Rigid connection: Bowed membrane + Acoustic tube

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1 Introduction

This document illustrates the approach to connecting a bowed membrane and an acoustic tube.

2 Model

2.1 Bowed membrane

The bowed membrane is described by the following differential equation:

$$\delta_{tt}u = c^2\delta_{\Delta}u - 2\sigma_0\delta_{t\cdot}u + 2\sigma_1\delta_{t-}\delta_{\Delta}u - J_B F_B \phi_{v_{rel}} \tag{1}$$

$$v_{rel} = I_B\delta_{t\cdot}u - v_B \tag{2}$$

where $u(x, y, t)$ is the transverse displacement of the membrane. The σ_0 and σ_1 terms model the frequency in-dependent and frequency dependent damping of the system.
v_{rel} is the relative velocity between the bow and the membrane. F_B and v_B are the bowing force and bowing velocity, while J_B is a spreading function of the bowing force over the membrane. It follows:

$$J_B = I_B 1/h^2 \tag{3}$$

where h, is the spatial discretization step of the membrane. I_B is an interpolation function (how the force is interpolated over the membrane), and the spreading function J_B is this distribution scaled with the discretization area of the membrane $(1/h^2)$. If the spreading function is for a 1-D element than it is scaled with the unit length instead $(1/h)$, an example will follow.

The bowing force, F_B as given above is scaled with the mass per unit area of the membrane $(\rho_M H)$. That is the force is not in given in $[N]$ but in $[\frac{N}{kg/m^2}]$.

Furthermore $\phi(v_{rel})$ is a non-linear (dimensionless) bowing characteristic given by:

$$\phi(v_{rel}) = \sqrt{2a}v_{rel}e^{-av_{rel}^2+1/2} \tag{4}$$

with $a = 100$.

2.2 Acoustic tube

The acoustic tube can be modelled as a 1-D damped wave and is described by the following differential equation:

$$\delta_{tt}w = c_{tube}^2\delta_{xx}w - 2\sigma_{0,tube}\delta_{t\cdot}w + 2\sigma_{1,tube}\delta_{t-}\delta_{xx}w \tag{5}$$

with $w(x,t)$ being the transverse displacement. Similarly to the membrane case, $\sigma_{0,tube}$ and $\sigma_{1,tube}$ model the frequency in-dependent and frequency dependent damping.

2.3 Adding the connection

A perfectly rigid connection is assumed, meaning that the displacement at the connection between the two elements is equal:

$$< u^n, J_M >_{D_M} = < w^n, J_T >_{D_T} \tag{6}$$

where $< f, g >_D$ represents the inner product of f and g over domain D. J_M is the distribution of the connection over the membrane and J_T is the distribution of the connection over the acoustic tube. If these distributions are very localized (a Dirac function for instance), then it means that the connection is only at 1 point.

This connection will result in additional force terms on the differential equations of the bowed membrane and acoustic tube respectively, as follows:

$$\delta_{tt}u = c^2\delta_\Delta u - 2\sigma_0\delta_{t\cdot}u + 2\sigma_1\delta_{t-}\delta_\Delta u - J_B F_B \phi(v_{rel}) + J_M F\frac{1}{\rho_M H} \tag{7a}$$

$$\delta_{tt}w = c_{tube}^2\delta_{xx}w - 2\sigma_{0,tube}\delta_{t\cdot}w + 2\sigma_{1,tube}\delta_{t-}\delta_{xx}w - J_T F\frac{1}{\rho_T A} \tag{7b}$$

with F being the connection force and J_M and J_T the connection distributions described above. $\rho_M H$ is the mass per square meter of the membrane and $\rho_T A$ is the mass per unit length of the tube.

2.4 Useful Identities

Two useful identities for solving the connection problem are the following:

$$< f, J_P >_D = I_P f \tag{8}$$

with J_P being a distribution function over some domain and I_P being its analogous interpolation function. Equation 8 follows from Equation 3.

Next, the time discretization operator δ_{tt} can be written as:

$$\delta_{tt} = \frac{2}{k}\left(\delta_{t\cdot} - \delta_{t-}\right) \tag{9}$$

with k being the discretization time step, $k = 1/f_{\mathrm{S}}$. f_{S} is the sampling rate.

3 Solving the connection

Looking at Equation 7a and Equation 2 it can be seen that the finite difference scheme needed to solve for the displacements of the membrane is implicit. This means that we need to solve for both v_{rel} and u at the same time. Furthermore, due to the connection with the acoustic tube and the condition necessary for rigid connection from Equation 6, we at the same time need to solve for F that satisfies the connection as well.

From Equation 7a and Equation 2, using the identity in Equation 9, one can rewrite Equation 7a as:

$$\begin{aligned}\frac{2}{k}\left(\delta_{t\cdot}u - v_B\right) - \frac{2}{k}\delta_{t-}u + 2\sigma_0\left(\delta_{t\cdot}u - v_B\right) + 2\sigma_0 v_B + \frac{2}{k}v_B = c^2\delta_\Delta u + \\ 2\sigma_1\delta_{t-}\delta_\Delta u - J_B F_B\phi(v_{rel}) + J_M F\frac{1}{\rho_M H}\end{aligned} \tag{10}$$

Since $v_{rel} = (\delta_{t\cdot}u - v_B)$ and applying the inner product over the bowing distribution, which is the same as applying the analogous interpolantion function operator of the bowing force, $< f, J_B >_{D_M} = I_B f$, it follows:

$$\begin{aligned} g_1\left(v_{rel}, F\right) = I_B J_B F_B\phi(v_{rel}) + \left(\frac{2}{k} + 2\sigma_0\right) - I_B J_M F\frac{1}{\rho_M H} + q = 0, \quad \text{with} \\ q = -\frac{2}{k}I_B\delta_{t-}u + 2\sigma_0 v_B + \frac{2}{k}v_B - c^2 I_B\delta_\Delta u - 2\sigma_1 I_B\delta_{t-}\delta_\Delta u\end{aligned} \tag{11}$$

where the function g_1 only depends on values at the current time step n.

Furthermore, from Equation 6, it follows that the same equality must be true at the next time step as well, $n+1$, i.e.:

$$< u^{n+1}, J_M >_{D_M} = < w^{n+1}, J_T >_{D_T} \tag{12}$$

Now, expanding the operators in Equation 7a and Equation 7b, we can solve for u^{n+1} and w^{n+1}:

$$u^{n+1} = \frac{k^2}{1+\sigma_0 k}\left[c^2\delta_\Delta u^n + \frac{2\sigma_1}{2}\left(\delta_\Delta u^n - \delta_\Delta u^{n-1}\right) - J_B F_B\phi(v_{rel}) + J_M F\frac{1}{\rho_M H} + \frac{2}{k^2}u^n - \frac{1-\sigma_0 k}{k^2}u^{n-1}\right] \tag{13a}$$

$$w^{n+1} = \frac{k^2}{1+\sigma_{0,tube}k}\left[c_{tube}^2\delta_{xx}w^n + \frac{2\sigma_{1,tube}}{2}\left(\delta_{xx}w^n - \delta_{xx}w^{n-1}\right) - J_T F\frac{1}{\rho_T A} + \frac{2}{k^2}w^n - \frac{1-\sigma_{0,tube}k}{k^2}w^{n-1}\right] \tag{13b}$$

We can now take the inner products of u^{n+1} and w^{n+1} with their respective connection distributions J_M and J_T over their respective domains:

$$
\begin{aligned}
&< u^{n+1}, J_M >_{D_M} = \frac{k^2}{1+\sigma_0 k}[c^2 I_M \delta_\Delta u^n + \frac{2\sigma_1}{2}\left(I_M\delta_\Delta u^n - I_M\delta_\Delta u^{n-1}\right) - \\
&I_M J_B F_B \phi(v_{rel}) + I_M J_M F\frac{1}{\rho_M H} + \frac{2}{k^2} I_M u^n - \frac{1-\sigma_0 k}{k^2} I_M u^{n-1}] \\
&< w^{n+1}, J_T >_{D_T} = \frac{k^2}{1+\sigma_{0,tube}k}[c_{tube}^2 I_T \delta_{xx} w^n + \frac{2\sigma_{1,tube}}{2}\left(I_T\delta_{xx}w^n - I_T\delta_{xx}w^{n-1}\right) - \\
&I_T J_T F\frac{1}{\rho_T A} + \frac{2}{k^2} I_T w^n - \frac{1-\sigma_{0,tube}k}{k^2} I_T w^{n-1}]
\end{aligned}
\tag{14}
$$

Having that $< u^{n+1}, J_M >_{D_M} = < w^{n+1}, J_T >_{D_T}$ results in:

$$
\begin{aligned}
&g_2(v_{rel}, F) = \frac{k^2}{1+\sigma_0 k}[c^2 I_M \delta_\Delta u^n + \frac{2\sigma_1}{2}\left(I_M\delta_\Delta u^n - I_M\delta_\Delta u^{n-1}\right) - \\
&I_M J_B F_B \phi(v_{rel}) + I_M J_M F\frac{1}{\rho_M H} + \frac{2}{k^2} I_M u^n - \frac{1-\sigma_0 k}{k^2} I_M u^{n-1}] - \\
&\{\frac{k^2}{1+\sigma_{0,tube}k}[c_{tube}^2 I_T \delta_{xx} w^n + \frac{2\sigma_{1,tube}}{2}\left(I_T\delta_{xx}w^n - I_T\delta_{xx}w^{n-1}\right) - \\
&I_T J_T F\frac{1}{\rho_T A} + \frac{2}{k^2} I_T w^n - \frac{1-\sigma_{0,tube}k}{k^2} I_T w^{n-1}]\} = 0
\end{aligned}
\tag{15}
$$

Now that we have two equations: g_1 and g_2 for our two unknowns (although nonlinear) at time step n: F and v_{rel} we can use a multivariate Newton-Raphson solver for the solution. Such a solver is illustrated in the following:

$$
\begin{bmatrix} v^n_{rel,(i+1)} \\ F^n_{(i+1)} \end{bmatrix} = \begin{bmatrix} v^n_{rel,(i)} \\ F^n_{(i)} \end{bmatrix} - \begin{bmatrix} \frac{\partial g_1}{\partial v_{rel}} & \frac{\partial g_1}{\partial F} \\ \frac{\partial g_2}{\partial v_{rel}} & \frac{\partial g_2}{\partial F} \end{bmatrix}^{-1} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}
\tag{16}
$$

where i is the iteration number.

The elements of the Jacobian matrix follow from simple derivations of the two functions g_1 and g_2 with respect to the unknown variables:

$$
\begin{aligned}
\frac{\partial g_1}{\partial v_{rel}} &= (2/k + 2\sigma_0) + I_B J_B F_B \phi'(v_{rel}) \\
\frac{\partial g_1}{\partial F} &= -\frac{I_B J_M}{\rho_M H} \\
\frac{\partial g_2}{\partial v_{rel}} &= -\frac{k^2}{1+\sigma_0 k} I_M J_B F_B \phi'(v_{rel}) \\
\frac{\partial g_2}{\partial F} &= k^2\left[\frac{I_M J_M}{(1+\sigma_0 k)\rho_M H} + \frac{I_T J_T}{(1+\sigma_{0,tube}k)\rho_T A}\right]
\end{aligned}
\tag{17}
$$