Recall: 2D ODEs

$$\begin{cases} \frac{dx}{dt} = f(x,y) \\ \frac{dy}{dt} = g(x,y) \end{cases} \rightsquigarrow \overset{\text{Linear}}{\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \underline{\underline{A}} \begin{bmatrix} x \\ y \end{bmatrix}}$$

suppose matrix $\underline{\underline{A}}^{2\times 2}$ has eigenvalues λ_1, λ_2

$\rightsquigarrow$ we know from lin alg that:

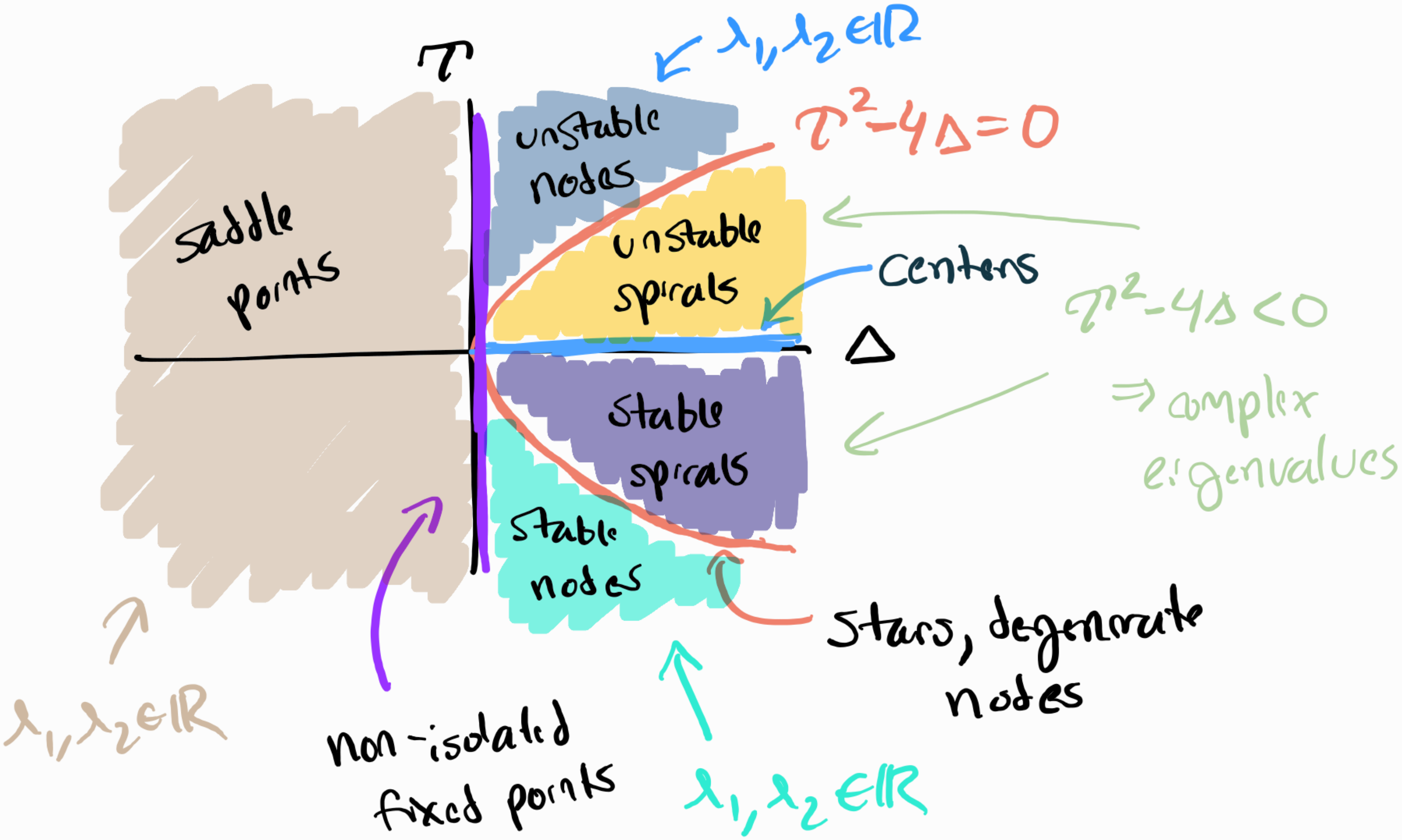
$$\begin{cases} tr(\underline{A}) = \tau = \lambda_1 + \lambda_2 \\ det(\underline{A}) = \Delta = \lambda_1 \lambda_2 \end{cases}$$

note: both $\tau, \Delta \in \mathbb{R}$

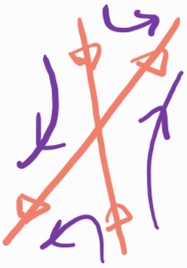
$\rightarrow$ Both λ_1, λ_2 and τ, Δ classify s.s.

Note also:

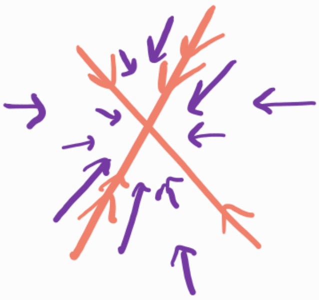
$$\lambda_{1,2} = \frac{1}{2}\left(\tau \pm \sqrt{\tau^2 - 4\Delta}\right)$$



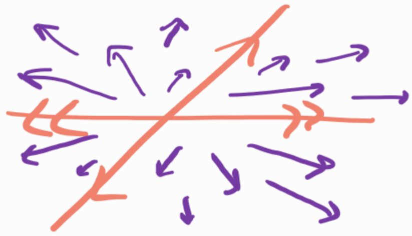
Saddle



stable node



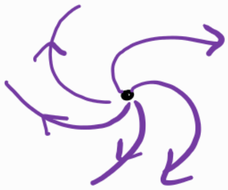
unstable node



center

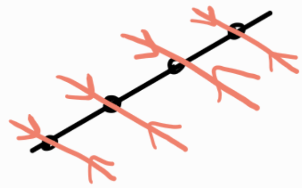
stable spiral

unstable spiral



non-isolated f.p.

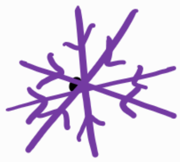
$\Delta = 0 \rightsquigarrow$ at least one $\lambda \equiv 0$



Stars

$\lambda_1 = \lambda_2 = \tau/2 \neq 0$

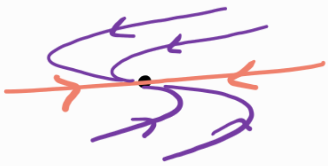
⊕ two distinct eigenvectors



all trajectories are straight lines

degenerate node

⊕ one distinct eigenvector



as $t \to \infty$, trajectories are $\parallel$ to eigenvector

Example

model for dynamics of love affairs (Strogatz 1988)

$$\begin{cases} \dot{R} = aR + bJ \\ \dot{J} = bR + aJ \end{cases} \quad \text{with } a<0,\ b>0$$

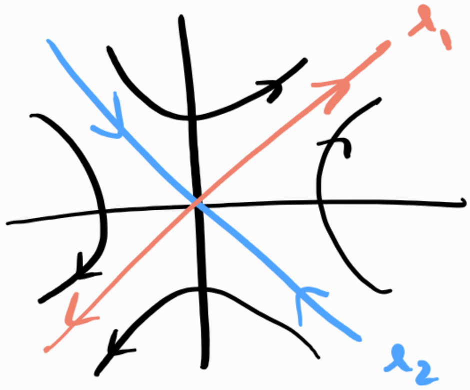
$$\Downarrow$$

$$\begin{bmatrix} a & b \\ b & a \end{bmatrix} \to \begin{cases} \tau = 2a < 0 \\ \Delta = a^2 - b^2 \end{cases}$$

$$\to \begin{cases} \lambda_1 = a+b,\ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 = a-b,\ v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{cases}$$

Case 1 $a^2 < b^2$

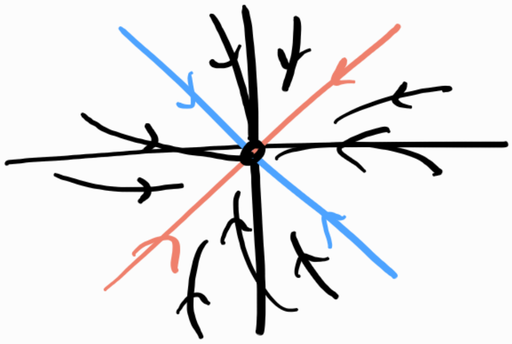
$\Rightarrow \Delta < 0 \Rightarrow$ saddle pt



Generally, either they become madly in love or hate each others' guts

Case 2 $a^2 > b^2$

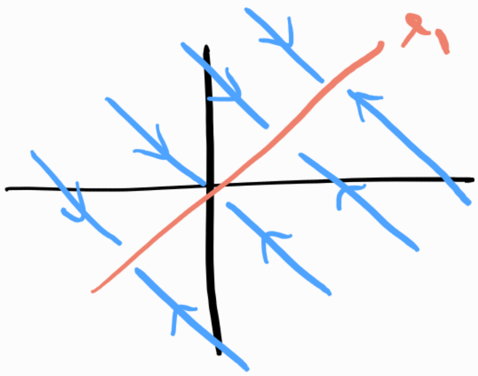
$\Rightarrow \Delta > 0 \Rightarrow$ stable node



The relationship will sizzle out

Case 3 $a^2 = b^2$

$\Rightarrow \Delta = 0 \Rightarrow$ nonisolated Sp



$\lambda_1 = 0$

$\lambda_2 = 2a = -2b$

Their relationship will stable with mutual feelings of love or hate

Linearization (analogous to 1D)

Let

$$u = x - x^*$$
$$v = y - y^*$$

u, v small

Taylor series

$$\Rightarrow \frac{\partial u}{\partial t} = \frac{\partial x}{\partial t} = f(x,y) = f(u + x^*, v + y^*)$$

$$\approx \cancel{f(x^*, y^*)}^{\,0} + u\frac{\partial f}{\partial x}(x^*, y^*) + v\frac{\partial f}{\partial y}(x^*, y^*)$$

$$\cancel{+ \frac{u^2}{2}\frac{\partial^2 f}{\partial x^2}(x^*, y^*) + 2uv\frac{\partial^2 f}{\partial x \partial y}(x^*, y^*)}$$

$$\cancel{+ \frac{v^2}{2}\frac{\partial^2 f}{\partial y^2}(x^*, y^*) + O(u^3, uv^2, u^2v, v^3)}$$

small!

$$\Rightarrow \begin{cases} \frac{\partial u}{\partial t} \approx u\,\underline{\frac{\partial f}{\partial x}(x^*, y^*)} + v\,\underline{\frac{\partial f}{\partial y}(x^*, y^*)} \\ \frac{\partial v}{\partial t} \approx u\,\underline{\frac{\partial g}{\partial x}(x^*, y^*)} + v\,\underline{\frac{\partial g}{\partial y}(x^*, y^*)} \end{cases}$$

linear system

constants!

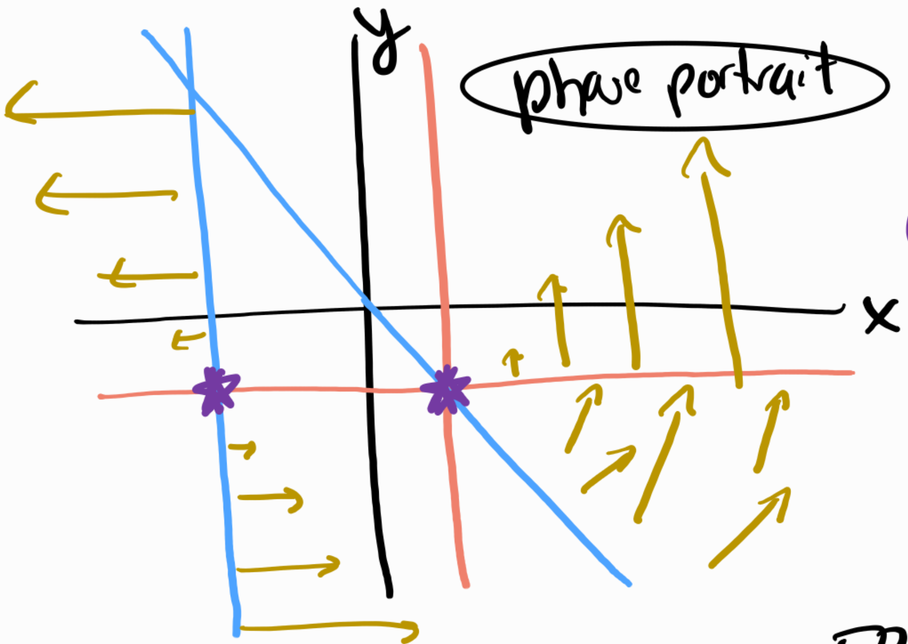
Jacobian

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} = J\Big|_{x^*, y^*}\begin{bmatrix} u \\ v \end{bmatrix} \qquad J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$\Longrightarrow$ find eigenvalues, classify them

Example

$$\begin{cases} \dfrac{dx}{dt} = (x-1)(y+1) \\[2ex] \dfrac{dy}{dt} = (y+x)(x+2) \end{cases} \Longrightarrow J = \begin{bmatrix} y+1 & x-1 \\ y+2x+2 & x+2 \end{bmatrix}$$



2 fixed pts:

$$(x^*, y^*) = (-2,-1)$$
$$= (1,-1)$$

$$J(-2,-1) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \begin{array}{l} \nearrow \tau = 0 \\ \searrow \Delta = -9 \end{array} \Rightarrow \underline{\underline{\text{saddle}}}$$

$$J(1,-1) = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \begin{array}{l} \nearrow \tau = 3 \\ \searrow \Delta = 0 \end{array} \Rightarrow \underline{\underline{\text{"non isolated f.p."}}}$$

linearization can be inconclusive

Example

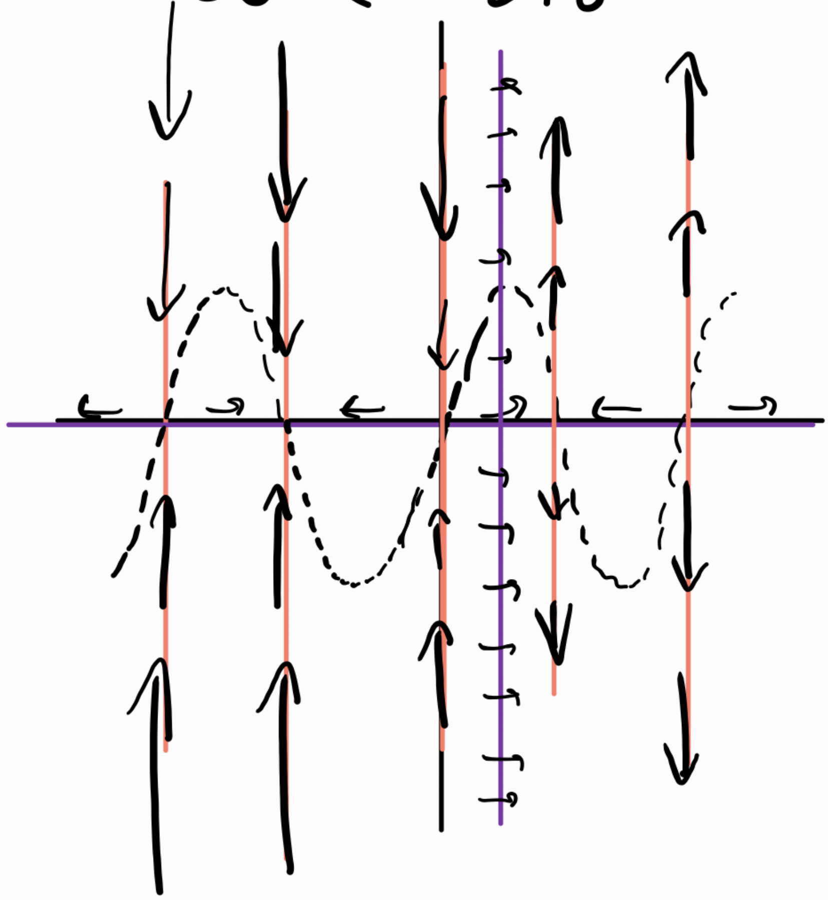
$\to x = \pi j, \ j \in \mathbb{Z}$

$$\begin{cases} \dfrac{dx}{dt} = \sin(x) \\[2mm] \dfrac{dy}{dt} = \left(x - \dfrac{\pi}{2}\right) y \end{cases}$$

$\to x = +\pi/2$

$y = 0$

$$V = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$



$$V(0,y) = \left\langle 0, -\frac{\pi}{2} y \right\rangle$$

$$V(\pi,y) = \left\langle 0, \frac{\pi}{2} y \right\rangle$$

$$V(-\pi,y) = \left\langle 0, -\frac{3\pi}{2} y \right\rangle$$

$$V\left(\frac{\pi}{2}, y\right) = \langle -1, 0 \rangle$$

$$V(x,0) = \langle \sin(x), 0 \rangle$$

fixed pts: $(x^*, y^*) = (\pi j, 0) \quad j \in \mathbb{Z}$

$$J = \begin{bmatrix} \frac{\partial}{\partial x}(\sin(x)) & \frac{\partial}{\partial y}(\sin(x)) \\ \frac{\partial}{\partial x}\left(\left(x + \frac{\pi}{2}\right) y\right) & \frac{\partial}{\partial y}\left(\left(x + \frac{\pi}{2}\right) y\right) \end{bmatrix} = \begin{bmatrix} \cos(x) & 0 \\ y & x - \pi/2 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -\pi/2 \end{bmatrix}$$

$\to \Delta = -\pi/2 < 0$ saddle

$\to \tau = 1 - \pi/2$

$$J(\mu,0) = \begin{bmatrix} -1 & 0 \\ 0 & \mu/2 \end{bmatrix} \rightarrow \Delta = -\mu/2 < 0, \quad \rightarrow \tau = \frac{\mu}{2} - 1$$

saddle

$$J(-\mu,0) = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{3\mu}{2} \end{bmatrix} \rightarrow \Delta = \frac{3\mu}{2} > 0, \quad \rightarrow \tau = -1 - \frac{3\mu}{2} < 0$$

$$\tau^2 - 4\Delta > 0$$

stable node

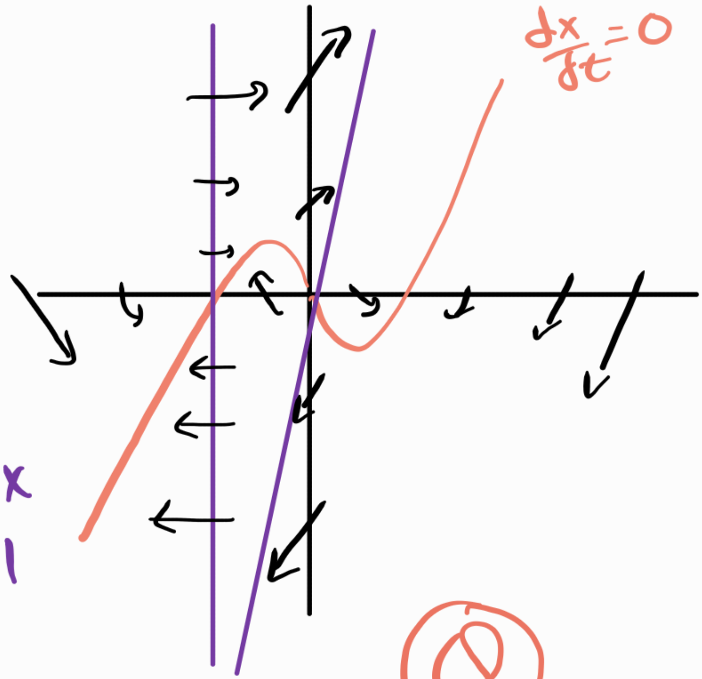
Example

$$\begin{cases} \dfrac{\partial x}{\delta t} = y - (x^3 - x) \\ \dfrac{\partial y}{\delta t} = (y-3x)(x+1) \end{cases}$$

$= yx + y - 3x^2 - 3x$

$y = 3x$

$x = -1$



$V(-1,y) = \langle y, 0 \rangle$

$V(x,0) = \langle -x(x^2-1), -3x(x+1) \rangle$

$x > 1$:	< 0	< 0
$0 < x < 1$:	> 0	< 0
$-1 < x < 0$:	< 0	> 0
$-1 < x$:	> 0	< 0

$V(0,1) = \langle 1, 1 \rangle \qquad V(0,-1) = \langle -1, -1 \rangle$

Q

What do you think is the f.p. classification?

$$J = \begin{bmatrix} -3x^2+1 & 1 \\ y-6x-3 & x+1 \end{bmatrix}$$

S.p.: $(-1,0), (0,0)$

$$J(-1,0) = \begin{bmatrix} -2 & 1 \\ 3 & 0 \end{bmatrix} \to \Delta = -3, \quad \tau = -2$$

saddle

$$J(0,0) = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \to \Delta = 4, \quad \tau = 2$$

$\tau^2 - 4\Delta < 0$

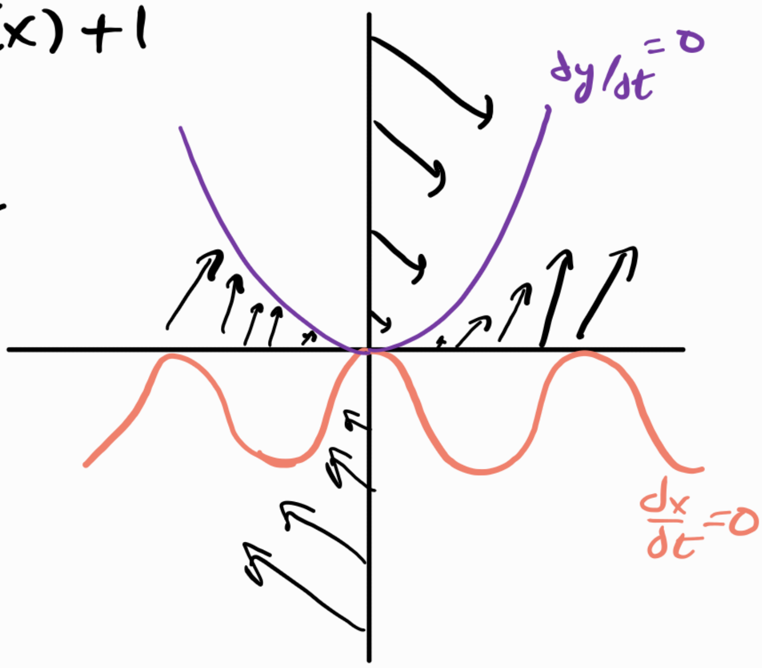
unstable spiral

Example

$$\begin{cases} \dfrac{dx}{dt} = y - \cos(x) + 1 \\ \dfrac{dy}{dt} = x^2 - y \end{cases}$$

$y = x^2$

$y = \cos(x) - 1$



$V(0,y) = \langle y, -y \rangle$

$V(x,0) = \langle \underbrace{-\cos(x)+1}_{\geq 0}, x^2 \rangle$

$$J = \begin{bmatrix} \sin(x) & 1 \\ 2x & -1 \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \to \Delta = 0, \quad \tau = -1$$

"non isolated S.p."

convergent

Consider the dynamical system:

Quiz

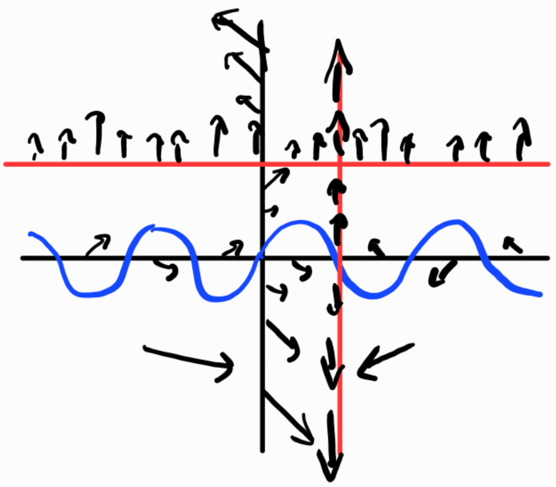
$$\begin{cases} \frac{dx}{\delta t} = (y-2)(x-\pi) \\ \frac{dy}{\delta t} = y - \sin(x) \end{cases}$$

a) Find and sketch the nullclines

X: $y=2$, $x=\pi$

b) Sketch the phase portrait

Y: $y=\sin(x)$



$v(x,y) = \langle (y-2)(x-\pi), y-\sin(x) \rangle$

$v(\pi, y) = \langle 0, y \rangle$

$v(x,2) = \langle 0, 2-\sin(x) \rangle$

$v(0,y) = \langle -\pi(y-2), y \rangle$

$v(x,0) = \langle -2(x-\pi), -\sin(x) \rangle$

$v(-\pi,-2) = \langle 8\pi, -2 \rangle$

$v(2\pi,-2) = \langle -4\pi, -2 \rangle$

c) Find the steady states

$(x^*, y^*) = (\pi, 0)$

d) Use the Jacobian to classify the steady state

$f = (y-2)(x-\pi)$

$g = y - \sin(x)$

$$\Rightarrow J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} y-2 & x-\pi \\ -\cos(x) & 1 \end{bmatrix}$$

$$J(\pi,0) = \begin{bmatrix} -2 & 0 \\ -1 & 1 \end{bmatrix} \begin{matrix} \nearrow \Delta = -2 \\ \searrow \tau = -1 \end{matrix}$$

saddle