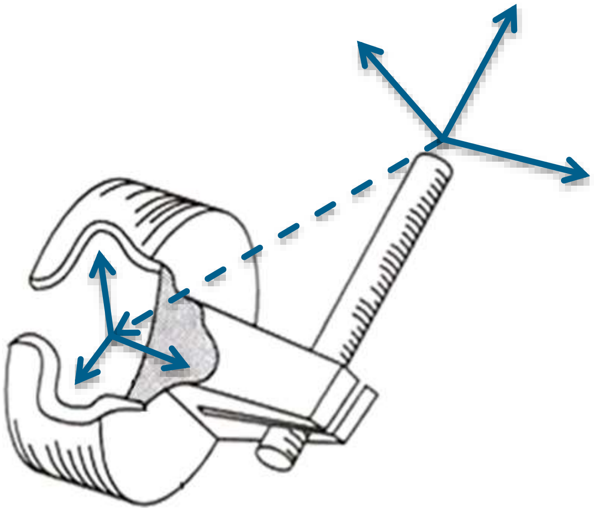
Robot modelling III



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Contents

- Speed analysis of the parts of a manipulator when the manipulated variable of the joints is changed
- Conversion of speeds to other CS
  - Coordinate systems are often referred to as frames
- Calculate velocity of a part/component as a superposition of translational and rotational velocity
- Relationship between joint and Cartesian speed of the end effector (Jacobian matrix)
- Investigation of forces and moments with a rigid kinematic chain

Velocity Vector

- Free vector (no starting point; only magnitude and direction)
  - Only rotation is considered
- Derivation of a position vector with respect to time:
  $^{B}\vec{v}_q = \frac{d}{dt}\,{}^{B}\vec{q}$
- Conversion into rotated CS: $^{A}\vec{v}_q = {}^{A}_{B}R \cdot {}^{B}\vec{v}_q$

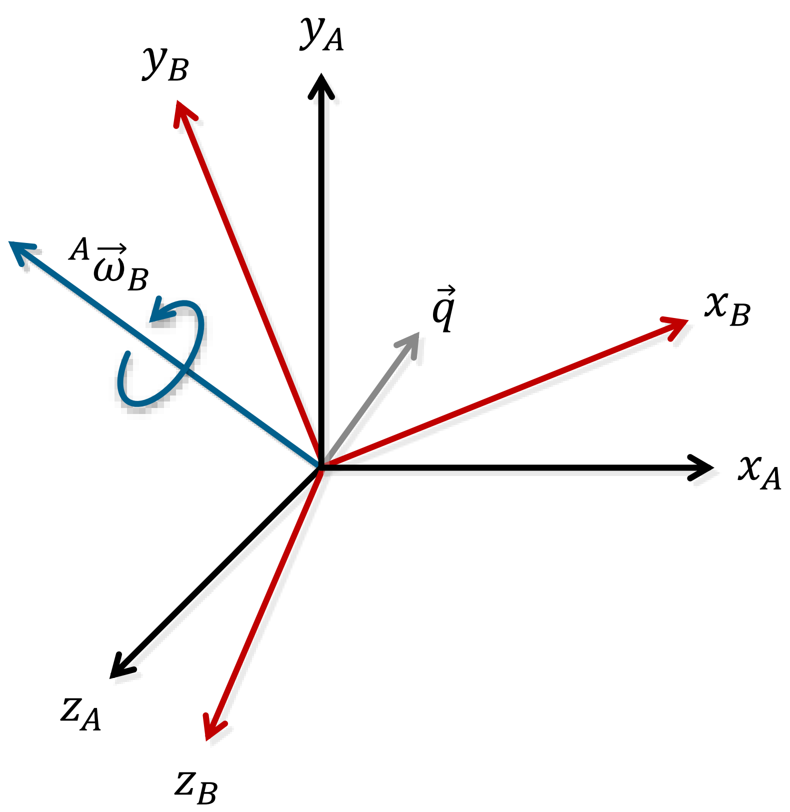
Linear Velocity

- Origin OB of the system B moves with a linear velocity ${}^{A}\vec{v}_{OB}$ relative to system A
- Point ${}^{B}\vec{q}$ represented in system B moves with a linear velocity ${}^{B}\vec{v}_{q}$
- System B was created from system A by rotation ${}^{A}_{B}R$
- Linear velocity of the point ${}^{B}\vec{q}$ relative to system A:

$$ {}^{A}\vec{v}_{q} = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_{q} $$

Rotational Velocity

- System A and system B share a common origin
- Linear velocity between the systems is 0: ${}^{A}\vec{v}_{OB} = 0$
- ${}^{B}\vec{q}$ is represented in system B
- System B rotates about an axis through the common origin of A and B at a rotational speed ${}^{A}\vec{\omega}_{B}$



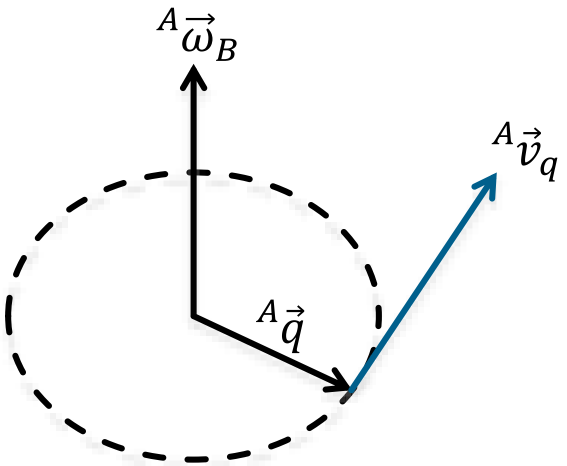
Rotational Velocity

- Speed of the point $\vec{q}$: ${}^{A}\vec{v}_q = {}^{A}\vec{\omega}_B \times {}^{A}\vec{q}$
- Considering the linear velocity

$$ {}^{A}\vec{v}_q = {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} $$

- Linear and rotational velocity

$$ {}^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} $$



Point Velocity in Another Reference System

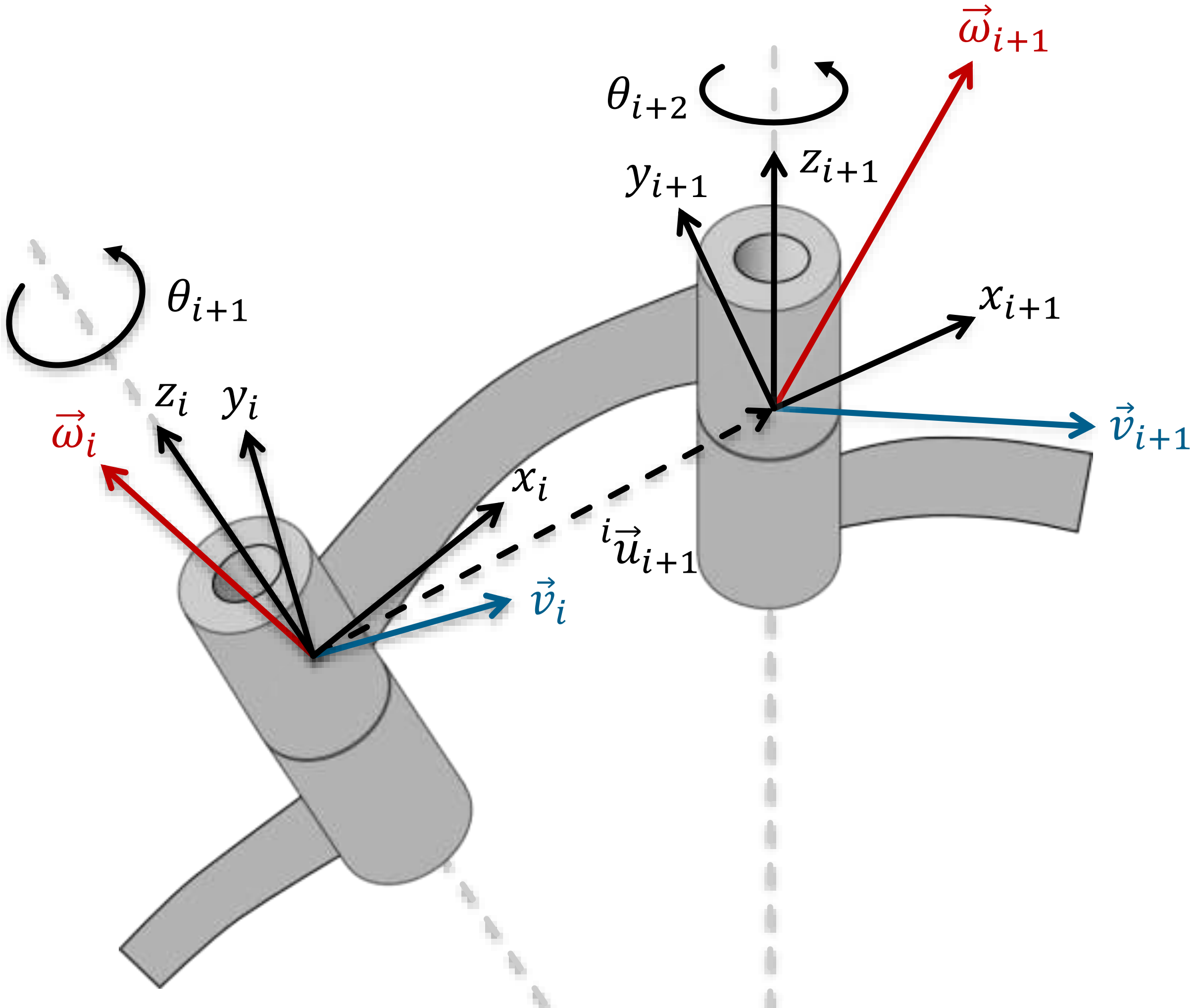
$${}^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R \cdot {}^{B}\vec{v}_q + {}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q}$$

- ${}^{A}\vec{v}_{OB}$: Translational velocity of origin OB in system A
- ${}^{A}_{B}R \cdot {}^{B}\vec{v}_q$: Translational velocity of the point ${}^{B}\vec{q}$ in the system B transformed to the reference system A
- ${}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q}$: Translational point velocity due to the rotation of the system B compared to A

Velocity of the Robot Parts

- Velocity of the end effector of a robot with n joints is calculated from the kinematic structure and all the members involved in the movement
- Velocity of a part consists of the velocity of its fixed CS and rotational and translational velocity of the part
- Velocity of the end effector in the base coordinate system is determined by successive calculation of the velocities of the parts from the base
- Velocity of the part $i + 1$ is the sum of the velocity of member i and the component resulting from relative motion between i and $i + 1$
  - Attention: Both summands must be in the same coordinate system!

Coordinate System and Identifiers



Rotational Velocity at Rotational Joints

- Let joint $i+1$ be a rotational join with degree of freedom θ_{i+1}
- ${}^{i}\vec{\omega}_{i+1} = {}^{i}\vec{\omega}_{i} + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i}$
  - ${}^{i}\vec{\omega}_{i}$: Rotational velocity of the part i
  - $\dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i}$: Component by rotation of joint $i+1$
  - $\dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} = (0 \quad 0 \quad \dot{\theta}_{i+1})^T$
- Transformation of ${}^{i}\vec{\omega}_{i+1}$ in the system $i+1$ by multiplying with ${}^{i+1}_{i}R$: ${}^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}\vec{\omega}_{i} + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} \right)$

Linear Velocity at Rotational Joints

- For the translational speed of the origin of coordinate system $i+1$ represented in system i it holds:

$$ {}^{i}\vec{v}_{i+1} = {}^{i}\vec{v}_{i} + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} $$

- Represented in system $i+1$

$$ {}^{i+1}\vec{v}_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}\vec{v}_{i} + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} \right) $$

Velocity of Linear Joints

- Let joint i be a translational joint with degree of freedom d_i

- Rotational velocity:

$$^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\vec{\omega}_i$$

- Translational velocity:

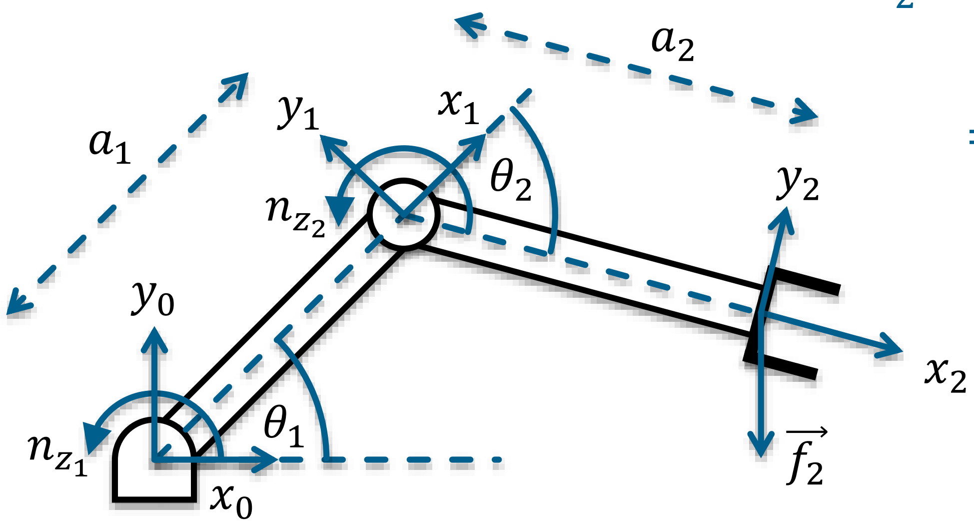
$$^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R \cdot \left({}^{i}\vec{v}_i + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} + \dot{d}_{i+1} \, {}^{i}\vec{e}_{z_i} \right)$$

Example: Planar Robot Arm

- Calculation of the rotation matrices required for the velocity ${}^{i+1}_{i}R = {}^{i}_{i+1}R^T$
- Rotations and translations separately

$$
{}^{0}_{1}A = T_{z_0}(0) \cdot R_{z_0}(\theta_1) \cdot R_{x_1}(0°) \cdot T_{x_1}(a_1)
= \begin{bmatrix} c_1 & -s_1 & 0 & c_1 a_1 \\ s_1 & c_1 & 0 & s_1 a_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{1}_{2}A = T_{z_1}(0) \cdot R_{z_1}(\theta_2) \cdot R_{x_2}(0°) \cdot T_{x_2}(a_2)
= \begin{bmatrix} c_2 & -s_2 & 0 & c_2 a_2 \\ s_2 & c_2 & 0 & s_2 a_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Example: Planar Robot Arm

- Specification: ${}^{0}\vec{v}_0 = \vec{0},\ {}^{0}\vec{\omega}_0 = \vec{0}$
- Derive by:

$$\boxed{\begin{aligned} {}^{i+1}\vec{\omega}_{i+1} &= {}^{i+1}_{i}R \cdot \left({}^{i}\vec{\omega}_i + \dot{\theta}_{i+1} \cdot {}^{i}\vec{e}_{z_i} \right) \\ {}^{i+1}\vec{v}_{i+1} &= {}^{i+1}_{i}R\left({}^{i}\vec{v}_i + {}^{i}\vec{\omega}_{i+1} \times {}^{i}\vec{u}_{i+1} \right) \end{aligned}}$$

$$\begin{aligned} {}^{0}\vec{\omega}_1 &= {}^{0}\vec{\omega}_0 + \dot{\theta}_1\, {}^{0}\vec{e}_{z_0} \\ &= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^{1}\vec{\omega}_1 &= {}^{1}_{0}R \cdot {}^{0}\vec{\omega}_1 \\ &= \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \end{aligned}$$

$$\begin{aligned} {}^{1}\vec{v}_1 &= {}^{1}_{0}R\left({}^{0}\vec{v}_0 + {}^{0}\vec{\omega}_1 \times {}^{0}\vec{u}_1 \right) \\ &= \begin{pmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} \times \begin{pmatrix} c_1 a_1 \\ s_1 a_1 \\ 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} 0 \\ a_1 \dot{\theta}_1 \\ 0 \end{pmatrix} \end{aligned}$$

Example: Planar Robot Arm

$$
\begin{aligned}
{}^{1}\vec{\omega}_2 &= {}^{1}\vec{\omega}_1 + \dot{\theta}_2\ {}^{1}\vec{e}_{z_1} \\
&= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}\vec{\omega}_2 &= {}^{2}_{1}R \cdot {}^{1}\vec{\omega}_2 \\
&= \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
{}^{2}\vec{v}_2 &= {}^{2}_{1}R\left({}^{1}\vec{v}_1 + {}^{1}\vec{\omega}_2 \times {}^{1}\vec{u}_2 \right) \\
&= {}^{2}_{1}R \cdot \left[\begin{pmatrix} 0 \\ a_1\dot{\theta}_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} \times \begin{pmatrix} c_2 a_2 \\ s_2 a_2 \\ 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -s_2 a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ a_1\dot{\theta}_1 + c_2 a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} s_2 a_1 \dot{\theta}_1 \\ a_1 c_2 \dot{\theta}_1 + a_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{pmatrix}
\end{aligned}
$$

Example: Planar Robot Arm

TCP linear velocity with respect to the base coordinate system

$$ {}^{0}_{2}R = {}^{0}_{1}R \cdot {}^{1}_{2}R = \begin{pmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} $$

$$ {}^{0}\vec{v}_2 = {}^{0}_{2}R \; {}^{2}\vec{v}_2 = \begin{pmatrix} -s_1 a_1 \dot{\theta}_1 - s_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ c_1 a_1 \dot{\theta}_1 + c_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{pmatrix} $$

$$ {}^{0}\vec{\omega}_2 = {}^{0}_{2}R \; {}^{2}\vec{\omega}_2 = \begin{pmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{pmatrix} $$

Application of Jacobian Matrix

Let $\vec{y} = f(\vec{x})$ with $\vec{x} \in \mathbb{R}^m, \vec{y} \in \mathbb{R}^n$.

$$\begin{aligned} y_1 &= f_1(x_1, x_2, \dots, x_m) \\ y_2 &= f_2(x_1, x_2, \dots, x_m) \\ &\vdots \\ y_n &= f_n(x_1, x_2, \dots, x_m) \end{aligned}$$

$$\begin{aligned} dy_1 &= \frac{df_1}{dx_1} dx_1 + \frac{df_1}{dx_2} dx_2 + \cdots + \frac{df_1}{dx_m} dx_m \\ dy_2 &= \frac{df_2}{dx_1} dx_1 + \frac{df_2}{dx_2} dx_2 + \cdots + \frac{df_2}{dx_m} dx_m \\ &\vdots \\ dy_n &= \frac{df_n}{dx_1} dx_1 + \frac{df_n}{dx_2} dx_2 + \cdots + \frac{df_n}{dx_m} dx_m \end{aligned}$$

Application of Jacobian Matrix

- Vector notation

$$\begin{pmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_n \end{pmatrix} = \begin{pmatrix} \frac{df_1}{dx_1} & \frac{df_1}{dx_2} & \cdots & \frac{df_1}{dx_m} \\ \frac{df_2}{dx_1} & \frac{df_2}{dx_2} & \cdots & \frac{df_2}{dx_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{df_n}{dx_1} & \frac{df_n}{dx_2} & \cdots & \frac{df_n}{dx_m} \end{pmatrix} \cdot \begin{pmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_m \end{pmatrix} = J(\vec{x})d\vec{x}$$

- $d\vec{y} = df(\vec{x}) = \frac{df(\vec{x})}{d\vec{x}} d\vec{x} = J(\vec{x})d\vec{x}$ with Jacobian Matrix $J(\vec{x}) = \frac{df(\vec{x})}{d\vec{x}}$

Application of Jacobian Matrix

- Derivation of the function $f(x)$ w.r.t. time yields
  $\frac{d\vec{y}}{dt} = \frac{df(\vec{x})}{dt} = J(\vec{x})\frac{d\vec{x}}{dt}$ or $\dot{\vec{y}} = J(\vec{x})\dot{\vec{x}}$
- Jacobian matrix (robotics): Relationship between end effector velocity $\dot{\vec{y}}$ and joint velocities $\dot{\vec{\theta}}$
  - $\dot{\vec{y}} = J(\vec{\theta})\dot{\vec{\theta}}$ with vector notation $\dot{\vec{y}} = \left(\dot{x}, \dot{y}, \dot{z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}\right)^T$
- Number of columns $m =$ movement/joint degrees of freedom
- Number of rows $n =$ degree of freedom in Cartesian space

Application of Jacobian Matrix

- Transformation of a square 6×6 Jacobian matrix in another CS:

$$ {}^{0}J(\vec{\theta}) = \underbrace{\begin{pmatrix} {}^{0}_{1}R & 0 \\ 0 & {}^{0}_{1}R \end{pmatrix}}_{6\times 6} \cdot \; {}^{1}J(\vec{\theta}) $$

- Rest of the procedure
  - Determine ${}^{m}\vec{v}_m$ and ${}^{m}\vec{\omega}_m$ as shown
  - Transform with the above equation in ${}^{0}\vec{v}_m$ and ${}^{0}\vec{\omega}_m$

Example: Jacobian Matrix

Using ${}^{0}\vec{v}_2$ from the example above:

$$\dot{\vec{y}} = {}^{0}\vec{v}_2 = \begin{pmatrix} -s_1 a_1 \dot{\theta}_1 - s_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ c_1 a_1 \dot{\theta}_1 + c_{12} a_2 (\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

with $J(\vec{\theta}) = \begin{pmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \\ 0 & 0 \end{pmatrix}$

Application of Jacobian Matrix

- Considering the angular velocity:

$$\dot{\vec{y}} = \begin{pmatrix} {}^0\vec{v}_2 \\ {}^0\vec{\omega}_2 \end{pmatrix} = \begin{pmatrix} {}^0v_{2_x} \\ {}^0v_{2_y} \\ {}^0v_{2_z} \\ {}^0\omega_{2_x} \\ {}^0\omega_{2_y} \\ {}^0\omega_{2_z} \end{pmatrix} = \begin{pmatrix} -s_1a_1 - s_{12}a_2 & -s_{12}a_2 \\ c_1a_1 + c_{12}a_2 & c_{12}a_2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix}$$

- Further possibility for the calculation of the Jacobian matrix: derivation of the forward kinematics

Inverse Jacobian Matrix

- Calculation of joint angular velocities from Cartesian velocities with inverse Jacobian matrix

$$\dot{\vec{\theta}} = J(\vec{\theta})^{-1}\dot{\vec{y}} \qquad \text{Solution, if } \det(J) \neq 0$$

- Not square → Cartesian degrees of freedom greater than joints degrees of freedom
  1. Elimination of linear dependent lines in J → Invertible matrix
  2. Least-square-method as an approximation

$$\dot{\vec{\theta}} = (J^T J)^{-1} J^T \dot{\vec{y}}$$

Inverse Jacobian Matrix

- Not square → Joint degrees of freedom greater than Cartesian degrees of freedom
  - There are a lot of solutions

1. Block degrees of freedom of movement so that J square
2. Introduce constraints (collision avoidance)

Determinant

- Assigns a scalar to a Matrix
- Definition for $n \times n$-Matrices
  (Laplace‵s formula for the i-th row)
  - $\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$
- Rule of thump for 2×2-Matrices: Rule of Sarrus
  - $\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$

Determinant

- $\det A = \det A^T$
- Swapping two rows changes the sign of the determinant
- Multiplication with a scaler λ:
  Determinant is multiplied by λ
- $\det(A^{-1}) = \frac{1}{\det A}$ for $\det A \neq 0$
- Determinant is 0, if
  - All elements of a row/column are 0
  - Two rows are linearly dependent

Determinant: Example

Expanding the determinant along row 1:

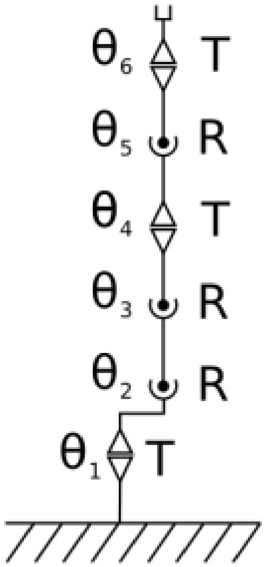
$$\det\begin{pmatrix}0 & 3 & 2\\ 1 & 2 & 3\\ 2 & 1 & 1\end{pmatrix} = 0\cdot\det\begin{pmatrix}2 & 3\\ 1 & 1\end{pmatrix} - 3\cdot\det\begin{pmatrix}1 & 3\\ 2 & 1\end{pmatrix} + 2\cdot\det\begin{pmatrix}1 & 2\\ 2 & 1\end{pmatrix}$$

$$= -3\cdot -5 + 2\cdot -3 = 15 - 6 = 9$$

Singularities

- Robot configuration often with singular Jacobian matrices, thus losing Cartesian degrees of freedom
- Types of singularities
  - At the edge of the working space
  - Inside the workroom

e.g. shown typical industry robot, where $\theta_5 = 0$, θ_4 and θ_6 act in the same direction, i.e. one degree of freedom is lost



- Attention: In the vicinity of singularities very big joint velocities can result from small Cartesian velocities

Singularities, Example: Planar Robot

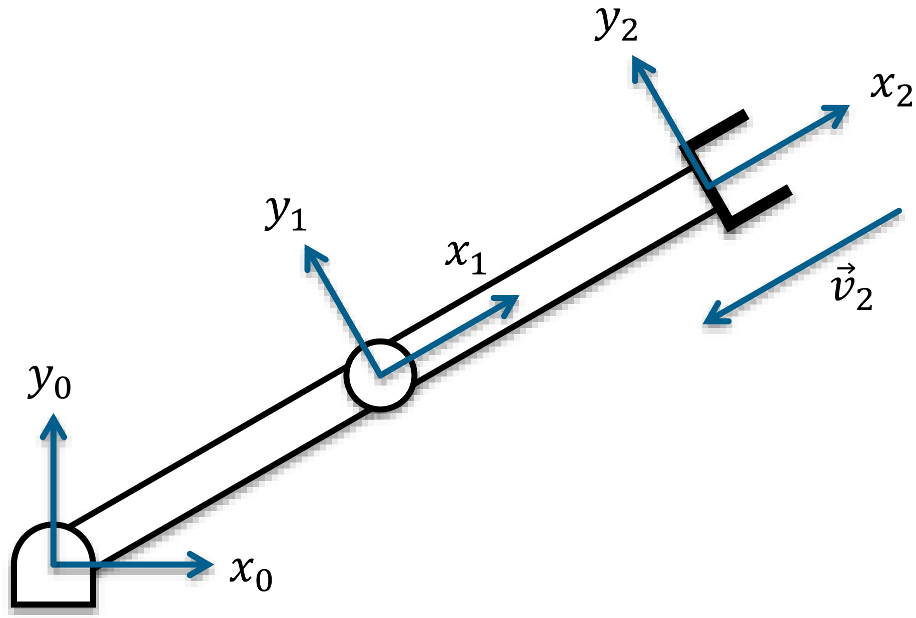
- Singular position of the planar robot
- Jacobian matrix: $J(\theta_1, \theta_2) = \begin{pmatrix} -s_1 a_1 - s_{12} a_2 & -s_{12} a_2 \\ c_1 a_1 + c_{12} a_2 & c_{12} a_2 \end{pmatrix}$
- Determinant: $\det(J) = a_1 a_2 \sin(\theta_2)$
- Singularity ($\det = 0$): $a_1 a_2 \sin(\theta_2) = 0 \rightarrow \theta_2 = 0$ and $\theta_2 = \pi$
- Relevant for practice: $\theta_2 = 0$, i.e. robotic arm fully extended (singularity at the edge of the workspace)

Singularities, Example: Planar Robot

- Inverse Jacobian matrix

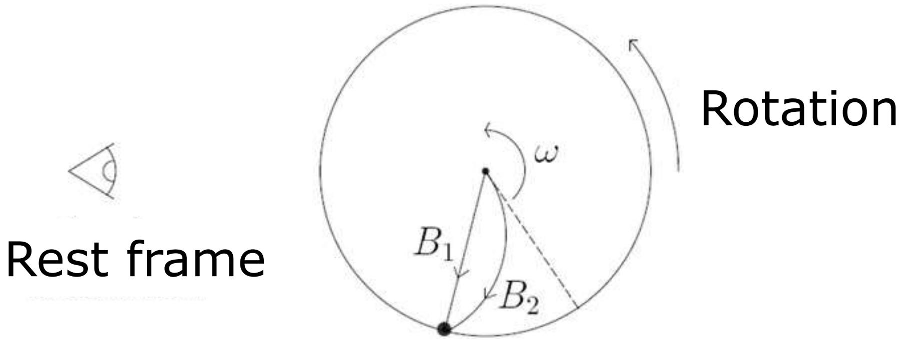
$$J^{-1}(\vec{\theta}) = \frac{1}{a_1 a_2 s_2} \begin{pmatrix} a_2 c_{12} & a_2 s_{12} \\ -a_1 c_1 - a_2 c_{12} & -a_1 s_1 - a_2 s_{12} \end{pmatrix}$$

- For $\theta_2 \rightarrow 0 \Rightarrow \sin\theta_2 \rightarrow 0 \Rightarrow \dot{\theta}_1$ and $\dot{\theta}_2 \rightarrow \infty$



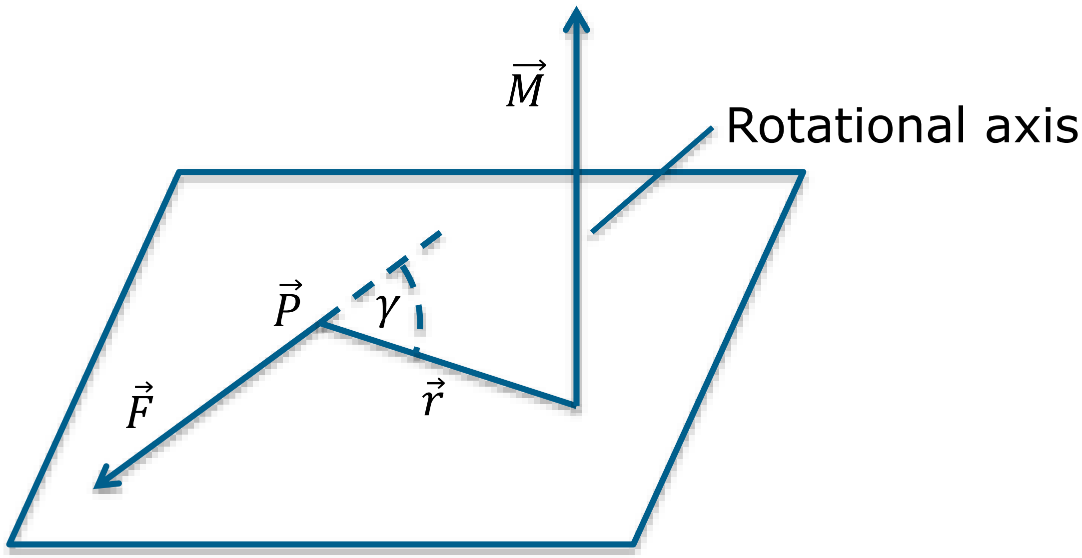
Physics Background: Forces

- Newton‘s second law: $\vec{F} = m \cdot \vec{a}$
- Weight $G = m \cdot g$ $(1\text{kg} \approx 9{,}81\text{N})$
- Centrifugal force $\vec{F_z} = -m \cdot \overrightarrow{a_r} = -m \cdot \vec{r} \cdot \omega^2$
- Coriolis force: Deflects radial moving bodies in a rotating frame of reference
  - Straight path B_1 with respect to a rest frame
  - Curved path B_2 with respect to a rotating frame
  - Therefore a force needs to be applied to keep a straight path



Physics Background: Torque

- Torque $\vec{M} = \vec{r} \times \vec{F}$ on a body with lever arm $\vec{r}$ and force $\vec{F}$
  - Distance r between point of mass and axis
- Equation for magnitude of torque $M = F \cdot r \cdot \sin\gamma$



Physics Background: Moment of Inertia

- Moment of inertia $dJ = r^2 dm$ for mass point with mass dm
- Moment of inertia $J = \int_{\text{Volume}} r^2 dm$ for a body
  - With mass distribution J relative to rotational axis
- Tensor: Inertia w.r.t x-y-z-System in homogeneous coordinates
- $M = \int \vec{r} \cdot \vec{r}^T dm = \begin{pmatrix} \int x^2\, dm & \int yx\, dm & \int xz\, dm & \int x\, dm \\ \int xy\, dm & \int y^2\, dm & \int yz\, dm & \int y\, dm \\ \int xz\, dm & \int yz\, dm & \int z^2\, dm & \int z\, dm \\ \int x\, dm & \int y\, dm & \int z\, dm & \int dm \end{pmatrix}$

Static Forces/Moments

- Calculation without consideration of movements
- Example: How high do torques have to be in order to keep an object of mass m in a certain position with TCP?
- Solution idea
  - Propagate powers and moments from link to link
  - Calculate a force/moment balance for each member
  - Start with the TCP
- $\vec{f}_i$: Force that attacks on link through link $i-1$
- $\vec{n}_i$: Torque (Moment) that attacks link through link $i-1$
- Forces/Moment equation
  (Influence of the next higher link)

$$ {}^{i}\vec{f}_i = {}^{i}\vec{f}_{i+1} \qquad {}^{i}\vec{n}_i = {}^{i}\vec{n}_{i+1} + {}^{i}\vec{u}_{i+1} \times {}^{i}\vec{f}_{i+1} $$

Static Forces/Moments: Propagation

- Static propagation of the forces /moments from link to link
- Forces at link i

$$ {}^{i}\vec{f}_i = {}^{i}_{i+1}R \cdot {}^{i+1}\vec{f}_{i+1} $$

- Moment at link i

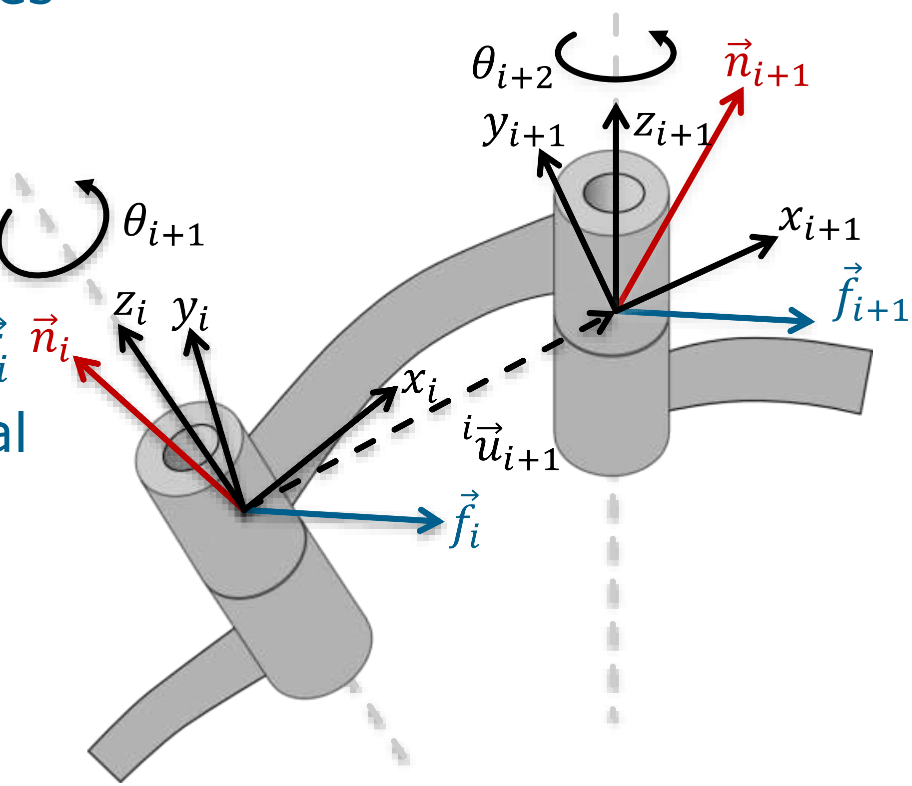
$$ {}^{i}\vec{n}_i = {}^{i}_{i+1}R \cdot {}^{i+1}\vec{n}_{i+1} + {}^{i}\vec{u}_{i+1} \times {}^{i}\vec{f}_i $$

- Required moment in rotational joints

$$ \tau_{i+1} = {}^{i}\vec{n}_i^{T} \cdot {}^{i}\vec{e}_{z_i} $$

- Required force in linear joints

$$ \tau_{i+1} = {}^{i}\vec{f}_i^{T} \cdot {}^{i}\vec{e}_{z_i} $$

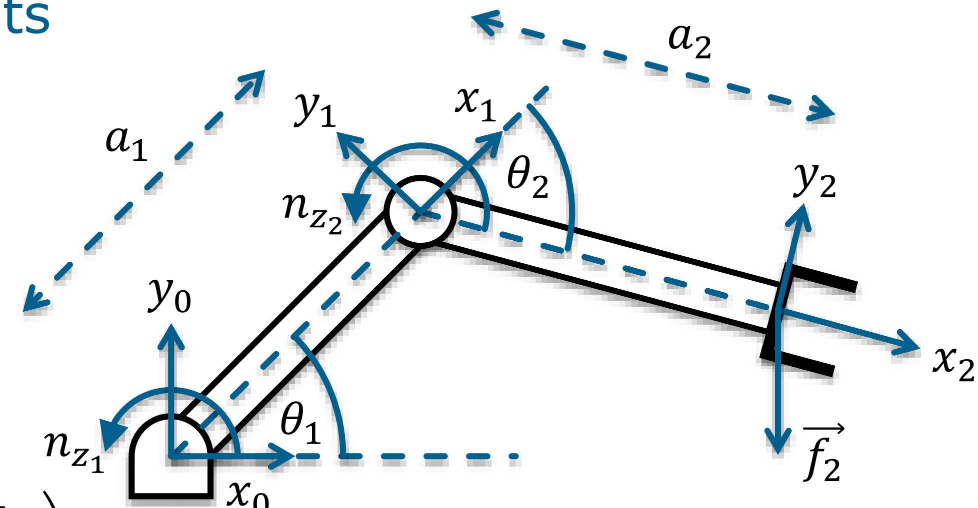


Static Forces/Moments: Example

- Given: Forces f, applied at the TCP
- Desired: Torques in the joints

$$
{}^{2}\vec{f}_2 = \begin{pmatrix} {}^{2}f_{2_x} \\ {}^{2}f_{2_y} \\ 0 \end{pmatrix} \qquad {}^{2}\vec{n}_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
{}^{1}\vec{n}_1 &= {}^{1}\vec{n}_2 + {}^{1}\vec{u}_2 \times {}^{1}\vec{f}_1 \\
&= {}^{1}\vec{u}_2 \times \left({}^{1}_{2}R \cdot {}^{2}\vec{f}_2 \right) \\
&= \begin{pmatrix} a_2 c_2 \\ a_2 s_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} c_2 \cdot {}^{2}f_{2_x} - s_2 \cdot {}^{2}f_{2_y} \\ s_2 \cdot {}^{2}f_{2_x} + c_2 \cdot {}^{2}f_{2_y} \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ a_2 \cdot {}^{2}f_{2_y} \end{pmatrix}
\end{aligned}
$$



Static Forces/Moments: Example

$$
{}^{0}\vec{n}_0 = {}^{0}_{1}R \cdot {}^{1}\vec{n} + {}^{0}\vec{u}_1 \times {}^{0}\vec{f}_0
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ a_2 \cdot {}^{2}f_{2_y} \end{pmatrix} + \begin{pmatrix} a_1 c_1 \\ a_1 s_1 \\ 0 \end{pmatrix} \times \begin{pmatrix} c_{12} \cdot {}^{2}f_{2_x} - s_{12} \cdot {}^{2}f_{2_y} \\ s_{12} \cdot {}^{2}f_{2_x} + c_{12} \cdot {}^{2}f_{2_y} \\ 0 \end{pmatrix}
$$

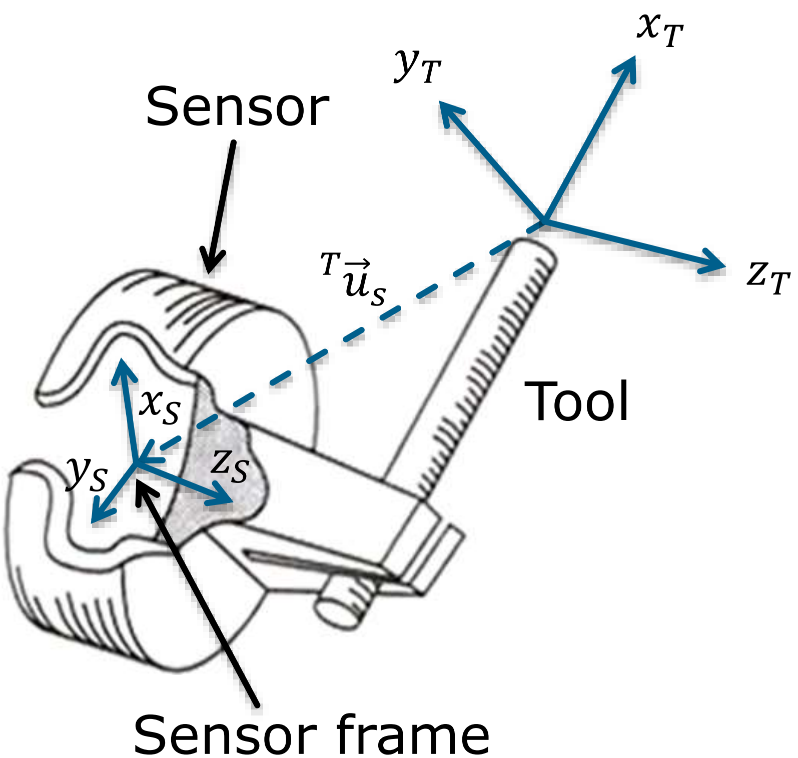
$$
= \begin{pmatrix} 0 \\ 0 \\ a_2 \cdot {}^{2}f_{2_y} + s_2 a_1 \cdot {}^{2}f_{2_x} + c_2 a_1 \cdot {}^{2}f_{2_y} \end{pmatrix}
$$

$$
\tau_1 = a_2 \cdot {}^{2}f_{2_y} + s_2 a_1 \cdot {}^{2}f_{2_x} + c_2 a_1 \cdot {}^{2}f_{2_y}
$$

$$
\tau_2 = a_2 \cdot {}^{2}f_{2_y}
$$

Transformation of Forces: Application Example

- TCP grips tool→ Load cell measures forces and moments in the wrist
- Desired: Forces and torques at the end of the staff



Force/Moment Calculation with Jacobian Matrix

- Contemplation of the virtual work in the Cartesian space - and in the configuration space
- Work, which is caused by the forces and moments $\vec{\eta}$ acting on the TCP, must be equal to the work that is applied in the joints by adjusting forces and setting moments $\vec{\tau}$
- $\vec{\eta}^T \cdot \delta\vec{y} = \vec{\tau}^T \cdot \delta\vec{\theta}$ (1)
  - $\vec{\eta} = \begin{pmatrix} \vec{f}_{TCP} \\ \vec{n}_{TCP} \end{pmatrix}$: 6×1, Cartesian force-/moment vector at TCP
  - $\delta\vec{y}$: 6×1, infinitesimal offset vector of TCP
  - $\vec{\tau}$: 6×1, force-/moment vector in joints
  - $\delta\vec{\theta}$: 6×1, change of joint positions

Force/Moment Calculation with Jacobian Matrix

- By inserting the relationship $\delta\vec{y} = J(\vec{\theta}) \cdot \delta\vec{\theta}$ (1) can be transformed into $\vec{\eta}^T \cdot J(\vec{\theta}) \cdot \delta\vec{\theta} = \vec{\tau}^T \cdot \delta\vec{\theta}$
- Thus, $\vec{\eta}^T \cdot J(\vec{\theta}) = \vec{\tau}^T$ and $\vec{\tau} = J^T(\vec{\theta}) \cdot \vec{\eta}$

Next Lecture

Robot modeling

- Dynamic modeling
- Moment of inertia
- Dynamics analysis