

# Vectorial approach to momentum and mass conservation equations

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# Overview

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  - Equations to be solved
- 2 Numerical solution
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- 3 Results
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# Momentum equation from N2L

- Have a volume element  $\omega$  with surface  $\partial\omega$
- Change in momentum due to internal stresses and external forces:

$$\frac{d}{dt} \int_{\omega} \mathbf{p} \, dv = \int_{\partial\omega} \mathbf{t} \cdot \mathbf{n} \, da + \int_{\omega} \mathbf{F}_{\text{ext}} \, dv \quad (1)$$

$$= \int_{\omega} \nabla \cdot \mathbf{t} \, dv + \int_{\omega} \mathbf{F}_{\text{ext}} \, dv \quad (2)$$

onto local form:

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} (\nabla \cdot \mathbf{v}) = \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \quad (3)$$

where

$$\mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} (\nabla \cdot \mathbf{v}) = \nabla \cdot (\mathbf{p} \mathbf{v}) \quad (4)$$

# To solve

Momentum equation

$$\frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot (\mathbf{p}\mathbf{v}) + \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \quad (5)$$

Mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{p} \quad (6)$$

# Solving in time

- Euler:

$$\mathbf{f}^t(\mathbf{x}) = \frac{\partial \mathbf{F}(t, \mathbf{x})}{\partial t} = \frac{\mathbf{F}(\mathbf{x}, t + \Delta t) - \mathbf{F}(\mathbf{x}, t)}{\Delta t} + \mathcal{O}(\Delta t^2) \quad (7)$$

giving

$$\mathbf{F}^{t+1}(\mathbf{x}) \approx \mathbf{F}^t(\mathbf{x}) + \mathbf{f}^t(\mathbf{x})\Delta t \quad (8)$$

- Runge-Kutta 4th order

$$\mathbf{F}^{t+1} = \mathbf{F}^t + \frac{\Delta t}{6} (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \quad (9)$$

where

$$\mathbf{k}_1 = \mathbf{F}^t(\mathbf{x})$$

$$\mathbf{k}_2 = \mathbf{F}^{t+1/2} \left( \mathbf{x} + \frac{\Delta t}{2} \mathbf{k}_1 \right)$$

$$\mathbf{k}_3 = \mathbf{F}^{t+1/2} \left( \mathbf{x} + \frac{\Delta t}{2} \mathbf{k}_2 \right)$$

$$\mathbf{k}_4 = \mathbf{F}^{t+1}(\mathbf{x} + \Delta t \mathbf{k}_3)$$

# Solving in space

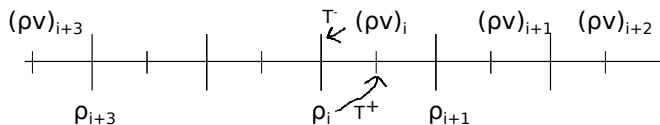


Figure: Staggered grid with cyclic boundary conditions

Interpolate forward half a step ( $\Delta x/2$ ):

$$T_x^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}(x + 1/2\Delta x, y, z, \dots) = \mathbf{g}_{x+1/2} \quad (10)$$

$$= a(\mathbf{g}_x + \mathbf{g}_{x+1}) + b(\mathbf{g}_{x-1} + \mathbf{g}_{x+2}) + c(\mathbf{g}_{x-2} + \mathbf{g}_{x+3}) \quad (11)$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \quad (12)$$

$$a = \frac{1}{2} - b - c; \quad b = -\frac{1}{24} - 5c; \quad c = \frac{3}{640}$$

# Solving in space

$$T_x^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}_{x+1/2} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \quad (13)$$

$$\partial_{,x}^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}'_{x+1/2} = \begin{bmatrix} a' & b' & c' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x - \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} - \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} - \mathbf{g}_{x+3} \end{bmatrix} \quad (14)$$

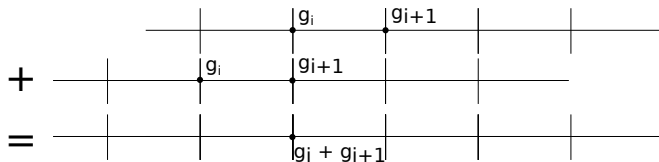


Figure: Numpy's `ROLL(array, steps)` causing cyclic boundaries, using this to interpolate and differentiate

# Stability check

▶ [http://77.237.250.152/wordpress/studier/stability\\_web.mp4](http://77.237.250.152/wordpress/studier/stability_web.mp4)





$i++i$

# Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

# Blocks of Highlighted Text

## Block 1

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## Block 2

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## Block 3

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# Multiple Columns

## Heading

- 1 Statement
- 2 Explanation
- 3 Example

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# Table

# Theorem

Theorem (Mass–energy equivalence)

$$E = mc^2$$

## Example (Theorem Slide Code)

```
\begin{frame}  
\frametitle{Theorem}  
\begin{theorem}[Mass--energy equivalence]  
$E = mc^2$  
\end{theorem}  
\end{frame}
```

# Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.



# Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].

# References



John Smith (2012)

Title of the publication

*Journal Name* 12(3), 45 – 678.

# The End