Mathematics: Exercise 1

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Exercise 1

We want to show that

$$W = \Big\{(x, y, z) \in \mathbb{R}^3 : x + 2y - z = 0\Big\}$$

is a vector space on the field $\mathbb{R}$.

We know from the lecture slides on page 47 that the euclidean space $\mathbb{R}^n$ with the standard operators $+$ and $*$ is a vector space. By applying this knowledge, we can reduce the problem into proving that W is a vector subspace of $\mathbb{R}^3$. To do this, I apply the two criteria from proposition 139, 3. as given in the lecture. When 1. and 2. are holing, W is a vector subspace of $\mathbb{R}^3$ and therefore a vector space on the field R.

1. $W \neq \emptyset$
   A simple calculation with the vector $(1, 1, 3)$ results in $1 + 2 - 3 = 0$, which is a true statement. We have shown that there exists at least one vector in W.

2. $(\forall \alpha, \beta \in \mathbb{R}) \cap (\forall x, y, z \in W)$ it needs to hold that $\alpha(x_1, y_1, z_1) + \beta(x_2, y_2, z_2) \in W$
   We can check this condition by applying the functional form $x + 2y + z = 0$ on the vector $\begin{pmatrix} \alpha x_1 + \beta x_2 \\ \alpha y_1 + \beta y_2 \\ \alpha z_1 + \beta z_2 \end{pmatrix}$, which itself is an expression of two random vector of W and two random scalars from R.
   This yields

$$\begin{aligned}(\alpha x_1 + \beta x_2) + 2(\alpha y_1 + \beta y_2) - (\alpha z_1 + \beta z_2) = 0 \leftrightarrow \\ \alpha(x_1 + 2y_1 - z_1) + \beta(x_1 + 2y_1 - z_1) = 0\end{aligned}$$

   $(x_1 + 2y_1 - z_1)$ and $(x_1 + 2y_1 - z_1)$ have to be zero by definition. We are left with $\alpha 0 + \beta 0 = 0 \leftrightarrow 0 + 0 = 0$ Since the statement is true, it is, that the second requirement holds.

Since both requirements are fulfilled, it is evident that W is a vector subspace of R and thereby a vector space on the field R with respect to the standard notations of addition and multiplication.

Exercise 2

To prove that V is a vector space on F, we can apply proposition 139, 3. Let V be defined as $(v^1, ..., v^n)$ such that $m \in \mathbb{N}$. Let an arbitrary set of vectors on the vector space V be defined as W. We want to show that $(W_1 \cap W_2)$ is a vector subspace of V.
Generally, there are three possible outcomes for an intersection.

1. $(W_1 \cap W_2) = \emptyset$
2. $(W_1 \cap W_2) = W_1 = W_2$
3. $(W_1 \cap W_2) = \exists w \in W_i | w \in W_i \cup w \notin W_{\neq i} \; \forall i = 1, 2$

Now, we can address each case and whether it applies to our specific question.

1. $(W_1 \cap W_2) = \emptyset$
   In our case this does not apply, since we W_1 and W_2 stem from the same vector space. By definition each vector space has to have a unique null element with respect to multiplication and addition, for example 0 and 1 for $\mathbb{R}$. Since each of them is unique in V, it has to be the same for W_1 and W_2 implying that the intersection includes at least these two null elements.
   By the same argument, we have shown that the intersection is non-empty set - the first condition of it being a vector subspace. For any other of the two cases, the non-emptyness holds by definition.

Exercise 3

(i)

$$\begin{gathered} x * 0 = 0 \\ x * (0 + 0) = 0 \text{ unter Zuhilfenahme des Kommutativgesetz} \\ x * 0 + x * 0 = x * 0 \\ x * 0 + x * 0 - x * 0 = x * 0 - x * 0 \text{ —}(-x * 0) \\ x * 0 = 0 \end{gathered}$$

(ii)

$$\begin{gathered} -x = (-1) * x \\ \text{Konkret, wir zeigen, dass -1 das additive inverse Element von erzeugt} \\ x + (-1) * x = (1) * x + (-1) * x = (1 - 1) * x = 0 * x, \text{ berücksichtige (i)} \\ x * 0 = 0 \end{gathered}$$

(iii)

$$\begin{gathered} x * y = 0, \text{ wenn } x = 0 \text{ oder } y = 0 \text{ muss nach (i) automatisch gelten} \\ \text{für } x, y \neq 0 \text{ muss es eine multiplikative Inverse geben, die x zu 0 transformiert} \\ x * y * y^{-1} = 0 * y^{-1} \\ x * 1 = 0, \text{ was ein Widerspruch ist.} \end{gathered}$$

(iv)

$$-(x+y) = (-x) + (-y) \Leftrightarrow -(x+y) = (-1) * (x) + (-1) * (y) \text{ durch (ii)}$$
$$-(x+y) = (-1) * (x+y)$$
$$-(x+y) = -(x+y) \text{ (ii)}$$

(v)

$$y^{-1} * x^{-1} = (x * y)^{-1}$$
$$1 = (x * y)^{-1} * y * x, \text{ wir wenden das Kommutativgesetz für Multiplikationen an}$$
$$1 = x^{-1} * y^{-1} * y * y \Leftrightarrow 1 = x^{-1} * 1 * x$$
$$1 = 1 * 1 = 1$$

(vi)

$$(-1) * (-x) = x, \text{ wende (ii) an}$$
$$0 = x - x$$
$$x^{-1} * x = 1$$
$$(x^{-1})^{-1} = x$$
$$x = x$$