

# Lecture 17 - Rotational Dynamics

## Torque

In the same way as a force causes linear acceleration,  $\vec{F} = m\vec{a}$ , there must be an analogous quantity and equation related to angular acceleration. From experience, for example when we [use a wrench](#) (or a spanner) we can expect that the angular equivalent of Force, which is called **Torque**, will depend on the amount of Force applied, the distance at which it is applied and the direction of its application.



We define the torque due to a force  $F$  applied at a distance  $R$  from a pivot point as either

$$\tau \equiv RF_{\perp} \text{ or } \tau \equiv R_{\perp}F$$

and we can calculate the magnitude of the torque using

$$\tau = RF \sin \theta$$

The units of torque are Nm

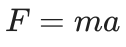
## Production of Torque in and engine

You have probably heard of the word torque mostly in the context of car engines.

See in [Wikipedia](#) how the torque on the crankshaft is produced in a [4 stroke engine](#).

## Acceleration Due to Torque

We now want to look at the relationship between the torque which is applied and the angular acceleration it generates. Let's look at the case of a force  $F$  applied at 90 degrees to an object of mass  $m$  attached to a pivot point at distance  $R$



Newton's second law tells us that

$$F = ma$$

If we multiply both sides by R we get

$$FR = maR$$

and by remembering that  $a = \alpha R$  we get

$$\tau = mR^2\alpha$$

## Moment of Inertia

In an object with a common angular acceleration which consists of many masses at various distances from the center of rotation, each of which may be subject to a torque we can say that

$$\sum_i \tau_i = (\sum_i m_i R_i^2) \alpha$$

As we discussed for forces, different masses can exert internal torques on each other, but when summed over all the masses these cancel due to Newton's Third Law, meaning that our sum over all the torques on the masses  $\sum_i \tau_i$  becomes the sum of the external torques

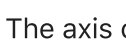
$\sum \tau$ . We call the quantity  $\sum_i m_i R_i^2$  the moment of inertia  $I$  and write the rotational equivalent of Newton's Second Law as

$$\sum \tau = I\alpha$$

This equation is valid for a rigid object around a fixed axis (ie. for this to be valid the masses can't move around with respect to the object during the motion as they would if the object was flexible, and the axis must stay in the same position with respect to the masses rotating around it).

We should note that the distance  $R$  is the distance of each mass with respect to the axis of rotation, which need not be the center of mass, and is not an intrinsic property of an object!

## Two Weights on a Thin Bar



The moment of inertia of two point weights  $m_1$  and  $m_2$  on a bar at distance  $R_1$  and  $R_2$  from the axis of rotation is

$$I = m_1 R_1^2 + m_2 R_2^2$$

The axis of rotation coincides with the center of mass only if  $m_1 R_1 = m_2 R_2$

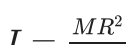
## Moment of Inertia for Extended Objects

So far our definition of moment of inertia is really only practical for systems comprised of one or more point like objects at a distance from an axis of rotation.

For extended objects we are much better served by considering an object as being made up of infinitesimally small mass elements, each a distance  $R$  from the axis of rotation, and integrating over these mass elements to find the moment of inertia.

$$I = \sum_i m_i R_i^2 \rightarrow I = \int R^2 dm$$

## Hoop, thin walled cylinder and solid cylinder



Considering rotation axis through the center of the circle

$$I = \int R^2 dm$$

Hoop

$$I = \frac{MR^2}{2\pi} \int_0^{2\pi} d\theta = MR^2$$

Thin walled cylinder

$$I = \frac{MR^2}{2\pi h} \int_0^h \int_0^{2\pi} d\theta dz = MR^2$$

Solid Cylinder

$$I = \frac{M}{\pi h R_C^2} \int_0^{R_C} \int_0^h \int_0^{2\pi} R^3 d\theta dz dR = \frac{M}{\pi h R_C^2} \frac{2\pi h R_C^4}{4} = \frac{1}{2} MR_C^2$$

## Parallel Axis Theorem

Usually a rotation axis that passes through the center of mass of an object will be one of the easiest to find the moment of inertia for, because as we saw in the last lecture the center of mass usually reflects the symmetry of the object. More moments of inertia on [wikipedia](#).

If we know the moment of inertia of an object around an axis that passes through its center of mass there is a theorem that can help us find the moment of inertia around a different axis parallel to the axis through the COM.

If the axis of rotation is a distance  $h$  from the axis through the COM then

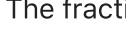
$$I = I_{COM} + Mh^2$$

where M is the total mass of the object.

As an example, a solid sphere stuck to a turning pole will have moment of inertia  
$$I = \frac{2}{5}MR^2 + MR^2 = \frac{7}{5}MR^2$$

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## When can we approximate a mass as a point?



Frequently we will want to approximate a mass at some distance from its center of rotation as a point mass. i.e we would like to simply write

$$I = mR_1^2$$

The parallel axis theorem tells us that in fact (if the mass is spherical and solid)

$$I = mR_1^2 + \frac{2}{5}mR_2^2$$

The fractional error introduced by the above approximation is

$$\frac{(I_{exact} - I_{approx})}{I_{exact}} = \frac{\frac{2}{5}mR_2^2}{mR_1^2 + \frac{2}{5}mR_2^2} = \frac{\frac{2}{5}}{\frac{R_1^2}{R_2^2} + \frac{2}{5}}$$

If we want to be accurate to say 1% we only need  $\frac{R_1}{R_2} \approx \sqrt{40} \approx 6$

At the end of the lecture	Rotational Kinetic Energy	Conservation of energy with rotation	Hoop and disk solution	Work done by a torque	Power and Torque	Next
</h>	Each piece of mass in a rotation problem that has velocity $v$ should have kinetic energy	Recall from the last lecture that for a rolling object $v = \omega r$	Hoop $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}mr^2\omega^2 = mv^2$ $mgh = mv^2$ $v = \sqrt{gh}$	$W = \int \vec{F} \cdot d\vec{l} = \int F_{\perp} d\theta = \int_{\theta_1}^{\theta_2} \tau d\theta$ $\tau = I\alpha$ $= I \frac{d\omega}{dt}$ $W = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2}I\omega^2 - \frac{1}{2}I\omega_1^2$	$P = \frac{dW}{dt} = \tau\omega$ This equation can help us understand the two "figures of merit" often given for a car engine: horsepower and torque	<ul style="list-style-type: none"><li>Next Lecture - Angular momentum</li><li>Next Lecture - Advantages (and interesting rotational motion exam</li></ul>
The surface of the torus on the pulley will be given by $\Sigma r$ and as we saw the moment of inertia of the pulley is $I = \frac{1}{2}MR^2$	$K = \frac{1}{2}mv^2$ and if we sum over all the masses: $K = \frac{1}{2}m_i v_i^2$	The kinetic energy of a rolling object is therefore $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2}I\frac{v^2}{r^2}$	Disk $K = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}mv^2 + \frac{1}{2} \frac{1}{2} m r^2 \omega^2 = \frac{3}{4}mv^2$ $mgh = \frac{3}{4}mv^2$ $v = \sqrt{\frac{4}{3}gh}$	Remember that for a sliding object (without friction) $v = \sqrt{2gh}$	Solid sphere $I = \frac{2}{5}mr^2$	Hollow sphere $I = \frac{2}{3}mr^2$
As we did before we noticed we shouldn't write Newton's Second Law for the two wheels	$m_2 \rightarrow T_2$ $m_1 \rightarrow T_1$					
$T_2 = \frac{1}{2}Mg$						
$a =$						
$a =$						
As we did before we noticed we shouldn't write Newton's Second Law for the two wheels						