intro to PDEs

Thus far, we have focused on (systems of) ODEs, where you have <u>one</u> independent variable

e.g. $\frac{\partial x}{\partial t} = f(x,t)$ $\Rightarrow$ soln $x(t)$ satisfies this eqn

← dependent

↖ independent

However, you may want to solve equations in $\mathbb{R}^n$ $n>1$ space, or something that varies in time & space.

e.g. modeling sound waves

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$ [wave eqn]

e.g. chlorine spread in a static pool

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}$$ [diffusion eqn]

e.g. algae in an active river

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(vu)$$ [advection eqn]

e.g. traffic flow

$$\frac{\partial u}{\partial t} = -u u_x$$ [Burger's eqn]

We will focus on eqns of the form:

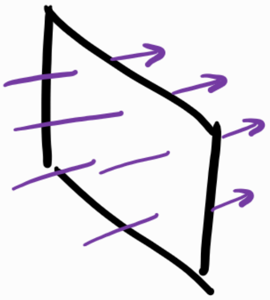
some concentration or density

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}$$

flux of u

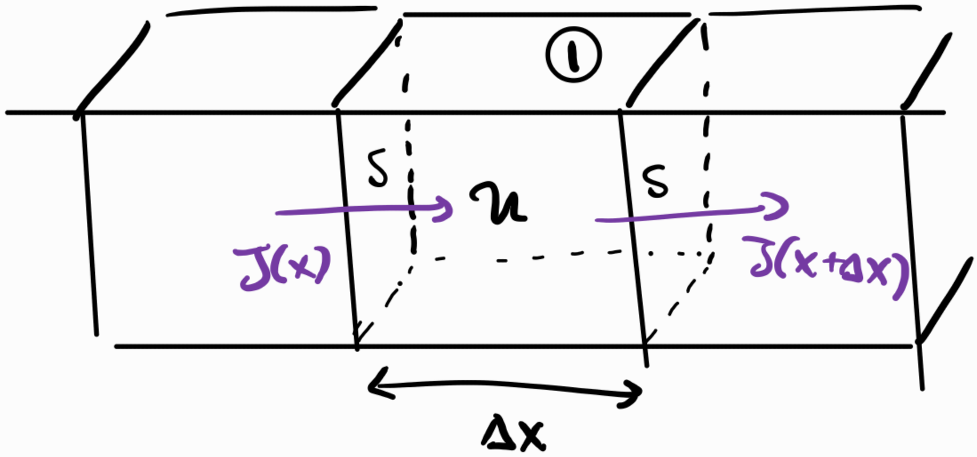
Continuity Eqn

Consider some surface S:



flux J = # u "particles" crossing S per unit area unit time Δt

Suppose flux is only in the x-direction.



Change in # particles in ① = $\Delta N = \Delta t\, S J(x) - \Delta t\, S J(x+\Delta x)$
over Δt time

$$\frac{\Delta N}{\Delta t} = -S\left(J(x+\Delta x) - J(x)\right)$$

$$\Downarrow$$

$$\frac{1}{\Delta t}\frac{\Delta N}{\Delta x S} = \frac{\Delta u}{\Delta t} = -\frac{\left(J(x+\Delta x) - J(x)\right)}{\Delta x}$$

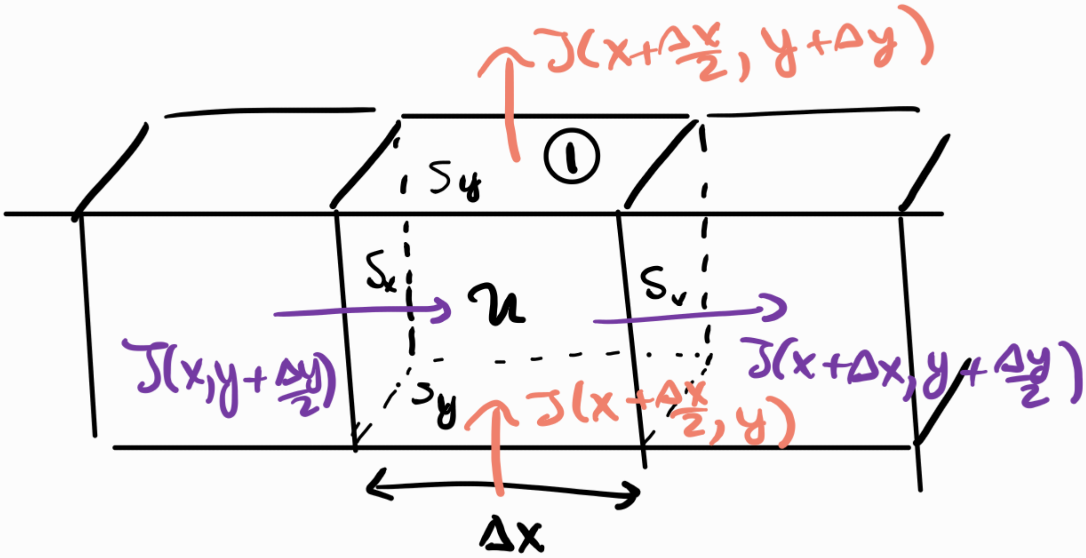
volume

$$\Big\downarrow \ \Delta t \to 0,\ \Delta x \to 0$$

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x}$$

partial derivatives
B/c $u = u(x,t)$

Suppose flow is also in y-direction:



$$\Delta N = \Delta t\, S_x\left[J(x, y+\tfrac{\Delta y}{2}) - J(x+\Delta x, y+\tfrac{\Delta y}{2})\right]$$

$$+\, \Delta t\, S_y\left[J(x+\tfrac{\Delta x}{2}, y) - J(x+\tfrac{\Delta x}{2}, y+\Delta y)\right]$$

$$= \Delta t\, \overset{V}{S_x \Delta x}\left[\frac{\text{"} \qquad \text{"}}{\Delta x}\right] + \Delta t\, \overset{V}{S_y \Delta y}\left[\frac{\text{"} \qquad \text{"}}{\Delta y}\right]$$

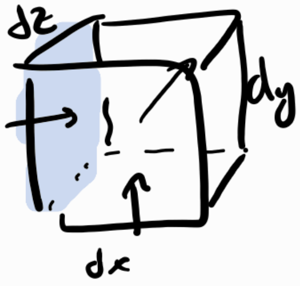
$$\frac{\Delta u}{\Delta t} = \frac{J(x, y+\frac{\Delta y}{2}) - J(x+\Delta x, y+\frac{\Delta y}{2})}{\Delta x} + \frac{J(x+\frac{\Delta x}{2}, y) - J(x+\frac{\Delta x}{2}, y+\Delta y)}{\Delta y}$$

$\downarrow$ $\Delta t \to 0$, $\Delta x \to 0$, $\Delta y \to 0$

$$\frac{\partial u}{\partial t} = -\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y}$$

A different way to think about it: (3D)

$\rho(\bar{x}, t)$ is mass density



Conservation of mass: mass cannot magically appear/disappear

$\Rightarrow$ it comes/goes at boundaries

Total mass:

$$M(t) = \int_V \rho(\vec{x},t)\,dV \longrightarrow \frac{dM}{dt} = \frac{\text{in} - \text{out}}{\text{time}}$$

change:

$J \sim \frac{\text{mass}}{\text{length}^2 \times \text{time}}$

$J \sim \frac{\text{density} \times \text{length}}{\text{time}}$

mass in during dt time:

$$= J|_x\, dy\,dz\,dt + J|_y\, dx\,dz\,dt + J|_z\, dx\,dy\,dt$$

mass out during dt time:

$$= J|_{x+\Delta x}\, dy\,dz\,dt + J|_{y+\Delta y}\, dx\,dz\,dt + J|_{z+\Delta z}\, dx\,dy\,dt$$

$\Downarrow$

$$\frac{dM}{dt} = \int_V \frac{\partial \rho}{\partial t}\,dV = (J|_x - J|_{x+\Delta x})\,dy\,dz + (J|_y - J|_{y+\Delta y})\,dx\,dz + (J|_z - J|_{z+\Delta z})\,dx\,dy$$

→ Fundamental Thm of calculus $= \int \frac{\partial J}{\partial x}\,dx$

$$\int_V \frac{\partial \rho}{\partial t}\,dV = \int \left(-\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y} - \frac{\partial J}{\partial z}\right) \underbrace{dx\,dy\,dz}_{dV}$$

since this is true $\forall V$:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x} - \frac{\partial J}{\partial y} - \frac{\partial J}{\partial z}$$

continuity eqn

Different fluxes

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$ [diffusion eqn]

$$\longrightarrow J = -D\frac{\partial u}{\partial x}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(vu)$$ [advection eqn]

$$\longrightarrow J = vu$$

$$\frac{\partial u}{\partial t} = -uu_x$$ [Burger's eqn]

$$\longrightarrow J = \frac{u^2}{2}$$

Initial conditions & boundary conditions

Recall: for ODEs $\frac{\partial x}{\partial t} = f(x,t)$, you needed to prescribe an initial condition to get a unique soln (data we know!)

e.g.

constants of integration

$$\frac{\partial x}{\partial t} = h \Rightarrow x = ht + c_0$$

$$\frac{\partial^2 x}{\partial t^2} = h \Rightarrow \frac{\partial x}{\partial t} = ht + c_0 \Rightarrow x(t) = \frac{ht^2}{2} + \underline{c_0}t + \underline{c_1}$$

2 this time

Since the only data we know is at $t=0$, we set constants of integration using knowledge of $x(t)$ at $t=0$ (e.g. $x(0)$, $x'(0)$, ...)

For PDEs, you similarly need conditions for spatial derivatives. You use data on the bdry

⇝ Boundary conditions

The # boundary conditions you need depends on the order of that variable in your PDE

e.g. wave eqn: $\underbrace{\frac{\partial^2 u}{\partial t^2}}_{\text{2nd order}} = c^2 \underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{2nd order}}$ on $0<x<L$

→ you could for example set:

$$\begin{cases} u(0,t) \overset{①}{=} u(L,t) \overset{②}{=} 0 & \text{boundary conditions} \\ u(x,0) = f(x) \; ① & \text{initial conditions} \\ \frac{\partial u}{\partial t}(x,0) = g(x) \; ② & \end{cases}$$

e.g. Burger's eqn: $\frac{\partial u}{\partial t} = -u u_x$ on $0<x<L$

→ you could for example set:

$$\begin{cases} \frac{\partial u}{\partial x}(L,t) = 0 & \text{boundary cond.} \\ u(x,0) = f(x) & \text{initial cond.} \end{cases}$$

Example

on $0 < X < L$ for $T \geq 0$

$$\frac{\partial C}{\partial T} = -\frac{\partial J}{\partial X} \quad \text{with } J = B\cos(X/L)\exp(T/\tau)$$

Nondimensionalize

$$\begin{cases} X = Lx \\ T = \tau t \\ C = \frac{B\tau}{L} c \end{cases}$$

$$\frac{[C]}{[T]} \sim \frac{[J]}{[L]} \sim \frac{[B]}{[L]}$$

$$\Rightarrow [C] \sim \frac{[B][T]}{[L]}$$

$$\frac{B\cancel{\tau}}{L\cancel{\tau}} \frac{\partial c}{\partial t} = -\frac{1}{L} \frac{\partial}{\partial x}\left(B\cos(x)\exp(t)\right)$$

$$\cancel{\frac{B}{L}} \frac{\partial c}{\partial t} = +\cancel{\frac{B}{L}} \sin(x)\exp(t)$$

$$\boxed{\frac{\partial c}{\partial t} = \sin(x)\exp(t)}$$ nondimensional eqn

Conditions

just need an I.C. for $c \Rightarrow c(x,0) = c_0(x)$

solve

$$\int \frac{\partial c}{\partial t} dt = \int \sin(x)\exp(t)\,dt$$

"constant" of integration!

gen soln $\quad c(x,t) = \sin(x)\exp(t) + h(x)$

to get unique soln, plug in I.C.:

$$c(x,0) = c_0(x) = \sin(x) + h(x)$$

$$\Rightarrow c(x,t) = \sin(x)\exp(t) - \sin(x) + c_0(x)$$

$$\boxed{c(x,t) = \sin(x)\left(\exp(t) - 1\right) + c_0(x)}$$

<u>Interpret</u>

The concentration increases over time exponentially but sinusoidally dependent on nondimensional position x. Concentration at $x=0$ will remain constant in time as such, and concentration will grow most at $x=L$

Quiz Consider $\frac{\partial C}{\partial T} = -\frac{\partial J}{\partial X}$

Let the time-dependent flux J be a function of X and T:

$$J = -\frac{A}{L} X^2 \cos(T/\tau)$$

Here, A and τ are constant parameters.

We aim to solve this on $0 < X < L$.

(a) Assuming C is measured in #/cm^3 and time in s, choose the scales and nondimensionalize

$$\begin{cases} T = \tau t \\ X = Lx \\ C = A\tau c \end{cases} \qquad \frac{[A]}{[L]}[L]^2 \sim \frac{[C][L]}{[T]} \Rightarrow [A] \sim [C]/[T]$$

$$\frac{\partial C}{\partial T} = -\frac{\partial J}{\partial X} = +\frac{2A}{L} X \cos(T/\tau)$$

$$\frac{A\tau}{\tau}\frac{\partial c}{\partial t} = +\frac{2A}{L} Lx\cos(t) \Rightarrow \underline{\frac{\partial c}{\partial t} = +2x\cos(t)}$$

(b) Do you need B.C.?

No because there are no x-derivatives

(c) Solve with $c(x,0) = g(x)$

$$c(x,t) = +2x\sin(t) + h_0(x)$$

"constant" of time integration

$$c(x,0) = +2x\sin(0) + h_0(x) = h_0(x) = g(x)$$

$$\Rightarrow \boxed{c(x,t) = +2x\sin(t) + g(x)}$$

(d) Explain in words the meaning of the soln

The concentration varies periodically around its initial condition with nondim period 2π. The magnitude of oscillation is $2x$, so close to $x=0$ variation is small and at $x=L$ it is maximum $2L$.

(e) [BONUS] Do you notice any limits on the I.C.?

Yes $\Rightarrow$ if $g(x) < 2x$, the soln is not physical, as concentration can end up being negative