On the Myriad Mathematical Traditions of Ancient Greece

David E. Rowe

Send submissions to David E. Rowe, Fachbereich 17—Mathematik, Johannes Gutenberg University, D55099 Mainz, Germany.

To exert one's historical imagination is to plunge into delicate deliberations that involve personal judgments and tastes. Historians can and do argue like lawyers, but their arguments are often made on behalf of an image of the past, and these historical images obviously change over time. Why should the history of mathematics be any different?

When we imagine the world of ancient Greek mathematics, the works of Euclid, Archimedes, and Apollonius easily spring to mind. Our dominant image of Greek mathematical traditions stresses the rigor and creative achievement that are found in texts by these three famous authors. Thanks to the efforts of Thomas Little Heath, the English-speaking world has long enjoyed easy access to this trio's major works and much else besides. Yet our conventional picture of Greek mathematics has drawn on little of this plentiful source material. Our image of Greek geometry, as conveyed in mathematical texts and most books on the history of mathematics, has tended to stress the formal structure and methodological sophistication found in a handful of canonical works—or, more accurately, in selected portions of them. Even the first two books of Euclid's *Elements*, which concern the congruence properties of rectilinear figures and culminate in theorem II 14 showing how to square such a figure, have often been trivialized. Many writers have distilled their content down to a few definitions, postulates, and elementary propositions, intended merely to illustrate the axiomatic-deductive method in classical geometry.

Talk of the origins of Greek mathematics shows similar selectivity. The discovery of incommensurables, though shrouded in mystery, presumably took place around the time of Plato's birth. Two younger contemporaries, Theaetetus and Eudoxus, both of whom had ties with the Academy, are credited with having developed theories that bear on this problem. These were the basis for the mature theories found in Euclid's *Elements:* Theaetetus's classification scheme for ratios of lines appears in Book X, the longest and most technically demanding of the thirteen books, whereas Book V presents Eudoxus's general theory of proportions, which elegantly skirts the problem of representing ratios of incommensurable magnitudes by providing a general criterion for determining when two ratios are equal (Definition V.5). A standard picture of the activity that led to this work has a group of mathematicians huddled over a diagram at Plato's Academy during the early fourth century. Some of these geometers have familiar names, and a few even appear in Plato's *Dialogues*, which contain several vivid scenes and vital clues for historians of mathematics. A few of its passages have provided some of the most tantalizing tidbits of information that have come down to us.

Particularly famous is the passage in Plato's *Theaetetus* where the young mathematician recounts how his teacher, Theodorus, had managed to prove the irrationality of the sides of squares with integral non-square areas, but only up to the square of area 17. Given that Theaetetus is credited with having solved this problem on his way to developing the massive theory of irrational lines that received its final form in Book X of Euclid's *Elements*, the significance of the historical events Plato alludes to in this passage has long been clear. Little wonder that experts like the late Wilbur Knorr were tempted to tease out of it as much as they could, beginning with the obvious question: why did Theodorus stop with the square of area 17? Knorr and numerous others have offered ingenious speculations about what went wrong with Theodorus's proof. Needless to say, such efforts to reconstruct Theodorus's argument on the basis of the meager remarks contained in the Platonic passage are driven by mathematical, not historical imagination. A mundane historical interrogation of the famous passage leads to quite a different thought. What if Theodorus simply gave up after finding separate proofs for the earlier cases? Maybe the number 17 had no special significance at all!

For David Fowler, these and other sources raised, but did not answer, a related historical question: how did the geometers of Plato's time (427?–347?) represent ratios of incommensurable magnitudes? Fowler was by no means the first to ask this question, but what interests us here is the way he went about answering it. He naturally reexamined the sources on the relevant prehistory. But inquisitive minds have a way of turning over new stones before all the old ones can be found, and so Fowler's inquiry became broader. What were the central problems that preoccupied the mathematicians in Plato's Academy? This world is lost, but it has left quite a few tempting mathematical clues, and Fowler makes the most of them in an imaginative attempt to restore the historical setting. In *The Mathematics of Plato's Academy*, he offers an unabashed reconstruction of mathematical life in ancient Athens, replete with fictional dialogues. Accepting the limitations imposed by the scanty sources, he gives both his historical and mathematical imagination free reign, and produces a new picture of mathematical life in ancient Athens.

Ironically, we seem to know more about the activities of the mathematicians affiliated with Plato's Academy than we do about those of any other time or place in the Greek world, even the museum and library of Alexandria, where many of the mathematical texts that have survived the rise and fall of civilizations and empires were first written. The Alexandrian mathematicians dedicated themselves to assimilating and systematizing the work of their intellectual ancestors. But we know next to nothing about their lives and how they went about their work. Even the famous author of the thirteen books known today as Euclid's *Elements* remains a shadowy figure. Was he a gifted creative mathematician or a mere codifier of the works of his predecessors? Is it even plausible that a single human being could have written all the numerous works that Pappus of Alexandria later attributed to Euclid? On the basis of internal evidence alone, it seems unlikely that the *Data* and the *Elements* were written by the same person. But what about all the other mostly nameless scholars who surely must have mingled with Euclid in Alexandria shortly after Alexander's death? Perhaps our Euclid was actually a gifted administrator who worked at the library and headed a research group to produce standard texts of ancient mathematical works. Is it too farfetched to imagine Euclid as the ancient Greek counterpart to the twentieth century's Bourbaki?

But leaving these biographical speculations aside, we can easily agree that the *Elements* established a paradigm for classical Greek geometry, or what came to be known as ruler-and-compass geometry. Indeed, synthetic geometry in the style of Euclid's *Elements* continued to serve as the centerpiece of the English mathematical curriculum until well into the nineteenth century. For Anglo-American gentlemen steeped in the classics, no formal education was complete without a sprinkling of Euclidean geometry. This mainly meant mimicking an old-fashioned style of deductive reasoning that many believed disciplined the mind and prepared the soul to understand and appreciate Reason and Truth. With David Hilbert's *Grundlagen der Geometrie*, published in 1899, the Euclidean style may be said to have made its peace with mathematical modernity. Hilbert upgraded its structure and redesigned its packaging, but most of all he gave it a new modernized system of axioms. Within this universe of "pure thought," Greek mathematics could still retain its honored place. Enshrined in the language of modern axiomatics, it took on new form in countless English-language texts that presented Greek geometry as a watered-down version of Heath's Euclid.

The history of mathematics abounds with examples of this kind: a good theorem, so the adage goes, is always worth proving twice (or thrice), just as a good theory is one worthy of being renovated. In the case of an old warhorse like Euclidean geometry, we take this for granted. But if mathematicians will never tire of modernizing older theories, we might still do well to ask what consequences this activity has for historical understanding. The reflection is required most urgently for Euclid's *Elements*, a work that has gone through more shifts of meaning and context than any other. Reading Euclid (carefully) had profound consequences for Isaac Newton, who soon thereafter immersed himself in the lesser-known works of ancient Greek geometers. He emerged a different mathematician, set on defending the Ancients against Moderns like René Descartes, who claimed to have found a methodology superior to Greek analysis. We need not puzzle over why Newton wrote his *Principia* in the language of geometry, once we understand his strong identification with what he understood by the problem-solving tradition of the ancient Greeks. Nothing rankled him more than Cartesian boasting about how this tradition had been supplanted by modern analysis.

For ourselves, looking from a post-Hilbertian perspective, the question can be posed like this: If we continue to view Greek mathematics through the prism of Euclid's *Elements*, and to view the *Elements* mainly as a model of axiomatic rigor, what effect will this have on our conception of the more remote past in which Greek mathematics grew? One of the more obvious consequences has been the glorification of the ancient Greeks at the expense of other ancient cultures. This theme has been the subject of much bickering ever since the publication of Martin Bernal's *Black Athena*. I will not enter this fracas here; it does suggest, however, that our pictures of ancient mathematics are in the process of change, and this applies to the indigenous traditions of Greece as well as to interaction with other cultures.

By accenting the plural in traditions, I mean to emphasize that there were several different currents of Greek mathematical thought. They continued to flourish in the Hellenistic world and beyond: we should not imagine Greek mathematics monolithically, as if a single mathematical style dominated all others.

Nor should we overestimate the unity of Greek mathematics even within the highbrow tradition of Euclid, Archimedes, and Apollonius. In his *Conica* and the other minor works, Apollonius systematically exploits an impressive repertoire of geometrical operations and techniques in order to derive a series of complex metrical theorems whose significance is often obscure. In this respect, his style contrasts sharply with Euclid's *Elements*.

When we compare the works of Apollonius and Euclid with those of Archimedes, whose inventiveness is far more striking than any single stylistic element, the contrasts only widen. Unlike Apollonius, Archimedes apparently had little interest in showcasing all possible variant results merely to demonstrate his arsenal of techniques. He was first and foremost a problem-solver, not a systematizer, and many of the problems he tackled were inspired by ancient mechanics. Ivo Schneider has suggested that Archimedes's early career in Syracuse was probably closer to what we would today call "mechanical engineering" than to mathematics. Not that this was unusual; practical and applied mathematics flourished in ancient Greece, and again in early modern Europe when Galileo taught these subjects as professor of mathematics at the University of Padua, which belonged to the Venetian Republic. Like Venice, Syracuse had an impressive navy, and we can be fairly sure that Archimedes spent a considerable amount of time around ships and the machines used to build them. From these, he must have learned the principles behind the various mechanical devices that Heron and Pappus of Alexandria would later describe and classify under the five classical types of machines for generating power.

Archimedes was neither an atomist nor a follower of Democritus. Nevertheless, the parallels between these two bold thinkers are both striking and suggestive. In one of his flights of fancy, Archimedes devised a number system capable of expressing the "atoms" in the universe. For this purpose he took a sand grain as the prototype for these tiny, indivisible corpuscles. Archimedes must have seen Democritus's atomic theory as at least a powerful heuristic device in mathematics. Democritus had introduced infinitesimals in geometry, and by so doing had found the volume of a cone, presumably arguing along lines similar to the ideas that led Bonaventura Cavalieri to his general principle for finding the volumes of solids of known cross-sectional area.

As is well known, Eudoxus is credited with having introduced the "method of exhaustion" in order to demonstrate theorems involving areas and volumes of curvilinear figures, including the results obtained earlier by Democritus. Archimedes used the Eudoxian method with impressive virtuosity, but because this technique could only be applied after one knew the correct result, he had to rely first on ingenuity to obtain provisional results. His inspiration came from mechanics. By performing sophisticated thought experiments with a fictitious balance, Archimedes could "weigh" various kinds of geometrical objects as if they were composed of "geometrical atoms"—indivisible slivers of lower dimension. As he clearly realized, this mechanical method was a definite no-no for a Eudoxian geometer, but he also knew that there was "method" to this madness, since it enabled him to "guess" the areas and volumes of curvilinear figures such as the segment of a parabola, cylinders, and spheres. As Heath once put it, here we gain a glimpse of Archimedes in his workshop, forging the tools he would need before he could proceed to formal demonstration.

Going one step further, he carried out thought experiments inspired by a problem of major importance to the economic and political welfare of Syracuse: the stability of ships. Archimedes's idealized vessels had hulls whose cross-sections were parabolic in shape, enabling him to determine the location of their centers of gravity precisely. Had he performed a similar service in seventeenth-century Sweden for King Gustav Adolfus, the latter might have been spared from witnessing one of the great blunders in maritime history: the disaster that befell his warship, the *Vassa*, which flipped over and sank in the harbor on her maiden voyage. (If you've ever visited the Vassa Museum in Stockholm, you'll realize that it wouldn't have taken an Archimedes to guess that this magnificent vessel was likely to keel over as soon as it caught its first strong gust of wind.)

Archimedes's work presumably was related to his other duties as an advisor to the Syracusan court, which later called upon him when the city was besieged by the Roman armies of Marcellus. Plutarch immortalized the story of how Archimedes single-handedly held back the Roman legions with all manner of strange, terrifying war machines. These legendary exploits inspired Italian Renaissance writers to elaborate on Archimedes's feats of prowess as a military engineer. No longer content with mechanical contraptions, the new-age Archimedes devises a system of mirrors that could focus the sun's rays on the sails of Roman ships, setting them all ablaze. These mythic elements reflect the imaginative reception of Archimedes during the Renaissance as a symbol of the power of human genius, a central motif in Italian humanism. Within the narrower confines of scientific thought, the reception of Archimedes's works underwent a long, convoluted journey during the Middle Ages, so that by Galileo's time they had begun to exert a deep influence on a new style of mathematics. By the seventeenth century, the Archimedean tradition had become strongly interwoven with the Euclidean tradition, but these two currents were by no means identical from their inception.

Another major significant tradition within Greek mathematics can be traced back to Pythagorean idealism, which continued to live on side-by-side with the rationalism represented by Euclid's *Elements*. If the Pythagorean dogma that "all is number" could no longer hold sway after the discovery of incommensurable magnitudes, this does not mean that all traces of Pythagorean mathematics vanished. Far from it: we have every reason to believe that the Pythagorean and Euclidean traditions interpenetrated one another, influencing both over a long period of time. Euclid's approach to number theory in Books VII–IX differs markedly from that found in the *Arithmetica* of Nicomachus of Gerasa, who continued to give expression to the Pythagorean tradition during the first century A.D. Still, the distinctive Pythagorean doctrine of number types (even and odd, perfect, etc.) can be found in both Euclid and Nicomachus, albeit in very different guises. Thabit ibn Qurra knew both works and assimilated these arithmetical traditions into Islamic mathematics. Finding Nicomachus's treatment of amicable numbers inadequate (Euclid ignores it completely), Thabit developed this topic further. Al-Kindi later translated the *Arithmetica* into Arabic and applied it to medicine. These two writers thus helped perpetuate and transform the Pythagorean mathematical tradition within the world of Islamic learning.

Taking Pythagorean cosmological thought into account, we seen an even deeper interpenetration of mythic elements into the Euclidean tradition. For Plutarch, a writer whose imagination often outran his critical judgment, Euclid's *Elements* was itself imbued with Pythagorean lore. He linked Euclid's beautiful Proposition VI 25 with the creation myth in Plato's *Timaeus*, a work rife with Pythagorean symbolism. Plato's Demiurge, the Craftsman of the universe, fashions his cosmos out of chaos following a metaphysical principle, one that Plutarch identified with theorem VI 25: given two rectilinear figures, to construct a third equal in area to the first figure and similar to the second. In other words, Euclid's geometrical craftsman must transform a given quantity of matter into a desired form.

But we need no Plutarchian wings of imagination to see that Euclid's *Elements* contain numerous and striking allusions to Pythagorean/Platonic cosmological thought, as noted by Proclus and other commentators. The theories of constructible regular polygons and polyhedra appear in Books IV and XIII, respectively, thereby culminating the first and last major structural divisions in the *Elements* (Books I–IV on the congruence properties of plane figures; Books XI–XIII on solid geometry). In both cases, the figures are constructed as inscribed figures in circles or spheres, the perfect celestial objects that pervade all of Greek astronomy and cosmology. Perhaps most striking of all, in Book XIII, which ends by proving that the five Platonic solids are the only regular polyhedra, Euclid determines the ratio of the side length to the radius of the circumscribed sphere according to the classification scheme presented in Book X for incommensurable lines. This body of mathematical knowledge shows its connection with the doctrine of celestial harmonies, an idea whose origins are obscure, but which undoubtedly stems from Pythagoreanism.

The doctrine that the heavens produce a sublime astronomical music through the movements of invisible spheres that carry the stars and planets continued to ring forth in the works of Plato and Cicero. Johannes Kepler went further, proclaiming in *Harmonice Mundi* (1619) the underlying musical, astrological, and cosmological significance of Euclid's *Elements*. For him, Book IV, on the theory of constructible polygons, contained the keys to the planetary aspects, the cornerstone of his "scientific" astrology. Historians of science have long overlooked the inspiration behind Kepler's self-acknowledged *magnum opus* from 1619, preferring instead to emphasize his "positive contributions" to the history of astronomy, namely Kepler's three laws. Few seem to have been puzzled about the connection between these laws and Kepler's cosmological views as first set forth in *Mysterium Cosmographicum* (1596), where he tries to account for the distances between the planets by a famous system of nested Platonic solids. Kepler published his first two laws (that the planets move around the sun in elliptical orbits, and that from the sun's position they sweep out equal areas in equal times) thirteen years later in *Astronomia Nova* (1609), which presents the astronomical results of his long struggle to grasp the motion of Mars. The third law (that for all planets the ratio of the square of their mean distance to the sun to the cube of their period is the same constant) only appeared another ten years later in *Harmonice Mundi*. Unlike the first two astronomical laws, the third had a deeper cosmic significance for Kepler, who never abandoned the cosmological views he advanced in 1596. Indeed, for him the third law vindicated his cosmology of nested Platonic solids by revealing the divine cosmic harmonies that God conceived for this system as elaborated by Kepler in Book V of *Harmonice Mundi*.

Kepler knew Euclid's *Elements* perhaps better than any of his contemporaries, and his imagination ran wild with it in *Harmonice Mundi*. Like so many early moderns, he saw his work as the continuation of a quest first undertaken by the ancient Greeks. Kepler believed that the Ancients had already discovered deep and immortal truths, none more important than those found in the thirteen books of the *Elements*. And since truth, for Kepler, meant Divine Truth, he saw his quest as inextricably interwoven with theirs. His historical sensibilities were shaped by a profound religious faith that led him to identify his Christian God with the Deity that pagan Greeks described in the mythic language of Pythagorean symbols. We gasp at the gulf that separates our post-historicist world from Kepler's naïve belief in a transcendent realm of bare truth. We can only marvel in the realization that it was Kepler's sense of a shared past that enabled him to compose his *Harmonice Mundi* while contemplating the truths he thought he saw in the works of ancient Greek writers.

These brief reflections suggest some broader conclusions for the history of mathematics: that mathematical knowledge, as a general rule, is related to various other types of knowledge, that its sources are varied, and that the form and content of its results are affected by the cultures within which it is produced. Those who have produced mathematics have done so in quite different societies, within which these producers have had quite varied functions. Western mathematics owes much of course to ancient Greek mathematicians, but even within the scope of the Greeks' traditions we encounter considerable variance in the styles and even the content and purposes of their mathematics. For this reason, we should avoid the temptation to reduce Greek mathematics to one dominant paradigm or style.