Last week's quiz

$A>0,\ B>0$

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial X^2} + AX^2 - B \quad \text{on } 0 \le X \le L$$

$$\frac{\partial C}{\partial X}(0,T) = 0 \quad \text{and} \quad C(L,T) = 0$$

Interpretation:

$D\frac{\partial^2 C}{\partial X^2} \to$ diffusion

$-B \to$ sink

$AX^2 \to$ spatially dependent source

Nondimensionalization:

$[A] \sim [C]/[T][X]^2$

$[B] \sim [C]/[T]$

$[D] \sim [X]^2/[T]$

$X = Lx$ (natural choice)

$T = \frac{L^2}{D}t$ (straightforward choice)

option 1: $C = \frac{AL^4}{D}c$

$$\frac{AL^4/\cancel{D}}{L^2/\cancel{D}}\frac{\partial c}{\partial t} = \frac{\cancel{D}AL^4}{\cancel{D}L^2}\frac{\partial^2 c}{\partial x^2} + AL^2x^2 - B$$

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + x^2 - \frac{B}{AL^2}$$

b

option 2: $C = \frac{BL^2}{D}c$

$$\frac{B\cancel{L^2}/\cancel{D}}{\cancel{L^2}/\cancel{D}}\frac{\partial c}{\partial t} = \frac{\cancel{D}B\cancel{L^2}}{\cancel{D}\cancel{L^2}}\frac{\partial^2 c}{\partial x^2} + AL^2x^2 - B$$

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} + \frac{AL^2}{B}x^2 - 1$$

a

Stationary soln:

$$\frac{\partial^2 c}{\partial x^2} + ax^2 - b = 0 \quad \begin{cases} \text{if option 1, } a = 1 \\ \text{if option 2, } b = 1 \end{cases}$$

$$\frac{\partial^2 c}{\partial x^2} = b - ax^2 \Rightarrow \frac{\partial c}{\partial x} = bx - \frac{ax^3}{3} + c_0$$

$$\Rightarrow c(x) = \frac{bx^2}{2} - \frac{ax^4}{12} + c_0 x + c_1$$

apply B.C.: $\frac{\partial c}{\partial x}(0) = 0$, $c(1) = 0$

① $\frac{\partial c}{\partial x}(0) = c_0 = 0$

② $c(1) = \frac{b}{2} - \frac{a}{12} + c_1 = 0 \Rightarrow c_1 = \frac{a}{12} - \frac{b}{2}$

$$c(x) = \frac{b}{2}x^2 - \frac{ax^4}{12} + \frac{a}{12} - \frac{b}{2}$$

in dimensions:

option 1: $a = 1$, $b = B/AL^2$

$$\frac{DC(X)}{AL^4} = \frac{B}{2AL^2}\frac{X^2}{L^2} - \frac{X^4}{12L^4} + \frac{1}{6} - \frac{B}{2AL^2}$$

$$C(X) = \frac{B}{2D}X^2 - \frac{A}{12D}X^4 + \frac{AL^4}{12D} - \frac{BL^2}{2D}$$

(same answer if option 2)

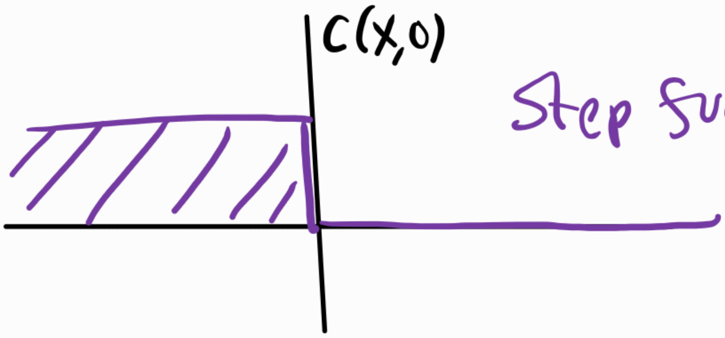
$$C(X) = \frac{B}{2D}(X^2 - L^2) - \frac{A}{12D}(X^4 - L^4)$$

HW this week

$$\frac{\partial C}{\partial T} = RC + D\frac{\partial^2 C}{\partial X^2} \quad \text{on} \quad X \in \mathbb{R}$$

$$\textcircled{+} \quad C(X, T=0) = \begin{cases} C_0, & X \le 0 \\ 0, & X > 0 \end{cases}$$

initial condition



Step function

$$[C_0] \sim [C]$$

$$\Downarrow$$

$$[C] \sim [C_0]$$

$$[T] \sim [1/R]$$

$$[X] \sim \left([D]/[R]\right)^{1/2}$$

Nondimensionalize

parameters: R, D, C_0

variables: X, T, C

$$\left[\frac{\partial C}{\partial T}\right] \sim [RC] \sim \left[D\frac{\partial^2 C}{\partial X^2}\right]$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\underbrace{\frac{[C]}{[T]} \sim [R][C]}_{\Downarrow} \sim [D]\frac{[C]}{[X]^2}$$

$$[R] \sim 1/[T]$$

$$[D] \sim [X]^2/[T]$$

$$[D] \sim [X]^2[R]$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} C &= C_0 c \\ T &= t/R \\ X &= \sqrt{D/R}\, x \end{aligned}}$$

$$\Rightarrow \cancel{C_0}\cancel{R}\frac{\partial c}{\partial t} = \cancel{C_0}\cancel{R}c + \cancel{D}\cancel{C_0}\cdot\frac{\cancel{R}}{\cancel{D}}\frac{\partial^2 c}{\partial x^2}$$

$$\boxed{\frac{\partial c}{\partial t} = c + \frac{\partial^2 c}{\partial x^2}}$$

Traveling wave soln

$z = x - vt$

$c(x,t) = c(x - vt) = c(z)$

velocity of the traveling wave

$$\begin{cases} \frac{\partial c}{\partial t} = \frac{\partial}{\partial t} c(x-vt) = -v \frac{\partial c}{\partial z} \\ \frac{\partial c}{\partial x} = \frac{\partial}{\partial x} c(x-vt) = \frac{\partial c}{\partial z} \\ \frac{\partial^2 c}{\partial x^2} = \frac{\partial^2}{\partial x^2} c(x-vt) = \frac{\partial^2 c}{\partial z^2} \end{cases}$$

$$\hookrightarrow \quad -v \frac{\partial c}{\partial z} = c + \frac{\partial^2 c}{\partial z^2}$$

Things to notice: ① c appears in all terms

② eqn is linear

(no c^2, $\frac{\partial c}{\partial z} \cdot c$, etc.)

Soln of the form

$c(z) = a_0 \exp(rz)$

Should solve the problem

$\Rightarrow$

$$-vr \cancel{a_0} \cancel{\exp(rt)} = \cancel{a_0} \cancel{\exp(rt)} + r^2 \cancel{a_0} \cancel{\exp(rt)}$$

$$-vr = 1 + r^2$$

$$\Rightarrow r^2 + vr + 1 = 0 \Rightarrow r = \frac{-v \pm \sqrt{v^2 - 4}}{2}$$

"2" general solns:

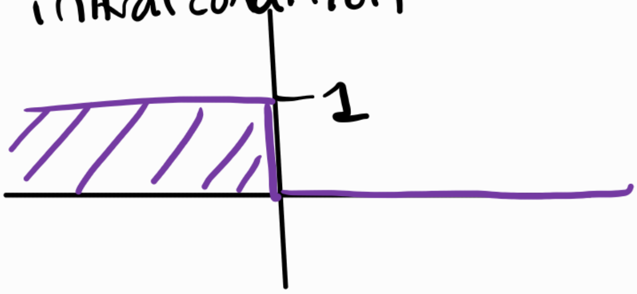
i.e. $$c_{+,-}(z) = a_0 \exp\left(\frac{-v \pm \sqrt{v^2-4}}{2} z\right)$$

$$c(z) = c_1 \exp\left(\frac{-v+\sqrt{v^2-4}}{2}(x-vt)\right) + c_2 \exp\left(\frac{-v-\sqrt{v^2-4}}{2}(x-vt)\right)$$

→ c_1 & c_2 determined by conditions

Q: Would such traveling waves evolve from the given initial condition?

initial condition



equation

$$\frac{\partial c}{\partial t} = c + \frac{\partial^2 c}{\partial x^2}$$

source   diffusion

Notice that ☆ is either pos or neg

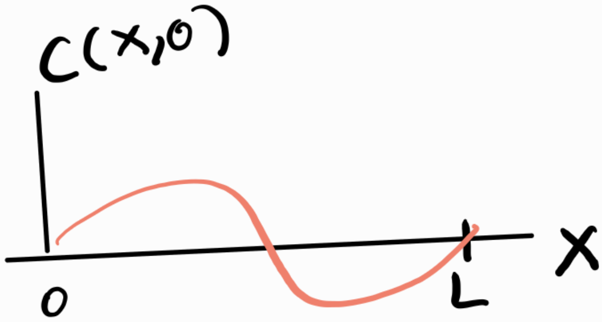
⇒ so at $x = +\infty$ or $x = -\infty$,
$c(z)$ should blow up (unless special c_1 & c_2)

⇒ this doesn't really make sense

Q: What do we expect to happen?

Example

$$\frac{\partial C}{\partial T} = -V\frac{\partial C}{\partial X} \qquad \begin{cases} \text{periodic on } 0 \le X \le L \\ \text{with } C(X,0) = C_0 \sin\left(2\pi \frac{X}{L}\right) \end{cases}$$



Nondimensionalization

parameters: V, C_0, L

variables: X, T, C

$$\frac{[C]}{[T]} \sim [V]\frac{[C]}{[X]} \Rightarrow [T] \sim [X]/[V]$$

$$\begin{cases} X = Lx \\ C = C_0 c \\ T = \frac{L}{V} t \end{cases} \rightsquigarrow \frac{C_0}{L/V}\frac{\partial c}{\partial t} = -V\frac{C_0}{L}\frac{\partial c}{\partial x}$$

$$\Downarrow$$

$$\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial x}$$

Ansatz: $c(x,t) = c(x-vt) = c(z)$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial z}\frac{\partial}{\partial t}(x-vt) = -v\frac{\partial c}{\partial z}$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial z}\frac{\partial}{\partial t}(x-vt) = \frac{\partial c}{\partial z}$$

$$\Rightarrow -v\frac{\partial c}{\partial z} = -\frac{\partial c}{\partial z}$$

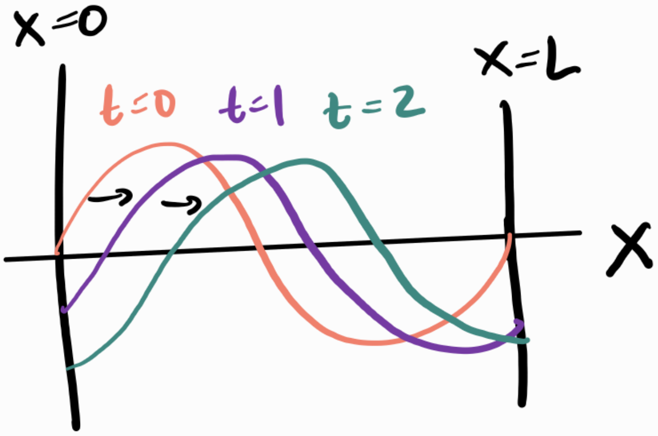
$$\Rightarrow \underline{\underline{v = -1}}$$

$$\Rightarrow c(x,t) = c(x-t)$$

$$\Rightarrow c(x,0) = c(x) = c_0 \sin(\pi x)$$

$$\Rightarrow \boxed{c(x,t) = c_0 \sin(\pi(x-t))}$$

as $t > 0$, soln shifts right by as much



In this case, a traveling wave soln makes sense?

Example

(already nondimensional)

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2} \quad \star$$

Ansatz $c(x,t) = a(x)\,b(t)$

$\Rightarrow$ derive conditions for $a(x)$ & $b(t)$

$$\star \Rightarrow a(x)\,b'(t) = a''(x)\,b(t)$$

assume $\lambda > 0$

$$\Rightarrow \frac{a''(x)}{a(x)} = \frac{b'(t)}{b(t)} = \text{constant} = -\lambda$$

$$\Rightarrow \begin{cases} a''(x) = -\lambda\, a(x) & (1) \\ b'(t) = -\lambda\, b(t) & (2) \end{cases}$$

Notice (1) is the simple harmonic oscillator:

$$\rightsquigarrow a(x) = A_1 \cos(\sqrt{\lambda}\, x) + A_2 \sin(\sqrt{\lambda}\, x)$$

For (2), try $b(t) = b_0 \exp(rt)$:

$$b'(t) = r\, b_0 \exp(rt) = r\,b(t) = -\lambda\, b(t) \Rightarrow r = -\lambda$$

$$\rightsquigarrow b(t) = b_0 \exp(-\lambda t)$$

General solns:

(λ, c_1, c_2 come from B.C. & I.C.)

$$c(x,t) = c_1 \cos(\sqrt{\lambda}\, x) \exp(-\lambda t) + c_2 \sin(\sqrt{\lambda}\, x) \exp(-\lambda t)$$

Quiz

Consider the reaction-advection equation:

$$\frac{\partial C}{\partial T} = -V\frac{\partial C}{\partial X} - BC$$

with initial condition

$$C(X, T=0) = \begin{cases} C_0, & X \le 0 \\ 0, & X > 0 \end{cases}$$

a) Choose the scales & nondimensionalize

parameters: C_0, V, B

variables: C, X, T $\Rightarrow$

$$\begin{cases} [C_0] \sim [C] \\ [V] \sim \frac{[X]}{[T]} \\ [B] \sim 1/[T] \end{cases}$$

4pt

$$\boxed{\begin{aligned} C &= C_0 c \\ T &= t/B \\ X &= \left(\frac{V}{B}\right) x \end{aligned}}$$

$$\frac{\partial C}{\partial T} = -V\frac{\partial C}{\partial X} - BC$$

$$\Rightarrow \frac{C_0}{1/B}\frac{\partial c}{\partial t} = -\frac{V C_0}{V/B}\frac{\partial c}{\partial x} - BC_0 c$$

$$\Rightarrow C_0 B\frac{\partial c}{\partial t} = -C_0 B\frac{\partial c}{\partial x} - BC_0 c$$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial x} - c}$$

b) assume there will be a "travelling wave" solution: $C(x,t) = C(x-vt)$. Derive the ODE for such a solution.

3pt

$$C(x,t) = C(x-vt) = C(z)$$

$$\Rightarrow \begin{cases} \frac{\partial c}{\partial t} = \frac{dc}{dz}\frac{\partial}{\partial t}(x-vt) = -v\frac{dc}{dz} \\ \frac{\partial c}{\partial x} = \frac{dc}{dz}\frac{\partial}{\partial x}(x-vt) = \frac{dc}{dz} \end{cases}$$

$$\Rightarrow -v\frac{dc}{dz} = -\frac{dc}{dz} - c$$

c) Find the general solution to your ODE in part (b), not constrained by I.C.

$$\frac{dc}{dz}(1-v) = -c$$

$$\Rightarrow \int \frac{dc}{c} = \int -\frac{1}{1-v} dz$$

$$\Rightarrow \ln(c) = -\frac{z}{1-v} + C_0$$

assume $C(z) = a_0 \exp(rt)$

$$\Downarrow$$

$$-vr = -r - 1$$

$$\Downarrow$$

$$r = \frac{-1}{1-v}$$

either way

3pt

$$c(z) = a_0 \exp\left(-\frac{1}{1-v}z\right)$$

$$\Rightarrow \boxed{c(x,t) = a_0 \exp\left(-\frac{(x-vt)}{1-v}\right)}$$

d) [BONUS] Argue qualitatively whether a traveling wave solution makes sense for this problem

2 pt

it does not make sense in this case, due to the reaction term eventually driving the soln to $c=0$