

# Vectorial approach to momentum and mass conservation equations

Marius Berge Eide

AST5110: ITA, UiO

*m.b.eide@astro.uio.no*

October 27, 2014

# Overview

- 1 Restating the problem
  - Momentum equation from N2L
  - Equations to be solved
- 2 Numerical solution
  - Solving in time
  - Solving in space
- 3 Results
  - Stability check
- 4 Second Section

# Momentum equation from N2L

- Have a volume element  $\omega$  with surface  $\partial\omega$
- Change in momentum due to internal stresses and external forces:

$$\frac{d}{dt} \int_{\omega} \mathbf{p} \, dv = \int_{\partial\omega} \mathbf{t} \cdot \mathbf{n} \, da + \int_{\omega} \mathbf{F}_{\text{ext}} \, dv \quad (1)$$

$$= \int_{\omega} \nabla \cdot \mathbf{t} \, dv + \int_{\omega} \mathbf{F}_{\text{ext}} \, dv \quad (2)$$

onto local form:

$$\frac{\partial \mathbf{p}}{\partial t} + \mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} (\nabla \cdot \mathbf{v}) = \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \quad (3)$$

where

$$\mathbf{v} \cdot (\nabla \mathbf{p}) + \mathbf{p} (\nabla \cdot \mathbf{v}) = \nabla \cdot (\mathbf{p} \mathbf{v}) \quad (4)$$

# To solve

Momentum equation

$$\frac{\partial \mathbf{p}}{\partial t} = -\nabla \cdot (\mathbf{p}\mathbf{v}) + \nabla \cdot \mathbf{t} + \nabla p_{\text{gas}} + \rho \mathbf{g} \quad (5)$$

Mass continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{p} \quad (6)$$

# Solving in time

- Euler:

$$\mathbf{f}^t(\mathbf{x}) = \frac{\partial \mathbf{F}(t, \mathbf{x})}{\partial t} = \frac{\mathbf{F}(\mathbf{x}, t + \Delta t) - \mathbf{F}(\mathbf{x}, t)}{\Delta t} + \mathcal{O}(\Delta t^2) \quad (7)$$

giving

$$\mathbf{F}^{t+1}(\mathbf{x}) \approx \mathbf{F}^t(\mathbf{x}) + \mathbf{f}^t(\mathbf{x})\Delta t \quad (8)$$

- Runge-Kutta 4th order

$$\mathbf{F}^{t+1} = \mathbf{F}^t + \frac{\Delta t}{6} (\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4) \quad (9)$$

where

$$\mathbf{k}_1 = \mathbf{F}^t(\mathbf{x})$$

$$\mathbf{k}_2 = \mathbf{F}^{t+1/2} \left( \mathbf{x} + \frac{\Delta t}{2} \mathbf{k}_1 \right)$$

$$\mathbf{k}_3 = \mathbf{F}^{t+1/2} \left( \mathbf{x} + \frac{\Delta t}{2} \mathbf{k}_2 \right)$$

$$\mathbf{k}_4 = \mathbf{F}^{t+1}(\mathbf{x} + \Delta t \mathbf{k}_3)$$

# Solving in space

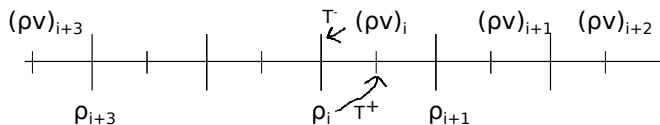


Figure: Staggered grid with cyclic boundary conditions

Interpolate forward half a step ( $\Delta x/2$ ):

$$T_x^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}(x + 1/2\Delta x, y, z, \dots) = \mathbf{g}_{x+1/2} \quad (10)$$

$$= a(\mathbf{g}_x + \mathbf{g}_{x+1}) + b(\mathbf{g}_{x-1} + \mathbf{g}_{x+2}) + c(\mathbf{g}_{x-2} + \mathbf{g}_{x+3}) \quad (11)$$

$$= \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \quad (12)$$

$$a = \frac{1}{2} - b - c; \quad b = -\frac{1}{24} - 5c; \quad c = \frac{3}{640}$$

# Solving in space

$$T_x^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}_{x+1/2} = \begin{bmatrix} a & b & c \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x + \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} + \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} + \mathbf{g}_{x+3} \end{bmatrix} \quad (13)$$

$$\partial_{,x}^+(\mathbf{g}(x, y, z, \dots)) = \mathbf{g}'_{x+1/2} = \begin{bmatrix} a' & b' & c' \end{bmatrix} \cdot \begin{bmatrix} \mathbf{g}_x - \mathbf{g}_{x+1} \\ \mathbf{g}_{x-1} - \mathbf{g}_{x+2} \\ \mathbf{g}_{x-2} - \mathbf{g}_{x+3} \end{bmatrix} \quad (14)$$

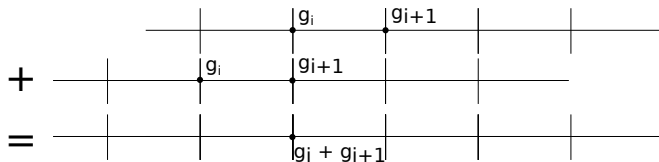


Figure: Numpy's `ROLL(array, steps)` causing cyclic boundaries, using this to interpolate and differentiate

# Stability check

▶ [http://77.237.250.152/wordpress/studier/stability\\_web.mp4](http://77.237.250.152/wordpress/studier/stability_web.mp4)





$i++i$

# Bullet Points

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus eu
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero
- Nam cursus est eget velit posuere pellentesque
- Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

# Blocks of Highlighted Text

## Block 1

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

## Block 2

Pellentesque sed tellus purus. Class aptent taciti sociosqu ad litora torquent per conubia nostra, per inceptos himenaeos. Vestibulum quis magna at risus dictum tempor eu vitae velit.

## Block 3

Suspendisse tincidunt sagittis gravida. Curabitur condimentum, enim sed venenatis rutrum, ipsum neque consectetur orci, sed blandit justo nisi ac lacus.

# Multiple Columns

## Heading

- 1 Statement
- 2 Explanation
- 3 Example

Lorem ipsum dolor sit amet, consectetur adipiscing elit. Integer lectus nisl, ultricies in feugiat rutrum, porttitor sit amet augue. Aliquam ut tortor mauris. Sed volutpat ante purus, quis accumsan dolor.

# Table

# Theorem

Theorem (Mass–energy equivalence)

$$E = mc^2$$

## Example (Theorem Slide Code)

```
\begin{frame}  
\frametitle{Theorem}  
\begin{theorem}[Mass--energy equivalence]  
$E = mc^2$  
\end{theorem}  
\end{frame}
```

# Figure

Uncomment the code on this slide to include your own image from the same directory as the template .TeX file.



# Citation

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2012].

# References



John Smith (2012)

Title of the publication

*Journal Name* 12(3), 45 – 678.

The End