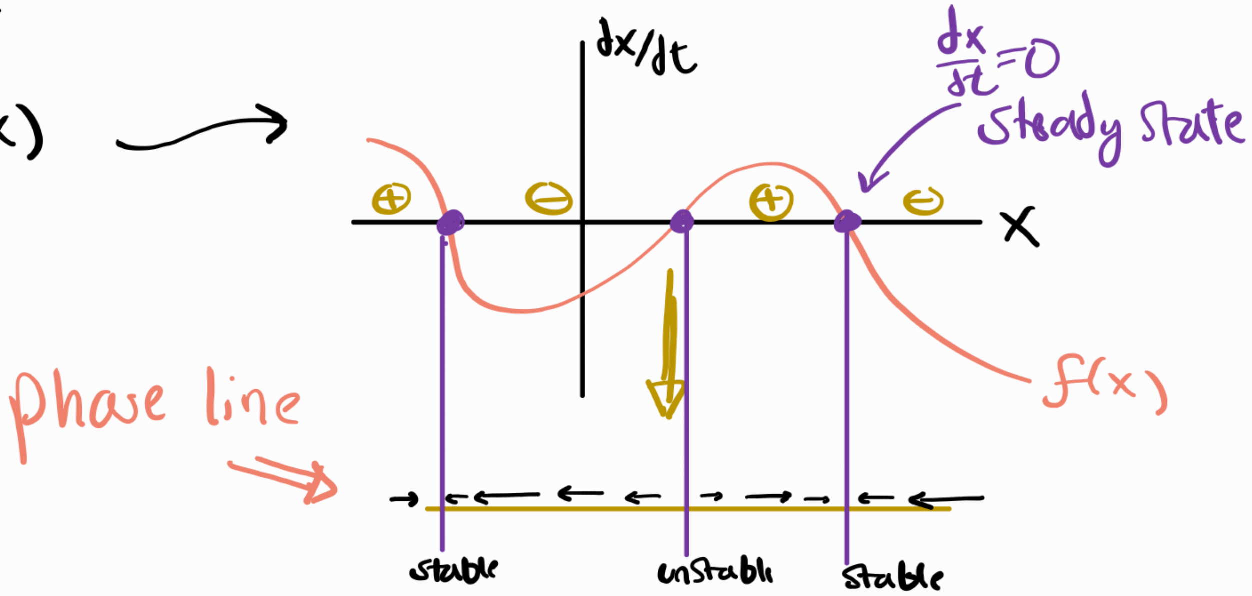
Last Week: Autonomous 1D ODEs

$$\frac{dx}{dt} = f(x) \longrightarrow$$



Linear Stability Analysis

We can determine fixed pt stability analytically:

⇒ let x^* be a fixed pt $\left(f(x^*)=0\right)$

⇒ small perturbation

$$\varepsilon(t) = x(t) - x^* \rightsquigarrow \text{how does it evolve?}$$

$$\frac{d}{dt}\varepsilon(t) = \frac{dx}{dt} = f(x(t)) = f(\varepsilon(t) + x^*)$$

↙ Taylor expansion

$$= \underbrace{f(x^*)}_{=0} + \varepsilon f'(x^*) + O(\varepsilon^2)$$

$$\frac{d\varepsilon}{dt} \approx \varepsilon f'(x^*) \xrightarrow[\text{soln}]{} \varepsilon(t) = \varepsilon_0 \exp(f'(x^*)t)$$

$f'(x^*) > 0 \Rightarrow$ perturbation grows $\Rightarrow$ unstable

$f'(x^*) < 0 \Rightarrow$ perturbation shrinks $\Rightarrow$ stable

[also, we see perturbations evolve w/ timescale $|1/f'(x^*)|$]

Example

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right) = f(N)$$

$r>0$
$K>0$

steady states: $N^*=0$, $N^*=K$

Stability: $f'(N) = r\left(1-\frac{N}{K}\right) - \frac{rN}{K}$

$\Rightarrow f'(0) = r > 0$ $\quad f'(K) = -r < 0$

unstable $\quad$ stable

Autonomous ODEs in 2D

General form in higher dimensions

$$\begin{cases} \frac{dx_1}{dt} = F_1(x_1, x_2, \dots, x_N) \\ \frac{dx_2}{dt} = F_2(x_1, x_2, \dots, x_N) \\ \vdots \\ \frac{dx_N}{dt} = F_N(x_1, x_2, \dots, x_N) \end{cases}$$

2D case

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases}$$

X-nullcline

$\{(x,y) \mid f(x,y) = 0\}$

Y-nullcline

$\{(x,y) \mid g(x,y) = 0\}$

Intersections are fixed points

1D phase line $\Rightarrow$ 2D phase plane

Example

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(y+1) \\ \dfrac{dy}{dt} = (y+x)(x+2) \end{cases}$$

$\Rightarrow$ plot nullclines in the phase plane

X-nullcline

$x=1$, y: any

OR

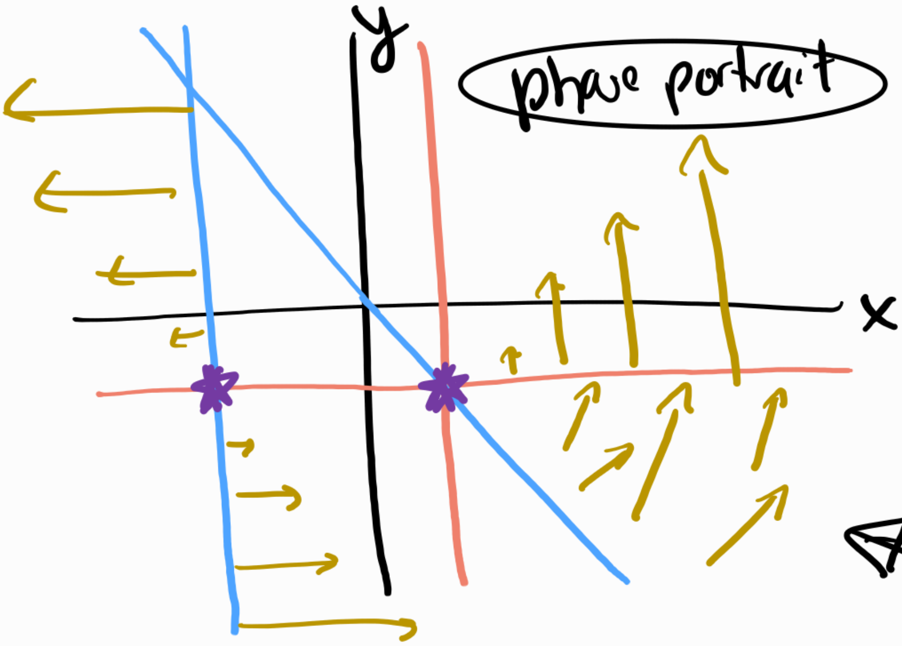
$y=-1$, x: any

Y-nullcline

$y=-x$

OR

$x=-2$, y: any



2 fixed pts:

$(x^*, y^*) = (-2, -1)$

$= (1, -1)$

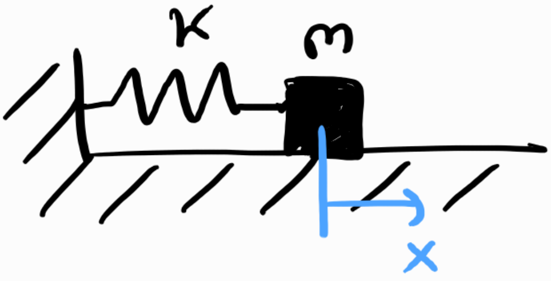
flow is now 2D

Direction field can help indicate stability, but for nonlinear problems the flow can change dramatically close vs. far

$\Rightarrow$ first, linear examples

$\Rightarrow$ then, linear stability analysis

Example Consider a mass on a spring on a smooth table. Let x be the deviation in position from rest.

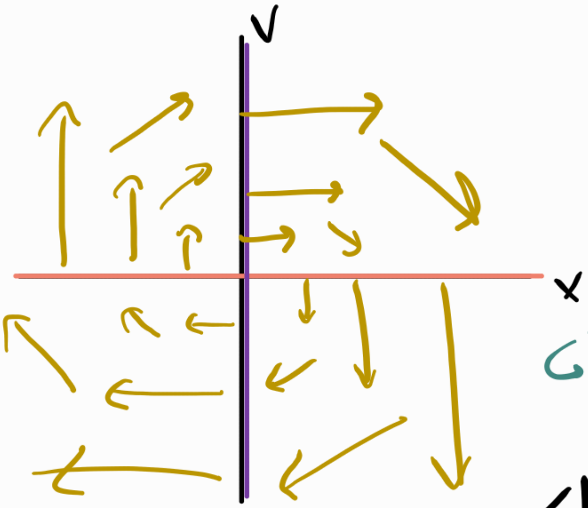


Newton's law tells us

$$m\ddot{x} = -kx \leftarrow \text{spring force}$$

We can rewrite this as

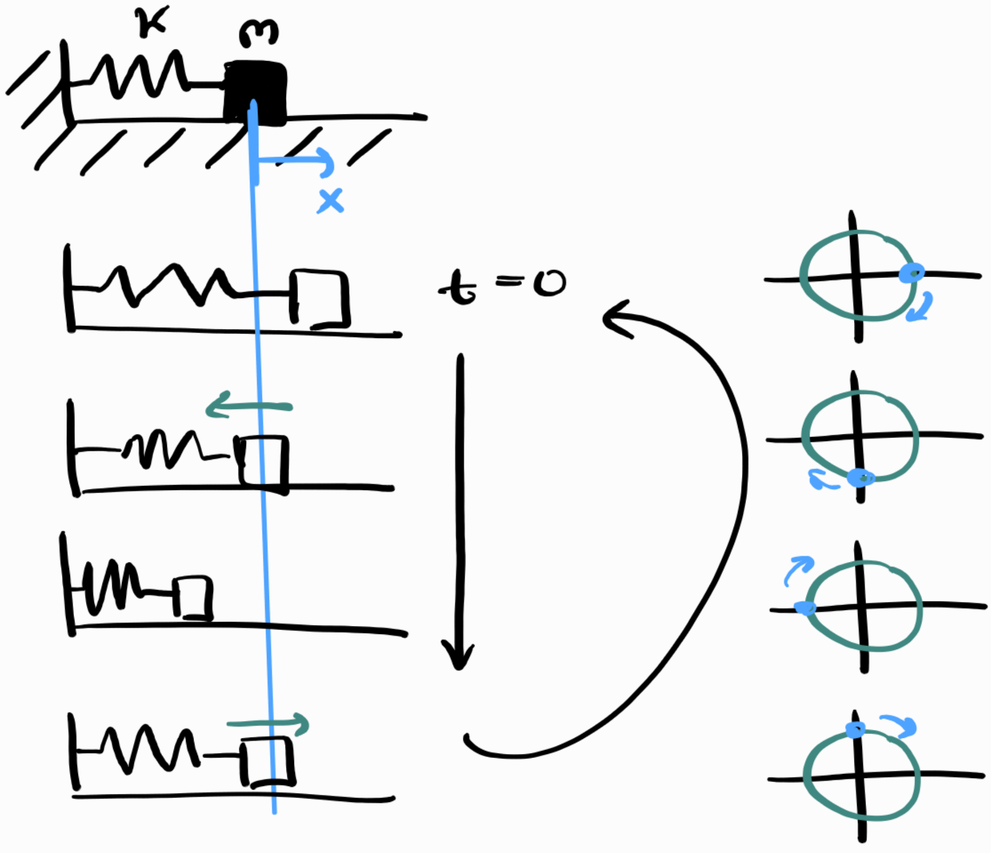
$$\begin{cases} \dfrac{dx}{dt} = v & \text{velocity} \\ \dfrac{dv}{dt} = -\omega^2 x & \text{acceleration} \end{cases} \qquad \omega^2 = k/m$$



x nullcline: $v = 0$

v nullcline: $x = 0$

Closed orbits around fixed point?



Example (Linear, saddle pt)

$$\begin{cases} \dfrac{dx}{dt} = x + y \\ \dfrac{dy}{dt} = 4x - 2y \end{cases} \Rightarrow \frac{d\bar{x}}{dt} = \underline{\underline{A}}\,\bar{x} \text{ with } \underline{\underline{A}} = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$$

X-nullcline: $y = -x$ Y-nullcline: $y = 2x$

$\boxed{Q}$ What do solns look like?

direction

$\Rightarrow$ Consider ansatz $\bar{x}(t) = \exp(\lambda t)\,\bar{v}$

growth rate

$$\frac{d\bar{x}}{dt} = \lambda \cancel{\exp(\lambda t)}\,\bar{v} = \underline{\underline{A}}\,\bar{x} = \cancel{\exp(\lambda t)}\,\underline{\underline{A}}\,\bar{v}$$

$$\Longrightarrow \boxed{\underline{\underline{A}}\,\bar{v} = \lambda \bar{v}} \rightsquigarrow \begin{pmatrix} \text{eigenvalue} \\ \text{eigenvector} \end{pmatrix} \text{ pair!} \quad \begin{matrix} v_1, \lambda_1 \\ v_2, \lambda_2 \end{matrix}$$

If you have initial condition

$$\bar{x}_0 = c_1 \bar{v}_1 + c_2 \bar{v}_2$$

$$\Downarrow$$

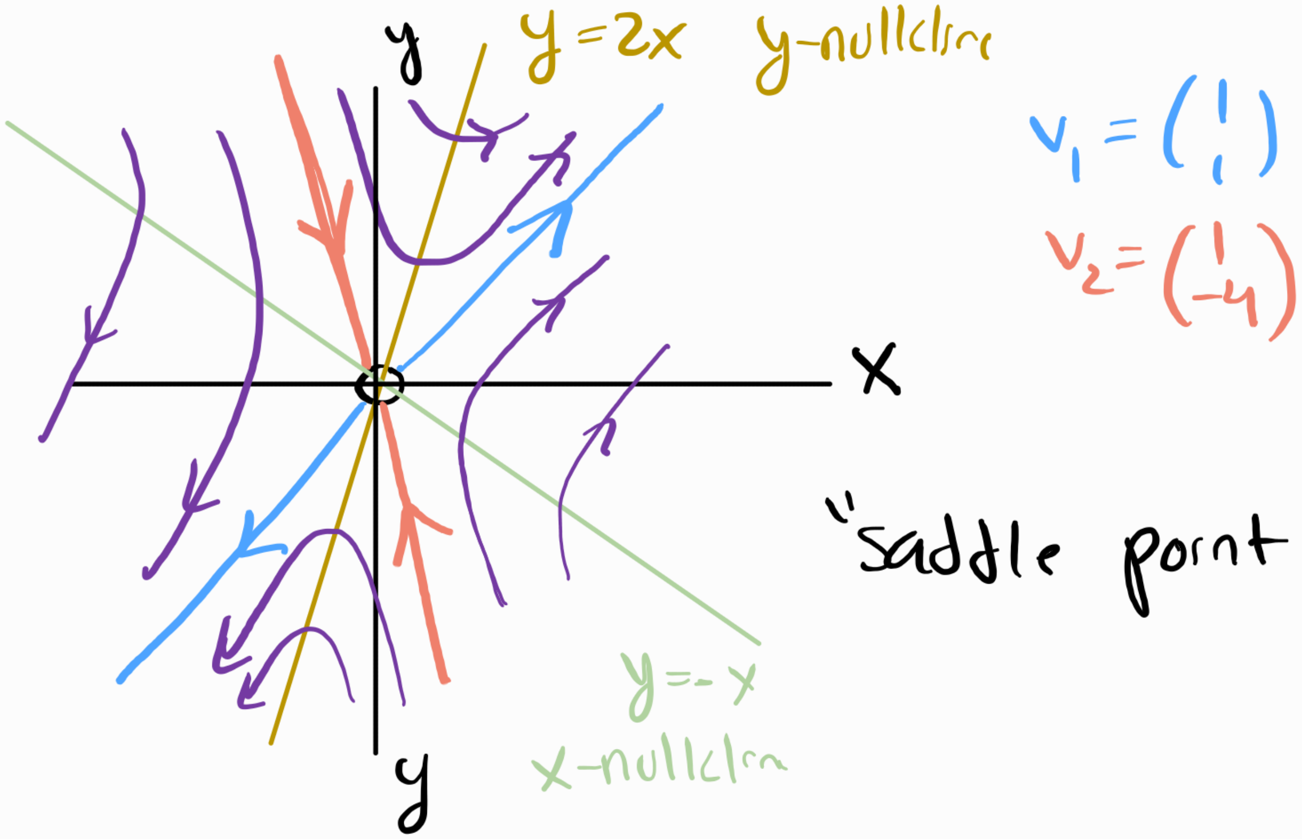
$$\bar{x}(t) = c_1 \exp(\lambda_1 t)\,\bar{v}_1 + c_2 \exp(\lambda_2 t)\,\bar{v}_2$$

$\Rightarrow$ By uniqueness, the only soln

note: $\bar{v}_1, \bar{v}_2$ are linearly independent

$\lambda_1 = 2$, $\bar{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\lambda_2 = -3$, $\bar{v}_2 = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$

growing decaying



"saddle point"

suppose matrix $\underline{\underline{A}}^{2\times 2}$ has eigenvalues λ_1, λ_2

$\leadsto$ we know from lin alg that:

$$\begin{cases} tr(\underline{\underline{A}}) = \tau = \lambda_1 + \lambda_2 \\ det(\underline{\underline{A}}) = \Delta = \lambda_1 \lambda_2 \end{cases}$$

note: both $\tau, \Delta \in \mathbb{R}$

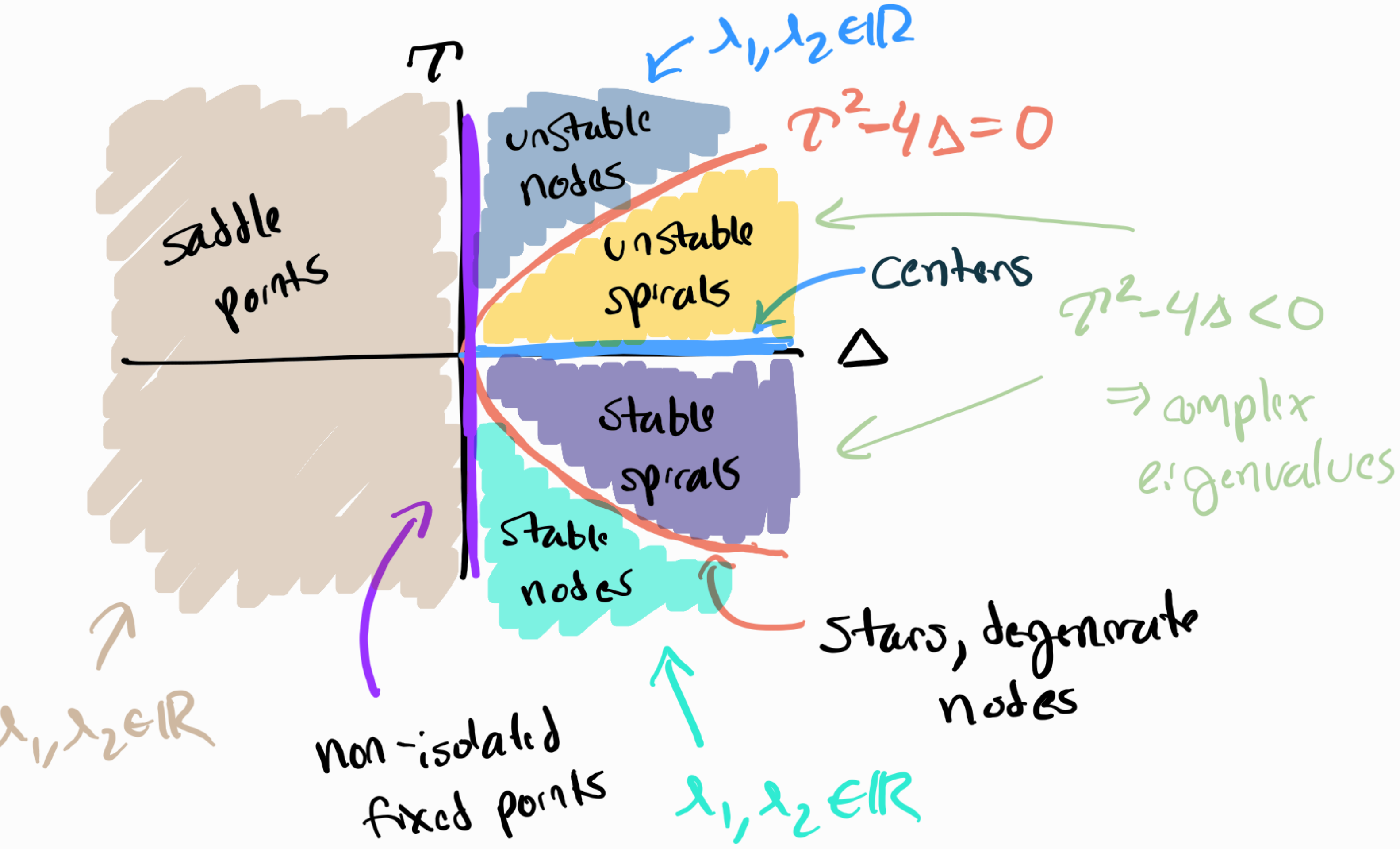
$\rightarrow$ Both λ_1, λ_2 and τ, Δ classify s.s.

Note also:

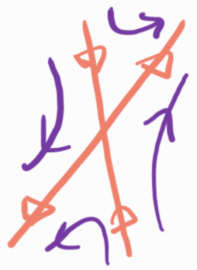
$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

$\Rightarrow$ if $\tau^2 - 4\Delta < 0 \Rightarrow$ complex eigenvalues

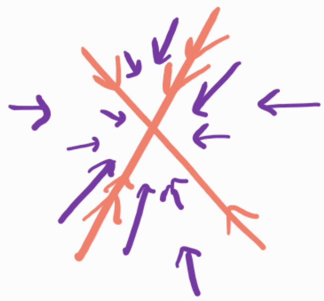
$\rightarrow$ if $\tau = 0, \Delta > 0 \Rightarrow$ purely imaginary eigenvalues



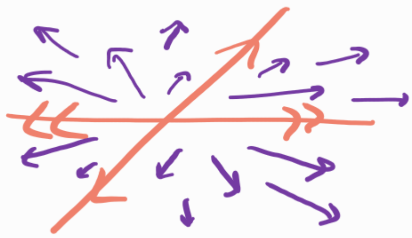
Saddle



stable node



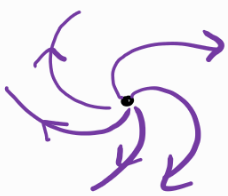
unstable node



center

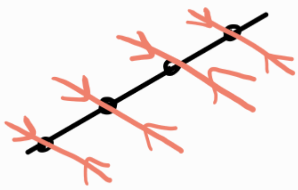
stable spiral

unstable spiral



non-isolated f.p

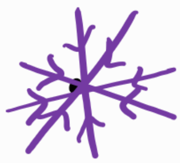
$\Delta = 0 \rightsquigarrow$ at least one $\lambda = 0$



Stars

$\lambda_1 = \lambda_2 = \tau/2 \neq 0$ $\longrightarrow$

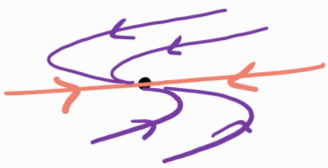
⊕ two distinct eigenvectors



all trajectories are straight lines

degenerate node

⊕ one distinct eigenvector



as $t \to \infty$, trajectories are $\parallel$ to eigenvector

Linearization (analogous to 1D)

Let

$u = x - x^*$

$v = y - y^*$

u, v small

$$\Rightarrow \frac{du}{dt} = \frac{dx}{dt} = f(x,y) = f(u+x^*, v+y^*)$$

Taylor series

$$\approx \cancel{f(x^*,y^*)}^{\,0} + u\frac{\partial f}{\partial x}(x^*,y^*) + v\frac{\partial f}{\partial y}(x^*,y^*)$$

$$\cancel{+ \frac{u^2}{2}\frac{\partial^2 f}{\partial x^2}(x^*,y^*)} + \cancel{2uv\frac{\partial^2 f}{\partial x \partial y}(x^*,y^*)}$$

$$\cancel{+ \frac{v^2}{2}\frac{\partial^2 f}{\partial y^2}(x^*,y^*)} + O(u^3, uv^2, u^2v, v^3)$$

small!

$$\Rightarrow \begin{cases} \dfrac{du}{dt} \approx u\,\underline{\dfrac{\partial f}{\partial x}(x^*,y^*)} + v\,\underline{\dfrac{\partial f}{\partial y}(x^*,y^*)} \\ \dfrac{dv}{dt} \approx u\,\underline{\dfrac{\partial g}{\partial x}(x^*,y^*)} + v\,\underline{\dfrac{\partial g}{\partial y}(x^*,y^*)} \end{cases}$$

linear system

constants!

Jacobian

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = J\Big|_{x^*,y^*}\begin{bmatrix} u \\ v \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$\Longrightarrow$ find eigenvalues, classify them

Example

S.p. $(-2,-1)$ & $(1,-1)$

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(y+1) \\ \dfrac{dy}{dt} = (y+x)(x+2) \end{cases} \Longrightarrow J = \begin{bmatrix} y+1 & x-1 \\ y+2x+2 & x+2 \end{bmatrix}$$

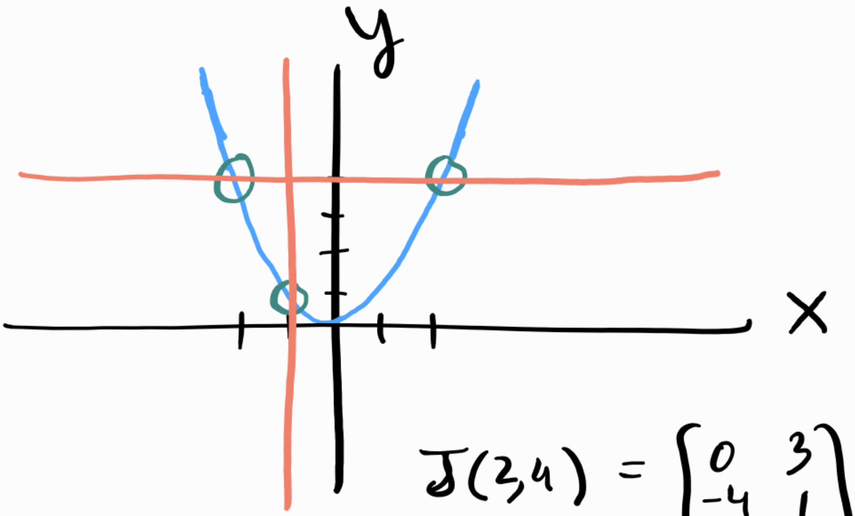
$$J(-2,-1) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \Rightarrow \begin{matrix} \tau = 0 \\ \Delta = -9 \end{matrix} \longrightarrow \text{saddle pt}$$

$$J(1,-1) = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \Rightarrow \begin{matrix} \tau = 3 \\ \Delta = 0 \end{matrix} \longrightarrow \text{'non isolated S.p.'}$$

☆ note: this classification is only local, because we linearized for small u, v

Example

$$\begin{cases} \dfrac{dx}{dt} = (x+1)(y-4) \\ \dfrac{dy}{dt} = y - x^2 \end{cases} \Longrightarrow J = \begin{bmatrix} y-4 & x+1 \\ -2x & 1 \end{bmatrix}$$



$(x^*, y^*) = (2, 4)$

$(x^*, y^*) = (-2, +4)$

$(x^*, y^*) = (-1, 1)$

$$J(2,4) = \begin{bmatrix} 0 & 3 \\ -4 & 1 \end{bmatrix} \Rightarrow \begin{matrix} \tau = 1 \\ \Delta = 12 \end{matrix} \Rightarrow \text{unstable spiral}$$

$$J(-1,1) = \begin{bmatrix} -5 & 0 \\ 2 & 1 \end{bmatrix} \Rightarrow \begin{matrix} \tau = -4 \\ \Delta = -5 \end{matrix} \Rightarrow \text{saddle}$$

$$J(-2,4) = \begin{bmatrix} 0 & -1 \\ 4 & 1 \end{bmatrix} \Rightarrow \begin{matrix} \tau = 1 \\ \Delta = 4 \end{matrix} \Rightarrow \text{unstable spiral}$$

Why spiral?

Suppose $\lambda = \alpha + i\beta$:

$$\exp(\lambda t) = \exp(\alpha t)\exp(i\beta t)$$
$$= \underbrace{\exp(\alpha t)}_{\text{exponential magnitude}} \left[\underbrace{\cos(\beta t) + i\sin(\beta t)}_{\text{oscillatory}}\right]$$

$\leadsto$ if $Re(\lambda) < 0 \Rightarrow$ stable spiral

$Re(\lambda) > 0 \Rightarrow$ unstable spiral

$Re(\lambda) = 0 \Rightarrow$ center

note: neither stable nor unstable!

Quiz

Consider the following "Romeo-Juliet" model:

$$\begin{cases} dR/dt = aR \\ dJ/dt = 2R - J \end{cases}$$

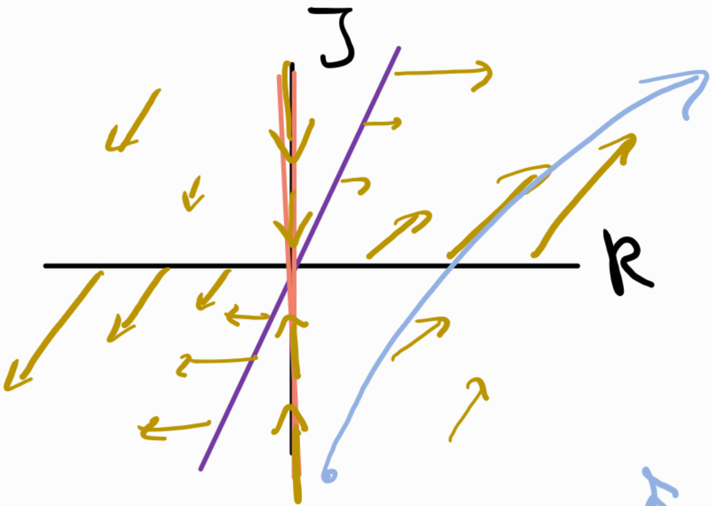
a) Juliet feels cautious of her own feelings but is responsive to Romeo's. Romeo does not care how Juliet feels.

If $a>0$, he is bolstered by his own feelings. If $a<0$, he is cautious. If $a=0$, nothing could ever change how he feels.

b) Find and sketch the nullclines when $a=2$

$\frac{dR}{dt}=0 \Rightarrow R=0$

$\frac{dJ}{dt}=0 \Rightarrow J=2R$



c) Sketch the phase portrait $(a=2)$

d) saddle node

e) eventually, they will be madly in love