Recall: 2D ODEs

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \rightsquigarrow \text{Linear: } \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{A}}\begin{bmatrix} x \\ y \end{bmatrix}$$

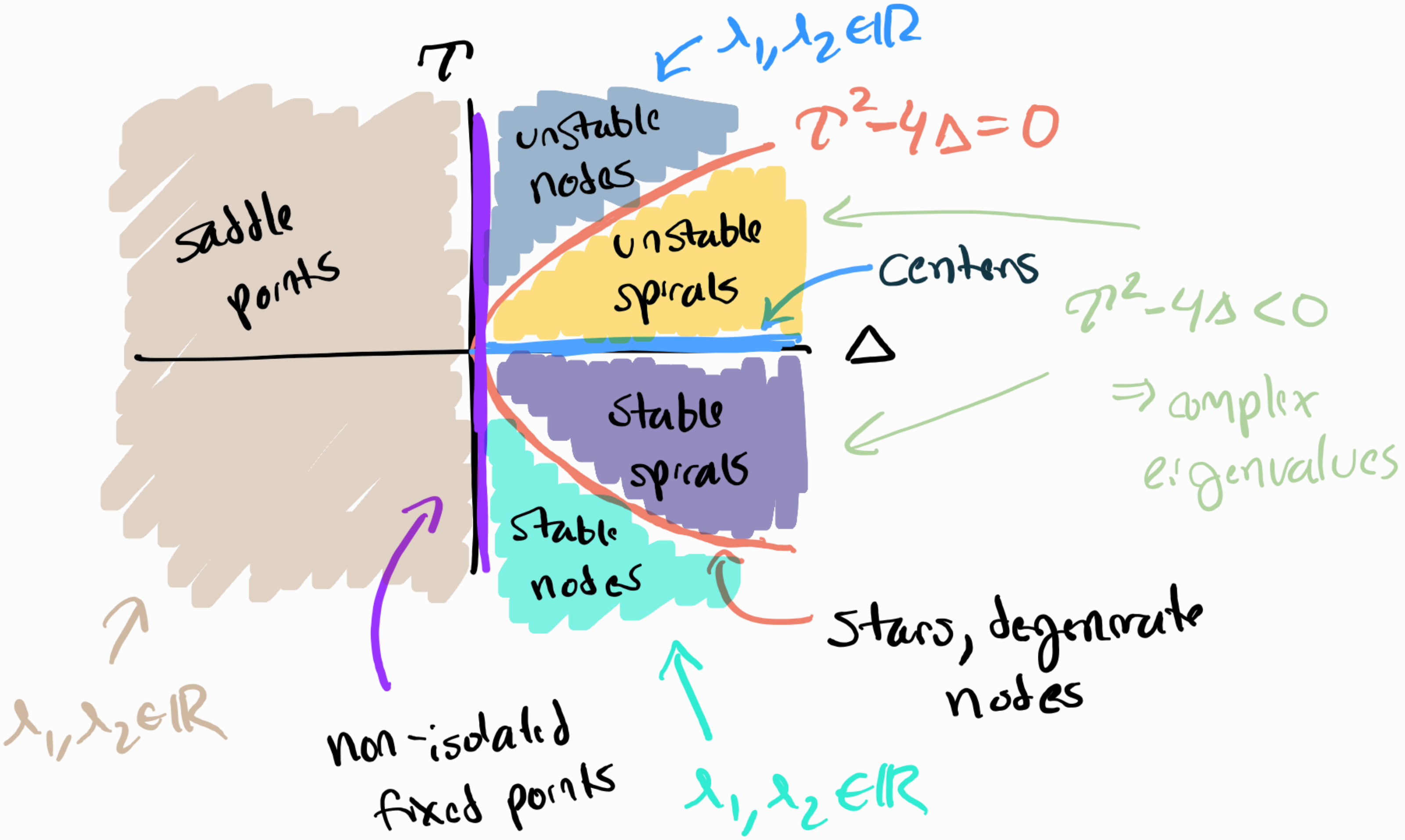
suppose matrix $\underline{\underline{A}}^{2\times 2}$ has eigenvalues λ_1, λ_2

$\rightsquigarrow$ we know from lin alg that:

$$\begin{cases} tr(\underline{A}) = \tau = \lambda_1 + \lambda_2 \\ det(\underline{A}) = \Delta = \lambda_1 \lambda_2 \end{cases}$$

note: both $\tau, \Delta \in \mathbb{R}$

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$

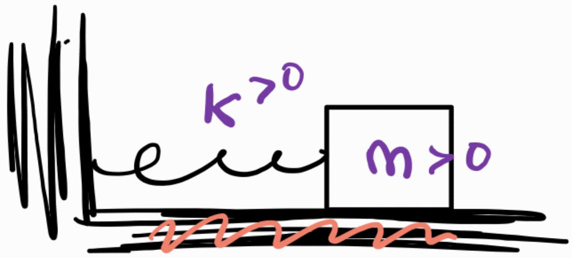


Example Damped Harmonic Oscillator

$$(ma = \Sigma \text{ forces})$$

$$m\ddot{x} = -kx - b\dot{x}$$

$$\Downarrow$$



friction $b>0$

$$\begin{cases} \dfrac{dx}{dt} = v \\[2ex] \dfrac{dv}{dt} = -\dfrac{k}{m}x - \dfrac{b}{m}v \end{cases}$$

$$\mathbb{J} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}$$

$$\hookrightarrow \begin{cases} \tau = -b/m \\ \Delta = k/m \end{cases}$$

$\boxed{Q}$ Can $(0,0)$ ever be a saddle node?

no! $\Delta > 0$

$\boxed{Q}$ What happens if $b=0$? What does this mean physically?

if $b=0 \Rightarrow$ no friction $\Rightarrow$ harmonic oscillator

$\Rightarrow \tau = 0 \Rightarrow$ <u>center</u>

$\boxed{Q}$ What if $b>0$?

$\Rightarrow$ Could be a <u>stable node</u> or <u>stable spiral</u>

spiral if $\tau^2 - 4\Delta < 0 \Rightarrow \dfrac{b^2}{m^2} - 4\dfrac{k}{m} < 0$

$\Rightarrow b^2 < 4km \Rightarrow b < 2\sqrt{km}$

$$\begin{cases} \text{center} & b = 0 \\ \text{stable spiral} & 0 < b < 2\sqrt{km} \\ \text{stable node} & 2\sqrt{km} < b \end{cases}$$

$\longleftarrow$ so much friction it stops immedietally

$\boxed{\text{Example}}$ Linearization in borderline cases

$$\begin{cases} \dfrac{dx}{dt} = -y + ax(x^2+y^2) \\ \dfrac{dy}{dt} = x + ay(x^2+y^2) \end{cases}$$

$\to$ centers, stars, degenerated nodes, nonisolated S.P.

$$J = \begin{bmatrix} a(x^2+y^2)+2ax^2 & -1+2axy \\ 1+2axy & 2ay^2+a(x^2+y^2) \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \to \begin{matrix} \tau = 0 \\ \Delta = 1 \end{matrix}$$ center? (or is it?)

$\Rightarrow$ change to polar coordinates:

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

① $2x\dot{x} + 2y\dot{y} = 2r\dot{r}$

$$\Rightarrow \dot{r} = \frac{1}{r}\left(-\cancel{x}y + ax^2(x^2+y^2) + \cancel{x}y + ay^2(x^2+y^2)\right)$$

$$= \frac{a(x^2+y^2)^2}{r} = ar^3 = \dot{r}$$

② $\dfrac{y}{x} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta \Rightarrow \theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\dot{y}}{x} - \frac{y}{x^2}\dot{x} = (\sec^2\theta)\,\dot{\theta}$$

$\to = \sec^2\left(\tan^{-1}\left(\frac{y}{x}\right)\right)$

$= \dfrac{y^2}{x^2} + 1$

$$\frac{\dot{y}x - y\dot{x}}{x^2}\left(\frac{x^2}{x^2+y^2}\right) = \frac{\dot{y}x - y\dot{x}}{r^2} = \dot{\theta}$$

$$= \frac{x^2 + \cancel{axyr^2} + y^2 - \cancel{axyr^2}}{r^2} = 1$$

In polar coordinates:

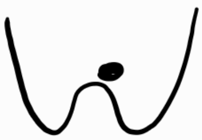
$$\begin{cases} \frac{dr}{dt} = ar^3 \\ \frac{d\theta}{dt} = 1 \end{cases}$$

$\leftarrow$ grows if $a > 0$, decays if $a < 0$

$\leftarrow$ oscillatory

Spirals!

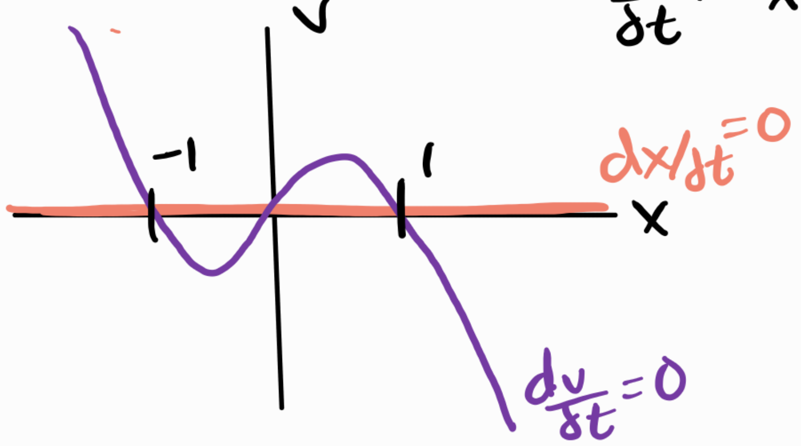
$\boxed{\text{Example}}$ Conservative system



Consider a particle of mass $m = 1$ moving in a double-well potential with potential energy

$$V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 \Rightarrow \underbrace{-\frac{dV}{dx}}_{\text{force}} = x - x^3$$

$$\ddot{x} = x - x^3 \Longrightarrow \begin{cases} \frac{dx}{dt} = v \\ \frac{dv}{dt} = x - x^3 \end{cases}$$



fixed points:

$(0,0)$, $(\pm 1, 0)$

Q what do they represent physically?

$$J = \begin{bmatrix} 0 & 1 \\ 1-3x^2 & 0 \end{bmatrix}$$

if not true, would have to be spirals

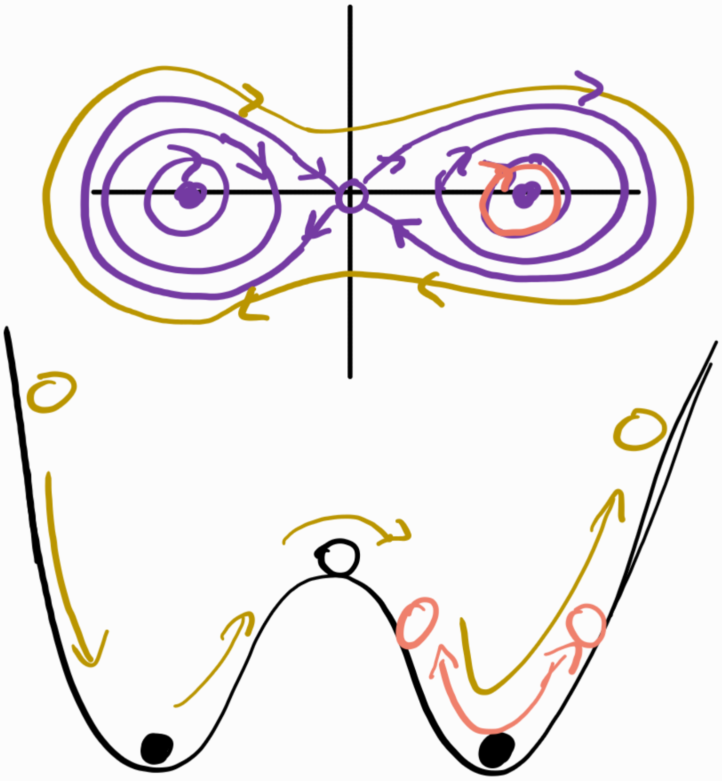
$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow \Delta = -1 < 0 \rightarrow \underline{\underline{\text{saddle}}}$$

$$J(\pm 1,0) = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} \rightarrow \Delta = 2 > 0, \quad \tau = 0 \rightarrow \underline{\underline{\text{centers}}}$$

true centers b/c of energy conservation!

$$E = KE + PE$$

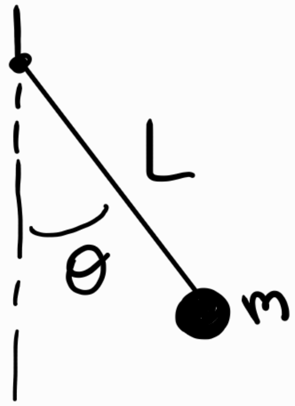
$$= \frac{1}{2}mv^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = \text{constant along a trajectory}$$



different closed loop trajectories

generally, centers in conservative systems are more robust

Example Pendulum



$\Rightarrow$ motion described by

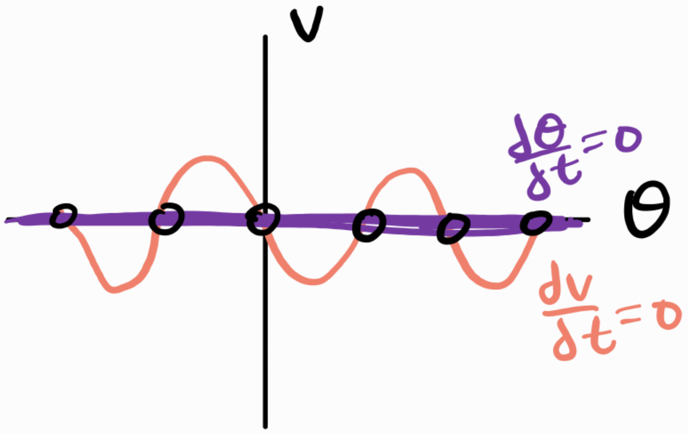
$$\frac{d^2\theta}{dT^2} + \frac{g}{L}\sin\theta = 0$$

Q what is a unit for time?

$$\omega = \sqrt{L/g} \Rightarrow T = \sqrt{L/g}\, t$$

nondimensional eqn:

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0 \Rightarrow \begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = -\sin\theta \end{cases}$$



s.p.: $\sin\theta = 0$ & $v=0$

$$\theta = n\pi \qquad v = 0$$

$$J = \begin{pmatrix} 0 & 1 \\ -\cos\theta & 0 \end{pmatrix}$$

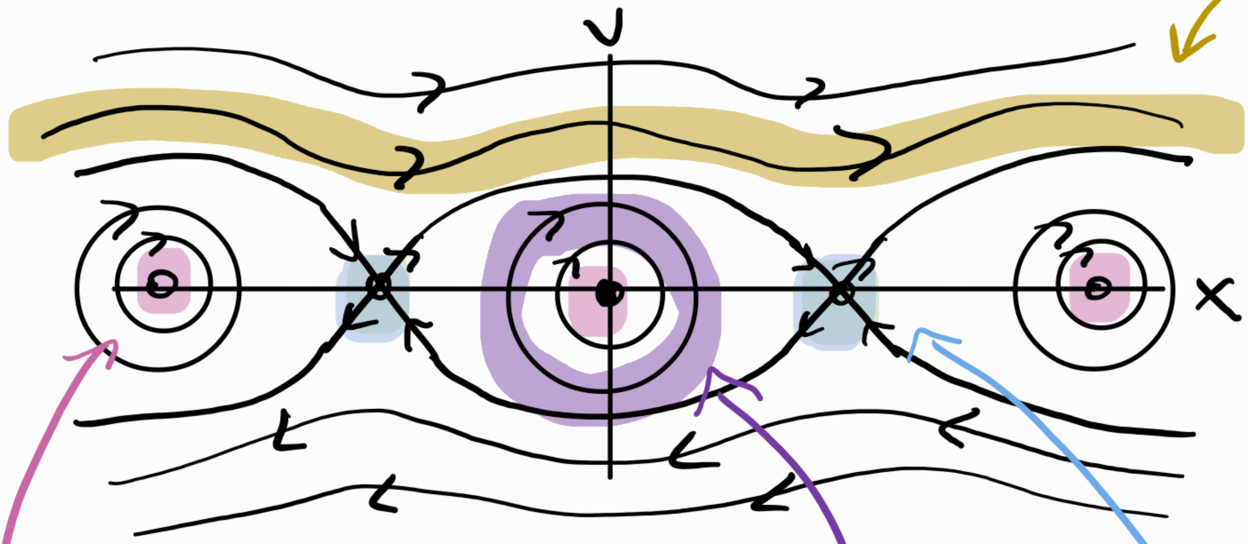
$\rightarrow J(0,0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ center

$\rightarrow J(\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ saddle

$\rightarrow J(-\pi,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ saddle

in this case, the conserved energy is $E = \frac{1}{2}v^2 - \cos\theta$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \begin{matrix} \lambda_1 = -1 & \bar{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ \lambda_2 = +1 & \bar{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{matrix}$$



Q what do different trajectories represent?

pendulum at rest on the bottom

oscillating around bottom equilibrium

unstable inverted pendulum

What if instead of $\frac{d^2\theta}{dt^2} + \sin\theta = 0$

we have $\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + \sin\theta = 0$?

$$\begin{cases} \frac{d\theta}{dt} = v \\ \frac{dv}{dt} = -bv - \sin\theta \end{cases} \Rightarrow J = \begin{bmatrix} 0 & 1 \\ -\cos\theta & -b \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} \rightarrow \begin{matrix} \Delta = 1 \\ \tau = -b \end{matrix}$$

$\tau^2 - 4\Delta = b^2 - 4 \Rightarrow 0 < b < 2$ stable spiral

Example Model of national economy (Jordan and Smith 1987)

"Keynesian cross"

$1 < \alpha < \infty$

$1 \le \beta < \infty$

$$\begin{cases} \frac{dI}{dt} = I - \alpha C \\ \frac{dC}{dt} = \beta(I - C - G) \end{cases}$$

national income I

rate of consumer spending C

rate of government spending

<u>fixed point:</u>

I nullcline: $I = \alpha C$

C nullcline: $I = C + G$

set equal

$$\Rightarrow \alpha C = C + G \Rightarrow \boxed{\begin{aligned} C &= G/\alpha - 1 \\ I &= \frac{\alpha}{\alpha - 1} C \end{aligned}}$$

$$J = \begin{pmatrix} 1 & -\alpha \\ \beta & -\beta \end{pmatrix}$$

$$\begin{cases} \Delta = -\beta + \alpha\beta \\ \tau = 1 - \beta \end{cases} \rightarrow$$ so long as $\beta > 1$, s.p. is stable

if $\beta = 1 \longrightarrow$ center, at least oscillatory

Suppose $G = G_0 + kI^2$

(government spending increases quadratically w/ I)

$$\begin{cases} \dfrac{dI}{dt} = I - \alpha C \\ \dfrac{dC}{dt} = \beta(I - kI^2 - C - G_0) \end{cases}$$

<u>fixed pt</u>

$\left(\frac{dI}{dt}=0\right)$ $\left(\frac{dC}{dt}=0\right)$

$I = \alpha C$ $\quad I - kI^2 = C + G_0$

$$\Rightarrow \alpha C - k\alpha^2 C^2 = C + G_0$$

$$\Rightarrow -k\alpha^2 C^2 + (\alpha - 1)C - G_0 = 0$$

$$\Rightarrow C^2 - \frac{(\alpha-1)}{k\alpha^2} C + \frac{1}{k\alpha^2} G_0 = 0$$

$$C = \frac{(\alpha-1) \pm \sqrt{(\alpha-1)^2 - 4k\alpha^2 G_0}}{2k\alpha^2}$$

$\boxed{Q}$ When are there 2, 1, or no roots?

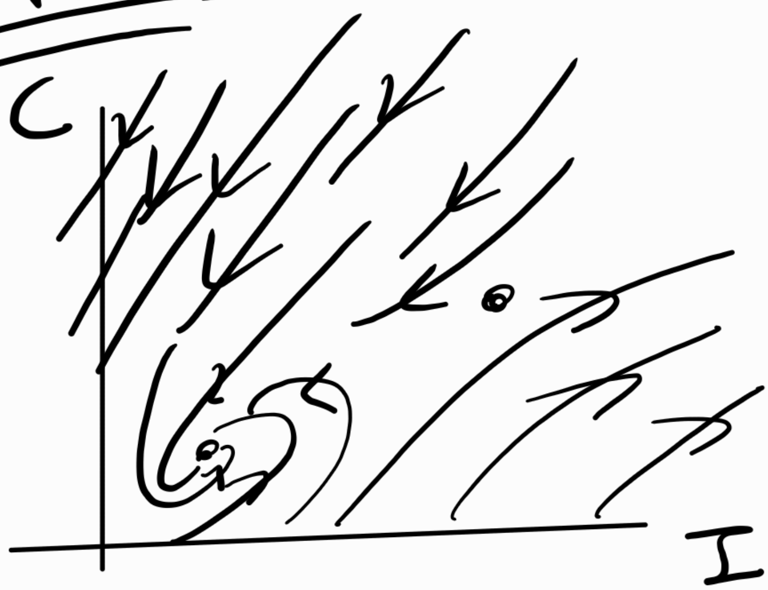
" " $> 0 \rightsquigarrow 2$ roots

$= 0 \rightsquigarrow 1$ root

$< 0 \rightsquigarrow 0$ roots

$\boxed{Q}$ What is the implication of this?

2 roots



Q Do we "want" fixed points here?

Q what happens at different initial conditions?

1 root

0 roots

$\boxed{\text{Quiz}}$ Suppose the concentrations of chemicals A and B can be modelled by:

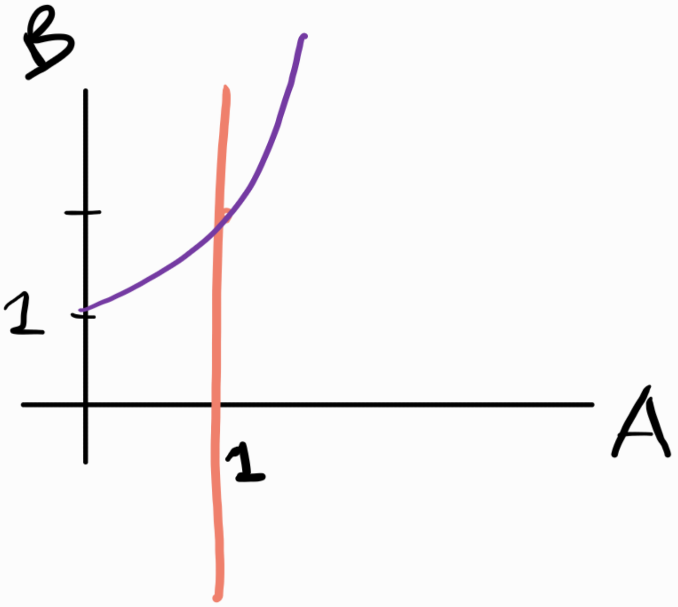
$$\begin{cases} \dfrac{dA}{dt} = -2A^2 + 2\gamma^2 \\ \dfrac{dB}{dt} = +A^2 - B + 1 \end{cases}$$

$\gamma > 0$

a) Find the nullclines.
Sketch them when $\gamma = 1$

$\dfrac{dA}{dt} = -2A^2 + 2\gamma^2 = 0 \Rightarrow A = \gamma$

$\dfrac{dB}{dt} = A^2 - B + 1 = 0 \Rightarrow B = A^2 + 1$



b) Find the non-negative steady state

$A = \gamma, \ B = \gamma^2 + 1$

c) Calculate the Jacobian at the Steady State

$$J = \begin{pmatrix} -4A & 0 \\ -2A & -1 \end{pmatrix}$$

$$J\Big|_{*} = \begin{pmatrix} -4\gamma & 0 \\ -2\gamma & -1 \end{pmatrix}$$

d) Using your answer in part (c), classify the Steady State when $\gamma = 1$

$$J(1,2) = \begin{pmatrix} -4 & 0 \\ -2 & -1 \end{pmatrix} \begin{matrix} \rightarrow \tau = -5 \\ \rightarrow \Delta = 4 \end{matrix}$$

$$\tau^2 - 4\Delta = 25 - 16 > 0$$

$\Rightarrow$ stable node