Quiz Last Week

$$\frac{\partial C}{\partial T} = \overbrace{-V\frac{\partial C}{\partial X}}^{\text{advection}} \overbrace{- BC}^{\text{reaction}} \quad \oplus \quad \text{initial condition} \quad C(X, T=0) = \begin{cases} C_0, & X \leq 0 \\ 0, & X > 0 \end{cases}$$

Nondimensionalization

parameters: C_0, V, B

variables: C, X, T

$$\Rightarrow \begin{cases} [C_0] \sim [C] \\ [V] \sim \dfrac{[X]}{[T]} \\ [B] \sim 1/[T] \end{cases} \Rightarrow \boxed{\begin{aligned} C &= C_0 c \\ T &= t/B \\ X &= \left(\frac{V}{B}\right)x \end{aligned}}$$

$$\frac{\partial C}{\partial T} = -V\frac{\partial C}{\partial X} - BC$$

$$\Rightarrow \frac{C_0}{1/B}\frac{\partial c}{\partial t} = -\frac{VC_0}{V/B}\frac{\partial c}{\partial x} - BC_0 c$$

$$\Rightarrow C_0 B\frac{\partial c}{\partial t} = -C_0 B\frac{\partial c}{\partial x} - BC_0 c$$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial x} - c}$$

Traveling wave soln

$$c(x,t) = c(x - vt) = c(z)$$

$$\begin{cases} \dfrac{\partial c}{\partial t} = \dfrac{dc}{dz}\dfrac{\partial}{\partial t}(x - vt) = -v\dfrac{dc}{dz} \\ \dfrac{\partial c}{\partial x} = \dfrac{dc}{dz}\dfrac{\partial}{\partial x}(x - vt) = \dfrac{dc}{dz} \end{cases} \Rightarrow \boxed{-v\frac{dc}{dz} = -\frac{dc}{dz} - c}$$

general soln → 2 ways:

① $\frac{dc}{dz}(1-v) = -c$

$\Rightarrow \int \frac{dc}{c} = \int -\frac{1}{1-v}\,dz$

$\rightarrow \ln(c) = -\frac{z}{1-v} + c_0$

② assume $c(z) = a_0 \exp(rt)$

$\Downarrow$

$-vr = -r - 1$

$\Downarrow$

$r = \frac{-1}{1-v}$

either way you get

$c(z) = a_0 \exp\left(-\frac{1}{1-v} z\right) \Rightarrow \boxed{c(x,t) = a_0 \exp\left(-\frac{(x-vt)}{1-v}\right)}$

Review of different PDEs

Continuity Eqn: $\frac{\partial c}{\partial T} = -\frac{\partial J}{\partial X}$, flux J

↳ $\frac{\partial c}{\partial T} = D \frac{\partial^2 c}{\partial X^2}$ [diffusion eqn] → $J = -D \frac{\partial c}{\partial X}$

↳ $\frac{\partial c}{\partial T} = -\frac{\partial}{\partial X}(vC)$ [advection eqn] → $J = vC$ ← flow to the right w/ velocity v

↳ $\frac{\partial c}{\partial T} = -C \frac{\partial c}{\partial X}$ [Burger's eqn] → $J = \frac{C^2}{2}$

Reaction systems: $\frac{\partial C}{\partial T} = -\alpha C$, $\frac{\partial D}{\partial T} = +\alpha C$

related

chemical reaction $C \xrightarrow{\alpha} D$

Sources & sinks:

$\frac{\partial c}{\partial T} = B \rightarrow C(T,X) = \underline{B}T + C_0(X)$

increase or decrease in C linearly w/ time

HW this week

$$\frac{\partial C}{\partial T} = KX + V\frac{\partial C}{\partial X} \quad \text{on} \quad 0 \le X \le L$$

(+) $C(X, T=0) = 0$

Q How many B.C. do you need? Where do you prescribe them?

$\frac{\partial C}{\partial X} \to$ order 1 derivative in $x \to$ 1 B.C. needed

$\frac{\partial C}{\partial T} = -\frac{\partial}{\partial X}(WC) \to$ advection eqn flowing right w/ velocity W

$\Rightarrow W = -V \longrightarrow$ flowing <u>left</u>

$\Rightarrow$ information coming from the <u>right</u>

$\leadsto$ this is where the B.C. should be

Choose $C(X=L, T) = 0$

Nondimensionalize

parameters: K, V, L

variables: C, X, T

$\Rightarrow$

$[L] \sim [X]$

$[K] \sim [C]/[X][T]$

$[V] \sim [X]/[T]$

$\Rightarrow X = Lx$

$T = \frac{L}{V}t$

$C = \frac{L^2 K}{V}c$

$C(X=L, T) = 0$

$\Downarrow$

$$\boxed{c(x=1, t) = 0}$$

$$\frac{L^2k/V}{L/V}\frac{\partial c}{\partial t} = kLx + V\frac{L^2k/V}{L}\frac{\partial c}{\partial x}$$

$$Lk\frac{\partial c}{\partial t} = kLx + kL\frac{\partial c}{\partial x} \Rightarrow \boxed{\frac{\partial c}{\partial t} = x + \frac{\partial c}{\partial x}}$$

Stationary distribution

general soln

$$\frac{\partial c}{\partial t} = 0 = x + \frac{\partial c}{\partial x} \Longrightarrow \frac{\partial c}{\partial x} = -x \Rightarrow c(x) = -\frac{x^2}{2} + a$$

specific soln

$$c(1) = 0 = -\frac{1}{2} + a \rightarrow a = 1/2 \Rightarrow \boxed{c(x) = -\frac{1}{2}(x^2 - 1)}$$

$\Longrightarrow$ put back in dimensions

$$\begin{cases} c = \frac{V}{L^2k}C \\ x = X/L \end{cases} \Rightarrow \frac{V}{L^2k}C(X) = -\frac{1}{2}\left(\frac{X^2}{L^2} - 1\right)$$

$$\boxed{C(X) = -\frac{k}{2V}\left(X^2 - L^2\right)}$$

Q How much time do we have to wait to see this stationary distribution?

$\rightarrow$ on the order of the timescale, $O(L/V)$

Example

$$\frac{\partial C}{\partial T} = -V\frac{\partial C}{\partial X} + D\frac{\partial^2 C}{\partial X^2}$$

Q: How many IC & BC do you need here?

1 I.C., 2 B.C.

Q: What B.C. could we choose here?

Let's pick $C(X=0,T)=0$, $\frac{\partial C}{\partial X}(X=L,T)=\frac{C_1}{L}$

Nondimensionalize

parameters: V, D, C_1, L

variables: C, X, T

$[V]=[X]/[T]$ $[L]=[X]$

$[D]=[X]^2/[T]$ $[C_1]=[C]$

one of these two determine the timescale

⤳ assume advection dominated

$\Rightarrow$ $X = Lx$, $C = C_1 c$, $T = \frac{L}{V} t$

$$\Rightarrow \frac{C_1}{L/V}\frac{\partial c}{\partial t} = -\frac{VC_1}{L}\frac{\partial c}{\partial x} + \frac{DC_1}{L^2}\frac{\partial^2 c}{\partial x^2}$$

$$\Rightarrow \boxed{\frac{\partial c}{\partial t} = -\frac{\partial c}{\partial x} + \underbrace{\frac{D}{VL}}_{d}\frac{\partial^2 c}{\partial x^2}}$$

B.C.: $C(X=0,T)=0 \Rightarrow c(x=0,t)=0$

$\frac{\partial C}{\partial X}(X=L,T)=\frac{C_1}{L} \Rightarrow \frac{\partial c}{\partial x}(x=1,t)=1$

Stationary soln

$$\frac{\partial c}{\partial t}=0=-\frac{\partial c}{\partial x}+d\frac{\partial^2 c}{\partial x^2} \rightarrow d\frac{d^2c}{dx^2}=\frac{dc}{dx}$$

2 options

① $\int d\frac{d^2c}{dx^2}dx=\int\frac{dc}{dx}dx$

$d\frac{dc}{dx}=c(x)+c_1$

$\frac{dc}{dx}=\frac{1}{d}(c+c_1)$

$\int\frac{dc}{(c+c_1)}=\int\frac{dx}{d}$

$\ln(c+c_1)=\frac{x}{d}+c_2$

$c+c_1=b\exp(x/d)$

$c(x)=a+b\exp(x/d)$

② Ansatz: $c(x)=c_0\exp(rx)$

$\Rightarrow \frac{dc}{dx}=c_0 r\exp(rx)$

$\Rightarrow \frac{d^2c}{dx^2}=r^2c_0\exp(rx)$

$\rightsquigarrow d r^2 c_0\exp(rx)=rc_0\exp(rx)$

$\rightsquigarrow dr^2=r$

$\rightsquigarrow r=0$ or $r=\frac{1}{d}$

$\Rightarrow c(x)=a\exp(0r)+b\exp(x/d)$

general soln

$$\Rightarrow \boxed{c(x)=a+b\exp(x/d)}$$

Apply B.C. $c(x=0)=a+b=0 \Rightarrow b=-a$

$\frac{\partial c}{\partial x}=a(1-\frac{1}{d}\exp(x/d))$

$\left.\frac{\partial c}{\partial x}\right|_{x=1}=a(1-\frac{1}{d}\exp(1/d))=1 \Rightarrow a=1/(1-\frac{1}{d}\exp(1/d))$

$$\boxed{c(x)=\frac{1-\exp(x/d)}{1-\frac{1}{d}\exp(1/d)}}$$

Specific soln

Quiz

constant source

Burgers

Consider
$$\frac{\partial C}{\partial T} = A - B\frac{\partial}{\partial X}\left(\frac{C^2}{2}\right) \quad (*)$$

on $0 \le X \le L$ with initial condition

$$C(X, T=0) = C_0$$

A) Explain why you only need one B.C. At which end should it be prescribed?

Only order 1 derivative in $C \Rightarrow$ one B.C.

$\frac{\partial}{\partial X}\left(\frac{C^2}{2}\right) = C\frac{\partial C}{\partial X}$ $\rightsquigarrow$ like advection w/ $V \propto C$

$\rightsquigarrow$ at inflow, left side

B) Choosing $C(X=0,T)=0$. Nondimensionalize

$[A] \sim [C]/[T]$ | $[B] \sim [X]/[T][C]$ | $[L] \sim [X]$ | $[C_0] \sim [C]$

$$\boxed{X = Lx, \quad C = C_0 c, \quad ① \; T = \frac{C_0}{A}t \quad \text{OR} \quad ② \; T = \frac{L}{BC_0}t}$$

$$\frac{C_0}{C_0/A}\frac{\partial c}{\partial t} = A - \frac{BC_0^2}{L}\frac{\partial}{\partial x}(c^2/2)$$

$$\frac{\partial c}{\partial t} = 1 - \frac{BC_0^2}{AL}\frac{\partial}{\partial x}(c^2/2)$$

$$\boxed{\frac{\partial c}{\partial t} = 1 - b\frac{\partial}{\partial x}(c^2/2)}$$

$$\frac{C_0}{L/BC_0}\frac{\partial c}{\partial t} = A - \frac{BC_0^2}{L}\frac{\partial}{\partial x}(c^2/2)$$

$$\frac{\partial c}{\partial t} = \frac{AL}{BC_0^2} - \frac{\partial}{\partial x}(c^2/2)$$

$$\boxed{\frac{\partial c}{\partial t} = a - \frac{\partial}{\partial x}(c^2/2)}$$

C) How much time to wait for the stationary distribution?

$\Rightarrow$ order of timescale, $\frac{C_0}{A}$ or $\frac{L}{BC_0}$

D) Find stationary distribution

$$1 - b\frac{\partial}{\partial x}\left(\frac{c^2}{2}\right) = 0 \qquad\qquad a - \frac{\partial}{\partial x}\left(\frac{c^2}{2}\right) = 0$$

$$\int \frac{\partial}{\partial x}\left(\frac{c^2}{2}\right)dx = \int \frac{1}{b}\,dx \qquad\qquad \int \frac{\partial}{\partial x}\left(\frac{c^2}{2}\right)dx = \int a\,dx$$

$$\frac{c^2}{2} = \frac{x}{b} + h_1 \qquad\qquad \frac{c^2}{2} = ax + h_1$$

$$c = \sqrt{\frac{2x}{b} + h_0} \qquad\qquad c = \sqrt{2ax + h_0}$$

+ B.C. $c(x=0) = 0 \Rightarrow h_0 = 0$

$$c = \sqrt{\frac{2x}{b}} \quad \text{or} \quad c = \sqrt{2ax}$$

E) Stationary distribution in dimensions

$$\frac{C}{C_0} = \sqrt{\frac{2X}{bL}} = \sqrt{\frac{2X}{L}\,\frac{AL}{BC_0^2}}$$

$$\boxed{C(X) = \sqrt{\frac{2XA}{B}}}$$