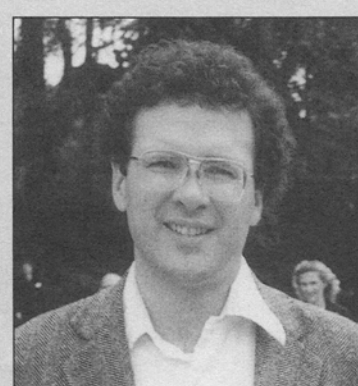
Gauss, Dirichlet, and the Law of Biquadratic Reciprocity

David E. Rowe

Gauss and Dirichlet are two of the most influential figures in the history of number theory. Gauss's monumental *Disquisitiones Arithmeticae*, first published in 1801, synthesized many earlier results and served as a point of departure for the modern approach to the subject [1]. The three principal sections of the book were devoted to the theory of congruences (where Gauss introduced the still standard notation $a \equiv b(\mathrm{mod}\ m)$), the classical subject of quadratic forms that had been studied by Fermat, Euler, Lagrange, and even Diophantos, and the theory of cyclotomic equations. Although the appearance of this masterpiece did much to establish Gauss's early mathematical fame, the sheer novelty of the work together with its rigidly formal exposition made it difficult for all but the greatest of his contemporaries to appreciate fully. One of those who did was Lagrange, the aged giant who, along with Euler, stood at the pinnacle of eighteenth-century mathematics. In 1804 he wrote the young protege of the Duke of Brunswick: "With your *Disquisitiones* you have at once arrayed yourself among the mathematicians of the first rank, and I see that your last section on cyclotomic equations contains the most beautiful analytic discoveries that have been made for a long time" [2].

Another mathematician who was deeply influenced by Gauss's *Disquisitiones* was Johann Peter Gustav Lejeune Dirichlet, who was born in 1805 in the town of Düren near Aachen in the Rhineland. Although he spent his formative years in Paris studying mathematics with some of the leading French mathematicians of the day, nothing attracted Dirichlet's interest so much as did Gauss's *magnum opus*. In the words of Ernst Eduard Kummer:

> This [work] exercised a much more significant influence on his whole mathematical education and development than his other Paris studies. Rather than having merely read through it once or even several times, his whole life long, over and again, he never stopped repeatedly studying the wealth of deep mathematical thoughts it



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P. G. Lejeune Dirichlet as a young man. (From the Porträtsammlung der Niedersächsischen Staats- und Universitätsbibliothek Göttingen).

contained. That is why it was never placed on his bookshelf, but rather always lay on the table at which he was working. One can well imagine the exertion it must have cost him in analyzing this extraordinary work, considering that more than twenty years after it had appeared there was still no one alive at that time who had studied all of it and understood it completely. Even Legendre, who dedicated a large part of his life to higher arithmetic, had to confess in the second edition of his *Théorie des nombres* that he would have liked to have enriched it with Gauss's results, but the methods of this author, being so peculiar, made it impossible to do so without the greatest digressions or without merely assuming the role of a translator. Dirichlet was the first not only to understand this work completely, but also to have made it accessible to others, in that he made its rigid methods, behind which the deepest thoughts lay hidden, fluid and transparent, and replaced many of the main points by simpler more genetic ones, without compromising the complete rigor of the proofs in the slightest. He was also the first to go beyond it and reveal the rich treasures and still deeper secrets of number theory [3].

One of the founders of analytic number theory, Dirichlet presented his first paper in this field in 1837. This contained the famous proof that there are infinitely many primes in every arithmetic progression $an + b$ where a and b are relatively prime. The following year he published the first portion of a two-part article entitled "Recherches sur diverses applications de l'analyse infinitésimale à la théorie des nombres," which introduced what have since become known as Dirichlet series. Although he published relatively little in his lifetime, Dirichlet, unlike Gauss, placed a high value not only on the discovery of new ideas but on their dissemination as well. In the course of his career at Berlin and Göttingen his students included some of the outstanding figures of the next generation—Eisenstein, Kronecker, Dedekind, and Riemann—and his *Vorlesungen über Zahlentheorie*, which only became accessible to the mathematical public in 1863 through the efforts of Richard Dedekind, also exerted a profound influence on number theory and algebra in the late nineteenth century.

In view of these circumstances, the intellectual and personal relationship of Gauss and Dirichlet would seem to possess an importance all its own for the history of mathematics. Nevertheless, this fascinating and significant chapter in nineteenth-century mathematics has, to the best of my knowledge, never been adequately dealt with by those who have written about either man. In part this is due to the fact that, while there is a plethora of literature on Gauss, no one has as yet produced even a small biography of Dirichlet [4]. In the English language practically nothing has been written about his life, work, and extraordinary influence [5]. The following contribution to this subject concentrates on the years 1825–1831, a formative period in Dirichlet's life when he and Gauss were working on closely related problems in the theory of biquadratic residues. Before turning to this story directly, however, a few introductory remarks regarding Gauss's prior arithmetical researches are called for here.

The second entry in Gauss's *Tagebuch*, dated April 8, 1796, indicates that he had already begun work on quadratic reciprocity during his first year as a student at Göttingen [6]. By June of that year he had obtained two entirely independent proofs of what he called the *Theorema Fundamentale*. Euler had already known this "fundamental theorem" which was later rediscovered by Legendre, but neither of them had given a complete demonstration of its validity [7]. For any prime p, an integer a is said to be a quadratic residue mod p if the congruence $x^2 \equiv a \pmod p$ has a solution. Gauss's "fundamental theorem," better known as the law of quadratic reciprocity, is then stated most succinctly by means of the Legendre symbol

$$\left(\frac{p}{q}\right) = \begin{cases} 1 \text{ if } p \text{ is a quadratic residue mod } q \\ -1 \text{ if not.} \end{cases}$$

If p and q are distinct odd primes, then the following reciprocal relationship holds between them:

$$\text{If } p, q \text{ are not both} \equiv 3 \pmod 4, \text{ then } \left(\frac{p}{q}\right) = \left(\frac{q}{p}\right).$$

$$\text{If } p \equiv q \equiv 3 (\text{mod } 4), \text{ then } \left(\frac{p}{q}\right) = -\left(\frac{q}{p}\right).$$

This relationship may be reformulated as follows:

$$\left(\frac{p}{q}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}\left(\frac{q}{p}\right).$$

The prime 2 can also be dealt with as a special case in this theory, since:

$$\left(\frac{2}{p}\right) = (-1)^{\frac{p^2-1}{8}}.$$

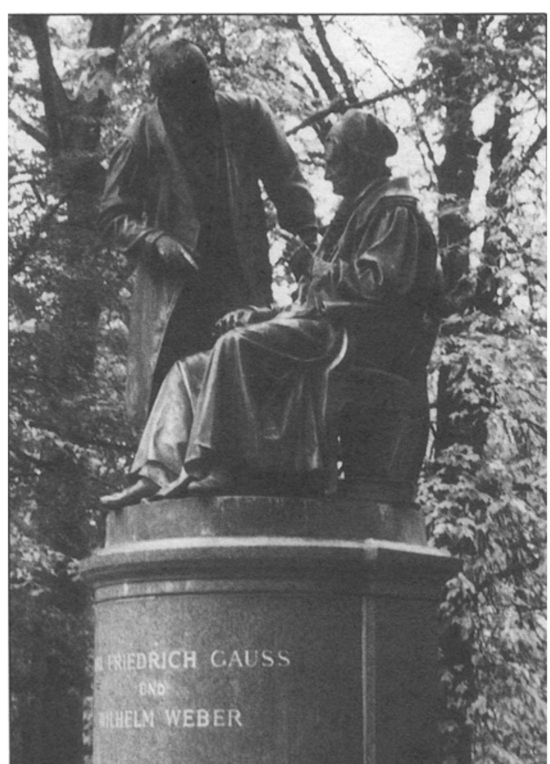
The two proofs Gauss discovered in 1796 were first presented to the world in the *Disquisitiones Arithmeticae*. The first of these (see § 133) is generally considered unreadable; the second, however, is clearer (see § 262) and makes use of the theory of quadratic forms developed earlier in the book. In 1808 Gauss published two new proofs, and nine years later two more —clearly this was one of his favorite theorems, and he returned to it often for inspiration. In fact, this result prompted him to search for an analogue in the theory of cubic and biquadratic residues. During February of 1807 he made the following entries in his *Tagebuch:* "Began the theory of cubic and biquadratic residues. . . . Further worked out and completed. Proofs thereto are still wanted. . . ." Out of this work arose a new proof (Gauss's sixth) of the "fundamental theorem," as he recorded on May 6: "We have discovered a totally new proof of the fundamental theorem based on totally elementary principles."

Following the death of his first wife, whom he loved dearly, on October 11, 1809, Gauss recorded nothing in his diary for more than two years. The next entry, dated February 29, 1812, began: "The preceding catalogue interrupted by misfortunate times resumed a second time. . . ." During the six-year period since recording his initial work on higher degree residues, Gauss only once made mention of further progress on this subject. In 1809 he noted: "the theorem for the cubic residue 3 proved with elegant special methods. . . ." From this one may surmise that he was still groping with special cases at this time, and had not yet proved a general relation for either cubic or biquadratic residues. Of course by now the number of entries in Gauss's diary had become sparse, so it is dangerous to make conjectures on this basis alone. Indeed, there is nothing in the diary that would prepare us for the following dramatic entry of October 23, 1813:

> The foundation of the general theory of biquadratic residues which we have sought for with utmost effort for almost seven years but always unsuccessfully at last happily discovered the same day on which our son is born. . . . This is the most subtle of all that we have ever accomplished at any time. It is scarcely worthwhile to intermingle it with mention of certain simplifications pertaining to the calculation of parabolic orbits.

These remarks were made just before Gauss's final diary entry, recorded July 9, 1814, when he wrote: "I have made by induction the most important observation that connects the theory of biquadratic residues most elegantly with the lemniscatic functions." Yet, as with so many other pioneering researches, Gauss published nothing whatsoever on biquadratic residues for many years to come. One may speculate on a number of plausible reasons for this; the demands of his astronomical and geodetic studies may have played a role here, for example, or his general reluctance to publish anything that was not in polished form. But perhaps an even more telling reason for Gauss's failure to present this work to the mathematical public was his instinctive conservatism when it came to matters of potential controversy. In this particular case, Gauss's work on biquadratic residues required a bold new approach to number theory in which the so-called Gaussian integers were introduced for the first time. As we shall have occasion to see later, Gauss was well aware that this represented a radical break with the number theory of the past, and he took pains to argue that this daring step was both natural and necessary. His defensive posture in this

The Gauss-Weber Denkmal in Göttingen.



matter brings to mind how he shunned all controversy regarding the status of Euclidean geometry. He once wrote his friend, the renowned astronomer Bessel, that he was reluctant to publish in this field because his views were bound to evoke the outcries of certain "Boeotians" [8]. Gauss kept his word, and even after the younger Bolyai presented arguments for the existence of non-Euclidean geometry in 1831, he maintained his stony silence on this subject.

Gauss had been aware of Dirichlet's work in number theory since 1826, when the young man sent him a copy of an earlier paper asking him if he would read it and then write a letter expressing his judgment of its contents to someone in Berlin. Over the course of the preceding four years Dirichlet had been studying under Fourier and Poisson at the Collège de France and the Faculté des Sciences. By the summer of 1825 his talent was already known to Alexander von Humboldt, who was preparing to use his influence to create a position for him within the Prussian university system. Like Dirichlet, Humboldt was about to return to Prussia after a prolonged stay in Paris, and he was well informed as to the high esteem in which the young Rhinelander was held by such leading French mathematicians as Fourier, Lacroix, and Poisson. Still, nothing would carry as much weight in the Berlin *Kulturministerium* as a letter of recommendation from Gauss, to whom Humboldt wrote the following lines on May 21, 1826:

> As you know, I cannot pretend to have a serious opinion when it comes to the higher regions of mathematics, but I do know through the great mathematicians that Paris possesses, and especially through my oldest friends Fourier and Poisson, that Herr Dirichlet has by nature the most brilliant talent, that he is progressing along the best Eulerian paths, and that one day Prussia will have in him (he is barely 21 years old!) an outstanding professor and academician. Grant my young friend, whose fortune interests me dearly, the protection of your great name [9].

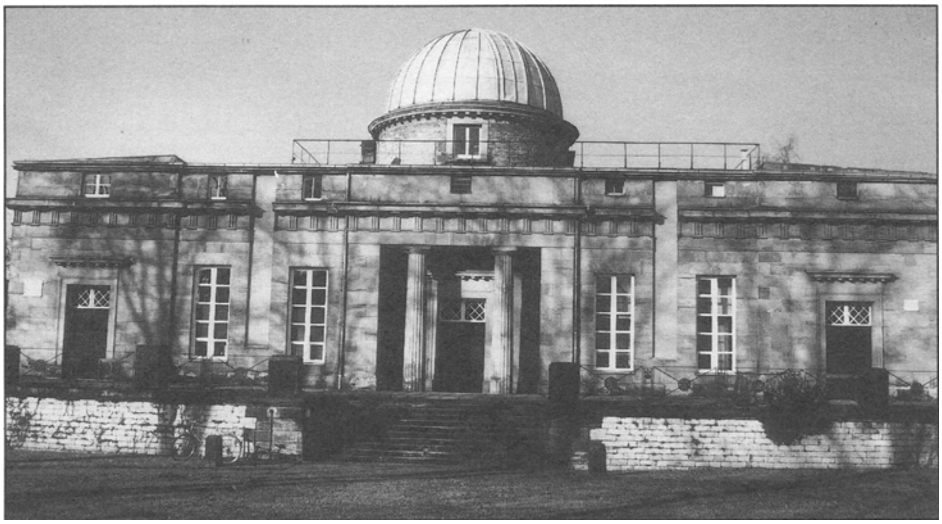
One week later Dirichlet sent Gauss the above-mentioned letter along with an offprint of his "*Mémoire sur l'impossibilité de quelques équations indeterminées du cinquième degré.*" This article had been accepted for publication by the French Academy in July 1825 after receiving a favorable referee's report from Legendre and Lacroix. It dealt with Diophantine equations of the form $x^5 + y^5 = Az^5$. Shortly after it appeared Legendre applied its techniques to prove that $x^5 + y^5 \neq z^5$, which was one of the first breakthroughs on Fermat's last theorem since Euler resolved the case $n = 3$ (Fermat himself proved the case $n = 4$). Even by the conventions of the day, Dirichlet's letter was written in a tone of exaggerated humility and almost embarrassing self-effacement. Of course his youth must be taken into account here, as well as his awareness of Gauss's reputation for being aloof and difficult [10]. Nevertheless, this letter gives a distinct impression of how truly daunted he was by Gauss's stature as a mathematician:

> At the kind recommendation of Baron von Humboldt I take the liberty of flattering myself with the hope that you will read and judge this first work of a young German with kind indulgence. If Your Honor should find it not altogether unworthy of your attention, I would dare to humbly beseech you with the request for permission to write you on occasion and to ask you for some indications that would guide my further scientific efforts. I would regard this permission as the greatest fortune, as the love with which I pursue indeterminate analysis makes me wish for nothing with more longing than that the author of the immortal *Disquisitiones Arithmeticae* might take some part in my efforts. Since I have busied myself primarily with higher arithmetic, I have completely followed my inclination without duly considering how little I may dare to hope of accomplishing anything substantial in this difficult area of mathematics given my restricted aptitude [11].

Six weeks passed before Gauss got around to writing his former student Johann Friedrich Encke, an astronomer in Berlin who was well connected with the Prussian Ministry of Culture. The "Prince of Mathematicians" was a reclusive sort of figure, and it was not often in the course of his career that he went out of his way to lend a helping hand to younger talents. In this case, however, when Dirichlet's future as a mathematician largely lay in his hands, he was unusually lavish in his praise of him. A few days after receiving Gauss's letter, Encke wrote to the Ministry urging that Prussia create a position for Dirichlet before the French won him for themselves. Speaking of Gauss's high regard for Dirichlet's work, Encke added: "In my eyes what gives this opinion of Gauss such high value is that this unique man has always sharply distinguished between works that are worthy through their diligence and those of true genius, and that so long as I have had the fortune to know him, he has never spoken of anyone with such warmth, however respectfully he may have spoken of the accomplishments of others" [12].

Two months later Dirichlet received a letter from Gauss, who excused himself for not having written earlier by saying that he had wanted to await further news from Berlin. Gauss wrote with uncharacteristic charm, the almost fatherly tone of his letter suggesting that Dirichlet's young talent may well have brought back memories of his own youth. He assured him that the prospects for an appointment did indeed look promising, and then went on to express his pleasure that Dirichlet had taken an interest in number theory while recounting his own love for the subject:

> It is all the more pleasing to me that you have a great attachment to that part of mathematics that has always been my favorite field of study, however seldom I have



The Sternwart in Göttingen where Gauss lived and worked from 1816 to 1855.

> pursued it. I dearly wish you a situation in which you will have as much control over your time and the choice of your work as possible. Immediately after the appearance of my *Disquisitiones*, I myself was very much hindered by other business, and later by my external circumstances, from following my inclinations to the degree that I would have wished [13].

He went on to say that he had given up his original plan of publishing a sequel to the *Disquisitiones* and instead would content himself with the publication of an occasional memoir on number theory. His plans called for a three-part study on biquadratic residues, the first of which would appear shortly. "The main materials for the rest," he added, "as well as for the related theory of cubic residues are essentially finished, although little of it has yet been written up adequately" [14].

By early November *Kultusminister* Altenstein informed Alexander von Humboldt that the state would agree to provide Dirichlet with a minimal salary of 400 *Taler* while he habilitated at Breslau. Humboldt pressed him for 600–700 *Taler* and also to have Dirichlet appointed as *außerordentlicher Professor*. This plea, however, was spurned by the Ministry. All the same, Dirichlet was delighted with this opportunity to commence his career as a mathematician, and, having received assurances of continued support from Humboldt, he had every reason to be optimistic about his future.

By this time the winter semester was already a month old, so Dirichlet requested permission to stay at home for the duration, proposing that he take up his duties at Breslau in the spring. This was granted, but shortly thereafter Altenstein learned that during his stay in Paris young Peter Gustav had been a private tutor to the children of General Maximilien Foy, the leader of the liberal opposition party in the Chamber of Deputies. By the standards of the day Altenstein was a reasonably liberal politician, but as K.-R. Biermann points out this was the era when the Carlsbad decrees were being carried out, a period when the "Metternichian reaction made it much too dangerous for Altenstein to place an unknown young man at a Prussian university simply on Humboldt's recommendation" [15]. He therefore had the Minister of the Interior investigate Dirichlet's character. Luckily his agents in Paris turned up nothing damaging, and subsequently he was allowed to teach at Breslau after all. In the meantime Dirichlet had been awarded a doctorate *honoris causa* by the Bonn faculty, and sometime during the middle of March he began the long journey from his home in the Rhineland to Breslau.

Along the way Dirichlet made stops in Berlin and in Göttingen, where he met Gauss for the first time. In all likelihood it was during the course of this visit that he first became aware of an announcement Gauss had published about two years earlier in the *Göttingische Gelehrte Anzeigen*. This notice conveyed the contents of *Theoria Residuorum Biquadraticorum, Commentatio prima*, the first article alluded to be Gauss in the letter cited above. Its main results were concerned with the biquadratic character of the number 2, which, as remarked earlier, forms a special case in the theory of quadratic residues. Now, in general, for a to be a biquadratic residue mod p (i.e., for $x^4 \equiv a(\mathrm{mod}\ p)$ to be soluble) it is clearly necessary that a also be a quadratic residue mod p. In the *Commentatio prima* Gauss offered criteria for determining whether or not $a = -1$ or $a = \pm 2$ will be a biquadratic residue mod p for some given prime p. These, he remarks, are special cases "that allow of being worked out without too much machinery and can serve as preparation for the general theory to be given in the future" [16]. The case $a = -1$ was already dealt with in the *Disquisitiones Arithme-*

ticae, where Gauss showed that -1 is a biquadratic residue mod p if and only if p is of the form $8n + 1$ (thus when $p = 8n + 5$ it is a quadratic but not a biquadratic residue).

For the case $a = \pm 2$, however, Gauss gave two distinct criteria for deciding this question. Since in the special case $a = 2$, p an odd prime, 2 is a quadratic residue mod p if and only if p is of the form $8k + 1$ or $8k + 7$, it follows that for p of the form $8k + 3$ or $8k + 5$, 2 is a biquadratic *nonresidue* mod p. Gauss, moreover, points out that it suffices to consider primes p of the form $4n + 1$, which means that there is no loss of generality in assuming p to be of the form $8k + 1$. Gauss's first criterion is obtained by writing $p = g^2 + 2h^2$, a representation that is possible and unique. Then ± 2 is a biquadratic residue mod p if and only if g is of the form $8n + 1$ or $8n + 7$. The second criterion uniquely represents p as $e^2 + f^2$, where e is odd and f even. Since $p = 8k + 1$, $f/2$ must be even. Then ± 2 is a biquadratic residue mod p if $f = 8m$ and a nonresidue if $f = 8m + 4$.

Whether directly from its author or through some other possible source, it was around this time that Dirichlet became aware of Gauss's announcement, and from this point forth he could not take his mind off it. How he struggled with these ideas and what resulted therefrom are best conveyed by the following excerpts from a letter (containing parts of another letter) Dirichlet wrote to his mother about seven months after he met Gauss in Göttingen. To the best of my knowledge, this is the first time this letter has been referred to in print [17]. The setting is as follows: his first semester as a *Dozent* having come to an end, Dirichlet had just returned to Breslau after spending a brief vacation in Dresden.

Breslau, October 29, 1827

Dearest mother!

Although I have already been back from Dresden 14 days, it was impossible until now for me to answer your two letters of which the first, dated September 11, only arrived the day before my return. You will find the reason in the following passage from a letter to Encke, which Professor Schoeller took with him to Berlin a few days ago. I quote this passage for you not so much to excuse myself as to enable you to judge for yourself whether the oft-mentioned love is serious or merely a jest, whereby it will now be altogether clear to you that (as you yourself said) one who is in love is incapable of proper work: [18]

"The signs of good-will with which you have favored me give me the courage to trouble you with a new request. I have been busy for some time with arithmetical investigations that were prompted by an article in the *Göttingische Gelehrte Anzeigen*. In this article (*Gelehrt. Anz.*, April 11, 1825) a till now unknown work by our Gauss is announced that was submitted to the Society there [the Göttinger Gesellschaft der Wissenschaften]. It concerns biquadratic residues and its two main results [the criteria mentioned above] are communicated *without proofs*. The elegance of these theorems set forth in me the urge to prove them, and after a number of vain efforts I succeeded in doing so a short time ago. My proofs appear to be completely different from those of the famous author of the *Disquisitiones arithmeticae*, for his, as they are described in the article, require a series of subtle preliminary investigations, whereas mine consist of a simple application and combination of results that have long been known. The method I have used, moreover, has led me to the discovery and proof of a *large* number of new theorems, some in the theory of *biquadratic* residues, some in related branches of higher arithmetic. I am now eagerly pursuing these things further and hope through my efforts to bring out a worthy piece of work that will attract the attention of mathematicians [19]. As, however, some time must pass before this work is completed, I wish to show that I am already in possession of a suitable method for the treatment of these matters. I have, therefore, written up the main features of this method and take the liberty of sending you this sketch by separate post in a sealed packet with the request that you kindly deposit it with your Academy, etc."

Regarding the above mentioned investigations I experienced a most peculiar fortune. Already in the course of the summer I had made a number of steps that brought me nearer to the goal I sought. Still, there always loomed one difficulty that needed to be overcome before I had the proof of Gauss's theorems. I concentrated on this matter incessantly, not only during my trip to the mountains but also in Dresden, and yet without gaining any real insight. One evening, as I wandered alone on the Elbe bridge (which, by the way, occurred only seldom, as I enjoyed too much being in the company of such kindly people as the Remer family) I had a few ideas that appeared to put me within grasp of the so long and zealously searched-for results. On the gorgeous Brühl terrace I let my thoughts go for several hours (till around ten o'clock), but still I could not see my way to the end of the matter (probably because nonmathematical ideas kept mixing together with the mathematical ones). With very weak hopes I went to bed and was extremely restless until around one o'clock when I finally fell asleep. But then I woke up again around four o'clock, and even awoke the health official, who slept in the same room, by hollering: "I've found it" [*"Ich habe es gefunden"*]. It took but a moment for me to get up, turn on the light, and, pen in hand, convince myself of its correctness. After this my investigations expanded every day, and fourteen days later I was in a position to send Herr Encke my six-page sketch. I have every reason to expect that this work will accomplish a good deal for my promotion, since Gauss announced his results, which indeed do not contain nearly so much as mine, with a certain pomp. I will certainly not neglect to mention in a tactful manner what he said about the difficulties that must be overcome in carrying out the proofs . . . [20].

The history of mathematics is filled with tales of remarkable discoveries: Poincaré saw the connection between non-Euclidean geometry and the Fuchsian functions in a moment's flash, Klein his *"Grenzkreistheorem"* in much the same way, and after weeks of intense deliberation Lie discovered the key property of

Dirichlet's Letter to His Mother

Breslau d. 29. Oct. 1827.

Innig geliebte Mutter!

Obgleich ich schon seit 14 Tagen von Dresden wieder zurück bin, so ist es mir bisher unmöglich gewesen, Deine beiden Briefe, von denen der erstere vom 11. Sept. datirt erst am Tage vor meiner Rückkehr hier angekommen ist, zu beantworten. Den Grund findest Du in folgender Stelle eines an Encke gerichteten Briefes, welchen Herr Prof. Schoeller vor einigen Tagen mit nach Berlin genommen hat. Ich theile Dir übrigens diese Stelle weniger zu meiner Entschuldigung mit, als um Dich in den Stand zu setzen, selbst zu beurtheilen, ob es mit der so häufig erwähnten Liebe Scherz oder Ernst ist, worüber Du jetzt völlig im klaren seyn wirst, da (es ist dies Deine eigene Ausserung) ein Verliebter zu keiner ordentlichen Arbeit fähig ist.

"Die Beweise von Wohlwollen, womit Sie mich beehrt haben, geben mir den Muth, Sie von Neuem mit einer Bitte zu belästigen. Ich bin seit einiger Zeit mit arithmetischen Untersuchungen beschäftigt, wozu mich ein in dem Göttingischen Gelehrten Anzeigen befindlicher Artikel veranlasst hat. In diesem Aufsatze (Gelehrt. Anzn. 11. April 1825) wird eine von unserm Gauss der dortigen Societät überreichte, bisher unbekannte Abhandlung über die biquadratischen Reste angekündigt und es werden zugleich die 2 Hauptsätze der Abhandlung *ohne Beweis* mitgeteilt. Die Eleganz dieser Sätze erregte bei mir den Wunsch, meinerseits Beweise für dieselben zu finden, was mir auch nach manchen vergeblichen Bemühungen vor einiger Zeit gelungen ist. Meine Beweise scheinen von denen des berühmten Verfassers der disquis. arith. ganz verschieden zu seyn, da die seinigen, wie sie in jenem Aufsatze gesagt wird, eine Reihe subtiler Hilfsuntersuchungen erfordert, die meinigen dagegen in einer einfachen Anwendung und Verbindung längst bekannter Sätze bestehen. Die von mir gebrauchte Methode hat mich ausserdem auf eine *grosse* Anzahl neuer theils in die Theorie der *biquad.* Reste, theils in verwandte Zweige der höheren Arithmetik gehörigen Sätze geführt und mir zugleich die Beweise derselben gegeben. Ich bin jetzt eifrig damit beschäftigt, diesen Gegenstand weiter zu verfolgen und hoffe durch meine Bemühungen eine der Aufmerksamkeit der Mathematiker würdige Abhandlung zu Stande zu bringen. Da aber bis zu Beendigung dieser Arbeit noch einige Zeit vergehen dürfte, so muss ich wünschen darzuthun, dass ich schon jetzt im Besitze einer zur Behandlung dieser Gegenstände geeigneten Methode bin. Ich habe daher die Hauptmomente dieser Methode zu Papier gebracht und bin so frei, Ihnen diesen Entwurf in beikommendem versiegelten Packet mit der gehorsamen Bitte zu übersenden, dasselbe gefälligst bei Ihrer Akademie deponieren zu wollen, etc."

Bei den ebenerwähnten Untersuchungen habe ich ein ganz eigenes Schicksal gehabt. Schon im Laufe dieses Sommers hatte ich einige Schritte getan, welche mich dem mir vorgesteckten Ziele näher brachten, allein es blieb noch immer eine Schwierigkeit zu überwinden, um die Beweise für die Gauss'schen Sätze ganz vollständig zu erhalten. Dieser Gegenstand beschäftigte mich unaufhörlich sowohl auf meiner Gebirgsreise als auf der nach Dresden, ohne dass ich mit der Sache ins Reine kam. Eines Abends, wo ich einsam auf der Elbbrücke wanderte (was übrigens sehr selten geschah, da ich gar zu gern in der Gesellschaft so liebenswürdiger Menschen war, wie die Familie Remer es ist) hatte ich einige Ideen, welche mich in den Besitz des so lange und so eifrig gesuchten [Satzes?] setzen zu müssen schienen. Auf der herrlichen Brühlschen Terrasse überliess ich mich mehrere Stunden lang (bis gegen 10 Uhr) meinen Gedanken, konnte aber dennoch mit der Sache nicht fertig werden (wahrscheinlich weil unmathematische Ideen sich mit den mathematischen mischten). Mit sehr geschwächter Hoffnung legte ich mich zu Bette und brachte die Nacht sehr unruhig zu, bis ich endlich gegen 1 Uhr in einen ordentlichen Schlaf fiel, aus dem ich aber um 4 Uhr schon wieder erwachte, indem ich den Medicinalrath, der in derselben Stube schlaf, mit dem Ausruf aufweckte: "Ich habe es gefunden." Aufstehen, Licht anzünden und mich mit der Feder in der Hand von der Richtigkeit der Sache überzeugen, war die Sache eines Augenblicks. Die Untersuchungen erweiterten sich nun jeden Tag und in 14 Tagen war ich im Stande, Herr Encke meinen 6 Bogen starken Entwurf zu übersenden. Ich habe allen Grund von dieser Abhandlung viel für meine Beförderung zu erwarten, da Gauss die seinige, welche doch bei weitem nicht soviel enthält, mit einem gewissen Pomp angekündigt hat. Ich werde gewiss nicht ermangeln, das, was er [gesagt hat,] über die Schwierigkeiten womit die Beweise geführt werden müssen, in meiner Abhandlung auf eine geschickte Weise anzuführen. . . .

Dresden in the mid-nineteenth century with a view of the Elbe Bridge and Brühl Terrace, where Dirichlet contemplated his proof of Gauss's results on biquadratic residues. (Furnished through the kind cooperation of Dr. Walter Purkert, Karl-Sudhoff-Institut, Leipzig.)

his line-to-sphere transformation while lying awake in bed one morning [21]. On July 10, 1796, Gauss had reason to remember the legendary reaction of Archimedes after discovering his hydrostatic law during a visit to the public baths [22]. It was on that day that Gauss wrote in his diary: "EUREKA. num $= \triangle + \triangle + \triangle$." Evidently he had found a proof that every number can be written as the sum of three triangular numbers, a conjecture first made by Fermat. Still for sheer drama, this "Eureka!" story, culminating with Dirichlet's impetuous cry *"Ich habe es gefunden,"* is practically in a class by itself.

It may at first seem hard to believe that an authentic episode like this, involving a mathematician of the stature of Dirichlet, could have become totally forgotten over the years. Yet I doubt that this phenomenon will surprise anyone who has spent a bit of time sifting through the letters and unpublished papers of some great mathematicians of the nineteenth century. Often such papers are untapped mines of information providing insights into their lives and work that simply cannot be found in published sources. In the case of this particular letter, my translation is based on a typewritten transcription of the original (which is included among Felix Klein's posthumous papers). As yet, I have been unable to ascertain whether or not the original version still exists. A portion of the letter also appears in the protocol book for Klein's seminar of 1909–1910 on mathematics and psychology. These notes indicate that the letter was presented to the seminar by the philosopher Leonard Nelson, who was a great-grandson of Dirichlet through his mother's side of the family. Apparently the original letter was at one time in the personal possession of the Nelson family, and it is possible that it is among the collection of letters (now located at the University Library in Kassel [23]) that Dirichlet wrote to his mother.

After discovering this letter, I was naturally curious about the surrounding circumstances and turned to Gauss's announcement in the *Göttingische Gelehrte Anzeigen* to see what it was that may have sparked Dirichlet's interest in the first place. The first sentence confirmed Dirichlet's assertion that Gauss had already submitted his memoir to the Scientific Society on April 5, 1825, although the *Commentatio prima* only appeared in print three years later. Another statement that appears in the introduction must have caught Dirichlet's attention: ". . . the present work is in no way intended as an exhaustive treatment of this rich topic. On the contrary, the development of the general theory, which requires an altogether special extension of the field of higher arithmetic, remains for the most part reserved for a future continuation" [24]. The "certain pomp" that Dirichlet referred to at the end of his letter to his mother appears in the following passage of Gauss's announcement:

> . . . as so often in higher arithmetic it is not so much the simplicity and beauty of the theorems as the difficulty of the proofs that distinguishes them so remarkably. As soon as one is prompted to conjecture the existence of a connection between the behavior of the number ± 2 and the two decompositions of the number p presented here, it is extremely simple to actually discover this connection through induction. Yet already with the first criterion it is not altogether easy to carry out the proof, and with the second the matter lies much deeper, as it is intimately bound with other subtle preliminary investigations, which for their part lead to a remarkable extension of cyclotomic theory. As has often been remarked, this wonderful chain of truths is primarily what gives higher arithmetic its so special attraction. Naturally these proofs themselves cannot be sketched here, and must be read in the monograph itself [25].

Dirichlet's simplified proofs of Gauss's criteria regarding the biquadratic character of the number 2 appeared in his article *"Recherches sur les diviseurs premiers d'une classe de formules du quatrième degré"* [26]. Regarding this work, Bessel wrote Alexander von Humboldt on April 14, 1828, that no one would have noticed the error had the name Lagrange stood at the top rather than Dirichlet's [27]. In addition to being simpler, Dirichlet's methods also turned out to be more powerful than those employed by Gauss. Not only did he succeed in handling the case $p = 2$, but he also answered the same question for an arbitrary prime p, thereby providing a complete analysis of those primes q for which p is a biquadratic residue mod q. Thus it appeared he was only a step away from establishing a reciprocity law for biquadratic residues analogous to Gauss's "fundamental theorem" in the theory of quadratic residues. It was probably only after Gauss announced his results on this subject three years later that Dirichlet began to realize why his own work on biquadratic reciprocity had reached an impasse at this point. Shortly after his article appeared, Dirichlet sent a copy to Gauss along with some remarks as to how he derived the results presented therein:

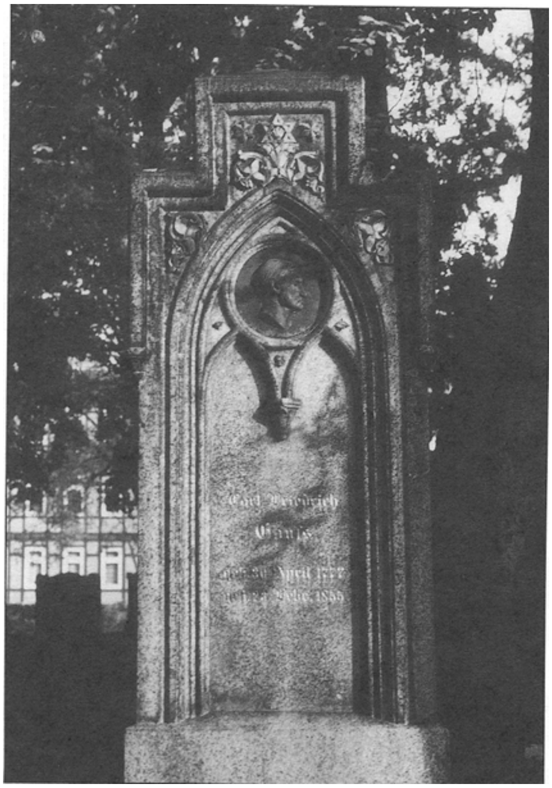
> . . . As soon as I had familiarized myself somewhat with my professional duties here, I began to busy myself with these things. My exertions remained for some time fruitless, until I succeeded in deriving your second criterion, whose proof by way of the path you have chosen appears to require a number of preliminary investigations, directly from the first. After this fortunate success my work again came to a standstill, and for a long time I could not find a suitable means for establishing the first criterion. Finally around the beginning of winter I came upon the proof presented in the enclosed article. It is so simple that it seems hard to comprehend that one would not grasp it immediately, just as soon as the proposition that requires proving became known. The continuation of my investigations, which are only partly contained in the article, has led me to a large number of results that one would certainly not have conjectured stood in any connection whatsoever with these matters. In the course of this work, I have encountered numerous remarkable examples of the often wonderful interconnections of arithmetical truths, which you regard as the main reason for the attraction that the investigations of indeterminate analysis afford us [28].

From this letter there would appear to be little doubt that the discovery Dirichlet made while in Dresden was related to the proof of Gauss's first criterion. Apparently, this was the discovery that gave him insight into a whole series of problems that animated his work during this formative period of his career. Surely this matter deserves some attention by experts in the history of number theory, as it may well be an important clue to understanding Dirichlet's intellectual development. In his letter to Gauss, Dirichlet also mentioned that he was still unable to obtain a copy of the *Commentatio prima,* despite numerous pleas to his bookdealer that he speed the order along. When Gauss wrote back seven weeks later, he mentioned that he would try to send an offprint of the article, but assumed that in the meantime it had arrived in Breslau and that Dirichlet was familiar with its contents. After expressing his pleasure with Dirichlet's latest effort, he added the following cryptic remarks to explain why his proofs differed so much from Dirichlet's own:

> I could have chosen a number of different forms of proof for the theorem there arising; it will not have escaped you, however, why I have preferred the one carried out here, namely primarily because the classification of 2 with respect to those modules for which it is a quadratic nonresidue (under B or D) must be regarded as an essentially integral part of the theorem for which most of the other forms of proof appear to be inapplicable.
>
> I have already had all of the material for this whole investigation in my possession for 23 years, except for the proof of the main theorem (to which that in the *Commentatio prima* is still not be counted), which I have had for about 14 years. I still hope and wish to be able to simplify the proofs of the latter somewhat, and plan to write approximately three works altogether on this subject. I have already made a beginning with the composition of the second, and hope to have it completed fairly soon, so long as the geodetic surveying that I have recently been assigned to again does not cause some delay [29].

Four months later Gauss and Dirichlet met in Berlin, where Alexander von Humboldt had invited them to

Dresden as it appeared before the Allied bombings at the close of World War II. (Furnished through the kind cooperation of Dr. Renate Tobies, Karl-Sundhoff-Institut, Leipzig.)



C. F. **Gauss's** gravestone in the Albani-Friedhof in Göttingen.

attend a meeting of the German Association of Scientists and Physicians. Somewhat earlier Dirichlet had been granted a leave of absence from Breslau; he never went back. For the next twenty-seven years he remained in Berlin, where he taught at the military school and as a *Privatdozent* at the University. In 1831, he was made an *außerordentlicher* and finally in 1839 and *ordentlicher Professor*. It was only after Gauss's death in 1855 that Dirichlet left Berlin for Göttingen, where he spent the last three and one-half years of his life as Gauss's successor.

It would appear unlikely that Dirichlet learned anything more about the nature of Gauss's "main theorem" when he saw him again in Berlin. Considering, moreover, that the *Commentatio prima* gave only the barest of hints as to what lay behind it, one must assume that Dirichlet (as well as other specialists) looked forward with great anticipation to the second of the three installments Gauss had planned. Both Dirichlet and Jacobi continued their investigations of biquadratic residues during these intervening years, and they must have grown rather impatient by the time Gauss's *Commentatio secunda* finally appeared in 1832. As before, it was preceded by a notice in the *Göttingische Gelehrte Anzeigen* describing the work's contents. This announcement was dated April 23, 1831, and in it Gauss indicated that the proofs for his earlier results on the congruence $x^4 \equiv k (\mathrm{mod}\ p)$ for $k = \pm 2$ and similar findings for $k = -3, 5, -7$, etc., could not be extended beyond a certain point. He then went on to make his first public pronouncement regarding the status of the so-called Gaussian integers and their place in the theory of biquadratic residues:

> One soon recognizes after this, that one can only break into this rich area of higher arithmetic by completely new paths. The author had already given an indication in the first work that to do so a remarkable extension of the whole field of higher arithmetic was essentially necessary, but without at the time explaining more closely what this consisted of: the present work has the intention of bringing this matter to light.
>
> For the true foundation of the theory of biquadratic residues, this is none other than the extension of the field of higher arithmetic, which has otherwise only been extended to the real whole numbers, to also include the imaginaries as well, for these must be granted exactly the same status as the others. As soon as this has once been observed, that theory appears in a completely new light and its results take on a highly surprising simplicity [30].

Gauss also mentioned that this idea had already been familiar to him for many years in another context. Nevertheless, he knew it would appear like a radical step to some, and he was probably hoping to ward off certain "Boeotians" when he wrote the following defense of this bold leap forward:

> Regarding the reality of negative numbers the situation has been clear for some time. It is only the imaginaries—formerly and sometimes still called *impossibles* [*unmögliche*]—which standing opposite the real numbers still remain less accepted. Merely tolerated, they thus appear more like an in itself contentless sign-game to which one categorically denies any intelligible substrate, without however wanting to disdain the rich tribute that this sign-game affords among the wealth of relationships between the real numbers.
>
> For many years the author has studied this highly important part of mathematics from a different standpoint, whereby the imaginary numbers can be used in connection with an object just as well as the negatives. Until now, however, there has been no appropriate occasion for publicly stating this in a definite manner, although the attentive reader can easily find traces of it in the work of 1799 on equations and in the *Preisschrift* on transformations of surfaces [31].

Gauss then went on to argue that much of the confusion surrounding imaginary numbers was merely a matter of terminology. He pointed out that had the quantities $+1$, -1, and $\sqrt{-1}$ been named direct, inverse, and lateral units—an allusion to their geometric properties under multiplication—the metaphysical doubts regarding the status of imaginaries never would have arisen.

Gauss's *Theoria Residuorum Biquadraticorum, Commentatio secunda* turned out to be his last major contribution to the theory of numbers. This paper was also one of the first to introduce and develop arithmetic in an algebraic number field. Much of it was devoted to establishing analogues of familiar number-theoretic ideas or techniques for the Gaussian integers. Thus, for example, § 37 gives the analogue to the fundamental theorem of arithmetic, showing that every Gaussian integer can be uniquely factored into complex primes. § 51 then presents a version of the Little Theorem of Fermat by utilizing a generalized Euler ϕ-function. The ring $Z[i]$ of Gaussian integers contains four units—$1, -1, i, -i$; numbers which differ by a unit multiple are termed associates. This theory is not a simple extension of ordinary arithmetic, since some numbers, like 3, remain prime, whereas others do not: e.g., $5 = (1 + 2i)(1 - 2i)$. Gauss introduced the norm $N(a + bi) = a^2 + b^2$ and classified the complex prime numbers according to three classes: real primes of the form $4n + 3$, non-real primes whose norm is a prime of the form $4n + 1$, and the special prime $1 + i$ that appears in the factorization of $2 = (1 + i)(1 - i)$. The fact that 2 is not a prime led Gauss to define an odd number to be one that is not divisible by $1 + i$. To simplify the statements of certain results, he further introduced the notion of a primary number. A nonunit z is said to be primary if $z \equiv 1(\text{mod } (1 + i)^3)$. In particular, primary numbers are odd, and if w is an arbitrary nonunit odd number then there exists a unique unit u such that uw is primary [32].

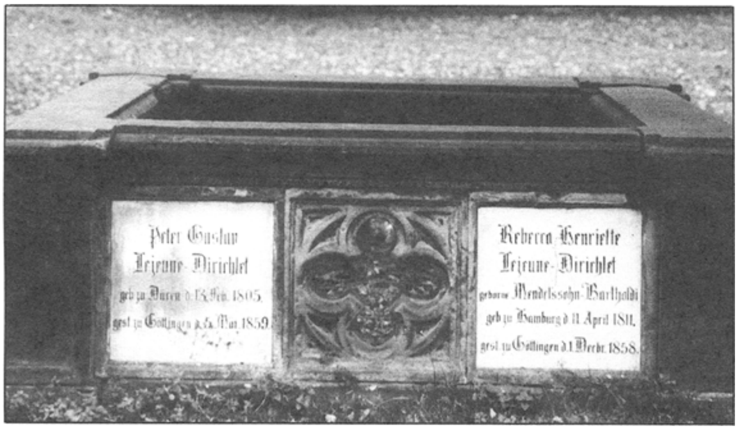
Rather than basing his analysis of biquadratic residues on three classes, depending on whether p is a biquadratic residue mod q, a quadratic residue but not a biquadratic residue, or neither, Gauss introduced four classes corresponding to the four units in $Z[i]$. In the classical theory of quadratic reciprocity the Legendre symbol $\frac{n}{p}$ is defined to be ± 1 according to which of the two congruences $n^{(p-1)/2} \equiv \pm 1(\text{mod } p)$ happens to hold. This definition is equivalent to the one given earlier, which merely depended on whether or not n happens to be a quadratic residue mod p. In the *Commentatio prima* Gauss used four equivalence classes to analyze the biquadratic character of $z = -1$ and $z = 2$. For a given prime p not dividing z, he showed that these four classes A, B, C, and D were determined by the fact that one of the following four congruences always holds: $z^{(p-1)/4} \equiv 1, f, f^2, f^3$ (mod p), where f satisfies $f^2 \equiv -1(\text{mod } p)$. To generalize to the Gaussian integers, however, he had to discover the following relation: For any complex prime p not dividing z, there exists a $k = 0, 1, 2, 3$ satisfying the congruence $z^{(N(p)-1)/4} \equiv i^k (\text{mod } p)$. This enables one to define a generalized Legendre symbol $(\frac{z}{p})_4 = i^k$, and by means of this the law of biquadratic reciprocity may then be stated in the following form [33]: If p and q are distinct primary primes, then

$$\left(\frac{p}{q}\right)_4 = (-1)^{\frac{N(p)-1}{4} \cdot \frac{N(q)-1}{4}} \left(\frac{q}{p}\right)_4$$

Gauss's formulation of this remarkable theorem appears in § 67 of the *Commentatio secunda*. It differs somewhat from the version given above, but the latter has the advantage of revealing more clearly the strong analogy between this result and the law of quadratic reciprocity. One of the principal aims of Gauss's third memoir was to present a proof of this theorem, which, in his words, "belongs among the most deeply-hidden truths of higher arithmetic." But, for reasons that remain unclear, he never published this final article. A sketch of a proof for the fundamental theorem of biquadratic reciprocity was found among his posthumous papers, and was published fifty years after his death in Volume X of Gauss's *Werke* along with tables he had compiled for computing the first $(N(m) - 1)/4$ powers of primitive roots of the module m. The proof he outlined was based on ideas similar to those utilized in his sixth proof of the "fundamental theorem." But since it has been impossible to date these documents reliably, one cannot be sure whether this proof was original with Gauss or based on other sources. The first proof of the law of biquadratic reciprocity was published by Eisenstein in 1844, although Jacobi had presented a proof somewhat earlier in his lectures at Königsberg [34].

In September 1838 Dirichlet again wrote to Gauss, forwarding him a copy of the article in which he proved that every arithmetic progression $a + bn$, with a and b relatively prime, contains infinitely many primes. In his letter Dirichlet remarked that he thought there might be some connection between his methods and those alluded to in the closing remarks at the end of the *Disquisitiones*, wherein Gauss indicated that his findings could be used to shed light on a number of areas in analysis. These remarks naturally aroused Dirichlet's curiosity, since the analytic methods he was beginning to introduce in number theory were intimately related to discontinuous functions represented by trigonometric series, and these, as he emphasized to Gauss, were "still completely unexplained at the time the *Disquisitiones* appeared" [35]. He then went on to say: "You will perhaps still remember that more than ten years ago when I was in Breslau you informed me of a criterion for deciding the question raised at the end of your first work on biquadratic residues (for the case $p = 8n + 5$), mentioning that you had derived it by means of the results announced at the end of the *Disquisitiones Arithmeticae*." And then Dirichlet proceeded to inform him about how this criterion could easily be derived from certain propositions he had discovered in the meantime.

Gauss's number-theoretic investigations remained a source of inspiration for Dirichlet throughout the remainder of his career. In fact, fate alone prevents us from reading about this inspiration firsthand. In the summer of 1858, while vacationing in Switzerland, Dirichlet was preparing a memorial speech on Gauss



The final resting place of Peter Gustav and Rebecca Lejeune Dirichlet.

that he was scheduled to deliver before the Göttingen Scientific Society. It was during this stay that he suffered a nearly fatal heart attack. At first he slowly began to regain his health, but the following winter his wife Rebecca, the sister of composer Felix Mendelssohn-Bartholdy, passed away unexpectedly. Filled with sorrow, Dirichlet's condition quickly worsened, and he followed her in death a few months later. They were buried together in Göttingen not far from the gravesite of the Prince of Mathematicians.

Acknowledgment

The author would like to thank Joseph W. Dauben and Harold M. Edwards for reading an earlier version of this article and offering a number of helpful suggestions for improving its style and substance.

Notes

1. An English translation of Gauss's *Disquisitiones Arithmeticae* was undertaken by Arthur A. Clarke (*DA*, New York & London: Yale University Press, 1966).
2. Quoted in Hans Wussing, *Carl Friedrich Gauss, Biographien hervorragender Naturwissenschaftler, Techniker und Mediziner*, 15, p. 32. Unless otherwise indicated, all translations from the German are the author's.
3. E. E. Kummer, "Gedächtnisrede auf Gustav Peter Lejeune Dirichlet," in *G. Lejeune Dirichlet's Werke*, vol. 2, ed. L. Kronecker and L. Fuchs, Berlin: Georg Reimer, 1897, pp. 311–344, on pp. 315–316.
4. For biographical information on Dirichlet, the two outstanding sources are Kummer's "Gedächtnisrede" cited above, and K.-R. Biermann, "Johann Peter Gustav Lejeune Dirichlet, Dokumente für sein Leben und Wirken," *Abhandlungen der Deutschen Akademie der Wissenschaften zu Berlin, Klasse für Mathematik, Physik und Technik*, 1959, Nr. 2, pp. 2–88.
5. A forthcoming study of Dirichlet's career by Uta Merzbach should do much to fill this gap in the literature.
6. The quotations from Gauss's diary that follow are taken from J. J. Gray, "A Commentary on Gauss's Mathematical Diary, 1796–1814, with an English Translation," *Expositiones Mathematicae*, 2(1984), 97–130.
7. Euler's approach to the theory of quadratic residues is discussed in Harold M. Edwards, "Euler and Quadratic Reciprocity," *Mathematics Magazine*, Vol. 56, No. 5 (1983), 285–291.
8. See Wussing, p. 57.
9. Quoted in Biermann, p. 13.
10. A letter written by Dirichlet to his mother expressed surprise that he had been received very cordially by Gauss in Göttingen. After meeting him, Dirichlet apparently had a much more favorable impression of the "Prince of Mathematicians" than he had had beforehand. See Kummer, p. 341.
11. *Dirichlet's Werke*, pp. 373–74.
12. Biermann, p. 14.
13. *Dirichlet's Werke*, p. 375.
14. *Ibid*.
15. Biermann, p. 12.
16. *Göttingische Gelehrte Anzeigen*, 59. St., April 11, 1825. Republished in C. F. Gauss, *Werke*, vol. II, Göttingen, Hildesheim: G. Ohms Verlag, 1973 reprint, on pp. 165–168.
17. A transcription of the original letter can be found in *Nachlass Klein XXIA*, Niedersächsische Staats- und Universitätsbibliothek Göttingen. The author wishes to thank this institution for permission to publish it here in translation.
18. These remarks and much of what follows reveal how eager Dirichlet was to persuade his mother that mathematics was a viable career choice for him. Like so many parents of famous mathematicians, Dirichlet's mother and father wanted him to take up something practical, and suggested that he study law at a German university. See Kummer, p. 314.
19. A marginal note reads: "La modestie est une bien belle chose."
20. A marginal note further reveals that Dirichlet was eager to leave Breslau at the earliest opportunity: "As much as I am satisfied with my present abode so far as the company is concerned, there is so little here to offer from the scientific side that I will mobilize all the forces I can to bring about my transfer to Berlin." ("So sehr ich auch mit meinem hierigen Aufenthalt, insofern von Umgang die Rede ist, zufrieden bin, so ist doch in wissenschaftlicher Hinsicht hier so wenig zu machen, daß ich alles aufbieten werde, um meine Versetzung nach Berlin zu bewerkstelligen.")
21. The circumstances surrounding Lie's discovery are described by Felix Klein in his *Gesammelte Mathematische Abhandlungen*, vol. 1, Berlin: Springer, 1973 reprint, on p. 97. Klein's discovery of the *Grenzkreistheorem* is discussed in his *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*, vol. 1, New York: Chelsea, 1967 reprint, on p. 379. The psychological implications of Poincaré's discoveries are discussed in Jacques Hadamard, *An Essay on the Psychology of Invention in the Mathematical Field*, New York: Dover, 1954 reprint, pp. 11–15, which examines many other interesting case studies as well.
22. The famous story of Archimedes running from the bath comes from Vitruvius's *De architectura*. See Ivor Thomas, *Selections Illustrating the History of Greek Mathematics*, vol. 2, Cambridge, Mass. & London, 1968, pp. 36–39.
23. For more information on Dirichlet's posthumous papers, see Gert Schubring, "The Three Parts of the Dirichlet *Nachlass*," *Historia Mathematica*, 13(1986), 52–56. Further details regarding Dirichlet's early career as revealed in the letters he wrote to his mother can be found in G.

Schubring, "Die Promotion von P. G. Lejeune Dirichlet. Biographische Mitteilungen zum Werdegang Dirichlets," *NTM* (*Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin*), 21, No. 1, 45–65.
24. *Göttingische Gelehrte Anzeigen*, p. 586.
25. *Ibid.*, p. 588.
26. Dirichlet's article first appeared in Crelle's *Journal für die reine und angewandte Mathematik*, 3(1828), 35–69. H. M. Edwards pointed out to me that the results on the biquadratic character of 2 mod p were already known to Euler. His work on this subject, however, was only published posthumously, and not until 1848!
27. Kummer, p. 323.
28. *Dirichlet's Werke*, pp. 376–78.
29. *Ibid.*, pp. 378–80.
30. *Göttingische Gelehrte Anzeigen*, April 23, 1831, republished in C. F. Gauss, *Werke*, vol. II, Hildescheim: G. Ohms Verlag, 1973 reprint, on p. 171.
31. *Ibid.*, p. 173
32. For more details, see Kenneth Ireland & Michael Rosen, *A Classical Introduction to Number Theory, Graduate Texts in Mathematics*, 84, New York: Springer, 1982
33. For more background on this see M. J. Collison, "The Origins of the Cubic and Biquadratic Reciprocity Laws," *Archive for History of Exact Sciences* 17(1977), 63–69; G. J. Rieger, "Die Zahlentheorie bei C. F. Gauss," in *C. F. Gauss Gedenkband anlässlich des 100. Todestages am 23. Februar 1955*, ed. Hans Reichardt, Leipzig: Teubner, 1957. Also Paul Bachmann, "Ueber Gauss' Zahlentheoretische Arbeiten," in *Materialen für eine wissenschaftliche Biographie von Gauss*, Heft 1, *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen, Math.-phys.* Klasse, 1911.
34. Eisenstein presented two proofs of the law of biquadratic reciprocity, one in the 1844 paper "Lois de réciprocité" (see Gotthold Eisenstein, *Mathematische Werke*, vol. 1, New York: Chelsea, 1975, pp. 126–140), and another in an article of the same year entitled "Einfacher Beweis und Verallgemeinerung des Fundamentaltheorems für die biquadratischen Reste" (*Ibid.*, pp. 141–163).
35. *Dirichlet's Werke*, p. 382.

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If others would but reflect on mathematical truths as deeply and continuously as I have, they would make my discoveries.

Carl F. Gauss