Robot Modelling I

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Content

- Degrees of freedom of a robotic system
- Geometric model
- Kinematic model
- Direct kinematic problem
- Inverse kinematic problem
- Dynamic model

Degree of Freedom (DoF) f of an Object in E_3

- Number of possible independent movements in relation to the BCS
  - Minimal number of translations and rotations for complete description of the object´s pose

- For objects with unconstrained movement in $3D$-space $f = 6$
  - 3 translations
  - 3 rotations

Kinematic Degrees of Freedom of a Robot F

- Degrees of freedom of a rotational joint: $F_R \leq 3$
  - Hinge joint
  - Cardan joint
  - Linear joint
  - Spherical joint
- DoF of a translational joint: $F_T = 1$
- Number of joints of robot: n (in general $n \geq 6$)
- Kinematic DoFs: $F = \sum_{i=1}^{n} \left(F_{R_i} + F_{T_i}\right)$

Relation between f and F

- The relationship holds: $F \geq f$
- Examples
  - 8-axis robot: DoF $f = 6$, kinematic DoF $F = 8$
  - Human hand: $f = 6,\ F = 22$
  - Human arm including shoulder: $f = 6,\ F = 12$

- In order to reach a DoF $f = 6$ for a robot‘s effector, it requires at least $F = 6$ axes of movement

Terminology

- Geometry: Mathematical description of robot form
- Kinematics: Geometric and analytic description of mechanical systems` states of movement
- Dynamics: Investigates movement of objects based on the forces and moments acting upon them

Geometric Model

- Displays bodies graphically
- Basis of movement calculation
- Identification of acting forces and moments
- Starting point of distance and collision measurement

Kinematic Model

- Describes the pose (position and orientation) of bodies in space with the help of the geometric model

- Kinematic chain: Several bodies, kinematically connected via joints (e.g. robot arm)
  - Closed kinematic chains
  - Open kinematic chains

- Purpose of kinematic model
  - Determining the relation between joint values and poses
  - Reachability analysis

Dynamic Model

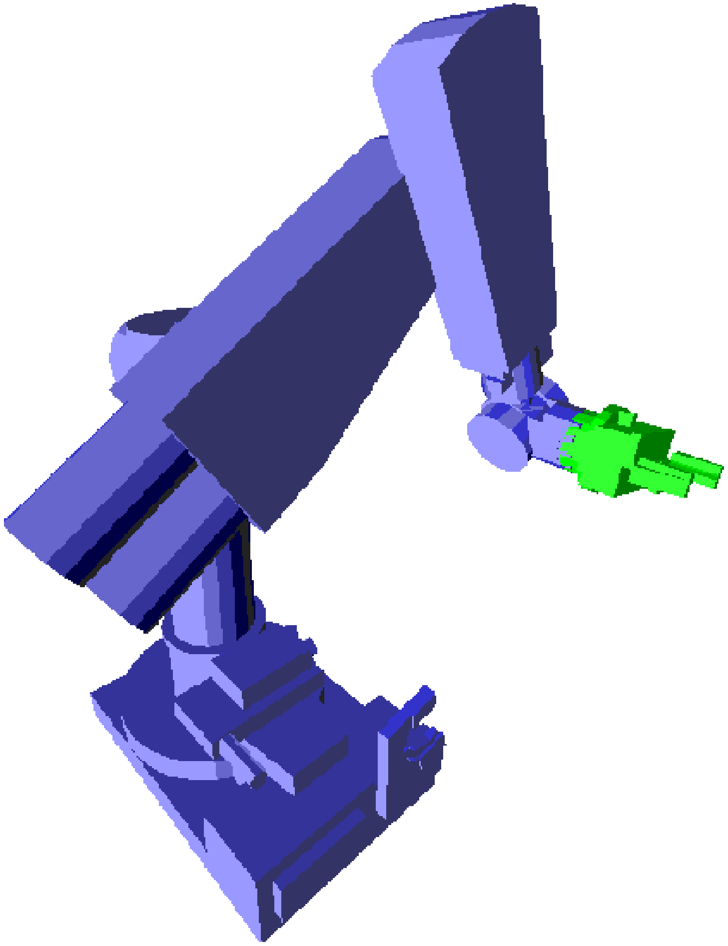
- Describes forces and moments acting in a mechanical multi-body system

- Purpose of dynamic model
  - Dimensioning of driving mechanism
  - Optimization of construction (light weight)
  - Consideration of bending and stiffness
  - Support for controller design

Geometric Model

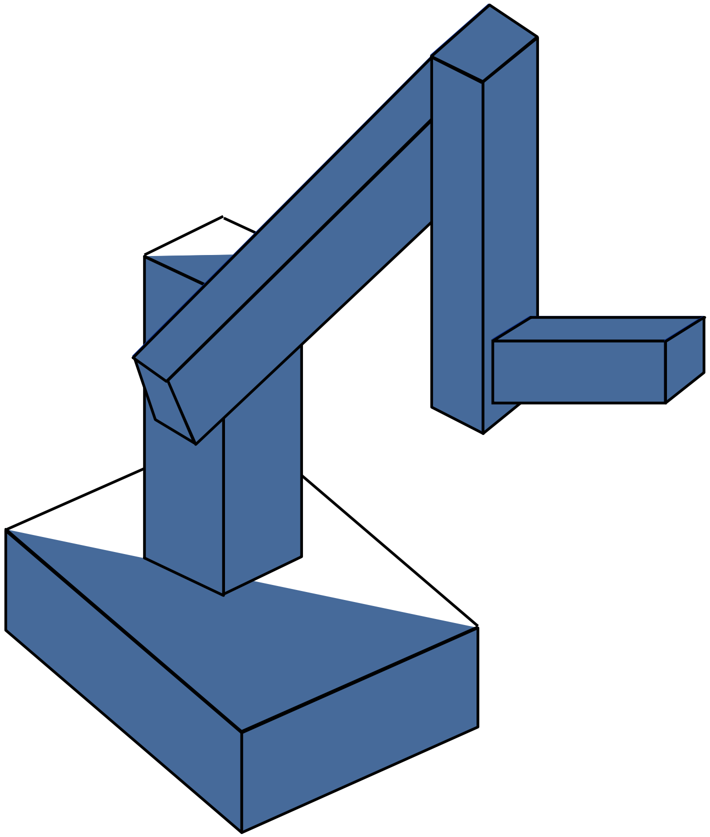
Classification:

- 2D-model
- 2,5D-model
- 3D-model
- Edge or wire-frame models
- Surface models
- Volumetric models



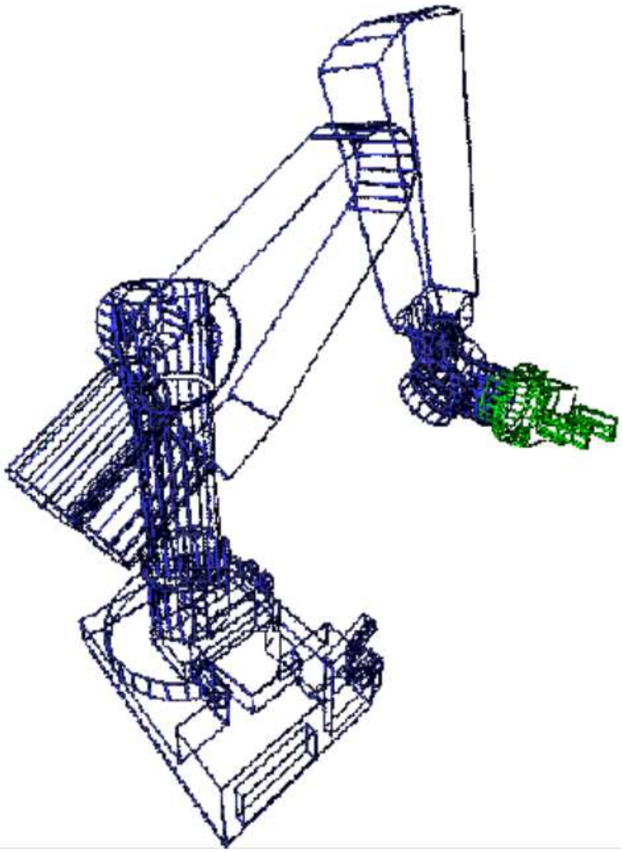
Geometric Model: Block World

- Bodies represented by enveloping cuboids (bounding boxes)
- Easy calculation with regard to collision avoidance



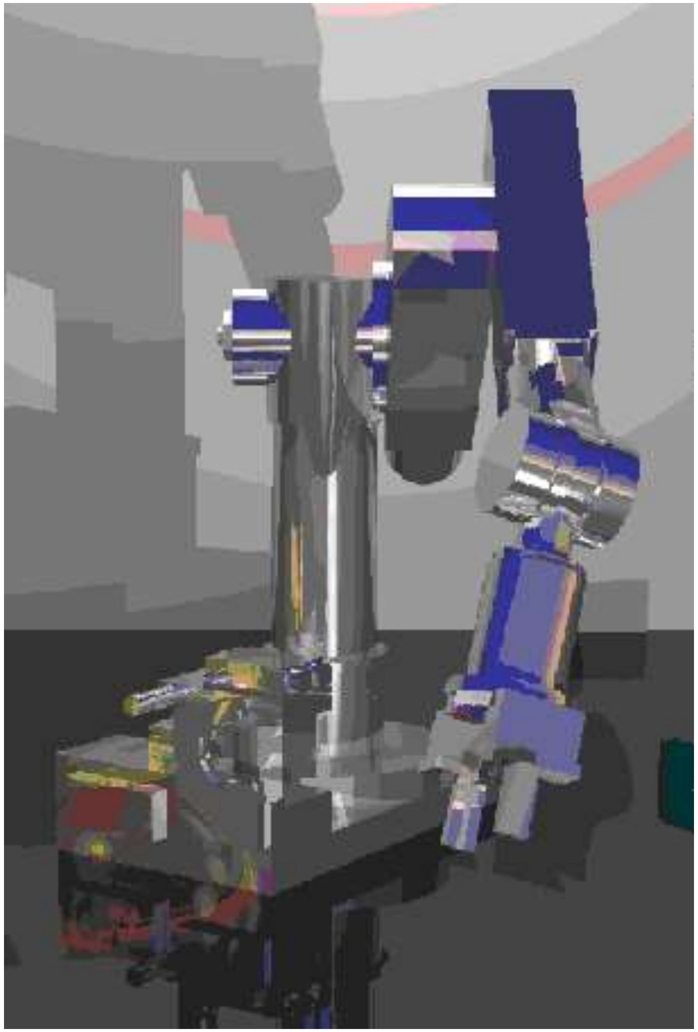
Geometric Model: Edge Model

- Bodies represented by polygons (edges)
- Quick visualization



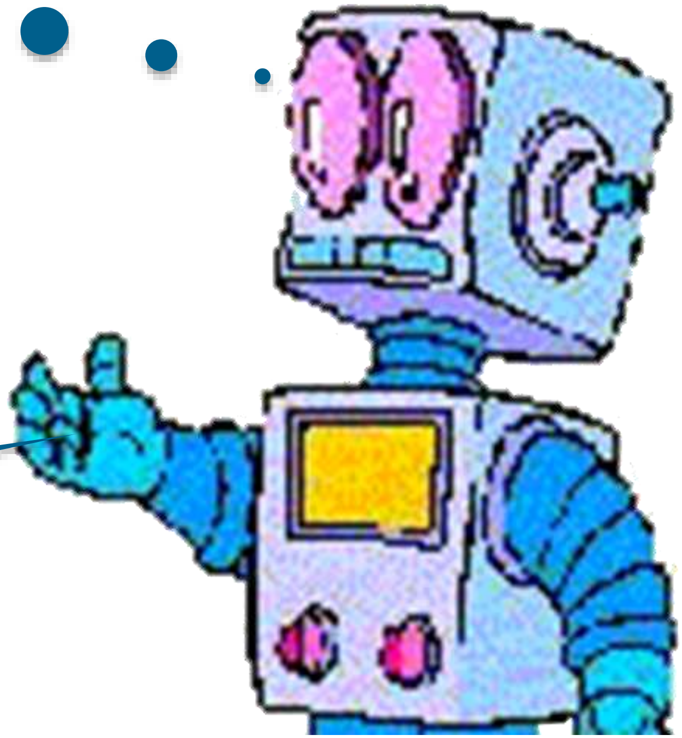
Geometric Models: Volumetric Models

- Precise representation of bodies
- Exact computation of contact points for collision avoidance
- Representation with animations



Direct Kinematic Problem

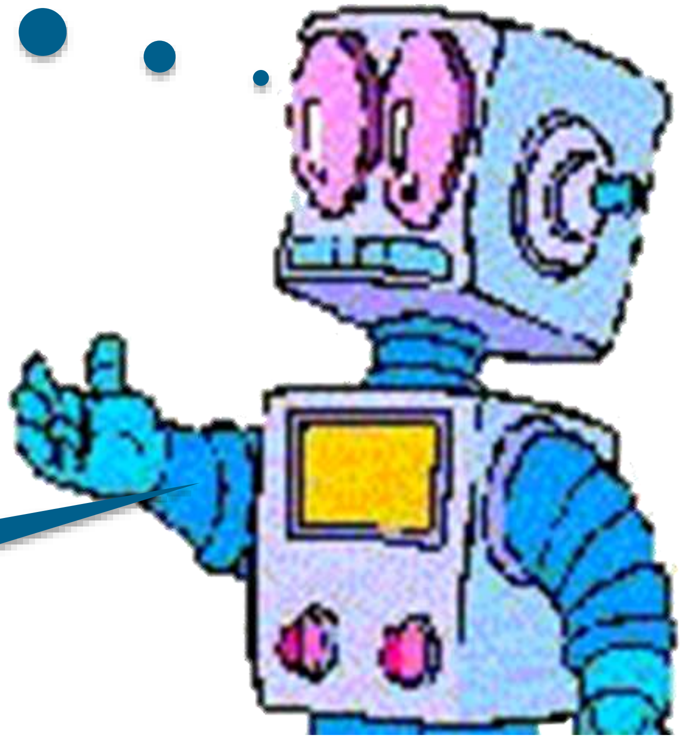
Where is my hand?



Find pose of TCP

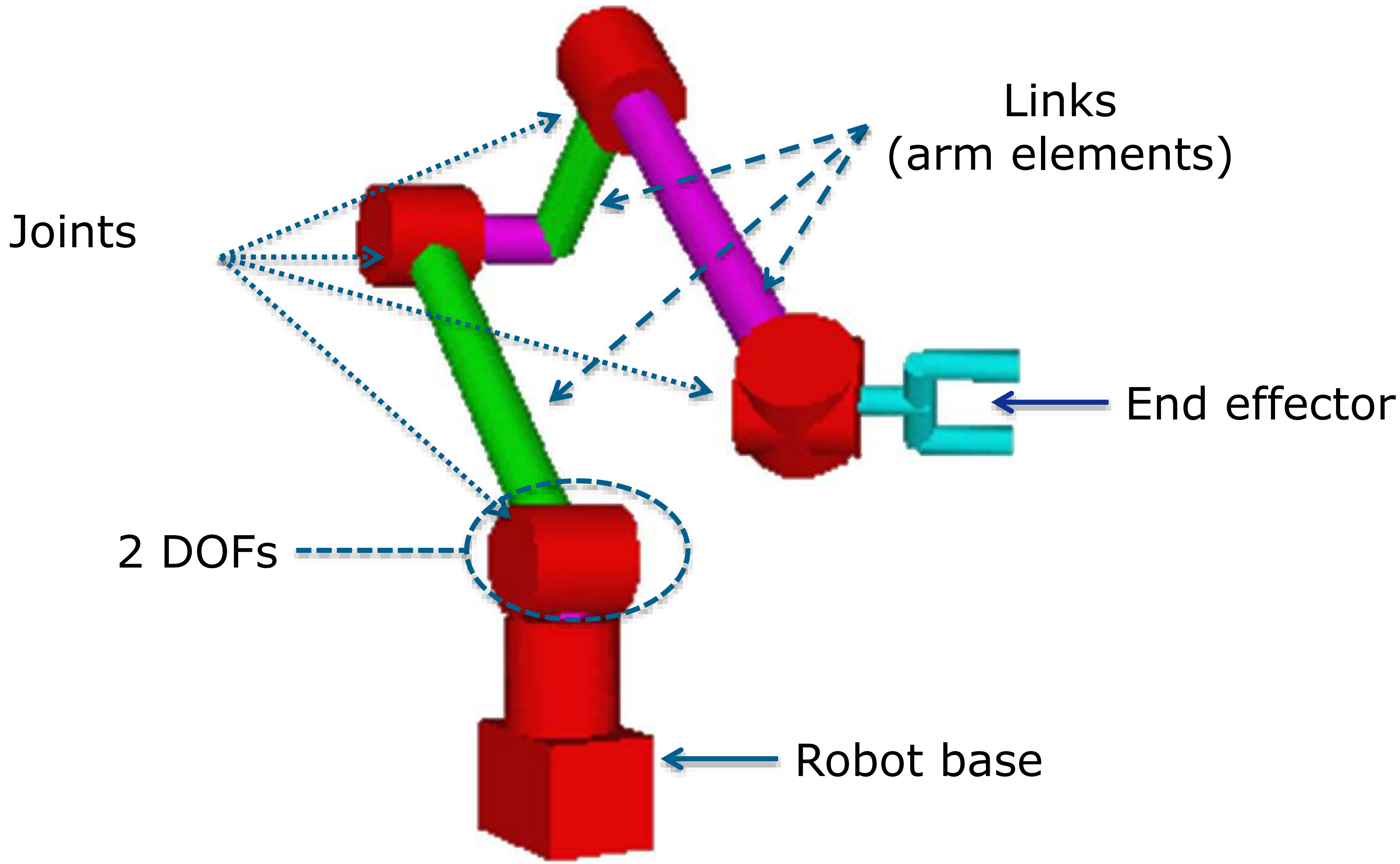
Inverse Kinematic Problem

How should I move my hand there?



Find the joint angles

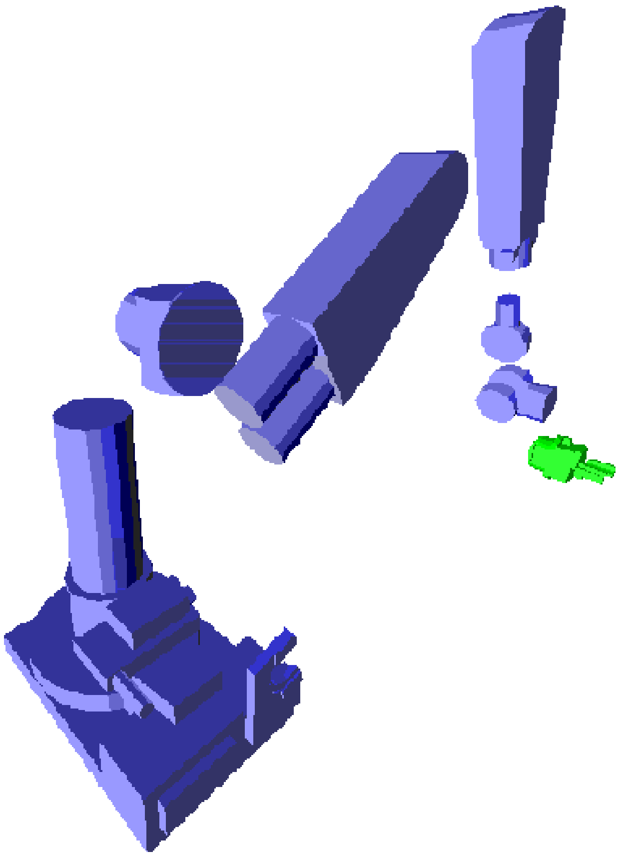
Kinematic Model: Links and Joints



Kinematic Model: Puma 260

Volumetric model

- Every arm element corresponds to a rigid body
- Every arm element is joined to the next via a linear or rotational joint
- Each joint has only one DoF (rot. or transl.)
- Kinematic pair = joint + joined arm elements



Puma 260

- 6-axis robot
- Basis and 6 arm elements (links)

Kinematic Model: Coordinate Systems

In order to describe the kinematics of a robot (kinematic chain), it is necessary to define the links' poses in relation to a reference coordinate system.

- Each link receives a fixed local CS
- Origin of each CS in joint responsible for moving given link
- A transformation matrix, relating the local CS to the reference system, needs to be determined for every link
- Transformation of local CS to reference CS via description vector or 4×4 homogenous transformation matrix.

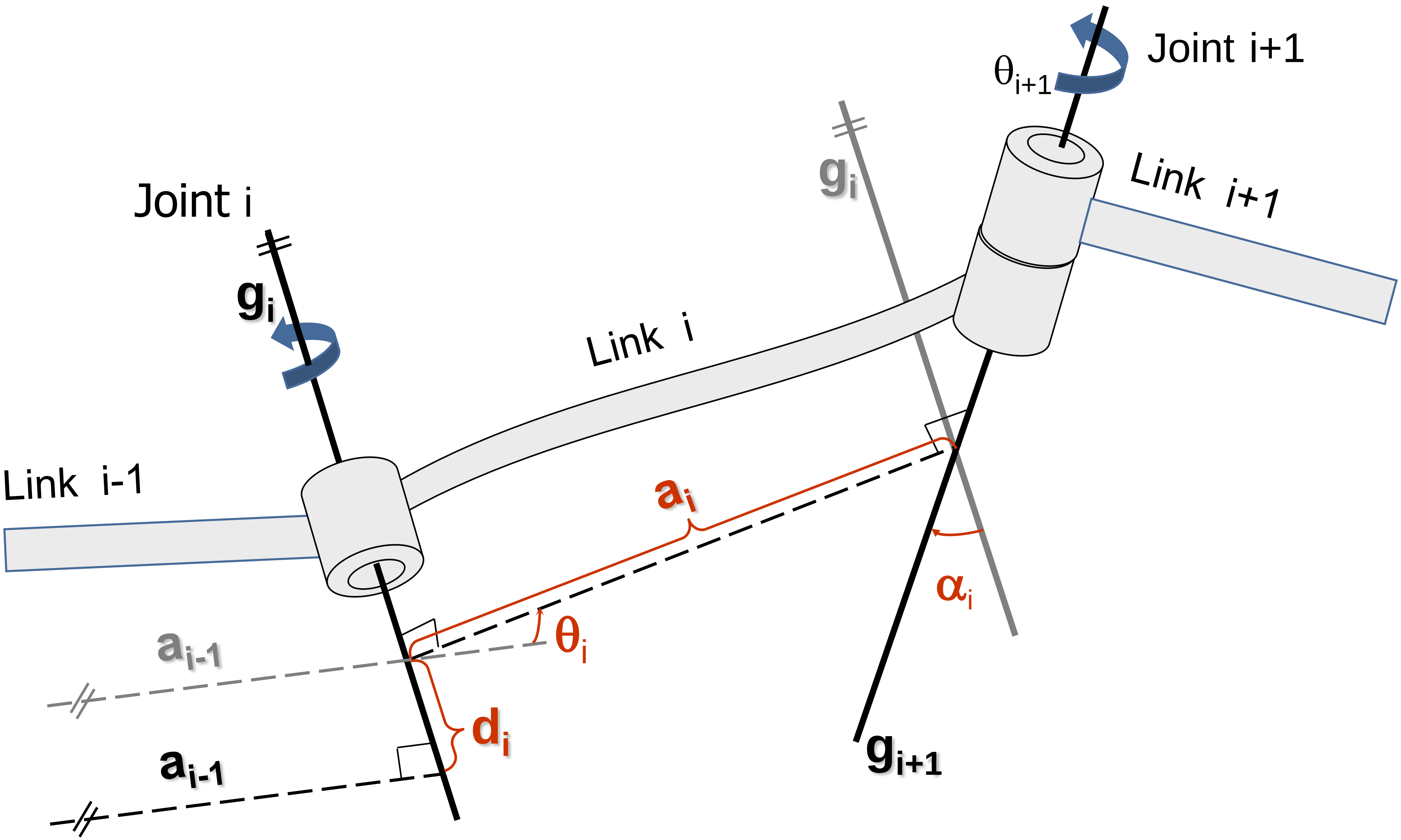
Link Parameters

- Every link i is connected through 2 confining joints i and $i+1$
- Let g_i and g_{i+1} be movement axes of joints (skewed to each other)
- Let a_i be the normal between g_i and g_{i+1}
- The distance of intersections of a_{i-1} and a_i with g_i is referred to as joint offset d_i
- Angle θ_i between a_{i-1} und a_i is referred to as joint angle
- Length of a_i (shortest distance between g_i and g_{i+1}) is called link length
- Angle α_i between g_i und g_{i+1} is called link twist

Parameters of Joints

Parameter	Symbol	Rotational joint	Prismatic joint
Link length	a	Invariant	Invariant
Link twist	α	Invariant	Invariant
Joint distance	d	Invariant	Variable
Joint angle	θ	Variable	Invariant

Derivation of Joint Distance and Angle

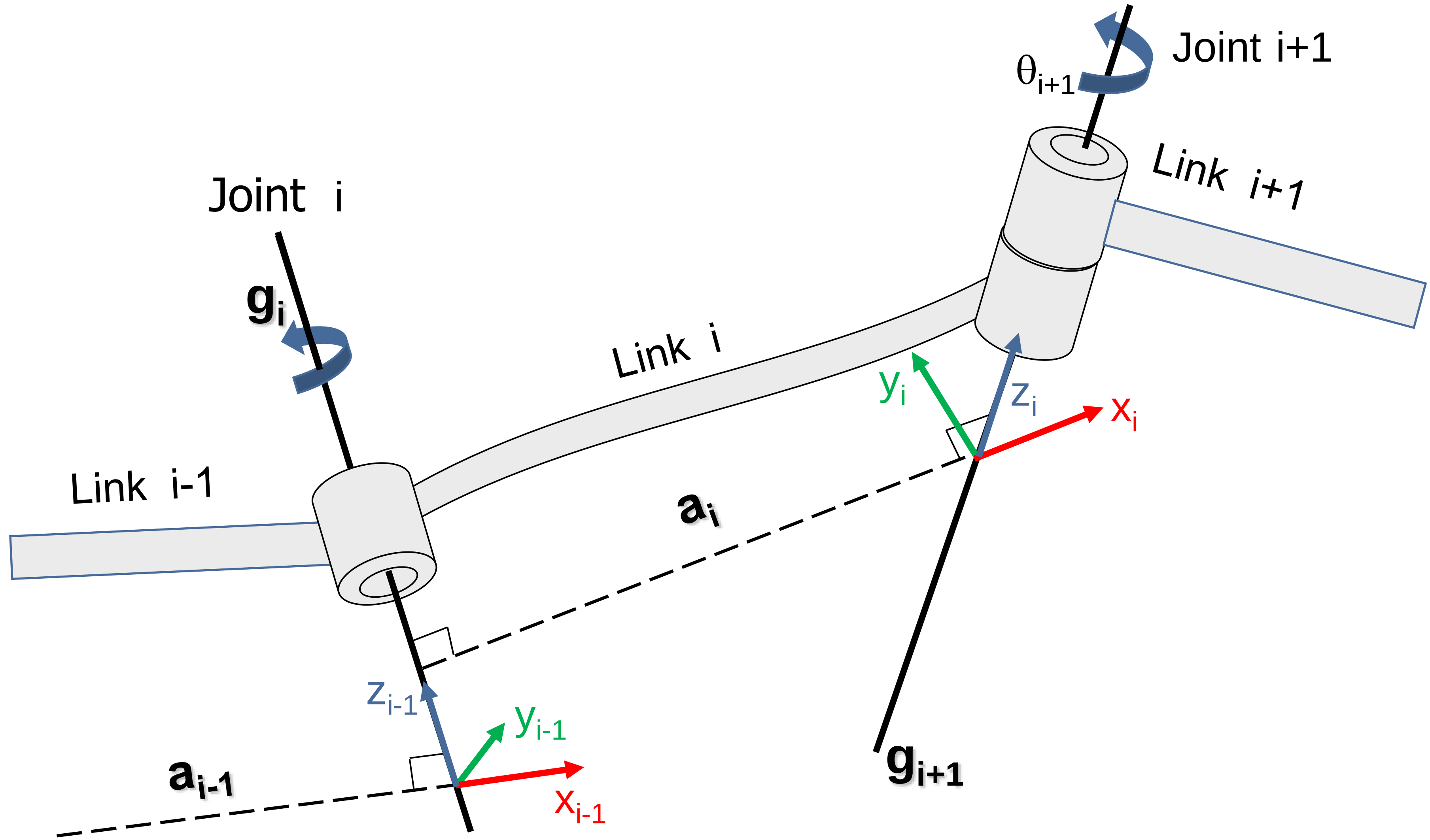


Denavit-Hartenberg-Convention

Definition of coordinate systems for every joint:

- Origin of CS i lies in intersection of a_i and g_{i+1}
- Axis z_i lies along the joint axis g_{i+1}
- Axis x_i follows as extension of normal a_i
- Complemented by axis y_i, so that a clockwise CS follows
- Origin of CS 0 can be freely placed along g_1
- The last coordinate system lies in the end effector

Denavit-Hartenberg-Convention



Kinematic Model

- Transformation from OCS of link i to OCS of link $i-1$ via Denavit-Hartenberg-Transformation

- Requirements
  - All CS follow the Denavit-Hartenberg-Convention
  - Parameters of link i are known

- Literature: Denavit, Hartenberg: „A Kinematic Notation for Lower-Pair Mechanisms Based on Matrices", Journal of Applied Mechanics, vol 77, pp 215-221

Denavit-Hartenberg-Transformation

Transformation from OCS_i to OCS_{i-1}

(1) Rotation θ_i around z_{i-1}-axis so that x_{i-1}-axis lies parallel to x_i-axis

$$R_{z_{i-1}}(\theta_i) = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(2) Translation d_i along z_{i-1}-axis to the intersection point of z_{i-1} and x_i

$$T_{z_{i-1}}(d_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg-Transformation

Transformation from OCS_i to OCS_{i-1}

(3) Translation a_i along x_i-axis in order to make the origins of the coordinate systems congruent

$$T_{x_i}(a_i) = \begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(4) Rotation α_i around x_i-axis, to transform the z_{i-1}-axis into z_i

$$R_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Denavit-Hartenberg-Transformation

In matrix notation, the D-H-Transformation of the coordinate system of link $i-1$ to i looks like:

$$ {}^{i-1}_{i}A = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(a_i) \cdot R_{x_i}(\alpha_i) $$

$$ = \begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Denavit-Hartenberg-Transformation (Inverse)

The inverse of transformation matrix ${}^{i-1}_{i}A$ corresponds to the transformation from CS_{i-1} to CS_i:

$${}^{i-1}_{i}A^{-1} = {}^{i}_{i-1}A$$

$$= \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & \sin\alpha_i & -d_i \sin\alpha_i \\ \sin\theta_i \cdot \sin\alpha_i & -\sin\alpha_i \cos\theta_i & \cos\alpha_i & -d_i \cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Direct Kinematic Model

To determine the pose of the manipulator (Tool Center Point, TCP) in relation to the BCS, the D-H-matrices are multiplied in order of the corresponding links:

$$ {}^{\mathrm{Basis}}_{\mathrm{TCP}}A = {}^{\mathrm{Basis}}_{1}A(\theta_1) \cdot {}^{1}_{2}A(\theta_2) \cdots {}^{n-2}_{n-1}A(\theta_{n-1}) \cdot {}^{n-1}_{n}A(\theta_n) $$

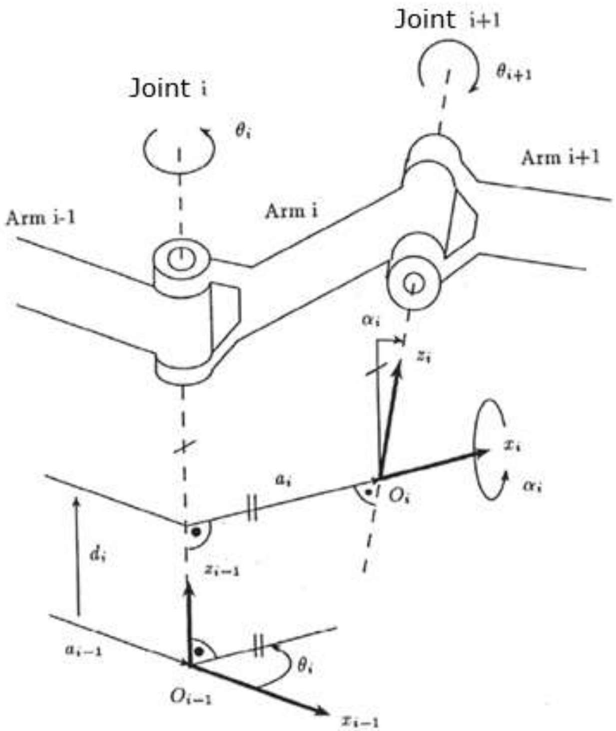
Direct Kinematics: Approach with D-H

(1) Finding the normal a_i

- Joint axes g_i
- a_i points from g_i to g_{i+1}

(2) Defining the CS

- Origin O_i in the intersection of a_i with g_{i+1}
- x_i lies on normal a_i and points in the same direction
- z_i lies on g_{i+1}, in the direction allowing joint rotation or translation in mathematically positive sense
- y_i completes the clockwise CS; at TCP, $y_i$indicates width of opening



Direct Kinematics: Approach with D-H

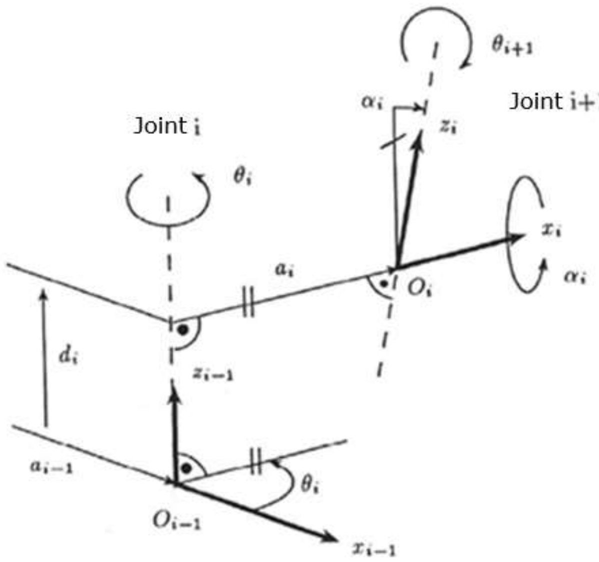
Special cases of $(1), (2)$

- g_i and g_{i+1} intersect
  - Direction of x_i not defined
  - x_i arises from x_{i-1} through smallest possible rotation around z_{i-1}
- g_i and g_{i+1} are parallel or collinear
  - Intersection of a_i and g_{i+1} not well defined
  - Determine the normals by backstepping (recursively)
  - Starting at next uniquely identifiable origin O_j with $j > i$
  - At last joint, the origin is in the center of TCP

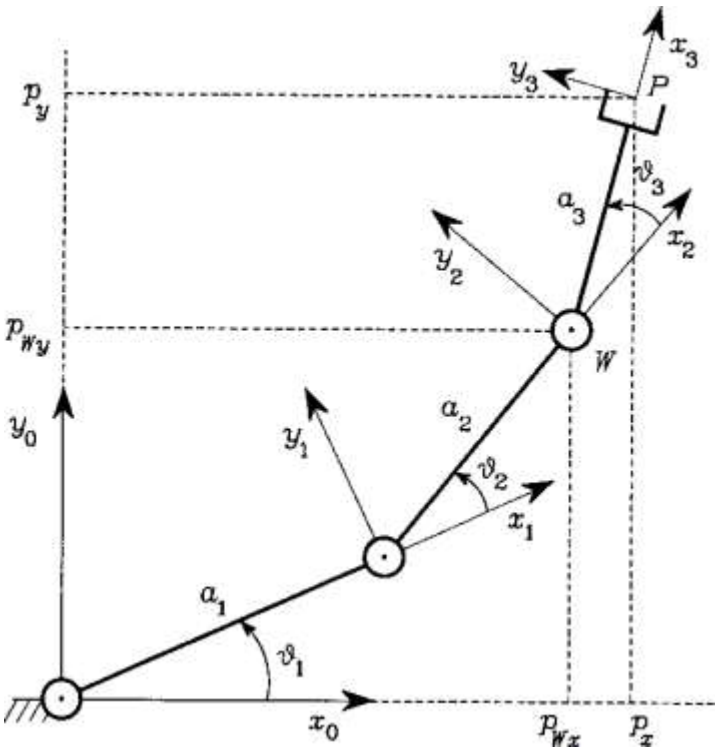
Direct Kinematics: Approach with D-H

(3) Determining the transformation matrix ${}^{i+1}_{i}A$

- Rotation of CS of joint i around z_{i-1} with joint angle θ_i
  $\rightarrow x'_{i-1}$ is parallel to x_i
- Translation by d_i along z_{i-1}
  $\rightarrow$ Origin lies in intersection of z_{i-1} and x_i
- Translation by $|a_i|$ along x_i
  $\rightarrow$ Origins are congruent
- Rotation around x_i with twist α_i
  $\rightarrow Z'_{i-1}$ is parallel to z_i
- ${}^{i-1}_{i}A = R_{z_{i-1}}(\theta_i) \cdot T_{z_{i-1}}(d_i) \cdot T_{x_i}(|a_i|) \cdot R_{x_i}(\alpha_i)$



Direct Kinematics: Example 1



Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

$$\begin{bmatrix} \cos\theta_i & -\sin\theta_i \cdot \cos\alpha_i & \sin\theta_i \cdot \sin\alpha_i & a_i \cdot \cos\theta_i \\ \sin\theta_i & \cos\theta_i \cdot \cos\alpha_i & -\cos\theta_i \cdot \sin\alpha_i & a_i \cdot \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

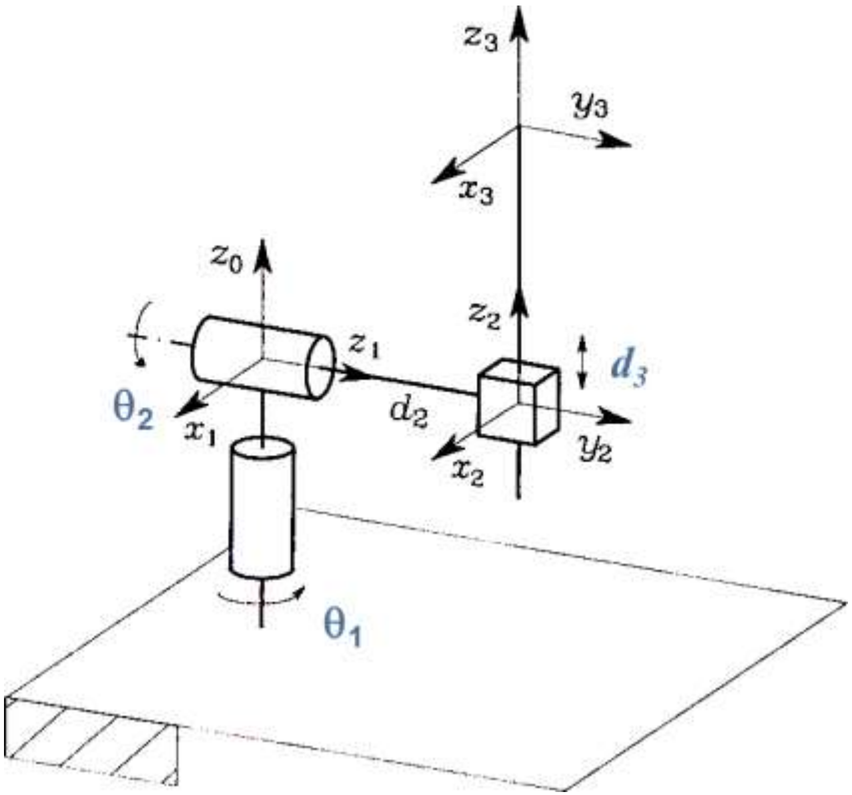
$$\Rightarrow {}_{i}^{i-1}A = \begin{bmatrix} c_i & -s_i & 0 & a_i \cdot c_i \\ s_i & c_i & 0 & a_i \cdot s_i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Direct Kinematics: Example 1

Result:

$$ {}^{0}_{3}A = {}^{0}_{1}A \cdot {}^{1}_{2}A \cdot {}^{2}_{3}A = \begin{bmatrix} c_{123} & -s_{123} & 0 & a_1 \cdot c_1 + a_2 \cdot c_{12} + a_3 \cdot c_{123} \\ s_{123} & c_{123} & 0 & a_1 \cdot s_1 + a_2 \cdot s_{12} + a_3 \cdot s_{123} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Direct Kinematics: Example 2



Joint	a_i	α_i	d_i	θ_i
1	0	$-90°$	0	θ_1
2	0	$90°$	d_2	θ_2
3	0	0	d_3	0

$$ {}_1^0A = \begin{bmatrix} c_1 & 0 & -s_1 & 0 \\ s_1 & 0 & c_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

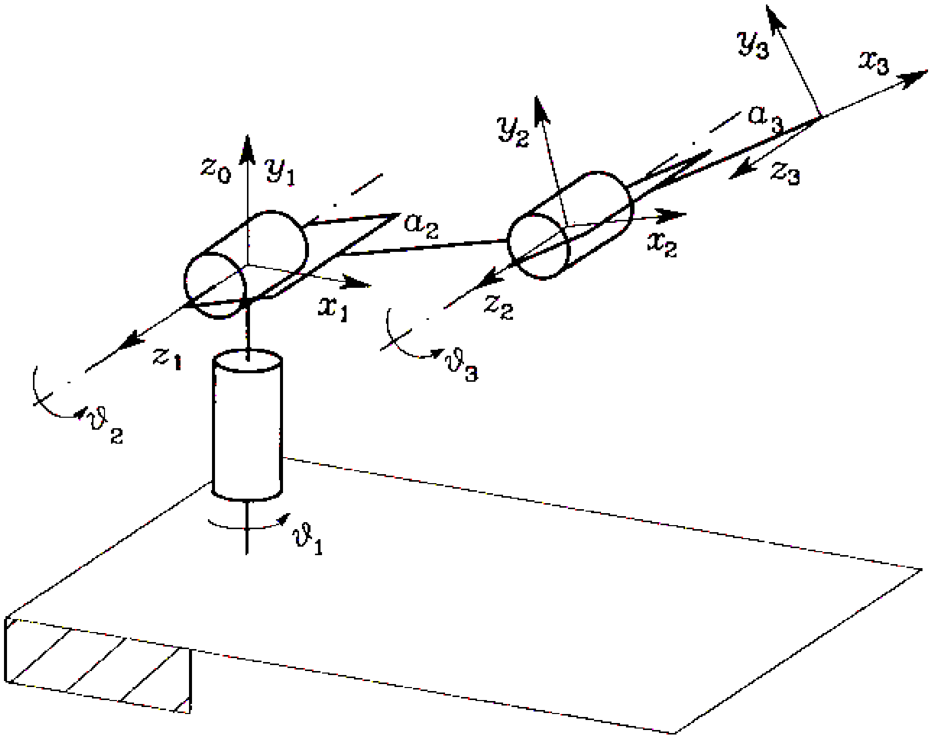
$$ {}_2^1A = \begin{bmatrix} c_2 & 0 & s_2 & 0 \\ s_2 & 0 & -c_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Direct Kinematics: Example 2

$$
{}^{2}_{3}A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

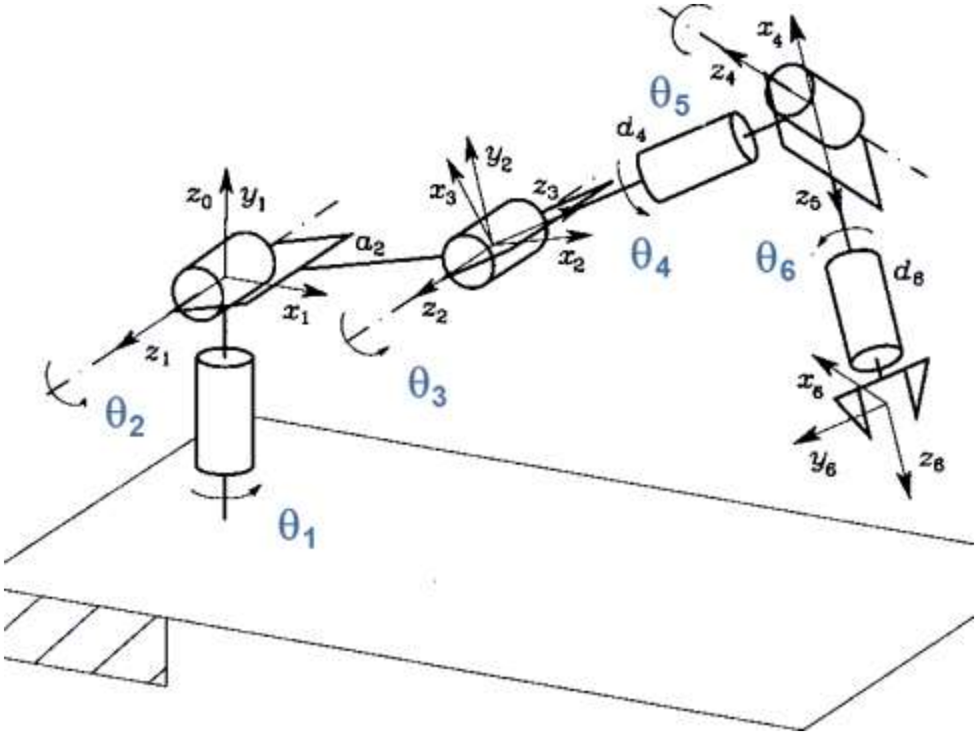
$$
\Rightarrow {}^{3}_{0}A = {}^{0}_{1}A \cdot {}^{1}_{2}A \cdot {}^{2}_{3}A = \begin{bmatrix} c_1 \cdot c_2 & -s_1 & c_1 \cdot s_2 & c_1 \cdot s_2 \cdot d_3 - s_1 \cdot d_2 \\ s_1 \cdot c_2 & c_1 & s_1 \cdot s_2 & s_1 \cdot s_2 \cdot d_3 + c_1 \cdot d_2 \\ -s_2 & 0 & c_2 & c_2 \cdot d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Direct Kinematics: Example 3



Joint	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Direct Kinematics: Example 4



Joint	a_i	α_i	d_i	θ_i
1	0	90°	0	θ_1
2	a_2	0	0	θ_2
3	0	90°	0	$\theta_3 + 90°$
4	0	−90°	d_4	θ_4
5	0	90°	0	$\theta_5 - 90°$
6	0	0	d_6	$\theta_6 + 90°$

Principle Screws

- For a central screw we have

$$A_S(h,\phi,\vec{g}) = \begin{bmatrix} R_{\vec{g}}(\theta) & h\vec{g} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times4}$$

- If $\vec{g} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, i.e. $\vec{g} = \vec{x}_{i-1}$

$$\Longrightarrow A(h,\phi,\vec{x}_{i-1}) = \begin{bmatrix} 1 & 0 & 0 & h \\ 0 & C\theta & -S\theta & 0 \\ 0 & S\theta & C\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Compound Screws

- Similar to homogeneous Matrices, screws can be combined:

$$ {}^{k-2}_{k}S\left(h,\phi,\vec{g},\vec{P}\right) = {}^{k-2}_{k-1}S\left(h_1,\phi_1,\vec{g}_1,\vec{P}_1\right){}^{k-1}_{k}S\left(h_2,\phi_2,\vec{g}_2,\vec{P}_2\right) $$

- General solution is lengthy, but not needed.

D&H with Screws

- We have θ_i, d_i, a_i and α_i from D&H for ${}^{i-1}_{i}A$.
- Then we have two central screws:

$$
{}^{i-1}_{i}A = S(d_i, \theta_i, z_{i-1}) \star S(a_i, \alpha_i, x_i) \quad (\star \text{ consecutive processing})
$$

$$
= \begin{bmatrix} C\theta_i & -S\theta_i C\theta_i & S\theta_i S\alpha_i & a_i C\theta_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\theta_i & a_i S\theta_i \\ 0 & S\alpha_i & C\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

D&H with Screws-compound

- From this, we can compute the compound $S\left(h,\phi,\vec{g},\vec{P}\right)$:
  - $\cos\phi = \frac{1}{2}\left(tr\left({}^{G}R_B\right) - 1\right)$

    $= \frac{1}{2}(\cos\theta_i + \cos\theta_i\cos\alpha_i + \cos\alpha_i - 1)$

    - $\vec{g} = \frac{1}{2S\phi}\begin{bmatrix} \sin\alpha_i + \cos\theta_i\sin\alpha_i \\ \sin\theta_i\sin\alpha_i \\ \sin\theta_i + \cos\alpha_i\sin\theta_i \end{bmatrix}$

D&H with Screws-compound

- As deduced before
- Screw $S(h, \phi, \vec{g}, \vec{P})$, where $P = (x, y, z)$ with $\mathrm{x} = 0$:

- $$\begin{bmatrix} h \\ y \\ z \end{bmatrix} = \begin{bmatrix} g_1 & -r_{12} & -r_{13} \\ g_2 & 1 - r_{22} & -r_{23} \\ g_3 & -r_{32} & 1 - r_{33} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & 1 - C\theta_i C\alpha_i & -C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & S\alpha_i & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

Consider that $\vec{g}$ is not parallel to x-axis

D&H with Screws-compound

- As deduced before
- Screw $S(h,\phi,\vec{g},\vec{P})$, where $P=(x,y,z)$ with $y=0$:

- $$\begin{bmatrix} h \\ x \\ z \end{bmatrix} = \begin{bmatrix} g_1 & 1-r_{11} & -r_{13} \\ g_2 & -r_{21} & -r_{23} \\ g_3 & -r_{31} & 1-r_{33} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & 1-C\theta_i & S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & S\theta_i & -C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & 0 & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

Consider that $\vec{g}$ is not parallel to y-axis

D&H with Screws-compound

- As deduced before
- Screw $S(h, \phi, \vec{g}, \vec{P})$, where $P=(x,y,z)$ with $z = 0$
  - $$\begin{bmatrix} h \\ x \\ y \end{bmatrix} = \begin{bmatrix} g_1 & 1 - r_{11} & -r_{12} \\ g_2 & -r_{21} & 1 - r_{22} \\ g_3 & -r_{31} & -r_{32} \end{bmatrix}^{-1} \begin{bmatrix} r_{14} \\ r_{24} \\ r_{34} \end{bmatrix}$$

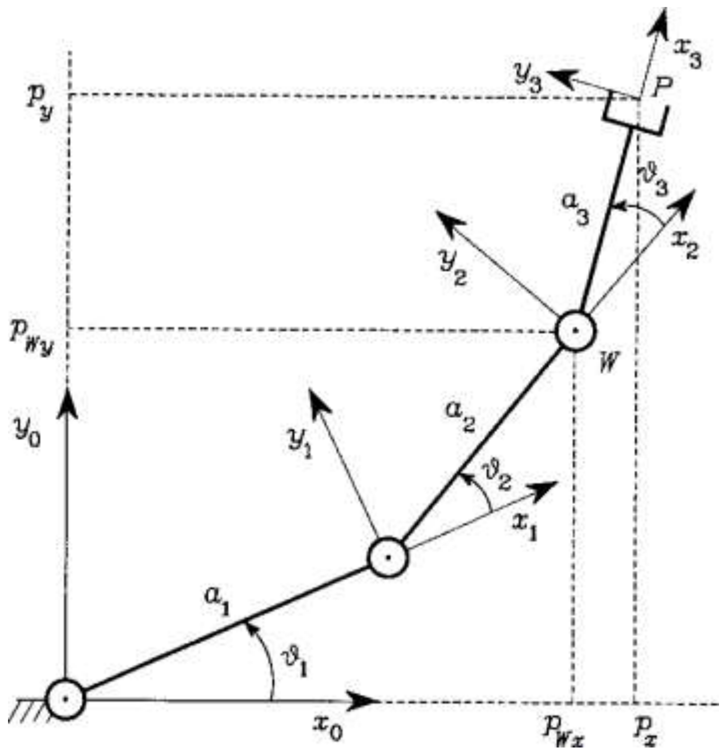
$$= \frac{1}{2S\phi} \begin{bmatrix} S\alpha_i + C\theta_i S\alpha_i & 1 - C\theta_i & -S\theta_i S\alpha_i \\ S\theta_i S\alpha_i & S\theta_i & 1 - C\theta_i S\alpha_i \\ S\theta_i + C\alpha_i S\theta_i & 0 & C\alpha_i \end{bmatrix}^{-1} \begin{bmatrix} a_i C\theta_i \\ a_i S\theta_i \\ d \end{bmatrix}$$

Consider that $\vec{g}$ is not parallel to z-axis

For usage check first y-axis!

Direct Kinematics: Example 1

- $\phi_i = \theta_i$
- $\hat{g}_i = (0,0,1)^T$
- $h_i = 0$
- $\vec{P}_i = \left(\frac{a_i(2C\theta-1)}{4S\theta}, \frac{a_i(2C\theta-1)}{4C\theta-1}, 0\right)^T$

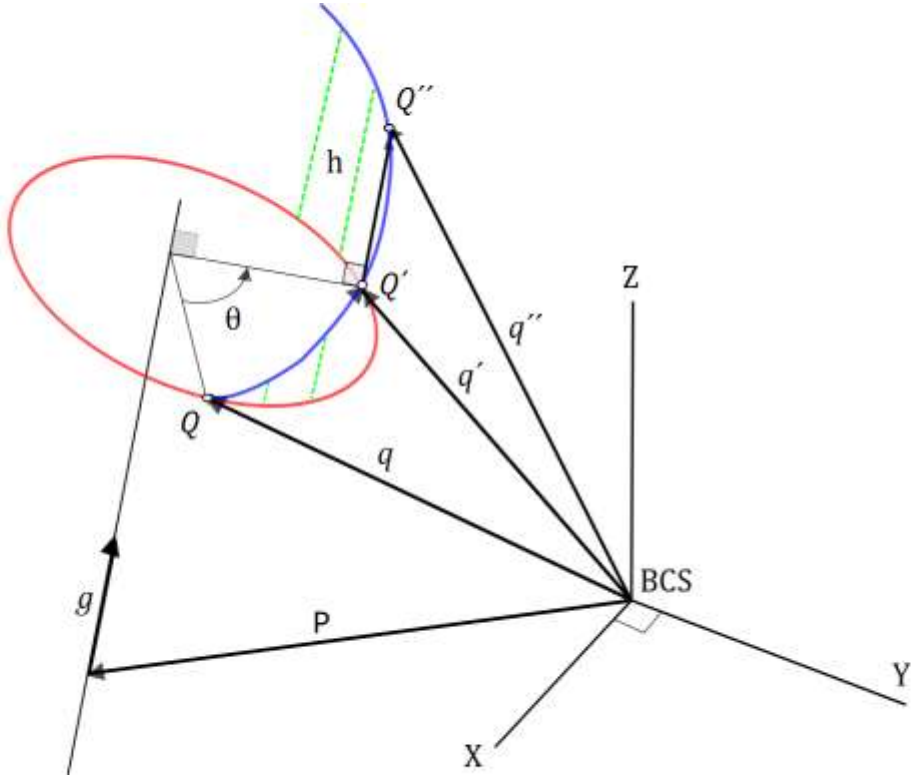


Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Exponential Representation of a Screw

- Similar to exponential coordinates for a rotation, a transformation can be notated.

- $$\vec{q}^{\,''} = \vec{P} + e^{\phi\hat{g}(\vec{q}-\vec{P})} + h\vec{g}$$
  $$=: [T]\begin{bmatrix}\vec{q}\\1\end{bmatrix}$$



Where,

- $$[T]=\begin{bmatrix}e^{\phi\vec{g}} & \left(I-e^{\phi\vec{g}}\right)\vec{P}+h\vec{g}\\ 0 & 1\end{bmatrix}$$

Exponential representation of central Screws

- For central screws $\vec{P} = \vec{0}$ the same holds:
- $[T] = \begin{bmatrix} e^{\phi \hat{\vec{g}}} & h\vec{g} \\ 0 & 1 \end{bmatrix}$

Note that:

$$e^{\begin{bmatrix} \phi \hat{g} & 0 \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} e^{\phi \hat{g}} & 0 \\ 0 & 1 \end{bmatrix}$$

and,

$$e^{\begin{bmatrix} 0 & h\vec{g} \\ 0 & 0 \end{bmatrix}} = \begin{bmatrix} I_3 & h\vec{g} \\ 0 & 1 \end{bmatrix}$$

Exponential representation of D&H

- Therefore:

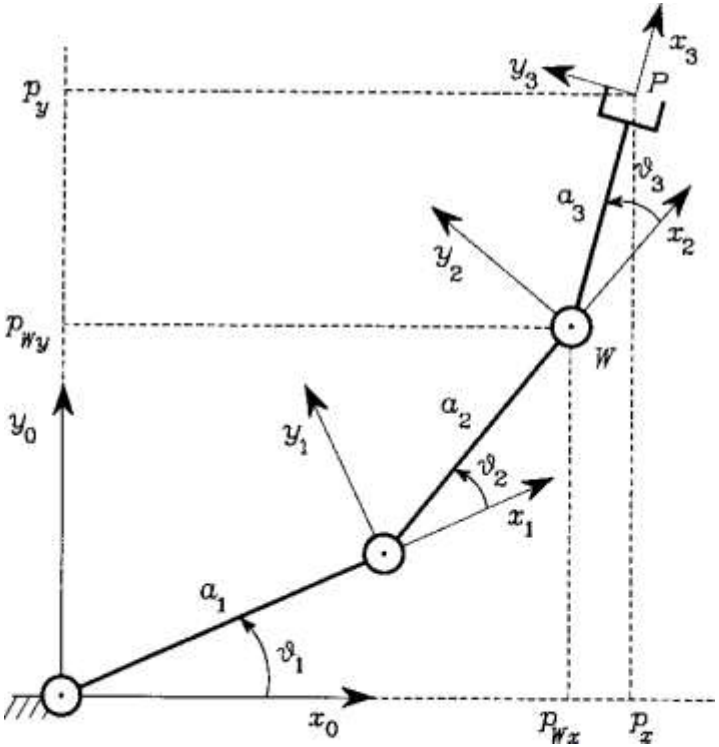
$$[T] = e^{\begin{bmatrix} \phi\hat{g} & h\vec{g} \\ 0 & 0 \end{bmatrix}} =: e^{\xi(h,\phi,\vec{g})}$$

- For D&H this means:

$$\begin{aligned} {}^{i-1}_{\ \ i}A &= S(d_i, \theta, z_{i-1}) \star S(a_i, \alpha_i, x_{i-1}) \\ &= e^{\xi(d_i,\theta_i,z_{i-1})} e^{\xi(a_i,\alpha_i,x_i)} \end{aligned}$$

Direct Kinematic: Example 1

- $$\begin{aligned} {}^{i-1}_{i}A &= e^{\xi(0,\theta_i,z_{i-1})} e^{\xi(a_i,0,x_i)} \\ &= e^{\begin{bmatrix} \theta_i \hat{z}_{i-1} & 0 \\ 0 & 0 \end{bmatrix}} e^{\begin{bmatrix} 0 & a_i x_i \\ 0 & 0 \end{bmatrix}} \\ &= \begin{bmatrix} e^{\theta_i \, z_{i-1}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I_3 & a_i x_i \\ 0 & 1 \end{bmatrix} \end{aligned}$$



Joint	a_i	α_i	d_i	θ_i
1	a_1	0	0	θ_1
2	a_2	0	0	θ_2
3	a_3	0	0	θ_3

Dual-Quaternion Arithmetic Operations

- The elementary arithmetic operations for dual-quaternions are (Reminder: $Q = (q_1, q_2, q_3, q_4), \quad q_i = q_{ri} + \varepsilon \cdot q_{di}, \quad \varepsilon^2 = 0 \;\; with$ r as real part/quartenion and d as dual part/quartenion):

  - Scalar multiplication: $sq = sq_r + sq_d\varepsilon$

  - Addition: $q_1 + q_2 = q_{r1} + q_{r2} + (q_{d1} + q_{d2})\varepsilon$

  - Multiplication: $q_1 q_2 = q_{r1} q_{r2} + (q_{r1} q_{d2} + q_{d1} q_{r2})\varepsilon$

  - Conjugate: $q^* = q_r^* + q_d^*\varepsilon$

Dual-Quaternion Arithmetic Operations

- Magnitude: $\|q\| = qq^*$
- Unit condition: $\|q\| = 1$

$$q_r^* q_d + q_d^* q_r = 0$$

- The unit dual-quaternion is our key concern as it can represent any rigid rotational and translational transformations.

Dual-Quaternion representation

- Only rotation around $\vec{g}$ with angle ϕ, then:
  - $q_{rot} = ((cos\left(\frac{\phi}{2}\right), 0), (g_1\, sin\left(\frac{\phi}{2}\right), 0), (g_2\, sin\left(\frac{\phi}{2}\right), 0), (g_3\, sin\left(\frac{\phi}{2}\right), 0))$
  - $= (cos\left(\frac{\phi}{2}\right), g_1\, sin\left(\frac{\phi}{2}\right), g_2\, sin\left(\frac{\phi}{2}\right), g_3\, sin\left(\frac{\phi}{2}\right))$
- Only translation by $\vec{t}$ with no rotation
  - $q_{trans} = \left((1,0), \left(0, \frac{t_1}{2}\right), \left(0, \frac{t_2}{2}\right), \left(0, \frac{t_3}{2}\right)\right)$
  - $= (1, \varepsilon\frac{t_1}{2}, \varepsilon\frac{t_2}{2}, \varepsilon\frac{t_3}{2})$

Dual-Quaternions representation

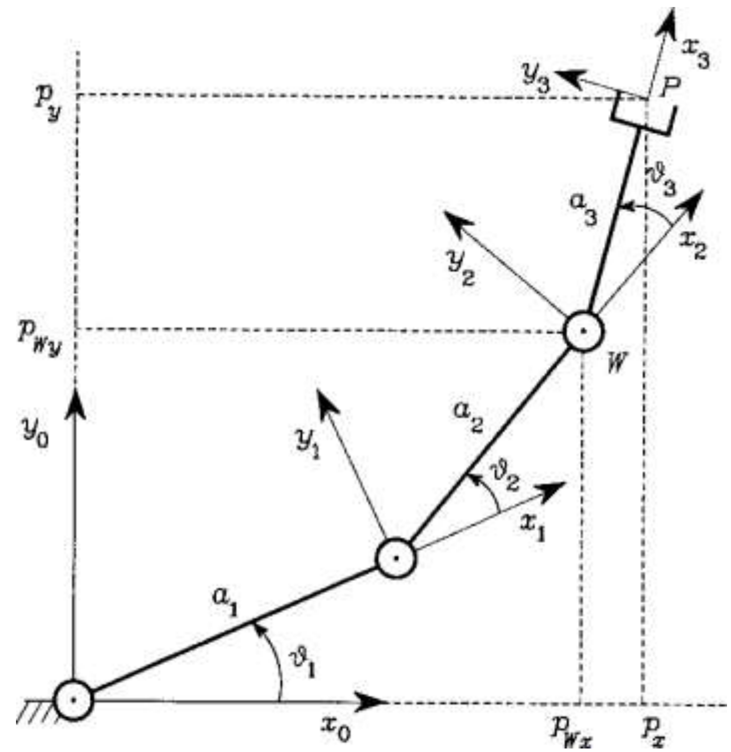
- Combination of both as q:

  - $q_r = q_{rot,r}$
  - $q_d = \frac{q_{trans,d}q_{rot,r}}{2}$

$$
\begin{aligned}
q = (&(C\phi - \varepsilon \frac{S\phi(g_1 t_1 + g_2 t_2 + g_3 t_3)}{2}),\\
&(g_1 S\phi + \varepsilon \frac{C\phi(t_2 g_3 - t_3 g_2)}{2}),\\
&(g_2 S\phi + \varepsilon \frac{C\phi(t_2 g_3 - t_3 g_2)}{2}),\\
&(g_3 S\phi + \varepsilon \frac{C\phi(t_2 g_3 - t_3 g_2)}{2}))
\end{aligned}
$$

Direct Kinematic: Example 1

- ${}^{i-1}_{i}q = (C\theta, \varepsilon a_i S\theta, \varepsilon a_i C\theta, S\theta)$
- And the transformation is then:
  - $P' = {}^{i-1}_{i}q \cdot P \cdot {}^{i-1}_{i}q^*$



Joint	a_i	α_i	d_i	θ_i
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Robot Kinematics

- Describes relations between joint angle space and the end effector‘s pose space in world coordinates
  - Joint angle space:
    Robot coordinates, configuration space
  - EE: Abbreviation for end effector
- Direct kinematic problem (forward kinematics)
- Inverse kinematic problem (backward kinematics)

Direct Kinematic Problem

- The manipulator‘s pose is to be determined from D-H-parameters and joint

- TCP pose in relation to the BCS (basis)

$$ {}^{\mathrm{BASIS}}_{\mathrm{TCP}}A = {}^{\mathrm{BASIS}}_{1}A(\theta_1) \cdot {}^{1}_{2}A(\theta_2) \cdots {}^{n-2}_{n-1}A(\theta_{n-1}) \cdot {}^{n-1}_{n}A(\theta_n) $$

- $\vec{\theta}$ given

Direct Kinematics Summary

- Sketch of the manipulator
- Enumerate links: Basis $= 0$, last link $= n$
- Identify and enumerate the joints
- Draw axes z_i for every joint i
- Determine parameters a_i between z_{i-1} and z_i
- Draw x_{i-1}-axes
- Determine parameter α_{i-1} (twist around x_{i-1}-axes)
- Determine parameter d_i (offset)
- Determine angle θ_i around z_i-axes
- Joint-transformation matrices $A_{i-1,1}$

Next Lecture …

Robot modelling

- Inverse kinematic problem
- Algebraic and geometric solutions
- Numerical methods
- Optimization