TMA4320 - Project 3: Compact stars

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1 Task 1

1.a

For a symmetrical star with outward pressure $P(r)$ and energy density ϵ we have

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\epsilon}\frac{dP}{dr}\right) = -\frac{4\pi G\epsilon}{c^4}. \tag{1}$$

With $P = K\epsilon^\gamma$, $\epsilon = \epsilon_0\theta^n$ and $r = a\xi$ we get

$$\frac{1}{(a\xi)^2}\frac{d}{ad\xi}\left[\frac{(a\xi)^2}{\epsilon_0\theta^n}\frac{d}{ad\xi}\left(K\epsilon_0^\gamma\theta^{n\gamma}\right)\right] = -\frac{4\pi G\epsilon_0\theta^n}{c^4}. \tag{2}$$

Rearranging the equation with $\gamma = 1 + \frac{1}{n}$ we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\frac{\xi^2\theta^n}{\theta^n}\frac{d\theta}{d\xi}\right] = -\frac{4\pi G\epsilon_0^{1-\frac{1}{n}}\theta^n}{(n+1)Kc^4}a^2, \tag{3}$$

and with $a = \left[\frac{(n+1)Kc^4\epsilon_0^{\frac{1}{n}-1}}{4\pi G}\right]^{1/2}$ we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n \tag{4}$$

1.b

From eq. (30) in the manual we have

$$M = 4\pi R^{(3-n)/(1-n)}\left[\frac{(n+1)Kc^4}{4\pi G}\right]^{n/(n-1)}\xi_1^{(3-n)/(1-n)}\xi_1^2\left|\theta'\left(\xi_1\right)\right|, \tag{5}$$

which is reduced to

$$M = 4\pi R^3\xi_1^5\left|\theta'(\xi_1)\right| \tag{6}$$

when $n = 0$. To find $\left|\theta'\left(\xi_1\right)\right|$ we solve (4) for $n = 0$:

$$\xi^2\frac{d\theta}{d\xi} = -\frac{\xi^3}{3} + C \tag{7}$$

From $\theta'(0) = 0$ we know that $C = 0$, and we get

$$|\theta'(\xi_1)| = |\xi_1|/3 \tag{8}$$

and, inserting into (6) and using $\xi_1 = R/a$ we get

$$M = \frac{4}{3}\pi R^3 \left(\frac{R}{a}\right)^6 \tag{9}$$

1.c

Inserting $\theta = \frac{u}{\xi}$ into (4) we get

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi\left(\frac{du}{d\xi} - \frac{u}{\xi}\right)\right) = -\frac{u}{\xi}, \tag{10}$$

which simplifies to

$$\frac{d^2u}{d\xi^2} + u = 0. \tag{11}$$

(11) has the general solution

$$u = C_1 \sin\xi + C_2 \cos\xi, \tag{12}$$

which gives

$$\theta = \frac{C_1 \sin\xi + C_2 \cos\xi}{\xi}. \tag{13}$$

Using the right side limit we find $\lim_{\xi\to 0^+} \theta(\xi) = 0$ gives gives $C_1 = 1$ and $C_2 = 0$. The limit $\lim_{\xi\to 0^+} \theta'(\xi) = 0$ (using L'Hopital on $\theta'(\xi) = (\xi\cos(\xi) - \sin(\xi))/\xi^2$), which is the value found in eq. (26) in the project manual. We conclude that

$$\theta(\xi) = \frac{u(\xi)}{\xi} = \frac{\sin\xi}{\xi} \tag{14}$$

is the solution of the Lane-Emden equation for $n = 1$.

2 Task 2

2.a

From eq. (19) in the manual we have

$$\frac{dP}{dr} = -\frac{G\epsilon(r)m(r)}{c^2r^2}\left[1 + \frac{P}{\epsilon(r)}\right]\left[1 + \frac{4\pi r^3 P(r)}{m(r)c^2}\right]\left[1 - \frac{2Gm(r)}{c^2 r}\right]^{-1}. \tag{15}$$

Splitting the equation into its multiplicative components and using

$\bar{P} = \dfrac{P}{\rho_0 c^2}$, $\alpha = \dfrac{r_s}{R} = \dfrac{2GM/c^2}{R}$, $x = \dfrac{r}{R}$, $m(r) = \dfrac{4}{3}\pi r^3 \rho_0 = M\dfrac{r^3}{R^3}$ and $\epsilon(r) = \rho_0 c^2$

we find that

$$
\begin{aligned}
\frac{dP}{dr} &= \frac{dP}{d\bar{P}}\frac{d\bar{P}}{dx}\frac{dx}{dr} = \rho_0 c^2 \frac{d\bar{P}}{dx}\frac{1}{R} && = \frac{d\bar{P}}{dx}\cdot\frac{\rho_0 c^2}{R} \\
-\frac{G\epsilon(r)m(r)}{c^2 r^2} &= -\frac{G\rho_0 c^2 \cdot Mr^3/R^3}{c^2 r^2} = -\frac{1}{2}\frac{2GM}{c^2 R}\frac{r}{R}\frac{\rho_0 c^2}{R} && = -\frac{1}{2}\alpha x \cdot \frac{\rho_0 c^2}{R} \\
1 + \frac{P}{\epsilon(r)} &= 1 + \frac{P}{\rho_0 c^2} && = 1 + \bar{P} \\
1 + \frac{4\pi r^3 P(r)}{m(r)c^2} &= 1 + \frac{4\pi r^3 P(r)}{4/3\pi r^3 \rho_0 c^2} = 1 + 3\frac{P}{\rho_0 c^2} && = 1 + 3\bar{P} \\
1 - \frac{2Gm(r)}{c^2 r} &= 1 - \frac{2GMr^3/R^3}{c^2 r} = 1 - \frac{2GM}{c^2 R}\left(\frac{r}{R}\right)^2 && = 1 - \alpha x^2
\end{aligned}
$$

and, dividing both sides of (15) by $\rho_0 c^2/R$, we get

$$
\frac{d\bar{P}}{dx} = -\frac{1}{2}\alpha x[1+\bar{P}][1+3\bar{P}]\left[1 - x^2\alpha\right]^{-1}. \tag{16}
$$

2.b

Using separation of variables and integrating we have

$$
\int \frac{d\bar{P}}{(1+\bar{P})(1+3\bar{P})} = -\frac{1}{2}\int \frac{\alpha x}{1-\alpha x^2}dx
$$

which after using partial fractions on the $\bar{P}$ integral and substituting $u = \alpha x^2$ in the other before integrating gives

$$
\frac{1}{2}\ln(1+3\bar{P}) - \frac{1}{2}\ln(1+\bar{P}) = \frac{1}{4}\ln(1-\alpha x^2) + C_1
$$

$$
\implies \frac{1+3\bar{P}}{1+\bar{P}} = C_2 \cdot \sqrt{1-\alpha x^2}
$$

$$
\implies 1 + 3\bar{P} = (1+\bar{P})C_2\sqrt{1-\alpha x^2}
$$

$$
\implies \left(\frac{3}{C_2} - \sqrt{1-\alpha x^2}\right)\bar{P} = \sqrt{1-\alpha x^2} - \frac{1}{C_2}
$$

$$
\implies \bar{P} = \frac{\sqrt{1-\alpha x^2} - C_3}{3C_3 - \sqrt{1-\alpha x^2}}
$$

with $C_3 = 1/C_2 = 1/e^{C_1}$. With the given condition $\bar{P}(x=1) = 0$ we finally find that

$$\bar{P}(x=1) = \bar{P} = \frac{\sqrt{1-\alpha \cdot 1} - C_3}{3C_3 - \sqrt{1-\alpha \cdot 1}} = 0 \implies C_3 = \sqrt{1-\alpha}$$

and thus

$$\bar{P} = \frac{\sqrt{1-\alpha x^2} - \sqrt{1-\alpha}}{3\sqrt{1-\alpha} - \sqrt{1-\alpha x^2}} \tag{17}$$

.

2.c

When $\alpha << 1$ we can use the taylor expansion of $\bar{P}$ with respect to α around 0 as a good approximation. Using the linear expansion

$$\bar{P} = \bar{P}\Big|_{\alpha=0} + \alpha \frac{d\bar{P}}{d\alpha}\Big|_{\alpha=0} + O(\alpha^2) \simeq \bar{P}\Big|_{\alpha=0} + \alpha \frac{d\bar{P}}{d\alpha}\Big|_{\alpha=0} \tag{18}$$

we only need to calculate $\frac{d\bar{P}}{d\alpha}\Big|_{\alpha=0}$. Letting

$$b = \sqrt{1-\alpha x^2} \qquad \implies \frac{db}{d\alpha} = b' = -\frac{1}{2}x^2 \frac{1}{\sqrt{1-\alpha x^2}} \tag{19}$$

$$c = \sqrt{1-\alpha} \qquad \implies \frac{dc}{d\alpha} = c' = -\frac{1}{2}\frac{1}{\sqrt{1-\alpha}} \tag{20}$$

we have

$$b(\alpha=0) = 1, \qquad b'(\alpha=0) = -\frac{1}{2}x^2 \tag{21}$$

$$c(\alpha=0) = 1, \qquad c'(\alpha=0) = -\frac{1}{2}. \tag{22}$$

Using b and c to express $\bar{P}$ we obtain

$$\frac{d\bar{P}}{d\alpha}\Big|_{\alpha=0} = \frac{d}{d\alpha}\left[\frac{\sqrt{1-\alpha x^2} - \sqrt{1-\alpha}}{3\sqrt{1-\alpha} - \sqrt{1-\alpha x^2}}\right]\Big|_{\alpha=0} = \frac{d}{d\alpha}\left[\frac{b-c}{3c-b}\right]\Big|_{\alpha=0} \tag{23}$$

$$= \frac{(b'-c')(3c-b) - (b-c)(3c'-b')}{(3c-b)^2}\Big|_{\alpha=0} \tag{24}$$

$$= \frac{(-\frac{1}{2}x^2 + \frac{1}{2})(3-1) - (1-1)(3c'-b')}{(3-1)^2} = \frac{1-x^2}{4}, \tag{25}$$

and thus, as $\bar{P}\Big|_{\alpha=0} = (b-c)/(3c-b) = (1-1)/(3-1) = 0$, our first order approximation valid for $\alpha << 0$ is

$$\bar{P} \simeq \frac{1}{4}\alpha(1-x^2). \tag{26}$$

3 Task 3

3.a

The tailor expansion around y_0 is

$$y(x) = y(x_0) + (x - x_0)y'(x_0) + O((x - x_0)^2). \tag{27}$$

For $x_1 = x_0 + h$, we get

$$y(x_1) = y(x_0) + hy'(x_0) + O(h^2). \tag{28}$$

We compare this with forwards Euler, and get

$$y_1 = y(x_0) + hy'(x_0) \implies \tau_1 = |y(x_1) - y_1| \propto O(h^2). \tag{29}$$

3.b

The local error generally is not a sum of the local errors, but each local error $\tau_i = a_i \cdot h^2$ where a_i is a function of the state of the numerical approximation at the point x_i. It can be shown that when the diff. eq. is Lipschitz continuous the factor a_i for any x_i is independent of h. Thereby, the global error as a function of h goes as

$$e_N = \sum_{i=1}^{N} [a_i \cdot O(h^2)] = \sum_{i=1}^{n} O(h^2) = N \cdot O(h^2) = \frac{x_N - x_0}{h} O(h^2) = O(h). \tag{30}$$

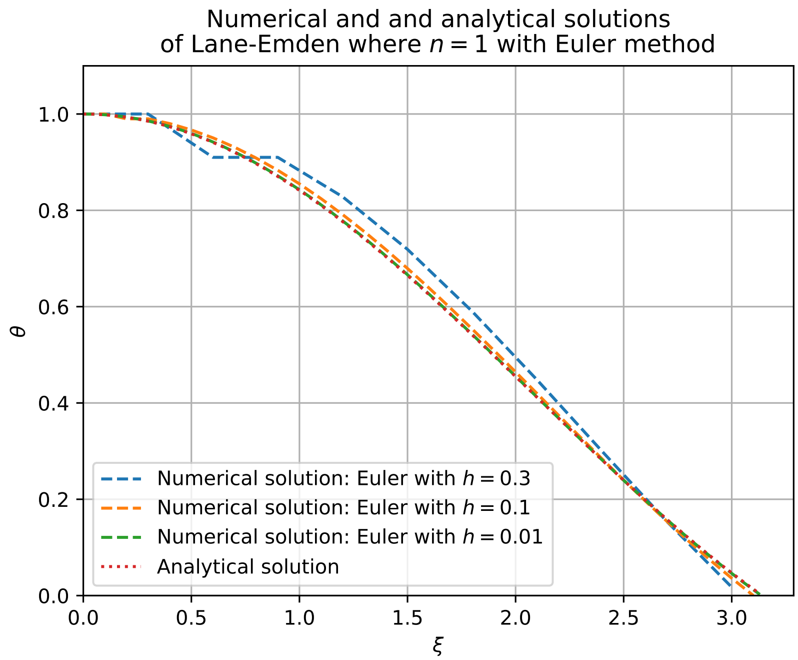
3.c

Using $\chi = d\theta/d\xi$ and the Lane-Emden equation (4) we get that

$$\frac{d\vec{\omega}}{d\xi} = f\left(\begin{bmatrix} \theta \\ \chi \end{bmatrix}\right) = \begin{bmatrix} \frac{d\theta}{d\xi} \\ \frac{d\chi}{d\xi} \end{bmatrix} = \begin{bmatrix} \chi \\ -\theta^n - \frac{2}{\xi}\chi \end{bmatrix}. \tag{31}$$

3.d

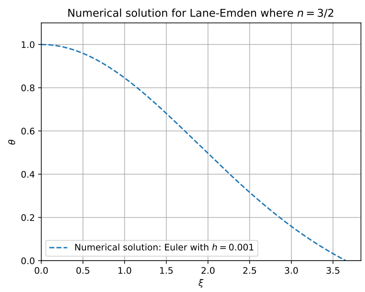
We plot the numerical solution with forwards Euler-method for $n = 1$, and the analytical solution from 1.c, of the Lane-Emden equation (4).

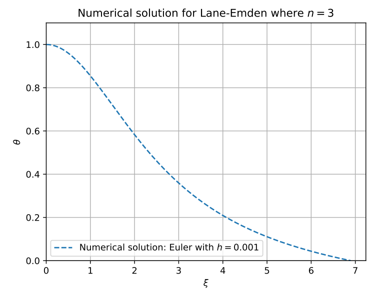


In the plot, the numerical solution with $h = 0.01$ is indistinguishable from the analytical solution, while when $h = 0.1$ there is a noticable difference. Using this plot as our guidance, we should use some step-length $h \leq 0.01$ when solving the dimesionless Lane-Emden equation. In the following tasks, we choose to use $h = 0.001$, as modern computers have no problem in solving the numerical equation in fractions of a second and the precision is ever so slightly better than with $h = 0.01$. If we were to run the same simulation several million times or if the results were to be analyzed further in some error-sensitive way, more effort would be put into the analysis of what precision is required.

3.e

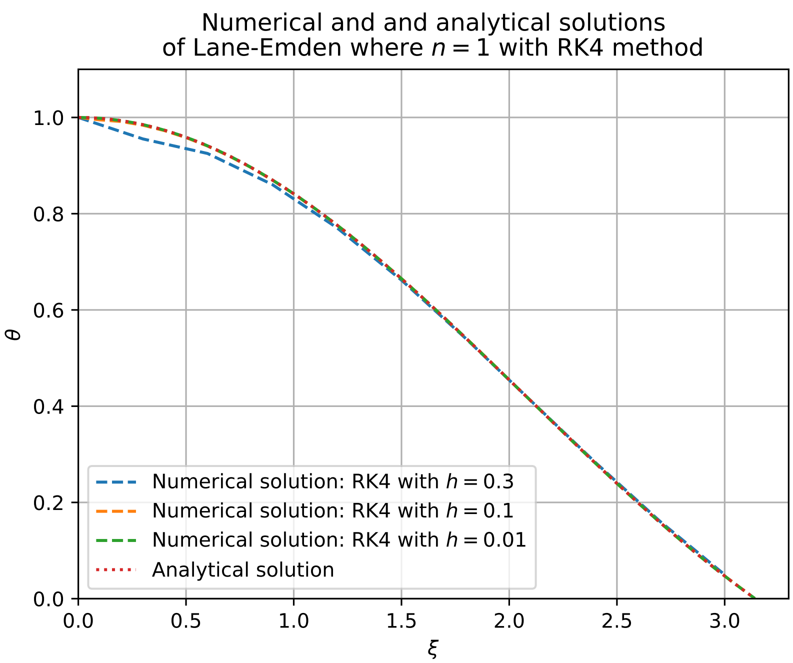
The following is the solution of the Lane-Emden equation with the forwards Euler-method for $n = 3/2$ and $n = 3$.





3.f

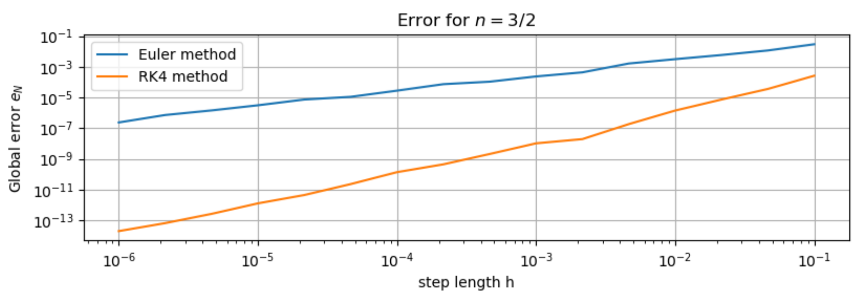
The following is the solution of the Lane-Emden equation with the RK4-method for $n = 1$.

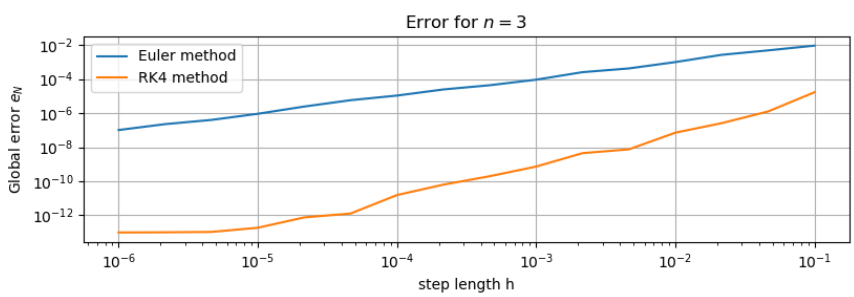


As expected the error is visibly smaller with the RK4-method than with the Euler method for the same step-lengths h. In this case, a step-length $h \leq 0.1$ would suffice for a graphical representation indistinguishable from the analytical solution, but as our computers have no problem in running a more precise simulation we would choose to use $h = 0.001$. If we had external requirements of either better efficiency or precision, we would alter our used step length to be larger or smaller respectively.

3.g

From the plot below it is obvious that the global error e_N decreases with smaller steps h, as is expected. The global error of the RK4-method is lower than the error for the Euler method for similar values h, and global error also converges faster towards 0 for the RK4-method. Using the plotted error for $n = 3/2$ as an example, we read from the graph that the global error e_N goes roughly as h for the Euler method while for RK4 e_N goes roughly as h^2.





Note that for $n = 3$ the error of the RK4-method seems to flatten out for $h < 10^{-5}$. This is because of lower precision in the given $\bar{\xi_N}$ than the numerical method is able to calculate. In essence, the numerical method is more precise than the given reference, and the error is due to the difference between the given $\bar{\xi_N}$ and the actual value.

The following is a code snippet from `task_i` in the function `global_error`. It can be noted that the constant error in the case where $n = 3$ is of the same magnitude as the rounding error introduced by rounding to the 13th decimal place. As $\bar{\xi_N}$ for $n = 3/2$ is more precise, the same error is not evident in the global error plot.

```python
def global_error(...):
    ...
    if n == 3/2:
        xi_final = 3.6537537362191657
        # Calculated with RK4 and h=1E-7, rounded to the 15th decimal place.
        # 3.65375 was given in the manual
    elif n == 3:
        xi_final = 6.89684861937482
        # Calculated with RK4 and h=1E-7, rounded to the 13th decimal place.
        # 6.89685 was given in the manual
    ...
```

3.h

We have already shown in task 2b that

$$\bar{P}(x) = \frac{\sqrt{1-\alpha} - \sqrt{1-\alpha x^2}}{\sqrt{1-\alpha x^2} - 3\sqrt{1-\alpha}}. \tag{32}$$

Letting $x = 0$ it is trivial that

$$\bar{P}(0) = \frac{\sqrt{1-\alpha} - 1}{1 - 3\sqrt{1-\alpha}}. \tag{33}$$

Furthermore, by using eq. 36 in the manual directly we find that

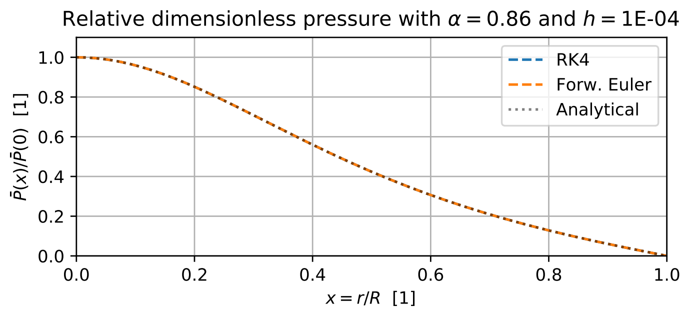
$$\left.\frac{d\bar{P}}{dx}\right|_{x=0} = -\frac{1}{2}\alpha x[1+\bar{P}][1+3\bar{P}]\left[1-x^2\alpha\right]^{-1}\Big|_{x=0} = 0 \tag{34}$$

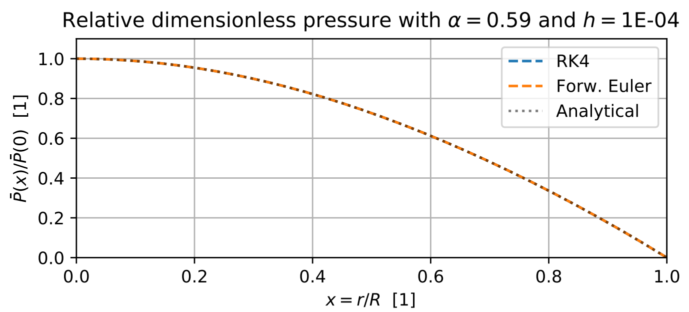
from the factor $[-\alpha x/2]_{x=0} = 0$.

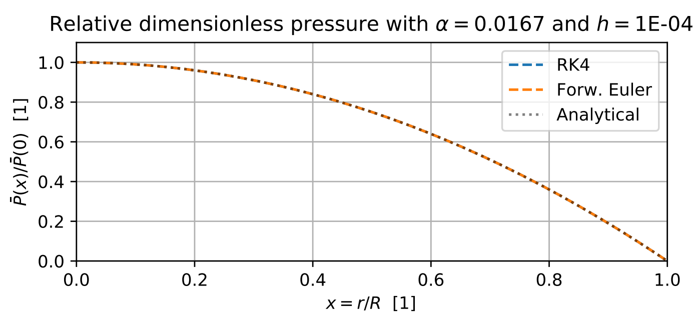
3.i

We see from the plot that with a step length $h = 10^{-4}$ our numerical approximation is indistinguishable from the analytical solution.

This value of h was chosen based on the global error at $x = r/R = 1$. At this point the numerical solution should give $\bar{P}(1) = 0$. For $h = 10^{-4}$ the RK4-method has a worst global error of $2.3 \cdot 10^{-13}$ (for $\alpha = 0.86$), while for $h = 3 \cdot 10^{-4}$ this error is $3.1 \cdot 10^{-4}$.

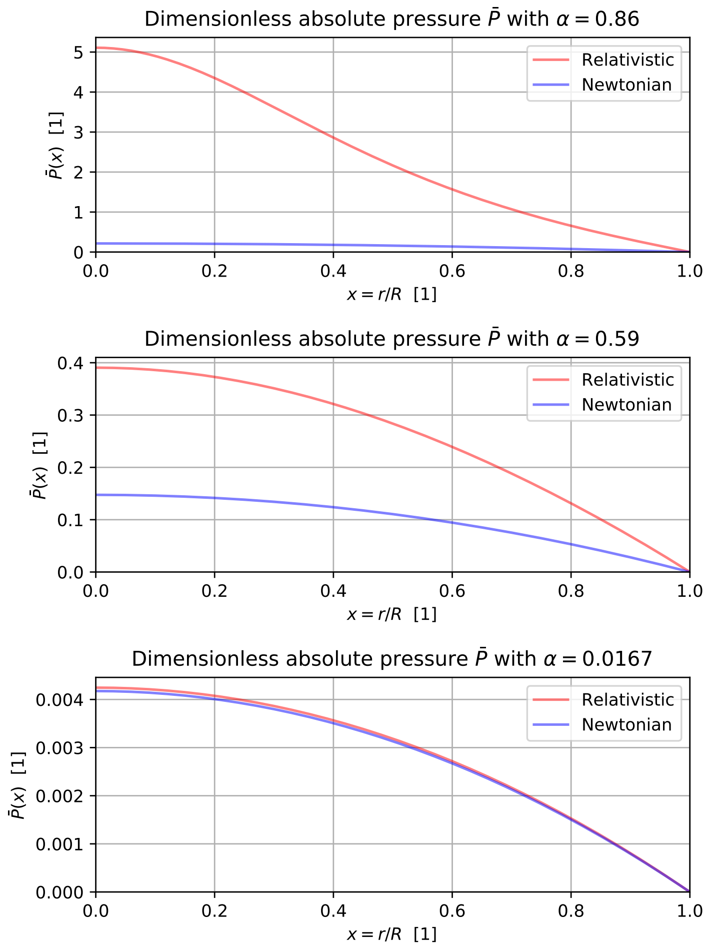






3.j

From the following plot we see that the Newtonian approximation works well when $\alpha = r_s/R$ is small, which fits with the intuition that a small α is tied to smaller relativistic effects.


Dimensionless absolute pressure $\bar{P}$ with $\alpha = 0.86$
Relativistic
Newtonian
$\bar{P}(x)$ [1]
$x = r/R$ [1]
Dimensionless absolute pressure $\bar{P}$ with $\alpha = 0.59$
Relativistic
Newtonian
$\bar{P}(x)$ [1]
$x = r/R$ [1]
Dimensionless absolute pressure $\bar{P}$ with $\alpha = 0.0167$
Relativistic
Newtonian
$\bar{P}(x)$ [1]
$x = r/R$ [1]