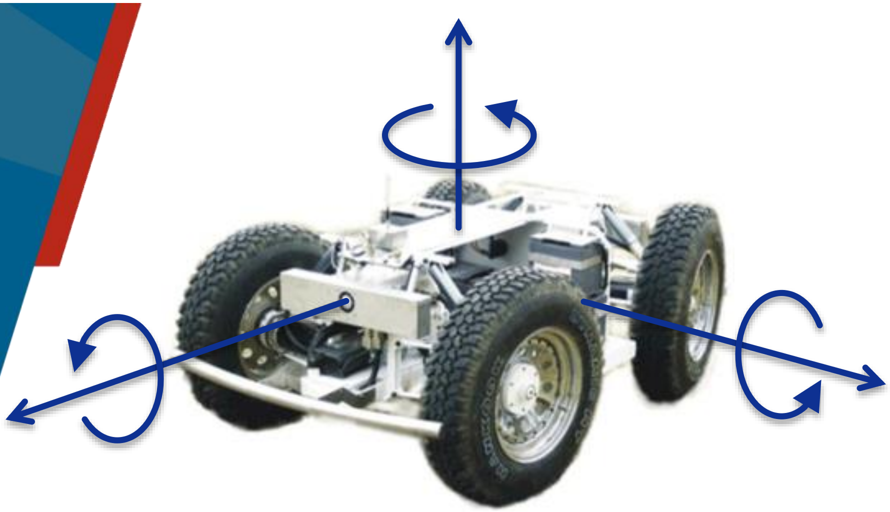
Autonomous Mobile Robots (AMR)

4. Modeling



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  - Geometric Solution
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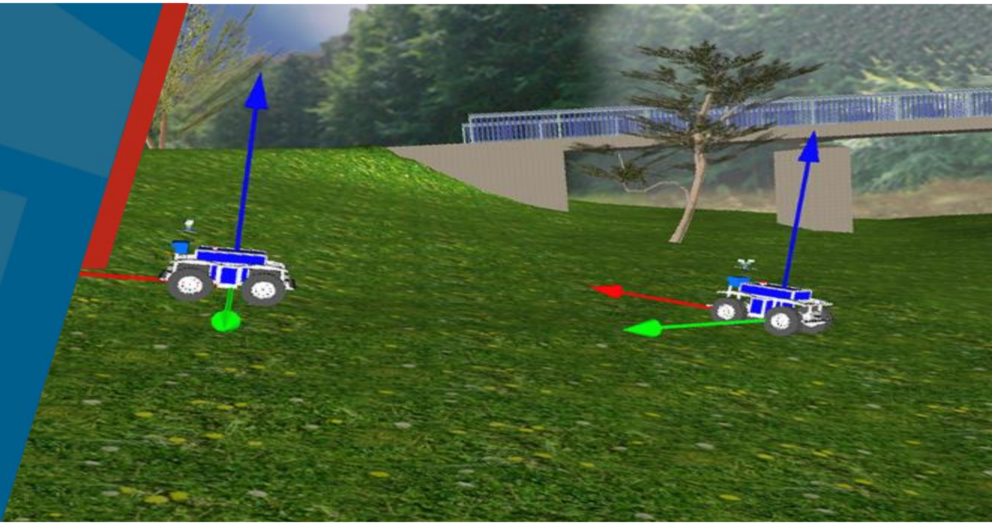
- Feedback-Control – an Example

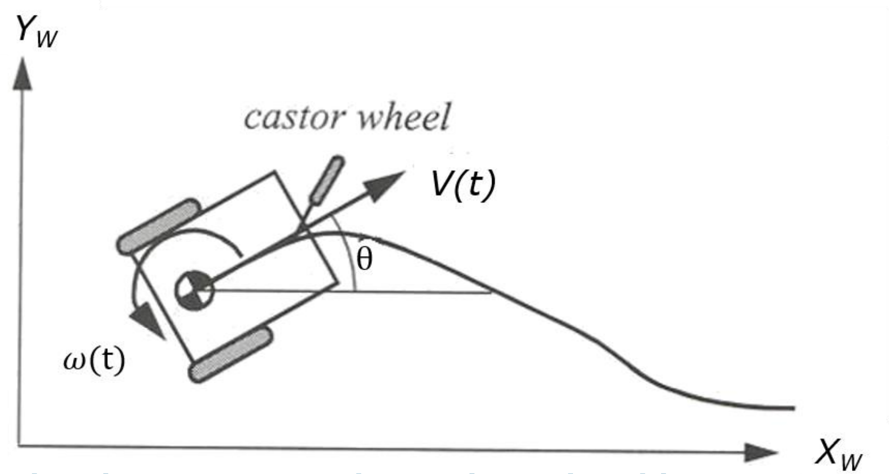
Modeling of AMRs

and its Operational Environment



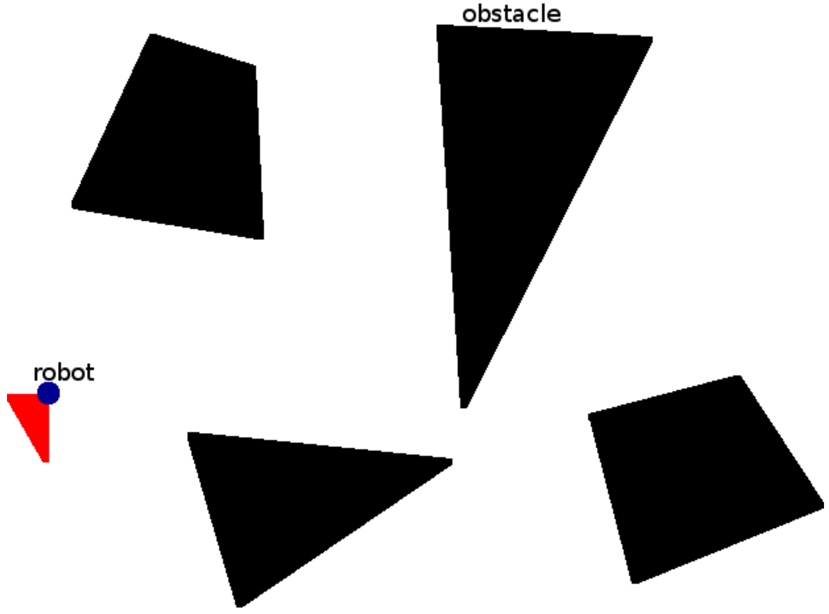
Modeling of AMRs and the Operational Environment

- Reduction of the robot complexity
- AMR movements as trajectories in a 3D Euclidian space
- Objects represented by boarder lines or geometric objects e.g.
  - polygon lines, grids
  - boxes
- AMR position represented by a single point in space
  - center of mass
  - kinematic center
- Orientation (heading) represented as a vector due to the world frame



Configuration Space

- Configuration space (C-space) spanned over the degrees of freedom that should be considered
- E.g. a mobile robot in a $2D$ space contains all possible positions and orientations (x, y, φ) and maps each point to reachable or not reachable



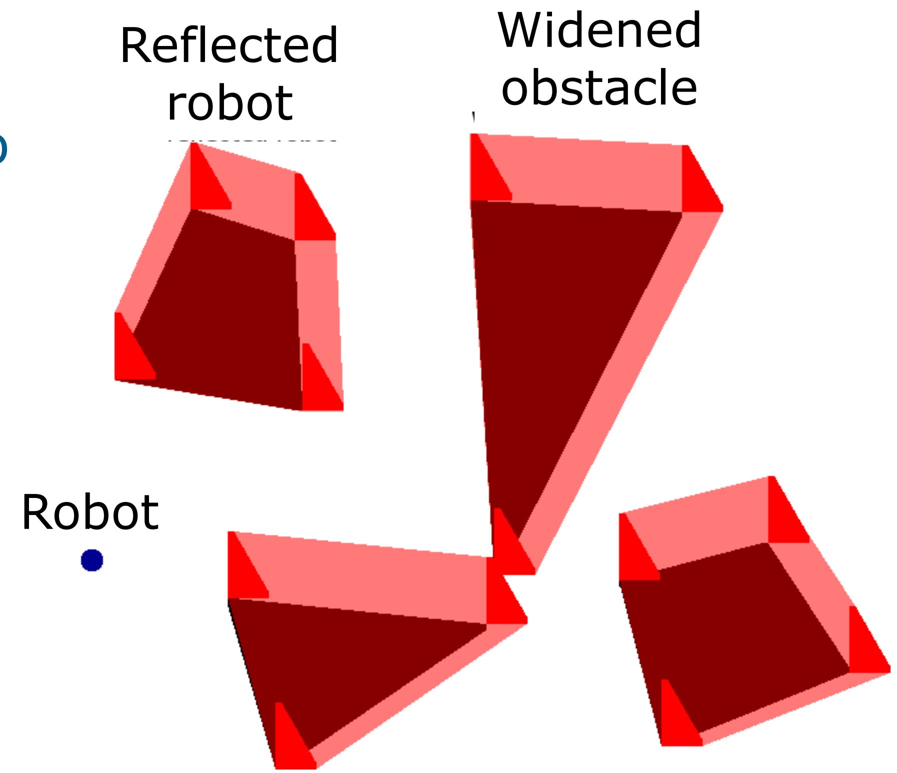
2D robot C-space

Configuration Space

Trajectory planning must take into account the AMR shape

=> widened obstacles by the reflected shape of the AMR

=> if point representation can reach a configuration, the same holds for the real robot



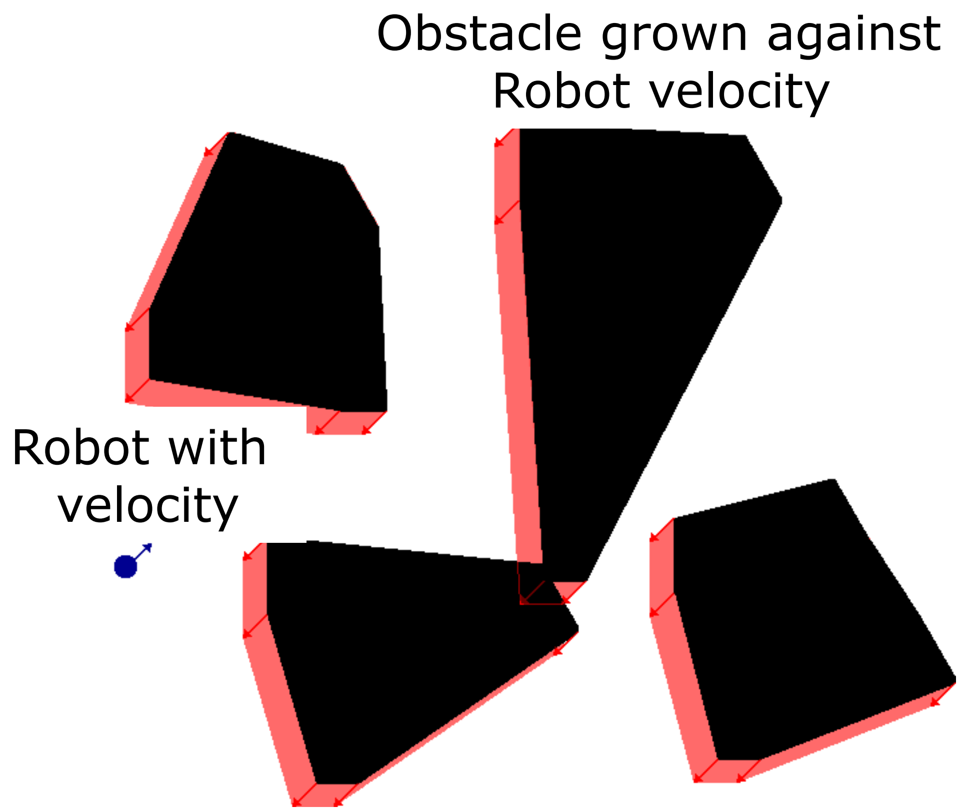
2D robot c-space

Configuration Space

- Extension of C-Space, considering the derivatives of each degree of freedom

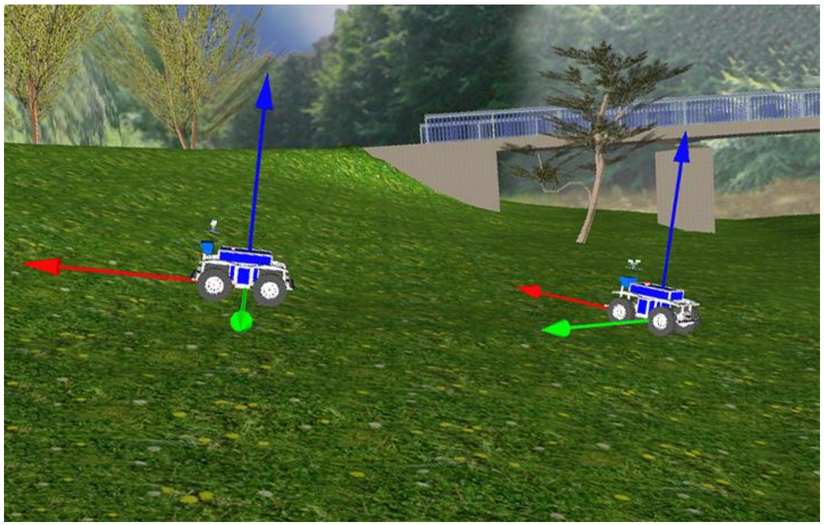
=> Extension space (X-space)

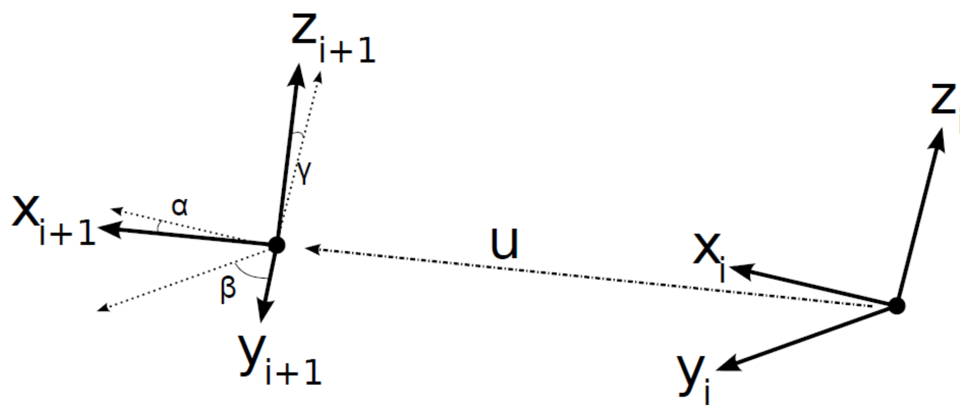
- E.g. X-Space for considering velocity
- Widen obstacles so that collision avoidance remains possible (braking deceleration)



2D robot X-space

Coordinate Systems





Modeling of a movement of an AMR by transformation of the robot coordinate systems in a 3D Euclidian space

Description of AMR Pose as Six-Tuple

- In general the pose of an arbitrary object in the Cartesian space can be described as a six-tuple $(x, y, z, \varphi, \psi, \theta)$
- The position vector ${}^{O}\vec{u}$ in an object frame O can be presented in base frame coordinates ${}^{B}\vec{u}$ by

$$
{}^{B}\vec{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} + {}^{B}_{O}R(\varphi, \psi, \theta)^{O}\,\vec{u}
$$

with $(x, y, z)^T$ being the translation vector between the origin of the two frames and R (3x3 matrix) the respective (combined) rotation matrix.

Homogeneous Transformation Matrix

- Another possibility to express the same as above are
  => homogeneous transformation matrices.
- Those 4×4 matrices (for 3D space) are composed as shown below

$$\begin{pmatrix} R & \vec{u} \\ \vec{p}^T & s \end{pmatrix}$$

- R: 3×3 rotation (orientation) matrix
- $\vec{u}$: Translation (position) vector $\vec{u} = \left(u_x, u_y, u_z\right)^T$
- $\vec{p}$: Perspective transformation (in general $\vec{p} = (0,0,0)^T$)
- s: Scaling factor (in general $s = 1$)

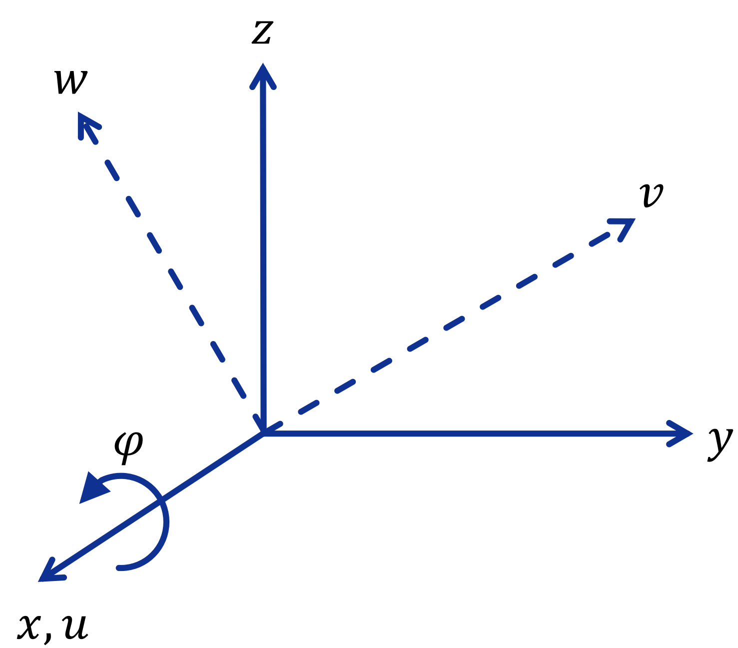
3D Rotation Matrices

- R is usually a combination of several rotations around the elementary axis
- $R_x(\varphi)$ describes a rotation around the x-axis of an arbitrary coordinate system via the angle φ. In an analog matter the other two required matrices are defined.

$$R_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}$$

$$R_y(\psi) = \begin{pmatrix} \cos\psi & 0 & \sin\psi \\ 0 & 1 & 0 \\ -\sin\psi & 0 & \cos\psi \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



Expression of Linked Rotations

- Two possible ways of expressing linked rotations
  - Euler angles (rotation around variable axes)

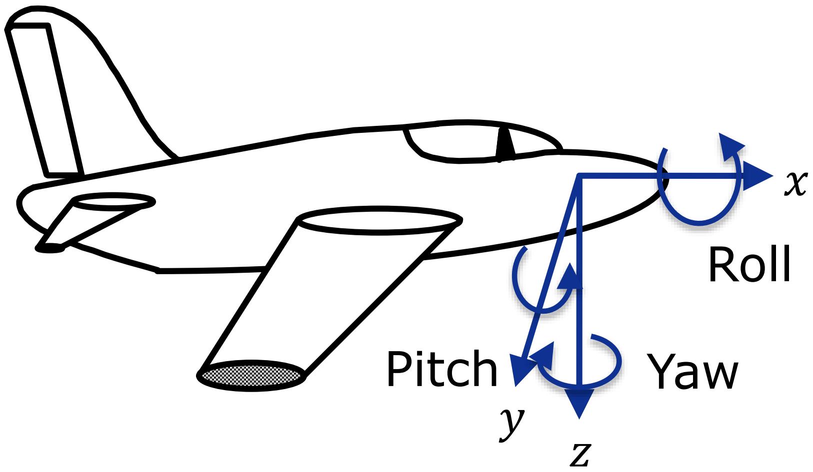
$$R = R_z(\varphi) \cdot R_x(\psi) \cdot R_{z'}(\theta)$$

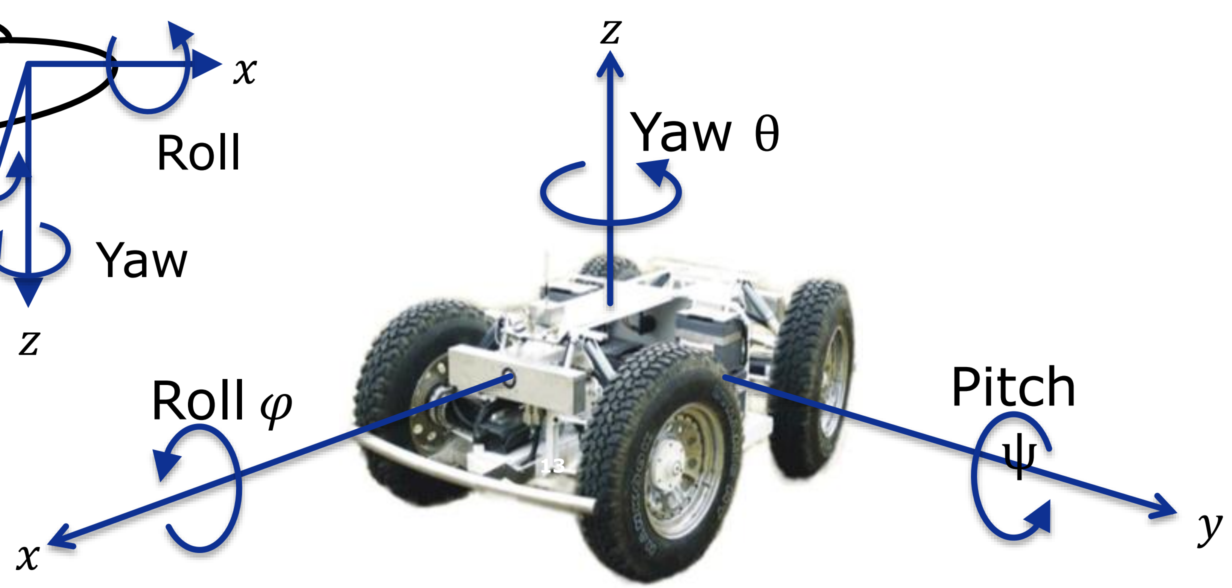
  - Tait-Bryan angles "Roll, Pitch, Yaw" (rotation around fixed axes)

$$R = R_z(\theta) \cdot R_y(\psi) \cdot R_x(\varphi)$$

Roll, Pitch, Yaw

- Euler: rotations along "new" axes
- Roll-Pitch-Yaw: rotations along fixed axes
- Terrestrial robotics the roll, pitch yaw system is used (change of AMR pose according to world coordinate system)





Combined Rotational Matrix (Roll, Pitch, Yaw)

- Arbitrary orientation in 3D space can be described using three rotations
- R_s as resulting matrix

  (abbreviations $s\varphi$ for $\sin\varphi$ and $c\varphi$ for $\cos\varphi$)

$$R_s = \begin{pmatrix} c\theta\, c\psi & c\theta\, s\psi\, s\varphi - s\theta\, c\varphi & c\theta\, s\psi\, c\varphi + s\theta\, s\varphi \\ s\theta\, c\psi & s\theta\, s\psi\, s\varphi + c\theta\, c\varphi & s\theta\, s\psi\, c\varphi - c\theta\, s\varphi \\ -s\psi & c\psi\, s\varphi & c\psi\, c\varphi \end{pmatrix}$$

Linear Velocity

- Transformation velocities between two frames is similar to pose transformation
- Transformation of a linear velocity vector ${}^{B}\vec{v}_q$ of an arbitrary point $\vec{q}$ presented in frame B to frame A

$$ {}^{A}\vec{v}_q = {}^{A}_{B}R\,{}^{B}\vec{v}_q $$

- If the origin of frame B has also a linear velocity relative to frame A then

$$ {}^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R\,{}^{B}\vec{v}_q $$

- If in addition point $\vec{q}$ is rotating around an arbitrary axis with the rotational velocity ${}^{A}\Omega_B$ then the linear velocity can be calculated with

$$ {}^{A}\vec{v}_q = {}^{A}\vec{v}_{OB} + {}^{A}_{B}R\,{}^{B}\vec{v}_q + {}^{A}\overrightarrow{\omega}_B \times {}^{A}_{B}R\,{}^{B}\vec{q} $$

Linear and Rotational Velocity

- A rotational vector ${}^{B}\vec{\omega}$ related to frame B can be transferred to frame A with ${}^{A}\vec{\omega} = {}^{A}_{B}R\,{}^{B}\vec{\omega}$
- Rotational and linear velocity of a sequence of segments connected by rotational or prismatic joints can stepwise be calculated
- Rotational velocity ${}^{i+1}\vec{\omega}_{i+1}$ and the linear velocity ${}^{i+1}\vec{v}_{i+1}$ relative to frame $i+1$ can be determined as

$$ {}^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R \cdot {}^{i}\vec{\omega}_i + \dot{\theta}^{i+1}_{i+1}\vec{e}_{z_{i+1}} $$

$$ {}^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\vec{v}_i + {}^{i}\vec{\omega}_i \times {}^{i}\vec{p}_{i+1}\right) $$

with ${}^{i}\vec{p}_{i+1}$ vector in direction of the segment i, $\vec{\omega}_i$ the rotational velocity and ψ_i the rotation of segment i around the elementary z-axis

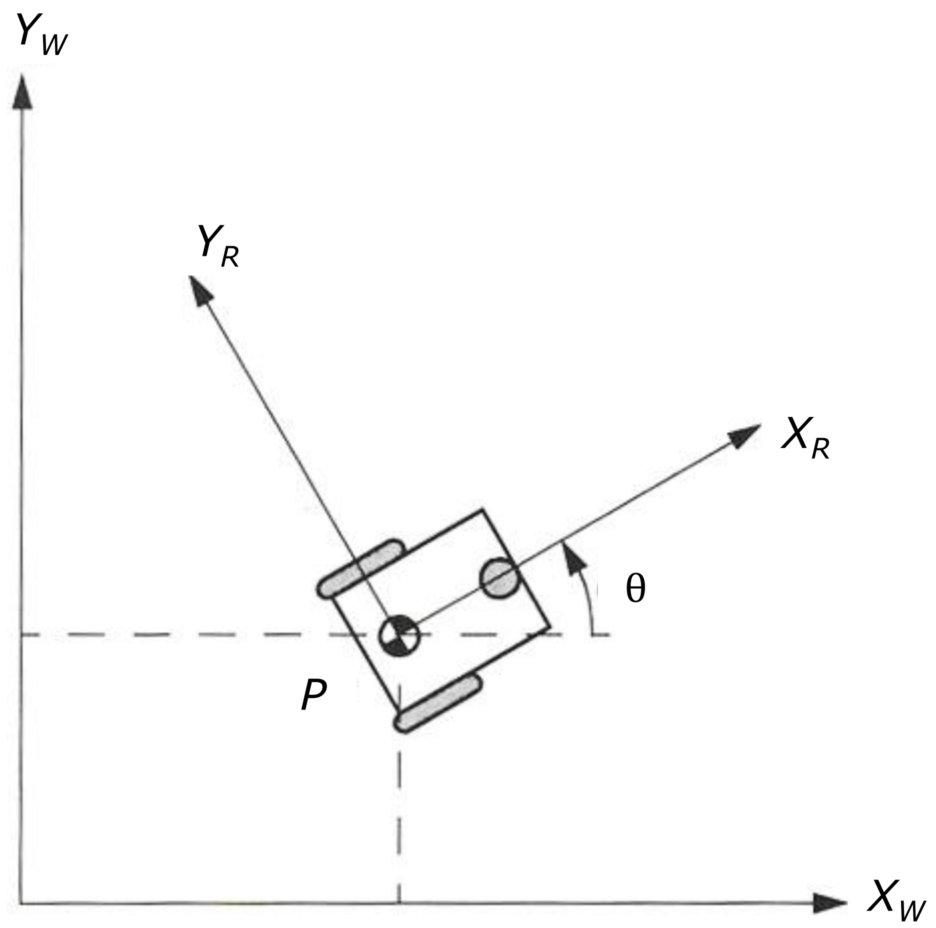
Linear and Rotational Velocity

- ${}^{i+1}_{i}R$ is the inverse of the orientation matrix ${}^{i}_{i+1}R$ from frame i to $i+1$
- ${}^{i}_{i+1}R$ is an orthonormal matrix

$\Rightarrow$ its inverse ${}^{i}_{i+1}R^{-1} = {}^{i+1}_{i}R$ is the transposed matrix ${}^{i}_{i+1}R^{T}$ (example in $2D$ space)

$$ {}^{i+1}_{i}R(\theta) = {}^{i}_{i+1}R^{-1}(\theta) = {}^{i}_{i+1}R^{T}(\theta) = \begin{pmatrix} c\theta & s\theta & x_w \\ -s\theta & c\theta & y_w \\ 0 & 0 & 1 \end{pmatrix} $$

World (Global) and Robot (Local) Coordinate System



World and Robot Coordinate System

- P describes the location of the kinematic center relative to the world coordinate system
- θ is the mathematically positive angle between X_W and X_R
- $O\{X_W, Y_W\} \mathrel{\widehat{=}}$ origin of the world coordinate system
- $O\{X_R, Y_R\} \mathrel{\widehat{=}} P \mathrel{\widehat{=}}$ origin of the robot's coordinate system
- X_R-axis is longitudinal axis of the robot through its kinematic center
- Y_R-axis is lateral axis of the robot through its kinematics center
- $\overrightarrow{\xi_W} = (x, y, \theta)^T$ coordinates of the kineamtic center in world coordinates

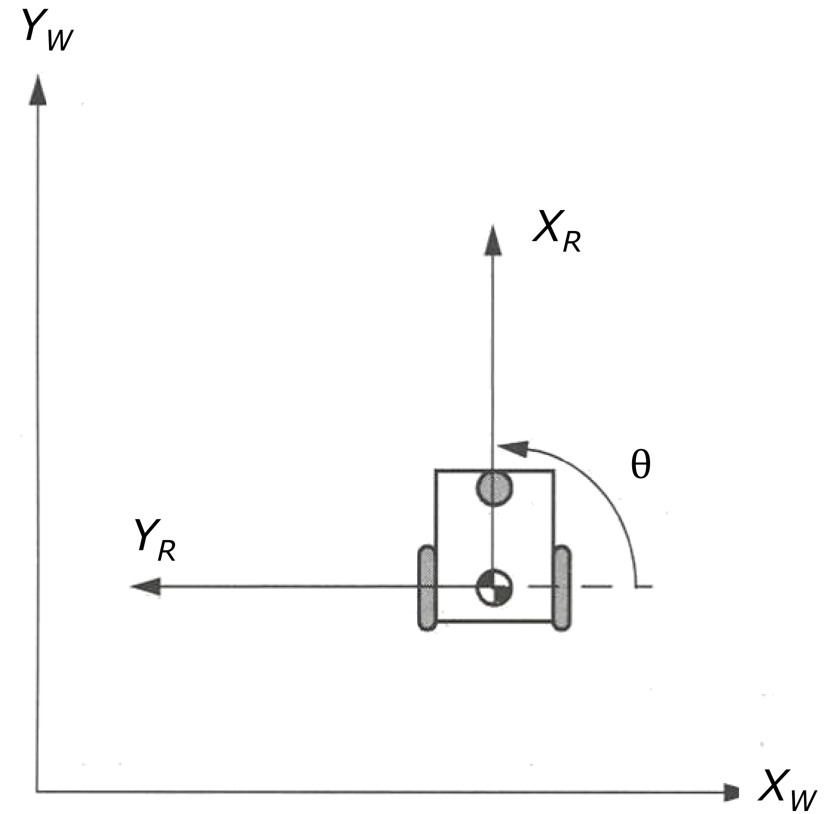
World and Robot Coordinate System

- Orientation via rotation around z-axis

$$R(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Speed transformation from world- to robot coordinate system

$$\dot{\vec{\xi}}_R = R(\theta)\, \dot{\vec{\xi}}_W$$



Coordinate transformation

World and Robot Coordinate System

- Rotation around z-axis with $\theta = 90\,°$

$$R\left(\frac{\pi}{2}\right) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

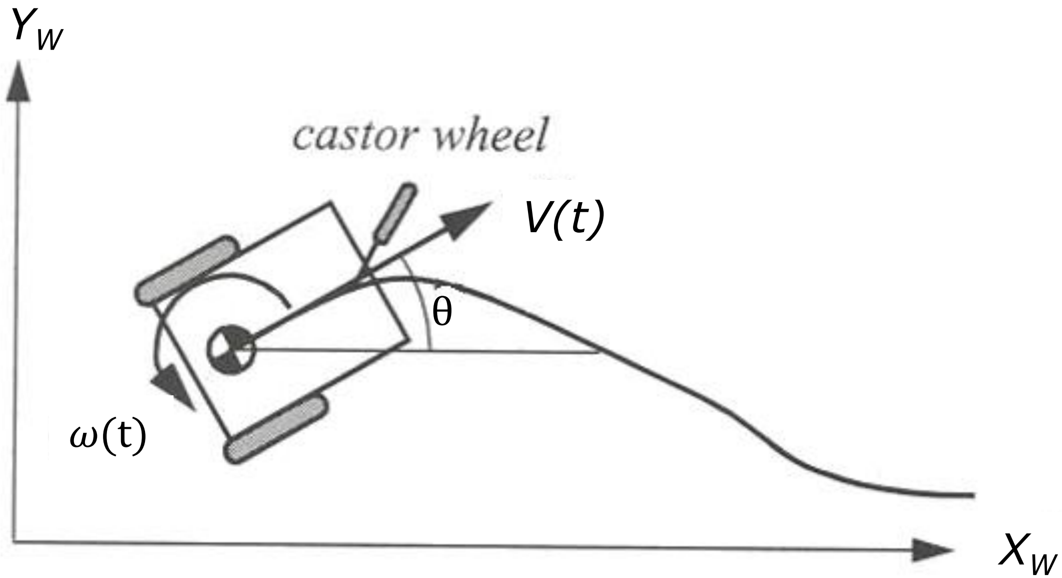
- Speed calculation in robot coordinates

$$\vec{\dot{\xi}}_R = R\left(\frac{\pi}{2}\right) \overrightarrow{\dot{\xi}_W} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{pmatrix}$$

Vehicle Kinematics - Analytical Solution

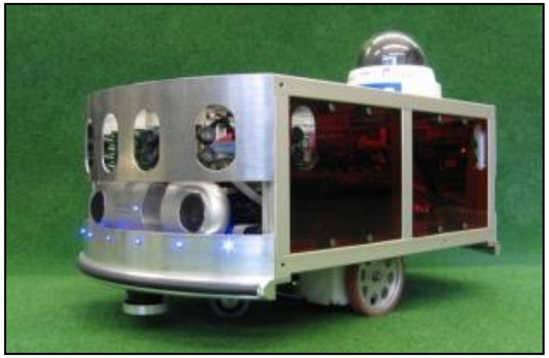
Forward Kinematics for Wheel-driven Robot

- What is the robot's trajectory, if wheel geometry and speed are known?



Example: Differential Drive

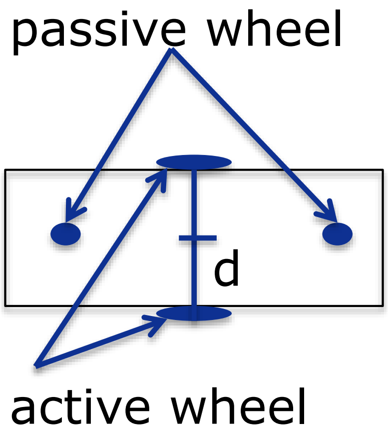
- r: radius of the wheel
- d: distance wheel center to kinematic center
- $\dot{\psi}_r, \dot{\psi}_l$: Angular velocity of left/right wheel
- $\dot{\vec{\xi}}_W$: speed calculation in world coordinates



$$\dot{\vec{\xi}}_W = (\dot{x}, \dot{y}, \dot{\theta})^T$$

$$= f(d, r, \theta, \dot{\psi}_l, \dot{\psi}_r)$$

$$= R(\theta)^{-1} \dot{\vec{\xi}}_R$$



Example: Differential Drive

- Calculation of the contribution of the left and right wheel to the speed along the longitudinal axis (X_R)

$$\dot{x}_{Rr} = \frac{r\dot{\psi}_r}{2} \qquad \dot{x}_{Rl} = \frac{r\dot{\psi}_l}{2}$$

- Calculation of the contribution of the left and right wheel to the angular velocity of kinematic center

$$\omega_r = \frac{r\dot{\psi}_r}{2d} \qquad \omega_l = \frac{-r\dot{\psi}_l}{2d}$$

- Summation of both results in the velocity and angular velocity along the X_R-axis, where

$$\dot{\vec{\xi}}_W = R(\theta)^{-1}\dot{\vec{\xi}}_r \qquad \dot{\vec{\xi}}_r = \left(\frac{r\dot{\psi}_r}{2} + \frac{r\dot{\psi}_l}{2}, 0, \frac{r\dot{\psi}_r}{2d} - \frac{r\dot{\psi}_l}{2d}\right)^T$$

Example: Differential Drive

- Since $R(\theta)$ is orthonormal:

$$R(\theta)^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

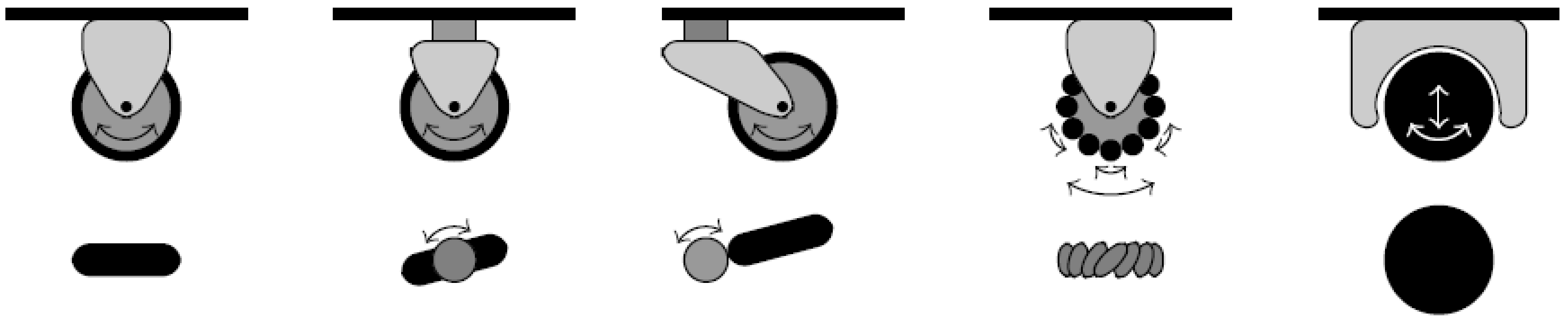
- Given $\theta = \frac{\pi}{2}$, $r = 1$, $d = 1$, $\dot{\psi}_r = 4$, $\dot{\psi}_l = 2$ and wheel speed notated in rounds per second $\frac{2\pi}{s}$:

$$\dot{\vec{\xi}}_W = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

Calculation of Wheel kinematics

- Stepwise calculation of the linear and the rotational velocity for wheeled vehicles in 2D environment
- Kinematic center moved with $(\dot{x}, \dot{y}, \dot{\theta})^T$
- Determine rotational velocity of each wheel
- Collect all equations and solve the equation system
- Consider:
  - Equation system must no be solvable
  - No dynamic aspects will be considered
- Calculation of the speed of the kinematic center based on the wheel speed can also be determined out of the equation system (inverse problem)

Basic Wheel Types



Standard wheel

Steerable standard wheel

Castor wheel

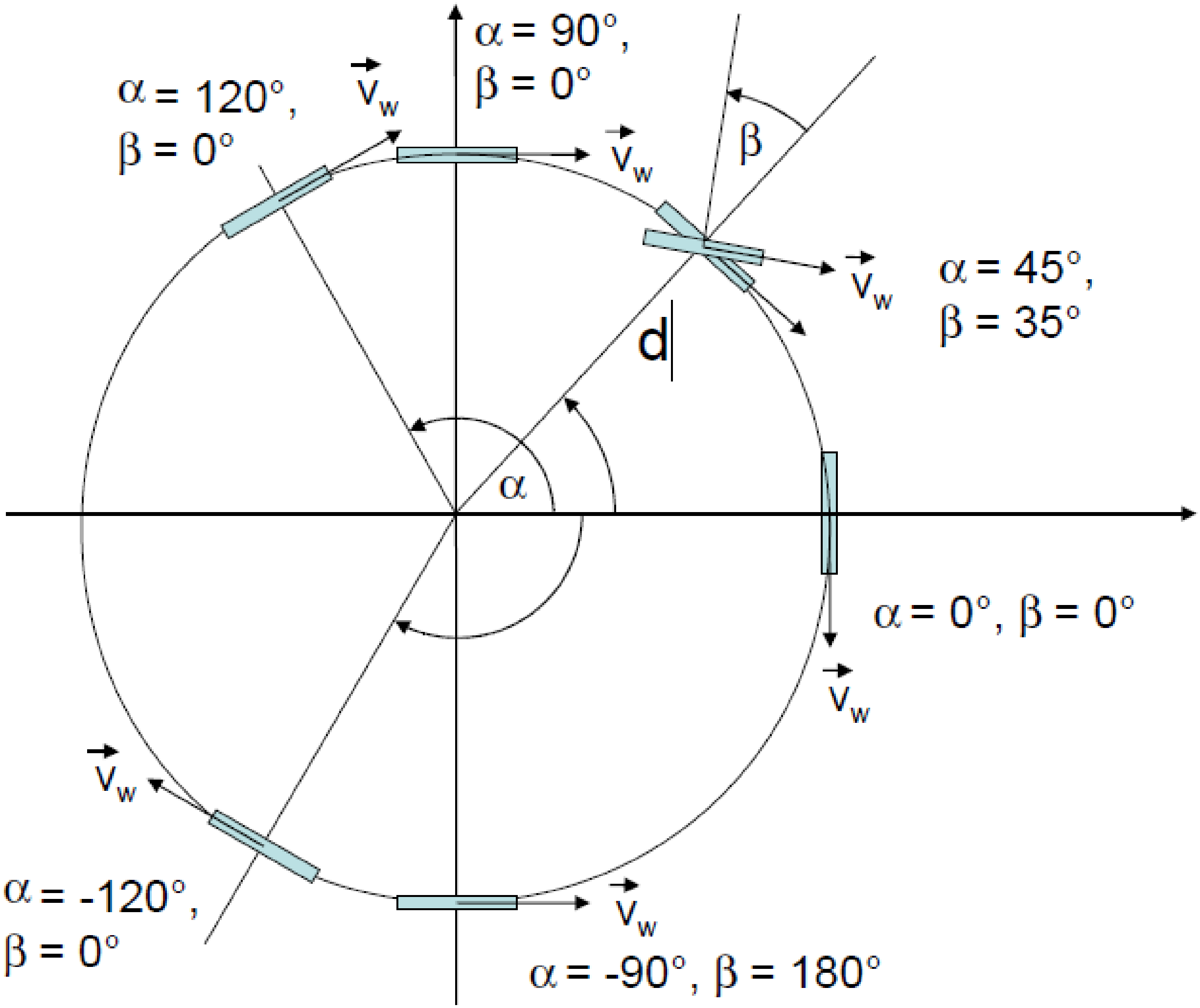
Swedish or Mecanum wheel

Ball or spherical wheel

Wheel Properties

- Standard and steerable wheel
  - Linear velocity along wheel plane: $\dot{x} = r\dot{\psi}$
  - No sliding orthogonal to wheel plane $\dot{y} = 0$
- Castor wheel
  - Linear velocity along wheel plane: $\dot{x} = r\dot{\psi}$
  - Linear perpendicular velocity: $\dot{y} = -d_c\dot{\beta}$
- Swedish/Mecanum wheel
  - Linear velocity along wheel plane: $\dot{x} = r\dot{\psi}\cos\gamma$
  - γ: angle of passive rollers ($45\,°$ or $90\,°$)
  - Linear perpendicular velocity: $\dot{y} = r\dot{\psi}\sin\gamma$

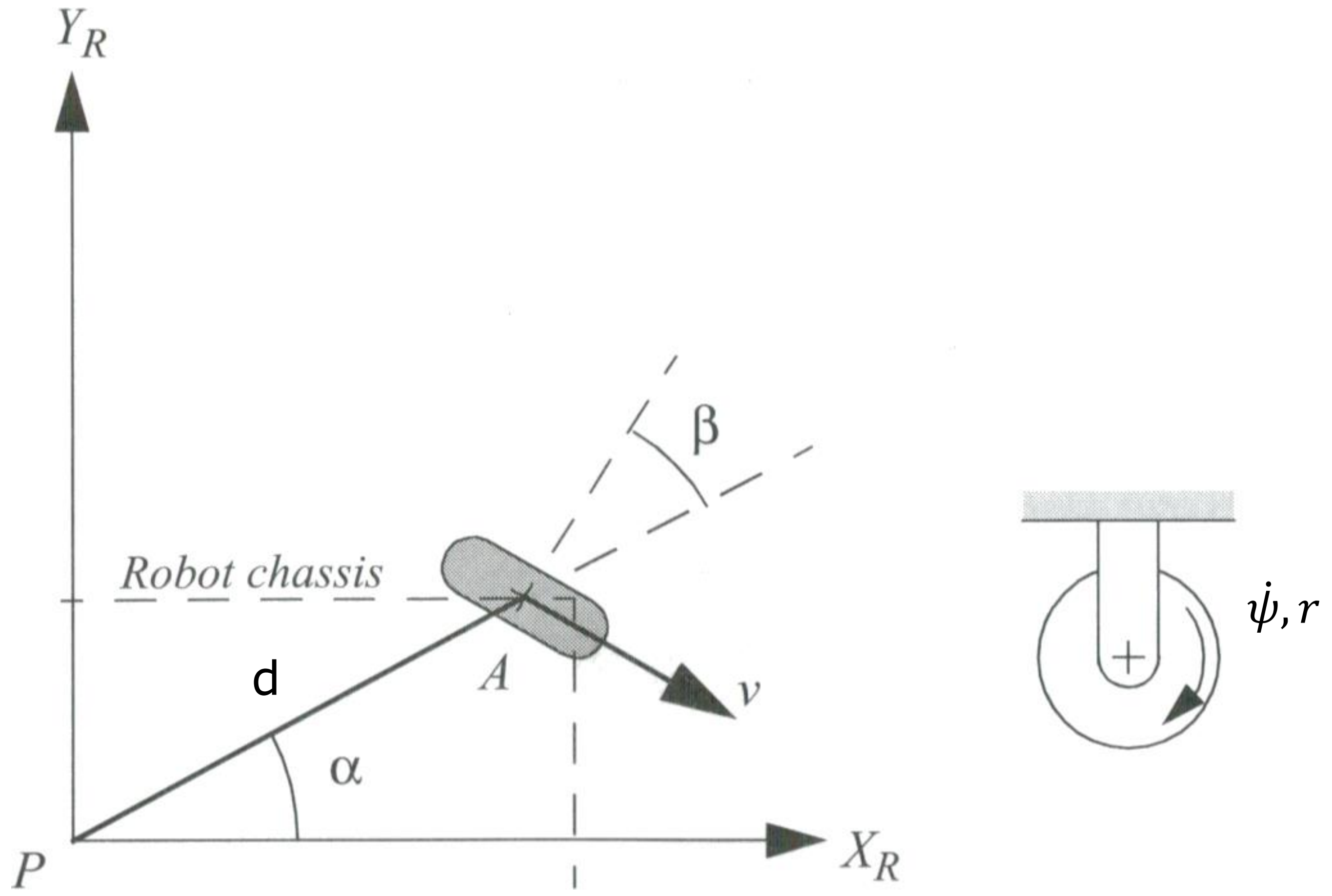
Kinematic Parameters



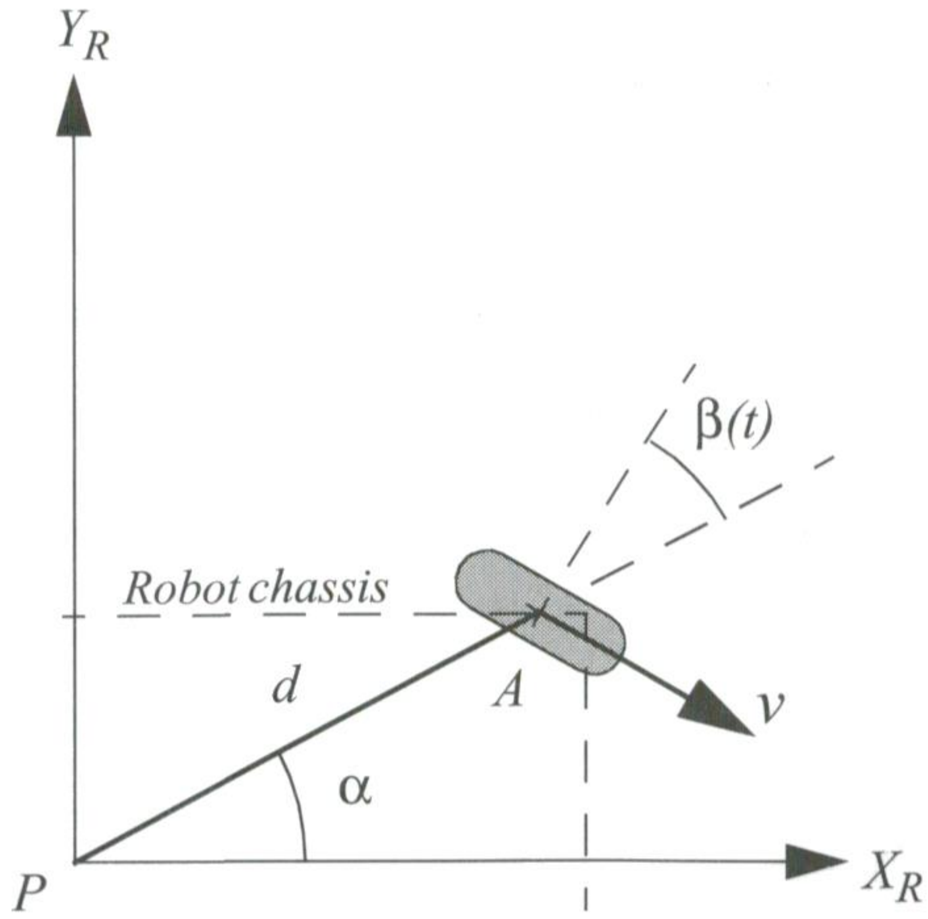
Kinematic Parameters

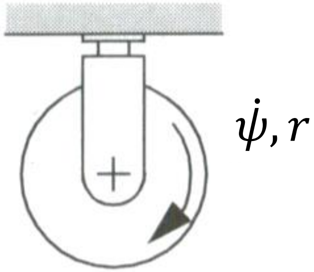
- α: Angle between x-axis and wheel mount point
- β: Angle between the straight line through the kinematic center and the fixing point of the wheel and the y-axis of the wheel frame
- d: Distance from the kinematic center to the wheel fixing point
- d_c: Distance from the wheel fixing point to the wheel supporting point (Castor wheel only)
- γ: Angle between the x-axis of the wheel and rolling direction of the rollers (Mecanum wheel only)

Kinematic Parameters: Standard Wheel

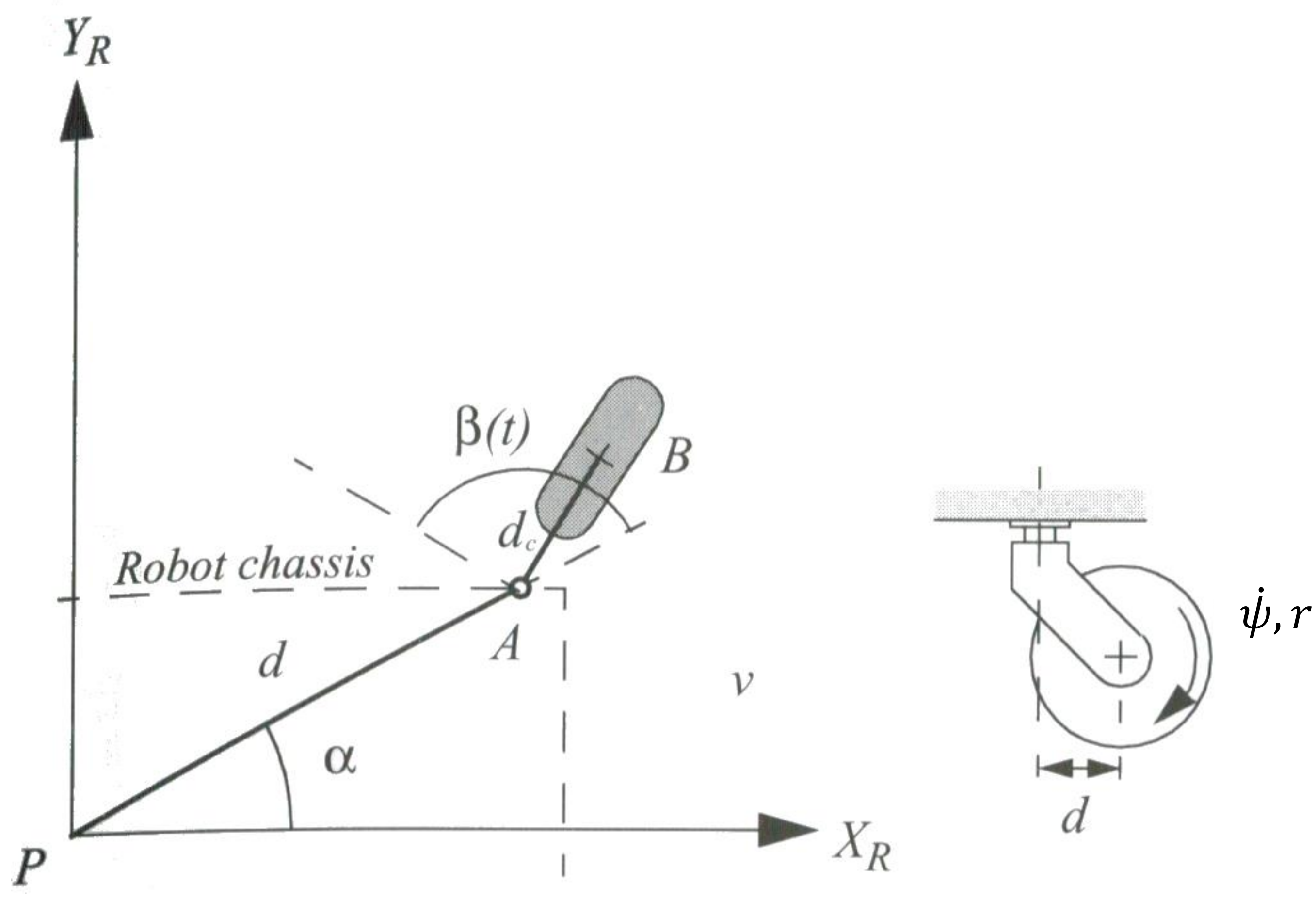


Kinematic Parameters: Steerable Wheel

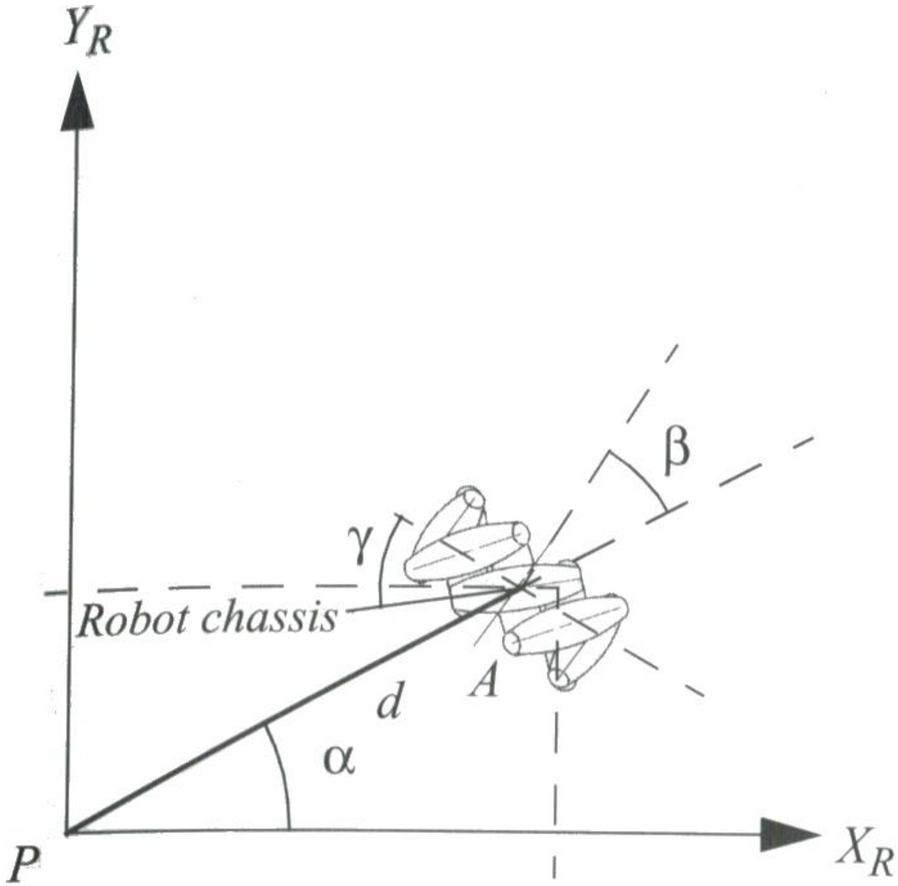


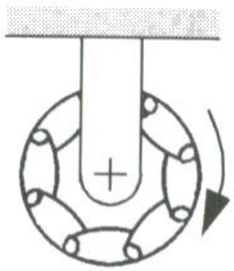


Kinematic Parameters: Castor Wheel

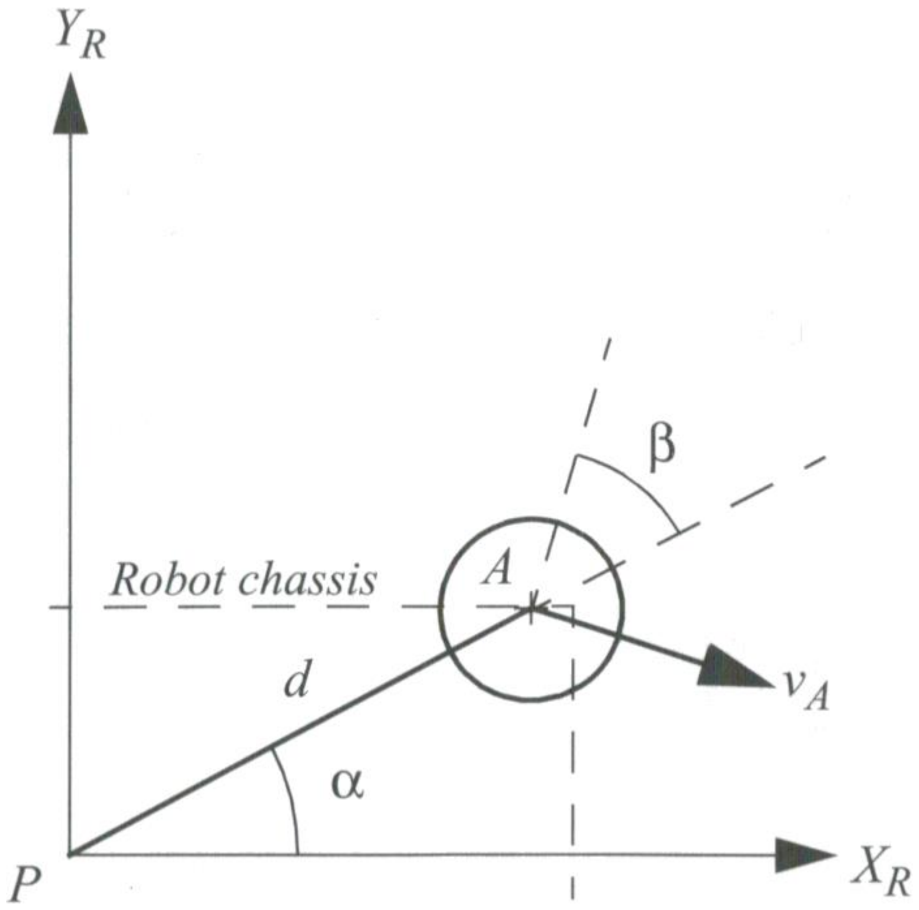


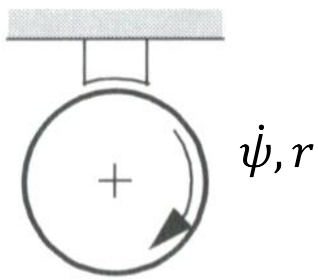
Kinematic Parameters: Mecanum Wheel





Kinematic Parameters: Ball Wheel





Calculation of Wheel Velocity

- Given: velocity vector $\vec{v} = \left(\dot{x}, \dot{y}, \dot{\theta}\right)^T$ of the kinematic center
- Calculation of linear velocity of standard wheel due to the speed of the kinematic center
- Stepwise apply the following equations:

$$^{i+1}\vec{\omega}_{i+1} = {}^{i+1}_{i}R \cdot \ {}^{i}\vec{\omega}_i + \dot{\theta}^{i+1}_{i+1}\vec{e}_{z_{i+1}}$$

$$^{i+1}\vec{v}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\vec{v}_i + {}^{i}\vec{\omega}_i \times {}^{i}\vec{p}_{i+1}\right)$$

- Initial parameters $^{1}\vec{\omega}_1 = \left(0{,}0, \dot{\theta}\right)^T$ and $^{1}\vec{v}_1 = (\dot{x}, \dot{y}, 0)^T$
- No additional rotational speed: all $^{i}\vec{\omega}_i = \left(0{,}0, \dot{\theta}\right)^T$

Calculation of Wheel Velocity

- After the rotation around the z-axis with angle α, ${}^{2}\vec{v}_2$ is calculated as

$$ {}^{2}\vec{v}_2 = \begin{pmatrix} c\alpha\,\dot{x} + s\alpha\,\dot{y} \\ -s\alpha\,\dot{x} + c\alpha\,\dot{y} \\ 0 \end{pmatrix} $$

- Due to the translation d ${}^{3}\vec{v}_3$ is

$$ {}^{3}\vec{v}_3 = \begin{pmatrix} c\alpha\,\dot{x} + s\alpha\,\dot{y} \\ -s\alpha\,\dot{x} + c\alpha\,\dot{y} + d\dot{\theta} \\ 0 \end{pmatrix} $$

Calculation of Wheel Velocity

- The last rotation around the z-axis with angle $\beta - 90°$ transfers the x-axis of the last frame to the rolling direction of the wheel.
- For the calculation of ${}^{4}\vec{v}_{4}$ in the next equation $\sin(\beta - 90°) = -\cos\beta$ and $\cos(\beta - 90°) = \sin\beta$ is used

$$
{}^{4}\vec{v}_{4} = \begin{pmatrix} s(\alpha+\beta)\dot{x} - c(\alpha+\beta)\dot{y} - c\beta\, d\dot{\theta} \\ c(\alpha+\beta)\dot{x} + s(\alpha+\beta)\dot{y} + s\beta\, d\dot{\theta} \\ 0 \end{pmatrix}
$$

Calculation of Wheel Velocity

- Last step is to equalize the linear velocity vector of the standard wheel to the velocity of the kinematic center represented in the wheel frame

$$\begin{pmatrix} s(\alpha+\beta) & -c(\alpha+\beta) & -c\beta\, d \\ c(\alpha+\beta) & s(\alpha+\beta) & s\beta\, d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r\dot{\psi} \\ 0 \\ 0 \end{pmatrix}$$

- 1st equation $r\dot{\psi} = s(\alpha+\beta)\dot{x} - c(\alpha+\beta)\dot{y} - c\beta\, d\dot{\theta}$ is called the rolling constraint of the wheel, because it describes the speed in the rolling direction of the wheel
- 2nd equation $c(\alpha+\beta)\dot{x} + s(\alpha+\beta)\dot{y} + s\beta\, d\dot{\theta} = 0$ is called the sliding constraint, which describes the speed perpendicular to wheel plane
- Sliding constraints are collected in Matrix S – add row $(c(\alpha +$

Wheel Type-specific Velocity Vector

- In case of the steerable standard wheel the fixed angle β is replaced by a function $\beta(t)$
- Equation could also be applied to the spherical wheel (based on the forces, which affect the wheel and change $\beta(t)$, only a linear velocity in the rolling direction exists)
- In case of castor wheels the y-component of the velocity vector is depending on the angular velocity $\dot{\beta}$ and the length of the rod

$$\begin{pmatrix} s(\alpha+\beta(t)) & -c(\alpha+\beta(t)) & -c\beta(t)d \\ c(\alpha+\beta(t)) & s(\alpha+\beta(t)) & s\beta(t)d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r\dot{\psi} \\ -d_c\dot{\beta}(t) \\ 0 \end{pmatrix}$$

- Sliding constraints are collected in Matrix S – add row $(c(\alpha +$

Wheel Type-specific Velocity Vector

Swedish or Mecanum wheel is able to move in an omnidirectional way

$$\begin{pmatrix} s(\alpha+\beta+\gamma) & -c(\alpha+\beta+\gamma) & -c(\beta+\gamma)d \\ c(\alpha+\beta+\gamma) & s(\alpha+\beta+\gamma) & s(\beta+\gamma)d \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r\dot{\psi}\cos\gamma \\ r\dot{\psi}\sin\gamma \\ 0 \end{pmatrix}$$

Vehicle Motion Calculation

Transformation of wheel velocities to velocity of kinematic center

- Determine parameters α, β, γ,d indicating the wheel pose
- Insert α, β, γ into appropriate wheel equation (as shown before) for each wheel
- Solve system of equations

Differential Drive Kinematics

- Two fixed standard wheels mounted on one axis
- Kinematic center located in the middle of the axis
- Distance between the kinematic center and each wheel: d
- Kinematic center is origin of coordinate frame
- One solution for modeling the wheel configuration is to place the wheels on the y-axis
- Kinematic parameters
  - $\alpha_l = 90\,°$
  - $\beta_l = 0\,°$
  - $\alpha_r = -90\,°$
  - $\beta_r = 180\,°$

Differential Drive Kinematics

- Apply equations for standard wheels (for the left and the right wheel)

$$s(\alpha_l + \beta_l)\dot{x} - c(\alpha_l + \beta_l)\dot{y} - c\beta_l\, d\dot{\theta} = r_l\dot{\psi}_l$$
$$c(\alpha_l + \beta_l)\dot{x} + s(\alpha_l + \beta_l)\dot{y} + s\beta_l\, d\dot{\theta} = 0$$
$$s(\alpha_r + \beta_r)\dot{x} - c(\alpha_r + \beta_r)\dot{y} - c\beta_r\, d\dot{\theta} = r_r\dot{\psi}_r$$
$$c(\alpha_r + \alpha_r)\dot{x} + s(\alpha_r + \beta_r)\dot{y} + s\beta_r\, d\dot{\theta} = 0$$

- If the above mentioned parameters are inserted the following equations will result

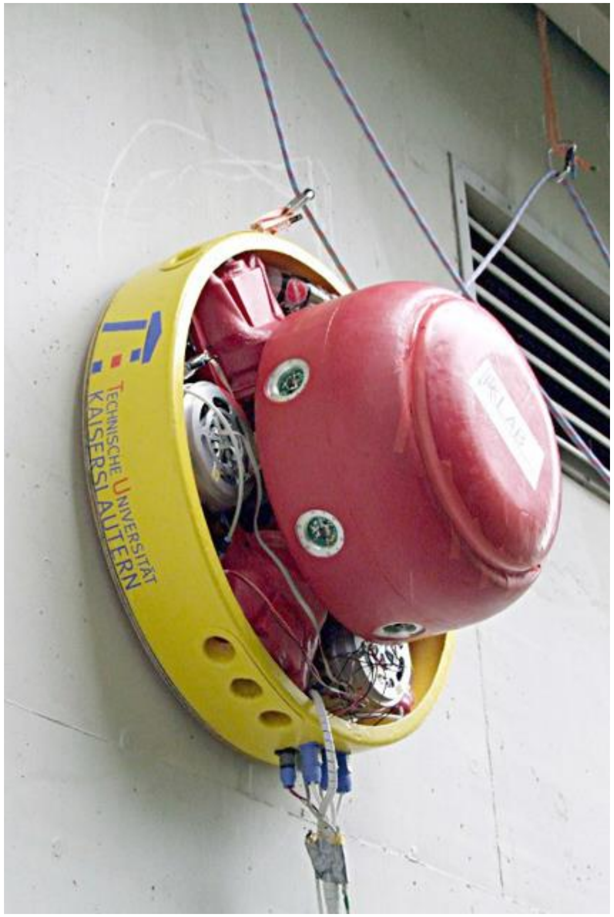
$$\dot{x} - d\dot{\theta} = r_l\dot{\psi}_l$$
$$\dot{y} = 0$$
$$\dot{x} + d\dot{\theta} = r_r\dot{\psi}_r$$
$$\dot{y} = 0$$

Differential Drive Kinematics

Solving the system of equations yields

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\left(r_l \dot{\psi}_l + r_r \dot{\psi}_r\right) \\ 0 \\ \frac{1}{2d}\left(-r_l \dot{\psi}_l + r_r \dot{\psi}_r\right) \end{pmatrix}$$

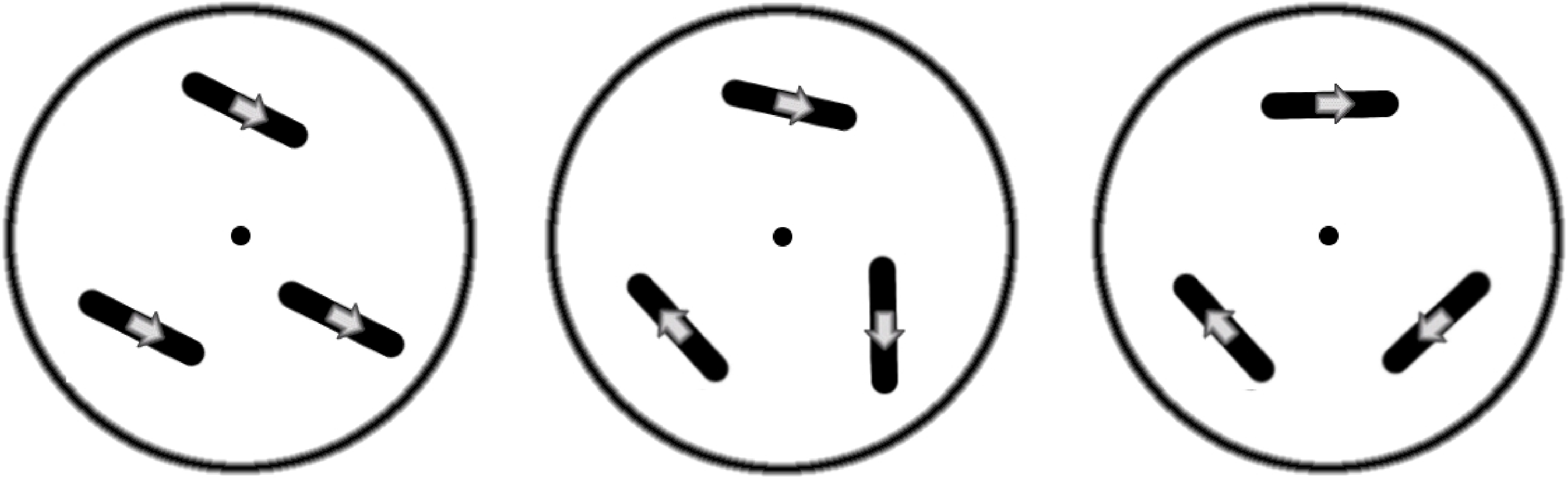
CROMSCI – Equipped with 3 Steerable Standard Wheels



Climbing robot CROMSCI

Wheel settings of CROMSCI

Omnidirectional Drive Kinematics



Typical orientations of the 3 steerable wheels of an omnidirectional vehicle

Omnidirectional Drive Kinematics

- Three steerable standard wheels
- Kinematic center located in the middle of the robot
- Distance between the kinematic center and each wheel: d
- Origin of coordinates frame is kinematic center
- Kinematic parameters
  - $\alpha_1 = 0\,°$
  - $\alpha_2 = 120\,°$
  - $\alpha_3 = -120\,°$

Omnidirectional Drive Kinematics

Applying equation for steerable standard wheel for each wheel leads to the following system of equations:

$$
\begin{aligned}
s(\alpha_1+\beta_1)\dot{x} - c(\alpha_1+\beta_1)\dot{y} - d\cdot c(\beta_1)\dot{\theta} &= r_1\dot{\psi}_1 \\
s(\alpha_2+\beta_2)\dot{x} - c(\alpha_2+\beta_2)\dot{y} - d\cdot c(\beta_2)\dot{\theta} &= r_2\dot{\psi}_2 \\
s(\alpha_3+\beta_3)\dot{x} - c(\alpha_3+\beta_3)\dot{y} - d\cdot c(\beta_3)\dot{\theta} &= r_3\dot{\psi}_3 \\
c(\alpha_1+\beta_1)\dot{x} + s(\alpha_1+\beta_1)\dot{y} + d\cdot s(\beta_1)\dot{\theta} &= 0 \\
c(\alpha_2+\beta_2)\dot{x} + s(\alpha_2+\beta_2)\dot{y} + d\cdot s(\beta_2)\dot{\theta} &= 0 \\
c(\alpha_3+\beta_3)\dot{x} + s(\alpha_3+\beta_3)\dot{y} + d\cdot s(\beta_3)\dot{\theta} &= 0
\end{aligned}
$$

Omnidirectional Drive Kinematics

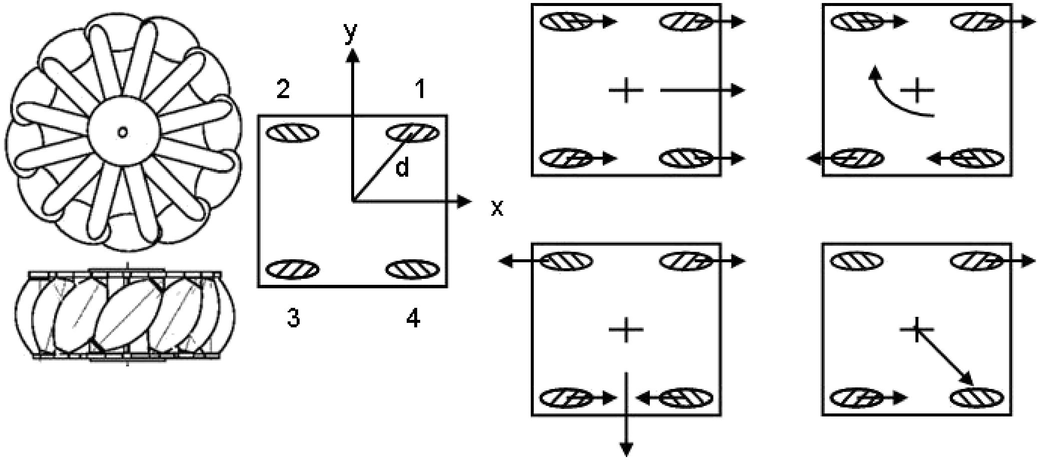
Based on these equation the steering angles $\beta_i, i = 1,2,3$ can be determined

$$
\begin{aligned}
c(\alpha_i + \beta_i) \cdot \dot{x} + s(\alpha_i + \beta_i) \cdot \dot{y} + d \cdot s(\beta_i) \cdot \dot{\theta} &= 0 \\
c(\alpha_i) \cdot c(\beta_i) \cdot \dot{x} - s(\alpha_i) \cdot s(\beta_i) \cdot \dot{x} + s(\alpha_i) \cdot c(\beta_i) \cdot \dot{y} + c(\alpha_i) \cdot s(\beta_i) \cdot \dot{y} + d \cdot s(\beta_i) \cdot \dot{\theta} &= 0 \\
c(\beta_i) \cdot (c(\alpha_i) \cdot \dot{x} + s(\alpha_i) \cdot \dot{y}) - s(\beta_i) \cdot \left(s(\alpha_i) \cdot \dot{x} - c(\alpha_i) \cdot \dot{y} - d \cdot \dot{\theta}\right) &= 0 \\
\frac{c(\alpha_i) \cdot \dot{x} + s(\alpha_i) \cdot \dot{y}}{s(\alpha_i) \cdot \dot{x} - c(\alpha_i) \cdot \dot{y} - d \cdot \dot{\theta}} &= \frac{s(\beta_i)}{c(\beta_i)} \\
\operatorname{atan2}\left(c(\alpha_i) \cdot \dot{x} + s(\alpha_i) \cdot \dot{y}, s(\alpha_i) \cdot \dot{x} - c(\alpha_i) \cdot \dot{y} - d \cdot \dot{\theta}\right) &= \beta_i
\end{aligned}
$$

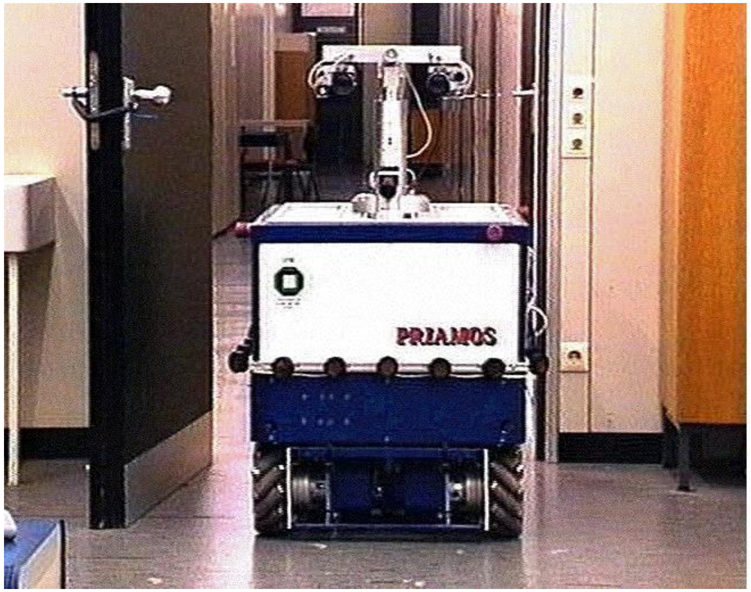
Also, the angular velocity of the wheel $\dot{\psi}_i$ can now be calculated

$$\dot{\psi}_i = \frac{1}{r_i}\left(s(\alpha_i + \beta_i)\dot{x} - c(\alpha_i + \beta_i)\dot{y} - d \cdot c(\beta_i)\dot{\theta}\right)$$

Mecanum Vehicle



Schematic configuration of a Mecanum wheel (Viewed from above)



The mobile robot PRIAMOS of the University of Karlsruhe driven by Mecanum wheels

Mecanum Kinematics

- To set up the kinematic equation first the parameters (α, β, γ) for each wheel must be determined

- The parameters for four wheels are
  - $\alpha_1 = 45\,°$, $\beta_1 = 45\,°$, $\gamma_1 = -45\,°$
  - $\alpha_2 = 135\,°$, $\beta_2 = -45\,°$, $\gamma_2 = 45\,°$
  - $\alpha_3 = -135\,°$, $\beta_3 = 225\,°$, $\gamma_3 = -45\,°$
  - $\alpha_4 = -45\,°$, $\beta_4 = 135\,°$, $\gamma_4 = 45\,°$

Mecanum Kinematics

- Assume all driven wheels have the same radius r, the same distance d from the kinematic center and the above mentioned parameters α, β, γ
- Using equation for Mecanum wheels leads to equations

$$s(45\,°)\dot{x} - c(45\,°)\dot{y} - d\dot{\theta} = r \cdot c(-45\,°)\dot{\psi}_1$$
$$s(135\,°)\dot{x} - c(135\,°)\dot{y} - d\dot{\theta} = r \cdot c(45\,°)\dot{\psi}_2$$
$$s(45\,°)\dot{x} - c(45\,°)\dot{y} - d\dot{\theta} = r \cdot c(-45\,°)\dot{\psi}_3$$
$$s(135\,°)\dot{x} - c(135\,°)\dot{y} - d\dot{\theta} = r \cdot c(45\,°)\dot{\psi}_4$$

Mecanum Kinematics

The velocities $\dot{x}, \dot{y}, \dot{\theta}$ of the kinematic center can be calculated

$$\dot{x} = \frac{r}{4}(\dot{\psi}_1 + \dot{\psi}_2 + \dot{\psi}_3 + \dot{\psi}_4)$$

$$\dot{y} = \frac{r}{4}(-\dot{\psi}_1 + \dot{\psi}_2 - \dot{\psi}_3 + \dot{\psi}_4)$$

$$\dot{\theta} = \frac{r}{d\sqrt{2}}(\dot{\psi}_1 - \dot{\psi}_2 - \dot{\psi}_3 + \dot{\psi}_4)$$

Mecanum Kinematics

The velocity vector of the kinematic center can be determined as

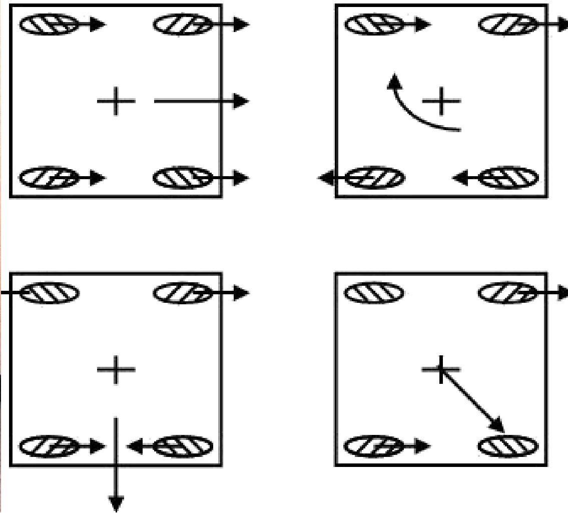
$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \frac{r_{wheel}}{4} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ a & -a & -a & a \end{pmatrix} \cdot \begin{pmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\psi}_3 \\ \dot{\psi}_4 \end{pmatrix}$$

with a $= \frac{2\sqrt{2}}{d}$.

Assuming the absence of slip the following must hold

$$\dot{\psi}_4 = \dot{\psi}_1 + \dot{\psi}_2 - \dot{\psi}_3$$

Maneuverability of AMRs



Choice of Wheel Configuration

- Choice of wheels and their configuration depends on …
  - Ability to maneuver
  - Controllability
  - Stability
- An optimum can only be achieved concerning a specific application

Stability

- Minimal number of wheels for static stability is 2, if center of gravity is located under the wheel axis (e. g. robot Cye)
- Under standard conditions it’s 3
- If more than 3 wheels are present, adjustment to rough terrain becomes necessary
- Typical setups
  - 2 driven wheels on a single axis with 1 or 2 passive castor wheels
  - Single driven and steered wheel with two passive fixed wheels

Maneuverability

- Robot’s degree of freedom (DOF)
- Outstanding ability to maneuver, when the only driven wheel can be rotated in an active way
- Example: Ackermann steering
  - Poor ability to maneuver
  - $5\ \mathrm{m}$ radius required for single rotation of a car
  - Lateral movement only possible trough forward/backward movement

Controllability

- Costs for locomotion of the robot on a given trajectory
- Contradiction to the robot's ability to maneuver
- Examples
  - Ackermann steering: easy to control
  - Mecanum wheels: very costly if high precision is required
- Costs for controllability depend on …
  - Wheel type
  - Drive geometry
  - Sensors to determine driving situation
  - Motor/transmission

Degree of Mobility

- The degree of mobility $f_m \in \{1,2,3\}$ points out how the sliding constraints (defined in Matrix S , see slide 42) affect the possibilities for robot movement $R(\theta)\,\dot{\xi}_I$
- The mobility can be calculated by using the rank of the matrix of sliding constraints of all standard wheels (steerable and fixed wheels) = number of independent constraints

$$f_m = 3 - \mathrm{rank}(S)$$

Degree of Mobility

- Example: Differential Drive
- Given only two fixed wheels geometrically we can set

$$d_1 = d_2, \qquad \beta_1 = \beta_2 = 0, \qquad \alpha_1 + \pi = \alpha_2$$

- Matrix S has two sliding constraints of the form $(c(\alpha + \beta) \;\; s(\alpha +$

Degree of Steerability

- The degree of steerability $f_s \in \{0,1,2\}$ quantifies the degree of controllable freedom and is determined via the number of independently controllable steering parameters
- Define S': Part of S defining sliding constraints of steerable wheels

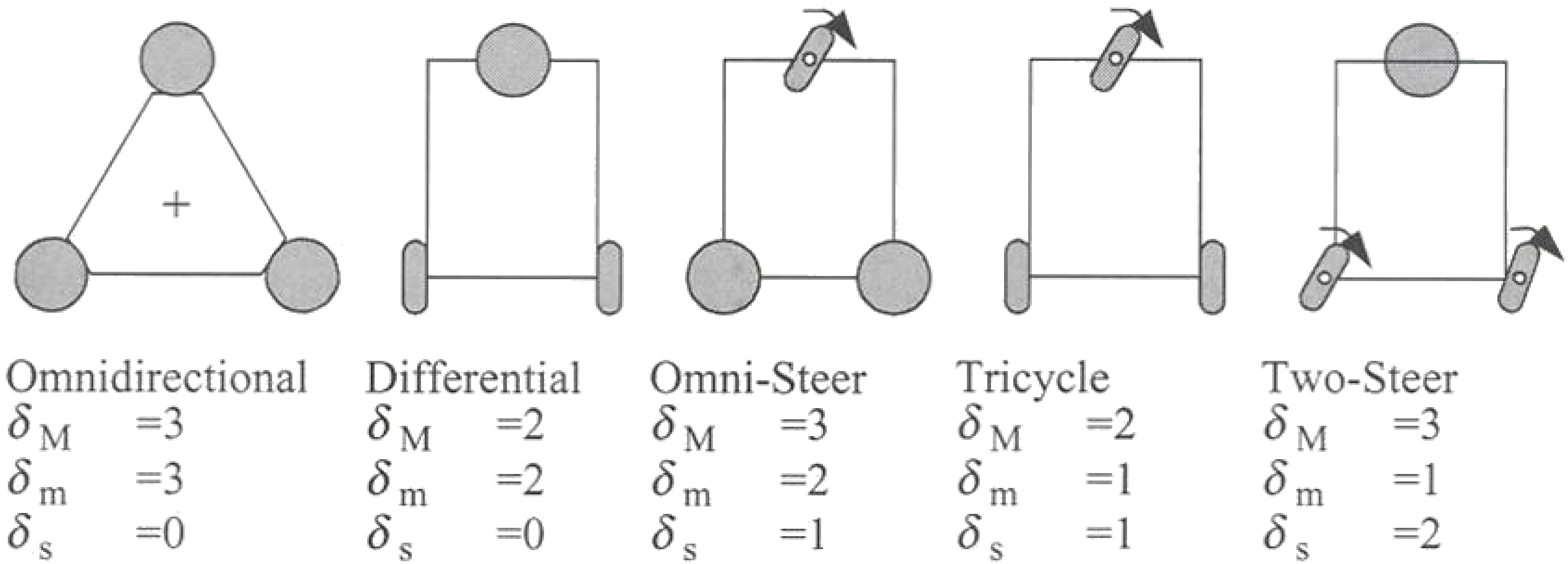
$$f_s = \mathrm{rank}(S')$$

- More possibilities in steering will result in lower mobility
  - $f_s = 0$: No steered wheel (differential drive)
  - $f_s = 1$: At least one steered wheel (Ackermann steering)
  - $f_s = 2$: At least two steered wheels, no fixed wheel

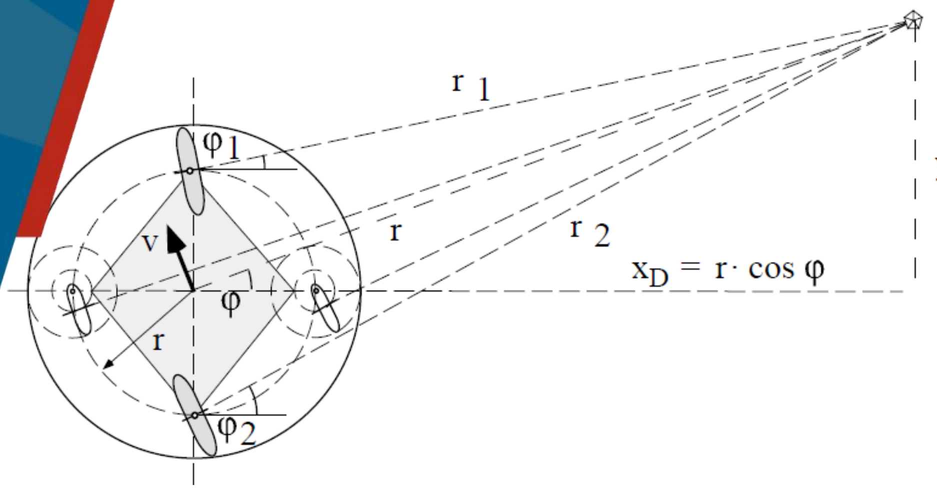
Degree of Maneuverability

The degree of maneuverability $f_M \in \{2,3\}$ points out the number of DOF the robot can manipulate

$$f_M = f_m + f_s$$



Vehicle Kinematics - Geometrical Solution



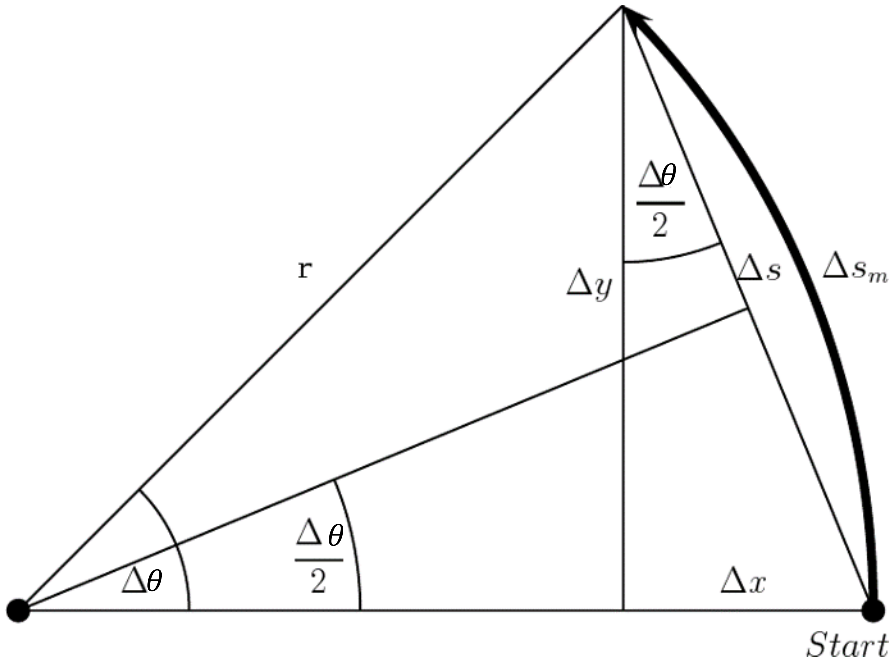
Differential Drive

- Given
  - Wheel velocities v_l, v_r
  - Time step Δt
- Calculation of the length of the driven path for each wheel and the kinematic center Δs_m

$$\Delta s_l = v_l \cdot \Delta t$$

$$\Delta s_r = v_r \cdot \Delta t$$

$$\Delta s_m = \frac{\Delta s_l + \Delta s_r}{2}$$



Geometrical solution of the differential drive kinematic

Differential Drive

- Based on this and the distance d between a wheel and the kinematic center the radius r can be derived

$$r = d \cdot \frac{\Delta s_r + \Delta s_l}{\Delta s_r - \Delta s_l}$$

- Keep in mind that we drive a left curve. Otherwise the radius will be negative (curve to the right). The change in orientation can be calculated using

$$\Delta\theta = \frac{\Delta s_m}{r} = \frac{\Delta s_r - \Delta s_l}{2 \cdot d}$$

Differential Drive

- For calculating translation changes one needs the length

$$\Delta s = 2 \cdot r \quad \cdot \sin\left(\frac{\Delta\theta}{2}\right)$$

- Based on Δs the final changes can be derived

$$\Delta x = \Delta s \cdot \sin\left(\frac{\Delta\theta}{2}\right)$$

$$\Delta y = \Delta s \cdot \cos\left(\frac{\Delta\theta}{2}\right)$$

Tricycle Drive

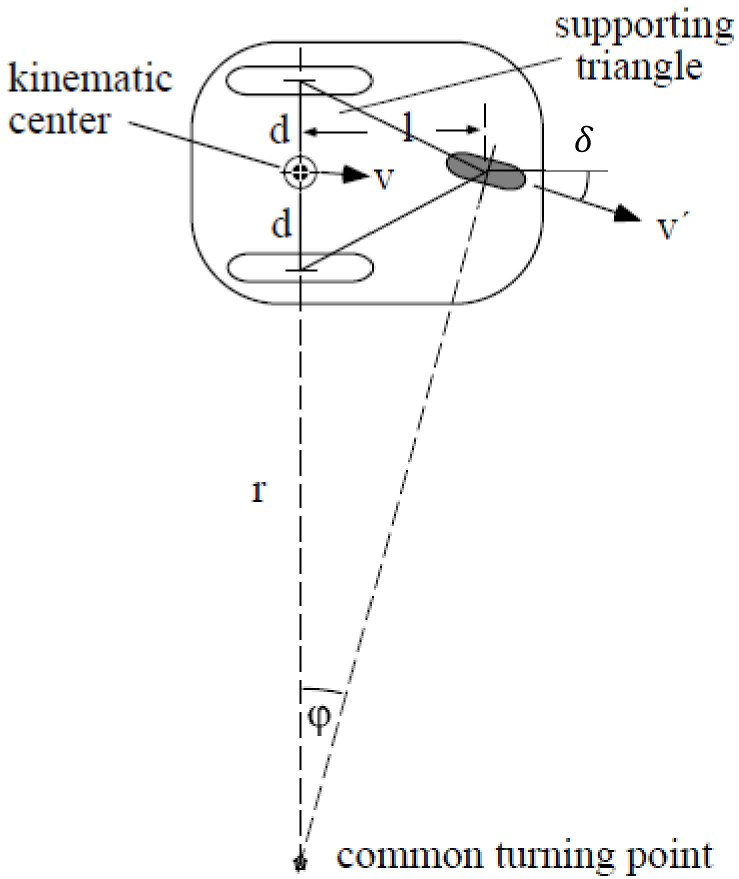
- Given: δ, v
- Unknown: v', v_l, v_r
- Solution

$$r = l \cdot \cot \delta$$

$$v' = \frac{v}{\cos \delta}$$

$$v_l = v \cdot \frac{r+d}{r}$$

$$v_r = v \cdot \frac{r-d}{r}$$

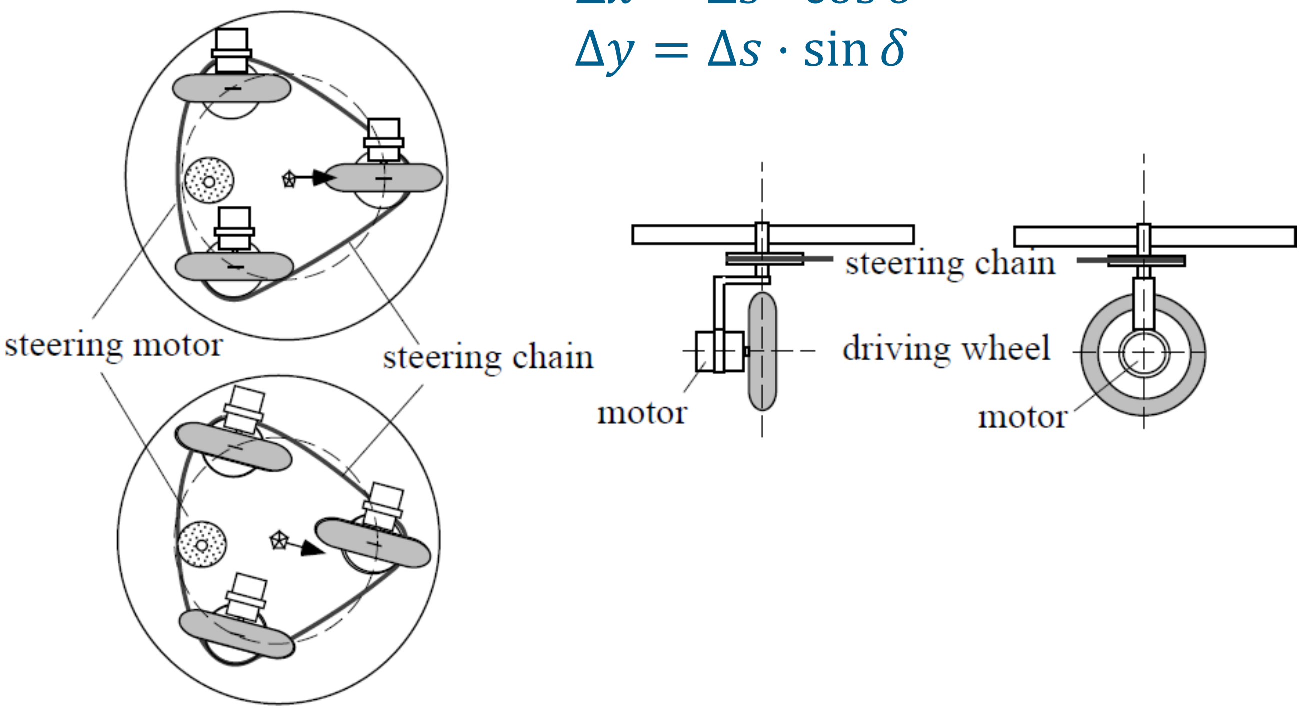


Tricycle drive kinematics

Synchro Drive

Let Δs denote the length of the traveled path of the driven wheels while δ is the steering angle. Thus we receive

$$\Delta x = \Delta s \cdot \cos \delta$$
$$\Delta y = \Delta s \cdot \sin \delta$$



Synchro Drive

If used the other way around, the above equation can be used to determine the desired parameters

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\delta = \arctan \frac{\Delta y}{\Delta x}$$

=> Synchro drive is unable to perform rotations

Ackermann Steering

- Given: v_D, δ, l, d
- Unknown: $v_{rr}, v_{lr}, v_{rf}, v_{lf}$
- Solution

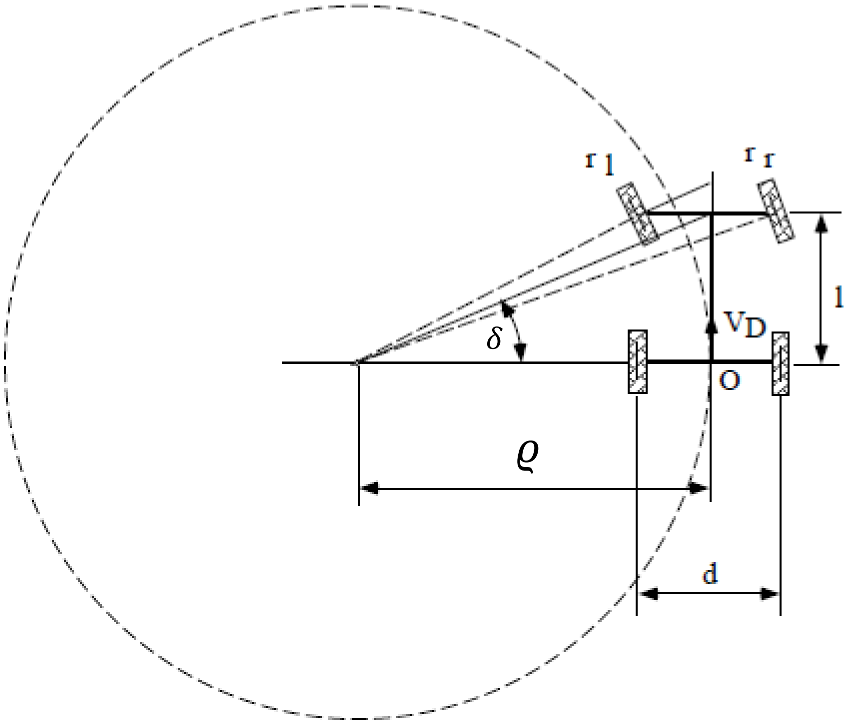
$$\varrho = \frac{l}{\tan\delta}$$

$$v_{lr} = \frac{(\varrho - d/2)\cdot v_D}{\varrho}$$

$$v_{rr} = \frac{(\varrho + d/2)\cdot v_D}{\varrho}$$

$$v_{lf} = \frac{\sqrt{(\varrho - d/2)^2 + l^2}\cdot|\tan\delta|}{l}\cdot v_D$$

$$v_{rf} = \frac{\sqrt{\varrho^2 + l^2}\cdot|\tan\delta|}{l}\cdot v_D$$

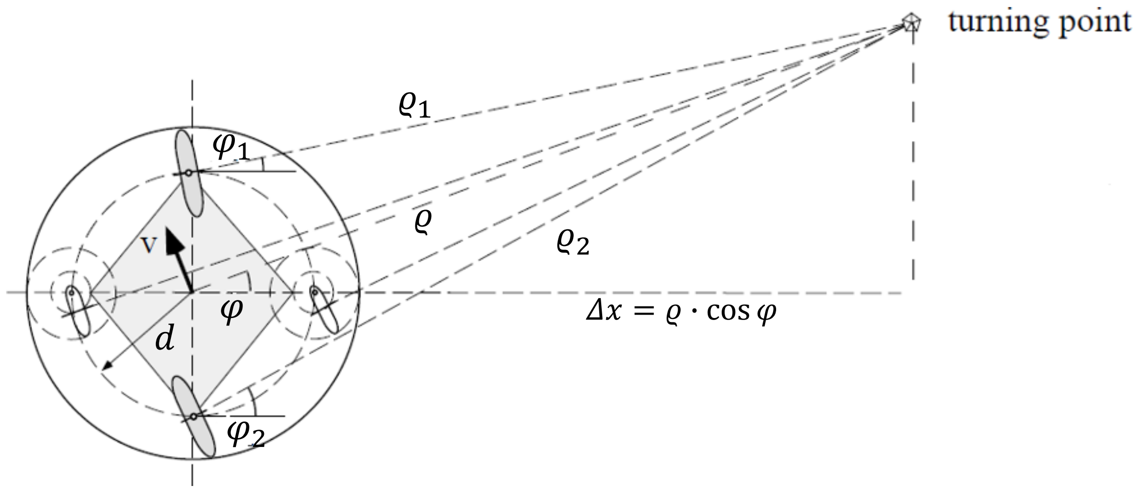


Kinematics of Ackermann steering

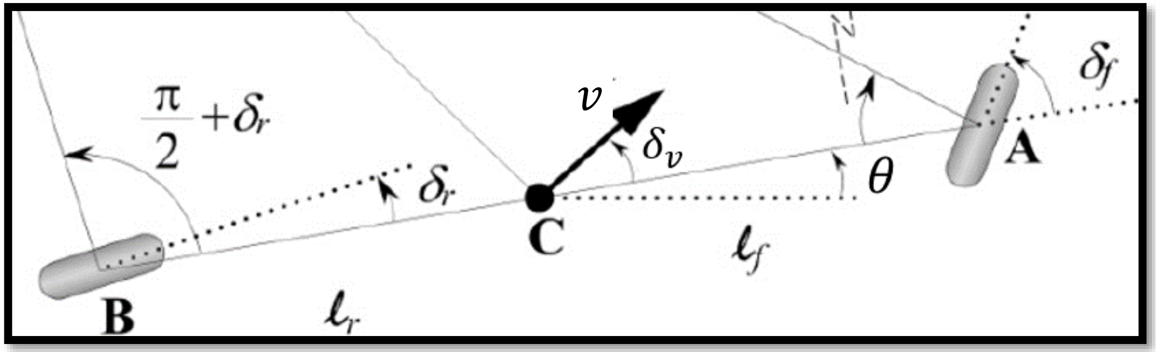
Omnidrive Kinematics

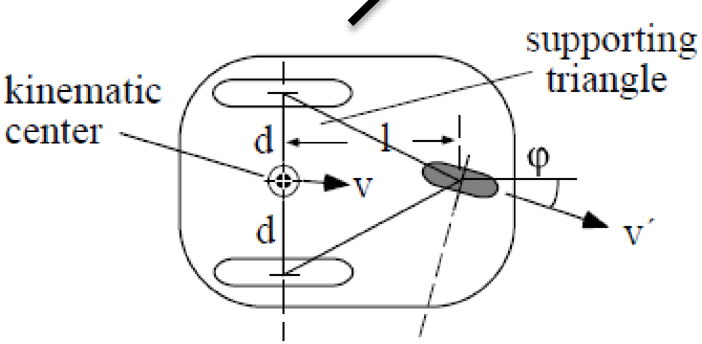
- Given: ϱ, φ, v, d
- Unknown: $\varphi_1, \varphi_2, v_1, v_2$
- Solution

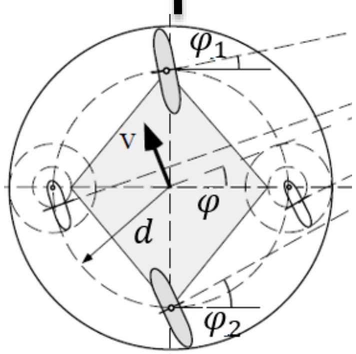
$$\Delta x = \varrho \cdot \cos\varphi$$
$$\Delta y = \varrho \cdot \sin\varphi$$
$$\varphi_1 = \arctan \Delta y - d / \Delta x$$
$$\varrho_1 = \varrho \cdot \cos\varphi \,/\, \cos\varphi_1$$
$$v_1 = v \cdot \varrho_1 / \varrho$$
$$\varphi_2 = \arctan \frac{\Delta y + d}{\Delta x}$$
$$\varrho_2 = \varrho \cdot \cos\varphi \,/\, \cos\varphi_2$$
$$v_2 = v \cdot \varrho_2 / \varrho$$

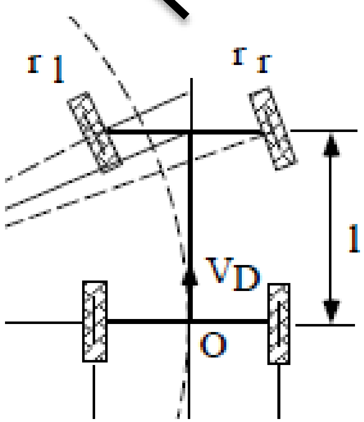


Bicycle Model









Bicycle Model

- Circular road of radius ϱ
- Low speed motion
  - Slip angles at both wheels are zero
  - Small total lateral force from both tires

$$F_y = \frac{mv^2}{\varrho}$$

- Point P is instantaneous rolling center
- Velocity at C is perpendicular to line PC
- Heading angle θ
- Slip angle δ_v
- Course angle $\gamma = \theta + \delta_v$

Bicycle Model

- Given: δ_f, δ_r, v, θ
- Unknown: $\dot{X}$, $\dot{Y}$, $\dot{\theta}$
- Solution

$$\delta_v = \tan^{-1}\left(\frac{l_f \tan\delta_r + l_r \tan\delta_f}{l_f + l_r}\right)$$

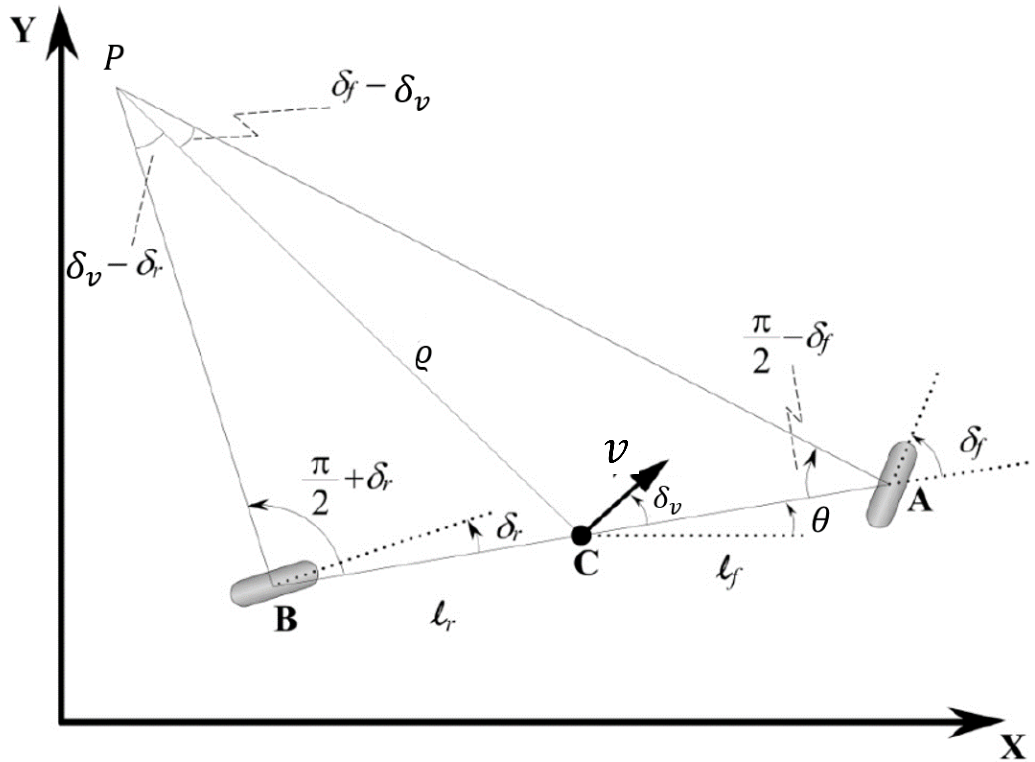
$$\dot{\theta} = \frac{v}{\varrho}$$

$$\varrho = \frac{l_f + l_r}{(\tan(\delta_f) - \tan(\delta_r))\cos(\delta_v)}$$

$$\dot{X} = v\cos(\theta + \delta_v)$$

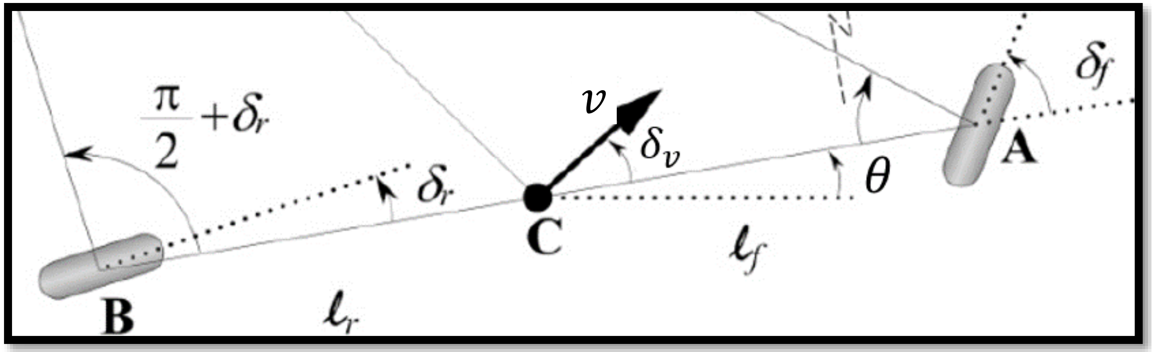
$$\dot{Y} = v\sin(\theta + \delta_v)$$

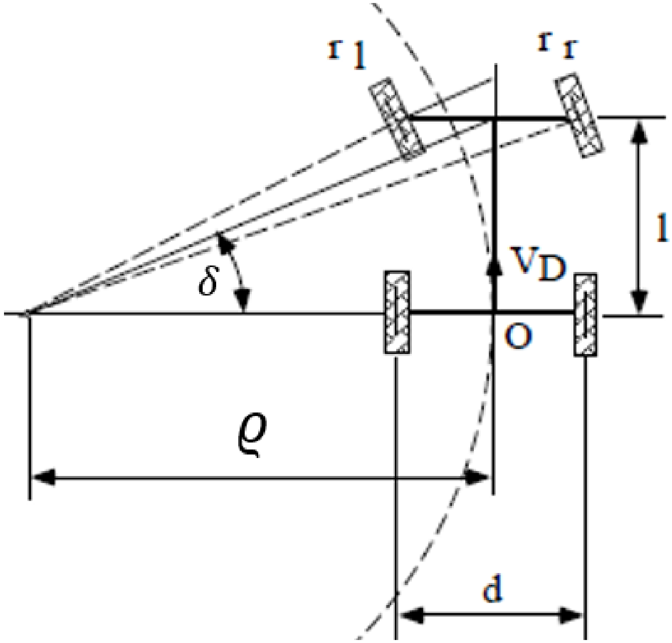
$$\dot{\theta} = \frac{v\cos(\delta_v)}{l_f + l_r}\left(\tan(\delta_f) - \tan(\delta_r)\right)$$



Ackermann Steering -> Bicycle Model

- $C = O$
- $\delta_r = 0$
- $\delta_f = \delta$
- $l_r = 0$
- $l_f = l$
- $\delta_v = 0$





$$\dot{\theta} = \frac{v}{\varrho}$$

$$\varrho = \frac{l}{\tan(\delta_f)}$$

$$\dot{X} = v\cos(\theta)$$

$$\dot{Y} = v\sin(\theta)$$

$$\dot{\theta} = \frac{v}{l_f}\tan(\delta_f)$$

Coming Next

Modeling - Mobile Robot Dynamics