# Some Elements of Learning Theory

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- A brief introduction to statistical learning
- From statistical learning to sequential decision making

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- Online convex optimization
- Contextual bandits

- We do some (short) proofs

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- Guidelines to practitioners (e.g., choice of learning bias, control of overfitting)
- Principled and successful algorithms (SVM, Boosting)

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- Learning algorithm: given a loss function, maps finite training sets to predictors

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- Zero-one loss: $f^*(\boldsymbol{x}) = 2\mathbb{I}\{\eta(\boldsymbol{x}) \geq 1/2\} - 1$ and $\ell_{\mathcal{D}}(f^*) = \mathbb{E}\Big[\min\{\eta(\boldsymbol{X}), 1 - \eta(\boldsymbol{X})\}\Big]$
  where $\eta(\boldsymbol{x}) = \mathbb{P}(Y = 1 \mid \boldsymbol{X} = \boldsymbol{x})$

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  - Show that $A$ can compress the training set (compression implies learning)

# Success stories: Characterization of sample complexity

What is the training set size $m_{\mathcal{H}}$ necessary and sufficient to ensure

$$\ell_{\mathcal{D}}(h_S) - \inf_{h \in \mathcal{H}} \ell_{\mathcal{D}}(h) \leq \varepsilon$$

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- Majority vote over a set of consistent predictors achieves upper bound in the realizable case

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## No Free Lunch Theorem

Let $a_1, a_2, \cdots > 0$ be any sequence of numbers slowly converging to zero.
For all binary classification algorithms $A$ there exists $\mathcal{D}$ such that the Bayes risk is zero and, simultaneously, $\mathbb{E}\big[\ell_{\mathcal{D}}\big(A(S_m)\big)\big] \geq a_m$ for all $m \geq 1$.

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## Curse of dimensionality

- Typical parametric rates for convergence to $\ell_{\mathcal{D}}(h^*)$: $m^{-1/2}$
- Typical nonparametric rates for convergence to Bayes risk: $m^{-1/d}$ for $d \geq 2$ (under assumptions on $\mathcal{D}$)

# Online learning


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- Data streams are ubiquitous: sensors, markets, user interactions
- New data is being generated all the time
- The train-test model of statistical learning is ill-suited for learning on data streams
- After observing a new data point, predictors should be incrementally adjusted at a constant cost

# History bits

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- Volodya Vovk independently develops a related framework (Aggregating strategies, 1990)
- Similar ideas also independently emerged in game theory and information theory

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- Computation of $h_{t+1}$ relies on local information

# The online learning protocol

The algorithm starts with a default model $h_1 \in \mathcal{H}$

For $t = 1, 2, \ldots$

1. The current model $h_t \in \mathcal{H}$ is tested on the next data point $(\boldsymbol{x}_t, y_t)$ in the stream
2. $A$ is charged with loss $\ell(y_t, h_t(\boldsymbol{x}_t))$
3. $h_{t+1} \in \mathcal{H}$ is computed based on $h_t$ and $(\boldsymbol{x}_t, y_t)$

- Computation of $h_{t+1}$ relies on local information
- No stochastic assumptions on the stream

# Regret

## Sequential risk

Given a convex loss $\ell$ and a stream $(\boldsymbol{x}_1, y_1), (\boldsymbol{x}_2, y_2), \ldots$, the sequential risk of $A$ is

$$\boxed{\sum_{t=1}^{T} \ell(y_t, h_t(\boldsymbol{x}_t))}$$

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$$R_T = \sum_{t=1}^{T} \ell(y_t, h_t(\boldsymbol{x}_t)) - \inf_{h \in \mathcal{H}} \sum_{t=1}^{T} \ell(y_t, h(\boldsymbol{x}_t))$$

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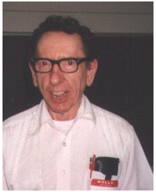
- A sequential counterpart to the variance error in statistical learning
- Can we ensure $\frac{R_T}{T} \to 0$ as $T \to \infty$ for all streams?

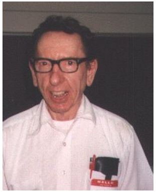
# Online learning as a repeated game



## Learning to play a game (1956)

- Theory of repeated games pioneered by James Hannan and David Blackwell

# Online learning as a repeated game



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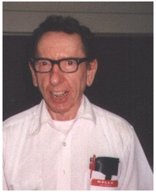

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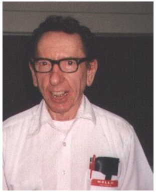

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- Let $\mathcal{H}$ be the $d$-dimensional simplex $\Delta_d$

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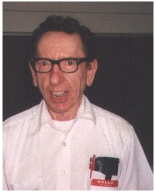

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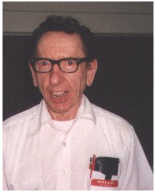

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- This is a linear loss with bounded coefficients $\ell_t(i) \in [0, 1]$

# Prediction with expert advice

## A sequential decision problem

- $d$ actions
- Unknown deterministic assignment of losses to actions $\boldsymbol{\ell}_t = (\ell_t(1), \ldots, \ell_t(d)) \in [0,1]^d$ for each time step $t$

? ? ? ? ? ? ? ? ? ?

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# Regret

$$R_T = \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p}_t - \min_{\boldsymbol{p} \in \Delta_d} \sum_{t=1}^{T} \boldsymbol{\ell}_t^\top \boldsymbol{p}$$

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- $\ell_t(i) \to L_t(i) \in \{0,1\}$ independent random coin flip
- For any player strategy $\quad \mathbb{E}\left[\sum_{t=1}^T L_t(I_t)\right] = \frac{T}{2}$

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## Lower bound using a statistical learning argument

- $\ell_t(i) \to L_t(i) \in \{0,1\}$ independent random coin flip
- For any player strategy $\mathbb{E}\left[\sum_{t=1}^{T} L_t(I_t)\right] = \frac{T}{2}$
- Then the expected regret is

$$\mathbb{E}\left[\max_{i=1,\dots,d} \sum_{t=1}^{T} \left(\frac{1}{2} - L_t(i)\right)\right] = (1-o(1))\sqrt{\frac{T \ln d}{2}}$$

for $d, T \to \infty$

# Exponentially weighted forecaster (Hedge)

At time $t$ pick action $I_t = i$ with probability proportional to

$$\exp\left(-\eta \sum_{s=1}^{t-1} \ell_s(i)\right)$$

the sum at the exponent is the total loss of action $i$ up to the previous time step

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- If $\eta = \sqrt{\frac{\ln d}{8T}}$ then $\boxed{R_T \le \sqrt{\frac{T \ln d}{2}}}$

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- We prove this later in a more general setting

# The bandit problem: playing an unknown game

- $d$ actions
- Unknown deterministic assignment of losses to actions $\boldsymbol{\ell}_t = (\ell_t(1), \dots, \ell_t(d)) \in [0,1]^d$ for each time step $t$

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?   .3   ?   ?   ?   ?   ?   ?   ?   ?

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# A growing range of applications

- Ad placement

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- Ad placement
- Dynamic content/layout optimization

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- Ad placement
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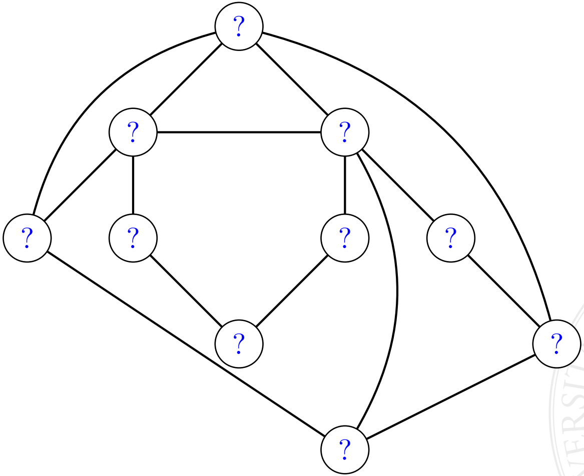
# An observability graph over actions



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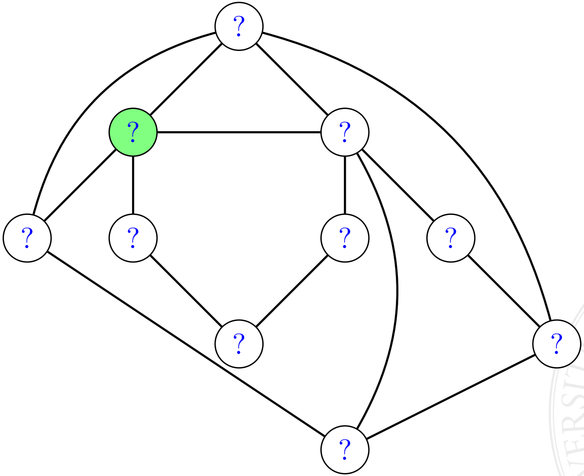

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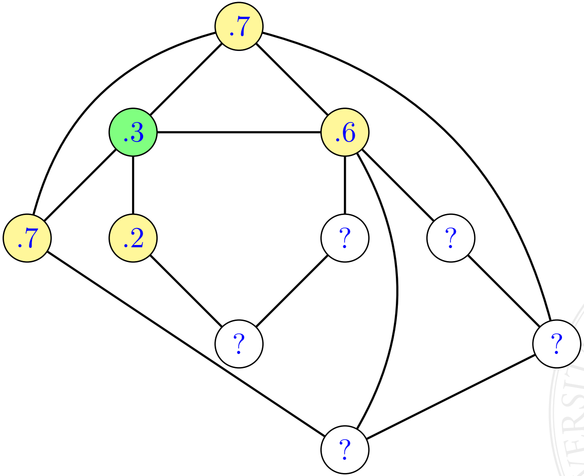
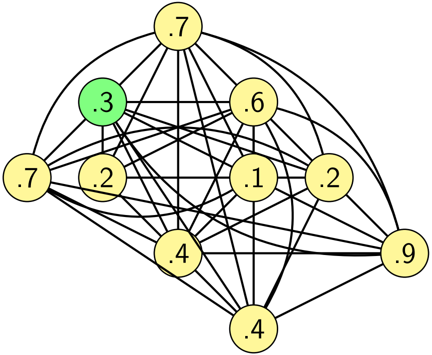


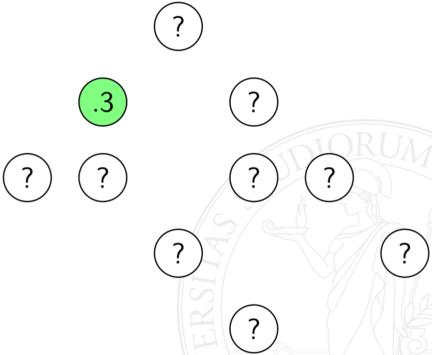

$\ell_t(i)$ is observed iff $I_t \in \{i\} \cup \mathcal{N}_G(i)$

# Recovering expert and bandit settings

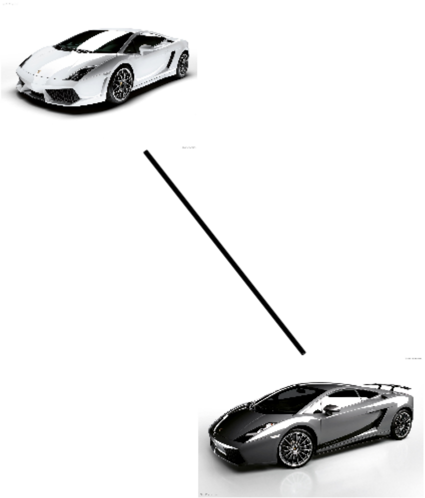
Experts: clique

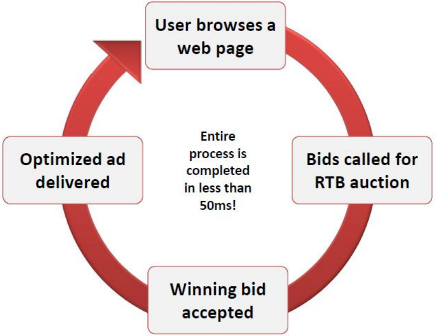



Bandits: edgeless graph



# Relationships between actions





# Hedge revisited on an observability graph $G$

## Player's strategy must use loss estimates

- $$p_t(i) \quad \propto \quad \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) \qquad i = 1, \dots, d$$

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Importance sampling estimator

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## Importance sampling estimator

$$\mathbb{E}_t\left[\widehat{\ell}_t(i)\right] = \frac{\ell_t(i)}{\mathbb{P}_t(\ell_t(i) \text{ observed})} \times \mathbb{P}_t(\ell_t(i) \text{ observed}) + 0 = \ell_t(i)$$

$$\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right] = \frac{\ell_t(i)^2}{\mathbb{P}_t(\ell_t(i) \text{ observed})^2} \times \mathbb{P}_t(\ell_t(i) \text{ observed}) + 0 = \frac{\ell_t(i)^2}{\mathbb{P}_t(\ell_t(i) \text{ observed})}$$

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# Regret analysis

$$\frac{W_{t+1}}{W_t} = \sum_{i=1}^{d} \frac{w_{t+1}(i)}{W_t} \qquad p_t(i) = \frac{1}{W_t} \exp\left(-\eta \sum_{s=1}^{t-1} \widehat{\ell}_s(i)\right) = \frac{w_t(i)}{W_t} \quad \text{is a r.v.!}$$

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$$\leq \sum_{i=0}^{d} p_t(i) \left(1 - \eta\, \widehat{\ell}_t(i) + \frac{\big(\eta\, \widehat{\ell}_t(i)\big)^2}{2}\right) \qquad \text{(using } e^{-x} \leq 1 - x + x^2/2 \text{ for all } x \geq 0\text{)}$$

# Regret analysis

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$$\leq 1 - \eta \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) + \frac{\eta^2}{2} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

# Regret analysis (cont.)

Taking logs, using $\ln(1+x) \leq x$, and summing over $t = 1, \dots, T$ yields

$$\ln \frac{W_{T+1}}{W_1} \leq -\eta \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) + \frac{\eta^2}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

## Regret analysis (cont.)

Taking logs, using $\ln(1+x) \le x$, and summing over $t = 1, \dots, T$ yields

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Moreover, for any fixed action $k$, we also have

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Putting together and dividing both sides by $\eta > 0$ gives

$$\sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i) - \sum_{t=1}^{T} \widehat{\ell}_t(k) \leq \frac{\ln d}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{d} p_t(i) \widehat{\ell}_t(i)^2$$

# Regret analysis (cont.)

Recall where we were:

$$\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\widehat{\ell}_t(i) - \sum_{t=1}^{T}\widehat{\ell}_t(k) \le \frac{\ln d}{\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\widehat{\ell}_t(i)^2$$

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Take expectation w.r.t. $I_1, \ldots, I_T$

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)\right] - \sum_{t=1}^{T}\mathbb{E}_t\left[\widehat{\ell}_t(k)\right]\right] \le \frac{\ln d}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right]\right]$$

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Loss estimates are unbiased:

$$\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\ell_t(i) - \sum_{t=1}^{T}\ell_t(k)\right] \le \frac{\ln d}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right]\right]$$

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This is just the regret

$$R_T = \mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\ell_t(i) - \sum_{t=1}^{T}\ell_t(k)\right] \leq \frac{\ln d}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right]\right]$$

# Regret analysis (cont.)

$$R_T \le \frac{\ln d}{\eta} + \frac{\eta}{2}\,\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} p_t(i)\mathbb{E}_t\left[\widehat{\ell}_t(i)^2\right]\right]$$

# Regret analysis (cont.)

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&\le \frac{\ln d}{\eta} + \frac{\eta}{2}\,\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} \frac{p_t(i)}{\mathbb{P}_t\big(\ell_t(i) \text{ is observed}\big)}\right] \qquad \text{(variance bound)}
\end{aligned}
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# Regret analysis (cont.)

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&= \frac{\ln d}{\eta} + \frac{\eta}{2}\,\mathbb{E}\left[\sum_{t=1}^{T}\sum_{i=1}^{d} \frac{p_t(i)}{p_t(i) + \sum_{j\in\mathcal{N}_G(i)} p_t(j)}\right] && \text{(observability condition)}
\end{aligned}
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# Regret analysis (cont.)

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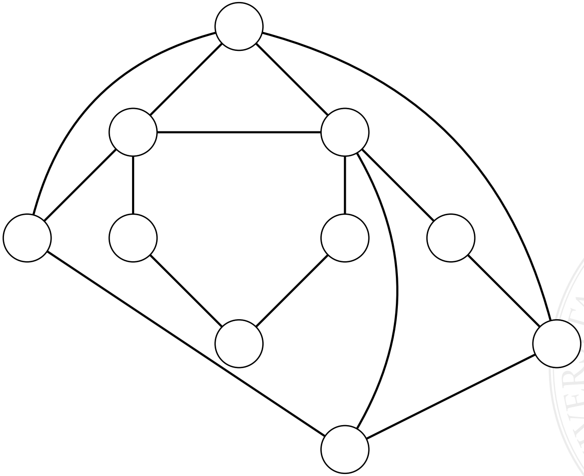
&\le \frac{\ln d}{\eta} + \frac{\eta}{2} T\,\alpha(G) && \text{(cool graph-theoretic fact)}
\end{aligned}
$$

$\alpha(G)$ is the independence number of $G$

## Independence number $\alpha(G)$

The size of the largest independent set in $G$



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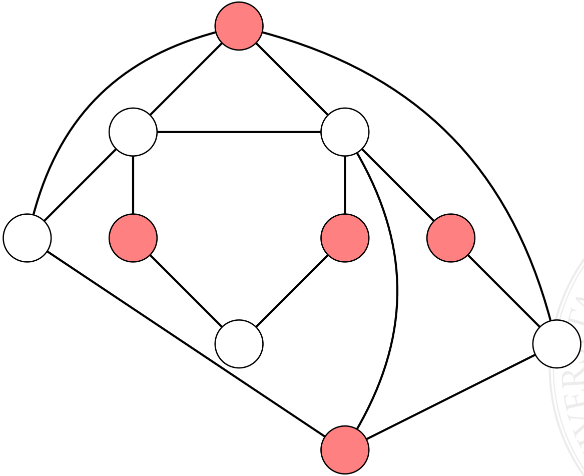

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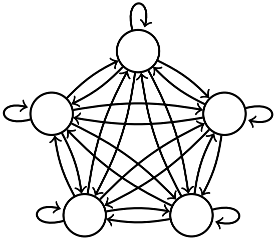
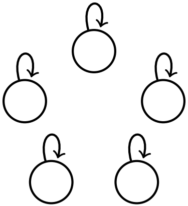
## Special cases

| | | | |
|---|---|---|---|
| Experts (clique): | $\alpha(G) = 1$ | $R_T \leq \sqrt{T \ln d}$ | Hedge algorithm |
| Bandits (edgeless graph): | $\alpha(G) = d$ | $R_T \leq \sqrt{Td \ln d}$ | Exp3 algorithm |

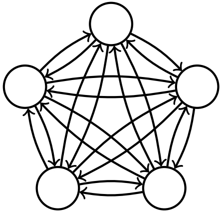
# More general feedback models



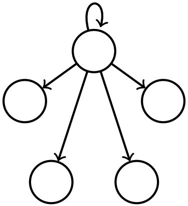
Experts



Bandits



Cops & Robbers



Revealing Action

# Partial monitoring: not observing your own loss

## Dynamic pricing: Perform as the best fixed price

1. Post a T-shirt price
2. Observe if next customer buys or not
3. Adjust price

Feedback does not reveal the player's loss

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 2 | 3 | 4 |
| 2 | $c$ | 0 | 1 | 2 | 3 |
| 3 | $c$ | $c$ | 0 | 1 | 2 |
| 4 | $c$ | $c$ | $c$ | 0 | 1 |
| 5 | $c$ | $c$ | $c$ | $c$ | 0 |

Loss

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0 | 1 | 1 | 1 | 1 |
| 3 | 0 | 0 | 1 | 1 | 1 |
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Feedback

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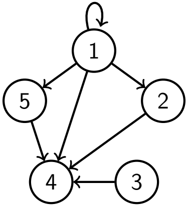
  3. Impossible games: $\Theta(T)$



# Contextual bandits

- A policy $\pi$ maps side information (e.g., feature vectors $\boldsymbol{x}_t$) to probabilistic decisions $\pi(\boldsymbol{x}_t) \in \Delta_d$

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- This holds for all loss sequences, sets of policies, and side information sequences
- Need time linear in $|\Pi|$ at each step

# Online convex optimization

Model space $\mathbb{V} \subseteq \mathbb{R}^d$ convex, closed, and nonempty

For $t = 1, 2, \ldots$

1. The current $h_t \in \mathcal{H}$ is tested on the next data point $(\boldsymbol{x}_t, y_t)$ in the stream
2. $A$ is charged with loss $\ell(y_t, h_t(\boldsymbol{x}_t))$
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Regret

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# Stochastic gradient descent

## Minimization of training error

$$\min_{\boldsymbol{w}\in\mathbb{V}} \sum_{i=1}^{m} \ell(\boldsymbol{w}, (\boldsymbol{x}_i, y_i))$$

$\ell(\boldsymbol{w}, (\boldsymbol{x}_i, y_i))$ measures the (convex) loss of $\boldsymbol{w}$ on the training example $(\boldsymbol{x}_i, y_i)$

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- Run online algorithm on the sequence of loss functions $\ell_t = \ell_t(\cdot,(\boldsymbol{X}_t,Y_t))$

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$$\mathbb{E}\left[\max_{\boldsymbol{u}\in\{\boldsymbol{v}_1,\boldsymbol{v}_2\}} R_T(\boldsymbol{u})\right] = \mathbb{E}\left[\max_{\boldsymbol{u}\in\{\boldsymbol{v}_1,\boldsymbol{v}_2\}} \sum_{t=1}^{T} L_t(\boldsymbol{u})\right] \qquad \text{(since } \mathbb{E}[L_t(\boldsymbol{w})] = 0\text{)}$$

# Lower bounds

- $\mathbb{V}$ is a bounded set of diameter $D$ and all $\ell_t$ are Lipschitz with constant $L$
- Take $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{V}$ such that $\|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_2 = D$ and set $\boldsymbol{z}_0 = (\boldsymbol{v}_1 - \boldsymbol{v}_2) / \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_2$
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&\geq LD\sqrt{\frac{T}{8}} && \text{(Khintchine inequality)}
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The geometry of $\mathbb{V}$ matters

# Gradient descent: from online to offline

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The Bregman divergence $B_\psi$ measures a generalized squared distance between $\boldsymbol{w}, \boldsymbol{w}_t \in \mathbb{V}$

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A differentiable $\psi : \mathbb{R}^d \to \mathbb{R}$ is $\mu$-strongly convex on $\mathbb{V}$ with respect to $\|\cdot\|$ if

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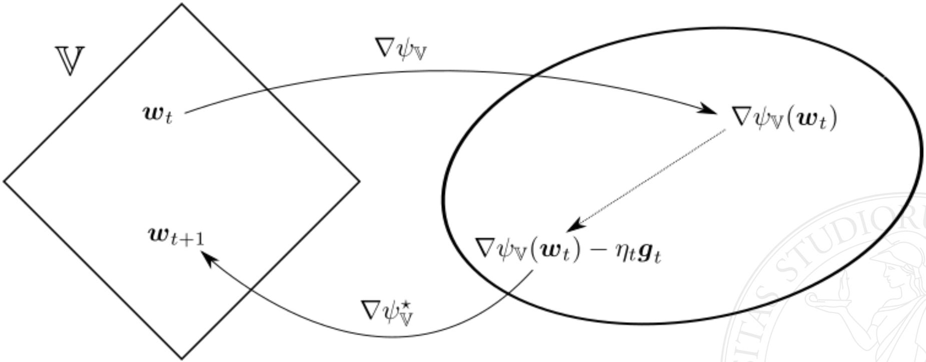
# The mirror step

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# Regret analysis

## Two basic inequalities

$\boxed{\boldsymbol{g}_t = \nabla \ell_t(\boldsymbol{w}_t)}$

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&\le \sum_{t=1}^T \boldsymbol{g}_t^\top(\boldsymbol{w}_t - \boldsymbol{u}) && \text{(linearized regret)} \\
&\le \sum_{t=1}^T \left(\frac{B_\psi(\boldsymbol{u}, \boldsymbol{w}_t)}{\eta_t} - \frac{B_\psi(\boldsymbol{u}, \boldsymbol{w}_{t+1})}{\eta_t}\right) + \frac{1}{2\mu}\sum_{t=1}^T \eta_t \|\boldsymbol{g}_t\|_\star^2 && \text{(Bregman's progress)}
\end{aligned}
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# Regret analysis (cont.)

$$\sum_{t=1}^{T} \left( \frac{B_\psi(\boldsymbol{u}, \boldsymbol{w}_t)}{\eta_t} - \frac{B_\psi(\boldsymbol{u}, \boldsymbol{w}_{t+1})}{\eta_t} \right) + \frac{1}{2\mu} \sum_{t=1}^{T} \eta_t \left\| \boldsymbol{g}_t \right\|_\star^2$$

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# Some remarks

- We can interpolate between OGD and EG using a $p$-norm as a mirror map:
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- Choosing $p = \frac{2\ln d}{2\ln d - 1}$ gives bound similar to EG without the tuning problem

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- OMD and FTRL have similar regret bounds and become identical in certain cases
- Time-dependent regularizers are generally more flexible than time-dependent learning rates

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Choose the model minimizing a second-order approximation of the true loss:

$$\boldsymbol{w}_{t+1} = \operatorname*{argmin}_{\boldsymbol{w}\in\mathbb{V}} \sum_{s=1}^{t} \widehat{\ell}_s(\boldsymbol{w}) \qquad \text{(Follow-the-Leader approach)}$$

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- $\mathcal{O}(d\ln T)$ matches the bound for strongly convex losses

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- This bound cannot be simultaneously achieved for all $\boldsymbol{u}$!

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- The length is learned using a parameterless $1$-dimensional online learning algorithm
- One such algorithm has regret $R_T(w) = \mathcal{O}\big(|w|\sqrt{T\ln(T)}\big)$ for all $w \in R$

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$1$-dimensional parameterless online algorithms extracted from investment strategies

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- A lower bound on $C_T$ implies an upper bound on $R_T(w)$ for all $w \in \mathbb{R}$

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- Matching upper bound obtained by using Hedge to aggregate $\mathcal{O}(\ln T)$ instances of OGD each tuned to a different $\Pi_T$

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- The set of intervals is carefully designed so that the overall number of instances to be run is $\mathcal{O}(\ln T)$

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- Linear prediction with convex loss $\ell(\boldsymbol{w}^\top \boldsymbol{x}, y_t)$

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- Then, by the bounded martingale concentration law,

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Regret: $$R_T^{\mathrm{cont}} = \sum_{t=1}^{T} \max_{\boldsymbol{x} \in C_t} \boldsymbol{w}^\top \boldsymbol{x} - \sum_{t=1}^{T} \boldsymbol{w}^\top \boldsymbol{x}_t$$

# The confidence ellipsoid

Fix a sequence of contexts $C_1, \dots, C_t$ and choices $\boldsymbol{x}_s \in C_s$, $s = 1, \dots, t$
RLS estimate

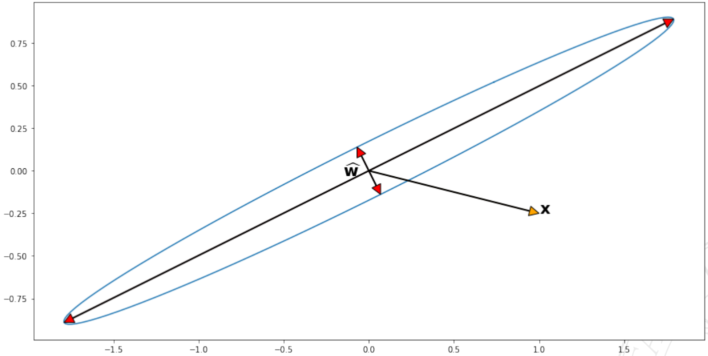
$$\widehat{\boldsymbol{w}}_t = V_t^{-1} \sum_{s=1}^{t} Y_s \boldsymbol{x}_s \qquad V_t = \lambda I_d + \underbrace{[\boldsymbol{x}_1, \dots, \boldsymbol{x}_t]}_{d \times t} [\boldsymbol{x}_1, \dots, \boldsymbol{x}_t]^\top$$

With high probability, $\boldsymbol{w} \in \mathcal{E}_t \equiv \left\{ \boldsymbol{u} \in \mathbb{R}^d : \|\boldsymbol{u} - \widehat{\boldsymbol{w}}\|_{V_t} \le \beta_t \right\}$

$\beta_t$ of order $D + R\sqrt{1 + d \ln\left(1 + \frac{t}{d}\right)}$

Think of $\mathcal{E}_t$ as a $d$-dimensional confidence interval

# The LinUCB/OFUL algorithm




Optimism in the face of uncertainty

$$\boldsymbol{x}_{t+1} = \operatorname*{argmax}_{\boldsymbol{x}\in C_{t+1}} \max_{\boldsymbol{u}\in\mathcal{E}_t} \boldsymbol{u}^\top \boldsymbol{x} = \operatorname*{argmax}_{\boldsymbol{x}\in C_t} \left( \widehat{\boldsymbol{w}}_t^\top \boldsymbol{x} + \beta_t \|\boldsymbol{x}\|_{V_t^{-1}} \right)$$

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- In this case, $R_T^{\mathrm{cont}} = \mathcal{O}\left((m \ln T)\sqrt{T}\right)$ for both algorithms

# Some references

List of wonderful references goes here