Vectorial approach to momentum and mass conservation equations

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Overview

Momentum equation from N2L

- Have a volume element ω with surface $\partial\omega$
- Change in momentum due to internal stresses and external forces:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\omega} \mathbf{p}\,\mathrm{d}v = \int_{\partial\omega} \mathbf{t}\cdot\mathbf{n}\,\mathrm{d}a + \int_{\omega} \mathbf{F}_{\mathrm{ext}}\,\mathrm{d}v \tag{1}$$

$$= \int_{\omega} \nabla\cdot\mathbf{t}\,\mathrm{d}v + \int_{\omega} \mathbf{F}_{\mathrm{ext}}\,\mathrm{d}v \tag{2}$$

onto local form:

$$\frac{\partial\mathbf{p}}{\partial t} + \mathbf{v}\cdot(\nabla\mathbf{p}) + \mathbf{p}\,(\nabla\cdot\mathbf{v}) = \nabla\cdot\mathbf{t} + \nabla p_{\mathrm{gas}} + \rho\mathbf{g} \tag{3}$$

where

$$\mathbf{v}\cdot(\nabla\mathbf{p}) + \mathbf{p}\,(\nabla\cdot\mathbf{v}) = \nabla\cdot(\mathbf{p}\mathbf{v}) \tag{4}$$

To solve

Momentum equation

$$\frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot (\mathbf{p}\mathbf{v}) + \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \tag{5}$$

Mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{p} \tag{6}$$

Solving in time

- Euler:

$$\mathbf{f}^t(\mathbf{x}) = \frac{\partial \mathbf{F}(t,\mathbf{x})}{\partial t} = \frac{\mathbf{F}(\mathbf{x}, t+\Delta t) - \mathbf{F}(\mathbf{x}, t)}{\Delta t} + \mathcal{O}(\Delta t^2) \tag{7}$$

  giving

$$\mathbf{F}^{t+1}(\mathbf{x}) \approx \mathbf{F}^t(\mathbf{x}) + \mathbf{f}^t(\mathbf{x})\Delta t \tag{8}$$

- Runge-Kutta 4th order

$$\mathbf{F}^{t+1} = \mathbf{F}^t + \frac{\Delta t}{6}\left(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4\right) \tag{9}$$

  where

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{F}^t(\mathbf{x}) \\
\mathbf{k}_2 &= \mathbf{F}^{t+1/2}\left(\mathbf{x} + \frac{\Delta t}{2}\mathbf{k}_1\right) \\
\mathbf{k}_3 &= \mathbf{F}^{t+1/2}\left(\mathbf{x} + \frac{\Delta t}{2}\mathbf{k}_2\right) \\
\mathbf{k}_4 &= \mathbf{F}^{t+1}\left(\mathbf{x} + \Delta t\mathbf{k}_3\right)
\end{aligned}$$

Solving in space

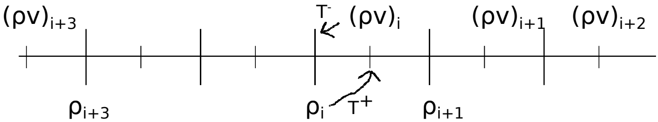

Figure: Staggered grid with cyclic boundary conditions

Interpolate forward half a step ($\Delta x/2$):

$$
\begin{aligned}
T_x^+(\mathbf{g}(x, y, z, \cdots)) &= \mathbf{g}(x + 1/2\Delta x, y, z, \cdots) = \mathbf{g}_{x+1/2} && (10)\\
&= a(\mathbf{g}_x + \mathbf{g}_{x+1}) + b\left(\mathbf{g}_{x-1}\mathbf{g}_{x+2}\right) + c\left(\mathbf{g}_{x-2} + \mathbf{g}_{x+3}\right) && (11)\\
&= \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} && (12)
\end{aligned}
$$

$$a = \frac{1}{2} - b - c; \qquad b = -\frac{1}{24} - 5c; \qquad c = \frac{3}{640}$$

Solving in space

$$T_x^+(\mathbf{g}(x,y,z,\cdots)) = \mathbf{g}_{x+1/2} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \tag{13}$$

$$\partial_{,x}^+(\mathbf{g}(x,y,z,\cdots)) = \mathbf{g}'_{x+1/2} = \begin{bmatrix} a' & b' & c' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x - \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} - \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} - \mathbf{g}_{x+3} \end{bmatrix} \tag{14}$$

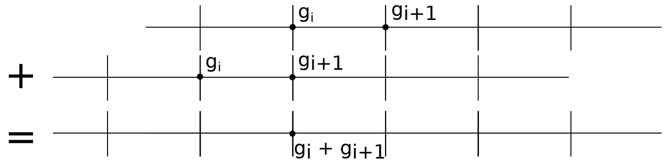

Figure: Numpy's `ROLL(array, steps)` causing cyclic boundaries, using this to interpolate and differentiate

Stability check

[http://77.237.250.152/wordpress/studier/stability_web.mp4]

Stable for small amplitudes

[http://77.237.250.152/wordpress/studier/working.mp4]

Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Blocks of Highlighted Text

Block 1

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Block 2

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Block 3

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Multiple Columns

Heading

1. Statement
2. Explanation
3. Example

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Table

Theorem

Theorem (Mass–energy equivalence)

$E = mc^2$

Verbatim

Example (Theorem Slide Code)

```
\begin{frame}
\frametitle{Theorem}
\begin{theorem}[Mass--energy equivalence]
$E = mc^2$
\end{theorem}
\end{frame}
```

Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.

Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].

References

John Smith (2012)
Title of the publication
Journal Name 12(3), 45 – 678.

The End