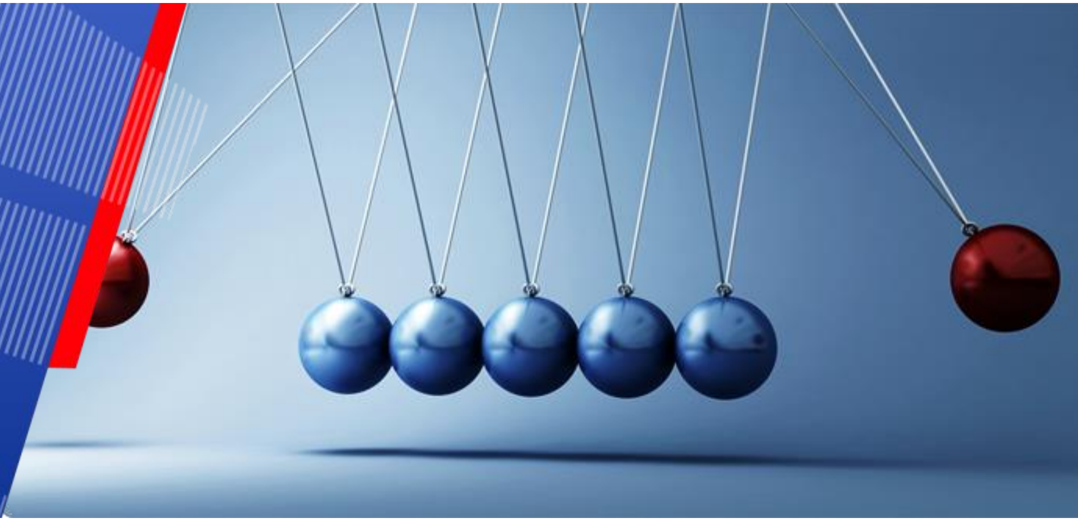
Autonomous Mobile Robots

4. Modeling - Mobile Robot Dynamics



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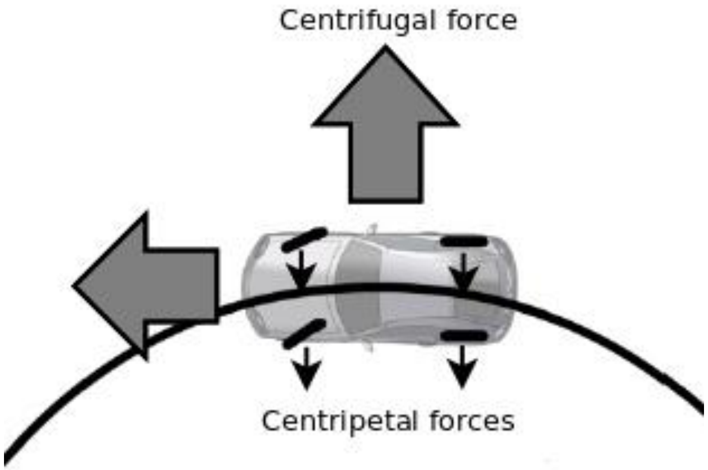
Why Regarding Dynamics?

- Kinematics do not consider any forces applied to the system
- Control solely based on kinematics only suitable if forces are small and can be neglected
- At higher speeds or instable systems dynamic forces are often crucial
  - Vehicles (at higher velocities)
  - Walking machines
  - ...



Kinematics vs. Dynamics

- **Kinematics** - motion of bodies and systems of bodies without consideration of the masses nor the forces

- **Dynamics** - relationship between motion of bodies and their causes based on
  - Inertia
  - Elasticity
  - Friction
- Application:
  - Analysis
  - Synthesis of mechanics
  - Control design

Dynamics – Equation of Motion

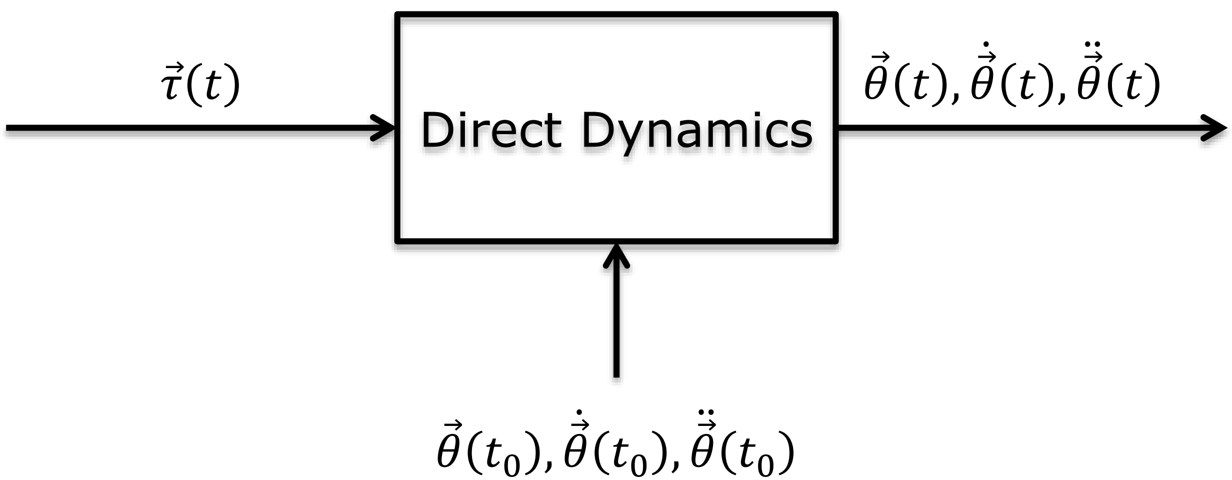
- Relationship between forces/moments, acceleration, velocity and position of objects

$$\vec{\tau} = I(\vec{\theta}) \cdot \ddot{\vec{\theta}} + n\left(\dot{\vec{\theta}}, \vec{\theta}\right) + g(\vec{\theta}) + R \cdot \dot{\vec{\theta}} \qquad (1)$$

- $\vec{\tau}$: $n \times 1$ vector of general forces and moments
- $I(\vec{\theta})$: $n \times n$ moment of inertia
- $n\left(\dot{\vec{\theta}}, \vec{\theta}\right)$: $n \times 1$ centrifugal and coriolis force
- $g(\vec{\theta})$: $n \times 1$ vector of gravitation component
- R: $n \times n$ diagonal-matrix describing friction
- $\vec{\theta}$: $n \times 1$ control variables

Direct Dynamics

- Calculate motion based on
  - external forces and moments
  - position
  - initial acceleration and velocity



Inverse Dynamics

- Calculate required forces and moments based on given motion

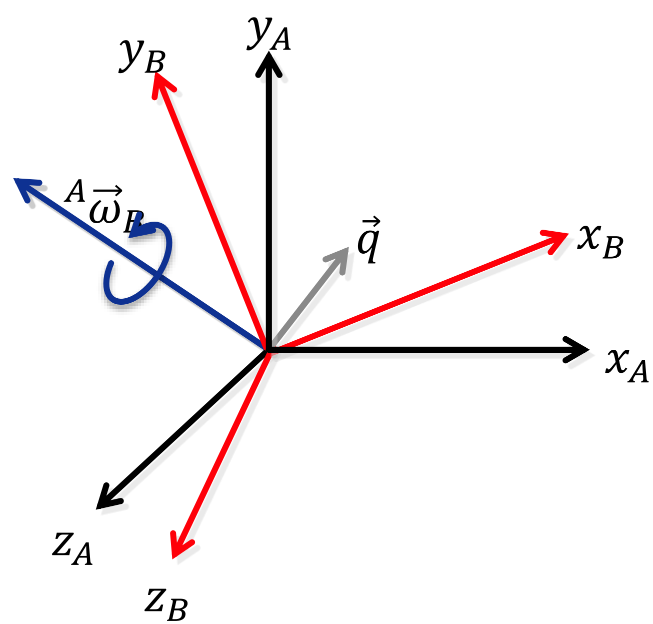
Acceleration of Rigid Bodies

- Linear Acceleration

$$ {}^{B}\dot{\vec{v}}_q = \frac{d}{dt}\,{}^{B}\vec{v}_q = \lim_{\Delta t \to 0} \frac{{}^{B}\vec{v}_q(t+\Delta t) - {}^{B}\vec{v}_q(t)}{\Delta t} \qquad (2) $$

- Angular Acceleration

$$ {}^{A}\dot{\vec{\omega}}_B = \frac{d}{dt}\,{}^{A}\vec{\omega}_B = \lim_{\Delta t \to 0} \frac{{}^{A}\vec{\omega}_B(t+\Delta t) - {}^{A}\vec{\omega}_B(t)}{\Delta t} \qquad (3) $$



Linear Acceleration

- Summary

$$
\begin{aligned}
{}^{A}\dot{\vec{v}}_q = {}^{A}_{B}R \cdot {}^{B}\dot{\vec{v}}_q + 2 \cdot \left({}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{v}_q \right) \\
+ {}^{A}\dot{\vec{\omega}}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} + {}^{A}\vec{\omega}_B \times \left({}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} \right)
\end{aligned}
\qquad (8)
$$

- General Case (Frames A, B common origin)

$$
\begin{aligned}
{}^{A}\dot{\vec{v}}_q = {}^{A}\dot{\vec{v}}_{OB} + {}^{A}_{B}R \cdot {}^{B}\dot{\vec{v}}_q + 2 \cdot \left({}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{v}_q \right) \\
+ {}^{A}\dot{\vec{\omega}}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} + {}^{A}\vec{\omega}_B \times \left({}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} \right)
\end{aligned}
\qquad (9)
$$

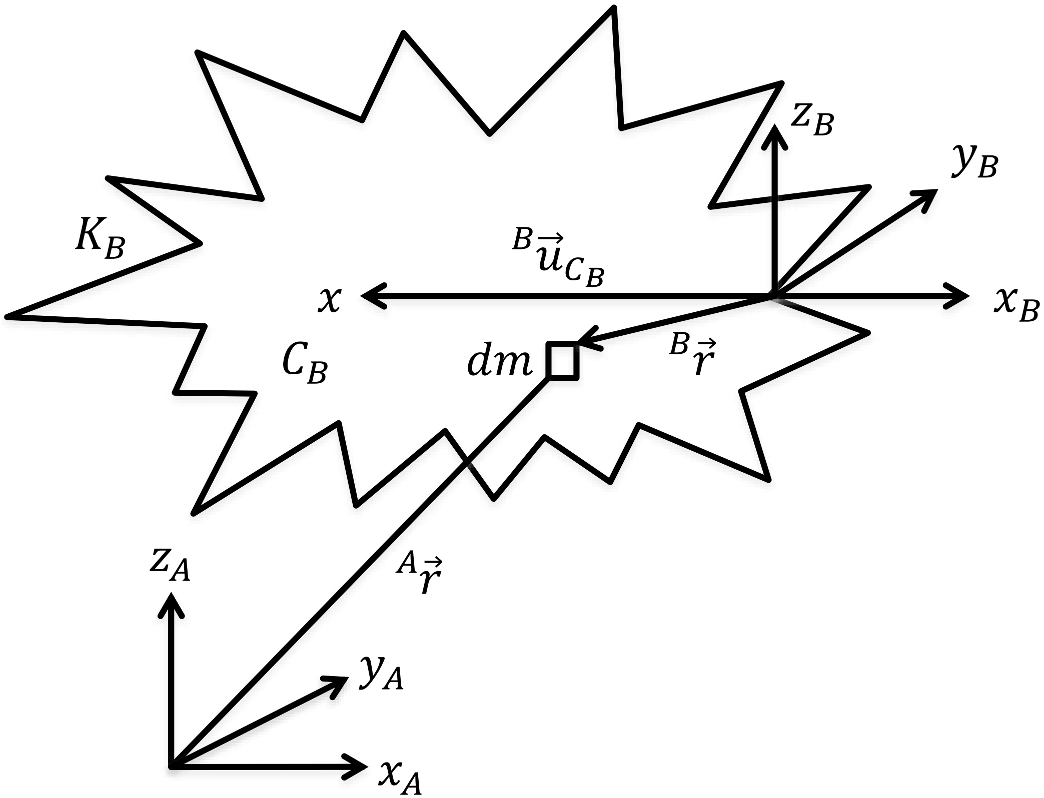
- ${}^{B}\vec{q}$ not moving

$$
{}^{B}\vec{v}_q = {}^{B}\dot{\vec{v}}_q = \vec{0}
$$

$$
\Rightarrow {}^{A}\dot{\vec{v}}_q = {}^{A}\dot{\vec{v}}_{OB} + {}^{A}\dot{\vec{\omega}}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} + {}^{A}\vec{\omega}_B \times \left({}^{A}\vec{\omega}_B \times {}^{A}_{B}R \cdot {}^{B}\vec{q} \right) \qquad (10)
$$

Mass Distribution: Preliminary Considerations

- dm: Mass particle
- C_B: Center of Gravity of Body B
- $\vec{u}_{C_B}$: Vector to C_B
- $\vec{r}$: Vector to the particle



Mass Distribution: Moment of Inertia Tensor

- Moment of Inertia Tensor in Frame A describes the rotatory inertia of an object

$$ {}^{A}I = \begin{bmatrix} {}^{A}i_{XX} & -{}^{A}i_{xy} & -{}^{A}i_{xz} \\ -{}^{A}i_{xy} & {}^{A}i_{yy} & -{}^{A}i_{yz} \\ -{}^{A}i_{xz} & -{}^{A}i_{yz} & {}^{A}i_{zz} \end{bmatrix} $$

- Scalar elements given by integration over mass distribution M
  - Axial moments of inertia

$$ {}^{A}i_{xx} = \iiint_M (y_A^2 + z_A^2)\,dm \qquad {}^{A}i_{yy} = \iiint_M \left(x_A^2 + z_A^2\right)dm \qquad {}^{A}i_{zz} = \iiint_M \left(x_A^2 + y_A^2\right)dm $$

  - moments of inertia product

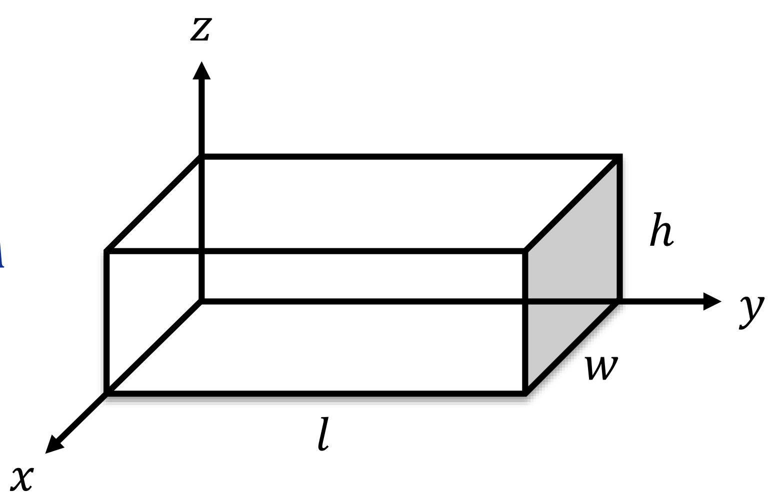
$$ {}^{A}i_{xy} = \iiint_M x_A y_A\,dm \qquad {}^{A}i_{xz} = \iiint_M x_A z_A\,dm \qquad {}^{A}i_{yz} = \iiint_M y_A z_A\,dm $$

- Moment of inertia tensor of a point mass is a zero matrix

Mass Distribution: Cubiod

- Moment of inertia of a cuboid of uniform density ρ
- $dm = \rho dx dy dz$ yields

$$
\begin{aligned}
{}^{A}i_{xx} &= \int_0^h \int_0^l \int_0^w (y_A^2 + z_A^2)\rho dx_A dy_A dz_A \\
&= \int_0^h \int_0^l (y_A^2 + z_A^2) w\rho dy_A dz_A \\
&= \int_0^h \left(\frac{l^3}{3} + z_A^2 l\right) w\rho dz_A \\
&= \left(\frac{hl^3w}{3} + \frac{h^3lw}{3}\right)\rho \\
&= \frac{m}{3}(l^2 + h^2) \quad \text{(with overall mass } m\text{)}
\end{aligned}
$$



- Analogously,

$$ {}^{A}i_{yy} = \frac{m}{3}(w^2 + h^2) $$

$$ {}^{A}i_{zz} = \frac{m}{3}(l^2 + w^2) $$

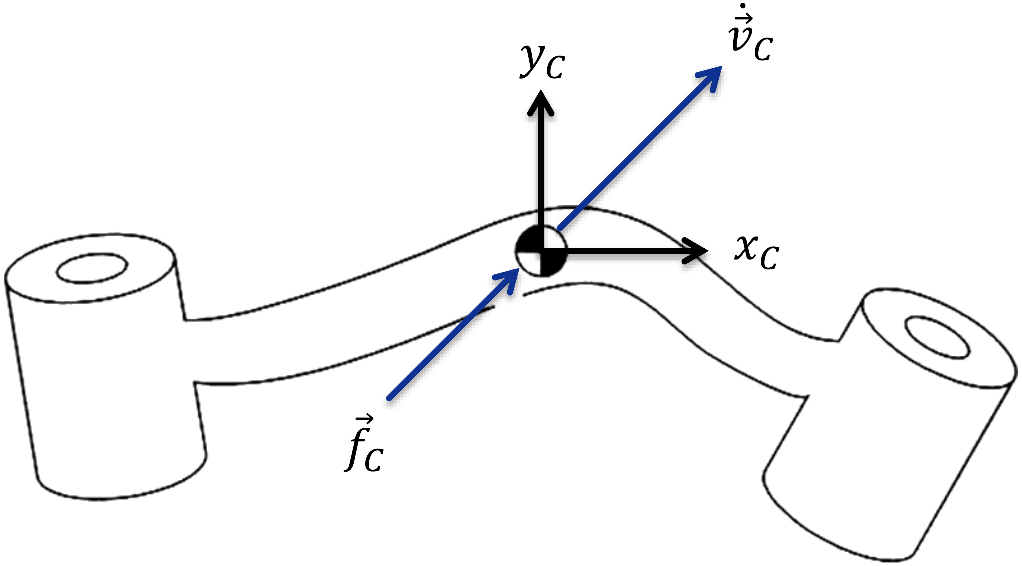
Derivation of the Dynamic Equations

- Synthetic Method (Newton-Euler):
  Free-body diagram
  - Law of conservation of (angular-) momentum
  - Constraint force elimination leads to motion compensation
  - Basis for more complicated multi-body modes describing systems of rigid bodies connected by joints

- Analytic Method (Lagrange):
  Application of the optimization principle
  - Energy consideration
  - Formal derivation yields motion equations

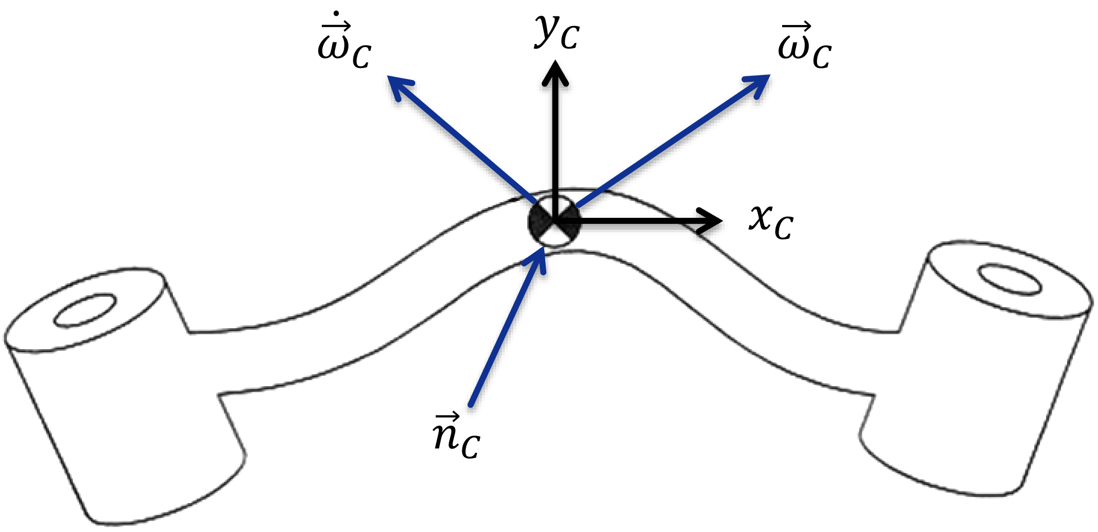
Newton-Euler: Base Equations

- Newton: $\vec{f}_C = m \cdot \dot{\vec{v}}_C$ (11)
  - m: overall mass
  - $\dot{\vec{v}}_C$: acceleration of the center of gravity C
  - $\vec{f}_C$: forces applying in C



Newton-Euler: Base Equations

- Euler: $\vec{n}_C = {}^C I \cdot \dot{\vec{\omega}}_C + \vec{\omega}_C \times {}^C I \cdot \vec{\omega}_C$ (12)
  - $\vec{\omega}_C$: angular velocity of the link
  - ${}^C I$: moment of inertia tensor in Frame C
  - $\vec{n}_C$: moment applying at center of mass which causes the rotation



Lagrange

- Motion equation according to Lagrange

$$\tau = \frac{d}{dt}\frac{\delta l}{\delta \dot{\theta}} - \frac{\delta l}{\delta \theta} \tag{13}$$

  - θ: rotation angle or translation path
  - $\dot{\theta}$: object velocity
  - τ: vector of force/moment applying to the object

- Lagrange function: $l = e_{kin} - e_{pot}$ (with regard to the base) (14)
  - Describes the difference between kinetic and potential energy of a mechanical system

Lagrange: Kinetic Energy

- Kinetic Energy

$$e_{kin} = \underbrace{\frac{1}{2} \mathrm{m} \cdot \vec{v}_C^T \cdot \vec{v}_C}_{\text{Linear Part}} + \underbrace{\frac{1}{2} \vec{\omega}^T \cdot {}^{C}I \cdot \vec{\omega}}_{\text{Rotational Part}} \qquad (15)$$

- $\vec{v}_C$ and $\vec{\omega}$ depend on position and velocity of the center of gravity

Lagrange: Kinetic Energy

- Kinetic energy depends on position and velocity

$$e_{kin}(\vec{\theta}, \dot{\vec{\theta}}) = \frac{1}{2} \dot{\vec{\theta}}^T \cdot M(\vec{\theta}) \cdot \dot{\vec{\theta}} \qquad (16)$$

  - $M(\vec{\theta})$: $n \times n$ mass-matrix, where each element is a complex function of $\vec{\theta}$
  - $M(\vec{\theta})$: positiv-definite, i.e. $\dot{\vec{\theta}}^T \cdot M(\vec{\theta}) \cdot \dot{\vec{\theta}}$ always yields a positive scalar

- Equation corresponds to the common equation of kinetic energy of a point-mass

$$e_{kin} = \frac{1}{2} m \cdot v^2 \qquad (17)$$

Lagrange: Potential Energy

- Potential Energy

$$e_{pot} = -m \cdot {}^{0}\vec{g}^{T} \cdot {}^{0}\vec{u}_C + e_{pot,ref} \tag{18}$$

  - ${}^{0}\vec{g}$: 3×1 gravity vector, with regard to the origin frame 0
  - ${}^{0}\vec{u}_C$: 3×1 vector to the center of mass
  - $e_{pot,ref}$: constant, such that $e_{pot} \geq 0$ holds

- Potential energy can be defined as function $e_{pot}(\vec{\theta})$ based on the joint angle

Lagrange: Modeling of Nonholonomic Robots

- Lagrange dynamic model of a nonholonomic robot has the form

$$E\tau = \frac{d}{dt}\frac{\delta l}{\delta \dot{\theta}} - \frac{\delta l}{\delta \theta} + D^T(\theta)\lambda \tag{19}$$

  - $D(\theta)$ is the m x n matrix of the m constraints introduced in the kinematic part

$$D(\theta)\dot{\theta} = 0$$

  - λ is the vector Lagrange multiplier
  - E is a non-singular transformation matrix
- This model leads to the motion equation

$$E\tau = I(\theta) \cdot \ddot{\theta} + n\left(\dot{\theta}, \theta\right) \cdot \dot{\theta} + g(\theta) + D^T(\theta)\lambda \tag{20}$$

Lagrange: Modeling of Nonholonomic Robots

- Elimination of constraint term by matrix $B(\theta)$ with

$$B^T(\theta)D^T(\theta) = 0 \tag{21}$$

- Due to the constraints we know that $v(t)$ exists with

$$\dot{\theta}(t) = B(\theta)v(t) \tag{22}$$

- Multiplying the motion equation by B^T

$$\bar{E}\tau = \bar{I}(\theta) \cdot \dot{v} + \bar{n}\left(\dot{\theta}, \theta\right) \cdot v + \bar{g}(\theta) \tag{23}$$
$$\bar{I} = B^T I B \tag{24}$$
$$\bar{n} = B^T I \dot{B} + B^T n B \tag{25}$$
$$\bar{g} = B^T g \tag{26}$$
$$\bar{E} = B^T E \tag{27}$$

Differential Drive Dynamics (Newton-Euler)

- Translational motion

$$F = m \cdot \dot{v} \qquad (28)$$

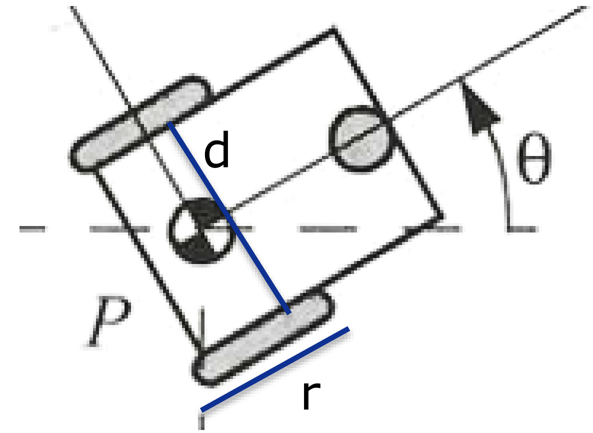
- Rotational motion

$$N = i \cdot \dot{\omega} \qquad (29)$$

- Assuming CoG coincides with P

$$F = F_r + F_l = \frac{1}{r}(\tau_r + \tau_l) \qquad (30)$$

$$N = (F_r - F_l) \cdot 2d = \frac{2d}{r}(\tau_r - \tau_l) \qquad (31)$$



$$\tau_r = r \cdot F_r \qquad (32)$$

$$\tau_l = r \cdot F_l \qquad (33)$$

- Dynamic model

$$\dot{\vec{\xi}}_W = R(\theta)^{-1}\dot{\vec{\xi}}_r \qquad (34)$$

$$\dot{\vec{\xi}}_r = \begin{pmatrix} \dot{v} \\ 0 \\ \dot{\omega} \end{pmatrix} \qquad (35)$$

$$\dot{v} = \frac{1}{mr}(\tau_r + \tau_l) \qquad (36)$$

$$\dot{\omega} = \frac{2d}{ir}(\tau_r - \tau_l) \qquad (37)$$

Differential Drive Dynamics (Lagrange)

- Assuming horizontal planar terrain and no slippage

$$g(\vec{\theta}) = 0 \qquad n\left(\dot{\vec{\theta}}, \vec{\theta}\right) = 0 \tag{38}$$

- Lagrange motion model

$$E\tau = I(\theta) \cdot \ddot{\theta} + D^T(\theta)\lambda \tag{39}$$

  - $I(\theta) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & i \end{bmatrix}$; $E = \frac{1}{r}\begin{bmatrix} cos\omega & cos\omega \\ sin\omega & sin\omega \\ 2d & -2d \end{bmatrix}$

- $B(\theta) = \begin{bmatrix} cos\omega & 0 \\ sin\omega & 0 \\ 0 & i \end{bmatrix}$

  - $\bar{I} = B^T I B = \begin{bmatrix} m & 0 \\ 0 & i \end{bmatrix}$; $\bar{E} = \frac{1}{r}\begin{bmatrix} 1 & 1 \\ 2d & 2d \end{bmatrix}$

Differential Drive Dynamics (Lagrange)

- Dynamic model becomes

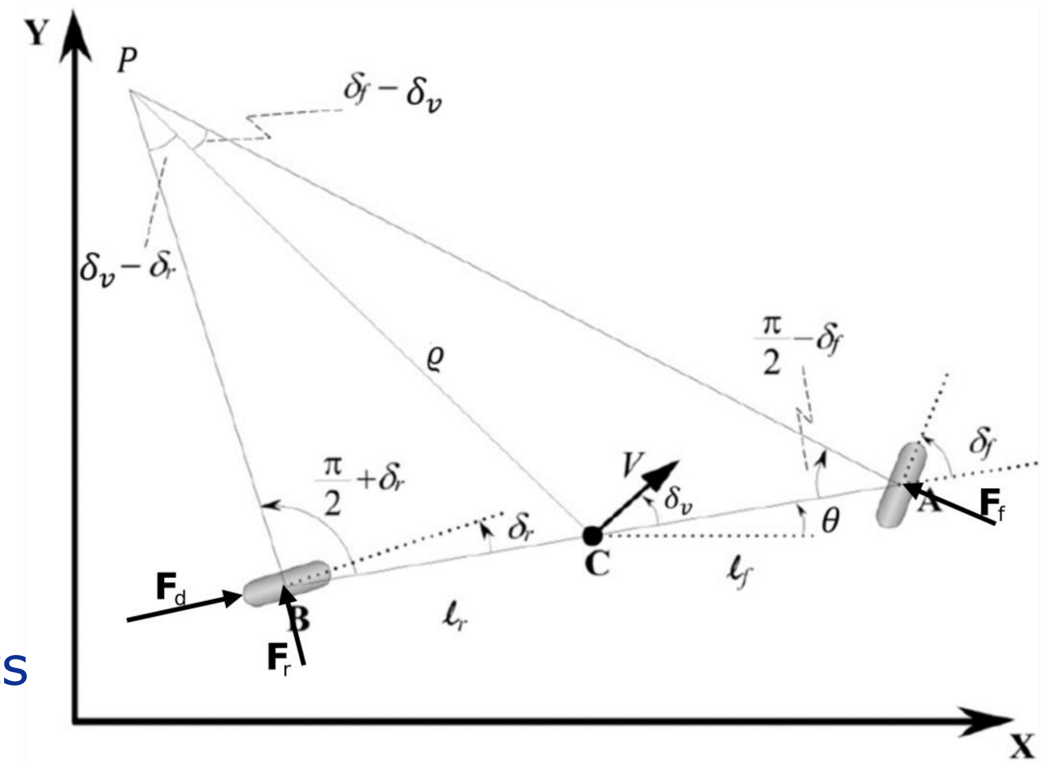
$$\begin{bmatrix} m & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} \dot{v} \\ \dot{\omega} \end{bmatrix} = \frac{1}{r} \begin{bmatrix} 1 & 1 \\ 2d & -2d \end{bmatrix} \begin{bmatrix} \tau_r \\ \tau_l \end{bmatrix} \tag{40}$$

- Identical to model by Newton-Euler method

$$\dot{v} = \frac{1}{mr}(\tau_r + \tau_l) \qquad \dot{\omega} = \frac{2d}{ir}(\tau_r - \tau_l) \tag{41}$$

Ackermann Dynamics

- Bicycle model
  - F_d driving force
  - F_r,F_f lateral slip forces
  - C center of gravity
  - $\delta_r = 0$



- Nonholonomic constraints according to kinematic
  - $\dot{x}_B sin(\theta) - \dot{y}_B cos(\theta) = 0$ (42)
  - $\dot{x}_A sin(\theta + \delta_f) - \dot{y}_A cos(\theta + \delta_f) = 0$ (43)

Ackermann Dynamics

- Need to express constrainst in terms of center of gravity
- Wheel coordinates in terms of center of gravity

$$
\begin{aligned}
x_B &= x_C - l_r cos(\theta) & x_A &= x_C + l_f cos(\theta) \\
y_B &= x_C - l_r sin(\theta) & y_A &= x_C + l_f sin(\theta) \\
\dot{x}_B &= \dot{x}_C + l_r \dot{\theta} sin(\theta) & \dot{x}_A &= \dot{x}_C - l_f \dot{\theta} sin(\theta) \\
\dot{y}_B &= \dot{y}_C - l_r \dot{\theta} cos(\theta) & \dot{y}_A &= \dot{y}_C + l_f \dot{\theta} cos(\theta)
\end{aligned} \tag{44}
$$

- Constrainst in terms of center of gravity

$$\dot{x}_C sin(\theta) - \dot{y}_C cos(\theta) + l_r \dot{\theta} = 0 \tag{45}$$
$$\dot{x}_C sin(\theta + \delta_f) - \dot{y}_C cos(\theta + \delta_f) - l_f \dot{\theta} cos(\theta) = 0 \tag{46}$$

Ackermann Dynamics

- Applying rotational trasformation

$$\begin{bmatrix} \dot{x}_C \\ \dot{y}_C \end{bmatrix} = \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{bmatrix} \begin{bmatrix} \dot{x}_c \\ \dot{y}_c \end{bmatrix} \quad (47)$$

  to the constraints yields ((47) in (45) and (46))

$$\dot{y}_c = l_r\dot{\theta} \text{ and } \dot{\theta} = \frac{tg\delta_f}{D}\dot{x}_c$$

$$\text{->} \quad \dot{y}_c = \frac{l_r}{D}(tg\delta_f)\dot{x}_c \quad (48)$$

  in the robot coordinate system.

- Need to regard acceleration for Newton-Euler base equations -> differentiation

$$\ddot{y}_c = l_r\ddot{\theta} \text{ and } \ddot{\theta} = \frac{tg\delta_f}{D}\ddot{x}_c + \frac{1}{Dcos^2(\delta_f)}\dot{x}_c\dot{\delta}_f$$

$$\text{->} \quad \ddot{y}_c = \frac{l_r}{D}(tg\delta_f)\ddot{x}_c + \frac{l_r}{Dcos^2(\delta_f)}\dot{x}_c\dot{\delta}_f \quad (49)$$

Ackermann Dynamics – Newton-Euler

- m = mass of the robot
- J = moment of inertia
- F_d = driving force
- F_f, F_r = wheel lateral force
- T = time constant of steering system
- u_s = steering control point
- K = const coefficient
- r = rear wheel radius
- τ_d = applied motor torque

$$m(\ddot{x}_c - \dot{y}_c\dot{\theta}) = F_d - F_f sin(\delta_f) \quad (50)$$

$$m(\ddot{y}_c - \dot{x}_c\dot{\theta}) = F_r - F_f cos(\delta_f) \quad (51)$$

$$J\ddot{\theta} = l_f F_f \cos(\delta_f) - l_r F_r \quad (52)$$

$$F_d = \left(\frac{1}{r}\right)\tau_d \quad (53)$$

$$\dot{\delta}_f = -\frac{1}{T}\delta_f + \frac{K}{T}u_s \quad (54)$$

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Ackermann Dynamics – State Space Form

- $x = [x_C \quad y_C \quad \theta \quad \dot{x}_c \quad \delta_f]^T$
- Solve (52) for F_r, with (49) in (51)

$$m(l_r\ddot{\theta} + \dot{x}_c\dot{\theta}) = F_f cos(\delta_f) + \frac{l_f}{l_r} F_f cos(\delta_f) - \frac{J}{l_r}\ddot{\theta}$$

$$= \frac{(l_f + l_r)F_f cos(\delta_f) - J\ddot{\theta}}{l_r}$$

<-> $$F_f = \left(\frac{m l_r{}^2 + J}{(l_f+l_r)cos(\delta_f)}\right)\ddot{\theta} + \left(\frac{m l_r \dot{x}_c}{(l_f+l_r)cos(\delta_f)}\right)\dot{\theta} \qquad (55)$$

- (48a,b),(49b),(55) in (50)

$$\ddot{x}_c = \frac{\dot{x}_c(m l_r{}^2 + J)tg\delta_f}{a}\dot{\delta}_f + \frac{(l_f + l_r)^2 cos^2(\delta_f)}{a} F_d$$

$$a = \left(cos^2(\delta_f)\right)\left(m(l_f + l_r)^2 + (m l_r{}^2 + J)tg^2\delta_f\right)$$

Ackermann Dynamics – State Space Form

- (48a,b) in (47) -> affine system with two inputs (τ_d, u_s)

$$\dot{x}_C = \left\{cos(\theta) - \left(\frac{\delta_f}{(l_f + l_r)}\right)(tg\delta_f)sin(\theta)\right\}\dot{x}_c$$

$$\dot{y}_C = \left\{sin(\theta) - \left(\frac{\delta_f}{(l_f + l_r)}\right)(tg\delta_f)cos(\theta)\right\}\dot{x}_c$$

$$\dot{\theta} = \left\{\left(\frac{1}{(l_f + l_r)}\right)(tg\delta_f)\right\}\dot{x}_c$$

$$\ddot{x}_c = \left(\frac{1}{a}\right)\{(m{l_r}^2 + J)(tg\delta_f)\dot{\delta}_f\dot{x}_c\} + \left(\frac{1}{a}\right)\left((l_f + l_r)^2 cos^2(\delta_f)\right)F_d$$

$$\dot{\delta}_f = -\frac{\delta_f}{T} + \frac{K}{T}u_s$$

$$F_d = \frac{\tau_d}{r}$$

Ackermann Dynamics – State Space Form

- Drift term

$$g_0(x) = \begin{bmatrix} \left\{cos(\theta) - \left(\frac{\delta_f}{(l_f + l_r)}\right)(tg\delta_f)sin(\theta)\right\}\dot{x}_c \\ \left\{sin(\theta) - \left(\frac{\delta_f}{(l_f + l_r)}\right)(tg\delta_f)cos(\theta)\right\}\dot{x}_c \\ \left\{\left(\frac{1}{(l_f + l_r)}\right)(tg\delta_f)\right\}\dot{x}_c \\ \left(\frac{1}{a}\right)\{(ml_r{}^2 + J)(tg\delta_f)\dot{\delta}_f\dot{x}_c\} \\ -\frac{\delta_f}{T} \end{bmatrix}$$

Ackermann Dynamics – State Space Form

- Two input fields
  - $g_1(x) = \begin{bmatrix} 0 & 0 & 0 & \frac{(l_f+l_r)^2 cos^2(\delta_f)}{ra} & 0 \end{bmatrix}^T$
  - $g_2(x) = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{K}{T} \end{bmatrix}^T$

- Dynamics State Space Model

$$\dot{x} = g_0(x) + g_1(x)\tau_d + g_2(x)u_s$$

Ackermann Dynamics – Simulation