Is Phase Really Needed for Weakly-Supervised Dereverberation ?

Detailed mathematical proofs

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1 Introduction

This document compiles some detailed mathematical proofs as supplementary material to the paper *Is Phase Really Needed for Weakly-Supervised Dereverberation ?* [3]. Section 2 recalls the statistical modeling and notations we use, as well as some additional details on the *Statistical Wave Field Theory* (hypotheses and physical interpretation). Then, complete descriptions of proofs are presented in Section 3.

2 Framework and notations

Introduced in [1], the Statistical Wave Field Theory (SWFT) describes the statistics of reverberation in the asymptotic regime of high frequencies and long times. Following the framework of SWFT, the statistics of a Room Impulse Response (RIR) are obtained by the randomization of the emitting source's position. Precisely, if h is the RIR, H its Fourier transform and V a 3D bounded domain representing the room, the following hypotheses are considered:

(H1) *High frequencies/long times:* $H(f)$ is studied for $f \gg f_s$ where f_s is the Schroeder frequency. Similarly, $h(t)$ is studied for $t \gg t_m$ where t_m is the mixing time.

(H2) *Uniform source distribution:* The source's position x_0 is a random variable, with uniform distribution over V. Moreover, the emitted impulse is isotropic.

(H3) *Spatial stationarity:* The image sources' distribution (or more specifically, the *B-function* [1]) has stationary and isotropic 1^{st} and 2^{nd} order statistics over space.

Hypotheses **(H1)**, **(H2)** and **(H3)** are linked to the properties of *mixing rooms* [2]. In practice, those assumptions are generally satisfied except for rooms presenting strong symmetries (shoebox, spherical or cylindrical rooms, etc).

Solving the waves equation under **(H1)**, **(H2)** and **(H3)**, SWFT leads to a time-frequency description of the RIR statistics, generalizing the so-called Polack model [2]. In formal words:

- $\forall t, h(t)$ is real, centered and Gaussian,
- The Wigner distribution of h is

$$W_h(t,f) := \int_{\mathbb{R}} \mathbb{E}\left[h\left(t+\frac{\tau}{2}\right) h\left(t-\frac{\tau}{2}\right)\right] e^{-2i\pi f\tau} \mathrm{d}\tau \tag{1}$$

$$= \mathbb{1}_{\mathbb{R}^+}(t)[B(\cdot)e^{-\alpha(\cdot)t} \star \operatorname{sinc}(4t\cdot)](f) \tag{2}$$

  where B is a frequency-dependent amplitude, α a frequency-dependent energy decay rate, and $\operatorname{sinc}(x) := \frac{\sin(\pi x)}{\pi x}$.

The sinc term is a border effect due to the discontinuity of h at $t = 0$. The following equation holds exactly for $t < 0$ and approximately for large values of $t > 0$:

$$W_h(t,f) \simeq \mathbb{1}_{\mathbb{R}^+}(t)B(f)e^{-\alpha(f)t}. \tag{3}$$

By definition of the Wigner distribution, the autocorrelation function is the inverse Fourier transform of W_h over f:

$$\gamma_h(t,\tau) = \mathbb{1}_{\mathbb{R}^+}(t) \int_{\mathbb{R}} B(f)e^{-\alpha(f)t+2i\pi f\tau} \mathrm{d}f \tag{4}$$

Moreover, because h is Gaussian and centered, any scalar product $\langle h, \varphi\rangle := \int h\varphi$ is a centered gaussian random variable in $\mathbb{R}$, and the following covariance formula holds:

$$\operatorname{Cov}(\langle h,\varphi\rangle, \langle h,\psi\rangle) := \iint_{\mathbb{R}^2} \varphi\left(t+\frac{\tau}{2}\right)\psi\left(t-\frac{\tau}{2}\right)\gamma_h(t,\tau)\mathrm{d}\tau\mathrm{d}t \tag{5}$$

$$= \int_0^{+\infty}\int_{\mathbb{R}}\int_{\mathbb{R}} \varphi\left(t+\frac{\tau}{2}\right)\psi\left(t-\frac{\tau}{2}\right) B(f)e^{-\alpha(f)t+2i\pi f\tau}\mathrm{d}f\mathrm{d}\tau\mathrm{d}t \tag{6}$$

3 Detailed Proofs

3.1 Asymptotic distribution of the RIR in the Fourier domain

The aim of this section is to provide detailed computations that stand behind the main result presented in [3].

Proposition 1. *If we consider the RIR h to be randomly sampled according to the generalized Polack model, as in Section 2, with parameters $\alpha(f)$ and $B(f)$, and H its Fourier transform, then, asymptotically when $f \to +\infty$,*

$$H(f) \sim \mathcal{N}_{\mathbb{C}}\left(0, \frac{B(f)}{\alpha(f)}\right). \tag{7}$$

Proof. First, we decompose the Fourier transform of h in real and imaginary parts:

$$H(f) = \langle h, c_f \rangle - i\langle h, s_f \rangle \tag{8}$$

where $c_f(t) := \cos(2\pi f t)$ and $s_f(t) := \sin(2\pi f t)$. As a bi-dimensional random variable in the complex plane, $H(f)$ is Gaussian, centered, with a 2×2 covariance matrix $\Sigma(f) = \begin{pmatrix} \sigma_+^2(f) & C(f) \\ C(f) & \sigma_-^2(f) \end{pmatrix}$. Its coefficients can be computed thanks to the covariance formula (6). For instance:

$$\sigma_+^2(f) = \mathrm{Var}(\langle h, c_f \rangle) = \int_0^{+\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} c_f\left(t + \frac{\tau}{2}\right) c_f\left(t - \frac{\tau}{2}\right) B(\xi) e^{-\alpha(\xi)t + 2i\pi\xi\tau} \mathrm{d}\xi \mathrm{d}\tau \mathrm{d}t. \tag{9}$$

Using the formula $\cos(a)\cos(b) = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$, it yields

$$\int_{\mathbb{R}} c_f\left(t + \frac{\tau}{2}\right) c_f\left(t - \frac{\tau}{2}\right) e^{2i\pi\xi\tau} \mathrm{d}\tau = \frac{1}{2}\left[\int_{\mathbb{R}} c_f(\tau) e^{-2i\pi\xi\tau} \mathrm{d}\tau + \int_{\mathbb{R}} c_f(2t) e^{-2i\pi\xi\tau} \mathrm{d}\tau\right] \tag{10}$$

$$= \frac{1}{4}\left[\delta(\xi + f) + \delta(\xi - f) + 2c_f(2t)\delta(\xi)\right]. \tag{11}$$

Injecting this expression in (9), and knowing that $B(-f) = B(f)$ and $\alpha(-f) = \alpha(f)$, $\sigma_+^2(f)$ boils down to:

$$\sigma_+^2(f) = \frac{B(f)}{2} \int_0^{+\infty} e^{-\alpha(f)t} \mathrm{d}t + \frac{B(0)}{2} \int_0^{+\infty} e^{-\alpha(0)t} c_f(2t) \mathrm{d}t. \tag{12}$$

The first integral can be computed directly: $\int_0^{+\infty} e^{-\alpha(f)t} \mathrm{d}t = \frac{1}{\alpha(f)}$. Similarly, since $c_f(2t) = \frac{e^{4i\pi f t} + e^{-4i\pi f t}}{2}$, we obtain for the second integral:

$$\int_0^{+\infty} e^{-\alpha(0)t} c_f(2t) \mathrm{d}t = \frac{1}{2}\left(\frac{1}{\alpha(0) - 4i\pi f t} + \frac{1}{\alpha(0) + 4i\pi f t}\right) \tag{13}$$

$$= \frac{\alpha(0)}{\alpha^2(0) + (4\pi f)^2} \tag{14}$$

$$= \frac{1}{\alpha(0)\left[1 + \left(\frac{4\pi f}{\alpha(0)}\right)^2\right]}. \tag{15}$$

In the end :

$$\sigma_+^2(f) = \frac{B(f)}{2\alpha(f)} + \frac{B(0)}{2\alpha(0)\left[1 + \left(\frac{4\pi f}{\alpha(0)}\right)^2\right]}. \tag{16}$$

With a similar reasoning, one can obtain:

$$\sigma_-^2(f) = \frac{B(f)}{2\alpha(f)} - \frac{B(0)}{2\alpha(0)\left[1 + \left(\frac{4\pi f}{\alpha(0)}\right)^2\right]} \tag{17}$$

and

$$C(f) = \frac{2\pi f B(0)}{\alpha^2(0)\left[1 + \left(\frac{4\pi f}{\alpha(0)}\right)^2\right]}. \tag{18}$$

All in all, when $f \to +\infty$, we have:

- $\sigma_\pm^2(f) \simeq \frac{B(f)}{2\alpha(f)}$,
- $C(f) \simeq 0$,

which means that $\Sigma(f) \simeq \frac{B(f)}{2\alpha(f)} I$, where I is the 2×2 identity matrix, and ends the proof. $\square$

3.2 Autocorrelation in the Fourier domain

In this section is demonstrated the second result used in [3]:

Proposition 2. *Let h be a general Polack RIR, with acoustics parameters α and B. Then its Fourier transform's autocorrelation function is equal to:*

$$\Gamma_H(f, \xi) := \mathbb{E}\left[H\left(f + \frac{\xi}{2}\right) H^*\left(f - \frac{\xi}{2}\right)\right] = \frac{B(f)}{\alpha(f)} \cdot \frac{1 - i\frac{2\pi\xi}{\alpha(f)}}{1 + \left(\frac{2\pi\xi}{\alpha(f)}\right)^2} \tag{19}$$

Proof. To prove this proposition, we use the properties of the Wigner distribution:

$$\Gamma_H(f, \xi) = \int_{\mathbb{R}} W_h(t, f) e^{-2i\pi\xi t} \mathrm{d}t \tag{20}$$

$$= B(f) \int_0^{+\infty} e^{-(\alpha(f) + 2i\pi\xi)t} \mathrm{d}t \tag{21}$$

$$= \frac{B(f)}{\alpha(f) + 2i\pi\xi} \tag{22}$$

$$= \frac{B(f)}{\alpha(f)} \cdot \frac{1 - i\frac{2\pi\xi}{\alpha(f)}}{1 + \left(\frac{2\pi\xi}{\alpha(f)}\right)^2} \tag{23}$$

$\square$

References

[1] Roland Badeau. "Statistical wave field theory". In: *J. Acoust. Soc. Am.* 156.1 (July 2024), pp. 573–599. ISSN: 0001-4966. DOI: 10.1121/10.0027914. eprint: https://pubs.aip.org/asa/jasa/article-pdf/156/1/573/20064911/573_1_10.0027914.pdf. URL: https://doi.org/10.1121/10.0027914.

[2] Jean-Dominique Polack. "La transmission de l'energie sonore dans les salles (in french)". PhD thesis. Université du Maine, 1988, 1 vol. (231 P.) URL: `http://www.theses.fr/1988LEMA1011`.

[3] Marius Rodrigues et al. *Is Phase Really Needed for Weakly-Supervised Dereverberation ?* Submitted to ICASSP 2026.