Quiz policy reminders

- 15 min
- open book, but <u>no electronic devices</u>
- usually, similar to HW, testing material up to and including Tues lecture
- 10 pts each, 10 best quizzes of 13
- 25% of class grade

Discrete Models (Review)

Difference equation:

$$Q_{t+1} = f(Q_t)$$

Simple growth model:

$$q_{t+1} = r q_t = r^t q_0$$

if $|r| > 1$, $|q_t| \to \infty$

if $|r| < 1$, $|q_t| \to 0$

Nondimensionalization:

interested more in dynamics than scale

$\Rightarrow$ sometimes different combinations of params yield same dynamics

$$Q_t = \bar{Q} q_t$$

without dimensions

with dimensions

Steady State:

when the solution does not change with time

$$\Rightarrow q_{t+1} = q_t = q^*$$

$$\Rightarrow q^* = f(q^*)$$ solve for q^*

Stability:

We want to understand the behavior <u>near</u> steady states

Consider a small perturbation ε_0 to q^*

$\Rightarrow$ How does the perturbation evolve?

$$q_t = q^* + \varepsilon_t$$

$$\Longrightarrow q^* + \varepsilon_{t+1} = f(q^* + \varepsilon_t)$$

$$\begin{cases} \text{Stable if } \varepsilon_t \to 0 \text{ as } t \to \infty \\ \text{Unstable if } \varepsilon_t \text{ grows as } t \to \infty \end{cases}$$

Quiz soln: (as example) $M_{t+1} = R\dfrac{M_t}{M_t + A}$

a) Choose a scale for M and nondimensionalize

$M = Am$ Case 1

$$A m_{t+1} = R\frac{A m_t}{A m_t + A}$$

$$m_{t+1} = \underbrace{\frac{R}{A}}_{=r}\,\frac{m_t}{m_t + 1}$$

$M = Rm$ Case 2

$$R m_{t+1} = R\frac{R m_t}{R m_t + A}$$

$$m_{t+1} = \frac{m_t}{m_t + \underbrace{A/R}_{a}}$$

b) Find the Steady states of your nondimensional eqn

$$m^* = r\frac{m^*}{m^*+1}$$

$$m^* = \frac{m^*}{m^*+a}$$

$\boxed{m^* = r-1 \text{ or } m^* = 0}$ Case 1

$\boxed{m^* = 1-a \text{ or } m^* = 0}$ Case 2

c) Assess the Stability for the Steady State $m^* = 0$

Let $m_t = m^* + \varepsilon_t = \varepsilon_t$

Case 1

$$\varepsilon_{t+1} = r\frac{\varepsilon_t}{\varepsilon_t+1} \approx r\varepsilon_t(1-\varepsilon_t)$$

$$\varepsilon_{t+1} = r\varepsilon_t - r\varepsilon_t^2 \approx r\varepsilon_t$$

$\boxed{\text{stable if } |r|<1 \text{ i.e. } |R/A|<1}$

Case 2

$$\varepsilon_{t+1} = \frac{\varepsilon_t}{\varepsilon_t+a} = \frac{\varepsilon_t/a}{\varepsilon_t/a+1} \approx \frac{\varepsilon_t}{a}\left(1-\frac{\varepsilon_t}{a}\right)$$

$$= \frac{\varepsilon_t}{a} - \left(\frac{\varepsilon_t}{a}\right)^2 \approx \varepsilon_t/a$$

$\boxed{\text{stable if } \left|\frac{1}{a}\right|<1 \text{ i.e. } |R/A|<1}$

Continuous Models in 1D

ODEs: $\frac{dy}{dt} = f(y,t)$

we will focus on this first

$\Rightarrow$ autonomous ODE: $\frac{dy}{dt} = f(y)$

Example Exponential growth

$$q_{t+1} = r q_t = r q_t + q_t - q_t$$

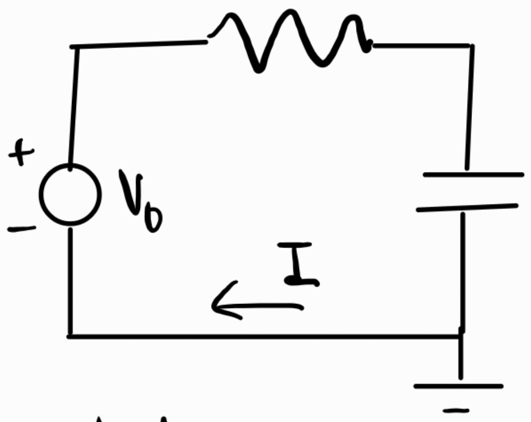
$$\frac{q_{t+1} - q_t}{\Delta t} = \frac{(r-1)}{\Delta t} q_t \xrightarrow[\lim \Delta t \to 0]{} \frac{dq}{dt} = \tilde{r} q$$

Q units of $\tilde{r}$?

soln: $q(t) = C\exp(\tilde{r}t)$

$\rightsquigarrow$ with I.C., $C = q_0$

Example Charge on a capacitor



Voltag drop across circuit:

$$-V_0 + RI + \frac{Q}{C} = 0$$

voltage resistance current charge capacitance

accumulation on capacitor from current:

$$I = \dot{Q} \Rightarrow \dot{Q} = \frac{V_0}{R} - \frac{Q}{RC} = \frac{dQ}{dT}$$

$Q \sim [C]$ charge coloumb

$I \sim [C]/[T]$

$V_0 \sim [V]$ volt

$R \sim [V][T]/[C]$ ohm

$C \sim [C]/[V]$ farad

nondimensionalization:

$$Q = CV_0 q \qquad T = RCt \implies \frac{dQ}{dT} = \frac{V_0}{R} - \frac{Q}{RC}$$

$$\frac{\cancel{C}V_0}{R\cancel{C}} \frac{dq}{dt} = \frac{V_0}{R} - \frac{\cancel{C}V_0 q}{R\cancel{C}}$$

$$\frac{dq}{dt} = 1 - q$$

Steady State

soln does not change with time

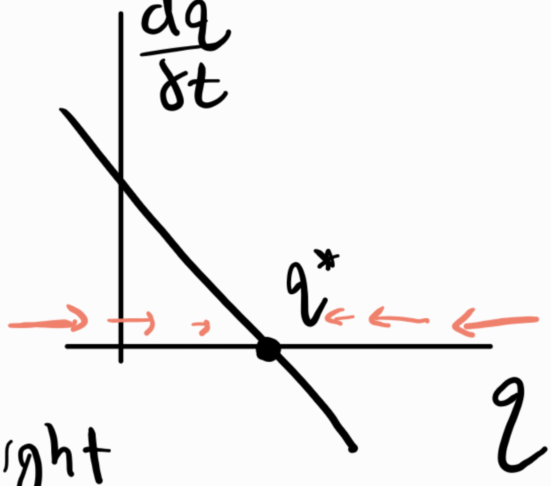
$\frac{dy}{dt} = f(y)$

$\implies y^*$ s.t. $f(y^*) = 0$

Example charge on a capacitor

$$\frac{dq}{dt} = 1 - q = 0 \implies q^* = 1$$

"phase-line"



Rrwe

Think of $\frac{dq}{dt}$ like flow of quantity $\implies \frac{dq}{dt} > 0$ to the right

Flow is always toward y^* $\Rightarrow$ globally stable

Stability

phase line informs direction and speed of a trajectory at that point

$\Rightarrow$ if flow around a steady state is always toward it, then the steady state is stable

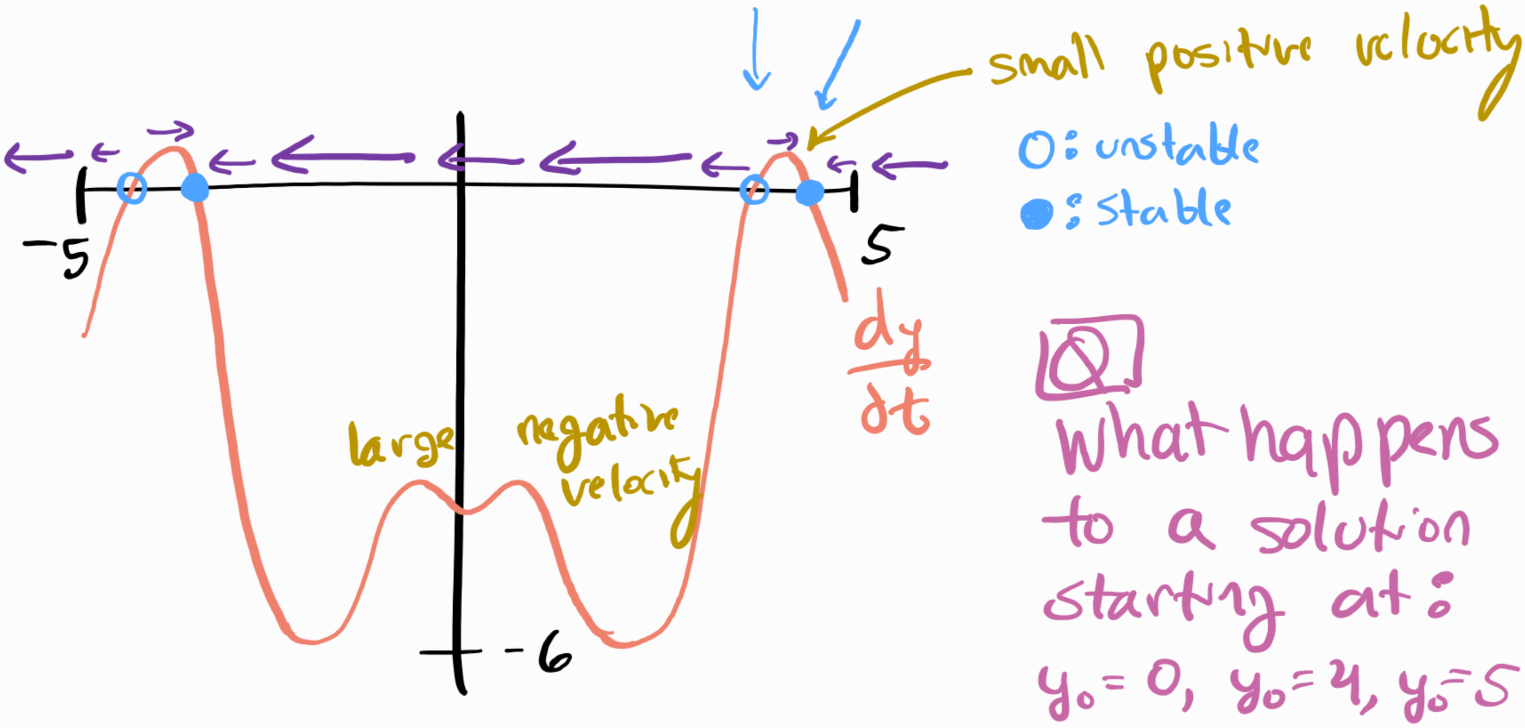
Example

(skip unless time allows)

$$\frac{dy}{dt} = y\cdot\sin(2y) + \cos(y) - 3$$

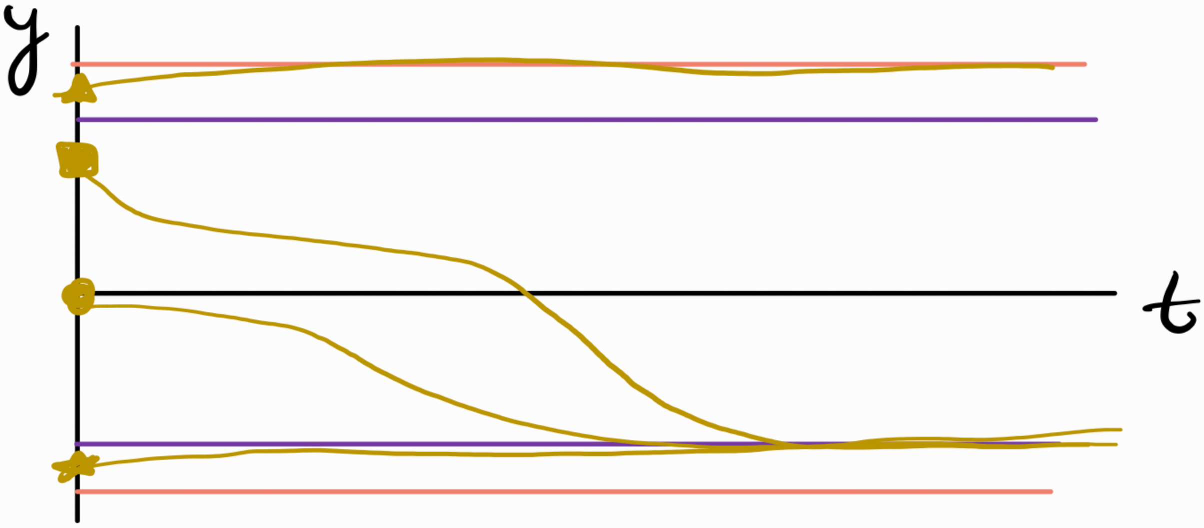
Think of it like: $y \sim$ position of ball

$\frac{dy}{dt} \sim$ velocity of ball



Q: What happens to a solution starting at: $y_0 = 0$, $y_0 = 4$, $y_0 = 5$

Sketch of soln:



Example charge over time in electric car

$$\frac{dY}{dT} = a\cos(Yb) + c$$

$a, c \sim [Y]/[T] \implies Y = y/b$

$b \sim Y[Y] \qquad T = \frac{1}{ab} t$

non dimensional eqn

$$\frac{Y/b}{Y/ab}\frac{dy}{dt} = a\cos(y) + c$$

$$a\frac{dy}{dt} = a\cos(y) + c$$

nondim parameter

$$\frac{dy}{dt} = \cos(y) + c/a = \cos(y) + h$$

Steady States

$$\frac{dy}{dt} = 0 = \cos(y) + h$$

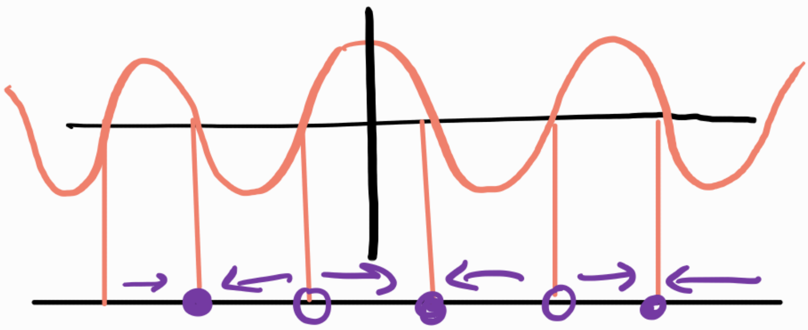
[Q] for what values of h are there ss?

$\Longrightarrow$ only exist if $|h| \leq 1$

$h = 0$

$$\frac{dy}{dt} = 0 = \cos(y) \Longrightarrow y_j^* = \frac{\pi}{2} + \pi j$$

$j = 0, \pm 1, \pm 2$



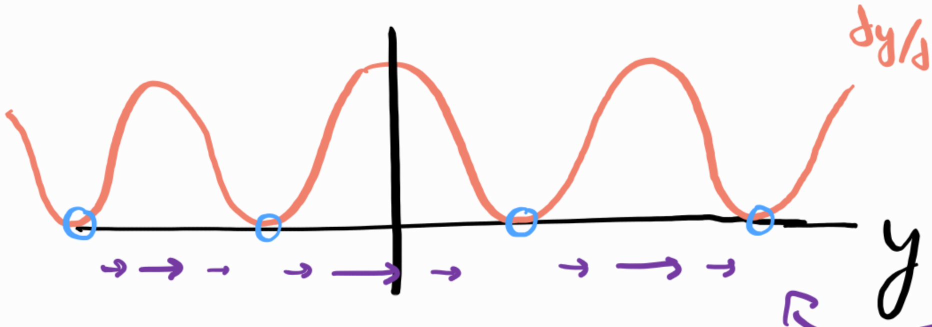
stable: j even

unstable: j odd

$h = 1$

$$\frac{dy}{dt} = \cos(y) + 1 = 0 \Rightarrow y_j^* = (2j+1)\pi$$

$j = 0, \pm 1, \pm 2, \ldots$



flow always to the right!

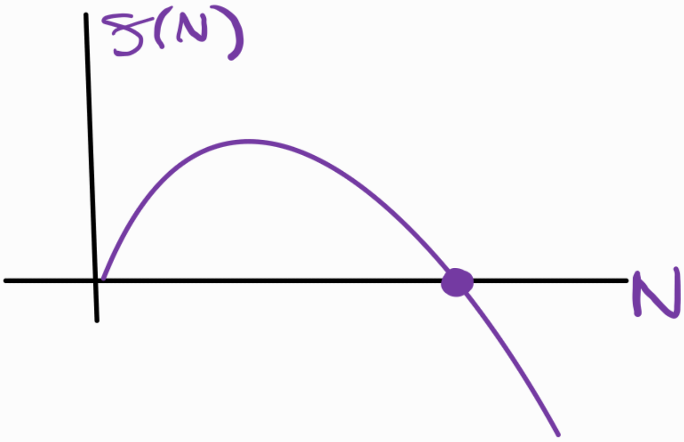
S.S. attracting from left, but repelling from right

$\Longrightarrow$ "half-stable"

Quiz — Gompertz model

$$\frac{dN}{dT} = -aN\ln(bN) = f(N)$$

\# tumor cells



a) independent variable: T, time

dependent variable: N, number

parameters: $a \sim [T]^{-1}$, $b \sim [N]^{-1}$

b)

$N = n/b$

$T = t/a$

non-dimensional variables

c)

$$\frac{d(n/b)}{d(t/a)} = -a\left(\frac{n}{b}\right)\ln\left(b\left(\frac{n}{b}\right)\right)$$

$$\frac{a}{b}\frac{dn}{dt} = -\frac{an}{b}\ln(n)$$

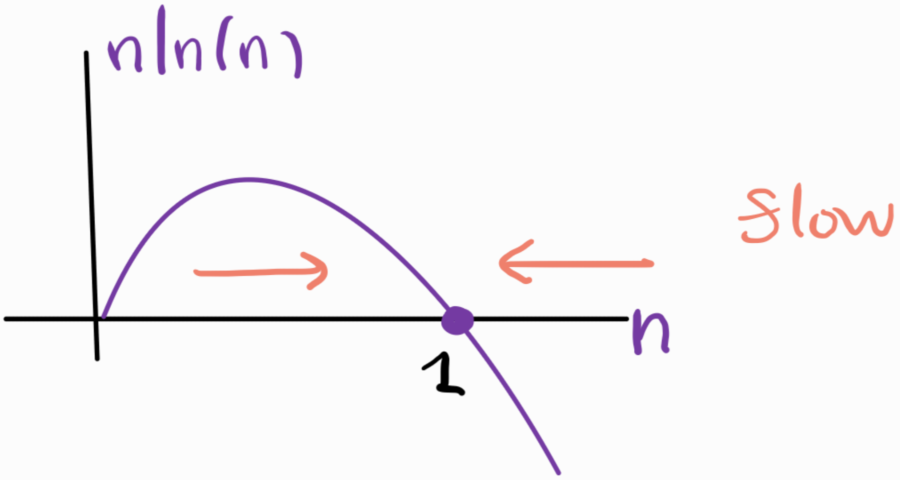
$$\boxed{\frac{dn}{dt} = -n\ln(n)}$$

d) $\frac{dn}{dt} = 0$ when $\ln(n) = 0$

$\Rightarrow \boxed{n^* = 1}$

note: $n\ln(n)$ not defined at $n=0$

e)



Since flow is toward $n=1$ around it $\Rightarrow$ it is a stable steady state