The Classification of Algebraic Surfaces by Castelnuovo and Enriques

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One of the lasting contributions to mathematics made by the Italian geometers of a century ago was the classification of algebraic surfaces. This article gives a brief account of the path taken by Enriques and Castelnuovo that culminated in the Enriques classification, as it is known. The classification was extended to all complex 2-manifolds by Kodaira in the early 1950s and has been much studied ever since.

An algebraic surface can be thought of as being defined in 3-space by a polynomial in three variables x, y, and z. As such, they were heavily studied throughout the nineteenth century, and many examples were known—the cubic surface with its 27 lines and Kummer's quartic surface with its 16 nodal points are among those that remain of interest (see [18]). In the 1870s, Max Noether created the outline of a general theory, which followed the outlines of the general theory of algebraic curves (itself a recent creation). But it was clear that much more remained to be done than had been solidly accomplished. The approach that appealed particularly to the Italian school of geometers was initiated by Veronese in his paper [29]. He used the method of projection and section to show how some singular curves in the plane and singular surfaces in 3-space could profitably be thought of as nonsingular objects in a high-dimensional space. To make a systematic attack on the study of surfaces, Corrado Segre then suggested that the best approach to surfaces would be to study them birationally and to look for families of curves sufficiently well behaved to yield an embedding of the surface in some suitable projective space.

To see how this can be done, at least in principle, it is worth looking at the simplest kind of birational transformation of the plane, and its effect on a mildly singular curve (described in Box 1).

A birational transformation replaces one surface with another, it suggests that singular surfaces might be blown up in ways that could simplify their singularities, and this was indeed to prove to be the case. The birational transformation in Box 1 was found by using a 3-parameter family of curves (in this case conics) with the property that at almost every point of the plane, there was a three-dimensional space of curves passing through the point. More general birational transformations are found by choosing other families of curves—it is a famous exercise in the subject to obtain the cubic surface with its 27 lines by considering all the plane cubics through 6 points.

A Quick Trip Through Algebraic Curves

After Riemann's death in 1866, leadership in the field of algebraic geometry passed to Clebsch; and when he died in 1872, to his former colleagues Alexander Brill and Max Noether. They gave the entire theory a firm twist in the direction of polynomial algebra. They thought of an algebraic curve as a plane curve and defined by a polynomial equation in two complex variables $F(z, w) = 0$ (or, equivalently, three homogeneous variables). Its genus, p, was defined in terms of the degree, n, of the defining equation and the nature of the singular points, according to the formula

$$\frac{1}{2}(n-1)(n-2) - \sum_i \alpha_i \frac{i(i-1)}{2} = p - 1,$$

where the curve has α_i points of order i. To distinguish it from other curves which enter the Brill–Noether theory, this curve will be called the ground curve. The everywhere-holomorphic-integrands on the curve, as Clebsch had shown, following Abel, are of the form $\phi\, dz/(\partial F/\partial w)$, where ϕ is of degree $n - 3$. Brill and Noether showed that a curve of order $n - 3$ that passes

Box 1

A Birational Transformation

A conic in the projective plane has equation

$$ax^2 + by^2 + cz^2 + dyz + ezx + fxy = 0.$$

The conics that pass through the points [1, 0, 0] and [0, 1, 0] have equations of the form

$$cz^2 + dyz + ezx + fxy = 0.$$

Each such conic defines a plane in projective space with coordinates $[t, u, v, w]$ with equation

$$ct + du + ev + fw = 0.$$

The coefficients of the conics that also pass through the point $[x_0, y_0, z_0]$, $z_0 \neq 0$, satisfy

$$cz_0{}^2 + dy_0z_0 + ez_0x_0 + fx_0y_0 = 0.$$

So all the planes in projective space corresponding to these conics pass through the point

$$[z_0{}^2, y_0z_0, z_0x_0, x_0y_0].$$

This allows us to define a map from the projective plane to projective space:

$$[x, y, z] \mapsto [z^2, yz, zx, xy].$$

The image of this map lies on the hyperboloid with equation $tw = uv$. Indeed, the image is the hyperboloid with the lines $[0, 0, v, w]$ and $[0, u, 0, w]$ deleted. Moreover, points on the line $z = 0$ joining [1, 0, 0] and [0, 1, 0] have coordinates of the form $[x, y, 0]$, and unless $x = 0$ or $y = 0$, they all map to the point in projective space with coordinates [0, 0, 0, 1]; the line is said to be blown down to the point. The map is not strictly defined at the points [1, 0, 0] and [0, 1, 0].

The line through the points $[x_0, y_0, z_0]$ and [1, 0, 0] has equation $z_0y = y_0z$ and consists of the points $[\sigma, \tau y_0, \tau z_0]$. They map to the points $[\tau z_0{}^2, \tau y_0z_0, \sigma z_0, \sigma y_0]$, which form a line through the image of the point $[x_0, y_0, z_0]$. This line of the ruling on the hyperboloid meets the line $[0, 0, v, w]$ at the point $[0, 0, z_0, y_0]$, so we say the line $[0, 0, v, w]$ is the blowup of the point [1, 0, 0]. The image of a line through [1, 0, 0] meets the blowup of [1, 0, 0] at the point determined by the coefficients in its equation. Similarly, for the other ruling of the hyperboloid, we say the line $[0, u, 0, w]$ is the blowup of the point [0, 1, 0]. In particular, the line $y = z$ maps to the line parametrised by $[\tau, \tau, \sigma, \sigma]$, and the line $y = -z$ maps to the line parametrised by $[\tau, -\tau, \sigma, -\sigma]$.

Because we shall be interested in the region of the hyperboloid corresponding to a neighbourhood of the point [0, 0, 1] in the plane, we can consider the region in which $v \neq 0$, and so without loss of generality, $v = 1$. So we consider the surface with equation $tw = u$.

Now, consider the folium of Descartes, familiarly written with equation

$$x^3 + y^3 = 3xyz,$$

which crosses itself at the point [0, 0, 1]. Its tangents there are the lines $x = 0$ and $y = 0$. We shall use the map just obtained to resolve this singular point. First, we move the singular point to the point [1, 0, 0], which we can do by writing the equation in the form $z^3 + y^3 = 3xyz$. The tangents at the singular point now have equations $z = 0$ and $y = 0$. To avoid problems caused by the fact that the tangent line $z = 0$ is blown down to a point, we rotate the curve through $\pi/4$, so that it has equation

$$2z^3 + 6zy^2 = 3x(z^2 - y^2).$$

Lines through the singular point have equations of the form $y = mz$, so they meet the curve again at the point $[x, mz, z]$, where $x = (2 + 6m^2)z/3(1 - m^2)$. This gives the parametrisation of the folium:

$$[x, y, z] = [2 + 6m^2, 3m(1 - m^2), 3(1 - m^2)].$$

Note that when $m = \pm 1$, this reduces to the point [1, 0, 0], as it should.

This point with general m is mapped to the point on the hyperboloid with coordinates $[z^2, mz^2, zx, mxz] = [z, mz, x, mx]$, which lies on the surface

$$2t^3 + 6tu^2 = 3v(t^2 - u^2).$$

So the image of the folium is the intersection of this surface with the hyperboloid $tw = uv$. Parametrically, this curve is given by $[3(1 - m^2), 3m(1 - m^2), 2 + 6m^2, m(2 + 6m^2)]$. In the region where $v \neq 0$, we can assume without loss of generality that $v = 1$, and the image of the folium is given parametrically by $[3(1 - m^2)/(2 + 6m^2), 3m(1 - m^2)/(2 + 6m^2), 1, m]$. When $m = 1$, this is the point [0, 0, 1, 1], and when $m = -1$, this is the point [0, 0, 1, −1]. The behaviour of the fourth coordinate makes it clear that this curve does not intersect itself. So the curve has been pulled apart at the singular point, and now is a nonsingular curve in projective space.

To obtain a plane curve, the hyperboloid is mapped to the projective plane by a stereographic projection. This will not reintroduce a double point (although the rulings of the hyperboloid through the vertex of projection map to points, they meet the curve in single points unless the vertex is badly chosen), so in this example, the original singular point has been resolved away. In general, the image of a more complicated curve might be resolved into simpler ones on the hyperboloid, and the hope would be that the final image would be a curve having less complicated singularities.

$(i - 1)$ times through each point on the ground curve of order i cuts the original curve in

$$n(n - 3) - \sum_i \alpha_i i(i - 1) = 2p - 2$$

further points, of which at least $p - 1$ can be chosen arbitrarily. Such a curve is called an adjoint curve; it cuts out what is called a canonical point-group on the original curve.

In particular, the version of the Riemann–Roch Theorem that Brill and Noether established in [2] says: Let an adjoint curve of order $n - 3$ be drawn through a set of Q points on the ground curve, and suppose q of these points determine the rest (and so the adjoint curve). If $q = Q - p + 1 + r$ (where $0 < r < p - 1$), then this curve meets the ground curve in $2p - 2 - Q = R$ further points that themselves belong to a set in which r points determine the rest. This version is connected to Riemann's by the simple observation that if ϕ_0 and ϕ_∞ are two adjoint curves (of any order), then their quotient ϕ_0/ϕ_∞ is a meromorphic function on the ground curve having prescribed zeros where the curve $\phi_0 = 0$ meets the ground curve and having poles where the curve $\phi_\infty = 0$ meets the ground curve. Interestingly, this observation was made explicitly for the first time by Klein in his lectures [22, p. 189n]. By omitting it, Brill and Noether made their preference for algebra over geometry known.

The study of algebraic curves taught the Italians that a curve other than a rational or elliptic curve can be embedded in a projective space. The first to appreciate this had been Kraus, who died at the age of 27 in 1881, and his approach was later taken up by Felix Klein. It rests on the insight that an algebraic curve of genus $p \geq 2$ has a p-dimensional space of holomorphic 1-forms. Kraus [23] had the happy idea that if a basis is chosen for these 1-forms, say $\omega_1, \ldots, \omega_p$, then the p-tuple $(\omega_1(z), \ldots, \omega_p(z)) = (f_1(z)dz, \ldots, f_p(z)dz)$, $p > 1$, gives rise to a map from the curve to a projective space of dimension $p - 1$: $z \to [f_1(z), \ldots, f_p(z)]$. The argument that this map is well defined (independent of the coordinate system used) and is a map into projective space (the f_i never simultaneously vanish) was ducked by him. It is a straighforward exercise in the Riemann–Roch Theorem, as is the observation that the degree of the map is $2p - 2$. Hyperelliptic curves also cause problems: as Kraus saw, one instead realises the curve as a branched double cover of the Riemann sphere. These cases aside, questions about algebraic curves have been reduced to questions in projective geometry. In particular, as Klein noted [22, p. 117], the absolute invariants of the normal curve are the moduli of the algebraic curve.

Castelnuovo and Enriques

Guido Castelnuovo and his close friend Federigo Enriques met in Rome in the autumn of 1892, when Castelnuovo was 27 and Enriques just 21. Castelnuovo was a former student of Corrado Segre, and Segre encouraged them to take up the study of algebraic surfaces, in which he was deeply interested. They approached the subject more obliquely than he had, and with a surer grasp of what the better developed theory of algebraic curves had to say. In particular, they pursued the idea that a Riemann–Roch Theorem for surfaces would be a powerful tool, and so it turned out. From Segre, they drew the lesson that the ideas of Kraus and Veronese could be harnessed to a suitable generalisation of the work of Brill and Noether.

They accepted that ideas from complex analysis should be played down, but they were more geometric and less algebraic than their German predecessors. They looked for a way of defining canonical curves, K, on a surface birationally. Projectively, these are curves on the surface of degree n cut out by adjoint surfaces of degree $n - 4$ that pass often enough through the singular points and singular curves of the surface (this had been Noether's approach). The adjoint, $A(C)$, of a curve C should be $C + K$. A suitable generalisation of the Riemann–Roch Theorem should apply to the surface and a curve C or the maximal linear system $|C|$ to which C belongs.

However, as Enriques observed in his first major paper, his *Ricerche* [10], Noether's definition of an adjoint surface invokes the degree, so it is projective but not birational. Another definition of the canonical curve must be sought. Moreover, as the example in Box 1 illustrates, whereas linear families of curves on the surface do yield maps to projective space, if there are points common to all the curves, the image of the surface that they provide will have new singularities (the base points will "blow up" into curves). It will be necessary to control, and ideally to eliminate, these exceptional curves.

This became a long-running story; interestingly, the definitions offered by the Italian algebraic geometers had nothing to do with holomorphic integrands. The topic of integrals was deliberately and happily left to the French, notably Picard but also Humbert (see [20]). It is quite striking that Picard in France and Castelnuovo and Enriques in Italy chose to develop their ideas in parallel, but not to merge their fields.

Castelnuovo and Enriques and the Riemann–Roch Theorem

One major aim of the *Ricerche* [10] was to establish a Riemann–Roch Theorem for surfaces. By analogy with algebraic curves, a Riemann–Roch Theorem for surfaces should concern an algebraic surface, a curve on it cut out by another surface, and a family of surfaces, one member of which cuts the given curve. To prove such a theorem, Enriques endeavoured to express the numbers involved in terms of quantities that apply to the given curve, and then to invoke the Riemann–Roch Theorem for the curve. He found it necessary to exclude certain types of surface, which was not necessarily a misfortune: it could mean that the excluded surfaces were in some way significantly different from the rest. He set aside ruled surfaces because they obstructed his characterisation of the canonical series (the sets of points cut out on a generic curve of $|C|$ by an adjoint system) as did surfaces of geometric genus 0. Ruled surfaces had already been shown by Cayley in 1871 [7] to have different arithmetic and geometric genera, so they already seemed to belong to a particular family of surfaces.

Given a linear system $|C|$, Enriques defined $|A(C)|$, the *adjoint linear system* to $|C|$, as the curves which cut out a canonical group on any generic plane section of the surface and which have $(i - 1)$-fold points at any i-fold point of the linear system $|C|$. The central difficulty lay in determining the dimension of the adjoint system of a linear system of curves of genus π, which, Enriques said, was of the greatest importance because of the many applications there were for the idea of an adjoint system. He was able to determine this dimension for regular surfaces (it is $\pi + p - 1$) and, therefore, in this, his

Box 2

The Geometric and Algebraic Genera of an Algebraic Surface

When Clebsch first considered algebraic surfaces, he sought to generalise the genus of a curve in two ways: as a number defined in terms of projective features of the surface (degree and nature of singularities), and in terms of the number of linearly independent integrands of double integrals the surface admits. At this point Cayley intervened. The second number is certainly non-negative. But the first one can be calculated explicitly for surfaces that are not too complicated, as Cayley knew very well because he had pioneered the general study of scrolls or ruled surfaces (they have a family of lines parametrised by a curve). When Cayley calculated the first number for certain scrolls having not too many singularities, and also for developable surfaces (those locally isometric to a plane), he found that it was negative. In fact, it was

$$-\tfrac{1}{2}(n-1)(n-2)+\delta+\kappa,$$

where δ is the number of double lines and κ the number of cuspidal lines the surface contains. It is therefore, he pointed out, "the deficiency [or, genus] of plane sections taken negatively." It follows that the two numbers introduced by Clebsch are different. Cayley went on to conjecture that if the deficiency is negative, the surface is rational; this turned out to be insufficient.

In Noether's approach, the number $(n-1)(n-2)(n-3)/6$ of independent ratios that can exist between the coefficients of a surface of degree $n-4$, together with numbers obtained from the singularities of the surface, should determine the projective properties of a surface. The simplest, and generally unavoidable, singularities on a surface in P^3 are of three kinds: pinch points, double curves with distinct tangent planes where the surface self-intersects, and isolated triple points on the double curve. They were first studied by Salmon [27], who was interested in evaluating the degree of the dual of a surface and wished to estimate the effect of singularities; I do not know who dubbed them "normal" singularities, but the name stuck.

In a memoir of 1871, Noether proposed a formula for the genus of a surface of degree n [24]. It was a number, obtained by counting coefficients, for the dimension of the space of surfaces of order $n-4$ passing $(i-1)$ times through each i-fold curve of F and $(k-2)$ times through each k-fold point. This number p_a, which became the arithmetic genus, had been shown by Zeuthen to be a birational invariant, and so one which would survive attempts to resolve the singularities.

In his paper [25], Noether defined geometric genus, p_g, as the number of linearly independent adjoint surfaces of degree $n-4$ to a surface F of degree n. The genus of the intersection of the surface with one of its adjoints he denoted $p^{(1)}$.

first major paper, Enriques restricted his attention to regular surfaces.

Enriques stated a Riemann–Roch theorem for regular algebraic surfaces of genus $p_g > 0$ in the following form. Suppose that the generic curve of $|C|$ has genus π, that, in general, a set of r points on the surface specifies a unique curve of $|C|$, and that any two members of $|C|$ meet in s points. Suppose also that the curves of $|C|$ are not contained in the canonical system (Enriques also dealt with the case where they are). Then,

$$r = p_g + \omega - \pi + 1 + s,$$

which can be rewritten as

$$r + \pi - 1 - s = p_g + \omega,$$

where ω is the so-called superabundance. He deduced it by an ingenious geometric argument from the Riemann–Roch Theorem on the curve C.

The superabundance of a linear system $|C|$ is defined as the number $\omega \geq 0$ such that through a group of points common to two curves of $|C|$, there pass $2p_g + \omega$ linearly independent curves of the adjoint system—this is ω more than you might expect. It was to prove an elusive concept. Writing to Castelnuovo in July 1894, Enriques said, "The superabundance ω has another significance which gives a reason for its name. If $|C|$ is cut out by means of adjoint surfaces, and the virtual dimension ρ of $\{C\}$ is calculated according to the postulation formula of Noether, one has, by the residue theorem,

$$r - \rho = \omega - i."$$

(In the case I'm describing, $i = 0$. The entire, fascinating collection of letters from Enriques to Castelnuovo has been published in [1].)

Plurigenera

Castelnuovo and Enriques soon became dissatisfied with the arithmetic and geometric genera. Not only were there two, which differed when a surface was irregular, but they did not characterise surfaces. In particular, a regular surface whose genera vanished was not necessarily a rational surface. This was illustrated by a remarkable discovery of Enriques in 1894. He considered a tetrahedron in $\mathbf{CP}^3$ and found a sextic surface (one of degree 6) that had the edges of the tetrahedron as its double curves. Any adjoint surface must be of degree 4 less than the surface, and so be of degree 2 and therefore a quadric surface, and it must pass through the double curves of the surface. But, plainly, there is no quadric surface through the six edges of a tetrahedron. So the sextic surface has no adjoints. However, there is a surface of degree $2(n-4) = 2(6-4) = 4$, which passes twice through the edges of the tetrahedron, namely the surface composed of the four planes that form the faces of the tetrahedron. So the surface of degree 6 has no adjoint surface and its genus is 0, but it does have a biadjoint surface, and its bigenus (which is one more than the dimension of the space of such surfaces, and is denoted P_2) is 1, not 0.

This inspired Castelnuovo to an investigation, and he was able to show in 1896 [3] that a surface with arithmetic and geometric genera equal to 0 and bigenus $P_2 = 0$ is indeed a rational surface. This was the first birational characterisation of a surface. It also marks the moment when the so-called plurigenera enter the analysis. They are related to the dimensions of the pluricanonical systems $|K^i|$ by the formula $P_i = \dim|K^i| + 1$.

The plurigenera P_i were studied by Enriques and Castelnuovo in their 1901 paper [5]. They showed that considering $|K^i| = |iA(C) - iC|$ and applying the Riemann–Roch Theorem to find the dimension of the linear system $|iA(C')|$, they could be expected to grow quadratically with i, but under certain conditions, they might only grow linearly. Particular classes of surfaces did less well; they may be only 0 and 1 or even always 0. Trivially, if $P_i > 0$, then $P_{ki} > 0$ for all integers k.

Enriques's *Introduzione* [11] marks a considerable advance on the *Richerche* in its level of generality. Irregular surfaces could now be treated, because of recent discoveries by Castelnuovo in his 1896 paper [3], one that drew strong praise from Severi in 1958 [28], toward the end of his life. Enriques was also helped by progress made by French workers, notably Humbert, in the function-theoretic study of surfaces. To emphasise that he was working systematically with birational properties, Enriques spoke now not of surfaces but of doubly infinite algebraic entities (*ente algebrico doppiamente infinito*)—a phrase that he took from Segre's influential paper on algebraic curves of 1894. It means a birational equivalence class of surfaces. Any representative of such a class he called an image (*immagine*) of the class.

1901: Partial Classifications

The last major innovation of Castelnuovo and Enriques in the 1890s concerned the theory of adjunction, which, as we have seen, was the most important single technique available to them. It was published in Enriques's *Intorno* [12], which is a prelude to the more thorough-going joint paper, the *Questioni* [5]. In the *Intorno*, the new idea of the Jacobian of a linear system leads to a simple account of adjoints. The Jacobian of a linear system $|C|$ of dimension greater than 2 is obtained as follows. Take a 2-dimensional subsystem, $|L|$, of $|C|$; its double points are the Jacobian of $|L|$. Enriques established that all the Jacobians of the 2-dimensional subsystems in $|C|$ lie in the same linear system, the complete Jacobian linear system, $|J(C)|$, associated to a linear system $|C|$. They then established the theorem that

$$|J(C + K)| = |J(C) + K|.$$

The concept of an adjoint system was recaptured in this framework by establishing that the adjoint linear system to $|C|$, $|A(C)|$, is given by

$$|A(C)| = |J(C) - 2C|.$$

This also simplified the important concept of the pluricanonical systems.

When does adjunction stop? Castelnuovo and Enriques tackled this problem in their *Questioni* [5] by showing that if it stops for one immersion of the surface in some projective space, it always stops. Then, they showed that if all the plurigenera vanish, then successive adjunction stops. Finally, they showed that stopping implies that all the plurigenera vanish. This was much harder and required them to distinguish the cases $p^{(1)} \leq 1$ and $p^{(1)} > 1$. Their handle on the growth of the plurigenera was of course provided by the Riemann–Roch Theorem, in line with Castelnuovo's earlier results.

Their paper concluded with a classification of surfaces:

- Surfaces for which $p^{(1)} \geq 1$ and for which all the plurigenera vanish are either rational or elliptic ruled surfaces (and $p^{(1)} = 1$).
- Surfaces for which $p^{(1)} \geq 1$ and for which some plurigenera do not vanish. In this case, either $p^{(1)} > 1$, which implies that for i sufficiently large no $P_i = 0$, or $p^{(1)} = 1$, in which case nothing could presently be said.
- Surfaces for which $p^{(1)} < 1$: all these surfaces were birationally equivalent to ruled surfaces of genus $p > 1$, and $p^{(1)} = -8(p - 1)$.

In this paper, Castelnuovo and Enriques also found, by making essential use of the Riemann–Roch Theorem and another invariant (the Zeuthen–Segre invariant, which I shall not define), that successive adjunction stops if and only if the surface is ruled. They also proved that exceptional curves can be eliminated on all but ruled surfaces. This important result, which they had hoped for and believed in through the 1890s, greatly simplified the theory.

1905–1907: More Classifications

After 1900, Castelnuovo and Enriques shifted their interest toward the classification of algebraic surfaces. In 1905, Enriques was pleased to publish a paper classifying certain kinds of irregular surface, those whose geometric genus $p_g = 0$ [13]. He showed that if the arithmetic genus $p_a < -1$, then the surface is birationally equivalent to a ruled surface, and if $p_a = -1$, then the surface possesses an elliptic group of automorphisms and is a pencil of elliptic curves. The plurigenera told these classes apart. For a ruled surface, all the plurigenera vanish ($p_g = P_2 = P_3 = \cdots = 0$), but for an elliptic surface not equivalent to a ruled one, either $P_4 \geq 1$ or $P_6 \geq 1$. It follows that the ruled surfaces are those for which $p_g = 0 = P_4 = P_6$. In particular, this theorem gives necessary and sufficient conditions for a polynomial equation in three variables $f(x, y, z) = 0$, to be written as a polynomial equation in two variables, $\phi(X, Y) = 0$, because a ruled surface is birationally equivalent to a cylinder. As he pointed out, this result was not what Castelnuovo had expected.

A more detailed examination of the elliptic case followed (see below), and then Enriques concluded the paper with an examination of the ruled, nonrational surfaces, which could now be characterised as those for which $p_g = P_2 = 0$ and $P_{12} \neq 0$ (the last condition is a consequence of either $P_4 \neq 0$ or $P_6 \neq 0$). They turned out to be of four kinds, three of which were elliptic pencils of elliptic curves.

In [14], Enriques took up the theme of surfaces for which $P_2 = 1$. His surface F_6 of degree 6 had shown a decade earlier that the equations $p_a = 0 = p_g$ were not sufficient conditions for a surface to be rational. Since then, other examples of nonrational surfaces of genus 0 had been found, he said, but F_6 differed from them in being the only known regular surface of bigenus 1, and in being the only surface whose biadjoint surface cut it precisely in its double curve. This suggested what Enriques could now prove: a regular surface is birationally equivalent to F_6 if and only if $p_a = 0 = P_3$ and $P_2 = 1$. This characterised a class of surfaces nowadays called Enriques surfaces.

In proving this result, Enriques was led to two classes of surfaces with different plurigenera. The first corresponds to a surface with $p_1 = P_3 = P_5 = \cdots = 0$ and $P_2 = P_4 = P_6 = \cdots = 1$ (the Enriques surface) and the other to surfaces with some $P_i > 1$, and, in fact, $P_6 > 1$. The paper concluded with further properties of the Enriques surface and examples of surfaces of the other kind. In particular, Enriques showed that the surfaces birationally equivalent to F_6 have an infinite discontinuous group of birational automorphisms.

In a second paper [15], Enriques analysed surfaces for which $p^{(1)} = 1$. He found that there were three cases:

1. $p_a = 1 = p_g$, in which case $P_2 \geq 1$;
2. $p_a = 0$ and $p_g = 1$, in which case $P_2 > 1$;
3. $p_a = -1$ and $p_g = 1$, in which case $P_4 \geq 1$, and the surface will not have an effective canonical curve if and only if $P_4 = 1$ (in which case all the plurigenera = 1).

Castelnuovo and Enriques next wrote the first of several surveys of the field [6], the *Résultats Nouveaux*, that appears as an appendix in the second volume of Picard and Simart [26]. They recapitulated the approach of 1901 as far as the introduction of the pluricanonical linear systems, and devoted the second part of their *Résultats* to the classification of algebraic surfaces. They now said that answers expressed in terms of numerical birational invariants belonged to the quantitative theory, whereas answers involving the indefinite continuation or eventual termination of the sequence of successive adjoints belonged to what they called the qualitative theory. This interestingly recalls Poincaré's contemporary distinction between the quantitative and the qualitative in his own work on differential equations. In both settings, the idea is that the qualitative approach precedes and guides the quantitative one. They then surveyed the older results in the field, which gave criteria for a surface to be rational, before listing their own results, divided into qualitative and quantitative. These included the following:

1. A surface is rational if and only if $p_a = P_2 = 0$ (which implies that $p_g = 0$).
2. Among irregular surfaces with $p_g = 0$ and $p_a < 0$, those with $p_a < -1$ are ruled, and the rulings are curves of genus $-p_a$, whereas those for which $p_a = -1$ form various sorts of elliptic surface.
3. A surface is ruled if and only if $P_4 = P_6 = 0$.

Joint work with Severi led Enriques in 1908 to establish more results about surfaces with $p_a = 0 = p_g$ and $P_2 = 1$. He showed that they are all double covers of the projective plane branched over a particular kind of curve of degree 8.

Various Kinds of "Elliptic" Surface

Two other approaches to surfaces were yielding results at the same time. A class of surfaces were found by means of automorphic functions in two variables; they were much studied by Picard and Humbert, and came to be called hyperelliptic surfaces. The parametrisation will be r to 1, where r is the rank of the surface; Kummer's quartic surface is of rank 2. Another class of surfaces was found by putting together pairs of curves. The Cartesian product of a curve X and a curve Y is a surface $X \times Y$. If X is the projective line, the surface is a ruled surface. If it is an elliptic curve, the surface is elliptic; if Y is also elliptic, the surface is sometimes called bielliptic. Because

Box 3

The Castelnuovo and Enriques Classification of Algebraic Surfaces

$P_{12} = 0$, ruled surfaces ($p^{(1)} \leq 0$)

- $p_a = 0$, rational surfaces ($p_a = P_2 = 0$),
- $p_a = -1$, elliptic ruled surfaces ($p_a = -1, P_4 = P_6 = 0$),
- $p_a < -1$, ruled surfaces of genus $-p_a > 1$.

$P_{12} = 1$ ($p^{(1)} = 1$), curve systems of genus π and grade $n = 2\pi - 2$

$p_g = 1$

- $p_a = 1$, an infinite set of families parametrised by the integers ($p_a = P_2 = 1$),
- $p_a = -1$, hyperelliptic surfaces of rank 1 and with divisor $\delta = 1, 2, \ldots$;

$p_g = 0$

- $p_a = 0$, surfaces of the sixth order which pass twice through the edges of a tetrahedron ($p_a = P_3 = 0, P_2 = 1$),
- $p_a = -1$, irregular hyperelliptic surfaces of rank > 1 ($P_2, P_3, P_4, P_6 = 0, 1$);

$P_{12} > 1$

$p^{(1)} = 1$

- $p_a = -1$, elliptic surfaces with a pencil of genus p_g of elliptic curves, parametrised by the integers
- $p_a \geq 0$, surfaces with a pencil of genus $p_g - p_a$ of elliptic curves, parametrised by pairs of integers

$p^{(1)} > 1$

Surfaces with an invariant image, canonical or pluricanonical surfaces; a finite number of families for each value of $p^{(1)}$.

elliptic curves have large automorphism groups, surfaces constructed in this way tend to have symmetries, quotients of them can be expected to be nonsingular surfaces. These surfaces should be expected to have a special place in any classification, akin to the elliptic curves in the classification of algebraic curves, and so they do. Indeed, surfaces other than these turned out to be either much simpler (ruled or even rational) or else much more complicated and inscrutable (the surfaces "of general type"). The hyperelliptic surfaces were the subject of long and detailed studies by Enriques and Severi, for which they were awarded the French Prix Bordin in 1907, and their younger Italian rivals Bagnera and de Franchis, who were awarded the Prix Bordin in 1909.

1914: The Full Castelnuovo and Enriques Classification

In 1914, Castelnuovo and Enriques published two articles in the *Encyklo-pädie der Mathematischen Wissen-schaften.* The second of these [17] culminated in a classification of algebraic surfaces (see Box 3). It is interesting to compare it with a modern version, taken from Barth, *et al.* [0], p. 188 (see Box 4).

Kodaira studied compact, complex 2-manifolds and found some that cannot be embedded in any projective space, and so escaped the Enriques–Castelnuovo classification. The integers which parametrise the families are also much better understood, but the qualitative behaviour of the plurigenera survives as the framework of the Kodaira classification.

The Enriques–Castelnuovo classification and many of the results obtained on the way represent a remarkable achievement. Over the years, Segre himself had attempted to tackle the singularities directly. This proved to be too hard, although a resolution theorem was obtained by the German mathematician H.W.E. Jung in 1908 [21]. Segre's former student Castelnuovo and his colleague Enriques got further, and the focus of algebraic geometry in Italy shifted to them.

Box 4

The Enriques–Kodaira Classification of Algebraic Surfaces

All $P_i = 0$

minimal rational surfaces:	$b_1 = 0, c_1^2 = 8, 9, c_2 = 4, 3,$
minimal surfaces[1] of class VII;	$b_1 = 1, c_1^2 \geq 0, c_2 \geq 0,$
ruled surfaces of genus $g \geq 1$;	$b_1 = 2g, c_1^2 = 8(1-g), c_2 = 4(1-g).$

All $P_i = 0, 1$ (the first nonzero plurigenus is indicated)

Enriques surfaces:	$b_1 = 0, c_1^2 = 0, c_2 = 12, P_2 = 1;$
hyperelliptic surfaces:	$b_1 = 2, c_1^2 = 0, c_2 = 0,$ at least one of $P_2, P_3, P_4, P_6 = 1;$
Kodaira surfaces[1]	
primary:	$b_1 = 3, c_1^2 = 0, c_2 = 0, P_1 = 1$
secondary:	$b_1 = 3, c_1^2 = 0, c_2 = 0,$ at least one of $P_2, P_3, P_4, P_6 = 1;$
K3-surfaces[2]:	$b_1 = 0, c_1^2 = 0, c_2 = 24, P_1 = 1,$
tori:	$b_1 = 4, c_1^2 = 0, c_2 = 0, P_1 = 1.$

The P_i grow linearly

minimal properly elliptic surfaces[2]: $c_1^2 = 0, c_2 \geq 0,$

The P_i grow quadratically

minimal surfaces of general type $b_1 \equiv 0(2), c_1^2 > 0, c_2 > 0.$

[1]None of these surfaces is projective.
[2]Some in this family are projective but others are not.
Note: The term "minimal" is used in the sense of minimal model, i.e., a birationally equivalent surface having no exceptional curves. The integers b_1, c_1, and c_2 are the first Betti number and the first and second Chern classes, respectively.

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