2.4 ITERATION METHOD

We have so far discussed root-finding methods which require an interval in which the root lies. We now describe methods which require one or more approximate values to start the solution and these values need not necessarily bracket the root. The first is the *iteration* method which requires one starting value of x.

To describe this method for finding a root of the equation

$$f(x) = 0, \tag{2.1}$$

we rewrite this equation in the form

$$x = \phi(x) \tag{2.8}$$

There are many ways of doing this. For example, the equation

$$x^3 + x^2 - 2 = 0$$

can be expressed in different forms

$$x = \sqrt{\frac{2}{1+x}}, \quad x = \sqrt{2 - x^3}, \quad x = (2 - x^2)^{1/3}, \text{ etc.}$$

Now, let x_0 be an approximate root of Eq. (2.8). Then, substituting in Eq. (2.8), we get the first approximation as

$$x_1 = \phi(x_0)$$

Successive substitutions give the approximations

$$x_2 = \phi(x_1), \quad x_3 = \phi(x_2), \ldots, \quad x_n = \phi(x_{n-1}).$$

The preceding sequence may not converge to a definite number. But if the sequence converges to a definite number ξ, then ξ will be a root of the equation $x = \phi(x)$. To show this, let

$$x_{n+1} = \phi(x_n) \tag{2.9}$$

be the relation between the nth and $(n + 1)$th approximations. As n increases, $x_{n+1} \rightarrow \xi$ and if $\phi(x)$ is a continuous function, then $\phi(x_n) \rightarrow \phi(\xi)$. Hence, in the limit, we obtain

$$\xi = \phi(\xi), \tag{2.10}$$

which shows that ξ is a root of the equation $x = \phi(x)$.

To establish the condition of convergence of Eq. (2.8), we proceed in the following way:
From Eq. (2.9), we have

$$x_1 = \phi(x_0) \tag{2.11}$$

From Eqs. (2.10) and (2.11), we get

$$\begin{aligned} \xi - x_1 &= \phi(\xi) - \phi(x_0) \\ &= (\xi - x_0)\,\phi'(\xi_0), \quad x_0 < \xi_0 < \xi, \end{aligned} \tag{2.12}$$

on using Theorem 1.5. Similarly, we obtain

$$\xi - x_2 = (\xi - x_1)\,\phi'(\xi_1), \quad x_1 < \xi_1 < \xi \tag{2.13}$$

$$\xi - x_3 = (\xi - x_2)\,\phi'(\xi_2), \quad x_2 < \xi_2 < \xi \tag{2.14}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\xi - x_{n+1} = (\xi - x_n)\,\phi'(\xi_n), \quad x_n < \xi_n < \xi \tag{2.15}$$

If we assume

$$|\phi'(\xi_i)| \le k \text{ (for all } i), \tag{2.16}$$

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$$|\phi'(\xi_i)| \le k \text{ (for all } i\text{)}, \tag{2.16}$$

then Eqs. (2.12) to (2.15) give

$$\left.\begin{aligned} |\xi - x_1| &\le k\,|\xi - x_0| \\ |\xi - x_2| &\le k\,|\xi - x_1| \\ |\xi - x_3| &\le k\,|\xi - x_2| \\ &\;\vdots \\ |\xi - x_{n+1}| &\le k\,|\xi - x_n| \end{aligned}\right\} \tag{2.17}$$

Multiplying the corresponding sides of the above equations, we obtain

$$|\xi - x_{n+1}| \le k^{n+1}\,|\xi - x_0| \tag{2.18}$$

If $k < 1$, i.e., if $|\phi'(\xi_i)| < 1$, then the right side of Eq. (2.18) tends to zero and the sequence of approximations $x_0, x_1, x_2, \ldots$ converges to the root ξ. Thus, when we express the equation $f(x) = 0$ in the form $x = \phi(x)$, then $\phi(x)$ must be such that

$$|\phi'(x)| < 1$$

in an immediate neighbourhood of the root. It follows that if *the initial approximation x_0 is chosen in an interval containing the root ξ, then the sequence of approximations converges to the root ξ.*

Geometrical Significance: Draw the graph of $y_1 = x$ and $y_2 = \phi(x)$ as shown below in the figure 3.2. Since $|\phi'(x)| < 1$ for convergence, the initial approximation of the curve $y_2 = \phi(x)$ must be less then 45° in the neighbourhood of x_0. This fact has been observed in constructing the graph.

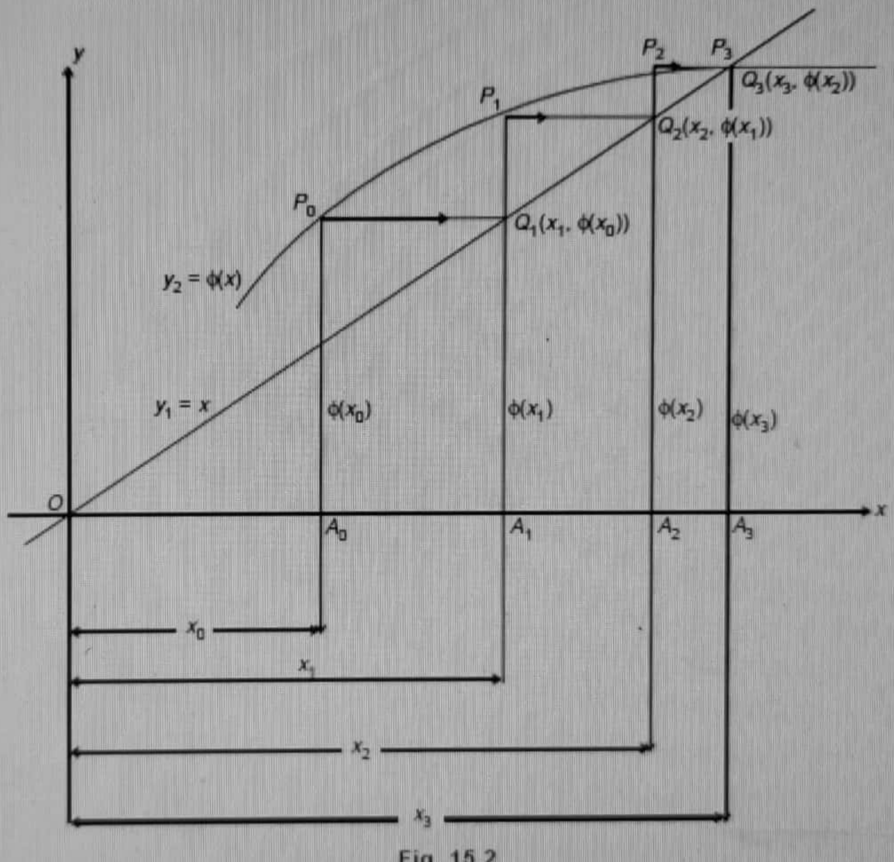

Fig. 15.2

For tracing the convergence of the iteration process, draw the ordinates $\phi(x_0)$. Then from point p_0, draw a line parallel to OX until it intersects the line $y_1 = x$ at the point $Q_1(x_1, \phi(x_0))$. This point Q_1 is the geometric representation of the first iteration $x_1 = \phi(x_0)$.

Then draw Q_1P_1, P_1Q_2, Q_2P_2, P_2Q_3 etc. as indicated by the arrows in geometry. The points $Q_1, Q_2, Q_3, \ldots\ldots\ldots$, thus approaches to the point of intersection of the curves $y_1 = x$ and $y_2 = \phi(x)$ as iteration proceeds and converges to the desired root.

Order of Convergence (Rate of Convergence):

Any method is said to have convergence of order p, if p is the largest positive real number such that $|\varepsilon_{n+1}| \leq k|\varepsilon_n|^p$ where k is a finite positive constant, and ε_n and ε_{n+1} are the errors in nth and $(n+1)$ th iterated value of the root α of the equation $f(x) = 0$ respectively. When $p = 1$, the convergence is said to be linear, when $p = 2$, the convergence is said to be quadratic.

Discussion on the Order of Convergence of Iteration Method:

Let α be the root of the equation $f(x) = 0$ expressible as $x = \phi(x)$, then

$$\alpha = \phi(\alpha) \qquad \ldots(1)$$

If x_n and x_{n+1} are the nth and $(n+1)$th approximation of the root α, then

$$x_{n+1} - \alpha = \phi(x_n) - \alpha \qquad \ldots(2)$$

From (1) and (2).

$$x_{n+1} = \phi(x_n) \qquad \ldots(3)$$

By mean value theorem of differential calculus,

$$\phi(x_n) - \alpha = (x_n - \alpha)\phi'(\theta), \text{ where } x_n < \theta < \alpha \qquad \ldots(4)$$

Using (4) in (3),

$$x_{n+1} - \alpha = (x_n - \alpha)\phi'(\theta) \qquad \ldots(5)$$

Let p be the maximum value of $|\phi'(\theta)|$ in the interval i.e. $|\phi'(\theta)| \leq p$ for all x in 1.

Then from (5),

$$|x_{n+1} - \alpha| \leq p|x_n - \alpha| \text{ i.e. } |\varepsilon_{n+1}| \leq p|\varepsilon_n| \qquad \ldots(6)$$

where ε_n and ε_{n+1} are errors in the nth and $(n+1)$th iterated value of the root.

Since the index of ε_n being 1, the rate of convergence of the iteration method $x_{n+1} = \phi(x_n)$ is linear.

Observations:

1. This method is particularly useful for finding the real root of an equation given in the form of an infinite series.
2. Smaller the value of $\phi'(x)$, the more rapidly it will converge.