial v}{\partial t} \cdot e^{\alpha x+\beta t} + \beta^2 v \cdot e^{\alpha x+\beta t} - 3\frac{\partial v}{\partial t} \cdot e^{\alpha x+\beta t} - 3\beta v \cdot e^{\alpha x+\beta t} -$$

$$-3x = \frac{\partial^2 v}{\partial x^2} \cdot e^{\alpha x+\beta t} + 2\alpha \frac{\partial v}{\partial x} e^{\alpha x+\beta t} + \alpha^2 v \cdot e^{\alpha x+\beta t} + 2\frac{\partial v}{\partial x} \cdot e^{\alpha x+\beta t} +$$

$$+2\alpha \cdot v \cdot e^{\alpha x+\beta t} + 2t - 3x - 2t \Leftrightarrow$$

$$\frac{\partial^2 v}{\partial t^2} + (2\beta - 3)\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + (2\alpha + 2)\frac{\partial v}{\partial x} + \left(2\alpha + \alpha^2 - \beta^2 + 3\beta\right) v.$$

Facem: $\alpha = -1,\ \beta = \frac{3}{2} \Rightarrow u(x,t) = v(x,t) \cdot e^{-x+\frac{3t}{2}} + xt \Rightarrow$

$$* \begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{5}{4} v \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = \sin x + \sin 3x,\ \frac{\partial v}{\partial t}(x,0) = \frac{-3}{2}(\sin x + \sin 3x). \end{cases}$$

Se aplică metoda separării variabilelor:

$$v(x,t) = \alpha(x) \cdot \beta(t) \Rightarrow$$
$$\alpha(x) \cdot \beta''(t) = \alpha''(x) \cdot \beta(t) + \tfrac{5}{4}\alpha(x) \cdot \beta(t) \Leftrightarrow$$

$$\Leftrightarrow \frac{\alpha''(x) + \frac{5}{4}\alpha(x)}{\alpha(x)} = \frac{\beta''(t)}{\beta(t)} = -\lambda$$

$$\Rightarrow \begin{cases} \underbrace{\alpha''(x) + \left(\lambda + \frac{5}{4}\right)\alpha(x) = 0}_{\text{are soluţie nenulă } \Leftrightarrow \lambda + \frac{5}{4} > 0} \\ \alpha(0) = \alpha(\pi) = 0 \end{cases} \qquad \beta''(t) + \lambda\beta(t) = 0$$

$$\lambda + \frac{5}{4} > 0 \Rightarrow \alpha(x) = C \cdot \cos\sqrt{\lambda + \frac{5}{4}}x + D\sin\sqrt{\lambda + \frac{5}{4}}x;$$

$$\alpha(0) = 0 \Rightarrow C = 0$$

$$\alpha(\pi) = 0 \Rightarrow \sin\sqrt{\lambda + \frac{5}{4}}\pi = 0 \Rightarrow \sqrt{\lambda + \frac{5}{4}}\pi = k\pi,\ k = 1, 2, ... \Rightarrow$$

$$\lambda_k = k^2 - \frac{5}{4},\ k \geq 1$$

$$\Rightarrow \alpha_k(x) = \sin kx,\ k \geq 1.$$

$$\beta''(t) + \left(k^2 - \frac{5}{4}\right)\beta(t) = 0$$

$k \geq 2 \Rightarrow$

$$\beta_k(t) = a_k \cos\sqrt{k^2 - \frac{5}{4}}t + b_k \sin\sqrt{k^2 - \frac{5}{4}}t,\ k \geq 2.$$

$k = 1 \Rightarrow$

$$\beta''(t) - \frac{1}{4}\beta(t) = 0 \Rightarrow \beta(t) = C_1 e^{\frac{t}{2}} + C_2 e^{\frac{-t}{2}} \Rightarrow$$

$$v(x,t) = \left(C_1 e^{\frac{t}{2}} + C_2 e^{\frac{-t}{2}}\right)\sin x +$$

$$+ \sum_{k=2}^{\infty}\left(a_k \cos\sqrt{k^2 - \frac{5}{4}}t + b_k \sin\sqrt{k^2 - \frac{5}{4}}t\right)\sin kx$$

$$v(x,0)=\sin x+\sin 3x\Rightarrow$$

$$\begin{cases} C_1+C_2=1\\ a_k=\begin{cases}1, & k=3\\ 0, & k\geq 2,\ k\neq 3.\end{cases}\end{cases}$$

$$\frac{\partial v}{\partial t}(x,t)=\left(\frac{C_1}{2}e^{\frac{t}{2}}+\frac{C_2}{2}e^{\frac{-t}{2}}\right)\sin x+$$

$$+\sum_{k=2}^{\infty}\left[-a_k\left(\sin\sqrt{k^2-\frac{5}{4}}t\right)\sqrt{k^2-\frac{5}{4}}+\right.$$

$$\left.+b_k\left(\cos\sqrt{k^2-\frac{5}{4}}t\right)\sqrt{k^2-\frac{5}{4}}\right]\sin kx$$

$$\frac{\partial v}{\partial t}(x,0)=\frac{C_1-C_2}{2}\sin x+\sum_{k=2}^{\infty}b_k\sqrt{k^2-\frac{5}{4}}\sin kx=$$

$$=\frac{-3}{2}\sin x-\frac{3}{2}\sin 3x$$

$$\Rightarrow\underbrace{\begin{cases}C_1-C_2=-3\\ C_1+C_2=1\end{cases}}_{\Downarrow}\ \text{şi } b_k=\begin{cases}\frac{-3}{\sqrt{31}}, & k=3\\ 0, & k\geq 2,\ k\neq 3.\end{cases}$$

$$C_1=-1,\quad C_2=2\Rightarrow$$

$$v(x,t)=\left(-e^{\frac{t}{2}}+2e^{\frac{-t}{2}}\right)\sin x+\left(\cos\frac{\sqrt{31}}{2}t-\frac{3}{\sqrt{31}}\sin\frac{\sqrt{31}}{2}t\right)\sin 3x.$$

Revenind la substituţie, obţinem:

$$u(x,t)=e^{-x+\frac{3t}{2}}\cdot v(x,t)+xt\Rightarrow$$

$$u(x,t)=\left(2e^{-x+t}-e^{2t-x}\right)\sin x+$$

$$+e^{-x+\frac{3t}{2}}\left(\cos\frac{\sqrt{31}}{2}t-\frac{3}{\sqrt{31}}\sin\frac{\sqrt{31}}{2}t\right)\sin 3x+xt.$$

Aplicaţia 3.32

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}+x+2t,\ 0<x<1\\ u\left(0,t\right)=0,\ u\left(1,t\right)=t\\ u\left(x,0\right)=e^x\sin\pi x.\end{cases}$$

Soluţie: Se face substituţia:

$$u\left(x,t\right)=e^{\alpha x+\beta t}\cdot v\left(x,t\right)+w\left(x,t\right),$$

unde:

$$w\left(x,t\right)=xt\Rightarrow u\left(x,t\right)=e^{\alpha x+\beta t}\cdot v\left(x,t\right)+xt\Rightarrow$$

$$\frac{\partial u}{\partial t}=\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t}+\beta v\cdot e^{\alpha x+\beta t}+x$$

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial x}\cdot e^{\alpha x+\beta t}+\alpha\cdot v\cdot e^{\alpha x+\beta t}+t\Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t}+2\alpha\frac{\partial v}{\partial x}e^{\alpha x+\beta t}+\alpha^2 v\cdot e^{\alpha x+\beta t}\Rightarrow$$

$$\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t}+\beta v\cdot e^{\alpha x+\beta t}+x=$$

$$=\frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t}+2\alpha\frac{\partial v}{\partial x}e^{\alpha x+\beta t}+\alpha^2 v\cdot e^{\alpha x+\beta t}-2\frac{\partial v}{\partial x}\cdot e^{\alpha x+\beta t}-$$

$$-2\alpha\cdot v\cdot e^{\alpha x+\beta t}-2t+x+2t\Rightarrow$$

$$\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t}=\frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t}+\left(2\alpha-2\right)\frac{\partial v}{\partial x}e^{\alpha x+\beta t}+$$

$$+\left(\alpha^2-2\alpha-\beta\right)v\cdot e^{\alpha x+\beta t}.$$

Facem: $\alpha = 1,\ \beta = 0 \Rightarrow$ ecuaţia devine:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v.$$

Deci, substituţia este:

$$u\left(x, t\right) = e^x v\left(x, t\right) + xt.$$

În urma acestei substituţii s-a obţinut problema mixtă:

$$* \begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v \\ v\left(0, t\right) = v\left(1, t\right) = 0 \\ v\left(x, 0\right) = \sin \pi x. \end{cases}$$

Pentru a găsi soluţia acestei probleme mixte se aplică metoda separării variabilelor.
Mai întâi se caută soluţii particulare, nenule, ale ecuaţiei

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v,$$

de forma:

$$v\left(x, t\right) = \alpha\left(x\right) \cdot \beta\left(t\right) \Rightarrow$$

$$\alpha\left(x\right) \cdot \beta'\left(t\right) = \alpha''\left(x\right) \cdot \beta\left(t\right) - \alpha\left(x\right) \cdot \beta\left(t\right) \Leftrightarrow$$

$$\Leftrightarrow \frac{\beta'\left(t\right)}{\beta\left(t\right)} = \frac{\alpha''\left(x\right) - \alpha\left(x\right)}{\alpha\left(x\right)} = -\lambda \Rightarrow$$

$$(1) \begin{cases} \alpha''\left(x\right) + \left(\lambda - 1\right)\alpha\left(x\right) = 0 \\ \alpha\left(0\right) = \alpha\left(1\right) = 0 \end{cases} \quad \text{şi} \quad (2)\, \beta'\left(t\right) + \lambda\beta\left(t\right) = 0.$$

Ecuaţia (1) are soluţii nenule $\Leftrightarrow$

$$\lambda - 1 > 0 \Rightarrow \alpha\left(x\right) = C_1 \cos\sqrt{\lambda - 1}x + C_2 \sin\sqrt{\lambda - 1}x.$$

$\alpha(0) = 0 \Rightarrow C_1 = 0 \Rightarrow \alpha(x) = C_2 \sin\sqrt{\lambda - 1}x; \quad \alpha(1) = 0 \Rightarrow$
$\sqrt{\lambda - 1} = k\pi,\ k = 1, 2, ... \Rightarrow \lambda_k = 1 + k^2\pi^2,\ k = 1, 2, ...$

Deci:

$$\alpha_k(x) = \sin k\pi x,\ k \in \mathbb{N}^*.$$

Înlocuind λ_k în (2) $\Rightarrow$

$$\beta'(t) + \left(1 + k^2\pi^2\right) \cdot \beta(t) = 0 \Rightarrow \beta_k(t) = C_k \cdot e^{-\left(1+k^2\pi^2\right)t},\ k \geq 1.$$

Am obţinut un şir de soluţii particulare, nenule, pentru ecuaţia:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} - v,$$

de forma:

$$v_k(x, t) = c_k e^{-\left(1+k^2\pi^2\right)t} \sin k\pi x,\ k \geq 1.$$

Se caută soluţie de forma:

$$v(x, t) = \sum_{k=1}^{\infty} c_k \cdot e^{-\left(1+k^2\pi^2\right)t} \sin k\pi x.$$

Pentru determinarea coeficienţilor c_k, se utilizează relaţia iniţială.

$$v(x, 0) = \sin \pi x = \sum_{k=1}^{\infty} c_k \cdot \sin k\pi x.$$

Cum sistemul $\left\{\sqrt{2}\sin k\pi x\right\}_{k\in\mathbb{N}^*}$ este ortogonal în $L^2((0, 1)) \Rightarrow$ "înmulţind scalar" cu $\sqrt{2}\sin k\pi x \Rightarrow$

$$c_1 = 1,\ \mathrm{c}_k = 0,\ (\forall)\, k \geq 2 \Rightarrow$$

$$v(x, t) = e^{-\left(1+\pi^2\right)t} \sin \pi x.$$

Revenind la substituţie, se obţine soluţia problemei mixte iniţiale:

$$\boxed{u(x,t) = xt + e^{x-t-\pi^2 t}\sin \pi x}.$$

Aplicaţia 3.33 Să se rezolve următoarea problemă mixtă:

$$\begin{array}{l} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 4\frac{\partial u}{\partial x} + x - 4t + 1 + e^{-2x} \cdot \cos^2 \pi x,\ 0 < x < 1 \\ u(0,t) = t,\ u(1,t) = 2t,\ u(x,0) = 0. \end{array}$$

Soluţie:

Se face substituţia:

$$u(x,t) = v(x,t) \cdot e^{\alpha x + \beta t} + t + x(2t - t) \Rightarrow$$

$$u(x,t) = tx + t + v(x,t) \cdot e^{\alpha x + \beta t} \Rightarrow$$

$$\frac{\partial u}{\partial t} = 1 + x + \frac{\partial v}{\partial t} \cdot e^{\alpha x + \beta t} + \beta \cdot v \cdot e^{\alpha x + \beta t}$$

$$\frac{\partial u}{\partial x} = t + \frac{\partial v}{\partial x} \cdot e^{\alpha x + \beta t} + v \cdot \alpha \cdot e^{\alpha x + \beta t} \Rightarrow$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2} \cdot e^{\alpha x + \beta t} + 2\alpha \frac{\partial v}{\partial x} \cdot e^{\alpha x + \beta t} + v \cdot \alpha^2 e^{\alpha x + \beta t} \Rightarrow$$

$$1 + x + \frac{\partial v}{\partial t} \cdot e^{\alpha x + \beta t} + \beta \cdot v \cdot e^{\alpha x + \beta t} =$$

$$= \frac{\partial^2 v}{\partial x^2} \cdot e^{\alpha x + \beta t} + 2\alpha \frac{\partial v}{\partial x} \cdot e^{\alpha x + \beta t} + v \cdot \alpha^2 e^{\alpha x + \beta t} + 4t +$$

$$+4\frac{\partial v}{\partial x} \cdot e^{\alpha x + \beta t} + 4v \cdot \alpha \cdot e^{\alpha x + \beta t} + x - 4t + 1 + e^{-2x}\cos^2 \pi x \Leftrightarrow$$

$$\frac{\partial v}{\partial t} \cdot e^{\alpha x + \beta t} = \frac{\partial^2 v}{\partial x^2} \cdot e^{\alpha x + \beta t} + (2\alpha + 4)\frac{\partial v}{\partial x} \cdot e^{\alpha x + \beta t} +$$

$$+\left(\alpha^2+4\alpha-\beta\right)\cdot v\cdot e^{\alpha x+\beta t}+e^{-2x}\cos^2\pi x.$$

Facem:

$$\left\{\begin{array}{l}2\alpha+4=0\\ \beta=0\end{array}\right.\Rightarrow\left\{\begin{array}{l}\alpha=-2\\ \beta=0\end{array}\right.\Rightarrow$$

$$\frac{\partial v}{\partial t}\cdot e^{-2x}=\frac{\partial^2 v}{\partial x^2}\cdot e^{-2x}-4v\cdot e^{-2x}+e^{-2x}\cos^2\pi x\Rightarrow$$

$$\Rightarrow\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-4v+\cos^2\pi x.$$

$$v\left(0,t\right)=v\left(1,t\right)=0,\ v\left(x,0\right)=0.$$

Deci, efectuând substituţia:

$$u\left(x,t\right)=v\left(x,t\right)\cdot e^{-2x}+x\left(t\right)+t,$$

am obţinut următoarea problemă mixtă:

$$*\left\{\begin{array}{l}\frac{\partial v}{\partial t}=\frac{\partial^2 v}{\partial x^2}-4v+\cos^2\pi x,\ 0<x<1\\ v\left(0,t\right)-v\left(1,t\right)=v\left(x,0\right)=0.\end{array}\right.$$

Pentru a găsi soluţia, căutăm soluţie de forma:

$$v\left(x,t\right)=\sum_{k=1}^{\infty}T_k\left(t\right)\cdot\sin k\pi x\Rightarrow$$

ecuaţia devine:

$$\sum_{k=1}^{\infty}\left[T_k'\left(t\right)+\left(k^2\pi^2+4\right)T_k\left(t\right)\right]\sin k\pi x=\sum_{k=1}^{\infty}g_k\left(t\right)\cdot\sin k\pi x$$

$\leftarrow$ dezvoltarea în serie de sinusuri a lui g pe intervalul $(0,1)$, unde $\xi\to g\left(t,\xi\right)=\cos^2\pi\xi$.

$$g_k\left(t\right)=2\int_0^1 g\left(t,\xi\right)\cdot\sin k\pi\xi d\xi=2\int_0^1\left(\cos^2\pi\xi\right)\cdot\sin k\pi\xi d\xi=$$

$$= \int_0^1 \sin k\pi\xi d\xi + \int_0^1 (\cos 2\pi\xi) \cdot \sin k\pi\xi d\xi =$$

$$= \left.\frac{-\cos k\pi\xi}{k\pi}\right|_0^1 + \frac{1}{\pi}\int_0^\pi (\cos 2\xi) \cdot \sin k\xi d\xi = \frac{1-(-1)^k}{k\pi} +$$

$$+ \frac{1}{2\pi}\int_0^\pi \left[\sin(k+2)\xi + \sin(k-2)\xi\right] d\xi =$$

$$= \frac{1-(-1)^k}{k\pi} + \frac{1}{2\pi} \cdot \left[\left.\frac{-\cos(k+2)\xi}{k+2}\right|_0^\pi - \left.\frac{\cos(k-2)\xi}{k-2}\right|_0^\pi\right] =$$

$$= \frac{1-(-1)^k}{k\pi} + \frac{1}{2\pi} \cdot \left[1-(-1)^k\right]\left(\frac{1}{k+2} + \frac{1}{k-2}\right) =$$

$$= \frac{1-(-1)^k}{2\pi}\left(\frac{2}{k} + \frac{1}{k+2} + \frac{1}{k-2}\right).$$

Deci:

$$g_k(t) = \frac{1-(-1)^k}{2\pi}\left(\frac{2}{k} + \frac{1}{k+2} + \frac{1}{k-2}\right) \overset{not.}{=} \gamma_k.$$

$$\sum_{k=1}^{\infty}\left[T_k'(t) + \left(k^2\pi^2+4\right)T_k(t)\right]\sin k\pi x = \sum_{k=1}^{\infty}\gamma_k \cdot \sin k\pi x \Rightarrow$$

$$\begin{cases} T_k'(t) + (k^2\pi^2+4)\,T_k(t) = \gamma_k \\ T_k(0) = 0 \end{cases}$$

$$v(x,0) = 0 \Rightarrow T_k(0) = 0, \ \ (\forall)\, k \geq 1.$$

Aplicăm metoda variaţiei constantelor

$$\Rightarrow T_k(t) = C_k(t) \cdot e^{-\left(k^2\pi^2+4\right)t} \Rightarrow$$

$$C_k'(t) \cdot e^{-\left(k^2\pi^2+4\right)t} - \left(k^2\pi^2+4\right) \cdot C_k(t) \cdot e^{-\left(k^2\pi^2+4\right)t} +$$

$$+\left(k^2\pi^2+4\right)\cdot C_k\left(t\right)\cdot e^{-\left(k^2\pi^2+4\right)t}=\gamma_k$$

$$\Rightarrow C_k'\left(t\right)=\gamma_k\cdot e^{\left(k^2\pi^2+4\right)t}\Rightarrow C_k\left(t\right)=\frac{\gamma_k}{k^2\pi^2+4}\cdot e^{\left(k^2\pi^2+4\right)t}+C_0.$$

$$T_k\left(0\right)=0\Rightarrow C_k\left(0\right)=0\Rightarrow C_0=\frac{-\gamma_k}{k^2\pi^2+4}\Rightarrow$$

$$\Rightarrow C_k\left(t\right)=\frac{\gamma_k}{k^2\pi^2+4}\left[e^{\left(k^2\pi^2+4\right)t}-1\right].$$

Deci:

$$T_k\left(t\right)=\frac{\gamma_k}{k^2\pi^2+4}\left[1-e^{-\left(k^2\pi^2+4\right)t}\right].$$

Deci:

$$v\left(x,t\right)=\sum_{k=1}^{\infty}T_k\left(t\right)\cdot\sin k\pi x=\sum_{k=1}^{\infty}\gamma_k\frac{1-e^{-\left(k^2\pi^2+4\right)t}}{k^2\pi^2+4}\cdot\sin k\pi x.$$

Prin urmare, soluţia generală este:

$$u\left(x,t\right)=t\left(x+1\right)+e^{-2x}\sum_{k=1}^{\infty}\gamma_k\frac{1-e^{-\left(k^2\pi^2+4\right)t}}{k^2\pi^2+4}\cdot\sin k\pi x,$$

unde

$$\gamma_k=\frac{1-\left(-1\right)^k}{2\pi}\left(\frac{2}{k}+\frac{1}{k+2}+\frac{1}{k-2}\right)=$$

$$=\begin{cases}0, & k=2m\\ \frac{1}{\pi}\left(\frac{2}{2m-1}+\frac{1}{2m+1}+\frac{1}{2m-3}\right), & k=2m-1,\ m=1,2,...\end{cases}$$

$$\Rightarrow u\left(x,t\right)=t\left(x+1\right)+\frac{e^{-2x}}{\pi}\sum_{m=1}^{\infty}\left(\frac{2}{2m-1}+\frac{1}{2m+1}+\frac{1}{2m-3}\right)\cdot$$

$$\cdot\frac{\left[1-e^{-\left[(2m-1)^2\pi^2+4\right]t}\right]\sin\left(2m-1\right)\pi x}{\left(2m-1\right)^2\pi^2+4}.$$

Aplicaţia 3.34 Să se rezolve problema mixtă:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - 7\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x} - 2t - 7x - e^{-x}\sin 3x,\ 0 < x < \pi \\ u(0,t) = 0,\ u(\pi,t) = \pi t,\ u(x,0) = 0,\ \frac{\partial u}{\partial t}(x,0) = x. \end{cases}$$

Soluţie:

Se face substituţia

$$u(x,t) = v(x,t)\cdot e^{\alpha x+\beta t} + 0 + \frac{x}{\pi}(\pi t - 0) = v(x,t)\cdot e^{\alpha x+\beta t} + xt \Rightarrow$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t} + \beta v\cdot e^{\alpha x+\beta t} + x$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2}\cdot e^{\alpha x+\beta t} + 2\beta\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t} + \beta^2 v\cdot e^{\alpha x+\beta t}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x}\cdot e^{\alpha x+\beta t} + \alpha\cdot v\cdot e^{\alpha x+\beta t} + t$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t} + 2\alpha\frac{\partial v}{\partial x}e^{\alpha x+\beta t} + \alpha^2 v\cdot e^{\alpha x+\beta t}.$$

Ecuaţia devine:

$$\frac{\partial^2 v}{\partial t^2}\cdot e^{\alpha x+\beta t} + 2\beta\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t} + \beta^2 v\cdot e^{\alpha x+\beta t} - 7\frac{\partial v}{\partial t}\cdot e^{\alpha x+\beta t} - 7\beta v\cdot e^{\alpha x+\beta t} -$$

$$-7x = \frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t} + 2\alpha\frac{\partial v}{\partial x}e^{\alpha x+\beta t} +$$

$$+\alpha^2 v\cdot e^{\alpha x+\beta t} + 2\frac{\partial v}{\partial x}\cdot e^{\alpha x+\beta t} + 2\alpha\cdot v\cdot e^{\alpha x+\beta t} + 2t - 2t - 7x - e^{-x}\sin 3x \Leftrightarrow$$

$$\frac{\partial^2 v}{\partial t^2}\cdot e^{\alpha x+\beta t} = \frac{\partial^2 v}{\partial x^2}\cdot e^{\alpha x+\beta t} + (2\alpha+2)\frac{\partial v}{\partial x}e^{\alpha x+\beta t} - (2\beta-7)\frac{\partial v}{\partial t}e^{\alpha x+\beta t} +$$

$$+\left(\alpha^2 - \beta^2 + 2\alpha + 7\beta\right)v\cdot e^{\alpha x+\beta t} - e^{-x}\sin 3x.$$

Alegem: $\alpha = -1,\ \beta = \frac{7}{2} \Rightarrow$

$$u(x,t) = e^{-x+\frac{7}{2}t} \cdot v(x,t) + xt \Rightarrow$$

$$\frac{\partial^2 v}{\partial t^2} \cdot e^{-x+\frac{7}{2}t} = \frac{\partial^2 v}{\partial x^2} \cdot e^{-x+\frac{7}{2}t} + + \frac{45}{4} v \cdot e^{-x+\frac{7}{2}t} - e^{-x} \sin 3x$$

$$\Leftrightarrow \begin{cases} \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2} + \frac{45}{4} v - e^{-x} \sin 3x \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = 0,\ \frac{\partial v}{\partial t}(x,0) = 0. \end{cases}$$

Se caută soluţie de forma:

$$v(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin kx \Rightarrow$$

$$\sum_{k=1}^{\infty} \left[T_k''(t) + \left(k^2 - \frac{45}{4}\right) T_k(t) \right] \sin kx = -e^{\frac{-7t}{2}} \sin 3x.$$

Sistemul: $\left\{ \frac{\sqrt{2}}{\pi} \sin kx \right\}_{k \geq 1}$ este sistem ortonormat în $L^2((0,\pi))$

$$\Rightarrow T_k''(t) + \left(k^2 - \frac{45}{4}\right) T_k(t) = -e^{\frac{-7t}{2}},\ k = 3.$$

$$T_k''(t) + \left(k^2 - \frac{45}{4}\right) T_k(t) = 0,\ k \neq 3. \quad T_k(0) = T_k'(0) = 0.$$

$$\begin{cases} T_k''(t) + \left(k^2 - \frac{45}{4}\right) T_k(t) = 0, \\ T_k(0) = T_k'(0) = 0, \end{cases} \quad k \neq 3 \Rightarrow$$

$$\Rightarrow T_k(t) = 0, \quad (\forall)\, k \in \mathbb{N}^*,\ k \neq 3.$$

Pentru $k = 3$, notăm $T_k(t) = T_3(t) = T(t)$. Avem problema Cauchy:

$$\begin{cases} T''(t) - \left(\frac{3}{2}\right)^2 T(t) = -e^{\frac{-7t}{2}} \\ T(0) = T'(0) = 0 \end{cases}$$

Aplicăm metoda variaţiei constantelor:

$$T(t) = C_1(t)\, e^{\frac{3t}{2}} + C_2(t)\, e^{\frac{-3t}{2}} \Rightarrow$$

$$T'(t) = \frac{3}{2}C_1(t)\, e^{\frac{3t}{2}} - \frac{3}{2}C_2(t)\, e^{\frac{-3t}{2}} + C_1'(t)\, e^{\frac{3t}{2}} + C_2'(t)\, e^{\frac{-3t}{2}}$$

$$\Rightarrow C_1'(t)\, e^{\frac{3t}{2}} + C_2'(t)\, e^{\frac{-3t}{2}} = 0 \quad (1)$$

$$T''(t) = \frac{9}{4}C_1(t)\, e^{\frac{3t}{2}} + \frac{9}{4}C_2(t)\, e^{\frac{-3t}{2}} + \frac{3}{2}C_1'(t)\, e^{\frac{3t}{2}} - \frac{3}{2}C_2'(t)\, e^{\frac{-3t}{2}} \Rightarrow$$

$$\frac{3}{2}\left(C_1'(t)\, e^{\frac{3t}{2}} - C_2'(t)\, e^{\frac{-3t}{2}}\right) = -e^{\frac{-7t}{2}} \qquad (2)$$

Din (1) şi (2) $\Rightarrow$

$$\begin{cases} C_1'(t) = \frac{1}{3}e^{-5t} \Rightarrow C_1(t) = \frac{-1}{15}e^{-5t} + C_1 \\ C_2'(t) = \frac{-1}{3}e^{-2t} \Rightarrow C_2(t) = \frac{1}{6}e^{-2t} + C_2 \end{cases}$$

$$\Rightarrow T(t) = \frac{-1}{15}e^{\frac{-7t}{2}} + C_1 \cdot e^{\frac{3t}{2}} + \frac{1}{6}e^{\frac{-7t}{2}} + C_2 \cdot e^{\frac{-3t}{2}} = \frac{e^{\frac{-7t}{2}}}{10} + \frac{1}{15}e^{\frac{3t}{2}} - \frac{1}{6}e^{\frac{-3t}{2}}$$

$$\Rightarrow v(x,t) = \left(\frac{e^{\frac{-7t}{2}}}{10} + \frac{1}{15}e^{\frac{3t}{2}} - \frac{1}{6}e^{\frac{-3t}{2}}\right)\sin 3x \Rightarrow$$

$$u(x,t) = e^{-x} \cdot e^{\frac{-7t}{2}}\left(\frac{e^{\frac{-7t}{2}}}{10} + \frac{1}{15}e^{\frac{3t}{2}} - \frac{1}{6}e^{\frac{-3t}{2}}\right)\sin 3x + xt \Rightarrow$$

$$u(x,t) = e^{-x}\left(\frac{1}{10} + \frac{1}{15}e^{5t} - \frac{1}{6}e^{2t}\right)\sin 3x + xt.$$

Aplicaţia 3.35 Să se rezolve următoarea problemă mixtă:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u + 2t\left(1-3t\right) - 6x + 2\cos x \cos 2x, \ 0 < x < \frac{\pi}{2} \\ \frac{\partial u}{\partial x}\left(0,t\right) = 1, \ u\left(\frac{\pi}{2}, t\right) = t^2 + \frac{\pi}{2}, \ u\left(x,0\right) = x. \end{cases}$$

Soluţie:

Se face substituţia:

$$u\left(x,t\right) = v\left(x\right) + w\left(x,t\right).$$

Ecuaţia devine:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + v''\left(x\right) + 6v\left(x\right) + 6w\left(x,t\right) + 2t\left(1-3t\right) - 6x + \cos x + \cos 3x$$

unde v şi w satisfac ecuaţia:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + 6w\left(x,t\right) + 2t\left(1-3t\right).$$

$$(1) \begin{cases} v''\left(x\right) + 6v\left(x\right) - 6x + \cos x + \cos 3x = 0 \\ v'\left(0\right) = 1, \ v\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \end{cases}$$

iar w satisface problema mixtă $\Rightarrow$

$$(2) \begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + 6w + 2t\left(1-3t\right) \\ \frac{\partial w}{\partial x}\left(0,t\right) = 0 \\ w\left(\frac{\pi}{2}, t\right) = t^2 \\ w\left(x,0\right) = x - v\left(x\right). \end{cases}$$

Rezolvând ecuaţia (1) cu coeficienţi constanţi, neomegenă şi cu condiţiile iniţiale $\Rightarrow$

$$v\left(x\right) = x - \frac{\cos x}{5} + \frac{\cos 3x}{3}.$$

Problema mixtă pe care o verifică w este:

$$\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + 6w + 2t\left(1-3t\right) \\ \frac{\partial w}{\partial t}\left(0,t\right) = 0,\ w\left(\frac{\pi}{2},t\right) = t^2 \\ w\left(x,0\right) = \frac{\cos x}{5} - \frac{\cos 3x}{3} \end{cases}$$

Pentru aceasta facem substituţia

$$w\left(x,t\right) = u\left(x,t\right) + t^2 \Rightarrow$$

$$\begin{cases} \frac{\partial u}{\partial t} + 2t = \frac{\partial^2 u}{\partial x^2} + 6u + 6t^2 + 2t - 6t^2 \\ \frac{\partial u}{\partial x}\left(0,t\right) = 0,\ u\left(\frac{\pi}{2},t\right) = t^2 \\ u\left(x,0\right) = \frac{\cos x}{5} - \frac{\cos 3x}{3} \end{cases} \Leftrightarrow$$

$$*\left(2\right)\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u,\ 0 < x < \frac{\pi}{2} \\ \frac{\partial u}{\partial x}\left(0,t\right) = 0,\ u\left(\frac{\pi}{2},t\right) = t^2 \\ u\left(x,0\right) = \frac{\cos x}{5} - \frac{\cos 3x}{3}. \end{cases}$$

Pentru a rezolva problema mixtă (2) apelăm la metoda separării variabilelor. Se caută soluţii particulare, nenule, de forma:

$$u\left(x,t\right) = \alpha\left(x\right)\cdot\beta\left(t\right) \Rightarrow$$

$$\alpha\left(x\right)\cdot\beta'\left(t\right) = \alpha''\left(x\right)\cdot\beta\left(t\right) + 6\alpha\left(x\right)\cdot\beta\left(t\right) \Leftrightarrow$$

$$\frac{\alpha''\left(x\right) + 6\alpha\left(x\right)}{\alpha\left(x\right)} = \frac{\beta'\left(t\right)}{\beta\left(t\right)} = -\lambda$$

$$\frac{\partial u}{\partial x} = \alpha'\left(x\right)\cdot\beta\left(t\right) \Rightarrow \begin{cases} \frac{\partial u}{\partial x}\left(0,t\right) = 0 \\ u\left(\frac{\pi}{2},t\right) = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha'\left(0\right) = 0 \\ \alpha\left(\frac{\pi}{2}\right) = 0 \end{cases}$$

Deci:

$$\begin{cases} \alpha''\left(x\right) + \left(\lambda+6\right)\alpha\left(x\right) = 0 \\ \alpha'\left(0\right) = 0,\ \alpha\left(\frac{\pi}{2}\right) = 0 \end{cases} \xrightarrow{\lambda+6>0}$$

$$\begin{cases} \alpha(x) = C_1 \cos\sqrt{\lambda+6}x + C_2 \sin\sqrt{\lambda+6}x \\ \alpha'(x) = -C_1\sqrt{\lambda+6}\cdot\sin\sqrt{\lambda+6}x + C_1\sqrt{\lambda+6}\cdot\cos\sqrt{\lambda+6}x \end{cases}$$

$$\alpha'(0) = 0 \Rightarrow C_2 = 0 \Rightarrow \alpha(x) = C_1\cos\sqrt{\lambda+6}x;$$

$$\alpha\left(\frac{\pi}{2}\right) = 0 \Rightarrow \cos\sqrt{\lambda+6}\cdot\frac{\pi}{2} = 0 \Rightarrow$$

$$\sqrt{\lambda_k+6}\cdot\frac{\pi}{2} = k\frac{\pi}{2} \Rightarrow \lambda_k + 6 = k^2 \Rightarrow \lambda_k = k^2 - 6,\ k = 1, 2, ...$$

Deci:

$$\alpha_k(x) = \cos kx,\ k = 1, 2, ...$$

Pentru $\lambda_k = k^2 - 6$ ecuaţia

$$\beta'(t) + \lambda\beta(t) = 0$$

devine:

$$\beta'(t) + \left(k^2-6\right)\beta(t) = 0 \Rightarrow \beta_k(t) = a_k\cdot e^{-\left(k^2-6\right)t},\ k\geq 1 \Rightarrow$$

$$u_k(x,t) = a_k\cdot\cos kx,\ k = 1, 2, ...$$

Se caută pentru problema mixtă (2) soluţie de forma:

$$u(x,t) = \sum_{k=1}^{\infty} a_k\cdot e^{-\left(k^2-6\right)t}\cdot\cos kx.$$

Pentru determinarea coeficienţiilor a_k se foloseşte condiţia iniţială.

$$u(x,0) = \frac{\cos x}{5} - \frac{\cos 3x}{3} = \sum_{k=1}^{\infty} a_k\cdot\cos kx.$$

$\left\{\frac{2}{\sqrt{\pi}}\cdot\cos k\pi\right\}_{k=1,2,...}$ este un sistem ortogonal în $L^2\left(\left(0,\frac{\pi}{2}\right)\right) \Rightarrow$

$$\Rightarrow a_1 = \frac{1}{5},\ a_3 = \frac{-1}{3},\ a_2 = 0,\ a_k = 0,\ k\geq 4.$$

$$\boxed{u\left(x,t\right)=\frac{e^{5t}\cos x}{5}-\frac{e^{-3t}}{3}\cos 3x}\Rightarrow$$

$$w\left(x,t\right)=t^2+e^{5t}\frac{\cos x}{5}-\frac{e^{-3t}\cos 3x}{3}.$$

Deci, soluţia problemei mixte iniţiale este:

$$u\left(x,t\right)=x-\frac{\cos x}{5}+\frac{\cos 3x}{3}+w\left(x,t\right)$$

$$\Rightarrow\boxed{u\left(x,t\right)=x+t^2+\frac{\cos x}{5}\left(e^{5t}-1\right)+\frac{\cos 3x}{3}\left(1-e^{-3t}\right)}.$$

Aplicaţia 3.36 Să se rezolve problema mixtă:

$$\begin{cases}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+6u+x^2\left(1-6t\right)-2\left(t+3x\right)+\sin^2 x,\ 0<x<\pi\\ \frac{\partial u}{\partial x}\left(0,t\right)=1,\ \frac{\partial u}{\partial x}\left(\pi,t\right)=2\pi t+1\\ u\left(x,0\right)=x.\end{cases}$$

Soluţie:

Se caută soluţie de forma:

$$u\left(x,t\right)=v\left(x\right)+w\left(x,t\right)\Rightarrow$$

$$\begin{cases}\frac{\partial w}{\partial t}=\frac{\partial^2 w}{\partial x^2}+v''\left(x\right)+6v\left(x\right)+6w\left(x\right)+x^2\left(1-6t\right)-2t+\\ \qquad+\sin^2 x-6x\\ v'\left(0\right)+\frac{\partial w}{\partial x}\left(0,t\right)=1,\ v'\left(\pi\right)+\frac{\partial w}{\partial x}\left(\pi,t\right)=2\pi t+1\\ w\left(x,0\right)+v\left(x\right)=x.\end{cases}$$

Obţinem următoarele ecuaţii:

$$(1)\begin{cases} v''(x) + 6v(x) - 6x = 0 \\ \underbrace{\begin{matrix} v'(0) = 1 \\ v'(\pi) = 1 \end{matrix}}_{\Downarrow} \\ \boxed{v(x) = x} \end{cases}$$

$$(2)\begin{cases} \frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + 6w + x^2 - 6tx^2 - 2t + \sin^2 x \\ \frac{\partial w}{\partial x}(0,t) = 0,\ \frac{\partial w}{\partial x}(\pi,t) = 2\pi t \\ w(x,0) = x - v(x) = 0 \end{cases}$$

Pentru rezolvarea problemei mixte (2) facem substituţia: $w(x,t) = u(x,u) + x^2 t$ şi rezultă următoarea problemă mixtă

$$\begin{cases} \frac{\partial u}{\partial t} + x^2 = \frac{\partial^2 u}{\partial x^2} + 2t + 6u + 6x^2 t + x^2 - 6x^2 t - 2t + \sin^2 x \\ \frac{\partial u}{\partial x}(0,t) = 0,\ \frac{\partial u}{\partial x}(\pi,t) = 0,\ u(x,0) = 0. \end{cases}$$

$$\Rightarrow (3)\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u + \sin^2 x \\ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(\pi,t) = u(x,0). \end{cases}$$

Căutăm soluţii de forma:

$$u(x,t) = \sum_{k=1}^{\infty} T_k(t) \cdot \sin kx \Rightarrow$$

$$\sum_{k=1}^{\infty} \left[T_k'(t) + \left(k^2 - 6\right) T_k(t)\right] \sin kx = \sum_{k=1}^{\infty} g_k(t) \cdot \sin kx,\ T_k(0) = 0$$

$$g_k(t) = \delta_k = \frac{2}{\pi}\int_0^{\pi} \sin^2 x \cdot \sin kx dx =$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin kx dx - \frac{1}{\pi}\int_0^{\infty} \cos 2x \cdot \sin kx dx =$$

$$= \left.\frac{-\cos kx}{k\pi}\right|_0^{\pi} - \frac{1}{2\pi}\int_0^{\pi}\sin(k+2)\,xdx - \frac{1}{2\pi}\int_0^{\pi}\sin(k-2)\,xdx =$$

$$= \frac{1-(-1)^k}{k\pi} + \frac{(-1)^k-1}{2\pi(k+2)} + \frac{(-1)^k-1}{2\pi(k-2)} \Rightarrow$$

$$\delta_k = \frac{1-(-1)^k}{2\pi}\left(\frac{2}{k} - \frac{1}{k+2} - \frac{1}{k-2}\right).$$

Obţinem ecuaţia

$$\begin{cases} T_k'(t) - (k^2-6)\,T_k(t) = \delta_k \\ T_k(0) = 0 \end{cases} \Rightarrow \begin{cases} T_k(t) = C_k \cdot e^{-\left(k^2-6\right)t} \\ C_k'(t) = \delta_k \cdot e^{\left(k^2-6\right)t} \\ C_k(t) = \frac{\delta_k}{k^2-6} e^{\left(k^2-6\right)t} + C_0 \end{cases} \Rightarrow$$

$$T_k(t) = \frac{\delta_k}{k^2-6} + C_0 \cdot e^{-\left(k^2-6\right)t}$$

$$T_k(0) = 0 \Rightarrow C_0 = \frac{-\delta_k}{k^2-6} \Rightarrow T_k(t) = \frac{\delta_k}{k^2-6}\left[1 - e^{-\left(k^2-6\right)t}\right].$$

Deci:

$$u(x,t) = \sum_{k=1}^{\infty} \frac{\delta_k}{k^2-6}\left[1 - e^{-\left(k^2-6\right)t}\right]\sin kx.$$

$$\delta_k = \begin{cases} 0 & , k = 2m \\ \frac{1}{\pi}\left(\frac{2}{2m-1} - \frac{1}{2m+1} - \frac{1}{2m-3}\right) & , k = 2m-1 \end{cases} , \ m = 1, 2, .. \Rightarrow$$

$$u(x,t) = \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{2}{2k-1} - \frac{1}{2k+1} - \frac{1}{2k-3}\right)\cdot$$

$$\cdot\frac{1 - e^{-\left[(2k-1)^2-6\right]t}}{(2k-1)^2-6}\cdot\sin(2k-1)\,x.$$

$$w(x,t) = x^2t + u(x,t)\,,\ u(x,t) = v(x) + w(x,t) = x + x^2t + \frac{1}{\pi} \Rightarrow$$

$$u(x,t) = x + x^2 t + \frac{1}{\pi}\sum_{k=1}^{\infty}\left(\frac{2}{2k-1} - \frac{1}{2k+1} - \frac{1}{2k-3}\right)\cdot$$

$$\cdot\frac{1 - e^{-\left[(2k-1)^2-6\right]t}}{(2k-1)^2 - 6}\cdot\sin(2k-1)x.$$

3.4 Problema lui Cauchy pentru operatorul undelor

3.4.1 Problema lui Cauchy clasică pentru operatorul undelor

Problema lui Cauchy clasică pentru ecuaţia undelor este:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - a^2\Delta u = f(x,t),\ x\in\mathbb{R}^n,\ t>0 \\ u\big|_{t=0} = u_0(x),\ \frac{\partial u}{\partial t}\big|_{t=0} = u_1(x) \end{cases} \tag{3.45}$$

unde $f \in C(t\geq 0)$, $u_0 \in C^1(\mathbb{R}^n)$, $u_1 \in C^1(\mathbb{R}^n)$. Soluţia problemei (3.45) este $u\in C^2(t>0)\bigcap C^2(t\geq 0)$.
Soluţia clasică a problemei Cauchy există, este unică şi este dată de:

- pentru $n=3$ avem formula lui Kirchhoff:

$$u(x,t) =$$

$$= \frac{1}{4\pi a^2}\int_{B_{at}(x)}\frac{f\left(\xi, t-\frac{1}{a}\|x-\xi\|\right)}{\|x-\xi\|}d\xi + \frac{1}{4\pi a^2 t}\int_{S_{at}(x)}u_1(\xi)\,d\sigma_\varepsilon +$$

$$+\frac{1}{4\pi a^2}\cdot\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{S_{at}(x)}u_0(\xi)\,d\sigma_\varepsilon\right] \tag{3.46}$$

- pentru $n=2$ avem formula lui Poisson:

$$u(x,t) = \frac{1}{2\pi a}\int_0^t \int_{B_{a(t-\tau)}(x)} \frac{f(\xi,\tau)}{\sqrt{a^2(t-\tau)^2 - \|x-\xi\|^2}} d\xi d\tau +$$

$$+\frac{1}{2\pi a}\int_{B_{at}(x)} \frac{u_1(\xi)}{\sqrt{a^2t^2 - \|x-\xi\|^2}} d\xi + \tag{3.47}$$

$$+\frac{1}{2\pi a}\cdot\frac{\partial}{\partial t}\int_{B_{at}(x)} \frac{u_0(\xi)}{\sqrt{a^2t^2 - \|x-\xi\|^2}} d\xi.$$

- pentru $n = 1$ avem formula lui D'Alambert:

$$u(x,t) = \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)\, d\xi d\tau + \frac{1}{2a}\int_{x-at}^{x+at} u_1(\xi)\, d\xi +$$

$$+\frac{1}{2}\left[u_0(x+at) + u_0(x-at)\right]. \tag{3.48}$$

Avem următoarele aplicaţii pentru problema Cauchy pentru operatorul undelor:
Problema Cauchy pentru operatorul undelor, caz n=3, formula lui Kirchhoff.

Aplicaţia 3.37

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 8\Delta u + t^2x^2 \\ u\big|_{t=0} = y^2 \\ \frac{\partial u}{\partial t}\big|_{t=0} = z^2. \end{cases}$$

$$u(x,y,z,t) =$$

$$= \frac{1}{4\pi a^2} \int_{B_{at(x,y,z)}} \frac{f\left(\xi, \eta, \zeta, t - \frac{1}{a} \left\|(x,y,z) - (\xi,\eta,\zeta)\right\|\right)}{\left\|(x,y,z) - (\xi,\eta,\zeta)\right\|} d\xi d\eta d\zeta +$$

$$+ \frac{1}{4\pi a^2 t} \int_{S_{at(x,y,z)}} u_1(\xi,\eta,\zeta)\, d\sigma +$$

$$+ \frac{1}{4\pi a^2} \cdot \frac{\partial}{\partial t} \left[\frac{1}{t} \int_{S_{at(x,y,z)}} u_0(\xi,\eta,\zeta)\, d\sigma \right].$$

$a = 2\sqrt{2};\quad f(x,y,z,t) = t^2 x^2;\quad u_0(x,y,z) = y^2;\quad u_1(x,y,z) = z^2.$

$$u(x,y,z,t) = \frac{1}{32\pi} \cdot$$

$$\cdot \int_{B_{2\sqrt{2}\cdot t}(x,y,z)} \frac{\xi^2 \left(t - \frac{1}{2\sqrt{2}} \cdot \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2} \right)^2}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} d\xi d\eta d\zeta +$$

$$+ \frac{1}{32\pi t} \int_{S_{2\sqrt{2}\cdot t}(x,y,z)} \zeta^2 d\sigma + \frac{1}{32\pi} \cdot \frac{\partial}{\partial t} \left[\frac{1}{t} \int_{S_{2\sqrt{2}\cdot t}(x,y,z)} \eta^2 d\eta \right].$$

Facem schimbare de variabilă:

$$\begin{cases} \xi = x + 2\sqrt{2} \cdot t \cdot u \\ \eta = y + 2\sqrt{2} \cdot t \cdot v \\ \zeta = z + 2\sqrt{2} \cdot t \cdot w \end{cases} \qquad \begin{cases} d\xi d\eta d\zeta = \left(2\sqrt{2}\right)^3 t^3 du dv dw \\ d\sigma_{(\xi,\eta,\zeta)} = \left(2\sqrt{2}\right)^2 t^2 d\sigma_{(u,v,w)} \end{cases}$$

$B_{2\sqrt{2}\cdot t}(x,y,z) \to B_1(0)$ - sfera unitate.

$$u(x,y,z) =$$

$$= \frac{1}{32\pi} \int_{B_1(0)} \frac{\left(x + 2\sqrt{2} \cdot t \cdot u\right)^2 \cdot \left(t - \frac{1}{2\sqrt{2}} \cdot 2\sqrt{2} \cdot t\sqrt{u^2 + v^2 + w^2} \right)^2}{2\sqrt{2} \cdot t\sqrt{u^2 + v^2 + w^2}} \cdot$$

$$\cdot\left(2\sqrt{2}\right)^3 t^3 dudvdw + \frac{1}{32\pi t}\int_{S_1(0)}\left(z+2\sqrt{2}\cdot t\cdot w\right)^2\cdot\left(2\sqrt{2}\right)^2 t^2 d\sigma +$$

$$+\frac{1}{32\pi}\cdot\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{S_1(0)}\left(y+2\sqrt{2}\cdot t\cdot v\right)^2\cdot\left(2\sqrt{2}\right)^2 t^2 d\sigma\right] =$$

$$= \frac{1}{32\pi}\cdot 8t^4\int_{B_1(0)}\frac{\left(x+2\sqrt{2}\cdot t\cdot u\right)^2\cdot\left(1-\sqrt{u^2+v^2+w^2}\right)}{\sqrt{u^2+v^2+w^2}}dudvdw +$$

$$+\frac{t}{4\pi}\int_{S_1(0)}\left(z+2\sqrt{2}\cdot t\cdot w\right)^2 d\sigma +$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\int_{S_1(0)}\left(y+2\sqrt{2}\cdot t\cdot v\right)^2 d\sigma\right] = \frac{t^4}{4\pi}\cdot$$

$$\cdot\int_{B_1(0)}\frac{\left(x^2+8t^2u^2+4\sqrt{2}\cdot x\cdot t\cdot u\right)\cdot\left(1-\sqrt{u^2+v^2+w^2}\right)}{\left(1-\sqrt{u^2+v^2+w^2}\right)}dudvdw +$$

$$+\frac{t}{4\pi}\int_{S_1(0)}\left(z^2+8t^2w^2\right)\cdot\left(4\sqrt{2}\cdot t\cdot z\cdot w\right)d\sigma +$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\int_{S_1(0)}\left(y^2+8t^2v^2+4\sqrt{2}\cdot t\cdot y\cdot v\right)d\sigma\right].$$

$$\int_{B_1(0)}\frac{4\sqrt{2}\cdot x\cdot t\cdot u\left(1-\sqrt{u^2+v^2+w^2}\right)}{\sqrt{u^2+v^2+w^2}}dudvdw = 0$$

$$\int_{S_1(0)}4\sqrt{2}\cdot t\cdot z\cdot w d\sigma = 4\sqrt{2}\cdot t\cdot z\int_{S_1(0)}w d\sigma =$$

$$= 4\sqrt{2}\cdot t\cdot z\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\cos\theta d\theta = 0$$

$$\int_{S_1(0)}4\sqrt{2}\cdot t\cdot y\cdot v d\sigma = 4\sqrt{2}\cdot t\cdot y\int_{S_1(0)}v d\sigma =$$

$$= 4\sqrt{2} \cdot t \cdot y \int_0^{2\pi} \sin\phi d\phi \int_0^{\pi} \sin^2\theta d\theta = 0.$$

Deci:

$$u(x,y,z,t) = \frac{t^4}{4\pi} \int_{B_1(0)} \frac{\left(x^2 + 8t^2u^2\right)\left(1 - \sqrt{u^2+v^2+w^2}\right)}{\sqrt{u^2+v^2+w^2}} dudvdw +$$

$$+ \frac{t}{4\pi} \int_{B_1(0)} \left(z^2 + 8t^2w^2\right) d\sigma +$$

$$+ \frac{1}{4\pi} \cdot \frac{\partial}{\partial t}\left[t \int_{S_1(0)} \left(y^2 + 8t^2v^2\right) d\sigma\right].$$

Coordonate sferice:

$$\begin{cases} x = \rho\cos\phi\sin\theta \\ v = \rho\sin\phi\sin\theta \\ w = \rho\cos\theta \end{cases} \qquad \begin{cases} dudvdw = \rho^2\sin\theta d\rho d\theta d\phi \\ \rho \in [0,1],\ \theta \in [0,\pi],\ \phi \in [0,2\pi] \end{cases}$$

$$\begin{cases} u = \cos\phi\sin\theta \\ v = \sin\phi\sin\theta \\ w = \cos\theta \end{cases} \qquad d\sigma = \sin\theta d\phi d\theta$$

Parametrizarea sferei unitate

$$u(x,y,z) =$$

$$= \frac{t^4}{4\pi} \int_0^1 d\rho \int_0^{2\pi} d\phi \int_0^{\pi} \left(x^2 + 8t^2\cos^2\phi\sin^2\theta\right) \frac{(1-\rho)^2}{\rho} \rho\sin\theta d\theta +$$

$$+ \frac{t}{4\pi} \int_0^{2\pi} d\phi \int_0^{\pi} z^2 ...$$

Sau:

$$u(x,y,z,t) = \frac{t^4}{4\pi} \int_{B_1(0)} \frac{x^2\left(1 - \sqrt{u^2+v^2+w^2}\right)^2}{\sqrt{u^2+v^2+w^2}} dudvdw +$$

$$+\frac{2}{\pi}t^6\int_{B_1(0)}\frac{u^2\left(1-\sqrt{u^2+v^2+w^2}\right)^2}{\sqrt{u^2+v^2+w^2}}dudvdw+$$

$$+\frac{t\cdot z^2}{4\pi}\int_{S_1(0)}d\sigma+\frac{2t^3}{\pi}\int_{S_1(0)}w^2d\sigma+$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\cdot y^2\int_{S_1(0)}d\sigma+8t^3\int_{S_1(0)}v^2d\sigma\right]=$$

$$=\frac{x^2t^4}{4\pi}\int_{B_1(0)}\frac{\left(1-\sqrt{u^2+v^2+w^2}\right)^2}{\sqrt{u^2+v^2+w^2}}dudvdw+$$

$$+\frac{2t^6}{\pi}\int_{B_1(0)}\frac{u^2\left(1-\sqrt{u^2+v^2+w^2}\right)^2}{\sqrt{u^2+v^2+w^2}}dudvdw+$$

$$+\frac{t\cdot z^2}{4\pi}\cdot 4\pi+\frac{2t^3}{\pi}\int_{B_1(0)}w^2d\sigma+$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\cdot y^2\cdot 4\pi+8t^3\int_{S_1(0)}v^2d\sigma\right]=$$

$$=\frac{t^4x^2}{4\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\int_0^1\frac{(1-\rho)^2}{\rho^2}\cdot\rho^2d\rho+$$

$$+\frac{2t^6}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta d\theta\int_0^1\frac{\rho^2\cos^2\phi\sin^2\theta\,(1-\rho)^2}{\rho}\cdot\rho^2d\rho+$$

$$+t\cdot z^2+\frac{2t^3}{\pi}\int_0^{2\pi}d\phi\int_0^{\pi}\sin\theta\cos^2\theta d\theta+$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\cdot y^2\cdot 4\pi+8t^3\int_0^{2\pi}d\phi\int_0^{\pi}\sin^2\phi\sin^2\theta\cdot\sin\theta d\theta\right]=$$

$$=\frac{t^2x^2}{4\pi}\left(-\cos\theta|_0^{\pi}\right)\int_0^1\left(\rho-2\rho^2+\rho^3\right)d\rho+$$

$$+\frac{2t^6}{\pi}\int_0^{2\pi}\underbrace{\cos^2\phi d\phi}_{\pi}\int_0^{\pi}\underbrace{\sin^3\theta d\theta}_{\frac{4}{3}}\int_0^1\left(\rho^3-2\rho^4+\rho^5\right)d\rho+$$

$$+t\cdot z^2+\frac{2t^3}{\pi}\cdot 2\pi\int_0^{\pi}\cos^2\theta\sin\theta d\theta+y^2+$$

$$+\frac{2}{\pi}\cdot\frac{\partial}{\partial t}\left[t^3\int_0^{2\pi}\underbrace{\sin^2\phi d\phi}_{\pi}\int_0^{\pi}\underbrace{\sin^3\theta d\theta}_{\frac{4}{3}}\right]=$$

$$=t^4x^2\left(\frac{\rho^2}{2}-\frac{2\rho^3}{3}+\frac{\rho^4}{4}\right)\Big|_0^1+\frac{2t^6}{\pi}\cdot\frac{4\pi}{3}\left(\frac{\rho^4}{4}-\frac{2\rho^5}{5}+\frac{\rho^6}{6}\right)\Big|_0^1+$$

$$+t\cdot z^2+4t^3\cdot\left.\frac{-\cos^3\theta}{3}\right|_0^{\pi}+y^2+\frac{2}{\pi}\cdot\frac{4\pi}{3}\cdot\frac{\partial}{\partial t}\left(t^3\right)=$$

$$=x^2t^4\left(\frac{1}{2}-\frac{2}{3}+\frac{1}{4}\right)+\frac{8t^6}{3}\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}\right)+t\cdot z^2+\frac{8t^3}{3}+y^2+\frac{8}{3}\cdot 3t^2=$$

$$=\frac{1}{12}x^2t^4+\frac{8t^6}{3}\cdot\frac{(15-24+10)}{60}+t\cdot z^2+\frac{8t^3}{3}+y^2+8t^2=$$

$$=\frac{x^2t^4}{12}+\frac{2t^6}{45}+t\cdot z^2+\frac{8t^3}{3}+y^2+8t^2.$$

Deci:

$$u\left(x,y,z,t\right)=\frac{x^2t^4}{12}+\frac{2}{45}t^6+t\cdot z^2+\frac{8t^3}{3}+y^2+8t^2.$$

$$\cdot\int_0^{\pi}\sin^3\theta d\theta=\frac{4}{3}$$

$$\cdot\cdot\int_0^{2\pi}\sin^2\theta\left(\cos^2\theta\right)d\theta=\pi.$$

Aplicaţia 3.38 Să se rezolve problema Cauchy pentru operatorul undelor ($n = 3$).

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + 2xyz \\ u\big|_{t=0} = x^2 + y^2 - 2z^2 \\ \frac{\partial u}{\partial t}\big|_{t=0} = 1. \end{cases}$$

$$a = 1,\ f(x, y, z, t) = 2xyz,$$
$$u_0(x, y, z) = x^2 + y^2 - 2z^2,\ u_1(x, y, z) = 1.$$

$$u(x, y, z, t) =$$
$$= \frac{1}{4\pi a^2} \int_{B_{at}(x,y,z)} \frac{f\left(\xi, \eta, \zeta, t - \frac{1}{a}\left\|(x, y, z) - (\xi, \eta, \zeta)\right\|\right)}{\|(x, y, z) - (\xi, \eta, \zeta)\|} d\xi d\eta d\zeta +$$
$$+ \frac{1}{4\pi a^2 t} \int_{S_{at}(x,y,z)} u_1(\xi, \eta, \zeta)\, d\sigma +$$
$$+ \frac{1}{4\pi a^2} \cdot \frac{\partial}{\partial t} \left[\frac{1}{t} \int_{S_{at}(x,y,z)} u_0(\xi, \eta, \zeta)\, d\sigma \right] =$$
$$= \frac{1}{4\pi} \int_{B_t(x,y,z)} \frac{2\xi\eta\zeta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} d\xi d\eta d\zeta +$$
$$+ \frac{1}{4\pi t} \int_{S_t(x,y,z)} d\sigma + \frac{1}{4\pi} \cdot \frac{\partial}{\partial t} \left[\frac{1}{t} \int_{S_t(x,y,z)} \left(\xi^2 + \eta^2 - 2\zeta^2\right) d\sigma \right].$$

Facem schimbarea de variabilă:

$$\begin{cases} \xi = x + t \cdot u \\ \eta = y + t \cdot v \\ \zeta = z + t \cdot w \end{cases} \qquad d\xi d\eta d\zeta = \frac{\Delta(\xi, \eta, \zeta)}{\Delta(u, v, w)} du dv dw =$$

$$= \begin{vmatrix} \frac{\partial \xi}{\partial u} & \frac{\partial \xi}{\partial v} & \frac{\partial \xi}{\partial w} \\ \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial u} \\ \frac{\partial \zeta}{\partial u} & \frac{\partial \zeta}{\partial u} & \frac{\partial \zeta}{\partial u} \end{vmatrix} du dv dw = \begin{vmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{vmatrix} du dv dw = t^3 du dv dw.$$

$$d\xi d\eta d\zeta = t^3 dudvdw;\ d\sigma_{(\xi,\eta,\zeta)} = t^2 d\sigma_{(u,v,w)}$$

$B_t(x,y,z) \to B_1(0)$.
$S_t(x,y,z) \to S_1(0)$.

$$u(x,y,z,t) = \frac{1}{4\pi}\int_{B_1(0)} \frac{2(x+t\cdot u)(y+t\cdot v)(z+t\cdot w)}{t\sqrt{u^2+v^2+w^2}} t^3 dudvdw +$$

$$+\frac{1}{4\pi t}\int_{S_1(0)} t^2 d\sigma +$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[\frac{1}{t}\int_{S_1(0)} (x+t\cdot u)^2 + (y+t\cdot v)^2 - 2(z+t\cdot w)^2\right] t^2 d\sigma =$$

$$= \frac{2t^2}{4\pi}\int_{S_1(0)} \left(\frac{xyz + t(xvz + yuz + zyw)}{\sqrt{u^2+v^2+w^2}} +\right.$$

$$\left. + \frac{t^2(uvz + vwx + uwy) + t^3 uvw}{\sqrt{u^2+v^2+w^2}}\right) dudvdw +$$

$$+\frac{t}{4\pi}\cdot aria\left(S_1(0)\right) +$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\int_{S_1(0)} \left[\left(x^2+y^2-2z^2\right) + 2t(xu+yv-2zw) +\right.\right.$$

$$\left.\left. + t^2\left(u^2+v^2-2w^2\right)\right] d\sigma\right] =$$

$$= \frac{t^2}{2\pi}\int_{B_1(0)} \frac{xyz}{\sqrt{u^2+v^2+w^2}} dudvdw +$$

$$+\frac{t^3}{2\pi}\int_{B_1(0)} xz\underbrace{\frac{v}{\sqrt{u^2+v^2+w^2}}}_{0} dudvdw +$$

$$+\frac{t^3}{2\pi}\int_{B_1(0)} yz\underbrace{\frac{u}{\sqrt{u^2+v^2+w^2}}}_{0} dudvdw +$$

$$+\frac{t^3}{2\pi}\int_{B_1(0)} xy\underbrace{\frac{w}{\sqrt{u^2+v^2+w^2}}}_{0}dudvdw+$$

$$+\frac{t^4}{2\pi}\int_{B_1(0)} z\underbrace{\frac{uv}{\sqrt{u^2+v^2+w^2}}}_{0}dudvdw+$$

$$+\frac{t^4}{2\pi}\int_{B_1(0)} y\underbrace{\frac{uw}{\sqrt{u^2+v^2+w^2}}}_{0}dudvdw+$$

$$+\frac{t^4}{2\pi}\int_{B_1(0)} x\underbrace{\frac{vw}{\sqrt{u^2+v^2+w^2}}}_{0}dudvdw+\frac{t}{4\pi}\cdot 4\pi+$$

$$+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\int_{S_1(0)}\left(x^2+y^2-2z^2\right)d\sigma+2t^2\int_{S_1(0)}\underbrace{x\cdot ud\sigma}_{0}+\right.$$

$$+2t^2\int_{S_1(0)}\underbrace{y\cdot vd\sigma}_{0}+4t^2\int_{S_1(0)}\underbrace{z\cdot wd\sigma}_{0}+$$

$$\left.+\,t^2\int_{S_1(0)}\left(u^2+v^2-2w^2\right)d\sigma\right]=$$

$$=\frac{xyzt^2}{2\pi}\int_{B_1(0)}\frac{dudvdw}{\sqrt{u^2+v^2+w^2}}+t+\frac{1}{4\pi}\cdot\frac{\partial}{\partial t}\left[t\left(x^2+y^2-2z^2\right)\cdot 4\pi\right]=$$

$$=\frac{xyzt^2}{2\pi}\int_{B_1(0)}\frac{dudvdw}{\sqrt{u^2+v^2+w^2}}+t+\left(x^2+y^2-2z^2\right).$$

$$\begin{cases} u=\rho\cos\phi\sin\theta \\ v=\rho\sin\phi sin\theta \\ w=\rho\cos\theta \end{cases} \qquad \begin{cases} \rho\in[0,1],\ \phi\in[0,2\pi],\ \theta\in[0,\pi] \\ dudvdw=\rho^2\sin\theta d\rho d\phi d\theta \end{cases}$$

$$\int_{B_1(0)} \frac{uv}{\sqrt{u^2+v^2+w^2}} dudvdw =$$

$$= \int_0^1 \int_0^\pi \int_0^{2\pi} \frac{\rho^2 \cdot \rho^2 \sin\phi \cos\phi \sin^3\theta}{\sqrt{\rho^2}} d\theta d\phi d\rho =$$

$$= \frac{1}{4} \int_0^{2\pi} \underbrace{(\sin\phi)'}_{\sin\phi|_0^{2\pi}} d\phi \int_0^\pi \sin^3\theta d\theta = 0.$$

Pentru $\int_{S_1(0)} (u^2+v^2-2w^2)\, d\sigma$ scriem reprezentarea parametrică a sferei exterioare:

$$\begin{cases} u = \cos\phi sin\theta, \\ v = \sin\phi \sin\theta, \\ w = \cos\theta, \end{cases} \qquad \begin{cases} \phi \in [0, 2\pi]\,,\ \theta \in [0, \pi]\,, \\ d\sigma = \sqrt{A^2+B^2+C^2} d\phi d\theta; \end{cases}$$

$$A = \frac{\Delta(v,w)}{\Delta(\phi,\theta)} = \begin{vmatrix} \cos\phi\sin\theta & \sin\phi\cos\theta \\ 0 & -\sin\theta \end{vmatrix} = -\cos\phi\sin^2\theta;$$

$$B = \frac{\Delta(w,u)}{\Delta(\phi,\theta)} = \begin{vmatrix} 0 & -\sin\theta \\ -\sin\phi\sin\theta & \cos\phi\cos\theta \end{vmatrix} = -\sin\phi\sin^2\theta;$$

$$C = \begin{vmatrix} -\sin\phi\sin\theta & \cos\phi\cos\theta \\ \cos\phi\sin\theta & \sin\phi\cos\theta \end{vmatrix} = -\sin\theta\cos\theta \Rightarrow$$

$$d\sigma = \sqrt{\cos^2\phi\sin^4\theta + \sin^2\phi\sin^4\theta + \sin^2\theta\cos^2\theta} d\phi d\theta = \sin\theta d\phi d\theta.$$

Deci $d\sigma = \sin\theta d\phi d\theta$.

$$\int_{S_1(0)} \left(u^2+v^2-2w^2\right) d\sigma = \int_0^{2\pi} d\phi \int_0^\pi \left(\sin^2\theta - 2\cos^2\theta\right) \sin\theta d\theta =$$

$$= 2\pi \left[\int_0^\pi \underbrace{\sin^3\theta d\theta}_{\frac{3\sin\theta - \sin 3\theta}{4}} + 2\int_0^\pi \cos^2\theta \left(\cos\theta\right)' d\theta \right] =$$

$$= \frac{-3\pi}{2}\cos\theta\Big|_0^\pi + \frac{4\pi}{3}\cos^3\theta\Big|_0^\pi + \frac{\pi}{6}\cos 3\theta\Big|_0^\pi = 0.$$

$$u\left(x,y,z,t\right) = \frac{xyzt^2}{2\pi}\int_{B_1(0)} \frac{dudvdw}{\sqrt{u^2+v^2+w^2}} + t + x^2 + y^2 - 2z^2 =$$

$$= \frac{xyzt^2}{2\pi}\int_0^1 d\rho \int_0^{2\pi} d\phi \int_0^\pi \frac{\rho^2\sin\theta}{\rho} d\theta =$$

$$= \frac{xyzt^2}{2\pi} \cdot 2\pi \cdot \frac{1}{2}\int_0^\pi \underbrace{\sin\theta d\theta}_{-\cos\theta|_0^\pi = 2} + t + x^2 + y^2 - 2z^2 =$$

$$= xyzt^2 + t + x^2 + y^2 - 2z^2.$$

Deci:

$$u\left(x,y,z,t\right) = xyzt^2 + t + x^2 + y^2 - 2z^2.$$

Soluţia a doua:

$$\Delta\phi = \Delta u_0 = \Delta u_1 = 0 \Rightarrow$$

$$u\left(x,y,z,t\right) = u_0\left(x,y,z\right) + t\cdot u_1\left(x,y,z\right) + \frac{t^2}{2}\cdot f\left(x,y,z\right) =$$

$$= x^2 + y^2 - 2z^2 + t + xyzt^2.$$

Problema Cauchy pentru operatorul undelor, caz $n = 2$. Formula lui Poisson.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = a^2\Delta u + f\left(x,y,t\right) \\ u\big|_{t=0} = u_0\left(x,y\right) \\ \frac{\partial u}{\partial t}\Big|_{t=0} = u_1\left(x,y\right) \end{cases}$$

$$u(x,y,t) =$$

$$= \frac{1}{2\pi a}\int_0^t \left[\int_{B_{a(t-\tau)}(x,y)} \frac{f(\xi,\eta,\zeta)}{\sqrt{a^2(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta\right] +$$

$$+\frac{1}{2\pi a}\int_{B_{at}(x,y)} \frac{u_1(\xi,\eta)}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta +$$

$$+\frac{1}{2\pi a}\cdot\frac{\partial}{\partial t}\int_{B_{at}(x,y)} \frac{u_0(\xi,\eta)\,d\xi d\eta}{\sqrt{a^2t^2-(x-\xi)^2-(y-\eta)^2}}.$$

$n = 2$ (Poisson)

Aplicaţia 3.39

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + 6xyt \\ u\big|_{t=0} = x^2 - y^2 \\ \frac{\partial u}{\partial t}\big|_{t=0} = xy. \end{cases}$$

Soluţie:

$a = 1,\ f(x,y,t) = 6xyt,\ \mathrm{u}_0(x,y) = x^2 - y^2,\ \mathrm{u}_1(x,y) = xy \Rightarrow$

$$u(x,y,z) =$$

$$= \frac{1}{2\pi}\int_0^t \left[\int_{B_{t-\tau}(x,y)} \frac{6\xi\eta\tau}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta\right] d\tau +$$

$$+\frac{1}{2\pi}\int_{B_t(x,y)} \frac{\xi\eta}{\sqrt{t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta +$$

$$+\frac{1}{2\pi}\cdot\frac{\partial}{\partial t}\int_{B_t(x,y)}\frac{\xi^2-\eta^2}{\sqrt{t^2-(x-\xi)^2-(y-\eta)^2}}d\xi d\eta.$$

$$\left.\begin{cases}\xi=x+(t-\tau)u\\ \eta=y+(t-\tau)v\end{cases}\Rightarrow\begin{array}{l}d\xi d\eta=(t-\tau)^2\,dudv\\ B_{t-\tau}(x,y)\to B_1(0)\end{array}\right\}\Rightarrow$$

$$\int_{B_{t-\tau}(x,y)}\frac{6\xi\eta\tau}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}}d\xi s\eta=$$

$$=6\tau\int_{B_1(0)}\frac{[x+(t-\tau)u]\cdot[y+(t-\tau)v]}{(t-\tau)\sqrt{1-u^2-v^2}}(t-\tau)^2\,dudv=$$

$$=6\tau(t-\tau)\int_{B_1(0)}\frac{xy+(t-\tau)(xv+yu)+(t-\tau)^2\,uv}{\sqrt{1-u^2-v^2}}dudv=$$

$$=6\tau(t-\tau)\,xy\int_{B_1(0)}\frac{dudv}{\sqrt{1-u^2-v^2}}+$$

$$+6\tau(t-\tau)^2\int_{B_1(0)}\frac{xv+yu}{\sqrt{1-u^2-v^2}}dudv+$$

$$+6\tau(t-\tau)^3\cdot\int_{B_1(0)}\frac{uv}{\sqrt{1-u^2-v^2}}dudv.$$

$$\begin{cases}u=\rho\cos\theta\\ v=\rho\sin\theta\end{cases},\quad\rho\in[0,1],\ \theta\in[0,2\pi]\Rightarrow dudv=\rho d\theta d\rho\Rightarrow$$

$$\int_{B_1(0)}\frac{dudv}{\sqrt{1-u^2-v^2}}=$$

$$=\int_0^1\frac{\rho}{\sqrt{1-\rho^2}}d\rho\int_0^{2\pi}d\theta=2\pi\left(-\sqrt{1-\rho^2}\right)\Big|_0^1=2\pi.$$

$$\int_{B_1(0)} \frac{xv+yu}{\sqrt{1-u^2-v^2}} dudv = 0.$$

$$\int_{B_1(0)} \frac{uv}{\sqrt{1-u^2-v^2}} dudv = \int_0^1 \int_0^{2\pi} \frac{\rho^2}{\sqrt{1-\rho^2}} \sin\theta\cos\theta d\phi d\theta =$$

$$= \frac{1}{2}\int_0^1 \frac{\rho^2}{\sqrt{1-\rho^2}} d\rho \int_0^{2\pi} \sin 2\theta d\theta = 0.$$

$$\Rightarrow \int_{B_{t-\tau}(x,y)} \frac{f(\xi,\eta,\tau)}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}} d\xi d\eta = 6xy\tau(t-\tau)2\pi.$$

$$\Rightarrow \int_0^t d\tau \int_{B_{t-\tau}(x,y)} \frac{f(\xi,\eta,\tau)}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\eta)^2}} d\xi d\eta =$$

$$= 6xy\cdot 2\pi \int_0^t \left(t\tau-\tau^2\right) d\tau =$$

$$= 6xy\cdot 2\pi \cdot \left(t\frac{\tau^2}{2}-\frac{\tau^3}{3}\right)\Big|_0^t = 6xy\cdot 2\pi\cdot\left(\frac{t^3}{2}-\frac{t^3}{3}\right) = xyt^3\cdot 2\pi.$$

$$\cdot\cdot \int_{B_t(x,y)} \frac{\xi\eta}{\sqrt{t^2-(x-\xi)^2-(y-\eta)^2}} d\xi d\eta =$$

$$= \int_{B_1(0)} \frac{(x+tu)(y+tv)}{t\sqrt{1-u^2-v^2}} t^2 dudv =$$

$$= t\int_{B_1(0)} \frac{xy}{\sqrt{1-u^2-v^2}} dudv + t^2 \int_{B_1(0)} \underbrace{\frac{xv+yu}{\sqrt{1-u^2-v^2}}}_{0} dudv +$$

$$+t^3 \int_{B_1(0)} \underbrace{\frac{uv}{\sqrt{1-u^2-v^2}}}_{0} dudv = 2\pi\cdot t\cdot xy.$$

$$\cdots \int_{B_1(0)} \frac{\xi^2-\eta^2}{\sqrt{t^2-(x-\xi)^2-(y-\eta)^2}} d\xi d\eta =$$

$$= \int_{B_1(0)} \frac{(x+tu)^2-(y-tv)^2}{t\sqrt{1-u^2-v^2}} \cdot t^2 \cdot dudv =$$

$$= t\int_{B_1(0)} \frac{x^2-y^2}{\sqrt{1-u^2-v^2}} dudv + 2t^2 \int_{B_1(0)} \frac{xu-yv}{\sqrt{1-u^2-v^2}} dudv +$$

$$+t^3 \int_{B_1(0)} \frac{\overbrace{u^2-v^2}^{0}}{\sqrt{1-u^2-v^2}} dudv =$$

$$= 2\pi t\left(x^2-y^2\right) + t^3 \int_0^1 \frac{\rho^3}{\sqrt{1-\rho^2}} d\rho \int_0^{2\pi} \underbrace{\cos 2\theta}_{0}\, d\theta = 2\pi t\left(x^2-y^2\right) \Rightarrow$$

$$u(x,y,t) = \frac{1}{2\pi} \cdot 2\pi xyt^3 + \frac{1}{2\pi} txy + \frac{1}{2\pi} Z \frac{\partial}{\partial t}\left[2\pi t\left(x^2-y^2\right)\right] \Rightarrow$$

$$\boxed{u(x,y,t) = xyt^3 + xyt + x^2 - y^2}.$$

$$\cdot \Delta\phi = \Delta u_0 = \Delta u_1 = 0 \Rightarrow$$

$$u(x,y,t) = u_0(x,y) + t \cdot u_1(x,y) + \int_0^t (t-\tau) \cdot f(x,y,\tau)\, d\tau =$$

$$= x^2 - y^2 + xyt + \int_0^t (t-\tau) \cdot 6xy\tau d\tau = xyt^3 + xyt + x^2 - y^2.$$

Aplicaţia 3.40 Cazul $n = 1$.

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} + xt \\ u(x,0) = x^2 \\ \frac{\partial u}{\partial t}(x,0) = x. \end{cases}$$

Aplicăm formula lui D'Alembert pentru: $a = 2$, $f(x,t) = xt$, $\mathrm{u}_0(x) = x^2$ şi $u_1(x) = x$. Avem:

$$u(x,t) =$$

$$= \frac{1}{4}\int_0^t \int_{x-2(t-\tau)}^{x+2(t-\tau)} \xi\tau d\xi d\tau +$$

$$+\frac{1}{4}\int_{x-2t}^{x+2t} \xi d\xi + \frac{1}{2}\left[(x+2t)^2 + (x-2t)^2\right] =$$

$$= \frac{1}{8}\int_0^t \tau\left[(x+2(t-\tau))^2 - (x-2(t-\tau))^2\right] d\tau +$$

$$+\frac{1}{8}\left[(x+2t)^2 + (x-2t)^2\right] +$$

$$+\frac{x^2+4xt+4t^2+x^2-4xt+4t^2}{2} =$$

$$= \int_0^t \tau(t-\tau)\, x d\tau + \frac{8xt}{8} + x^2 + 4t^2 =$$

$$= x^2 + xt + 4t^2 + x\left(t\frac{\tau^2}{2} - \frac{\tau^3}{3}\right)\Bigg|_0^t = \frac{1}{6}xt^3 + xt + x^2 + 4t^2.$$

Remarca 3.41 Problema Cauchy pentru operatorul undelor se poate rezolva şi cu ajutorul dezvoltării în serie Taylor în jurul lui 0, după puterile lui t, dacă datele iniţiale sunt funcţii analitice.

Căutăm

$$u(x,t) = u_h(x,t) + u_p(x,t), \tag{3.49}$$

unde:

$$u_h(x,t) = \sum_{k\geq 0} \frac{t^{2k}}{(2k)!} \cdot a^{2k} \cdot \Delta^k u_0(x) + \sum_{k\geq 0} \frac{t^{2k+1}}{(2k+1)!} \cdot a^{2k} \cdot \Delta^k u_1(x), \tag{3.50}$$

unde: $\Delta^2 u_0 = \Delta(\Delta u_0), \ldots., \Delta^k u_0 = \Delta\left(\Delta^{k-1} u_0\right).$

$$u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds, \tag{3.51}$$

unde:

$$\widetilde{u}(x,t,s) = \sum_{k\geq 0} \frac{t^{2k+1}}{(2k+1)!} \cdot a^{2k} \cdot \Delta^k f(x,s). \tag{3.52}$$

În cele ce urmează vom aplica această metodă pentru următoarele probleme Cauchy pentru ecuaţia undelor.

Folosind dezvoltarea în serie Taylor rezolvaţi următoarele probleme Cauchy pentru operatorul undelor:

Aplicaţia 3.42

$$\text{i)} \begin{cases} \frac{\partial^2 u}{\partial t^2} = 8\Delta u + t^2 x^2 \\ u\big|_{t=0} = y^2 = u_0(x,y,z) \\ \frac{\partial u}{\partial t}\big|_{t=0} = z^2 = u_1(x,y,z) \end{cases} \qquad (n=3)$$

$$\text{ii)} \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + x \cdot e^t \cdot \cos(3y+4z) \\ u\big|_{t=0} = xy\cos z \\ \frac{\partial u}{\partial t}\big|_{t=0} = yz \cdot e^x \end{cases} \qquad (n=3)$$

$$\text{iii)} \begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + \cos x \cdot \sin y \cdot e^z \\ u\big|_{t=0} = x^2 e^{y+z} \\ \frac{\partial u}{\partial t}\big|_{t=0} = \sin x \cdot e^{y+z} \end{cases} \qquad (n=3)$$

$$\text{iv)}\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + (x^2+y^2+z^2)\cdot e^t \\ u\big|_{t=0} = \frac{\partial u}{\partial t}\big|_{t=0} = 0 \end{cases} \qquad (n=3)$$

$$v)\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + 6t\cdot e^{x\sqrt{2}}\sin y\cdot\cos z \\ u\big|_{t=0} = e^{x+y}\cos z\sqrt{2} \\ \frac{\partial u}{\partial t}\big|_{t=0} = e^{3y+4z}\sin 5x \end{cases} \qquad (n=3)$$

$$*vi)\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + x^3 - 3xy^2 \\ u\big|_{t=0} = e^x\cos y \\ \frac{\partial u}{\partial t}\big|_{t=0} = e^y\sin x \end{cases} \qquad (n=2)$$

$$vii)\begin{cases} \frac{\partial^2 u}{\partial t^2} = \Delta u + t\sin y \\ u\big|_{t=0} = x^2 \\ \frac{\partial u}{\partial t}\big|_{t=0} = \sin y \end{cases} \qquad (n=2)$$

$$viii)\begin{cases} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 6 \\ u\big|_{t=0} = x^2 \\ \frac{\partial u}{\partial t}\big|_{t=0} = 4x \end{cases} \qquad (n=1)$$

Soluţie: *i)* Avem $f(t,x,y,z) = t^2x^2$, $u_0(x,y,z) = y^2$ şi $u_1(x,y,z) = z^2$.

$$u_h(x,y,z,t) = \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\cdot 8^n\cdot\Delta^n\left(y^2\right) + \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\cdot 8^n\cdot\Delta^n\left(z^2\right) =$$

$$= \left(y^2 + \frac{t^2}{2}\cdot 8\cdot 2\right) + tz^2 + \frac{t^3}{6}\cdot 8\cdot 2 = y^2 + tz^2 + 8t^2 + \frac{8}{3}t^3.$$

$$\widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\cdot 8^n\cdot\Delta^n\left(s^2x^2\right) = \frac{t}{1!}s^2x^2 + \frac{t^3}{6}8\cdot 2\cdot s^2 =$$

$$= ts^2x^2 + \frac{8}{3}s^2t^3,$$

de unde avem:

$$\widetilde{u}_p(x,y,z,t)=$$

$$=\int_0^t \widetilde{u}(x,y,z,t-s,s)\,ds=$$

$$=x^2\int_0^t s^2(t-s)\,ds+\frac{8}{3}\int_0^t s^2(t-s)^3\,ds=$$

$$=x^2\left(t\frac{s^3}{3}-\frac{s^4}{4}\right)\Big|_0^t+\frac{8}{3}\int_0^t\left(s^2t^3-3t^2s^3+3ts^4-s^5\right)ds=$$

$$=x^2\frac{4t^4-3t^4}{12}+\frac{8}{3}\cdot\left(t^3\frac{s^3}{3}-3t^2\frac{s^4}{4}+3t\frac{s^5}{5}-\frac{s^6}{6}\right)\Big|_0^t=x^2\frac{t^4}{12}+\frac{t^6}{10}.$$

Deci:

$$u(x,y,z,t)=u_p(x,y,z,t)+u_h(x,y,z,t)=$$

$$=x^2\frac{t^4}{12}+y^2++tz^2+8t^2+\frac{8}{3}t^3+\frac{t^6}{10}.$$

ii) Avem

$$f(x,y,z,t)=x\cdot e^t\cdot\cos(3y+4z)\,,$$

$$u_0(x,y,z)=xy\cos z,\ u_1(x,y,z)=e^x\cdot yz.$$

$$u(x,y,z,t)=u_h(x,y,z,t)+u_p(x,y,z,t)\,.$$

$$u_h(x,y,z,t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\cdot\Delta^n u_0(x,y,z)+\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\cdot\Delta^n u_1(x,y,z)\,.$$

Calculăm $\Delta^n u_0,\ \Delta^n u_1$ şi $\Delta^n f$.

$$\Delta^0 u_0 = u_0 = xy\cos z,\ \Delta u_0 = -xy\cos z = -u_0;\ \Delta^2 u_0 = u_0....,$$

$$\Delta^{(n)} u_0 = (-1)^n \cdot u_0 = (-1)^n \cdot xy\cos z.$$

$$\Delta^0 u_1 = u_1 = yze^x,\ \Delta u_1 = u_1, ...,\ \Delta^n u_1 = u_1 = yze^x.$$

$$\Delta^0 f(x,y,z,s) = f(x,y,z,s) = xe^s\cos(3y+4z),$$

$$\Delta^1 f = -3^2 xe^s\cos(3y+4z) - 4^2 xe^s\cos(3y+4z) =$$

$$= -25xe^s\cos(3x+4y) = -25f$$

$$\Delta^2 f = (-25)^2 f, ...,\ \Delta^n f = (-25)^n f.$$

$$\cdot u_h(x,y,z,t) = \sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!} \cdot (-1)^n xy\cos z + \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} yz \cdot e^x =$$

$$= xy\cos z \cdot \cos t + yze^x \mathrm{sh} t.$$

$$\cdot \cdot \widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} (-25)^n xe^s\cos(3y+4z) =$$

$$= \frac{x}{5} e^s \cos(3y+4z) \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (5t)^{2n+1} =$$

$$= \frac{x}{5} \cdot e^s \sin 5t.$$

$$\cdot \cdot \cdot u_p(x,y,z,t) = \int_0^t \widetilde{u}(x,y,z,t-s,s)\,ds =$$

$$= \frac{1}{5} x \cdot \cos(3y+4z) \int_0^t e^s \sin 5(t-s)\,ds =$$

$$= \frac{x}{26} \left(e^t - \cos 5t - \frac{\sin 5t}{5} \right) \cos(3y+4z).$$

$$u(x,y,z,t) = xy\cos z \cdot \cos t + yze^x \mathrm{sh} t +$$

$$+\frac{x}{26}\left(e^t-\cos 5t-\frac{\sin 5t}{5}\right)\cos\left(3y+4z\right).$$

iii) Avem $f\left(x,y,z,t\right)=\cos x\sin y\cdot e^z,\ u_0\left(x,y,z\right)=x^2\cdot e^{y+z}$ şi $u_1\left(x,y,z\right)=\sin x\cdot e^{y+z}$. Căutăm

$$u_h\left(x,y,z,t\right)=\sum_{n\geq 0}\frac{t^{2n}}{(2n)!}\Delta^n u_0\left(x,y,z\right)+\sum_{n\geq 0}\frac{t^{2n+1}}{(2n+1)!}\Delta^n u_1\left(x,y,z\right).$$

Calculăm

$$\Delta^0 u_0,\ \Delta u_0=2\left(x^2+1\right)e^{y+z},\ \Delta^2 u_0=2\left(x^2+3\right)e^{y+z},$$

$$\Delta^3 u_0=2\left(x^2+5\right)e^{y+z},$$

$$\Delta^4 u_0=2\left(x^2+7\right)e^{y+z},...,\ \Delta^n u_0=2\left(x^2+2n-1\right)e^{y+z},\ \ (\forall)\,n\geq 1.$$

Avem:

$$\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\Delta^n u_0=x^2e^{y+z}+2x^2t^2e^{y+z}\sum_{n=1}^{\infty}\frac{t^{2n+1}}{(2n-1)!}-2e^{y+z}\sum_{n=1}^{\infty}\frac{t^{2n}}{(2n)!}=$$

$$=x^2e^{y+z}+2x^2t\cdot e^{y+z}\mathrm{sh}t-2e^{y+z}\left(\mathrm{ch}t-1\right)=$$

$$=x^2e^{y+z}\left(1+2t\mathrm{sh}t\right)+2e^{y+z}\left(1-\right)\mathrm{ch}t.\quad \text{(I)}$$

Calculăm $\Delta^n u_1,\ n\geq 0$.

$$\Delta^0 u_1=u_1=\sin x\cdot e^{y+z},\ \Delta u_1=\sin x\cdot e^{y+z},...,\ \Delta^n u_1=\sin x\cdot e^{y+z},$$

deci:

$$\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\cdot\Delta^n u_1\left(x,y,z\right)=\sin x\cdot e^{y+z}\cdot\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}=$$

$$= \sin x \cdot e^{y+z} \cdot \mathrm{sh} t. \quad (II)$$

Din (I) şi (II) avem:

$$u_h(x,y,z,t) = e^{y+z}\left(x^2 + 2x^2 t \cdot \mathrm{sh} t + 4\mathrm{sh}^2 \frac{t}{2} + \sin x \cdot \mathrm{sh} t\right). \quad \text{(III)}$$

Calculăm:

$$\widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \Delta^n f(x,y,z,s)$$

$\Delta^0 f(x,y,z,s) = f(x,y,z,s) = \cos x \cdot \sin y \cdot e^z$
$\Delta^1 f(x,y,z,s) = -\cos x \cdot \sin y \cdot e^z$, $\Delta^2 f(x,y,z,s) = \cos x \cdot \sin y \cdot e^z, \ldots,$
$\Delta^n f(x,y,z,s) = (-1)^n \cos x \cdot \sin y \cdot e^z,$
deci:

$$\widetilde{u}(x,y,z,t,s) =$$

$$= \left[\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!}\right] \cos x \cdot \sin y \cdot e^z = e^z \cdot \cos x \cdot \sin y \cdot \sin t.$$

$$u_p(x,y,z,t) =$$

$$= \int_0^t \widetilde{u}(x,y,z,t-s,s)\, ds = e^z \cdot \cos x \cdot \sin y \int_0^t \sin(t-s)\, ds =$$

$$= e^z \cdot \cos x \cdot \sin y \cdot (1 - \cos t) = 2e^z \cdot \cos x \cdot \sin y \sin^2 \frac{t}{2}.$$

$$\cdot u(x,y,z,t) = u_h + u_p = 2e^z \cdot \cos x \cdot \sin y \sin^2 \frac{t}{2} +$$

$$+ e^{y+z}\left(x^2 + 2x^2 t \cdot \mathrm{sh} t + 4 \cdot \mathrm{sh}^2 \frac{t}{2} + \sin x \cdot \mathrm{sh} t\right).$$

iv)

$$\widetilde{u}(x,y,z,t,s)=\sum_{n\geq 0}\frac{t^{2n+1}}{(2n+1)!}\Delta^n f(x,y,z,s),$$

unde: $f(x,y,z,s)=(x^2+y^2+z^2)\,e^s$. Avem:

$$\Delta^0 f=\left(x^2+y^2+z^2\right)e^s,\ \Delta^1 f=6e^s,$$

$$\Delta^2 f=0,...,\ \Delta^n f=0,\ \ (\forall)\,n\geq 2.$$

Deci:

$$\widetilde{u}(x,y,z,t,s)=t\left(x^2+y^2+z^2\right)e^s+\frac{t^3}{3!}6e^s=t\left(x^2+y^2+z^2\right)e^s+t^3e^s.$$

Avem:

$$u(x,y,z,t)=\int_0^t\widetilde{u}(x,y,z,t-s,s)\,ds=$$

$$=\left(x^2+y^2+z^2\right)\int_0^t(t-s)\left(e^s\right)'ds+\int_0^t(t-s)^3\left(e^s\right)'ds=$$

$$=\left(x^2+y^2+z^2+6\right)\left(e^t-t-1\right)-t^3-3t^2.$$

v)

$$u_0(x,y,z)=$$

$$=e^{x+y}\cos z\sqrt{2},\ u_1(x,y,z)=e^{3y+4z}\sin 5x,\ f(x,y,z,t)=$$

$$=6t\cdot e^{x\sqrt{2}}\sin t\cdot\cos z.$$

$$u_h(x,y,z,t)=\sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\Delta^n u_0+\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\Delta^n u_1.$$

Avem:

$$\Delta^0 u_0 = u_0 = e^{x+y}\cos z\sqrt{2}, \ -\Delta u_0 = 0, ..., \ \Delta^n u_0 = 0, \ n \geq 2.$$

$$\Delta^0 u_1 = u_1 = e^{3y+4z}\sin 5x, \ \Delta \mathrm{u}_1 = 0, .., \ \Delta^n u_1 = 0, \ n \geq 2.$$

Deci:

$$u_h(x,y,z,t) = u_0 + tu_1 = e^{x+y}\cos z\sqrt{2} + t\cdot e^{3y+4z}\sin 5x.$$

$$\widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\Delta^n f(x,y,z,s) = 6t\cdot s\cdot e^{x\sqrt{2}}\sin y\cdot\cos z.$$

$$\Delta^0 f = f, \ \Delta^1 f = 0, ..., \ \Delta^n f = 0.$$

De unde:

$$u_p(x,y,z,t) =$$

$$= \int_0^t \widetilde{u}(x,y,z,t-s,s)\,ds = 6e^{x\sqrt{2}}\sin y\cdot\cos z\cdot\int_0^t s(t-s)\,ds =$$

$$= t^3\cdot e^{x\sqrt{2}}\sin y\cdot\cos z.$$

Prin urmare,

$$u(x,y,z,t) =$$

$$= u_p + u_h = t^3\cdot e^{x\sqrt{2}}\sin y\cdot\cos z + e^{y+x}\cos z\sqrt{2} + t\cdot e^{3y+4z}\sin 5x.$$

vi)

$$u_0(x,y) = e^x\cos y, \ u_1(x,y) = e^y\sin x, \ f(x,y,t) = x^3 - 3xy^2.$$

$$u_h(x,y,t) = \sum_{n=0}^{\infty}\frac{t^{2n}}{(2n)!}\Delta^n u_o(x,y) + \sum_{n=0}^{\infty}\frac{t^{2n+1}}{(2n+1)!}\Delta^n u_1(x,y)$$

$\Delta^0 u_0(x,y) = e^x \cos y,\ \Delta u_0(x,y) = 0$
$\Delta^0 u_1(x,y) = e^y \sin x,\ \Delta u_1(x,y) = 0.$

$$u_h(x,y,t) = e^x \cos y + t \cdot e^y \sin x.$$

$\Delta^0 f = x^3 - 3xy^2,\ \Delta f(x,y,s) = 0 \Rightarrow$

$$\widetilde{u}(x,y,t,s) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!} \Delta^n f(x,y,s) = t\left(x^3 - 3xy^2\right).$$

Deci:

$$u_p(x,y,t) = \int_0^t \widetilde{u}(x,y,t-s,s)\,ds = \left(x^3 - 3xy^2\right) \int_0^t (t-s)\,ds =$$

$$= \left(x^3 - 3xy^2\right)\left(t \cdot s - \frac{s^2}{2}\right)\Big|_0^t = \frac{t^2}{2}\left(x^3 - 3xy^2\right).$$

$$u(x,y,t) =$$

$$= u_p(x,y,t) + u_h(x,y,t) = \frac{t^2}{2}\left(x^3 - 3xy^2\right) + e^x \cos y + t \cdot e^y \sin x$$

vii)

$$u_0(x,y) = x^2 \Rightarrow \Delta^0 u_0 = x^2,\ \Delta u_0 = 2,\ \Delta^n u_0 = 0,\ (\forall)\, n \geq 2.$$

$$\Delta^0 u_1(x,y) = u_1(x,y) = \sin y,\ \Delta u_1 = -\sin y,$$

$$\Delta^2 u_1 = \sin y, ...,\ \Delta^n u_1 = (-1)^n \sin y.$$

Deci:

$$\cdot u_h(x,y,t) = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!}\Delta^n u_0 + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!}\Delta^n u_1 =$$

$$= x^2 + \frac{2t^2}{2!} + \left[\sum_{n\geq 0} (-1)^n \frac{t^{2n+1}}{(2n+1)!}\right] \sin y = x^2 + t^2 + \sin t \cdot \sin y.$$

$$\cdot \cdot \widetilde{u}(x,y,t,s) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{(2n+1)!}\Delta^n f(x,y,s)$$

$\Delta^0 f(x,y,s) = s \cdot \sin y$, $\Delta f = -s \cdot \sin y, ..., \Delta^n f = s \cdot (-1)^n \sin y$,

deci:

$$\widetilde{u}(x,y,t,s) = s \cdot \sin y \cdot \sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{(2n+1)!} = s \cdot \sin y \cdot \sin t.$$

De unde, obţinem:

$$u_p(x,y,t) = \int_0^t \widetilde{u}(x,y,t-s,s)\,ds = \sin y \cdot \int_0^t s \cdot \sin(t-s)\,ds =$$

$$= \sin y \cdot \left[s \cdot \cos(t-s)\big|_0^t - \int_0^t \cos(t-s)\,ds \right] =$$

$$= \sin y \left[t + \sin(t-s)\big|_0^t \right] = (t - \sin t) \cdot \sin y.$$

Deci, soluţia problemei este:

$$u(x,y,t) = x^2 + t^2 + \sin t \cdot \sin y + (t - \sin t)\sin y = x^2 + t^2 + t \sin y.$$

viii) $u = u_h + u_p$ unde:

$$u_h(x,t) = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!} \Delta^n u_0(x) + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \Delta^n u_1(x) =$$

$$= x^2 + 2\frac{t^2}{2!} + t \cdot 4x = x^2 + t^2 + 4xt.$$

$$\widetilde{u}(x,t,s) = \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \Delta^n f(x,s) = 6t \Rightarrow$$

$$u_p(x,t) = \int_0^t \widetilde{u}(x,t-s,s)\, ds = 6\int_0^t (t-s)\, ds =$$

$$= \frac{-6}{2} \cdot (t-s)^2\Big|_0^t = 3t^2.$$

Deci:

$$u(x,t) = u_h(x,t) + u_p(x,t) = x^2 + t^2 + 4xt + 3t^2 = (x+2t)^2.$$

3.4.2 Problema Cauchy pentru ecuaţia undelor în distribuţii

n=3, Kirchhoff

I. Soluţia fundamentală a ecuaţiei undelor

Definiţia 3.43 Fie ecuaţia diferenţială liniară cu coeficienţi constanţi: $L(D) = f$, unde

$$L(D) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ [\alpha] \leq m}} a_\alpha D^\alpha \tag{3.53}$$

$f \in D'(\mathbb{R}^n)$ şi G un domeniu din $\mathbb{R}^n$.

i) Fie $f \in C(G)$; se numeşte *soluţie clasică* a ecuaţiei (3.53) în G orice funcţie $u \in C^m$ pentru care:

$$L(D)u(x) = f(x), (\forall)x \in G.$$

ii) se numeşte *soluţie generalizată* (soluţie în distribuţii) a ecuaţiei (3.53) în G orice distribuţie $u \in D'(\mathbb{R}^n)$ astfel încât

$$< L(D)u, \phi >=< f, \phi >, (\forall)\phi \in D(\mathbb{R}^n).$$

iii) Fie $f = \delta_0$ distribuţia lui Dirac $\langle \delta_0, \varphi \rangle = \varphi(0)$, $\forall \varphi \in D^n(\mathbb{R})$; soluţia generalizată a ecuaţiei (3.53) în $\mathbb{R}^n$ - notată E - se numeşte *soluţie fundamentală* (sau funcţie de influenţă).

Proprietatea 3.44 Dacă există $E * f$ în $D'(\mathbb{R}^n) \Rightarrow u = E * f$ este unica soluţie generalizată.
u soluţie clasică $\Rightarrow u$ soluţie generalizată;
$f \in C(G), u$ soluţie generalizată şi $u \in C^m(G) \Rightarrow u$ soluţie clasică.

Proprietatea 3.45 Soluţia fundamentală a ecuaţiei undelor pentru $n = 3$

$$\frac{\partial^2 E_3}{\partial t^2} - a^2 \Delta E_3 = \delta_0(x, t)$$

este:

$$E_3(x, t) = \frac{H(t)}{4\pi a^2 t} \delta_{S_{at}}(x) \tag{3.54}$$

care acţionează astfel:

$$\langle E_3(x,t), \phi(x,t)\rangle = \frac{1}{4\pi a^2}\int_0^\infty \langle \delta_{S_{at}}(x), \phi(x,t)\rangle \frac{dt}{t} =$$

$$= \frac{1}{4\pi a^2}\int_0^\infty \int_{S_{at}(x)} \phi(x,t) d\sigma_x \frac{dt}{t}, (\forall)\phi \in S(\mathbb{R}^{3+1}). \qquad (3.55)$$

Demonstraţie. Fie ecuaţia undelor pentru orice $n \geq 1$:

$$\frac{\partial^2 E_n}{\partial t^2} - a^2 \Delta E_n = \delta_0(x,t)$$

căreia îi aplicăm transformata Fourier parţială F_x şi găsim:

$$F_x\left[\frac{\partial^2 E_n}{\partial t^2}(x,t)\right](\xi) - a^2 F_x\left[\Delta E_n(x,t)\right](\xi) = F_x\left[\delta_0(x,t)\right](\xi).$$

Notăm: $\tilde{E}_n(\xi,t) = F_x\left[E_n(x,t)\right](\xi)$ şi ţinând cont de proprietatea transformatei Fourier în distribuţii avem:

$$F_x\left[\frac{\partial^2 E_n}{\partial t^2}(x.t)\right](\xi) = \frac{\partial^2}{\partial t^2} F_x\left[E_n(x,t)\right](\xi) = \frac{\partial^2 \tilde{E}_n}{\partial t^2}(\xi.t);$$

$$F_x\left[\Delta E_n(x,t)\right](\xi) = -\left\|\xi\right\|^2 F_x\left[E_n(x,t)\right](\xi) = -\left\|\xi\right\|^2 \cdot \tilde{E}_n(\xi,t);$$

$$F_x\left[\delta_0(x,t)\right](\xi) = F_x\left[\delta_0(x)\cdot\delta_0(t)\right](\xi) =$$

$$= F_x\left[\delta_0(x)\right](\xi)\cdot\delta_0(t) = 1\cdot\delta_0(t) = \delta_0(t).$$

Ecuaţia anterioară devine:

$$\frac{\partial^2 \tilde{E}_n}{\partial t^2}(\xi,t) + a^2\left\|\xi\right\|^2 \tilde{E}_n(\xi,t) = \delta_0(t),$$

care în $S'(\mathbb{R}^n)$ are soluţia: $\tilde{E}_n(\xi, t) = H(t) \cdot \frac{\sin at\|\xi\|}{a\|\xi\|}$. Fie $n = 3$. Aplicăm transformata Fourier inversă şi ştim:

$$F\left[\overbrace{\delta \underbrace{S_a}_{\text{sfera de rază } a \text{ şi centru } 0 \text{ în } \mathbb{R}^3} (x)}^{\text{distribuţia simplu strat pentru } S_a}\right](\xi) = 4\pi a \cdot \frac{\sin a\|\xi\|}{\|\xi\|},$$

de unde găsim:

$$E_3(x,t) = \frac{H(t)}{a} \cdot F_\xi^{-1}\left[\frac{\sin at\,\|\xi\|}{\|\xi\|}\right](x) = \frac{H(t)}{a\pi a^2 t}\delta_{S_{at}}(x).$$

$$E_3(x,t) = \frac{H(t)}{a\pi a^2 t}\delta_{S_{at}}(x).$$

□

II. Problema Cauchy generalizată pentru ecuaţia undelor:

Fiind dată $F \in D'(\mathbb{R}^{3+1})$ cu supp$F \subset \mathbb{R}^3 \times [0, \infty)$ să se găsească $u \in D'(\mathbb{R}^{3+1})$ astfel încât $\left(\frac{\partial^2}{\partial t^2} - a^2\Delta\right) u = F(x,t)$ în $D'(\mathbb{R}^{3+1})$.

i) $\exists! u \in D'(\mathbb{R}^{3+1})$ soluţie generalizată, $u = E_n * F$, u dependent continuu de F în $D'(\mathbb{R}^{3+1})$.

ii) Dacă $F(x,t) = f(x,t) + u_1(x) \cdot \delta_0(t) + u_0(x) \cdot \delta_0'(t)$ cu $f \in D'(\mathbb{R}^{3+1})$, supp$f \in \mathbb{R}^3 \times [0, \infty)$ şi $u_0, u_1 \in D'(\mathbb{R}^3) \Rightarrow$

$u(x,t) = V_3(x,t) + V_3^1(x,t) + V_3^{(0)}(x,t)$, unde: $V_3 = f * E_3$ potenţial retardat de densitate f; $V_3^1 = [u_0(x) \cdot \delta_0'(t)] * E_3 = \frac{\partial}{\partial t}([u_0(x) \cdot \delta_0(t)] * E_3)$, şi se numeşte *potenţial retardat superficial de dublu strat* cu densitatea u_0; $V_3^{(0)} = [u_1(x) \cdot \delta_0(t)] * E_3$ se numeşte *potenţial retardat superficial de simplu strat* cu densitatea u_1.

Teorema 3.46 *Dacă* $f \in L^1_{loc}(\mathbb{R}^{3+1})$ *atunci*

$$V_3 \in L^1_{loc}(\mathbb{R}^{3+1})$$

şi

$$V_3(x,t) = \frac{H(t)}{4\pi a^2} \int_{B_{at}(x)} \frac{f\left(\xi, t - \frac{1}{a}\|x-\xi\|\right)}{\|x-\xi\|} d\xi.$$

Demonstraţie. Fie $\phi \in D(\mathbb{R}^3)$ şi avem:

$$< V_3, \phi > = < f(x,t) * E_3(y,\tau), \phi > =$$

$$= < f(x,t) \cdot E_3(y,\tau), \eta(\tau)\eta(t)\eta(a^2\tau^2 - \|y\|^2) \cdot \phi(x+y, t+\tau) > =$$

$$= < E_3(y,\tau), \eta(\tau)\eta(a^2\tau^2 - \|y\|^2) < f(x,t), \eta(t) \cdot \phi(x+y, t+\tau) >> =$$

$$= < E_3(y,\tau), \eta(\tau)\eta(a^2\tau^2 - \|y\|^2) \int_{\mathbb{R}^4} f(x,t) \cdot \eta(t) \cdot \phi(x+y, t+\tau) dxdt > =$$

$$= < E_3(y,\tau), \eta(\tau)\eta(a^2\tau^2 - \|y\|^2) \int_{\mathbb{R}^4} f(x-y, t-\tau) \cdot \eta(t-\tau) \cdot \phi(x,t) dxdt > =$$

$$= \frac{1}{4\pi a^2} \int_{\mathbb{R}^3} \frac{1}{\|y\|} \cdot \eta\left(\frac{\|y\|}{a}\right) \eta(0) \cdot$$

$$\cdot \left[\int_{\mathbb{R}^4} f\left(x - y, t - \frac{\|y\|}{a}\right) \cdot \eta\left(t - \frac{\|y\|}{a}\right) \cdot \phi(x,t) dxdt\right] dy \overset{*}{=}$$

$$< E_3(x,t), \phi(x,t) > = \frac{1}{4\pi a^2} \int_{\mathbb{R}^{3+1}} \frac{H(t)}{4\pi a^2 t} \cdot \delta_{S_{at}}(x) \cdot \phi(x,t) dxdt =$$

$$= \frac{1}{4\pi a^2} \int_0^\infty \frac{1}{t} \int_{S_{at}} \phi(x,t) d\sigma_x dt =$$

$$= \frac{1}{4\pi a^2} \int_{\mathbb{R}^3} \frac{\phi\left(x, \frac{\|x\|}{a}\right)}{\|x\|} dx$$

$$\stackrel{*}{=} \frac{1}{4\pi a^2} \int_{\mathbb{R}^4} \left[\int_{\mathbb{R}^3} \frac{1}{\|y\|} \cdot f\left(x-y, t-\frac{\|y\|}{a}\right) dy \right] \cdot \phi(x,t) dx dt =$$

$$= \int_{\mathbb{R}^4} \left[\frac{H(t)}{4\pi a^2} \cdot \int_{B_{at}(0)} \frac{1}{\|y\|} \cdot f\left(x-y, t-\frac{\|y\|}{a}\right) dy \right] \cdot \phi(x,t) dx dt \Rightarrow$$

$$\Rightarrow V_3(x,t) = \frac{H(t)}{4\pi a^2} \cdot \int_{B_{at}(0)} \frac{1}{\|y\|} \cdot f\left(x-y, t-\frac{\|y\|}{a}\right) dy =$$

dar $x - y = \xi$

$$= \frac{H(t)}{4\pi a^2} \cdot \int_{B_{at}(0)} \frac{f\left(\xi, t - \frac{1}{a}\|x-\xi\|\right)}{\|x-\xi\|} d\xi.$$

□

Teorema 3.47 *Dacă $u_1 \in L^1_{loc}(\mathbb{R}^n) \Rightarrow V_3^{(0)} \in L^1_{loc}(\mathbb{R}^4)$ şi pentru orice $\phi \in D(\mathbb{R}^4)$ avem:*

$$< V_3^{(0)}, \phi >=< [u_1(x) \cdot \delta(t)] * E_3(x,t), \phi >=$$

$$=< u_1(x) \cdot \delta(t) \cdot E_3(y,\tau), \eta(t) \cdot \eta(\tau) \cdot \eta(a^2\tau^2 - \|y\|^2) \cdot \phi(x+y, t+\tau) >=$$

$$=< E_3(y,\tau), \eta(a^2\tau^2 - \|y\|^2) \cdot \eta(\tau) < u_1(x) \cdot \delta(t), \eta(t) \cdot \phi(x+y, t+\tau) >>=$$

$$=< E_3(y,\tau), \eta(\tau) \cdot \eta(a^2\tau^2 - \|y\|^2) \cdot \int_{\mathbb{R}^3} u_1(x) \cdot \phi(x+y, \tau) dx >=$$

$$= \frac{1}{4\pi a^2} \int_0^\infty \frac{\eta(0)}{\tau} \int_{S_{a\tau}(0)} \left[\int_{\mathbb{R}^3} u_1(x) \cdot \phi(x+y, \tau) dx \right] d\sigma_y d\tau =$$

$$= \frac{1}{4\pi a^2}\int_0^{\infty}\frac{1}{\tau}\cdot\int_{\mathbb{R}^3}\int_{S_{a\tau}(0)} u_1(x)\cdot\phi(x+y,\tau)d\sigma_y d\tau dx =$$

$$\frac{1}{4\pi a^2}\int_0^{\infty}\int_{\mathbb{R}^3}\int_{S_{a\tau}(0)}\frac{1}{\tau}\cdot u_1(x)\cdot\phi(x+y,\tau)d\sigma_y dx d\tau =$$

$$= \frac{1}{4\pi a^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}}\frac{H(t)}{t}\left[\int_{S_{at}(0)} u_1(x-y)d\sigma_y\right]\cdot\phi(x,t)dxdt =$$

$$= \int_{\mathbb{R}^4}\frac{H(t)}{4\pi a}\int_{S_{at}(0)} u_1(x-y)d\sigma_y \Rightarrow V_3^{(0)} = \frac{H(t)}{4\pi a^2 t}\cdot\int_{S_{at}(0)} u_1(x-y)d\sigma_y$$

$\underline{\underline{\xi = x - y}}$

$$= \frac{H(t)}{4\pi a^2 t}\cdot\int_{S_{at}(0)} u_1(\xi)d\sigma_\xi.$$

Analog avem:

$$V_3^{(1)} = [u_0(x)\cdot\delta_0'(t)] * E_3 = \frac{\partial}{\partial t}\{[u_0(x)\cdot\delta_0(t)] * E_3\} \Rightarrow$$

$$< V_3^{(1)}, \phi >= \frac{\partial}{\partial t} < [u_0(x)\cdot\delta_0(t)] * E_3, \phi >=$$

conform teoremei 2

$$= \frac{\partial}{\partial t}\left\{\int_{\mathbb{R}^4}\left[\int_{S_{at}(x)} u_0(\xi)d\sigma_\xi\right]\cdot\phi(x,t)dxdt\right\} \Rightarrow$$

$$V_3^{(1)}(x,t) = \frac{\partial}{\partial t}\left[\frac{H(t)}{4\pi a^2 t}\cdot\int_{S_{at}(x)} u_0(\xi)d\sigma_\xi\right]$$

Soluţia clasică a problemei Cauchy pentru $n = 3$:

$$u(x,t) = V_3 + V_3^{(0)} + V_3^{(1)} =$$

$$\frac{1}{4\pi a^2}\cdot\int_{B_{at}(0)}\frac{f\left(\xi, t - \frac{1}{a}\|x-\xi\|\right)}{\|x-\xi\|}d\xi + \frac{1}{4\pi a^2 t}\cdot\int_{S_{at}(0)} u_1(\xi)d\sigma_\xi +$$

$$+\frac{1}{4\pi a^2}\cdot\frac{\partial}{\partial t}\left[\frac{1}{t}\cdot\int_{S_{at}(x)} u_0(\xi)d\sigma_\xi\right],$$

pentru:

$$\begin{cases} f\in C^2(t>0);\\ u_0\in C^3(\mathbb{R}^3);\\ u_1\in C^2(\mathbb{R}^3).\end{cases}$$

(formula lui Kirchhoff)

Problema Cauchy pentru ecuaţia undelor, cazul $n = 2$.

Fie ecuaţia undelor în distribuţii pentru $n = 2$.

$$\frac{\partial^2 E_2}{\partial t^2} - a^2\Delta E_2 = \delta_0(x,t) \leftarrow E_2 \tag{3.56}$$

se numeşte *soluţia fundamentală* a ecuaţiei undelor.
Aplicăm Fourier parţială F_x asupra ecuaţiei anterioare. Ecuaţia devine:

$$\frac{\partial^2 \tilde{E}_2}{\partial t^2}(\xi,t) - a^2\left\|\xi\right\|^2 \underbrace{\tilde{E}_2(\xi,t)}_{F_x[E_2(x,t)](\xi,t)} = \delta_0(t) \Rightarrow$$

în $S'(\mathbb{R})$ avem soluţia:

$$\tilde{E}(\xi,t) = H(t)\frac{\sin at\left\|\xi\right\|}{a\left\|\xi\right\|}.$$

Folosim metoda coborârii: din $E_3(x,x_3,t)$ găsim $E_2(x,t)$. Fie $\{\eta_k\}_k \subset D(\mathbb{R})$, $\eta_k \xrightarrow{k\to\infty} 1$ pe $\mathbb{R}$ şi $\phi(x,t)\in D(\mathbb{R}^{2+1}) \Rightarrow$

$$< E_2(x,t),\phi(x,t) >= \lim_{k\to\infty} < E_3(x,x_{\_3},t),\phi(x,t)\cdot\eta_k(x_3) >=$$

$$= \lim_{k\to\infty} \frac{1}{4\pi a^2} \int_0^\infty \int_{S_{at}} \phi(x,t)\cdot \eta_k(x_3) d\sigma_{x,x_3} \frac{dt}{t} =$$

$$= \int_0^\infty \int_{S_{at}} \phi(x,t) d\sigma_{x,x_3} \frac{dt}{t}$$

S_{at} are ecuaţia:

$$\|x\|^2 + x_3^2 = a^2t^2 \Rightarrow x_3 = \pm\sqrt{a^2t^2 - \|x\|^2} \Rightarrow$$

$$d\sigma_{x,x_3} = \sqrt{1 + \left(\frac{\partial x_3}{\partial x_1}\right)^2 + \left(\frac{\partial x_3}{\partial x_2}\right)^2} = \frac{at}{\sqrt{a^2t^2 - \|x\|^2}} dx \Rightarrow$$

$$\Rightarrow < E_2(x,t), \phi(x,t) >=$$

$$= \frac{1}{4\pi a^2} \int_0^\infty 2 \cdot \int_{\substack{B_{at}(0) \\ \|x\| \le at}} \frac{at\cdot\phi(x,t)}{\sqrt{a^2t^2 - \|x\|^2}} dxdt =$$

$$= \frac{2}{4\pi a^2} \int_0^\infty \int_{\|x\|\le at} \frac{at\cdot\phi(x,t)}{\sqrt{a^2t^2 - \|x\|^2}} dxdt =$$

$$= \frac{1}{2\pi a^2} \int_{\mathbb{R}^3} \frac{H(at - \|x\|)}{\sqrt{a^2t^2 - \|x\|^2}} \cdot \phi(x,t) dxdt \Rightarrow$$

$$E_2(x,t) = \frac{1}{2\pi a^2} \cdot \frac{H(at - \|x\|)}{\sqrt{a^2t^2 - \|x\|^2}}$$

$V_2 = f * E_2$ şi fie $\phi \in D(\mathbb{R}^{2+1})$ avem:

$$< V_2, \phi >=< f * E_2, \phi >=$$

$$=< f(x,t) \cdot E_2(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2 - \|y\|^2) \cdot \phi(x+y, t+\tau) >=$$

$$=< E_2(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2-\|y\|^2)\cdot\int_{\mathbb{R}^3} f(x-y,t-\tau)\cdot\phi(x,y)dxdt >=$$

$$= \int_0^\infty \int_{B_{a\tau}(0)} \frac{1}{2\pi a\sqrt{a^2\tau^2-\|y\|^2}}\cdot$$

$$\cdot\left[\int_{\mathbb{R}^3} f(x-y,t-\tau)\cdot\phi(x,y)dxdt\right] dyd\tau =$$

$$= \int_{\mathbb{R}^3}\left[\int_0^\infty \frac{f(x-y,t-\tau)}{2\pi a\sqrt{a^2\tau^2-\|y\|^2}} dyd\tau\right]\phi(x,t)dxdt \Rightarrow$$

$$V_2(x,t) = H(t)\int_0^t \int_{B_{a\tau}(0)} \frac{f(x-y,t-\tau)}{2\pi a\sqrt{a^2\tau^2-\|y\|^2}} dyd\tau \overset{t-\tau=\tau'}{\underset{\xi=x-y}{=}}$$

$$= \frac{H(t)}{2\pi a}\int_0^\infty \int_{B_{a(t-\tau)}(x)} \frac{f(\xi,\tau)d\xi d\tau}{\sqrt{a^2(t-\tau)^2-\|x-\xi\|^2}} \Rightarrow$$

$$V_2(x,t) = \frac{H(t)}{2\pi a}\int_0^t \int_{B_{a(t-\tau)}(x)} \frac{f(\xi,\tau)}{\sqrt{a^2(t-\tau)^2-\|x-\xi\|^2}} d\xi d\tau$$

$V_2^0 = [u_1(x)\cdot\delta(t)] * E_2(x,t)$ şi fie $\phi\in D(\mathbb{R}^3)\Rightarrow$

$$\langle V_2^{(0)}, \phi\rangle =$$

$$=< E_2(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2-\|y\|^2) < \delta(t)u_1(x), \phi(x+y,t+\tau) >>=$$

$$=< E_2(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2-\|y\|^2)\cdot\int_{\mathbb{R}^2} u_1(x), \phi(x+y,\tau)dx >=$$

$$= \int_0^\infty \int_{B_{a\tau}(0)} \frac{1}{2\pi a\sqrt{a^2\tau^2-\|y\|^2}}\cdot\underbrace{\int_{\mathbb{R}^2} u_1(x)\cdot\phi(x+y,\tau)dx}_{=\int_{\mathbb{R}^2} u_1(x-y)\cdot\phi(x,\tau)dx} \overset{*}{=}$$

$$< E_2(x,t), \phi(x,t) >=< \frac{H\left(at - \|x\|\right)}{2\pi a\sqrt{a^2\tau^2 - \|y\|^2}}, \phi(x,t) >=$$

$$= \int_0^\infty \int_{B_{at}(0)} \frac{\phi(x,t)}{2\pi a\sqrt{a^2\tau^2 - \|y\|^2}} dxdt$$

$$\stackrel{*}{=} \int_{\mathbb{R}^3} \int_0^\infty \left[\int_{B_{a\tau}(0)} \frac{1}{2\pi a\sqrt{a^2\tau^2 - \|y\|^2}} \cdot u_1(x-y)dy \right] \phi(x,\tau)d\tau dx =$$

$$\int_{\mathbb{R}^3} \left[\frac{H(\tau)}{2\pi a} \int_{B_{a\tau}(0)} \frac{u_1(x-y)}{2\pi a\sqrt{a^2\tau^2 - \|y\|^2}} dy \right] \phi(x,\tau)dxd\tau \stackrel{t=\tau}{\Rightarrow}$$

$$V_2^{(0)} = \frac{H(t)}{2\pi a} \int_{B_{at}(0)} \frac{u_1(x-y)}{2\pi a\sqrt{a^2t^2 - \|y\|^2}} dy =$$

$$= \frac{H(t)}{2\pi a} \int_{B_{at}(x)} \frac{u_1(\xi)}{2\pi a\sqrt{a^2t^2 - \|x-\xi\|^2}} d\xi \rightarrow \text{ Poisson pentru soluţia}$$

clasică, la fel pentru

$$V_2^1(x,t) = \frac{\partial}{\partial t} \left[\frac{H(t)}{2\pi a} \int_{B_{at}(x)} \frac{u_0(\xi)}{\sqrt{a^2t^2 - \|x-\xi\|^2}} d\xi \right]$$

$$u(x,t) = V_2(x,t) + V_2^0(x,t) + V_2^1(x,t),$$

de unde se confirmă formula Poisson pentru soluţia clasică.

Problema undelor. Cazul $n = 1$.

$$\tilde{E}_1(\xi, t) = H(t) \cdot \frac{\sin at\xi}{a\xi} \Rightarrow$$

$$E_1(x,t) = \frac{H(t)}{a} F_\xi^{-1}\left[\frac{\sin at\xi}{\xi}\right](x) = \frac{H(at - |x|)}{2a} = E_1(x,t)$$

Fie $\phi \in D(\mathbb{R})$. Avem:

$$< V_1, \phi >=$$

$$=< E_1(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2 - |y|^2)\cdot \int_{\mathbb{R}^2} f(x-y, t-\tau)\phi(x,t)dxdt >=$$

$$= \int_0^\infty \int_{-a\tau}^{a\tau} \frac{1}{2a}\left[\int_{\mathbb{R}^2} f(x-y, t-\tau)\cdot\phi(x,y)dxdt\right] dyd\tau =$$

$$= \int_{\mathbb{R}^2}\left[\int_0^\infty \int_{-a\tau}^{a\tau} \frac{1}{2a} f(x-y, t-\tau)dyd\tau\right]\phi(x,t)dxdt =$$

$$= \frac{H(t)}{2a}\int_0^t \int_{-a\tau'}^{a\tau'} f(x-y, t-\tau')dyd\tau' \underset{t-\tau'=\tau}{\overset{x-y=\xi}{\Rightarrow}} V_1(x,t) =$$

$$= \frac{H(t)}{2a}\int_0^t \int_{-a(t-\tau)}^{a(t-\tau)} f(\xi,\tau)d\xi d\tau$$

$$\phi \in D(\mathbb{R}^2) \Rightarrow \left\langle V_1^{(0)}, \phi\right\rangle =$$

$$= \left\langle E_1(y,\tau), \eta(\tau)\cdot\eta(a^2\tau^2 - |y|^2)\cdot\int_{\mathbb{R}} u_1(x)\phi(x+y,\tau)dx\right\rangle =$$

$$= \frac{1}{2a}\int_0^\infty \int_{-a\tau}^{a\tau}\int_{\mathbb{R}} u_1(x)\phi(x+y,\tau)dxdyd\tau =$$

$$= \frac{1}{2a}\int_0^\infty \int_{\mathbb{R}}\left[\int_{-a\tau}^{a\tau} u_1(x-y)\,dy\right]\phi(x,\tau)\,dxd\tau =$$

$$= \int_{\mathbb{R}^2} \left[\frac{H(t)}{2a} \int_{-a\tau}^{a\tau} u_1 \underbrace{(x-y)}_{\xi} dy \right] \phi(x,t)\, dxdt \Rightarrow$$

$$\boxed{V_1^{(0)}(x,t) = \frac{H(t)}{2a} \int_{x-at}^{x+at} u_1(\xi)\, d\xi}.$$

$$\cdot \boxed{V_1^{(1)}(x,t) = \frac{1}{2a} \cdot \frac{\partial}{\partial t} \left[H(t) \cdot \int_{x-at}^{x+at} u_1(\xi)\, d\xi \right]} \Rightarrow$$

formula lui D'Alembert:

$$u = V_1 + V_1^{(0)} + V_1^{(1)}.$$

3.5 Problema Cauchy pentru ecuaţia căldurii

3.5.1 Problema Cauchy pentru ecuaţia căldurii în distribuţii

1. Soluţia fundamentală a ecuaţiei căldurii:

$$\frac{\partial E}{\partial t} - a^2 \Delta E = \delta_0(x,t)$$

este:

$$E(x,t) = \frac{H(t)}{(2a\sqrt{\pi t})} \cdot e^{-\frac{\|x\|^2}{4a^2 t}}.$$

Demonstraţie. Aplicăm transformata Fourier parţială F_x asupra ecuaţiei căldurii şi obţinem:

$$F_x \left[\frac{\partial E}{\partial t}(x,t) \right](\xi) - a^2 F_x [\Delta E(x,t)](\xi) = F_x [\delta_0(x,t)](\xi).$$

Notăm

$$\widetilde{E}(\xi,t) = F_x[E(x,t)](\xi,t) = \int_{\mathbb{R}^{n+1}} E(x,t)\cdot e^{ix\xi}dx$$

şi avem, folosind proprietăţile transformatei Fourier parţială:

$$F_x\left[\frac{\partial E}{\partial t}(x,t)\right](\xi) = \frac{\partial \widetilde{E}}{\partial t}(\xi,t);$$

$$F_x[\Delta E(x,t)](\xi) = -\|\xi\|^2\widetilde{E}(\xi,t)$$

şi

$$F_x[\delta_0(x,t)](\xi) = F_x[\delta_0(x)\cdot\delta_0(t)](\xi) =$$
$$= F_x[\delta_0(x)](\xi)\cdot\delta_0(t) = 1\cdot\delta_0(t) = \delta_0(t).$$

Ecuaţia căldurii devine:

$$\frac{\partial \widetilde{E}}{\partial t}(\xi,t) + a^2\|\xi\|^2\widetilde{E}(\xi,t) = \delta_0$$

şi în $S'(\mathbb{R})$ are soluţia:

$$\widetilde{E}(\xi,t) = H(t)\cdot e^{-a^2\|\xi\|^2t}$$

şi aplicăm transformata Fourier parţială în raport cu ξ şi obţinem, folosind inversa transformatei Fourier

$$E(x,t) = F_\xi^{-1}\left[\widetilde{E}(\xi,t)\right](x,t) = (2\pi)^{-n}F_\xi\left[\widetilde{E}(-\xi,t)\right](x,t) =$$

$$= \frac{H(t)}{(2\pi)^n}F_\xi\left[e^{-a^2\|\xi\|^2t}\right](x,t) = \frac{H(t)}{(2\pi)^n}F_\xi\left[e^{-2a^2t\frac{\|\xi\|^2}{2}}\right](x,t) =$$

$$= \frac{H(t)}{(2\pi)^n}F_\xi\left[e^{\frac{\|a\sqrt{2}t\xi\|^2}{2}}\right](x,t) = \boxed{F_\xi[f(a\xi)](x) = a^{-n}F_\xi[f(\xi)]\left(\frac{x}{a}\right)}$$

$$= \frac{H(t)}{(2\pi)^n} F_\xi \left[e^{-\frac{\|a\sqrt{2t}\xi\|^2}{2}} \right](x,t) =$$

în cazul nostru $a \to a\sqrt{2}t$

$$= \frac{H(t)}{(2\pi)^n} \cdot \frac{2}{\left(a\sqrt{2t}\right)^n} F_\xi \left[e^{-\frac{\|\xi\|^2}{2}} \right] \left(\frac{x}{a\sqrt{2t}} \right) =$$

$$= \frac{H(t)}{(2\pi)^n} \cdot \frac{1}{\left(a\sqrt{2t}\right)^n} \cdot (2\pi)^{\frac{n}{2}} \cdot e^{-\frac{\|\xi\|^2}{4a^2 t}}.$$

Avem următoarea transformare:

$$F_p \left[e^{-\frac{\|\xi\|^2}{2}} \right](x) = (2\pi)^{\frac{n}{2} \cdot e^{-\frac{\|x\|^2}{2}}} .$$

Deci

$$\boxed{E(x,t) = \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot e^{-\frac{\|x\|^2}{4a^2 t}}}.$$

□

2. Avem rezultatele:

i) $f(x,t) \in \mathrm{D}'(\mathbb{R}^{n+1})$ cu *supp* $f \subset \mathbb{R}^n \times [0,\infty]$ şi $u_0 \in \mathrm{D}'(\mathbb{R}^n)$; dacă există $V = E * f$ (potenţialul termic de densitate f) şi $V^0 = E(x,t) * [u_0 \cdot \delta_0(t)]$ (potenţialul termic superficial de densitate u_0) atunci:

$$\left(\frac{\partial}{\partial t} - a^2 \Delta \right) V = f(x,t), \left(\frac{\partial}{\partial t} - a^2 \Delta \right) V^{(0)} = u_0(x) * \delta_0(t)$$

în $u_0 \in \mathrm{D}'(\mathbb{R}^{n+1})$.

ii) Problema Cauchy generalizată pentru ecuaţia căldurii este: Fiind date $F, u_0 \in \mathrm{D}'(\mathbb{R}^n)$ cu *supp*$F \subset \mathbb{R}^n \times [0,\ \infty)$ să se gasească $u \in \mathrm{D}'(\mathbb{R}^{n+1})$ astfel încât $\frac{\partial u}{\partial t} - a^2 \Delta u = F(x,t)$ în $\mathrm{D}'(\mathbb{R}^{n+1})$.

Dacă există $E * F$ atunci unica soluţie generalizată este $u = E * F$. În particular, dacă $F(x,t) = f(x,t) + u_0 \cdot \delta_0(t)$, atunci $u = V + V^0$.

Demonstraţie.

$$u = E * F = F * E = f * E + [u_0(x) \cdot \delta_0(t)] * E(x,t) = V + V^0.$$

□

3. Avem pentru orice $\phi \in \mathrm{D}(\mathbb{R}^{n+1})$:

$$\langle V, \phi \rangle = \langle f(x,t) \cdot E(y,\tau), \eta(t) \cdot \eta(\tau) \cdot \phi(x+y, t+\tau) \rangle =$$

$$\langle E(y,\tau), \eta(t) \langle f(x,t), \eta(t) \cdot \phi(x+y, t+\tau) \rangle \rangle =$$

$$= \left\langle E(y,\tau), \eta(\tau) \int_{\mathbb{R}^{n+1}} f(x,t) \cdot \eta(t) \cdot \phi(x+y, t+\tau)\, dxdt \right\rangle =$$

$$= \left\langle E(y,\tau), \eta(\tau) \int_{\mathbb{R}^{n+1}} f(x-y, t-\tau) \cdot \eta(t-\tau) \cdot \phi(x,t)\, dxdt \right\rangle =$$

$$\langle f(x,t) \cdot E(y,\tau), \phi(x+y, t+\tau) \rangle =$$

$$= \langle E(y,t), \langle f(x,t), \phi(x+y, t+\tau) \rangle \rangle =$$

$$\left\langle E(y,\tau), \int_{\mathbb{R}^{n+1}} f(x,t) \cdot \phi(x+y, t+\tau)\, dxdt \right\rangle =$$

$$= \left\langle E(y,\tau), \int_{\mathbb{R}^{n+1}} f(x-y, t-\tau) \cdot \phi(x,t)\, dxdt \right\rangle =$$

$$= \int_{\mathbb{R}^{n+1}} E(y,\tau) \left[\int_{\mathbb{R}^{n+1}} f(x-y, t-\tau)\, \phi(x,t)\, dxdt \right] dyd\tau =$$

Avem relaţia $*$:

$$*E(x,t) = \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot e^{-\frac{\|x\|^2}{4a^2 t}} \Rightarrow$$

$$\langle E(x,t), \phi(x,t)\rangle =$$

$$= \int_{\mathbb{R}^{n+1}} \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot e^{-\frac{\|x\|^2}{4a^2 t}} \cdot \phi(x,t)\, dxdt =$$

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{\|x\|^2}{4a^2 t}}}{\left(2a\sqrt{\pi t}\right)^n} \cdot \phi(x,t)\, dxdt.$$

Cu relaţia * avem în continuare

$$= \int_{\mathbb{R}^{n+1}} \left[\int_{\mathbb{R}^{n+1}} \frac{H(\tau)}{\left(2a\sqrt{\pi t}\right)^n} \cdot e^{-\frac{\|y\|^2}{4a^2 \tau}} \cdot f(x-y, t-\tau)\, dyd\tau \right] \cdot$$

$$\cdot \phi(x,y)\, dxdt \Rightarrow$$

$$V(x,y) = \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{\|y\|^2}{4a^2 \tau}}}{\left(2a\sqrt{\pi \tau}\right)^n} \cdot f\left(\underbrace{x-y}_{\xi}, \underbrace{t-\tau}_{\alpha}\right) \underbrace{dyd\tau}_{d\xi d\alpha}$$

cu

$$\begin{array}{l} x-y=\xi;\ t-\tau=\alpha \\ \hline\hline y = x-\xi \\ \tau = t-\alpha;\ \alpha \to \tau \end{array}$$

avem

$$\int_0^\infty \int_{\mathbb{R}^n} \frac{f(\xi,\tau)}{2a\sqrt{\pi(t-\tau)}} \cdot e^{-\frac{\|x-\xi\|^2}{4a^2(t-\tau)}} d\xi d\tau.$$

$$\cdot \boxed{V(x,y) = \int_0^\infty \int_{\mathbb{R}^n} \frac{f(\xi,\tau)}{2a\sqrt{\pi(t-\tau)}} \cdot e^{-\frac{\|x-\xi\|^2}{4a^2(t-\tau)}} d\xi d\tau.}$$

4. $V^{(0)} = [u_0(x) \cdot \delta_0(t)] * E$ potenţialul termic superficial de densitate $u_0 \in \mathrm{D}(\mathbb{R}^n)$. Fie $\phi \in \mathrm{D}(\mathbb{R}^{n+1})$

$$\left\langle V^{(0)}, \phi \right\rangle = \left\langle \left[u_0(x) \cdot \delta_0(t)\right] * E(x,t), \phi(x,t) \right\rangle =$$

$$= \left\langle u_0(x) \cdot \delta_0(t) \cdot E(y,\tau), \eta(t) \cdot \eta(\tau) \cdot \phi(x+y, t+\tau) \right\rangle =$$

$$= \left\langle E(y,\tau) \cdot \left\langle u_0(x) \cdot \delta_0(t), \eta(t) \cdot \eta(\tau) \cdot \phi(x+y, t+\tau) \right\rangle \right\rangle =$$

$$= \left\langle E(y,\tau), \int_{\mathbb{R}^{n+1}} u_0(x) \cdot \phi(x+y,\tau)\, dx d\tau \right\rangle =$$

$$= \int_{\mathbb{R}^{n+1}} \left[\int_0^{\infty} \int_{\mathbb{R}^n} \frac{e^{-\frac{\|y\|^2}{4a^2\tau}}}{\left(2a\sqrt{\pi\tau}\right)^n} \cdot u_0(x) \cdot \phi(x+y,\tau)\, dy d\tau \right] dx d\tau =$$

$$= \left\langle E(y,\tau), \int_{\mathbb{R}^{n+1}} u_0(x-y)\, \phi(x,\tau)\, dx d\tau \right\rangle =$$

$$= \int_{\mathbb{R}^{n+1}} \left[\int_0^{\infty} \int_{\mathbb{R}^n} \frac{e^{-\frac{\|y\|^2}{4a^2\tau}}}{\left(2a\sqrt{\pi\tau}\right)^n} \cdot u_0(x-y)\, \phi(x,\tau)\, dy d\tau \right] dx d\tau =$$

$$= \int_{\mathbb{R}^{n+1}} \left[\int_0^{\infty} \int_{\mathbb{R}^n} \frac{e^{-\frac{\|y\|^2}{4a^2\tau}}}{\left(2a\sqrt{\pi\tau}\right)^n} \cdot u_0(x-y)\, dy d\tau \right] \phi(x,\tau)\, dx d\tau$$

Deci

$$V^{(0)}(x,t) = \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot \int_{\mathbb{R}^n} u_0(\xi) \cdot e^{-\frac{\|x-\xi\|^2}{4a^2 t}} d\xi.$$

Soluţia generalizată este:

$$u(x,t) = V(x,t) + V^{(0)}(x,t) =$$

$$= \int_0^{\infty} \int_{\mathbb{R}^n} \frac{f(\xi,\tau)}{\left[2a\sqrt{\pi(t-\tau)}\right]^n} \cdot e^{-\frac{\|x-\xi\|^2}{4a^2(t-\tau)}} d\xi d\tau +$$

$$+ \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot \int_{\mathbb{R}^n} u_0(\xi) \cdot e^{-\frac{\|x-\xi\|^2}{4a^2 t}} d\xi.$$

Proprietăţi 3.48 Proprietăţi ale soluţiei fundamentale pentru ecuaţia căldurii:

i) E este nenegativă, nulă pentru $t < 0$, indefinit diferenţiabilă pentru orice (x, t) şi local integrabilă pe $\mathbb{R}^{n+1}$;

ii) $\int_{\mathbb{R}^n} E(x,t)\,dx = 1, \ \ (\forall)\, t > 0$;

iii) $E(x,t) \to \delta_0(t)$ în $\mathrm{D}'(\mathbb{R}^n)$ pentru $t \to 0_+$.

3.5.2 Problema Cauchy clasică pentru operatorul căldurii

Teorema 3.49 *Considerăm datele* $f \in C(t \geq 0)$ *şi* $u_0 \in C(\mathbb{R}^n)$ *astfel încât* $u \in C^2(t > 0) \bigcap C(t \geq 0)$ *să verifice problema Cauchy clasică pentru ecuaţia căldurii:*

$$\begin{cases} \frac{\partial u}{\partial t} - a^2 \Delta u = f(x,t) \ \ \text{în } \mathbb{R}^n \times (0, \infty) \\ u\big|_{t=0} = u_0(x) \end{cases}. \tag{3.57}$$

Atunci problema (3.57) *are soluţie unică de forma:*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \frac{f(\xi,\tau)}{\left(2a\sqrt{\pi(t-\tau)}\right)^n} \cdot e^{-\frac{\|x-\xi\|^2}{4a^2(t-\tau)}} d\xi d\tau +$$

$$+ \frac{H(t)}{\left(2a\sqrt{\pi t}\right)^n} \cdot \int_{\mathbb{R}^n} u_0(\xi) \cdot e^{-\frac{\|x-\xi\|^2}{4a^2 t}} d\xi, \tag{3.58}$$

unde $H(t) = \begin{cases} 1, \ t \geq 0 \\ 0, \ t < 0. \end{cases}$

Exemplul 3.50 Să se rezolve problema Cauchy:

$$\begin{cases} \frac{\partial u}{\partial t} = 2\Delta u + t\cos x \\ u(x,0) = \cos y \cdot \cos z \end{cases} \quad (n=3)$$

Soluţie. Avem $a=\sqrt{2}$, $f(x,y,z,t) = t\cos x$, $u_0(x,y,z) = \cos y \cdot \cos z$. Cu formula (3.58) avem:

$$u(x,y,z,t) =$$

$$= \int_0^t \int_{\mathbb{R}^3} \frac{\tau\cos\xi}{\left(2\sqrt{2}\cdot\sqrt{\pi(t-\tau)}\right)^3} \cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\sigma)^2}{8(t-\tau)}} d\xi d\eta d\sigma d\tau +$$

$$+ \frac{1}{\left(2\sqrt{2t}\right)^3} \int_{\mathbb{R}^3} \cos\eta\cos\sigma \cdot e^{-\frac{(x-\xi)^2+(y-\eta)^2+(z-\sigma)^2}{8(t-\tau)}} d\xi d\eta d\tau =$$

$$= \int_0^t \frac{\tau}{\left[2\sqrt{2}\cdot\sqrt{\pi(t-\tau)}\right]^3} \cdot \left(\int_{\mathbb{R}} \cos\xi \cdot e^{-\frac{(x-\xi)^2}{8(t-\tau)}} d\xi\right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}} e^{-\frac{(y-\eta)^2}{8(t-\tau)}} d\eta\right) \cdot \left(\int_{\mathbb{R}} e^{-\frac{(z-\sigma)^2}{8(t-\tau)}} d\sigma\right) +$$

$$+ \int_0^t \frac{\tau}{\left[2\sqrt{2}\cdot\sqrt{\pi t}\right]^3} \cdot \left(\int_{\mathbb{R}} \cos\xi \cdot e^{-\frac{(x-\xi)^2}{8t}} d\xi\right) \cdot$$

$$\cdot \left(\int_{\mathbb{R}} e^{-\frac{(y-\eta)^2}{8t}} d\eta\right) \cdot \left(\int_{\mathbb{R}} e^{-\frac{(z-\sigma)^2}{8t}} d\sigma\right) \tag{3.59}$$

Facem schimbarea de variabilă:

$$\frac{x-\xi}{2\sqrt{2}\cdot\sqrt{t-\tau}} = u \Rightarrow d\xi = -2\sqrt{2}\sqrt{t-\tau}du, \xi = x - 2\sqrt{2}\sqrt{t-\tau}u.$$

Avem:

$$\int_{\mathbb{R}} \cos\xi \cdot e^{-\frac{(x-\xi)^2}{8(t-\tau)}} d\xi =$$

$$= \int_{\mathbb{R}} 2\sqrt{2}\sqrt{t-\tau}\cos\left(x - 2\sqrt{2}\sqrt{t-\tau}u\right) e^{-u^2} du =$$

$$= 2\sqrt{2}\sqrt{t-\tau}\cos x \cdot \int_{\mathbb{R}} \cos\left(2\sqrt{2}\sqrt{t-\tau}u\right) e^{-u^2} du \qquad (3.60)$$

Pentru a calcula ultima integrală din (3.60) introducem funcţia definită printr-o integrală improprie cu parametru:

$$F(\alpha) = \int_{\mathbb{R}} \cos(2\alpha u) \cdot e^{-u^2} du$$

şi derivând sub semnul integralei în raport cu α, obţinem:

$$F'(\alpha) = \int_{\mathbb{R}} (-2u) \cdot e^{-u^2} \cdot \sin(2\alpha u)\, du =$$

$$= e^{-u^2} \cdot \sin 2\alpha u\big|_{\mathbb{R}} - 2\alpha \int_{\mathbb{R}} e^{-u^2} \cdot \cos 2\alpha u du = -2\alpha F(\alpha)$$

de unde, rezolvând ecuaţia cu variabile separabile:

$$\frac{dF(\alpha)}{F(\alpha)} = -2\alpha d\alpha \Rightarrow \begin{cases} F(\alpha) = C \cdot e^{-\alpha^2} \\ F(0) = C = \sqrt{\pi} \end{cases},$$

de unde

$$F(\alpha) = \int_{\mathbb{R}} \cos(2\alpha u) \cdot e^{-u^2} du = \sqrt{\pi} e^{-\alpha^2},$$

de unde, punând $\alpha = \sqrt{2} \cdot \sqrt{t-\tau}$ obţinem pentru relaţia (3.60):

$$\int_{\mathbb{R}} \cos\xi \cdot e^{-\frac{(x-\xi)^2}{8(t-\tau)}} d\xi = 2\sqrt{2} \cdot \sqrt{\pi(t-\tau)} \cdot e^{-2(t-\tau)} \cos x \qquad (3.61)$$

În continuare, facem substituţia $\frac{y-\eta}{2\sqrt{2}\sqrt{t-\tau}} = v\ \eta = y - 2\sqrt{2}\sqrt{t-\tau}v$ $d\eta = -2\sqrt{2}\sqrt{t-\tau}dv$ şi respectiv substituţia $\frac{z-\sigma}{2\sqrt{2}\sqrt{t-\tau}} = w$ $\sigma = z - 2\sqrt{2}\sqrt{t-\tau}w$
$d\sigma = -2\sqrt{2}\sqrt{t-\tau}dw$ obţinând:

$$\begin{cases} \int_{\mathbb{R}} e^{-\frac{(y-\eta)^2}{8(t-\tau)}} d\eta = 2\sqrt{2}\sqrt{t-\tau} \int_{\mathbb{R}} e^{-v^2} dv = 2\sqrt{2}\sqrt{\pi(t-\tau)} \\ \int_{\mathbb{R}}^{-\frac{(z-\sigma)^2}{8(t-\tau)}} d\sigma = 2\sqrt{2}\sqrt{\pi(t-\tau)}. \end{cases} \tag{3.62}$$

Procedând analog, obţinem:

$$\begin{cases} \int_{\mathbb{R}} e^{-\frac{(x-\xi)^2}{8t}} d\xi = 2\sqrt{2\pi} \ \left(\text{cu substituţia } \frac{x-\xi}{2\sqrt{2t}} u \right) \\ \int_{\mathbb{R}} \cos\eta \cdot e^{-\frac{(y-\eta)^2}{8t}} d\eta = \int_{\mathbb{R}} 2\sqrt{2t} \cos y \cos\left(2\sqrt{2t}v\right) \cdot e^{-v^2} dv \\ \qquad = 2\sqrt{2t} \cos y \cdot e^{-2t} \left(\text{cu substituţia } \frac{y-\eta}{2\sqrt{2t}} = v \right) \\ \int_{\mathbb{R}} \cos\sigma \cdot e^{-\frac{(z-\sigma)^2}{8t}} d\sigma = 2\sqrt{2\pi t} \cos z \cdot e^{-2t} \\ \qquad \left(\text{cu substituţia } \frac{z-\sigma}{2\sqrt{2t}} = w \right). \end{cases} \tag{3.63}$$

Înlocuim relaţiile (3.61), (3.62) şi (3.63) în (3.60) şi obţinem:

$$u(x,y,z,t) =$$

$$= \int_0^t \frac{\tau \cos x}{\left(2\sqrt{2}\sqrt{\pi(t-\tau)}\right)^3} \left(2\sqrt{2}\sqrt{(t-\tau)}\right)^3 \cdot e^{-2(t-\tau)} d\tau +$$

$$+ \frac{1}{\left(2\sqrt{2}\sqrt{\pi t}\right)^3} \left(2\sqrt{2\pi t}\right)^3 \cdot \cos y \cos z \cdot e^{-4t} =$$

$$= \cos x \cdot e^{-2t} \int_0^t \tau \cdot e^{2\tau} d\tau + \cos y \cos z \cdot e^{-4t} =$$

$$= \cos x \left(\frac{t}{2} - \frac{1}{4} + \frac{1}{4} \cdot e^{-2t} \right) + \cos y \cos z \cdot e^{-4t}.$$

Exemplul 3.51

$$\begin{cases} \frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} + t + e^t \\ u(x,0) = 2 \end{cases} \qquad (n=1).$$

Cu formula (3.58) avem pentru datele $a = 2$, $f(x,t) = t + e^t$, $u_0(x) = 2$,

$$u(x,t) =$$

$$= \int_0^{\infty} \int_{\mathbb{R}} \frac{\tau + e^{\tau}}{4\sqrt{\pi(t-\tau)} \cdot e^{-\frac{(x-\xi)^2}{16(t-\tau)}} d\xi d\tau} + \frac{1}{4\sqrt{\pi t}} \int_{\mathbb{R}} 2e^{-\frac{(x-\xi)^2}{16t}} d\xi.$$

Facem schimbarea de variabilă:

$$u = \frac{x-\xi}{4\sqrt{t-\tau}}, du = \frac{-d\xi}{4\sqrt{t-\tau}}$$

de unde prima integrală din partea dreaptă a expresiei lui $u(x,t)$ este:

$$\int_0^t \int_{\mathbb{R}} \frac{\tau + e^{\tau}}{4\sqrt{\pi(t-\tau)}} \cdot e^{-\frac{(x-\xi)^2}{16(t-\tau)}} d\xi d\tau =$$

$$= \left[\int_0^t (\tau + e^{\tau}) d\tau\right] \cdot \left(\int_{-\infty}^{+\infty} \frac{1}{4\sqrt{\pi(t-\tau)}} \cdot e^{-u^2} \left(-4\sqrt{t-\tau}\right) du\right) =$$

$$= \left[\int_0^t (\tau + e^{\tau}) d\tau\right] \left(\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\pi}} \cdot e^{-u^2} du\right) =$$

$$= \left(\frac{\tau^2}{2} + e^{\tau}\right)\Bigg|_0^t \cdot \frac{1}{\sqrt{\pi}}\sqrt{\pi} = \frac{t^2}{2} + e^t - 1.$$

Analog, pentru a doua integrală, facem schimbarea de variabilă

$$u = \frac{x-\xi}{4\sqrt{t}}, d\xi = -4\sqrt{t} du$$

şi deci:

$$\int_{\mathbb{R}} 2 \cdot e^{-\frac{(x-\xi)}{16t}} d\xi = 2 \int_{\mathbb{R}} e^{-u^2} 4\sqrt{t} du = 8\sqrt{t} \int_{\mathbb{R}} e^{-u^2} du = 8\sqrt{\pi t}.$$

În concluzie:

$$u(x,t) = \frac{t^2}{2} + e^t - 1 + \frac{1}{4\sqrt{\pi t}} 8\sqrt{\pi t} = \frac{t^2}{2} + e^t + 1.$$

Remarca 3.52 Analog secţiunii (3.4) soluţia problemei Cauchy pentru operatorul căldurii este, în cazul în care datele $f(x,t)$ şi $u_0(x)$ sunt funcţii analitice:

$$u(x,t) = u_h(x,t) + u_p(x,t) \tag{3.64}$$

unde:

$$u_h(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot a^{2n} \cdot \Delta^n u_0(x) \tag{3.65}$$

$$u_p(x,t) = \int_0^t \widetilde{u}(x, t-s, s)\, ds, \tag{3.66}$$

unde:

$$\widetilde{u}(x,t,s) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot a^{2n} \cdot \Delta^n f(x,s).$$

Cu această metodă vom rezolva următoarele trei aplicaţii:

Aplicaţia 3.53

$$\begin{cases} \frac{\partial u}{\partial t} = 3\Delta u + e^t, \\ u\big|_{t=0} = \sin(x-y-z), \end{cases} \quad (n=3).$$

Avem: $a = 3,\ f(x,y,z,t) = e^t,\ u_0(x,y,z) = \sin x - y - z.$

$$\Delta^0 u_0 = \sin(x-y-z), \Delta u_0 = -3\sin(x-y-z) = -3u_0,$$

$$\Delta^2 u_0 = \Delta(\Delta u_0) = -3\Delta u_0 = (-3)^2 u_0$$

şi prin inducţie:

$$\Delta^n u_0 = (-3)^n u_0, \ (\forall)\, n \leq 1.$$

Avem cu (3.65):

$$u_h(x,y,z,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot (-9)^n \sin(x-y-z) = e^{-9t} \sin(x-y-z).$$

Apoi cu (3.66):

$$\widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot 3^n \Delta^n (e^s), u_p(x,y,z,t) = \int_0^t e^s ds = e^t - 1.$$

Deci:

$$u(x,y,z,t) = e^t - 1 + e^{-9t} \sin(x-y-z).$$

Aplicaţia 3.54

$$\begin{cases} \frac{\partial u}{\partial t} = 2\Delta u + t\cos, \\ u\big|_{t=0} = \cos y \cos z, \end{cases} \quad (n=3)$$

Acest exemplu a fost rezolvat şi cu ajutorul formulei (3.58). Datele sunt: $a = 2$, $f(x,y,z,t) = t\cos x$ şi $u_0(x,y,z) = \cos y \cos z$.

Cu (3.65) avem:

$$u_h(x,y,z,t) = \sum_{n \geq 0} 2^n \cdot \frac{t^n}{n!} \cdot \Delta^n u_0(x,y,z);$$

$$\Delta^0 u_0 = u_0, \Delta u_0 = -2\cos y \cos z = -2u_0, \Delta^2 u_0 = \Delta(\Delta u_0) =$$
$$= (-2)^2 u_0, \Delta^n u_0 = (-2)^n u_0 = (-2)^n \cos y \cos z.$$

Deci:

$$u_h(x,y,z,t) = \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} \cdot s\cos x = e^{-2t}\cos y \cos z.$$

Deoarece

$$f(x,y,z,s) =$$
$$= s \cdot \cos x, \Delta^0 f = f, \Delta f = -f, \Delta^2 f = f, \Delta^n f = (-1)^n f.$$

Deci:

$$\widetilde{u}(x,y,z,t,s) = \sum_{n=0}^{\infty} \frac{(-2t)^n}{n!} \cdot s\cos x = e^{-2t} \cdot s\cos x,$$

prin urmare:

$$u_p(x,y,z,t) = \int_0^t \widetilde{u}(x,y,z,t-s,s)\,ds = e^{-2t}\cos x \int_0^t s \cdot e^{2s} ds =$$
$$= \cos x \left(\frac{t}{2} - \frac{1}{4} + \frac{1}{4} \cdot e^{-2t}\right).$$

Deci:

$$u(x,y,z,t) = e^{-4t}\cos y \cos z + \cos x \left(\frac{t}{2} - \frac{1}{4} + \frac{1}{4} \cdot e^{-2t}\right).$$

Aplicaţia 3.55

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + e^{-t}\cos x, \\ u|_{t=0} = \cos x, \end{cases} \quad (n=1).$$

$a = 1, f(x,t) = e^{-t}\cos x, u_0(x) = \cos x$. Avem:

$$u_h(x,t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot (\cos x)^{(2n)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} (-1)^n \cos x =$$

$$= \cos x \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = e^{-t}\cos x.$$

$$\widetilde{u}(x,t,s) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{\partial^{2n} f}{\partial x^{2n}}(x,s) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot \frac{\partial^{2n}}{\partial x^{2n}} \left(e^{-s}\cos x\right) =$$

$$= e^{-s}\cos x \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = e^{-s-t}\cos x.$$

Conform (3.66) obţinem:

$$u_p(x,t) = \int_0^t \widetilde{u}(x,t-s,s)\,ds = \cos x \int_0^{\infty} e^{-s-(t-s)}ds =$$

$$= e^{-t}\cos x \int_0^t ds = t \cdot e^{-t}\cos x.$$

Deci:

$$u(x,t) = u_h(x,t) + u_p(x,t) = e^{-t}\cos x + t \cdot e^{-t}\cos x =$$

$$= (t+1)\,e^{-t}\cos x.$$

3.6 Probleme la limită pentru ecuaţii eliptice

3.6.1 Ecuaţia Laplace. Problema Dirichlet (interioară, exterioară) şi problema Neumann (interioară, exterioară) pentru ecuaţia Laplace

Fie domeniul $D_r(0)$- discul centrat în zero, de rază r în $\mathbb{R}^2$ şi fie ecuaţia Laplace $\Delta u = 0$ pe $D_r(0)$ sau pe $\mathbb{R}^2 \backslash \overline{D_r}(0)$.

Teorema 3.56 *Considerăm problema Dirichlet interioară pentru ecuaţia Laplace:*

$$\begin{cases} \Delta u = 0 & \text{în} \quad D_r(0) \\ u = u_0^- & \text{pe} \quad U_r(0) \end{cases}$$

Atunci, făcând schimbările:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}, u(x,y) = \widetilde{u}(\rho,\theta)$$

obţinem:

$$\widetilde{u}(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cdot\cos(\phi-\theta)} \cdot \widetilde{u_0^-}(\phi)\,d\phi$$

(formula lui Poisson)
unde:

$$\widetilde{u_0^-}(\phi) = u_0^-(r\cos\phi, r\sin\phi),$$

respectiv:

$$u(z) = Re\frac{1}{2\pi i}\int_{U_r(0)} u_0^-(\xi)\,\frac{\xi+z}{\xi-z}\frac{d\xi}{\xi}$$

(formula lui Schwartz).

Demonstraţie. Deoarece $D_r(0)$ este un domeniu simplu conex, iar u este funcţie armonică există o funcţie olomorfă $f : D_r(0) \to \mathbb{C}$ astfel încât $u = Ref$.
Cum f se poate dezvolta în serie Tazlor în jurul lui 0 avem:

$$f(z) = \frac{1}{2}c_0 + \sum_{k=1}^{\infty} c_k z^k, (\forall)\, z \in D_r(0).$$

Pentru $z = x + iy = \rho(\cos\theta + i\sin\theta) = \rho \cdot e^{i\theta}$ şi $c_k = a_k - ib_k$ cu $a_k, b_k \in \mathbb{R}$ obţinem:

$$\widetilde{u}(\rho, \theta) = Ref\left(\rho \cdot e^{i\theta}\right) =$$

$$= Re\left[\frac{1}{2}(a_0 - ib_0) + \sum_{k=1}^{\infty} \rho^k (a_k - ib_k)(\cos k\theta + i\sin k\theta)\right] =$$

$$= \frac{a_0}{2} + \sum_{k=1}^{\infty} \rho^k (a_k \cos k\theta + b_k \sin k\theta) \tag{3.67}$$

Condiţia pe frontieră $u = u_0^-$ în coordinate polare devine $\widetilde{u}(r, \theta) = u_0^-(\theta)$, $(\forall)\, \theta \in [0, 2\pi]$ şi folosind formula (3.67) avem:

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) = u_0^-(\theta), (\forall)\, \theta \in [0, 2\pi]. \tag{3.68}$$

Ţinem cont de relaţiile:

$$\int_0^{2\pi} 1 d\theta = 2\pi, \int_0^{2\pi} \cos k\theta \; d\theta = \int_0^{2\pi} \sin k\theta \; d\theta = 0,$$

$$\int_0^{2\pi} \cos k\theta \; \mathrm{cosl}\theta \; d\theta = \int_0^{2\pi} \sin k\theta \; \mathrm{sinl}\theta \; d\theta = \pi\delta_{kl},$$

$$\int_0^{2\pi} \sin k\theta \, \mathrm{cosl}\theta \, d\theta = 0$$

Atunci, obţinem:

$$a_0 = \frac{1}{\pi} \int_0^{\pi} u_0^- (\phi) \, d\phi, a_k = \frac{1}{\pi r^k} \int_0^{2\pi} u_0^- (\phi) \cdot \cos k\phi \, d\phi, b_k =$$

$$= \frac{1}{\pi r^k} \int_0^{2\pi} u_0^- (\phi) \cdot \sin k\phi \, d\phi$$

Înlocuim coeficienţii găsiţi în formula (3.67) şi găsim:

$$\widetilde{u} (\rho, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 2 \sum_{k=1}^{\infty} \left(\frac{\rho}{r} \right)^k (\cos k\theta \cdot \cos k\phi \, + \right.$$

$$\left. + \, \sin k\theta \cdot \sin k\phi) \right] \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 2 \sum_{k=1}^{\infty} \left(\frac{\rho}{r} \right)^k \cos k (\theta - \phi) \right] \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \left[1 + 2Re \sum_{k=1}^{\infty} \left(\frac{\rho}{r} \right)^k \cdot e^{ik(\theta - \phi)} \right] \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} Re \left[1 + 2 \frac{\frac{\rho}{r} \cdot e^{ik(\theta-\phi)}}{1 - \frac{\rho}{r} \cdot e^{ik(\theta-\phi)}} \right] \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} Re \frac{1 + \frac{\rho}{r} \cdot e^{ik(\theta-\phi)}}{1 - \frac{\rho}{r} \cdot e^{ik(\theta-\phi)}} \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} Re \frac{r \cdot e^{i\phi} + \rho e^{i\theta}}{r \cdot e^{i\phi} - \rho e^{i\theta}} \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi} \int_0^{2\pi} Re \frac{\left(r \cdot e^{i\phi} + \rho e^{i\theta} \right) \left(r \cdot e^{-i\phi} - \rho e^{-i\theta} \right)}{\left(r \cdot e^{i\phi} - \rho e^{i\theta} \right) \left(r \cdot e^{i\phi} - \rho e^{-i\theta} \right)} \cdot u_0^- (\phi) \, d\phi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Re\frac{r^2-\rho^2-i\cdot 2r\rho\sin(\phi-\theta)}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}\cdot u_0^-(\phi)\,d\phi =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Re\frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}\cdot u_0^-(\phi)\,d\phi$$

Deci, am obţinut formula lui Poisson care dă soluţia problemei Dirichlet interioară în coordonate polare:

$$\widetilde{u}(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{(r^2-\rho^2)\cdot u_0^-(\phi)}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}d\phi.$$

În

$$\widetilde{u}(\rho,\theta) = \frac{1}{2\pi}\int_0^{2\pi} Re\frac{re^{i\phi}+\rho\cdot e^{i\theta}}{re^{i\phi}-\rho\cdot e^{i\theta}}\cdot u_0^-(\phi)\,d\phi$$

facem schimbarea de variabilă $\xi = r\cdot e^{i\phi} \Rightarrow d\phi = \frac{1}{i\xi}d\xi$ şi obţinem $\left(z=\rho e^{i\theta}\right)$:

$$u(z) = Re\frac{1}{2\pi i}\int_{U_r(0)} u_0^-(\xi)\cdot\frac{\xi+z}{\xi-z}\cdot\frac{d\xi}{\xi}$$

(Schwartz). $\square$

Teorema 3.57 *Considerăm problema Neumann interioară*

$$\begin{cases} \Delta u = 0 & \text{în } D_r(0) \\ \frac{\partial u}{\partial n} = u_1^- & \text{pe } \mathrm{U}_r(0) \end{cases}$$

Atunci:

$$\widetilde{u}(\rho,\theta) = \frac{a_0}{2} + \frac{r}{2\pi}\int_0^{2\pi}\ln\frac{1}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}\cdot\widetilde{u_1^-}(\phi)\,d\phi$$

unde: $\widetilde{u_1^-}(\phi) = u_1^-(r\cos\phi, r\sin\phi)$ *(formula lui Poisson),*

$$u(z) = \frac{a_2}{2} + Re\frac{r}{\pi i}\int_{U_r(0)} \ln\frac{1}{z-\xi}\cdot u_1^- \frac{d\xi}{\xi}$$

(Schwartz)

Demonstraţie. Cu schimbarea din teorema (3.56) avem:

$$\widetilde{u}(\rho,\theta) = \frac{a_0}{2} + \sum_{k\geq 1}\rho^k\left(a_k\cos k\theta + b_k\sin k\theta\right).$$

Condiţia limită, devine în coordonate polare:

$$\frac{\partial u}{\partial n}(r\cos\theta, r\sin\theta) = \frac{\partial\widetilde{u}}{\partial\rho}(r,\theta) = u_1^-(r\cos\theta, r\sin\theta) = \widetilde{u_1^-}(\theta).$$

Deci:

$$\frac{\partial\widetilde{u}}{\partial\rho}(r,\theta) = \widetilde{u_1^-}(\theta),$$

obtinând:

$$\sum_{k=1}^{\infty} kr^{k-1}\left(a_k\cos k\theta + b_k\sin k\theta\right) = \widetilde{u_1^-}(\theta) \Rightarrow$$

$$\begin{array}{l} a_k = \frac{1}{\pi k r^{k-1}}\int_0^{2\pi}\widetilde{u_1^-}(\phi)\cos k\phi d\phi; \\ b_k = \frac{1}{\pi k r^{k-1}}\int_0^{2\pi}\widetilde{u_1^-}(\phi)\sin k\phi d\phi;\ k\geq 1. \end{array} \tag{3.69}$$

Din (3.69) deducem:

$$\int_0^{2\pi}\widetilde{u_1^-}(\theta)\,\mathrm{d}\theta. \tag{3.70}$$

Înlocuim coeficienţii a_k şi b_k, $k\geq 1$ şi obţinem:

$$\widetilde{u}(\rho,\theta) = \frac{a_0}{2} + \sum_{k\geq 1}\frac{r}{\pi k}\cdot\left(\frac{\rho}{r}\right)^k\cdot\left[\int_0^{2\pi}\cos k(\theta-\phi)\cdot\widetilde{u_1^-}(\phi)\,d\phi\right] =$$

$$= \frac{a_0}{2} + \frac{r}{\pi} \cdot \int_0^{2\pi} \sum_{k=1}^{\infty} \frac{1}{k} \cdot Re \left[\frac{\rho}{r} \cdot e^{i(\theta-\phi)} \right]^k \cdot \widetilde{u_1^-}(\phi)\, d\phi =$$

$$= \frac{a_0}{2} + \frac{r}{\pi} \cdot \int_0^{2\pi} Re \left(\sum_{k=1}^{\infty} \frac{1}{k} \cdot \left[\frac{\rho}{r} \cdot e^{i(\theta-\phi)} \right]^k \right) \cdot \widetilde{u_1^-}(\phi)\, d\phi \quad (3.71)$$

- $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$; $|z| < 1$ şi integrând:

$$\sum_{k\geq 0} \frac{z^{k+1}}{k+1} = \sum_{k\geq 0} \frac{z^k}{k} = -\ln(z-1) = \ln \frac{1}{z-1} \Rightarrow$$

$$Re \left(\sum_{k\geq 0} \frac{z^k}{k} \right) = Re \ln \frac{1}{z-1} = Re \ln \frac{1}{|z-1|}.$$

Înlocuim pe z cu $\frac{\rho}{r} \cdot e^{i(\theta-\phi)}$ şi obşinem în (3.71):

$$\widetilde{u}(\rho, \theta) = \frac{a_0}{2} + \frac{r}{\pi} \cdot \int_0^{2\pi} \ln \frac{1}{\left| \frac{\rho}{r} \cdot e^{i(\theta-\phi)} - 1 \right|} \cdot \widetilde{u_1^-}(\phi)\, d\phi =$$

ţinem cont de (3.69)

$$= \frac{a_0}{2} + \frac{r}{\pi} \cdot \int_0^{2\pi} \ln \frac{1}{|\rho e^{i\theta} - r e^{i\phi}|} \cdot \widetilde{u_1^-}(\phi)\, d\phi =$$

$$= \frac{a_0}{2} + \frac{r}{2\pi} \cdot \int_0^{2\pi} \ln \frac{1}{|\rho e^{i\theta} - r e^{i\phi}|^2} \cdot \widetilde{u_1^-}(\phi)\, d\phi \quad (3.72)$$

Dar:

$$\left| r e^{i\phi} - \rho e^{i\theta} \right|^2 = |r \cos\phi - \rho \cos\theta + i (r \sin\phi - \rho \cos\theta)|^2 =$$

$$= r^2 + \rho^2 - 2r\rho \cos(\theta - \phi)$$

Deci, formula (3.72) devine:

$$\widetilde{u}(\rho,\theta)=\frac{a_0}{2}+\frac{r}{2\pi}\cdot\int_0^{2\pi}\ln\frac{1}{\left|\rho e^{i\theta}-re^{i\phi}\right|^2}\cdot\widetilde{u_1^-}(\phi)\,d\phi=$$

$$=\frac{a_0}{2}+\frac{r}{2\pi}\cdot\int_0^{2\pi}\ln\frac{1}{\left|\rho e^{i\theta}-re^{i\phi}\right|^2}\cdot\widetilde{u_1^-}(\phi)\,d\phi=$$

$$=\frac{a_0}{2}+\frac{r}{2\pi}\cdot\int_0^{2\pi}\ln\frac{1}{r^2+\rho^2-2r\rho\cos(\theta-\phi)}\cdot\widetilde{u_1^-}(\phi)\,d\phi.$$

$$(Poisson)$$

În formula lui Poisson care ne dă soluţia $\widetilde{u}(\rho,\theta)$ facem substituţia $\xi=r\cdot e^{i\phi}$, de unde lungimea arcului de curbă este:

$$d\gamma=\sqrt{\left[(r\cos\phi)'\right]^2+\left[(r\sin\phi)'\right]^2}d\phi=rd\phi.$$

Deci:

$$d\gamma=rd\phi\Rightarrow d\phi=\frac{d\gamma}{r}.$$

Atunci:

$$u(z)=\frac{a_0}{2}+\frac{r}{\pi}\int_{U_r(0)}\ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\,\frac{d\gamma}{r}=$$

$$=\frac{a_0}{2}+\frac{1}{\pi}\int_{U_r(0)}\ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\,d\gamma.$$

Pe de altă parte: $\xi=r\cdot e^{i\phi}\Rightarrow d\xi=ire^{i\phi}d\phi\Rightarrow d\phi=\frac{d\xi}{i\xi}$ şi din

$$\frac{d\gamma}{r}=d\phi\Rightarrow d\gamma=rd\phi=r\cdot\frac{1}{\pi}\cdot\frac{d\xi}{\xi}=\frac{r}{\pi}\cdot\frac{d\xi}{\xi}.$$

Deci:

$$u(z)=\frac{a_0}{2}+\frac{1}{\pi}\int_{U_r(0)}\ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\,d\gamma=$$

$$= \frac{a_0}{2} + Re\frac{1}{\pi}\int_{U_r(0)} \ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\,d\gamma =$$

$$= \frac{a_0}{2} + Re\frac{1}{\pi i}\int_{U_r(0)} \ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\cdot\frac{d\xi}{\xi}$$

Deci:

$$u(z) = \frac{a_0}{2} + Re\frac{1}{\pi i}\int_{U_r(0)} \ln\frac{1}{|z-\xi|}\cdot u_1^-(\xi)\cdot\frac{d\xi}{\xi}. \quad (Schwartz)$$

□

Teorema 3.58 *i) Problema Dirichlet exterioară*

$$\begin{cases} \Delta u = 0 & \hat{in}\ \mathbb{R}^2\backslash\overline{\Delta_r(0)} \\ u = u_0^+ & \text{pe } U_r(0) \end{cases}$$

are soluţia

$$\widetilde{u}(\rho,\theta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cdot\cos(\phi-\theta)}\cdot\widetilde{u}_0^+(\phi)\,d\phi$$

$$(Poisson)$$

unde $\widetilde{u_0^+}(\phi) = u_0^+(r\cos\phi, r\sin\phi)$;

$$u(z) = -Re\frac{1}{2\pi i}\int_{U_r(0)} u_0^+(\xi)\,\frac{\xi+z}{\xi-z}\frac{d\xi}{\xi} \quad (Schwartz).$$

ii) Problema Neumann exterioară

$$\begin{cases} \Delta u = 0 & \hat{in}\ \mathbb{R}^2\backslash\overline{\Delta_r(0)} \\ \frac{\partial u}{\partial n} = u_1^+ & \text{pe } U_r(0) \end{cases}$$

are soluţia

$$\widetilde{u}(\rho,\theta) =$$

$$= -\frac{r}{2\pi}\int_0^{2\pi} \ln \frac{1}{r^2+\rho^2-2r\rho\cdot\cos(\phi-\theta)} \cdot \widetilde{u}_1^+(\phi)\,d\phi + \frac{a_0}{2}(Poisson)$$

unde $\widetilde{u_1^+}(\phi) = u_1^+(r\cos\phi, r\sin\phi)$;

$$u(z) = \frac{a_0}{2} - Re\frac{1}{\pi i}\int_{U_r(0)} \ln\frac{1}{(\xi - z)} \cdot u_1^+(\xi)\frac{d\xi}{\xi} \quad (Schwartz).$$

Demonstraţie. Dacă $\Delta u = 0$ pe $\mathbb{R}^2 \backslash \overline{D_r(0)}$, atunci există o funcţie derivabilă $f : \mathbb{R}^2 \backslash \overline{D_r(0)} \to \mathbb{C}$ astfel încât $u = Ref$. Atunci f se poate dezvolta în serie de puteri în jurul punctului de la infinit:

$$f(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k \cdot z^{-k}, \qquad (\forall)\, z \in \mathbb{R}^2 \backslash \overline{D_r(0)}.$$

Punem:

$$z = \rho e^{i\theta} = \rho(\cos\theta + i\sin\theta), \quad c_k = a_k + ib_k$$

şi deci

$$\widetilde{u}(\rho,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \rho^{-k}(a_k\cos k\theta + b_k \sin k\theta)$$

i) şi ii) se demonstrează analog teoremelor (3.56) şi (3.57). □

Aplicaţia 3.59 Să se rezolve următoarea problemă Dirichlet interioară:

$$\begin{cases} \Delta u = 0 \text{ în } D_1(0) \\ u = x^2 \text{pe} U_1(0) \end{cases}$$

Metoda I-a:

Cu schimbările din teorema (3.56) avem:

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases},$$

$$\widetilde{u}(r,\theta) = u(r\cos\theta + r\sin\theta),$$

$$\widetilde{u}(r,\theta) = \frac{a_0}{2} + \sum_{k\geq 1} r^k (a_k\cos k\theta + b_k\sin k\theta).$$

Condiţia pe frontieră devine:

$$\widetilde{u}(1,\theta) = \cos^2\theta = \frac{1}{2} + \frac{\cos 2\theta}{2}\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k\cos k\theta + b_k\sin k\theta)$$

de unde rezultă:

$$a_0 = 1,\ a_2 = \frac{1}{2},\ a_k = 0,\ \ (\forall)\, k \geq 1,\ k \neq 2,\ b_k = 0,\ \ (\forall)\, k \geq 1.$$

Deci:

$$\begin{array}{l} \widetilde{u}(r,\theta) = \frac{1}{2} + \frac{r^2}{2}\cos 2\theta \Rightarrow \\ u(x,y) = \frac{1}{2} + \frac{1}{2}Rez^2 = \frac{1}{2} + \frac{1}{2}Re\,(x+iy)^2 = \\ \qquad = \frac{1}{2} + \frac{1}{2}Re\,(x^2 - y^2 + i2xy) = \frac{1}{2}(1 + x^2 - y^2). \end{array}$$

Deoarece:

$$z = r\cos\theta + i\sin\theta \Rightarrow r^2\cos 2\theta = Rez^2.$$

Metoda a II-a:
Folosim formula lui Schwartz:

$$u(z) = Re\frac{1}{2\pi i}\int_{U_r(0)} u_0^-(\xi)\,\frac{\xi + z}{\xi - z}\frac{d\xi}{\xi}$$

$$u_0^-(\xi) = (Re\xi)^2 = \left(\frac{\xi+\overline{\xi}}{2}\right)^2 = \left(\frac{\xi^2+1}{2\xi}\right)^2 = \frac{(\xi^2+1)^2}{4\xi^2}$$

$|\xi| = 1$ pentru că $\xi \in U_1(0)$.
Deci:

$$u(z) = Re\frac{1}{2\pi i}\int_{U_r(0)} \frac{(\xi^2+1)^2(\xi+z)}{4\xi^2(\xi-z)}d\xi =$$

$$= Re\left[Rez\left[f,0\right] + Rez\left[f,z\right]\right],$$

unde:

$$f(\xi) = \frac{(\xi^2+1)^2(\xi+z)}{4\xi^2(\xi-z)}$$

are $\xi = 0$ pol de ordinul trei, $\xi = z$ pol de ordinul unu în $D_1(0)$ ($|\xi| < 1$ pentru că $\xi \in D_1(0)$).
Calculăm:

$$Rez\ \ [f,0] = \frac{1}{2!}\lim_{\xi\to 0}\left[\frac{(\xi^2+1)^2(\xi+z)}{4\xi^3(\xi-z)}\right]'' =$$

$$= \frac{1}{8}\lim_{\xi\to 0}\left[4\left(\xi^2+1\right) + 8\xi^2 + 8z\frac{(3\xi^2+1)(\xi-z)-(\xi^3+\xi)}{(\xi-z)^2} -\right.$$

$$\left. -4z\cdot\frac{\xi+1}{\xi-z}\cdot\frac{\xi-z-\xi-1}{(\xi-z)^2}\right] =$$

$$= \frac{1}{8}\left[4 + 8z\cdot\frac{-z}{z^2} + \frac{4z}{z}\cdot\frac{-1-z}{z^2}\right] = -\frac{1+z^2}{2z^2.}$$

$$Rez\ \ [f,z] = \lim_{\xi\to z}\frac{(\xi^2+1)^2(\xi+z)}{4\xi^3(\xi-z)}\cdot(\xi-z) =$$

$$= \frac{\left(1+z^2\right)^2 \cdot 2z}{4z^3} = \frac{\left(1+z^2\right)^2}{2z^2.}$$

$$u\left(z\right) = Re\left[\frac{\left(1+z^2\right)^2 - \left(1+z^2\right)}{2z^2}\right] = Re\frac{1+z^2}{2} =$$

$$= Re\frac{1+x^2-y^2+i2xy}{2} = \frac{\left(1+x^2-y^2\right)}{2}.$$

Aplicaţia 3.60 Să se rezolve problema Dirichlet exterioară:

$$\begin{cases} \Delta u = 0 \text{ în } \mathbb{C}\backslash\overline{D_1\left(0\right)}, \\ u = x^2 \text{ pe } U_1\left(0\right). \end{cases}$$

Metoda I-a:

$$\left.\begin{array}{l} \widetilde{u}\left(r,\theta\right) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{1}{r^k}\left(a_k \cos k\theta + b_k \sin k\theta\right) \\ \widetilde{u}\left(1,\theta\right) = \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta \end{array}\right\} \Rightarrow$$

$$a_0 = 1,\ a_2 = \frac{1}{2},\ a_k = 0,\ k \neq 0,2;\ b_k = 0,\ k \geq 1.$$

$$\widetilde{u}\left(r,\theta\right) = \frac{1}{2} + \frac{\cos 2\theta}{2r^2};\ z = r \cdot e^{i\theta} \Rightarrow z^{-2} = \frac{e^{-i2\theta}}{r^2} \Rightarrow$$

$$\frac{\cos 2\theta}{2r^2} = Re z^{-2} = Re\frac{1}{z^2}.$$

$$u\left(x,y\right) = \frac{1}{2} + \frac{1}{2}Re\frac{1}{z^2} = \frac{1}{2} + \frac{1}{2}Re\frac{1}{\left(x+iy\right)^2} = \frac{1}{2} + \frac{1}{2}Re\frac{\left(x-iy\right)^2}{\left(x^2+y^2\right)^2} =$$

$$= \frac{1}{2}\left[1 + \frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right].$$

Metoda a II-a:
Folosim formula lui Schwartz:

$$u(z) = -Re\frac{1}{2\pi i}\int_{U_1(0)} u_0^+(\xi)\frac{\xi+z}{\xi-z}\frac{d\xi}{\xi}.$$

Unde:

$$u_0^+(\xi) = (Re\xi)^2 = \left(\frac{\xi+\overline{\xi}}{2}\right)^2 = \frac{\left(\xi^2+1\right)^2}{4\xi^2} \qquad \left(\overline{\xi} = \frac{1}{\xi},\ |\xi| = 1\right)$$

$$u(z) = -Re\frac{1}{2\pi i}\int_{U_1(0)} \frac{\left(1+\xi^2\right)^2(\xi+z)}{4\xi^3(\xi-z)}d\xi$$

$$f(\xi) = \frac{\left(\xi^2+1\right)^2(\xi+z)}{4\xi^2(\xi-z)}$$

are în $\xi = 0$ pol de ordinul trei situat în $D_1(0)$, $\xi_1 = z$ pol de ordinul unu care nu este în $D_1(0)$.
Deci cu teorema reziduurilor şi aplicaţia anterioară:

$$u(z) = -ReRez[f,0] = -Re\left[-\frac{1+z^2}{2z^2}\right] = \frac{1}{2}Re\left(1+\frac{1}{z^2}\right) =$$

$$= \frac{1}{2}Re\left[1+\frac{1}{(x+iy)^2}\right] = \frac{1}{2}Re\left[1+\frac{(x-iy)^2}{\left(x^2+y^2\right)^2}\right] =$$

$$= \frac{1}{2}\left[1+\frac{x^2-y^2}{\left(x^2+y^2\right)^2}\right].$$

Aplicaţia 3.61 Rezolvaţi problema Neumann interioară:

$$\begin{cases} \Delta u = 0 \text{ în } D_1(0) \\ \frac{\partial u}{\partial \overline{n}} = y^3 \text{pe} U_1(0) \end{cases}$$

Metoda I-a:

$$\widetilde{u}\,(r,\theta)=\frac{a_0}{2}+\sum_{k\geq 1} r^k\,(a_k\cos k\theta+b_k\sin k\theta)$$

Condiţia pe frontieră devine:

$$\left.\frac{\partial u}{\partial r}\right|_{r=1}=(\sin\theta)^3=\frac{3}{4}\sin\theta-\frac{1}{4}\sin 3\theta=$$

$$=\sum_{k\geq 1}[ka_k\cos k\theta+kb_k\sin k\theta]\Rightarrow$$

$$b_1=\frac{3}{4},\ b_3=-\frac{1}{12},\ \mathrm{b}_k=0,\ k\neq 1,3;\ ,\ \mathrm{a}_k=0,\ k\geq 1\Rightarrow$$

$$\widetilde{u}\,(r,\theta)=\frac{3}{4}r\sin\theta-\frac{r^3}{12}\sin 3\theta+C$$

Deci:

$$u\,(x,y)=\frac{1}{4}\left[3y-\frac{1}{3}\left(3x^2y-y^3\right)\right]+C.$$

Metoda a II-a:
Folosim formula lui Schwartz:

$$u\,(z)=Re\frac{1}{2\pi i}\int_{U_1(0)}u_1^-\,(\xi)\,\frac{1}{z-\xi}\frac{d\xi}{\xi}+C$$

$$\overline{u}_1\,(\xi)=(Im\xi)^3=\left(\frac{\xi^2-1}{2i\xi}\right)^2=-\frac{1}{8i}\cdot\frac{\left(\xi^2-1\right)^3}{\xi^3}$$

$$u\,(z)=Re\frac{1}{\pi i}\int_{U_1(0)}-\frac{1}{8i}\cdot\frac{\left(\xi^2-1\right)^3}{\xi^3}\cdot\ln\frac{1}{z-\xi}\cdot\frac{d\xi}{\xi}+C=$$

$$= \frac{1}{8\pi} Re \int_{U_1(0)} \frac{\left(\xi^2 - 1\right)^3}{\xi^4} \cdot \ln \frac{1}{z - \xi} d\xi + C,$$

Notăm: $h(z) = \int_{U_1(0)} \frac{\left(\xi^2-1\right)^3}{\xi^4} \cdot \ln \frac{1}{z-\xi} \cdot d\xi$ şi derivând sub semnul integralei în raport cu z obţinem:

$$h'(z) = -\int_{U_1(0)} \frac{\left(\xi^2 - 1\right)^3}{\xi^4 (z - \xi)} d\xi;$$

notăm $g(\xi) = \frac{\left(\xi^2+1\right)^3}{4\xi^2(z-\xi)}$ care are: $\xi = 0$ pol de ordinul patru situate în $U_1(0)$, $\xi_1 = z$ pol de ordinul unu.
Cu teorema rezidurilor găsim:

$$h'(z) = -2\pi i \left[Rez\,[g, 0] + Rez\,[g, z]\right]$$

$$\bullet Rez\ [g, 0] = \frac{1}{3!} \lim_{\xi \to 0} \left[\frac{\left(\xi^2 - 1\right)^3}{z - \xi}\right]''' = \frac{1}{6}\left(-\frac{6}{z^4} + \frac{18}{z^2}\right) = \frac{3}{z^2} - \frac{1}{z^4}$$

$$\bullet\bullet\ Rez\ [g, z] = \lim_{\xi \to z} (\xi - z) \frac{\left(\xi^2 - 1\right)^3}{\xi^4 (\xi - z)} = -\frac{\left(z^2 - 1\right)^3}{z^4}.$$

Deci:

$$h'(z) = -2\pi i \left(\frac{3}{z^2} - \frac{1}{z^4} - \frac{z^6 - 3z^4 + 3z^2 - 1}{z^4}\right) =$$

$$= -2\pi i \left(3 - z^2\right) \Rightarrow h(z) = -2\pi i \left(3z - \frac{z^3}{3}\right) + C.$$

Deci:

$$u(z) = \frac{1}{8\pi} Reh(z) + C = -\frac{1}{4} Re\left[i3z + \frac{(iz)^3}{3}\right] + C =$$

$$= -\frac{1}{4} Re\left[3xi - 3y + \frac{1}{3}(ix - y)^3\right] + C =$$

$$= -\frac{1}{4} Re\left[3ix - 3y + \frac{1}{3}\left(-ix^3 - y^3 + 3x^2y - 3ixy^2\right)\right] \Rightarrow$$

$$\Rightarrow u(x,y) = \frac{1}{4}\left[3y - \frac{1}{3}\left(3x^2y - y^3\right)\right] + C.$$

Aplicaţia 3.62 Rezolvaţi problema Neumann exterioară:

$$\begin{cases} \Delta u = 0 \text{ în } \mathbb{R}^2 \backslash \overline{D_1(0)} \\ \frac{\partial u}{\partial \overline{n}} = y^3 \text{ pe } U_1(0) \end{cases}$$

Metoda I-a:

$$\widetilde{u}(r,\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \frac{1}{r^k}\left(a_k \cos k\theta + b_k \sin k\theta\right).$$

Condiţia pe frontieră devine în coordonate polare:

$$\left.\frac{\partial \widetilde{u}}{\partial r}\right|_{r=1} = \sum_{k \geq 1} -k\left(a_k \cos k\theta + b_k \sin k\theta\right) = (\sin\theta)^3 =$$

$$= \frac{3}{4}\sin\theta - \frac{\sin 3\theta}{4}.$$

De unde obţinem: $a_k = 0$, $a_0 = 0$, $\mathrm{b}_1 = -\frac{3}{4}$, $b_3 = \frac{1}{12}$, $b_k = 0$, $k \neq 1, 3$.

$$\widetilde{u}\left(r,\theta\right)=-\frac{3}{4}\frac{\sin\theta}{r}+\frac{1}{12}\frac{\sin 3\theta}{r^3}+C.$$

De unde avem:

$$u\left(z\right)=-\frac{1}{4}Im\left[\frac{3}{z}-\frac{1}{3z^3}\right]+C=\frac{1}{4}Re\left[\frac{3i}{z}-\frac{i}{3z^3}\right]+C=$$

$$=-\frac{1}{4}Im\left[\frac{3\overline{z}}{\left|z\right|^2}-\frac{\overline{z}^3}{3\left|z\right|^6}\right]+C=$$

$$=-\frac{1}{4}Im\left(\frac{3x-i3y}{x^2+y^2}-\frac{\left(x-iy\right)^3}{\left(x^2+y^2\right)^3}\right)=$$

$$=-\frac{1}{4}\left(\frac{-3y}{x^2+y^2}+\frac{3x^2y-y^3}{\left(x^2+y^2\right)^3}\right)+C=$$

$$=\frac{1}{4}\left(\frac{3y}{x^2+y^2}-\frac{3x^2y-y^3}{\left(x^2+y^2\right)^3}\right)+C.$$

Metoda a II-a:

Cu formulele lui Schwartz găsim:

$$u\left(z\right)=Re\frac{1}{\pi i}\int_{U_1(0)}u_1^+\ln\frac{1}{z-\xi}\frac{d\xi}{\xi}+C=$$

$$=\frac{1}{8\pi}Re\int_{U_1(0)}u_1^+\frac{\left(\xi^2-1\right)^3}{\xi^4}\ln\frac{1}{z-\xi}d\xi+C.$$

Deoarece:

$$u_1^+\left(\xi\right)=\left(Im\xi\right)^3=\left(\frac{\xi-\overline{\xi}}{2i}\right)^2=-\frac{1}{8i}\cdot\frac{\left(\xi^2-1\right)^3}{\xi^3}$$

$$h\left(z\right)=\int_{U_1(0)}\frac{\left(\xi^2-1\right)^3}{\xi^4}\ln\frac{1}{-\xi+z}d\xi\Rightarrow$$

$$h'(z) = -\int_{U_1(0)} \frac{\left(\xi^2-1\right)^3}{\xi^4 (z-\xi)} d\xi = -2\pi i Rez\left[g,0\right].$$

Unde: $g(\xi) = \frac{\left(\xi^2-1\right)^3}{\xi^4(z-\xi)}$ are $\xi = 0$ pol de ordinul patru situate în interiorul lui $U_1(0)$; polul $\xi = z \in \mathbb{R}^2 \backslash \overline{D_1(0)}$, deci nu se ia în calcul. Cu teorema reziduurilor avem folosind problema anterioară:

$$\begin{aligned}
&h'(z) = -2\pi i Rez\left[g,0\right] = -\tfrac{2\pi i}{3!} \lim_{\xi\to 0} \left[\tfrac{\left(\xi^2-1\right)^3}{(z-\xi)}\right]''' = \\
&= -2\pi i\left(\tfrac{3}{z^2} - \tfrac{1}{3z^4}\right) + C \\
&\qquad \Rightarrow h(z) = -2\pi i\left(-\tfrac{3}{z} + \tfrac{1}{3z^3}\right) + C = 2\pi i\left(\tfrac{3}{z} + \tfrac{1}{3z^3}\right) + C.
\end{aligned}$$

Revenind la formula lui Schwartz:

$$u(x,y) = \frac{1}{8\pi} Re\left[2\pi i\left(\frac{3}{z} - \frac{1}{3z^3}\right)\right] + C =$$

$$= \frac{1}{4} Re\left[3i\frac{\overline{z}}{|z|^2} - \frac{i}{3}\cdot\frac{\overline{z}^3}{|z|^6}\right] + C =$$

$$= \frac{1}{4}\left(\frac{3y}{x^2+y^2} - \frac{3x^2y - y^3}{\left(x^2+y^2\right)^3}\right) + C.$$

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