Policies & Logistics

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Office Hrs: Wed 10:00-11:00a CIWW 805

Thurs 2:30-3:30p CIWW 805

[or by appointment in office 907]

Quizzes:
- end of recitation, 15 min
- open book, but <u>no electronic devices</u>
- usually, similar to HW, testing material up to and including Tues lecture
- 10 pts each, 10 best quizzes
- 25% of class grade

Recitation notes posted by following Tues, including soln to quiz

Discrete Models

a state or quantity q defined at individual moments in time

$$q_t = \text{value of } q \text{ at time } t = 0, 1, 2, \ldots$$

$\Rightarrow$ you can also think of t as an index, with a timestep Δt between t & $t+1$

It's evolution is described by a difference eqn

$$q_{t+1} = f(q_t)$$

Example

Every hour, the number N of fruit flies in my lab sink doubles.

Q What difference eqn describes this system?

$$N_{t+1} = 2N_t \qquad t = 0, 1, 2, \text{ hours}$$

More generally, a simple growth model is

$$q_{t+1} = r q_t$$

Notice:

$$q_{t+1} = r(r q_{t-1}) = r^2 q_{t-1} = r^3 q_{t-2} = \ldots$$

$$q_{t+1} = r^t q_0$$

$\leftarrow$ initial condition

Example: Logistic growth

growth parameter

assume $R = 1/2$

deer population $\rightarrow$

$$b_{t+1} = b_t\left[1 + R\left(1 - \frac{b_t}{K}\right)\right]$$

carrying capacity

Notice if $b_t = K$:

$$b_{t+1} = b_t\left[1 + R\left(1 - K/K\right)\right] = b_t \Rightarrow \text{no change in population}$$

if $b_t = 2K$:

$$b_{t+1} = b_t\left[1 + R(1-2)\right] = b_t\left[1 - R\right] < b_t$$

$\Rightarrow$ decrease in population

if $b_t = K/2$:

$$b_{t+1} = b_t\left[1 + R(1 - 0.5)\right] = b_t\left[1 + R/2\right] > b_t$$

$\Rightarrow$ increase in population

"carrying capacity": defines a "cap" on population size, determined by environmental factors

$\boxed{Q}$ What are some examples?

$\Rightarrow$ available food, water, land, etc

units

[N]: number [L]: length [T]: time [M]: mass

$\boxed{Q}$ What are the units of R, k?

$$b_{t+1} = b_t\left[1 + R\left(1 - \frac{b_t}{k}\right)\right]$$

$$\Rightarrow \cancel{[N]} = \cancel{[N]}\left(1 + R\left(1 - \frac{[N]}{k}\right)\right)$$

$$\Rightarrow \underbrace{R\left(1 - \frac{[N]}{k}\right)}_{\text{no-units}} = \text{no units}$$

$\to R \sim$ unit-less

$\to k \sim [N]$

$\boxed{\text{Example}}$ mass concentration of chlorine in pool

$$C_{t+1} = \frac{C_t^2 H}{k} + Z C_t^3$$

(Suppose $H = 3\ m^{-1}$)

$$\begin{cases} [M]/[L]^3 \sim [M]^2/[L]^6 \times [L]^{-1} \times \frac{1}{k} \\ [M]/[L]^3 \sim Z\frac{[M]^3}{[L]^9} \end{cases}$$

can only add things of same units

$\Rightarrow$ both have units of concentration

$$\Rightarrow \begin{cases} k \sim [L]^{-4}[M]^1 \\ Z \sim [L]^6[M]^{-2} \end{cases}$$

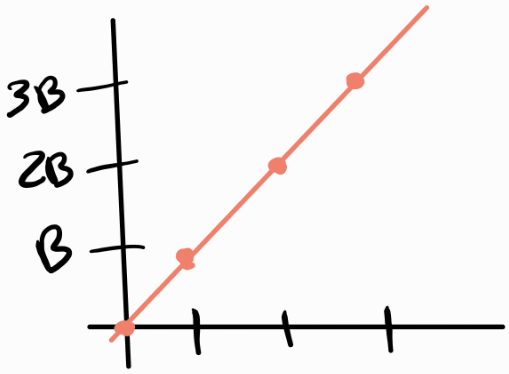
Non-dimensionalization

Often, we are interested more in dynamics than we are in scale of solutions

Example

Consider $\begin{cases} C_{t+1} = C_t + B \\ C_0 = 0 \end{cases}$

Soln: $C_t = Bt$



$\Rightarrow$ shape/dynamics do not depend on value of B

$\rightsquigarrow$ only scale!

$\Rightarrow$ non-dimensionalize: $C = Bc$

with dimensions

without dimensions

$$\Downarrow$$

$$Bc_{t+1} = Bc_t + B \Rightarrow \boxed{c_{t+1} = c_t + 1}$$

Example

$$C_{t+1} = \frac{C_t^2 H}{K} + \underline{\underline{Z C_t^3}} \Rightarrow \begin{cases} K \sim [L]^{-4}[M]^{1} \\ Z \sim [L]^{6}[M]^{-2} \end{cases}$$

$$\sim [1/C_t^2]$$

dominating term

$$C_t = (Z^{-1/2})\, c_t$$

$$Z^{-1/2}$$

$$Z^{-1/2} c_{t+1} = \left(Z^{-1} H/K\right) c_t^2 + Z \times Z^{-3/2} c_t^3$$

$$c_{t+1} = \underbrace{\left(\frac{Z^{-1/2}H}{K}\right)} c_t^2 + c_t^3$$

check this has no dim

$$\frac{\left([L]^6[M]^{-2}\right)^{-1/2}[L]^{-1}}{[L]^{4}[M]^{1}} = \frac{[L]^{-3}[M][L]^{-1}}{[L]^{-4}[M]} = 1 \checkmark$$

$$\boxed{c_{t+1} = h c_t^2 + c_t^3} \quad \text{where } h = \frac{Z^{-1/2}H}{K}$$

Steady State

→ when the solution does not change with time

$$\Rightarrow q_{t+1} = q_t = q^*$$ ← Steady State

$\boxed{\text{Example}}$

Consider a dimensionless equation

$$y_{t+1} = y_t \sin(y_t)$$

note: if y_t had dimensions, this would not work!

y^* steady-state defined by

$$y^* = y^* \sin(y^*) \Rightarrow y^* \text{ s.t. } \sin(y^*) = 1$$

$$\Rightarrow y^* = \frac{\pi}{2} + 2\pi\mathbb{Z}$$

Stability

We want to understand the behavior near steady states

Consider a small perturbation ε_0 to q^*

$\Rightarrow$ How does the perturbation evolve?

$$q_t = q^* + \varepsilon_t$$

$\boxed{\text{Example}}$ $\quad y_{k+1} = y_k \exp(r(1-y_k))$

1) First, find steady state:

$$y^* = y^* \exp(r(1-y^*))$$

$$\Rightarrow 1 = \exp(r(1-y^*)) \Rightarrow y^* = 1$$

2) Now, evolve a small perturbation from y^*

$$y_k = 1 + \varepsilon_k \Rightarrow 1 + \varepsilon_{k+1} = (1+\varepsilon_k)\exp(-r\varepsilon_k)$$

Recall: $\exp(x) \approx 1 + x + \frac{x^2}{2} + \ldots$

$$\Rightarrow 1 + \varepsilon_{k+1} \approx (1+\varepsilon_k)(1 - r\varepsilon_k) = 1 + \varepsilon_k - r\varepsilon_k - \cancel{r\varepsilon_k^2}$$

too small

$$\Rightarrow \varepsilon_{k+1} = (1-r)\varepsilon_k = (1-r)^k \varepsilon_0$$

$|1-r| > 1$: ε_k grows as $t \to \infty$, y_t diverges

Unstable

$|1-r| < 1$: ε_k shrinks as $t \to \infty$, $y_t \to y^*$

Stable

$\boxed{\text{Example}}$ Suppose the mass of Goblins in your dungeon evolves like:

$$G_{t+1} = \frac{C}{B+G_t} + RG_t + L$$

$\boxed{Q}$ What are the units of C, R, B?

$R \sim$ no units, $B \sim [M]$, $C \sim [M]^2$, $L \sim [M]$

$\boxed{Q}$ What is a "good" scale for G_t?

options: $G = Bg$ ← B and G_t have same units & L

$G = Lg$

$G = \sqrt{C}\, g$

$G = \frac{C}{B} g$ ← if $B \gg G_t$, first term $\approx C/B$

try this!

$$\frac{C}{B} g_{t+1} = \frac{C}{B + \frac{C}{B} g_t} + R\frac{C}{B} g + L$$

$$g_{t+1} = \frac{B}{B + \frac{C}{B} g_t} + g_t + \frac{BL}{C} = \frac{1}{1 + b g_t} + g_t + \ell$$

$$b = C/B^2 \quad \ell = BL/C$$

$\boxed{Q}$ What are the Steady State(s)?

$$g^* = \frac{1}{1+bg^*} + g^* + \ell \Rightarrow \frac{1}{1+bg^*} = -\ell$$

$$\Longrightarrow g^* = -\left(\frac{1}{b} + \frac{1}{b\ell}\right)$$

$\boxed{Q}$ Is it stable? $\quad (g_t = g^* + \varepsilon_t)$

$$\cancel{-\frac{1}{b}} - \cancel{\frac{1}{b\ell}} + \varepsilon_{t+1} = \frac{1}{1+b(-1)\left(\frac{1}{b}+\frac{1}{b\ell}\right)+b\varepsilon_t} + \cancel{-\left(\frac{1}{b} + \frac{1}{b\ell}\right)} + \varepsilon_t + \ell$$

$$\varepsilon_{t+1} = \varepsilon_t + \ell + \frac{1}{b\varepsilon_t - \frac{1}{\ell}}$$

$$\frac{1}{1-x} \approx 1 + x + x^2$$

$$= \varepsilon_t + \ell - \ell\left(\frac{1}{1-b\varepsilon_t\ell}\right)$$

$$\approx \varepsilon_t + \ell - \ell(1+b\varepsilon_t\ell) = \varepsilon_t + \ell - \ell - b\ell^2\varepsilon_t$$

$$\varepsilon_{t+1} = \underline{\underline{(1-b\ell^2)}}\,\varepsilon_t$$

$\hookrightarrow$ stable if $|1-b\ell^2| < 1$

With dimensions:

$$\left|1 - \frac{C}{B^2}\cdot\frac{B^2L^2}{C^2}\right| = \left|1 - \frac{L^2}{C}\right| < 1$$

Example $y_{t+1} = y_t \sin(y_t)$

$\Rightarrow$ consider $y^* = \frac{\pi}{2} \Longrightarrow y_t = \frac{\pi}{2} + \varepsilon_t$ perturbation

plug in

$$\frac{\pi}{2} + \varepsilon_{t+1} = \left(\frac{\pi}{2} + \varepsilon_t\right) \underline{\sin\left(\frac{\pi}{2} + \varepsilon_t\right)}$$

$\approx 1 - \frac{\varepsilon_t^2}{2} + O(\varepsilon_t^4)$ for small ε_t

$$\Downarrow$$

$$\approx \left(\frac{\pi}{2} + \varepsilon_t\right)\left(1 - \frac{\varepsilon_t^2}{2}\right) = \frac{\pi}{2} + \varepsilon_t - \cancel{\frac{\pi}{4}\varepsilon_t^2} - \cancel{\frac{\varepsilon_t^3}{2}}$$

very small!

$\Rightarrow \varepsilon_{t+1} = \varepsilon_t$ inconclusive!

$\rightsquigarrow$ perturbation, <u>to first order,</u> does not grow or shrink

$$M_{t+1} = R\frac{M_t}{M_t + A}$$

a) Choose a scale for M and nondimensionalize

$\underset{\sim\sim\sim}{M = Am}$

$$Am_{t+1} = R\frac{Am_t}{Am_t + A}$$

$$m_{t+1} = \underbrace{\frac{R}{A}}_{=r}\frac{m_t}{m_t + 1}$$

$\underset{\sim\sim\sim}{M = Rm}$

$$Rm_{t+1} = R\frac{Rm_t}{Rm_t + A}$$

$$m_{t+1} = \frac{m_t}{m_t + \underbrace{A/R}_{a}}$$

b) Find the steady states of your nondimensional eqn

$$m^* = r\frac{m^*}{m^* + 1}$$

$$\boxed{\begin{gathered} m^* = r - 1 \\ \text{or} \\ m^* = 0 \end{gathered}}$$

$$m^* = \frac{m^*}{m^* + a}$$

$$\boxed{\begin{gathered} m^* = 1 - a \\ \text{or} \\ m^* = 0 \end{gathered}}$$

c) Assess the Stability for the Steady State $m^* = 0$, recalling the Taylor series

$$\frac{1}{1+x} = 1 - x + O(x^2)$$

Let $m = m^* + \varepsilon = \varepsilon$

Case 1

$$\varepsilon_{t+1} = r\frac{\varepsilon_t}{\varepsilon_t + 1} \approx r\varepsilon_t(1-\varepsilon_t)$$

$$\varepsilon_{t+1} = r\varepsilon_t - r\varepsilon_t^2 \approx r\varepsilon_t$$

$\boxed{\text{stable if } |r| < 1 \text{ i.e. } |R/A| < 1}$

Case 2

$$\varepsilon_{t+1} = \frac{\varepsilon_t}{\varepsilon_t + a} = \frac{\varepsilon_t/a}{\varepsilon_t/a + 1} \approx \frac{\varepsilon_t}{a}\left(1 - \frac{\varepsilon_t}{a}\right)$$

$$= \frac{\varepsilon_t}{a} - \left(\frac{\varepsilon_t}{a}\right)^2 \approx \varepsilon_t/a$$

$\boxed{\text{stable if } \left|\frac{1}{a}\right| < 1 \text{ i.e. } |R/A| < 1}$