Homework (Complexity Analysis)

1. (**20 points**) Consider the following algorithm.

ALGORITHM *Mystery(n)*

//Input: A nonnegative integer n

S ← 0

for *i* ←1 **to** ***n*** **do**

S ← S + i*i

return S

a. What does this algorithm compute?

The sum of squares from 1 to n.

b. What is its basic operation?

Multiplication and then addition

c. How many times is the basic operation executed?

n times

d. What is the worst case running time of this algorithm in asymptotic notation?

O(n)

2. (**20 points**) Consider the following algorithm.

ALGORITHM *Secret*(A[0…n-1])

//Input: An array A[O..n - 1] of *n* real numbers

minval ←A[O]; *maxval* ←A[O]

for i ←1to *n*- 1 **do**

if (A[i] < *minval)*

minval ←A[i]

if *(A[i]* > *maxval)*

maxval ←*A[i]*

return *maxval* - *minval*

a. What does this algorithm compute?

Difference between the maximum and minimum values of the input.

b. What is its basic operation?

Comparison.

c. How many times is the basic operation executed?

2(n-1) times as two comparisons occur in each iteration of the loop.

d. What is the worst case running time of this algorithm in asymptotic notation?

O(n)

3. (**10 points**) Compute the following sums.

a. 1 + 3 + 5 + 7…. 999

The i^{th} term in this series is $a_i = 2i - 1$. If S(n) denotes the sum with n elements, we can form the recurrence relation as

$S(n) = S(n-1) + 2n-1$

Solving the above equation, we get the sum of the series (assuming n terms) as: $S(n) = n(n-1) + n = n^2$. From n^{th} item = 2n-1 = 999, we get n = 500. Substituting n = 500, we get $S(500) = 1 + 3 + 5 + 7 \ldots + 999 = 500^2 =$ 250,000.

b. 2 + 4 + 8 + 16 + . . . + 1024

The i^{th} term in this series is $a_i = 2^i$. We get the recurrence relation as follows:

$S(n) = 2 + 4 + 8 + 16 + \ldots + 1024 = 2(1 + 2 + 4 + \ldots + 512) =$

$2 + 2(2 + 4 + 8 \ldots + 512) = 2 + 2S(n-1)$, $S(1) = 2$.

Therefore the sum $S(n) = 2^{(n+1)} – 2$, where n = the number of terms in the sequence.

Prove by induction (not required for the solution):

Assume $S(n-1) = 2^{(n-1+1)} – 2 = 2^n – 2$.

$S(n) = 2 + 2S(n-1) = 2 + 2 (2^n – 2) = 2 + 2^{(n+1)} – 4 = 2^{(n+1)} -2$

Substituting n = 10, we get

$S = 2^{(10+1)} – 2$

$= 2^{11} – 2$

$= 2048 – 2$

$= 2046$

4. (**20 points**) Climbing stairs Problem: Find the number of different ways to climb an n-stair staircase if each step is either one or two stairs.

This is a Fibonacci series. F(n) = F(n-1) + F(n-2) with F(1) = 1 and F(2) = 2.

5. (**30 points**) Solve the following recurrence relations and prove by induction.

a. $x(n) = x(n-1) + 5$ for $n > 1$, $x(1) = 0$

$$
\begin{aligned}
x(n) &= x(n-1) + 5 \\
&= [x(n-2) + 5] + 5 \\
&= [x(n-3) + 5] + 5 + 5] \\
&= x(n-3) + 5*3 \\
&= x(n-i) + 5*i
\end{aligned}
$$

For, $i = (n - 1)$

$$x(n) = x(n - (n - 1)) + 5 * (n - 1)$$

$$
\begin{aligned}
&= x(1) + 5(n - 1) \\
&= 0 + 5(n - 1) \\
&= 5(n - 1)
\end{aligned}
$$

Prove by induction:

Step 1: for $n = 1$, $x(n=1) = 5(n-1) = 5(1-1)=0$. Since $X(1)=1$ is given, the formula is true

Step 2: assume the formula is true for n-1, that is, $x(n-1) = 5((n-1)-1)$

Step3: to show the formula is also true for n, $x(n) = x(n-1) + 5 = 5((n-1)-1) + 5 = 5(n-2) + 5 = 5n - 10 + 5 = 5n - 5 = 5(n-1)$, so the formula is also true.

b. $x(n) = 3x(n-1)$ for $n > 1$, $x(1) = 4$

$$
\begin{aligned}
x(n) &= 3x(n - 1) \\
&= 3[3 x (n - 2] \\
&= 3^2 * x(n - 2) \\
&= 3 * 3[3 x (n - 3)] \\
&= 3^3 * x(n - 3) \\
&= 3^i * x(n - i)
\end{aligned}
$$

For, $i = n-1$,

$$
\begin{aligned}
x(n) &= 3^{(n-1)} x(n - (n - 1)) \\
&= 3^{(n-1)} x(1) \\
&= 3^{(n-1)} * 4
\end{aligned}
$$

Prove by induction:

Step 1: for n=2, $x(2) = 3x(2-1) = 3x(1) = 3*4 = 12$, $3^{(2-1)}*4 = 3*4 = 12$. So the formula is true for n=2

Step 2: assume the formula is true for n-1

Step 3: to show the formula is also true for n, $x(n) = 3x(n-1) = 3 * 3^{((n-1)-1)} * 4 = 3*3^{(n-2)} * 4 = 3^{(n-1)} * 4$, so the formula is also true for n

c. $x(n) = x(n-1) + n$ for $n > 0$, $x(0) = 0$

$$\begin{aligned} x(n) &= x(n - 1) + n \\ &= [x(n - 2) + (n -1)] + n \\ &= [x(n - 3) +(n -2)] + (n-1) + n \\ &\ldots\ldots\ldots\ldots \\ &=[x(n - i) +(n – (i - 1))] + (n-(i-2)) + (n-(i-3)) + \ldots.. + n \end{aligned}$$

For, $i = n$,

$$\begin{aligned} x(n) &= x(n - n) + n *(n+1)/2 \\ &= x(0) + n * (n+1)/2 \\ &= 0 + n * (n+1)/2 \end{aligned}$$

Prove by induction:

Step 1: for n=1, $x(1) = x(0) + 1 = 0 + 1 = 1$. from the formula, $x(1) = 0+1*(1+1)/2 = 1$, so the formula is true

Step 2: assume the formula is true for n-1

Step 3: to show the formula is also true for n, $x(n) = x(n-1) + n = (n-1)*(n-1 + 1)/2 + n = (n-1)*n/2 + n = n*n/2 –n/2 + n = n*n/2 + n/2 = (n + 1)*n/2 = n*(n+1)/2$, so the formula is also true for n.