Hipparchus, Plutarch, Schröder, and Hough

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1. HIPPARCHUS AND PLUTARCH. Plutarch was a Greek biographer and philosopher from Chaeronea, who was born before A.D. 50 and died after A.D. 120. He is best known for his *Parallel Lives*, which inspired such Renaissance writers as Montaigne, Shakespeare, Dryden, and Rousseau. His many other works have been gathered together under the name *Moralia*, "a collection of comparatively short treatises and dialogues which cover an immense range of subjects, literary, ethical, political, and scientific" [**21**, p. 8]. Part of the *Moralia* consists of the *Table-Talk*, "a collection of dialogues purporting to reproduce the after-dinner conversation of Plutarch and his friends and relatives on various occasions" [**20**, p. 2]. In the *Table-Talk* [**20**, VIII.9, 732] appears the following statement:

> Chrysippus says that the number of compound propositions that can be made from only ten simple propositions exceeds a million. (Hipparchus, to be sure, refuted this by showing that on the affirmative side there are 103,049 compound statements, and on the negative side 310,952.)

Chrysippus (*c*. 280–207 B.C.) came to Athens around 260 and became a leading Stoic philosopher. Hipparchus was a Greek astronomer (*c*. 190–after 127 B.C.) from Nicaea in Bithynia (now Iznik, Turkey) who spent much of his life at Rhodes. He was perhaps the greatest astronomer of antiquity. He is most famous for his discovery of the precession of the equinoxes, based on his own observations and those of Timocharis 160 years earlier. For further information on the work of Hipparchus, see [**19**, Book I, E], [**32**]. Hipparchus was an excellent mathematician (though for a contrary view see [**33**, p. 211]); he was the first person to make systematic use of trigonometry, and he was probably the inventor of stereographic projection. However, for many centuries no one was able to make sense of the statement of Plutarch. For instance, T. L. Heath [**12**, vol. 2, p. 256], a standard older authority on Greek mathematics, says of Plutarch's statement that "it seems impossible to make anything of these figures," while the more recent authority O. Neugebauer [**19**, p. 338] states that Plutarch's statement "[has], however, so far eluded a satisfactory explanation." Similarly W. and M. Kneale [**16**, p. 162], authorities on the history of logic, remark that "It is difficult to make any satisfactory sense of the passage." N. L. Biggs [**2**, p. 113] notes the paucity of combinatorial computations by the ancient Greeks and referring to Plutarch's passage says that "the most interesting of them is also the most mysterious." A number of eminent mathematicians and historians of mathematics, such as M. Cantor, J. Tropfke, S. Günther, and E. Artin, have attempted to understand Plutarch's statement without success. An attempt to reconstruct Hipparchus' procedure appears in [**1**], though it will be apparent from our discussion that this attempt is incorrect. Another incorrect speculation appears in [**30**, p. 63].

2. SCHRÖDER. Friedrich Wilhelm Karl Ernst Schröder was a German logician who was born in Mannheim on November 25, 1841, and died in Karlsruhe on June

16, 1902. He passed the doctoral exam at the University of Heidelberg in 1862 and had positions in Zurich (at the Eidgenössische Polytechnikum), Karlsruhe, Pforzheim, and Baden-Baden, before accepting a post as full professor at Karlsruhe in 1876. Schröder worked mainly on the foundations of mathematics, notably with combinatorics, the theory of functions of a real variable, and mathematical logic. He was one of the first persons to accept Cantor's ideas in set theory and was one of the developers of mathematical logic in the second half of the nineteenth century. Schröder is best known to combinatorialists for his paper **[25]**, in which he discusses four "bracketing problems." The first two problems concern the bracketing or parenthesization of a string of letters that we may assume to be all identical, say the letter x. The second two problems are analogues of the first two where the string of letters is replaced by a set of elements. We will discuss only the first two problems here.

The formal definition of a bracketing is the following. First, x itself is considered to be a bracketing. Recursively define a bracketing to be a sequence $B = (B_1, \dots, B_k)$, where $k \geq 2$ and each B_i is a bracketing. We represent the bracketing B as a parenthesized string of x's. Thus, think of B as a k-ary product $(B_1)(B_2) \cdots (B_k)$. If some B_i is the single letter x, then we remove the parentheses surrounding B_i for clarity of notation. Thus, for example, the bracketing

$$(xx)((xxxxx)x(xx))(xx(xx)) \tag{1}$$

represents a way of multiplying 14 x's whose last operation was a ternary operation $(B_1)(B_2)(B_3)$, where $B_1 = xx$, $B_2 = (xxxxx)x(xx)$, and $B_3 = xx(xx)$, and similarly for B_1, B_2, and B_3. There are exactly eleven bracketings of four letters, namely,

$$xxxx \quad (xx)xx \quad x(xx)x \quad xx(xx) \quad (xxx)x \quad x(xxx)$$
$$((xx)x)x \quad (x(xx))x \quad (xx)(xx) \quad x((xx)x) \quad x(x(xx)).$$

Note that the last five of these are built up entirely from *binary* operations and are therefore called *binary bracketings*.

There are three fundamental equivalent ways to represent a bracketing in addition to a parenthesized string discussed above: as *plane trees*, *polygon dissections*, and *Łukasiewicz words*. We now briefly describe these alternative representations. If B is a bracketing, then we first define the plane tree $\tau(B)$ corresponding to B. If B consists of a single letter, then $\tau(B)$ is a single root vertex. If $B = (B_1, \dots, B_k)$ then $\tau(B)$ consists of a root vertex (drawn at the top), with subtrees $\tau(B_1), \dots, \tau(B_k)$, drawn in that order from left to right. Thus, the key property defining a plane tree is that the subtrees of every vertex are linearly ordered. For instance, the plane tree corresponding to the bracketing of equation (1) is shown in Figure 1. Note that a binary bracketing corresponds to a *binary plane tree*, i.e., a plane tree for which every non-endpoint vertex has exactly two successors.

Next we consider polygon dissections. Let P be a convex polygon. A *dissection* of P is obtained by drawing some diagonals that don't intersect in their interiors. Thus, P is divided up into regions that are themselves convex polygons. In particular, if P has m sides and we draw $m - 3$ such diagonals (the maximum number possible), then we obtain a dissection for which every region is a triangle; such dissections are called *triangulations*. We now explain how to associate a plane tree $\tau(D)$ with a polygon dissection D. We associate with the "degenerate" polygon with just two vertices a single root vertex. Now fix once and for all an edge e of the polygon P, called the *root edge*. In a given dissection D, the edge e is contained in a unique polygon Q that is a region of D. Let $k + 1$ be the number of

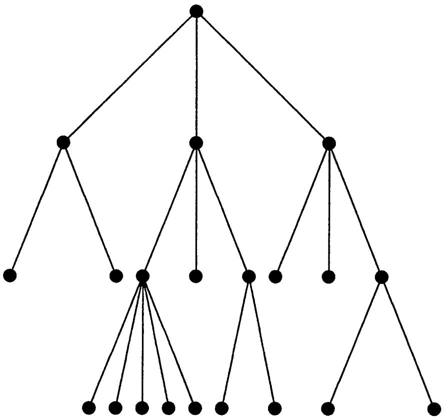

Figure 1. A plane tree.

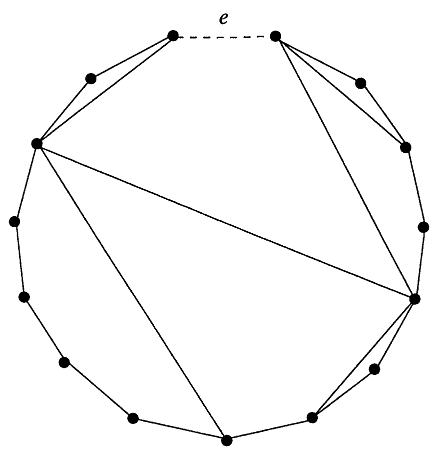

Figure 2. A polygon dissection.

edges of Q. If we remove the edge e and the interior of Q from D, then we are left with dissections $D_1, D_2, \ldots, D_k$ of k polygons (some possibly with just two vertices), reading counterclockwise from e along the boundary of Q, such that D_i and D_{i+1} intersect at a single vertex for $1 \leq i \leq k-1$. Define recursively $\tau(D)$ to be the plane tree whose subtrees of the root are $\tau(D_1), \ldots, \tau(D_k)$ in that order. Note that if P has $n+1$ vertices, then $\tau(D)$ has n endpoints. Figure 2 shows the polygon dissection corresponding to the tree of Figure 1.

Finally we consider Łukasiewicz words. The letters of such words come from the alphabet $A = \{x_0, x_1, x_2, \ldots\}$. The *weight* $\delta(x_i)$ of a letter x_i is defined by $\delta(x_i) = i - 1$. A word $y_1 y_2 \cdots y_m$ made of letters from A is said to be a *Łukasiewicz word* if $\delta(y_1) + \cdots + \delta(y_j) \geq 0$ for $1 \leq j \leq m-1$, and $\delta(y_1) + \cdots + \delta(y_m) = -1$. Thus, $y_m = x_0$. The set of all Łukasiewicz words is called the *Łukasiewicz language* [**17**, Ch. 11.3]. To obtain a Łukasiewicz word $\omega(\tau)$ from a plane tree τ, do a depth-first (preorder) search through the tree. By definition, this is a linear ordering $\delta(\tau) = v_1, v_2, \ldots, v_p$ of the vertex set of τ defined recursively by $\delta(\tau) = v, \delta(\tau_1), \ldots, \delta(\tau_k)$, where v is the root of τ, and $\tau_1, \ldots, \tau_k$ are the subtrees of v (in that order). Define

$$\omega(\tau) = x_{\deg(v_1)} x_{\deg(v_2)} \cdots x_{\deg(v_k)},$$

where $\deg(v_i)$ denotes the degree (number of successors or children) of vertex v_i. For instance, the Łukasiewicz word corresponding to the plane tree of Figure 1 is

$$x_3 x_2 x_0^2 x_3 x_5 x_0^6 x_2 x_0^2 x_3 x_0^2 x_2 x_0^2.$$

Note that since our bracketings B do not allow unary operations, the plane tree $\tau(B)$ has no vertices of degree one, and the corresponding Łukasiewicz word does not involve the letter x_1.

The correspondences we have established are easily seen to yield the following result.

Proposition. (a) *Let $s(n)$ denote the total number of bracketings of a string of n letters. Then $s(n)$ is also equal to (i) the number of plane trees with no vertex of degree one and with n endpoints, (ii) the number of dissections of a convex $(n+1)$-gon, and (iii) the number of Łukasiewicz words with no x_1's and with n x_0's.*

(b) *Let $b(n)$ denote the number of binary bracketings of a string of n letters. Then $b(n)$ is also equal to (i) the number of binary plane trees with n endpoints (and hence with $2n-1$ vertices), (ii) the number of triangulations of a convex $(n+1)$-gon, and (iii) the number of Łukasiewicz words with n x_0's and $n-1$ x_2's (and with no other letters); such words, usually with the last x_0 deleted, are sometimes called* Dyck words.

We are now ready to explain the contribution of Schröder to these bracketing problems. Schröder's first problem asks for the number $b(n)$ of binary bracketings of a string of n letters. Using a generating function argument, Schröder derives the formula (stated slightly differently)

$$b(n) = \frac{1}{n}\binom{2n-2}{n-1}.$$

Thus $b(n)$ is just the *Catalan number* C_{n-1}, for which an enormous literature exists. For some further information and references, see **[11]**, **[14]**. A list of about fifty combinatorial interpretations of Catalan numbers will appear in [**31**, Exercise 6.17] and is available on the World Wide Web at
http://www-math.mit.edu/~rstan/ec/ec.html.

Schröder's second problem asks for the total number $s(n)$ of bracketings of a string of n letters. Schröder's main result on his second problem is the generating function

$$\sum_{n\geq 1} s(n)x^n = \frac{1}{4}\left(1 + x - \sqrt{1 - 6x + x^2}\right). \tag{2}$$

He also gives the values (with the typographical error 145 for $s(5) = 45$)

$$(s(1), \ldots, s(10)) = (1,1,3,11,45,197,903,4279,20793,103049). \tag{3}$$

Perhaps the quickest way to obtain equation (2) is the following. Let y denote the left-hand side. The recursive definition of bracketing is equivalent to the formula

$$y = x + y^2 + y^3 + y^4 + \cdots = x + \frac{y^2}{1-y}. \tag{4}$$

Multiplying by $1 - y$ yields the quadratic equation

$$2y^2 - (1 + x)y + x = 0. \tag{5}$$

One of the solutions is spurious, and the other one is just the right-hand side of (2).

The numbers $s(n)$ are now called *Schröder numbers*. Schröder does not mention any other combinatorial interpretations of Schröder numbers, nor does he give a single outside reference. Let us point out some additional references. The problem of counting the triangulations of a convex polygon was raised by Segner [**26**] and solved (anonymously) by Euler [**9**]. The connection between bracketings and plane trees was known to Cayley [**4**]. The bijection between plane trees and polygon dissections appears in Etherington [**8**], with a sequel by Erdélyi and Etherington in [**7**]. The bijection between bracketings and Łukasiewicz works is essentially the "reverse Polish notation" or "parenthesis-free notation" developed by the Polish logician Jan Łukasiewicz (1878–1956). He came upon the idea of this notation in 1924 and first published it in 1929, as explained in [**18**, p. 180, footnote 3]. The connection between reverse Polish notation and enumerative combinatorics appears in a pioneering paper of George Raney [**22**].

There is now a considerable literature on Schröder numbers and related numbers. To get into this literature, see [**3**], [**15**, p. 55], [**23**], [**27**], and [**34**]. Let us also mention that it is easy to obtain a simple recurrence relation [**5**], [**6**, p. 57] for the Schröder numbers that allows them to be computed rapidly. Namely, differentiate (5) with respect to x and solve for y' to obtain

$$y' = \frac{y - 1}{4y - 1 - x} = \frac{(x - 3)y - x + 1}{x^2 - 6x + 1},$$

the latter equality a consequence of the quadratic equation (5). Hence

$$(x^2 - 6x + 1)y' - (x - 3)y + x - 1 = 0.$$

Expanding the left-hand side in a power series in x and setting the coefficient of x^n equal to 0 yields

$$(n + 2)s(n + 2) - 3(2n + 1)s(n + 1) + (n - 1)s(n) = 0, \quad n \geq 1. \tag{6}$$

No direct combinatorial proof of this formula was known until D. Foata and D. Zeilberger, after reading an earlier version of this paper, found such a proof [**10**].

3. HOUGH. The stage is now set for the *dénouement*. The astute reader may have already anticipated it by comparing Plutarch's cryptic statement with the values (3) of the Schröder numbers. In January 1994 David Hough (1949–), a graduate student at George Washington University (who decided only in 1992 that he would pursue a career in mathematics), noticed that the mysterious number 103,049 of Plutarch, i.e., the number of compound propositions that can be formed from ten simple propositions, is just the tenth Schröder number! Hough learned about Plutarch's statement from [**30**, Exercise 1.45]. Hough's discovery strongly suggests that Hipparchus was carrying out a calculation equivalent to the modern calculation of the number of bracketings of a string of ten letters. However, it remains to determine exactly what Hipparchus and Plutarch meant by a "compound proposition." In Stoic logic, compound propositions are built up from simple ones using such connectives as "and," "or," and "if... then" [**16**, Ch. III.5]. This does not seem like enough information to pinpoint precisely what Hipparchus had in mind.

We can also ask how Hipparchus computed the number 103,049. As noted in [**24**, p. 101], this number is much too large to have been computed by a direct enumeration of all the cases. Moreover, it is highly unlikely that Hipparchus was aware of the sophisticated recurrence (6). More probable is that Hipparchus used the "obvious" recurrence (equivalent to equation (4))

$$s(n) = \sum_{i_1 + \cdots + i_k = n} s(i_1) \cdots s(i_k), \quad n \geq 2, \tag{7}$$

where the sum ranges over all ways to write n as an (ordered) sum of $k \geq 2$ positive integers. The sum on the right-hand side of equation (7) in the case $n = 10$ has 511 terms. There are only 41 "essentially different" terms, corresponding to the 41 partitions of 10 into a least two parts, i.e., the 41 ways to write 10 as an *unordered* sum of at least two positive integers. If the terms of the sum are grouped according to the partition of 10 to which they correspond, it is still necessary to count the number of ways of ordering each partition. For instance, the partition $3 + 2 + 2 + 1 + 1 + 1$ has 60 orderings of its terms, thus contributing the amount $60s(3)s(2)^2s(1)^3$ to the sum (7). We cannot but admire Hipparchus' ability to compute the Schröder number $s(10)$ at a distant time when not even a remotely similar accurate computation is known. For further information about combinatorics in ancient times, see [**2**], [**24**].

The number 310,952 in Plutarch's statement, i.e., the number of compound propositions that can be formed from ten simple propositions "on the negative side," remains an enigma. Many possible variants of plane trees have been looked at without success. Moreover, Neil Sloane has verified that the numbers 310,952 and 103,049 + 310,952 = 414,001 do not appear anywhere in the valuable tables [**28**]. Thus the mystery of Plutarch's statement remains at most half solved.

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REFERENCES

1. K.-R. Biermann and J. Mau, Überprüfung einer frühen Anwendung der Kombinatorik in der Logik, *J. Symbolic Logic* **23** (1958), 129–132.
2. N. L. Biggs, The roots of combinatorics, *Historia Mathematica* **6** (1979), 109–136.
3. J. Bonin, L. W. Shapiro, and R. Simion, Some q-analogues of the Schröder numbers arising from combinatorial statistics on lattice paths, *J. Stat. Planning and Inference* **34** (1993), 35–55.
4. A. Cayley, On the analytical form called trees, Part II, *Philos. Mag.* (4) **18** (1859), 374–378.
5. L. Comtet, Calcul pratique des coefficients de Taylor d'une fonction algébrique, *Enseignement Math.* **10** (1964), 267–270.
6. L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht/Boston, 1974.
7. A. Erdélyi and I. M. H. Etherington, Some problems of non-associative combinations (2), *Edinburgh Math. Notes* **32** (1940), 7–12.
8. I. M. H. Etherington, Some problems of non-associative combinations (1), *Edinburgh Math. Notes* **32** (1940), 1–6.
9. L. Euler, Summarium, *Novi Commentarii academiae scientiarum Petropolitanae* **7** (1758/59), 13–15. Reprinted in *Opera Omnia* (1) **26** (1953), xvi–xviii.
10. D. Foata and D. Zeilberger, A classic proof of a recurrence for a very classical sequence, preprint. Available from the website http://www.math.temple.edu/~zeilberg/mamarim/mamarimhtml/classic.html.
11. M. Gardner, Catalan numbers, Mathematical Games, *Scientific American* **234** (June, 1976), pp. 120–125, and bibliography on p. 132. Reprinted (with an Addendum) in Chapter 20 of *Time Travel and Other Mathematical Bewilderments*, W. H. Freeman, New York, 1988.

12. T. L. Heath, *A History of Greek Mathematics*, Oxford University Press, Oxford; 1921; reprinted by Dover, New York, 1981.
13. T. L. Heath, *A Manual of Greek Mathematics*, Oxford University Press, Oxford, 1931; reprinted by Dover, New York, 1963.
14. P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, *Math. Intelligencer* **13** (Spring, 1991), 64–75.
15. M. Klazar, On *abab*-free and *abba*-free set partitions, *Europ. J. Combinatorics* **17** (1996), 53–68.
16. W. Kneale and M. Kneale, *The Development of Logic*, Oxford University Press, Oxford, 1962, 1971.
17. M. Lothaire, *Combinatorics on Words*, Encyclopedia of Mathematics and Its Applications, vol. 17, Addison-Wesley, Reading, Massachusetts, 1983.
18. J. Łukasiewicz, *Selected Works* (L. Borkowski, ed.), North-Holland, Amsterdam, 1970.
19. O. Neugebauer, *A History of Ancient Mathematical Astronomy*, vol. 1, Springer-Verlag, Berlin/Heidelberg/New York, 1975.
20. Plutarch, *Moralia*, vol. IX (introduction by E. L. Minar, Jr.), Loeb Classical Library, Harvard University Press, Cambridge, Massachusetts, 1961.
21. Plutarch, *The Rise and Fall of Athens*: *Nine Greek Lives*, translated by I. Scott-Kilvert, Penguin, London, 1960.
22. G. N. Raney, Functional composition patterns and power series reversion, *Trans. Amer. Math. Soc.* **94** (1960), 441–451.
23. D. G. Rogers and L. W. Shapiro, Deques, trees and lattice paths, in *Combinatorial Mathematics VIII* (K. L. McAvaney, ed.), Lecture Notes in Mathematics, no. 884, Springer-Verlag, Berlin, pp. 293–303.
24. A. Rome, Procédés anciens de calcul des combinaisons, *Ann. Soc. sci. Bruxelles*, ser. A **50** (1930), 97–104.
25. E. Schröder, Vier combinatorische Probleme, *Z. für Math. Physik* **15** (1870), 361–376.
26. A. de Segner, Enumeratio modorum, quibus figurae planae rectilineae per diagonales dividuntur in triangula, *Novi Commentarii academiae scientiarum Petropolitanae* **7** (1758/59), 203–209.
27. L. W. Shapiro and A. B. Stephens, Bootstrap percolation, the Schröder numbers, and the n-kings problem, *SIAM J. Discrete Math.* **4** (1991), 275–280.
28. N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, 1995.
29. R. Stanley, Differentiably finite power series, *European J. Combinatorics* **1** (1980), 175–188.
30. R. Stanley, *Enumerative Combinatorics*, vol. 1, Wadsworth & Brooks/Cole, Belmont, CA, 1986; second printing, Cambridge University Press, 1996.
31. R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, Cambridge, in preparation.
32. G. J. Toomer, Hipparchus, in *Dictionary of Scientific Biography* (C. C. Gillipsie, editor in chief), vol. XV, supplement I, Scribner's, New York, 1978, pp. 207–224.
33. B. L. van der Waerden, *Geometry and Algebra in Ancient Civilizations*, Springer-Verlag, Berlin, 1983.
34. J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.* **146}** (1995), 247–262.

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