HW

$$\frac{\partial C}{\partial T} = \overbrace{kC}^{\text{reaction}} + \underbrace{D\frac{\partial^2 C}{\partial X^2}}_{\text{diffusion}} \quad \text{on } -L \leq X \leq L$$

$$\oplus \begin{cases} C(-L,T) = 0 \\ C(+L,T) = 0 \end{cases}$$

Nondimensionalize

parameters: k, D, L $\nearrow$ $[k] \sim 1/[T]$, $[D] \sim [X]^2/[T]$, $[L] \sim [X]$

$$X = Lx, \quad T = \frac{1}{k}t, \quad C = C_0 c$$

$$\Rightarrow \frac{C_0}{1/k}\frac{\partial c}{\partial t} = kC_0 c + \frac{DC_0}{L^2}\frac{\partial^2 c}{\partial x^2}$$

$$\longrightarrow \boxed{\frac{\partial c}{\partial t} = c + \delta \frac{\partial^2 c}{\partial x^2}} \quad \text{where } \delta = \frac{D}{kL^2}$$

$$\oplus \begin{cases} c(-1,t) = 0 \\ c(+1,t) = 0 \end{cases} \quad \text{B.C.}$$

Homogeneous soln: a soln s.t. $\frac{\partial c}{\partial x} = 0$, $c(x,t) = c(t)$

Stationary soln: a soln s.t. $\frac{\partial c}{\partial t} = 0$, $c(x,t) = c(x)$

$\Rightarrow$ homogeneous, stationary soln: $c(x,t) = c_1$ [constant]

$\rightsquigarrow$ B.C. require $c_1 = \underline{\underline{c(x,t) = 0}}$

Stability

Assume small perturbation $\varepsilon(x,t) = \varepsilon_0 \cos(qx)\exp(\lambda t)$

$\rightsquigarrow$ B.C. require $\varepsilon(-1,t) = 0$ & $\varepsilon(+1,t) = 0$

$\Rightarrow q = \frac{\pi}{2} + \pi n,\ n \in \mathbb{Z}$ satisfies this

finding λ:

$$\frac{\partial c}{\partial t} = c + \delta \frac{\partial^2 c}{\partial x^2} \rightarrow \lambda \varepsilon(x,t) = \varepsilon(x,t) - \delta q^2 \varepsilon(x,t)$$

$$\Rightarrow \boxed{\lambda = 1 - \delta q^2}$$ Dispersion relation

Stable if $\underbrace{\lim_{t\to\infty} \varepsilon(x,t) = 0}$, else unstable

$\Updownarrow$

need $\exp(\lambda t) \to 0$

$\Updownarrow$

need $\lambda < 0 \rightsquigarrow 1 < \delta q^2$ ✩

Many choices for $q \rightarrow$ which one are we worried about the most?

Problem if **any** q exceed ✩

$\Rightarrow$ if smallest q satisfies ✩, all O.K.

$\Rightarrow$ need to see when ✩ holds for $q = \frac{\pi}{2}$

$$1 < \delta \frac{\mu^2}{4} \xrightarrow[\text{back in dimensions}]{} 1 < \frac{D}{k}\left(\frac{\mu}{2L}\right)^2$$

Q How can we explain this?

$$\Rightarrow \boxed{D > k\left(\frac{2L}{\mu}\right)^2}$$

Stability condition

In order to get a nontrivial soln to grow from close to $c(x,t)=0$, you need small diffusion, large reaction rate, and/or a large domain.

This comes from how at the boundary, concentration is absorbed ~ if diffusion is too fast, any increase in concentration quickly "hits" the boundary and is absorbed.

Example Kuramoto-Sivashinsky eqn

$$\frac{\partial u}{\partial t} = -\frac{\partial^4 u}{\partial x^4} - \frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 \quad ☆$$

(this equation describes flame front propagation)

Stationary, homogeneous soln:

$u = \text{constant} \rightsquigarrow$ let us consider $u=0$,
the case without fire

Stability: consider a perturbation

$$\varepsilon(x,t) = \varepsilon_0 \cos(kx)\exp(\lambda t)$$

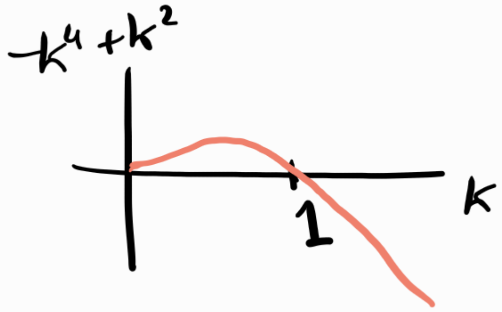
$$\frac{\partial \varepsilon}{\partial t} = \lambda\varepsilon,$$

$$\frac{\partial \varepsilon}{\partial x} = -k\varepsilon, \quad \frac{\partial^2 \varepsilon}{\partial x^2} = -k^2\varepsilon, \quad \frac{\partial^3 \varepsilon}{\partial x^3} = +k^3\varepsilon, \quad \frac{\partial^4 \varepsilon}{\partial x^4} = +k^4\varepsilon$$

$$\Longrightarrow ☆ = \lambda\varepsilon = -k^4\varepsilon + k^2\varepsilon + \cancel{O(\varepsilon^2)}^{\,0}$$

$$\Longrightarrow \boxed{\lambda = -k^4 + k^2} \quad \text{Dispersion Relation}$$

Stable if $\varepsilon \to 0$ as $t \to \infty \Longrightarrow$ if $\lambda < 0 \ \forall k$



$\Longrightarrow$ for $k<1$, $\lambda>0$

$\Longrightarrow$ Unstable equilibrium

Example sine-Gordon eqn
(used initially to study surfaces of negative curvature)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} - \sin(u) \quad \star$$

Stationary homogeneous soln: $u \equiv 0$

Stability: $u = \varepsilon(x,t)$

linearize $\star$: $\sin(\varepsilon) \sim \varepsilon + O(\varepsilon^2)$

$$\rightsquigarrow \frac{\partial^2 \varepsilon}{\partial t^2} = c^2 \frac{\partial^2 \varepsilon}{\partial x^2} - \varepsilon$$

wave-train solns

Let $\varepsilon(x,t) = \varepsilon_0 \exp(i(kx + \lambda t))$

$$\Rightarrow -\lambda^2 \varepsilon = -c^2 k^2 \varepsilon - \varepsilon$$

$$\Rightarrow \lambda^2 = c^2 k^2 + 1 \rightsquigarrow \boxed{\lambda = \pm\sqrt{1 + c^2 k^2}}$$

note: $\frac{\lambda(k)}{k}$ is nonconstant $\Rightarrow$ different wavelengths travel at diff speeds $\Rightarrow$ dispersion

Quiz Consider the diffusion eqn for mass concentration C:

$$\frac{\partial C}{\partial T} = D\frac{\partial^2 C}{\partial X^2} \quad \text{on } 0 \le X \le L$$

with reflecting boundary conditions

$$\frac{\partial C}{\partial X}(X=0,T)=0 \quad \& \quad \frac{\partial C}{\partial X}(X=L,T)=0$$

Assume that initially, the total mass inside the domain is M_0, i.e.

$$\int_0^L C(X,T=0)\,dX = M_0$$

a) Scale and nondimensionalize the equation

Parameters: $M_0, L, D \rightarrow [D] \sim [X]^2/[T]$

Variables: C, X, T $\qquad [M_0] \sim [C][L]$

$$\begin{cases} X = Lx \\ T = \frac{L^2}{D}t \\ C = \frac{M_0}{L}c \end{cases} \Rightarrow \frac{M_0/L}{L^2/D}\frac{\partial c}{\partial t} = \frac{D\,M_0/L}{L^2}\frac{\partial^2 c}{\partial x^2}$$

$$\boxed{\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}} \oplus \boxed{\begin{array}{l}\frac{\partial c}{\partial x}(0,t)=0 \\ \frac{\partial c}{\partial x}(1,t)=0\end{array}}$$

$$\int_0^1 \frac{M_0}{L}c(x,t=0)\,L\,dx = M_0 \Rightarrow \underline{\int_0^1 c(x,t=0)\,dx = 1}$$

b) Find the homogeneous stationary soln to the nondimensional system

$\Rightarrow C(x,t) = \text{constant} = C_0$

$\oplus \int_0^1 C_0\, dx = 1 \Rightarrow C_0 = 1$

c) Consider a perturbation of the form

$$\varepsilon(x,t) = \varepsilon_0 \cos(kx)\exp(\lambda t)$$

Use the boundary conditions to find k.

$\frac{\partial \varepsilon}{\partial x}(x,t) = -\varepsilon_0 k \sin(kx)\exp(\lambda t)$

$\frac{\partial \varepsilon}{\partial x}(0,t) = 0$

$\frac{\partial \varepsilon}{\partial x}(1,t) = -\sin(k)\left[\varepsilon_0 k \exp(\lambda t)\right] = 0$

$k = \pi \mathbb{Z}$

d) Find the dispersion relation between λ & k [use the nondimensional eqn you derived in part (a)]

$c = 1 + \varepsilon$

$\Rightarrow \frac{\partial c}{\partial t} = \lambda \varepsilon(x,t)$

$\frac{\partial^2 c}{\partial x^2} = -k^2 \varepsilon(x,t)$

$\Rightarrow \lambda = -k^2$

e) [Bonus] Is your solution from (b) stable? Why?

Yes b/c $\forall$ k, $\lambda < 0 \Rightarrow$ all $\varepsilon(x,t)$ decay