Symmetry. By Hermann Weyl. Princeton University Press, 1952. vii+168 pp. \$3.75.

This little book, artistically printed and beautifully illustrated, consists of four lectures given at Princeton as the author's "swan song" on the occasion of his retirement from the Institute for Advanced Study. It is so full of interesting material that, after beginning to read, one can hardly put it down. Although it is written for laymen, there are very few experts who will not learn something. For, besides mathematics and crystallography, it deals with philosophy, history, poetry, sculpture, painting, architecture, theology, physics, chemistry, physiology, embryology, and botany. The author has resisted the temptation to include an account of his own research on symmetry in space of many dimensions (see, *e.g.*, his contribution to the reviewer's *Regular Polytopes*, New York, 1949, pp. 204–207). Instead, he has collected a multitude of facts and theories, observations and quotations, drawings and photographs, blending them in his own inimitable fashion. The arrangement of text and illustrations is reminiscent of Steinhaus's *Mathematical Snapshots* (New York, 1950), but there is little overlapping of material.

The first lecture, on "Bilateral symmetry," contains an abundance of unusual information. We read, for example, that as many as one person in every five thousand is born with *situs inversus:* the consistent inversion of left and right throughout all the asymmetrical organs of the body. Turning from anatomy to biochemistry, the author gives a dramatic account of Pasteur's separation of laevo- and dextro-tartaric acid, of which only the latter occurs in organic nature. "Nature," he writes, "in giving us the wonderful gift of grapes so much enjoyed by Noah, produced only one of the forms, and it remained for Pasteur to produce the other!" But he soberly adds, "If there is a difference in principle between life and death, it does not lie in the chemistry of the material substratum."

The lecture on "Translatory, rotational, and related symmetries" includes a very readable introduction to the theory of vectors and to groups of transformations. On page 45, he makes the remark (too often overlooked in elementary instruction): "A vector is really the same thing as a translation, although one uses different phraseologies for vectors and translations." He ascribes to Leonardo da Vinci the enumeration of finite groups of congruent transformations in the plane: the cyclic groups C_1, C_2, $\cdots$ and the dihedral groups D_1, D_2, $\cdots$ These groups are illustrated by many examples from nature; *e.g.*, D_6 by snowflakes, D_5 by the geranium, and the rarer C_5 by *vinca herbacea*. At this point he might well have reproduced the 17-gonal section of a strand of equisetum from Hans Meierhofer, *Die Augen auf in unseres Herrgotts Garten!* (Zürich, 1947), page 48.

In connection with the occurrence of dihedral groups in animals, he refers to Sir D'Arcy W. Thompson's *Growth and Form*, "a masterpiece of English literature, which combines profound knowledge in geometry, physics, and biology with humanistic erudition and scientific insight of unusual originality."

One of the photographs reproduced from Thompson's book shows the giant sunflower, *Helianthus maximus*, whose florets form 34 right-handed and 55 left-handed spirals (p. 71). Since 34 and 55 are members of the Fibonacci sequence

$$1, 1, 2, 3, 5, 8, 13, 21, \cdots,$$

this is an instance of the phenomenon called "phyllotaxis." A simpler instance is provided by any well-formed pineapple. The author cites an article of 1872 by P. G. Tait for the best attempt at an explanation.

He reproduces a page from Haeckel's *Challenger monograph*, which shows the skeletons of several Radiolarians, including an octahedron, an icosahedron and a dodecahedron "in astonishingly regular form." The same page had been used by Thompson (*Growth and Form*, Fig. 340), who made the following comment, in a letter to the reviewer, dated 9th March, 1947: "As to Haeckel, I wouldn't trust him round the corner, and I have the gravest doubt whether his pentagonal dodecahedron and various others ever existed outside his fertile fancy. I believe I may safely say that no type-specimens of these exist in the British Museum, or anywhere else. He was an artist, a pattern-designer, a skilled draughtsman. He had a minute professorial salary in a small University. The *Challenger* paid eight guineas apiece for as many plates as he chose to draw; and he kept on drawing them, and lived on the proceeds (so they used to say) till the end of his life. He represents a thoroughly bad period in Natural Science."

In the lecture on "Ornamental symmetry" we see that the plane tessellation of regular hexagons occurs not only on the floors of bathrooms but in the bees' honeycomb, in the parenchyma of maize, in the retinal pigment of our eyes, and on the surface of certain diatoms. There is an account of the enumeration (by Polya and Niggli, 1924) of the seventeen two-dimensional crystallographic groups, which were unconsciously used by the Egyptians, Moors and Chinese in their ornaments. In this connection, some illustrations are reproduced from Owen Jones' *The Grammar of Ornament* (London, 1868). In the same lecture we find an introduction to the theory of positive definite quadratic forms and their representation by lattices. There is also a proof that the only possible periods for rotational symmetry of a lattice are 2, 3, 4, and 6. A still neater proof was given long ago by William Barlow, Philosophical Magazine (6 ser., 1, 1901, p. 17).

The final lecture, on "Crystals: the general mathematical idea of symmetry" begins with an outline of the enumeration of the 230 space-groups, which was carried out independently by Fedorov in Russia (1885), Schoenflies in Germany (1891) and Barlow in England (1894). Then the notion of a congruent transformation in Euclidean 3-space is extended to that of a Lorentz transformation in Minkowskian space-time. This "world" is an affine 4-space with a real cone of isotropic lines through each point. It is unfortunate that the author says (on p. 132): "The light cone . . . at a definite world-point O, 'here-now,' . . . divides the world into future and past." He seems to have overlooked the third region, exterior to the cone, which one could reasonably call "the present."

From relativity theory he deftly turns to quantum theory, Galois theory, and cyclotomy, showing the importance of the group of automorphisms in the investigation of any "structure-endowed entity."

The author's command of language is truly amazing. Who else would say (p. 127), "Temperature is the environmental factor *kat'exochen*"? Only twice is he at a loss for the English equivalent of a German word: on page 77 he writes "Umklappung" for *half-turn*, and on page 96 "modul" for *modulus*.

The four lectures are supplemented by two mathematical appendices: "Determination of all finite groups of proper rotations in 3-space" and "Inclusion of improper rotations." The latter is especially neat. Finally, there is a list of acknowledgments for the 72 figures, and a good index (except that the entry "Thompson" confuses Sir D'Arcy with Lord Kelvin, whose space-filling of truncated octahedra is displayed on page 92).

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