Assignment 2

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Group 3

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2.1.1

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 8 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}$$

Eigen values and vectors can be calculated from:

$$\begin{aligned} Av &= \lambda v \\ Av - \lambda v &= 0 \\ (A - \lambda I) \cdot v &= 0 \\ Det(A - \lambda I) &= 0 \end{aligned}$$

Where λ is the Eigen value and v is the Eigen vector.

Matrix A first:

$$Det(A) = \begin{bmatrix} 3-\lambda & 4 \\ 5 & 8-\lambda \end{bmatrix}$$

$$(3-\lambda) \cdot (8-\lambda) - (4 \cdot 5) = 0$$

$$\lambda^2 - 11\lambda + 4 = 0$$

$$\lambda_1 = \frac{-\sqrt{105}+11}{2}$$

$$\lambda_2 = \frac{\sqrt{105}+11}{2}$$

$$A - \lambda_1 I = \begin{bmatrix} \frac{\sqrt{105}-5}{2} & 4 \\ 5 & \frac{\sqrt{105}+5}{2} \end{bmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} \frac{-\sqrt{105}-5}{2} & 4 \\ 5 & \frac{-\sqrt{105}+5}{2} \end{bmatrix}$$

Having both Eigen values λ_1 and λ_2 we can solve each homogeneous system to find the respective Eigen vectors:

Calculating the Eigen vector for λ_1 first:

$$\left[\begin{array}{cc|c} \frac{\sqrt{105}-5}{2} & 4 & 0 \\ 5 & \frac{\sqrt{105}+5}{2} & 0 \end{array}\right] \begin{array}{c} r_1 / \frac{\sqrt{105}-5}{2} \\ r_2 - 5r_1 \end{array} \left[\begin{array}{cc|c} 1 & \frac{\sqrt{105}+5}{10} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \dfrac{-\sqrt{105} - 5}{10} \cdot x_2$

$x_2 = x_2$

Eigen vector $\vec{v_1} = \begin{bmatrix} \frac{-\sqrt{105}-5}{10} \\ 1 \end{bmatrix}$

Calculating the Eigen vector for λ_2 now:

$$\left[\begin{array}{cc|c} \frac{-\sqrt{105}-5}{2} & 4 & 0 \\ 5 & \frac{-\sqrt{105}+5}{2} & 0 \end{array}\right] \begin{array}{c} r_1/\frac{-\sqrt{105}-5}{2} \\ r_2 - 5r_1 \end{array} \left[\begin{array}{cc|c} 1 & \frac{-\sqrt{105}+5}{10} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \dfrac{\sqrt{105} - 5}{10} \cdot x_2$

$x_2 = x_2$

Eigen vector $\vec{v_2} = \begin{bmatrix} \frac{\sqrt{105}-5}{10} \\ 1 \end{bmatrix}$

Normalizing both vectors:

$$\text{Normalized Eigen vector } \vec{v_1} = \frac{\begin{bmatrix} \frac{-\sqrt{105}-5}{10} \\ 1 \end{bmatrix}}{\sqrt{(\frac{-\sqrt{105}-5}{10})^2 + 1^2}} = \begin{bmatrix} -0.836 \\ 0.548 \end{bmatrix}$$

$$\text{Normalized Eigen vector } \vec{v_2} = \frac{\begin{bmatrix} \frac{\sqrt{105}-5}{10} \\ 1 \end{bmatrix}}{\sqrt{(\frac{\sqrt{105}-5}{10})^2 + 1^2}} = \begin{bmatrix} 0.465 \\ 0.885 \end{bmatrix}$$

Matrix B now:

$Det(B) = \begin{bmatrix} 4-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$

$(4-\lambda) \cdot (1-\lambda) - (2 \cdot 3) = 0$

$\lambda^2 - 5\lambda - 2 = 0$

$\lambda_1 = \dfrac{-\sqrt{33} + 5}{2}$

$\lambda_2 = \dfrac{\sqrt{33} + 5}{2}$

$B - \lambda_1 I = \begin{bmatrix} \frac{\sqrt{33}-+3}{2} & 2 \\ 3 & \frac{\sqrt{33}-3}{2} \end{bmatrix}$

$B - \lambda_2 I = \begin{bmatrix} \frac{-\sqrt{33}+3}{2} & 2 \\ 3 & \frac{-\sqrt{33}-3}{2} \end{bmatrix}$

Having both Eigen values λ_1 and λ_2 we can solve each homogeneous system to find the respective Eigen vectors:

Calculating the Eigen vector for λ_1 first:

$$\left[\begin{array}{cc|c} \frac{\sqrt{33}+3}{2} & 2 & 0 \\ 3 & \frac{\sqrt{33}-3}{2} & 0 \end{array}\right] \begin{array}{c} r_1/\frac{\sqrt{33}+3}{2} \\ r_2 - 3r_1 \end{array} \left[\begin{array}{cc|c} 1 & \frac{\sqrt{33}-3}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \dfrac{-\sqrt{33}+3}{6} \cdot x_2$

$x_2 = x_2$

Eigen vector $\vec{v_1} = \begin{bmatrix} \frac{-\sqrt{33}+3}{6} \\ 1 \end{bmatrix}$

Calculating the Eigen vector for λ_2 now:

$$\left[\begin{array}{cc|c} \frac{-\sqrt{33}+3}{2} & 2 & 0 \\ 3 & \frac{-\sqrt{33}-3}{2} & 0 \end{array}\right] \begin{array}{c} r_1/\frac{-\sqrt{33}+3}{2} \\ r_2 - 3r_1 \end{array} \left[\begin{array}{cc|c} 1 & \frac{-\sqrt{33}-3}{6} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = \dfrac{\sqrt{33}+3}{6} \cdot x_2$

$x_2 = x_2$

Eigen vector $\vec{v_2} = \begin{bmatrix} \frac{\sqrt{33}+3}{6} \\ 1 \end{bmatrix}$

Normalizing both vectors:

Normalized Eigen vector $\vec{v_1} = \dfrac{\begin{bmatrix} \frac{-\sqrt{33}+3}{6} \\ 1 \end{bmatrix}}{\sqrt{(\frac{-\sqrt{33}+3}{6})^2 + 1^2}} = \begin{bmatrix} -0.416 \\ 0.909 \end{bmatrix}$

Normalized Eigen vector $\vec{v_2} = \dfrac{\begin{bmatrix} \frac{\sqrt{33}+3}{6} \\ 1 \end{bmatrix}}{\sqrt{(\frac{\sqrt{33}+3}{6})^2 + 1^2}} = \begin{bmatrix} 0.824 \\ 0.566 \end{bmatrix}$

2.1.2

Firstly, lets calculate the mean and variance of each student group:

Groups	N	Mean	Variance
High	9	7.67	1.12
Medium	6	7.08	0.54
Low	5	6.60	0.67

Calculate the degrees of freedom:

$df(Between) = k - 1$(where k = number of groups)
$df(Between) = 3 - 1 = \mathbf{2}$
$df(Within) = n - k$(where n = number of occurrences)
$df(Within) = 9 + 6 + 5 - 3 = \mathbf{17}$
$df(Total) = df(Between) + df(Within)$
$df(Total) = 17 + 2 = \mathbf{19}$

Calculate the sum of squares between and within groups:

$SS(Between) = \sum_{i=1}^{k} m_i(\bar{x} - \bar{x}_i)^2$ with m_i = number of samples in the group
$SS(Between) = 9 \cdot (7.67 - 7.22)^2 + 6 \cdot (7.08 - 7.22)^2 + 5 \cdot (6.60 - 7.22)^2 = \mathbf{3.83}$
$SS(Within) = \sum_{i=1}^{k} v_i(m_i - 1)$ with v_i = variance of the group
$SS(Within) = (1.12 \cdot 8) + (0.54 \cdot 5) + (0.67 \cdot 4) = \mathbf{14.41}$
$SS(Total) = SS(Between) + SS(Within)$
$SS(Total) = 3.83 + 14.41 = \mathbf{18.24}$

With the sum of squares calculated we can now calculate the mean square:

$MS(Between) = \frac{SS(Between)}{k - 1}$
$MS(Between) = 3.83 \ / \ (3 - 1) = \mathbf{1.91}$
$MS(Within) = \frac{SS(Within)}{n - k}$
$MS(Within) = 14.40 \ / \ (20 - 3) = \mathbf{0.84}$

And lastly we calculate the value of F-stat:

F-stat $= \frac{MS(Between)}{MS(Within)}$
F-stat $= 1.91/0.84 = \mathbf{2.27}$

With all the values calculated we can fill the ANOVA table as such:

Source	SS	DF	MS	F
Between	3.83	2	1.91	2.26
Within	14.41	17	0.84	NA
Total	18.24	19	NA	NA