Vectorial approach to momentum and mass conservation equations

Marius Berge Eide

AST5110: ITA, UiO

m.b.eide@astro.uio.no

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Overview

Momentum equation from N2L

- Have a volume element ω with surface $\partial\omega$
- Change in momentum due to internal stresses and external forces:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\omega} \mathbf{p} \, \mathrm{d}v = \int_{\partial\omega} \mathbf{t} \cdot \mathbf{n} \, \mathrm{d}a + \int_{\omega} \mathbf{F}_{\mathrm{ext}} \, \mathrm{d}v \quad (1)$$

$$= \int_{\omega} \nabla \cdot \mathbf{t} \, \mathrm{d}v + \int_{\omega} \mathbf{F}_{\mathrm{ext}} \, \mathrm{d}v \quad (2)$$

onto local form:

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} \left(\nabla \cdot \mathbf{v} \right) = \nabla \cdot \mathbf{t} + \nabla p_{\mathrm{gas}} + \rho \mathbf{g} \quad (3)$$

where

$$\mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} \left(\nabla \cdot \mathbf{v} \right) = \nabla \cdot (\mathbf{p} \mathbf{v}) \quad (4)$$

To solve

Momentum equation

$$\frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot (\mathbf{p}\mathbf{v}) + \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \tag{5}$$

Mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{p} \tag{6}$$

Solving in time

- Euler:

$$\mathbf{f}^t(\mathbf{x}) = \frac{\partial \mathbf{F}(t, \mathbf{x})}{\partial t} = \frac{\mathbf{F}(\mathbf{x}, t + \Delta t) - \mathbf{F}(\mathbf{x}, t)}{\Delta t} + \mathcal{O}(\Delta t^2) \qquad (7)$$

giving

$$\mathbf{F}^{t+1}(\mathbf{x}) \approx \mathbf{F}^t(\mathbf{x}) + \mathbf{f}^t(\mathbf{x})\Delta t \qquad (8)$$

- Runge-Kutta 4th order

$$\mathbf{F}^{t+1} = \mathbf{F}^t + \frac{\Delta t}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \qquad (9)$$

where

$$\mathbf{k}_1 = \mathbf{F}^t(\mathbf{x})$$
$$\mathbf{k}_2 = \mathbf{F}^{t+1/2}\left(\mathbf{x} + \frac{\Delta t}{2}\mathbf{k}_1\right)$$
$$\mathbf{k}_3 = \mathbf{F}^{t+1/2}\left(\mathbf{x} + \frac{\Delta t}{2}\mathbf{k}_2\right)$$
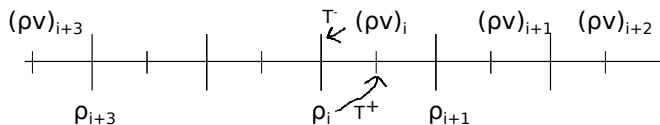$$\mathbf{k}_4 = \mathbf{F}^{t+1}(\mathbf{x} + \Delta t\mathbf{k}_3)$$

Solving in space

Figure: Staggered grid with cyclic boundary conditions

Interpolate forward half a step $(\Delta x/2)$:

$$T_x^+(\mathbf{g}(x, y, z, \cdots)) = \mathbf{g}(x + 1/2\Delta x, y, z, \cdots) = \mathbf{g}_{x+1/2} \tag{10}$$

$$= a(\mathbf{g}_x + \mathbf{g}_{x+1}) + b(\mathbf{g}_{x-1}\mathbf{g}_{x+2}) + c(\mathbf{g}_{x-2} + \mathbf{g}_{x+3}) \tag{11}$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \tag{12}$$

$$a = \frac{1}{2} - b - c; \qquad b = -\frac{1}{24} - 5c; \qquad c = \frac{3}{640}$$

Solving in space

$$T_x^+(\mathbf{g}(x,y,z,\cdots)) = \mathbf{g}_{x+1/2} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \qquad (13)$$

$$\partial_{,x}^+(\mathbf{g}(x,y,z,\cdots)) = \mathbf{g}_{x+1/2}' = \begin{bmatrix} a' & b' & c' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x - \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} - \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} - \mathbf{g}_{x+3} \end{bmatrix} \qquad (14)$$
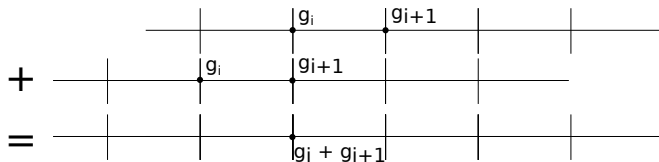

Figure: Numpy's ROLL(array, steps) causing cyclic boundaries, using this to interpolate and differentiate

Stability check

Stable without pressure

Stable for small amplitudes

Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Blocks of Highlighted Text

Block 1
Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

Block 2
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Heading

1. Statement
2. Explanation
3. Example

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Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
\frametitle{Theorem}
\begin{theorem}[Mass--energy equivalence]
$E = mc^2$
\end{theorem}
\end{frame}
```

Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

An example of the \cite command to cite within the presentation:

This statement requires citation [Smith, 2012].

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 − 678.

The End