Recall: 2D ODEs

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \rightsquigarrow \text{Linear: } \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} x \\ y \end{bmatrix}$$

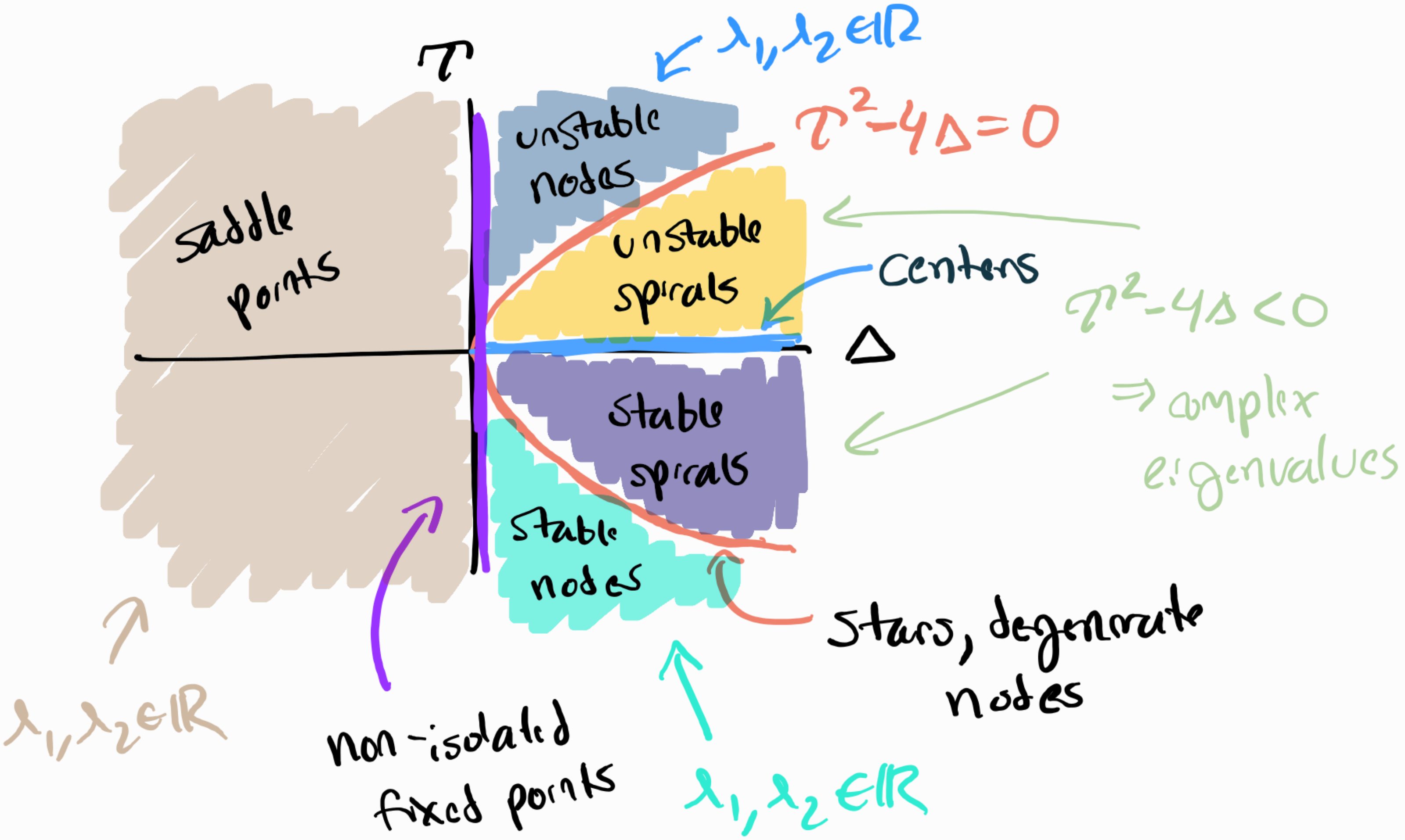
suppose matrix $\underline{\underline{A}}^{2\times 2}$ has eigenvalues λ_1, λ_2

$\rightsquigarrow$ we know from lin alg that:

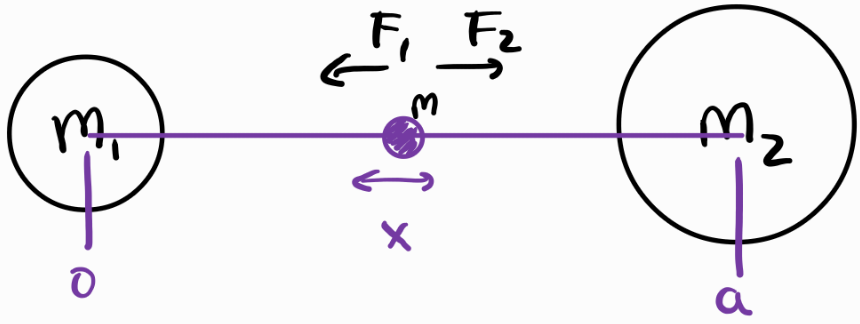
$$\begin{cases} tr(\underline{\underline{A}}) = \tau = \lambda_1 + \lambda_2 \\ det(\underline{\underline{A}}) = \Delta = \lambda_1 \lambda_2 \end{cases}$$

note: both $\tau, \Delta \in \mathbb{R}$

$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



Example Gravitational equilibrium



$$m\ddot{x} = -F_1 + F_2 = -\frac{Gm_1 m}{x^2} + \frac{Gm_2 m}{(x-a)^2}$$

$$\Rightarrow \begin{cases} \dfrac{dx}{dt} = v \\[2mm] \dfrac{dv}{dt} = -\dfrac{Gm_1}{x^2} + \dfrac{Gm_2}{(x-a)^2} \end{cases}$$

<u>fixed point</u>

$\frac{dx}{dt} = 0 \Leftrightarrow \underline{\underline{v=0}}$ & $\frac{dv}{dt} = 0 \Longleftrightarrow -\frac{Gm_1}{x^2} + \frac{Gm_2}{(x-a)^2} = 0$

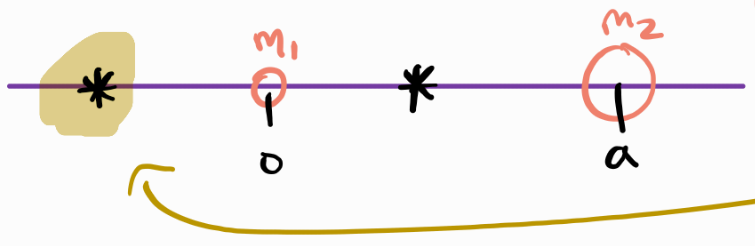
$$\Rightarrow \frac{m_1}{x^2} = \frac{m_2}{(x-a)^2} \Rightarrow m_1(x^2 - 2ax + a^2) = m_2 x^2$$

$$\Rightarrow x^2\left(1 - \frac{m_2}{m_1}\right) - 2ax + a^2 = 0$$

$$\Rightarrow x = \frac{+2a \pm \sqrt{4a^2 - 4a^2\left(1 - \frac{m_2}{m_1}\right)}}{2\left(1 - \frac{m_2}{m_1}\right)}$$

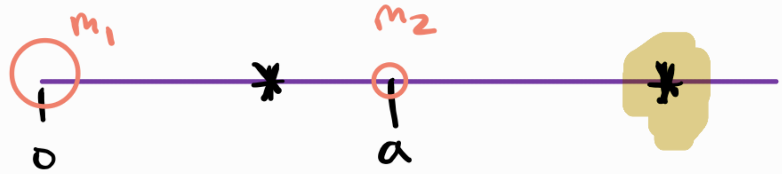
$$= a\left(\frac{1 \pm \sqrt{+m_2/m_1}}{1 - \frac{m_2}{m_1}}\right) = a\left(\frac{m_1 \pm \sqrt{m_1}\sqrt{m_2}}{m_1 - m_2}\right)$$

if $m_2 > m_1 \Rightarrow$ one pos, one neg f.p.



these ones are not real
⇒ direction of force changes

if $m_1 > m_2 \Rightarrow$ two positive s.p.



Stability

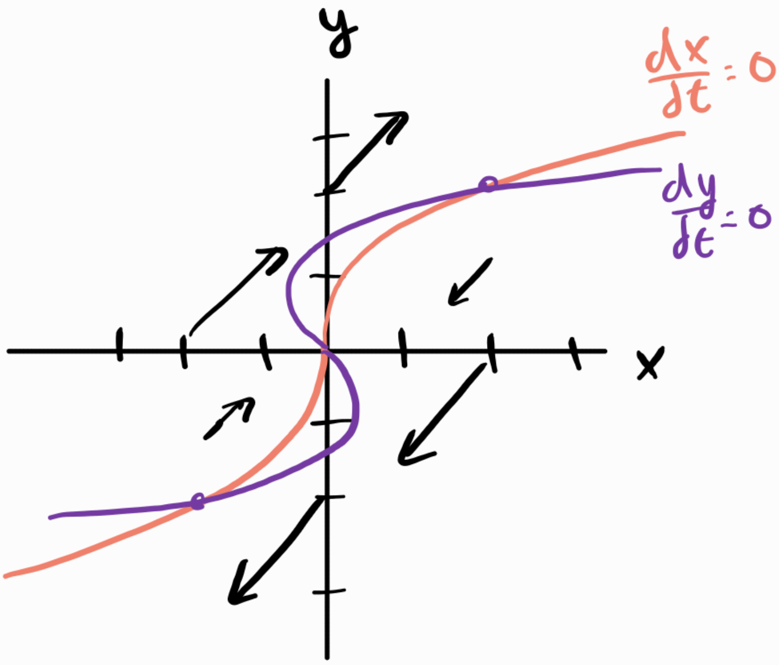
$$J = \begin{pmatrix} 0 & 1 \\ +\frac{2Gm_1}{x^3} - \frac{2Gm_2}{(x-a)^3} & 0 \end{pmatrix}$$

$$J\left(a\left(\frac{m_1 - \sqrt{m_1}\sqrt{m_2}}{m_1 - m_2}\right)\right) = \begin{pmatrix} 0 & 1 \\ \frac{2G(\sqrt{m_1}+\sqrt{m_2})^4}{a^3\sqrt{m_1}\sqrt{m_2}} & 0 \end{pmatrix}$$

$$\tau = 0, \quad \Delta = -\frac{2G(\sqrt{m_1}+\sqrt{m_2})^4}{a^3\sqrt{m_1}\sqrt{m_2}} < 0$$ saddle

Example

$$\begin{cases} \frac{dx}{dt} = y^3 - 4x \\ \frac{dy}{dt} = y^3 - y - 3x \end{cases}$$



fixed points

$(-2,-2), (0,0), (2,2)$

phase portrait

$v(x,y) = \langle y^3 - 4x, y^3 - y - 3x \rangle$

$v(2,0) = \langle -8, -6 \rangle$ $\quad v(0,-2) = \langle -8, -6 \rangle$

$v(-2,0) = \langle 8, 6 \rangle$ $\quad v(2,1) = \langle 4, 3 \rangle$

$v(0,2) = \langle 8, 6 \rangle$ $\quad v(-2,-1) = \langle -4, -3 \rangle$

Stability

$$J = \begin{bmatrix} -4 & 3y^2 \\ -3 & 3y^2-1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} -4 & 0 \\ -3 & -1 \end{bmatrix} \to \tau = -5, \quad \Delta = 4 \to \tau^2 - 4\Delta = 25 - 16 > 0$$

Stable node

$$J(\pm 2, \pm 2) = \begin{bmatrix} -4 & 12 \\ -3 & 11 \end{bmatrix} \to \tau = 7, \quad \Delta = -8$$

saddles

an invariant trajectory

Consider if $x = y$. Then $v(x,y) = v(x,x) = (x^3 - 4x, x^3 - 4x)$

$\rightsquigarrow$ you stay on line $x = y$. It is invariant!

What happens long time?

Define: $V = x - y$

$$\frac{dv}{dt} = \frac{dx}{dt} - \frac{dy}{dt} = y^3 - 4x - y^3 + y + 3x = y - x$$

$$\Rightarrow \frac{dv}{dt} = -V \rightsquigarrow \text{so as } t \to \infty, \ V \to 0$$

$\Rightarrow$ This means $|x(t) - y(t)| \to 0$ for all trajectories

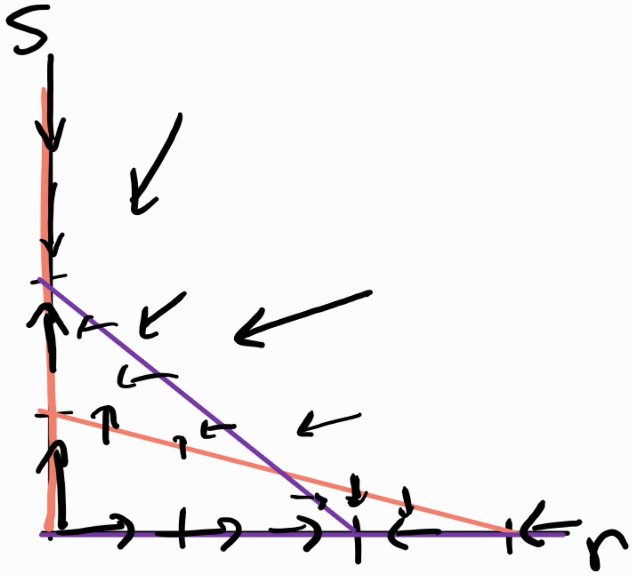
Example Rabbits versus Sheep

$$\frac{dr}{dt} = r(3-r-3s) = \underbrace{r(3-r)}_{\text{logistic growth with carrying capacity}} \underbrace{-3rs}_{\text{competition!}}$$

$$\frac{ds}{dt} = s(2-r-s)$$

$$= s(2-s) - sr$$

sheep consume the resource more efficiently



nullclines

$\frac{dr}{dt} = 0$ when $r = 0$

$s = -\frac{r}{3} + 1$

$\frac{ds}{dt} = 0$ when $s = 0$

$s = 2 - r$

$$V(r,s) = \langle r(3-r-3s), s(2-r-s) \rangle$$

$$v(0,s) = \langle 0, s(2-s) \rangle$$

$$v(r,0) = \langle r(3-r), 0 \rangle$$

$$v(2,2) = \langle -10, -4 \rangle$$

$$v(2,1) = \langle -4, -1 \rangle$$

$$V(1,2) = \langle -4, -2 \rangle$$

$$v(1,3) = \langle -9, -6 \rangle$$

fixed pts

$$(0,2), (3,0), (3/2, 1/2), (0,0)$$

stability

$$J = \begin{pmatrix} 3-2r-3s & -3r \\ -s & 2-r-2s \end{pmatrix}$$

unstable node

$$J(0,0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\Delta = 6, \quad \tau = 5$$

$$J(0,2) = \begin{bmatrix} -3 & 0 \\ -2 & -2 \end{bmatrix} \rightarrow \tau = -5, \quad \rightarrow \Delta = 6$$

$\tau^2 - 4\Delta = 25 - 24 > 0$

stable node

$$J(3,0) = \begin{bmatrix} -3 & -9 \\ 0 & -1 \end{bmatrix} \rightarrow \tau = -4, \quad \rightarrow \Delta = 3$$

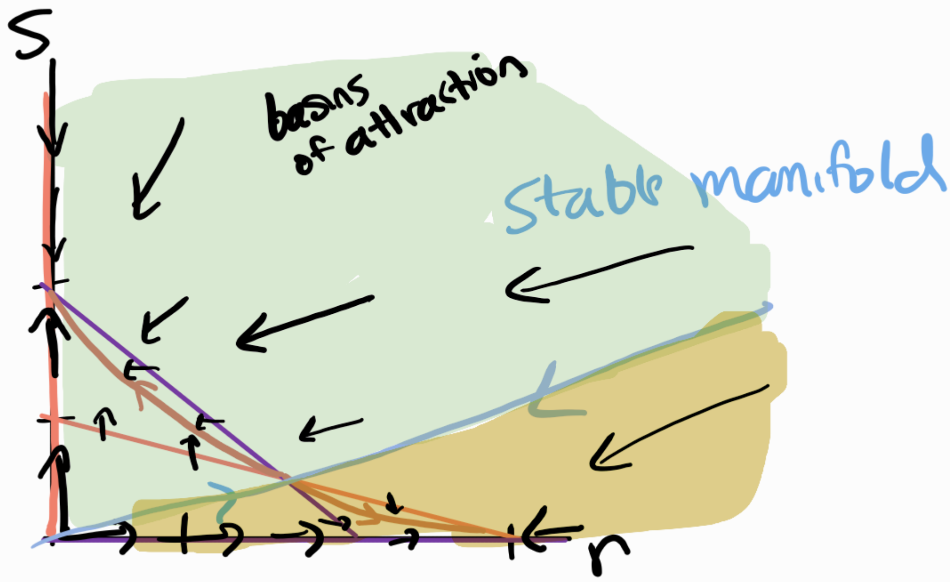
$\tau^2 - 4\Delta = 16 - 12 > 0$

stable node

$$J(3/2, 1/2) = \begin{bmatrix} -3/2 & -9/2 \\ -1/2 & -1/2 \end{bmatrix} \rightarrow \tau = -2, \quad \rightarrow \Delta = -3/2$$

saddle

Q What does this mean for the rabbits & sheep long term in most I.C.?



Example Epidemic model

$k, l > 0$

Let $\begin{cases} x: \text{number of healthy people} \\ y: \text{number of sick people} \end{cases}$ z: number of dead ppl

Consider the following epidemic model (Kermack & McKendrick 1927)

$$\begin{cases} \frac{dx}{dt} = -kxy \\ \frac{dy}{dt} = +kxy - ly \\ \frac{dz}{dt} = +ly \end{cases}$$

$\boxed{Q}$ What do these terms represent?

Healthy ppl get sick at a rate proportional to x & y. This assumes healthy & sick ppl encounter each other at a rate $\propto$ their numbers $\oplus$ every encounter has some chance of disease transmission

$$Y + X \xrightarrow{k} 2Y$$

$\boxed{Q}$ What do these terms represent?

A sick person has some constant probability of dying $Y \xrightarrow{l} Z$

Notice that $\frac{dx}{dt} + \frac{dy}{dt} + \frac{dz}{dt} = 0 \Rightarrow$ $\boxed{Q}$ What does this mean?

Total population $(x+y+z)$ is constant

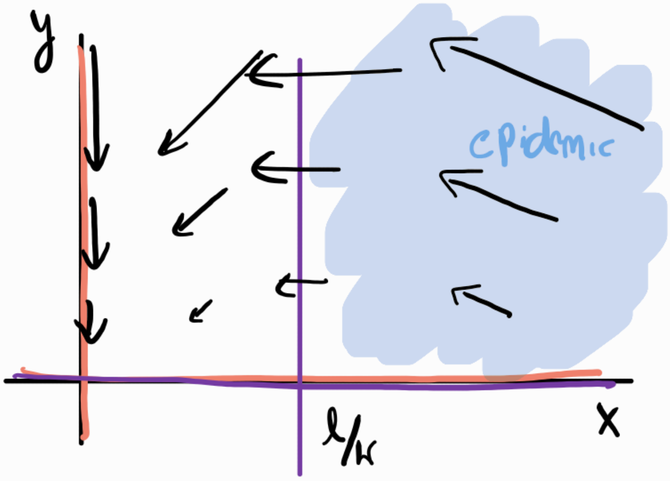
$\Rightarrow$ Let's ignore z, since its dynamics do not matter

$$\begin{cases} \frac{dx}{dt} = -kxy \\ \frac{dy}{dt} = +kxy - ly \end{cases}$$

→ x nullclines: $x=0$, $y=0$

→ y nullclines: $x = \frac{l}{k}$, $y=0$

Q What does it mean biologically that all of the line $y=0$ are s.s.?



If there are no sick ppl ($y=0$), no one can get sick! Number of healthy ppl is constant

Phase portrait

$v(0,y) = \langle 0, -ly \rangle$

$v(\frac{l}{k}, y) = \langle -ly, 0 \rangle$

$v(\frac{l}{2k}, y) = \langle -\frac{l}{2}y, -\frac{l}{2}y \rangle$

$v(\frac{2l}{k}, y) = \langle -2ly, ly \rangle$

fixed points

$y=0$, $x \geq 0$

$$J = \begin{bmatrix} -ky & -kx \\ ky & kx - l \end{bmatrix}$$

Stability

$$J(x,0) = \begin{bmatrix} 0 & -kx \\ 0 & kx - l \end{bmatrix}$$

→ $\tau = kx - l$

→ $\Delta = 0$ ⇒ non isolated f.p.

Q what kind?

If $0 < x < \frac{l}{k}$, $\tau < 0$, attracting

If $\frac{l}{k} < x$, $\tau > 0$, repelling

Q In which x_0, y_0 cases do you have an epidemic where y initially increases?

Quiz

Consider a population model where $x(t)$ is the number of healthy people and $y(t)$ is the number of sick people. Suppose they follow the dynamical system:

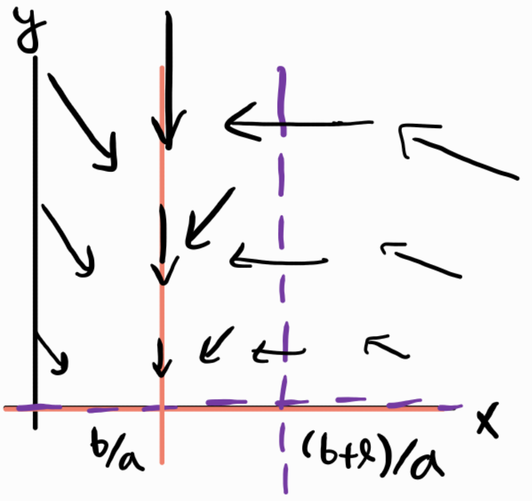
$$\begin{cases} \frac{dx}{dt} = by - axy \\ \frac{dy}{dt} = axy - by - ly \end{cases} \qquad a, b, l \geq 0$$

a) In words, explain what these equations say about the healthy and sick people.

Healthy ppl become sick with some constant probability when they encounter sick ppl. Sick ppl get better with some probability. Sick ppl die with some constant rate

b) Find and sketch the nullclines.

c) Sketch the phase portrait



$$\begin{cases} \frac{dx}{dt} = 0 \rightarrow y = 0 \text{ or } x = \frac{b}{a} \\ \frac{dy}{dt} = 0 \rightarrow y = 0 \text{ or } x = \frac{b+l}{a} \end{cases}$$

phase portrait

$V(0,y) = \langle by, -(b+\ell)y \rangle$

$V(b/a, y) = \langle 0, -\ell y \rangle$

$V(\frac{b+\ell}{a}, y) = \langle -\ell y, 0 \rangle$

$V(\frac{2b+\ell}{a}, y) = \langle -(b+\ell)y, by \rangle$

$V(\frac{b+\ell/2}{a}, y) = \langle -\frac{\ell}{2}y, -\frac{\ell}{2}y \rangle$

d) Find the steady states and use the Jacobian to classify them.

$x = \text{any},\ y = 0$

$$J = \begin{bmatrix} -ay & b-ax \\ ay & ax-b-\ell \end{bmatrix} \xrightarrow{y=0} \begin{bmatrix} 0 & b-ax \\ 0 & ax-b-\ell \end{bmatrix} \to \Delta = 0$$

$$\Rightarrow \begin{cases} 0 < x < \frac{+\ell}{a} \Rightarrow \text{attracting nonisolated s.p} & \tau < 0 \\ \frac{b+\ell}{a} < x \Rightarrow \text{repelling nonisolated s.p} & \tau > 0 \end{cases}$$

e) Explain qualitatively what would happen if you started with $x_0 = \frac{2b+\ell}{a}$ healthy ppl & $y_0 = \frac{\ell}{a}$ sick ppl.

First more ppl would get sick, then after there are less than $\frac{b+\ell}{a}$ healthy ppl, the number of sick people falls until there are none.