CS450: Numerical Analysis
Solutions to HW 5

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Problem 1:

a

$$p(t) = x_1 + x_2 t + x_3 t^2$$

$$Ax = \begin{bmatrix} 1 & t_1 & t_1^2 \\ 1 & t_2 & t_2^2 \\ 1 & t_3 & t_3^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to p(t) = t^2$$

b

$$p(t) = y_1 \frac{(t-t_2)(t-t_3)}{(t_1-t_2)(t_1-t_3)} + y_2 \frac{(t-t_1)(t-t_3)}{(t_2-t_1)(t_2-t_3)} + y_3 \frac{(t-t_1)(t-t_2)}{(t_3-t_1)(t_3-t_2)} = \frac{(t-1)t}{2} + 0 + \frac{t(t+1)}{2} = t^2$$

c

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & t_2 - t_1 & 0 \\ 1 t_3 - t_1 & (t_3-t_1)(t_3-t_2) & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \to p(t) = 1 + (-1)(t+1) + 1(t+1)(t) = t^2$$

Problem 2:

a

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 29 \\ 65 \\ 125 \end{bmatrix} \to x = \begin{pmatrix} 5 \\ 2 \\ 3 \\ 1 \end{pmatrix}$$

$$p(t) = 5 + 2t + 3t^2 + t^3$$

b

$$p(t) = y_1 \frac{(t-t_2)(t-t_3)(t-t_4)}{(t_1-t_2)(t_1-t_3)(t_1-t_4)} + y_2 \frac{(t-t_1)(t-t_3)(t-t_4)}{(t_2-t_1)(t_2-t_3)(t_1-t_4)}$$
$$+ y_3 \frac{(t-t_1)(t-t_2)(t-t_4)}{(t_3-t_1)(t_3-t_2)(t_3-t_4)} + y_4 \frac{(t-t_1)(t-t_2)(t-t_3)}{(t_4-t_1)(t_4-t_2)(t_4-t_3)} = 5 + 2t + 3t^2 + t^3$$

Problem 3:

Since derivatives of $sin(t)$ are trigonometric functions, which vary from -1 to 1: $M = |cos(t)| = 1$

$$h = \frac{(\frac{\pi}{2})}{4} = \frac{\pi}{8}$$
$$error = \frac{Mh^n}{4n}$$

For $n = 5 \rightarrow error = \frac{1(\frac{\pi}{8})^5}{4*5} \approx 4.7 * 10^{-4}$
Output:

```
Ax=y
A=
[[ 0.          0.          0.          0.          1.        ]
 [ 0.02378152  0.06055913  0.15421257  0.39269908  1.        ]
 [ 0.38050426  0.48447307  0.61685028  0.78539816  1.        ]
 [ 1.92630283  1.63509662  1.38791312  1.17809725  1.        ]
 [ 6.08806819  3.87578459  2.4674011   1.57079633  1.        ]]
y=
[ 0.          0.38268343  0.70710678  0.92387953  1.        ]
x
[ 0.02871423 -0.20358546  0.01995143  0.99631698  0.        ]
error points
[ 0.08156991  0.41617974  0.16104572  0.40593701  0.96036883  0.45437729
  0.24161395  0.90819928  0.28901118  0.43435661]
errors
[  1.86469987e-04   1.94357425e-05   2.11471087e-04   1.12468444e-05
   7.11447481e-05   4.56490243e-05   1.54886499e-04   5.82384784e-05
   1.05610800e-04   3.27894494e-05]
max error
0.000211471086827
```

A is a vandermonde matrix. x represents the polynomial coefficients in decreasing order starting with t^4. The max error on the points sampled is $2.1 * 10^{-4}$ which is below the calculated value. To find the number of points required for an error of 10^{-10}:

$$10^{-10} = \frac{1(\frac{\pi}{8})^n}{4n}$$

Solving for n yields: $n \approx 19.94 \rightarrow n = 20$

Problem 4:

$$Q_n(f) = \sum_{i=1}^{n} w_i f(x_i)$$

$$= \sum_{i=1}^{n} (\int_a^b l_i(x)dx) f(x_i)$$

$$= \int_a^b (\sum_{i=1}^{n} l_i(x) f(x_i)) dx$$

But $p(x) = \sum_{i=1}^{n} l_i(x) f(x_i) \approx f(x)$

$$= \int_a^b (p(x))dx \approx \int_a^b (f(x))dx = I(f)$$

Problem 5:

$$avg = \frac{f(x+h) - f(x)}{2h} + \frac{f(x) - f(x-h)}{2h} = \frac{f(x+h) - f(x-h)}{2h}$$

$$f(x+h) \approx f(x) + f'(x)h + f''(x)h^2 + \frac{f'''(x)h^3}{6} + ...$$

$$f(x-h) \approx f(x) - f'(x)h + f''(x)h^2 - \frac{f'''(x)h^3}{6} + ...$$

$$\frac{f(x+h) - f(x-h)}{2h} = \frac{f(x+h) - f(x-h)}{2h} - \frac{f'''(x)h^2}{6} + ...$$

$$= \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

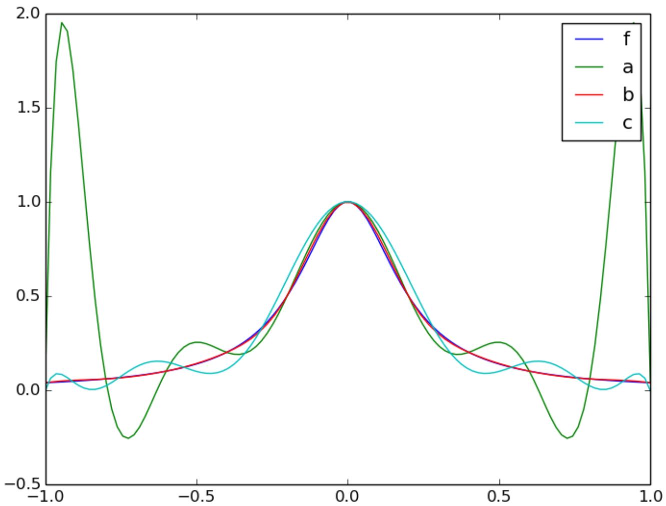
Problem 6:

For forward difference: $F(h) = a_0 + a_1 h + O(h^2)$, so $p = 1$.
$q = \frac{.2}{.1} = 2$

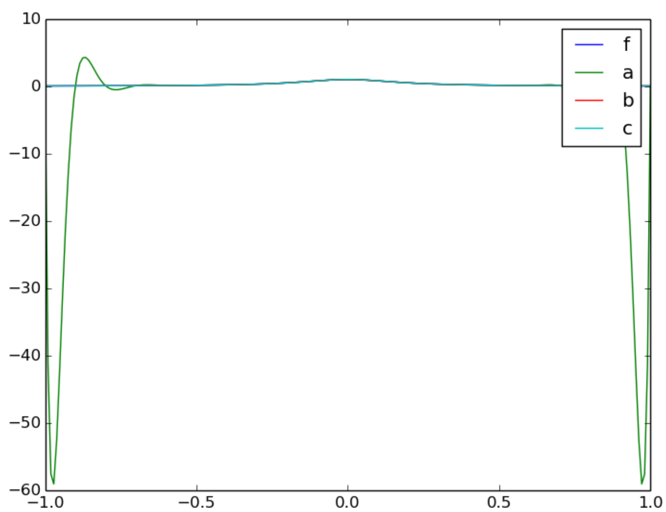
$$F(0) = a_0 = F(h) + \frac{F(h) - F(\frac{h}{2})}{q^{-p} - 1} = F(h) + \frac{F(h) - F(\frac{h}{2})}{2^{-1} - 1} = 2F(\frac{h}{2}) - F(h) = 2(-.9091) - (-.8333) = -.9849$$

Problem 7:

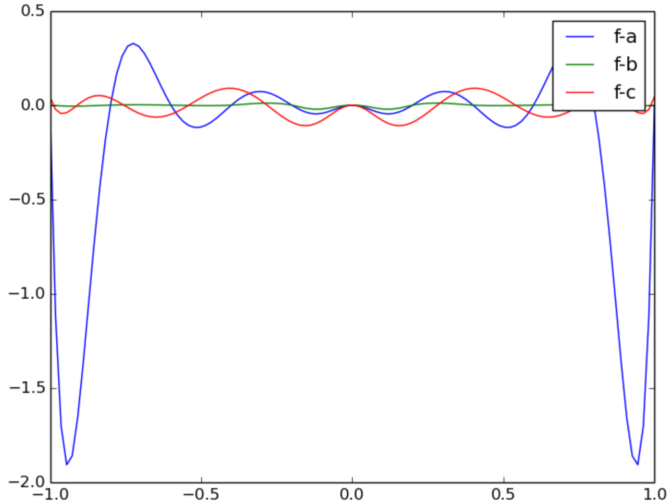
Interpolants and function (n = 11)



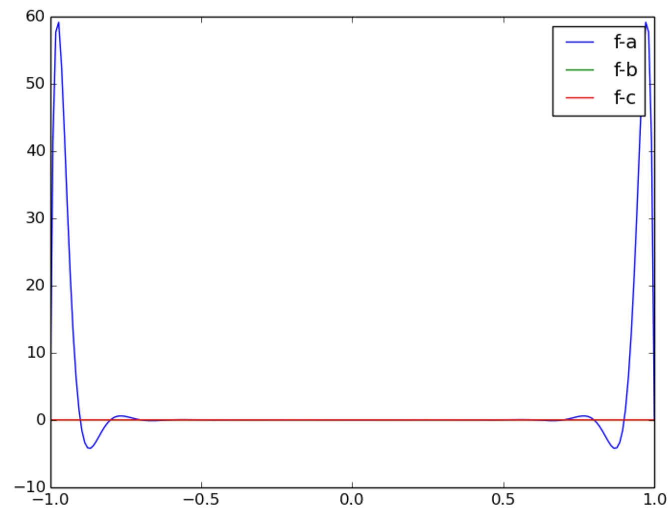
Interpolants and function (n = 21)



Error (n = 11)



Error (n = 21)



Max errors

```
1.9079060982 #a n = 11
0.0215283590725 #b n = 11
0.109147304176 #c n = 11
59.1223686715 #a n = 21
0.00312636097646 #b n = 21
0.0152923024513 #c n = 21
```

Method b with n = 21 is the most accurate. Method a with n = 21 is the least accurate.