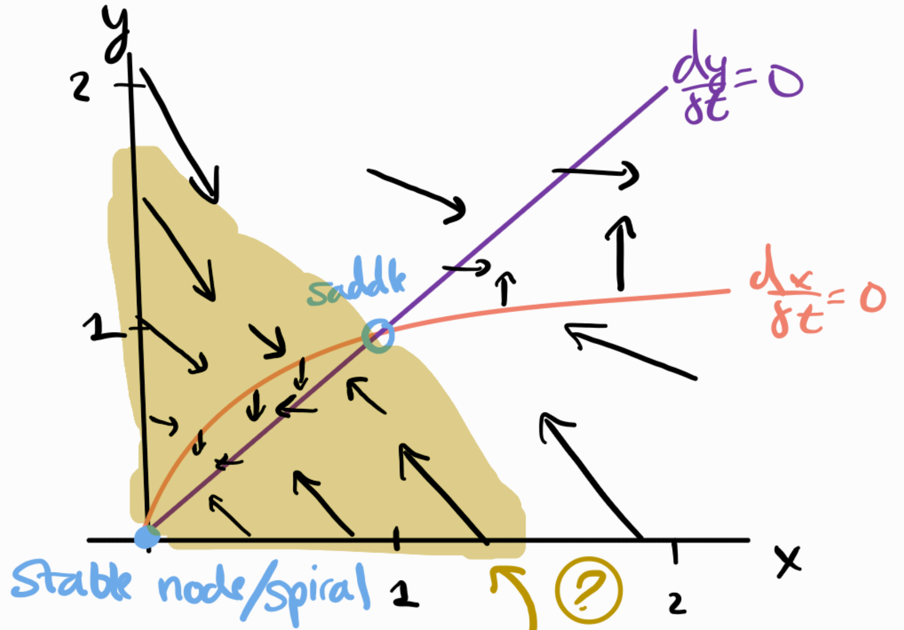
Example Two chemical concentrations (MT2)

$$\begin{cases} \dfrac{dx}{dt} = y^2 - x \\ \dfrac{dy}{dt} = x - y \end{cases}$$



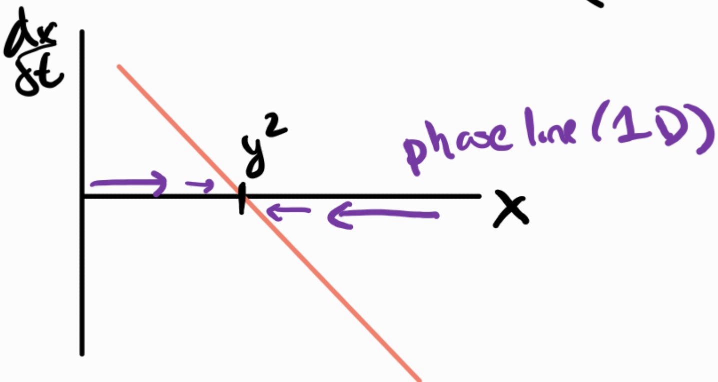
Q What is the basin of attraction for $(0,0)$?

Instead, consider

$$\begin{cases} \dfrac{dx}{dt} = y^2 - x \\ \dfrac{dy}{dt} = \varepsilon(x - y) \end{cases}$$

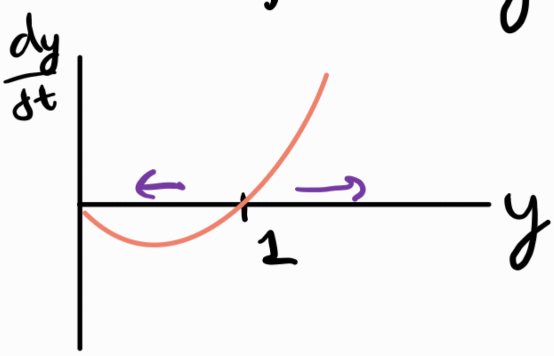
with $\varepsilon \ll 1 \Rightarrow y$ changes very slowly

Think about x first: (treating y as constant)



$\forall x_0, \; x \to y^2$

meanwhile, if $x = y^2$ then $\dfrac{dy}{dt} = \varepsilon(y^2 - y)$



if $y_0 \in (0,1), \; y \to 0$

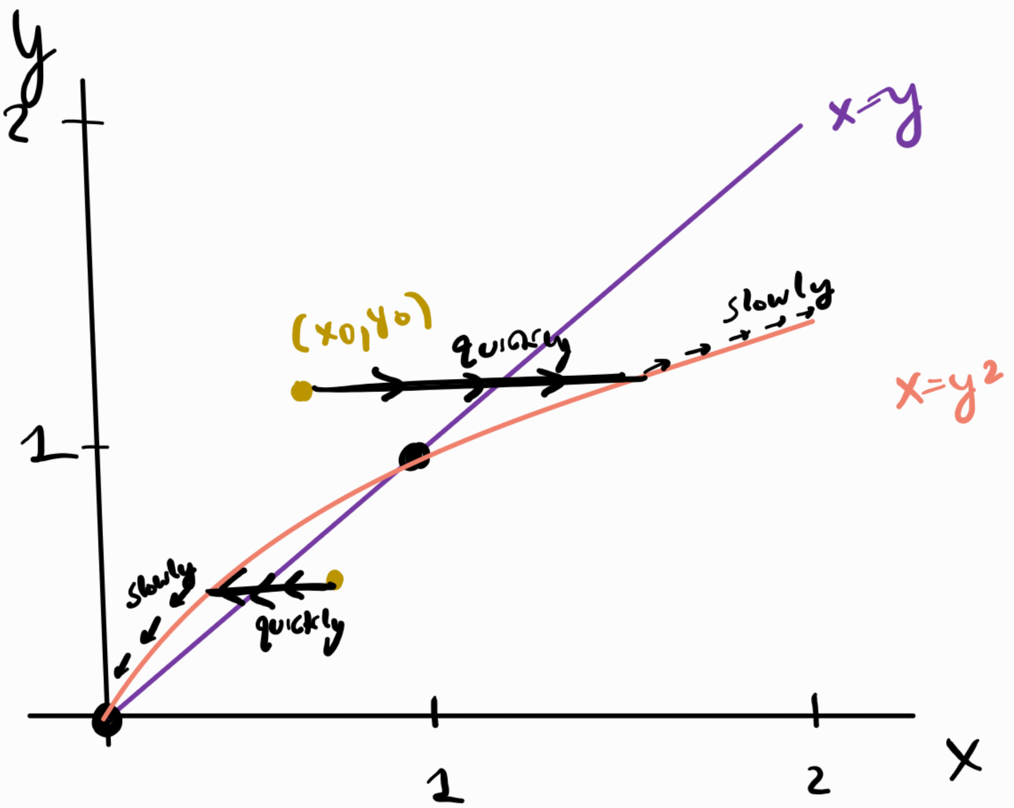
$y_0 \in (1, \infty), \; y \to \infty$

So, given initial condition (x_0, y_0),

① $x \to y_0^2$ quickly

② If $y_0 < 1$, both $x, y \to 0$ slowly

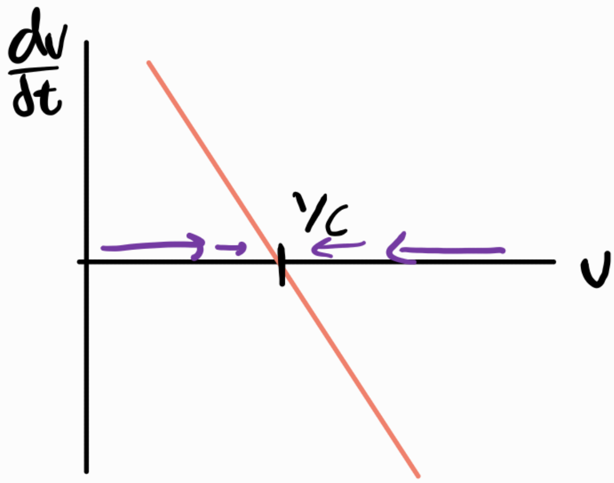
If $y_0 > 1$, both $x, y \to \infty$ slowly



Example (Homework)

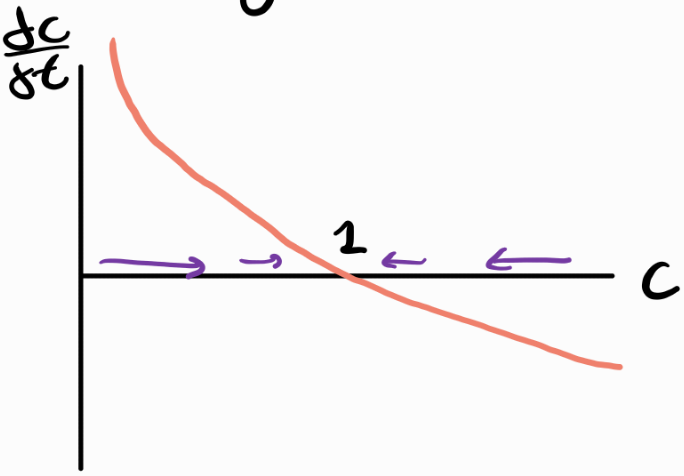
$$\begin{cases} \dfrac{dv}{dt} = \dfrac{1}{c} - v \\[2ex] \dfrac{dc}{dt} = \varepsilon (v - c) \end{cases}$$

assume c constant:



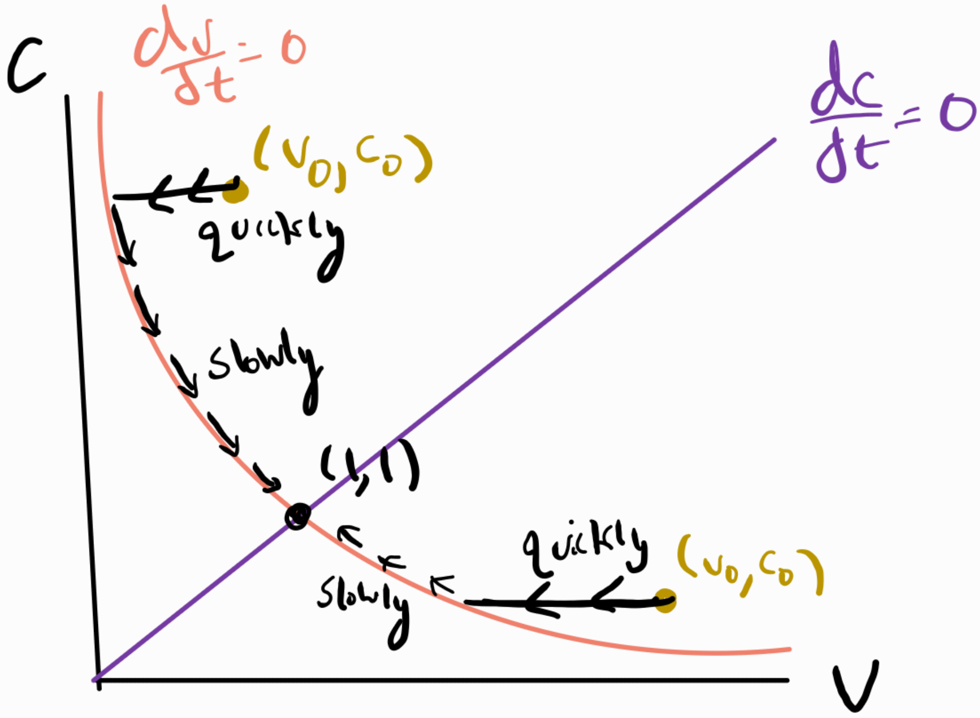
$\forall\, v_0,\ v \to 1/c$ quickly

assuming $v = 1/c$:



$\forall\, c_0,\ c \to 1$ slowly

So given initial c_0, v_0, first $v \to 1/c_0$ very quickly, then $c, v \to 1, 1$ slowly



$\boxed{\text{Example}}$ Van der Pol oscillator

$$\ddot{x} + \mu(x^2-1)\dot{x} + x = 0$$

$\mu >> 1$

(nonlinear electric circuits used in the first radios)

Usually, we would write this as 2 variable system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x + y(1-x^2) \end{cases}$$

Instead, let:

$$w = \dot{x} + \mu\left(\tfrac{1}{3}x^3 - x\right)$$

$$\Rightarrow \dot{w} = \ddot{x} + \mu(x^2\dot{x} - \dot{x}) = \ddot{x} + \mu\dot{x}(x^2-1) = -x$$

So

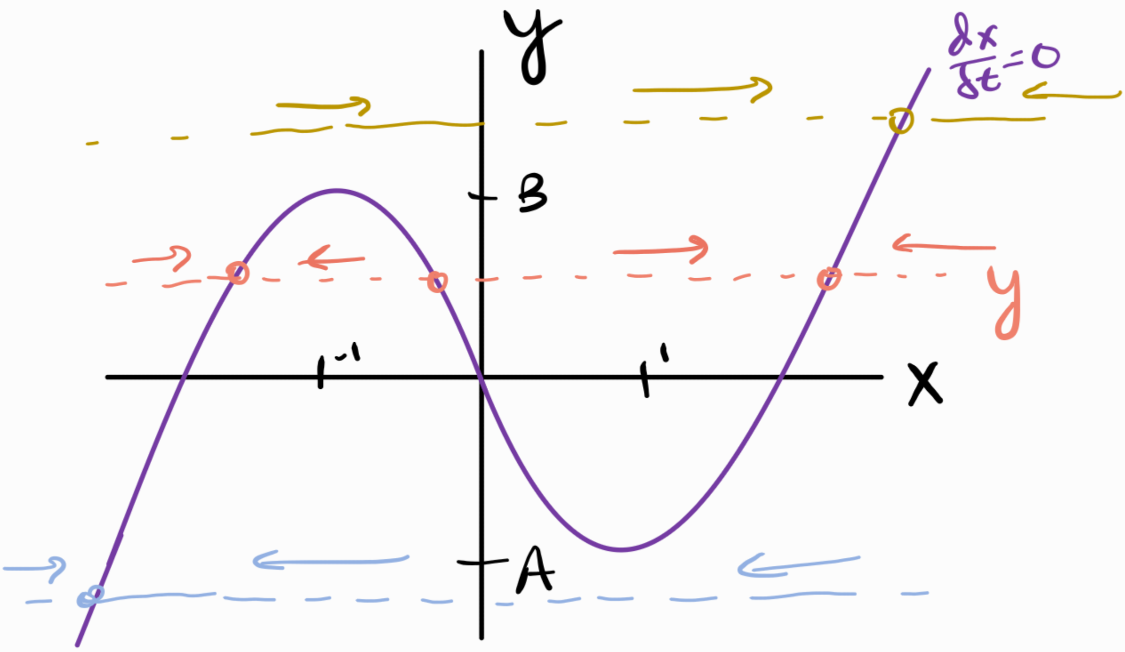
$$\begin{cases} \dot{x} = w - \mu\left(\tfrac{1}{3}x^3 - x\right) \\ \dot{w} = -x \end{cases}$$

large

let us further consider $y = w/\mu$ change of variables

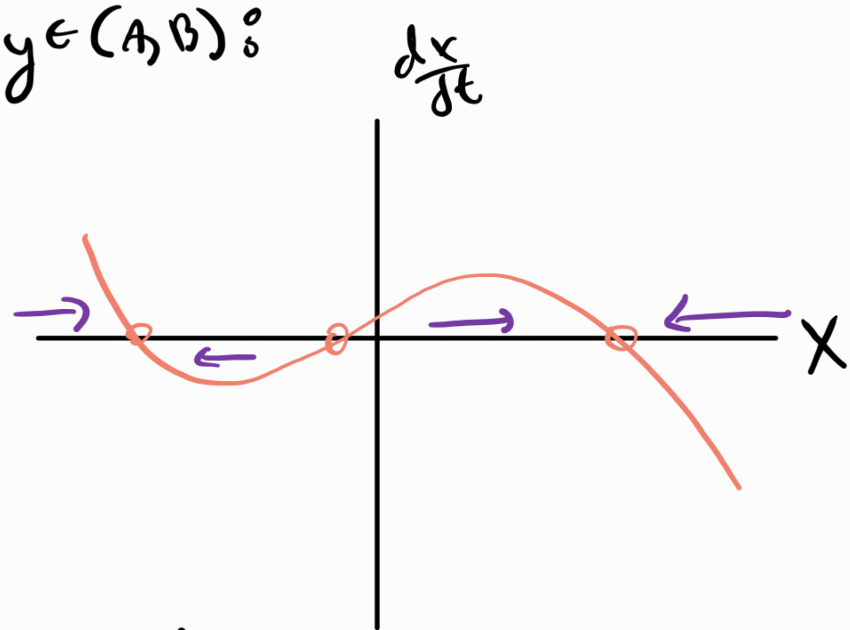
$$\begin{cases} \dot{x} = \mu\left[y - \left(\tfrac{1}{3}x^3 - x\right)\right] & \underline{\underline{\text{fast}}} \quad O(\mu) \\ \dot{y} = -\frac{1}{\mu}x & \underline{\underline{\text{slow}}} \quad O(\mu^{-1}) \end{cases}$$

widely separated timescales

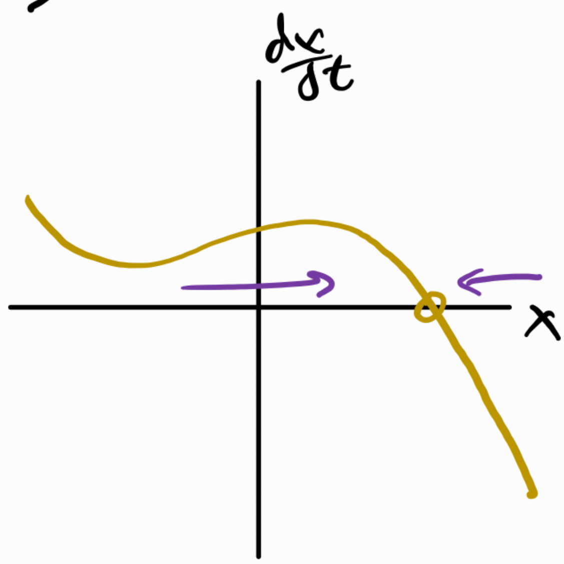


first analyze $\dot{x}$ for constant y:

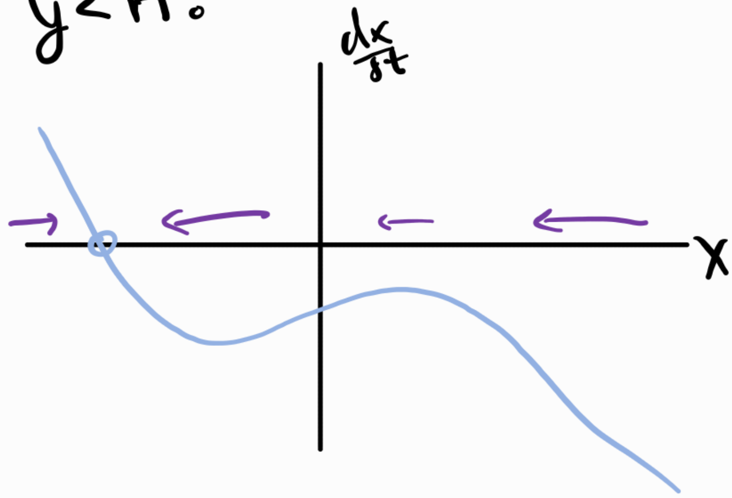
$y \in (A, B)$:



$y > B$



$y < A$:



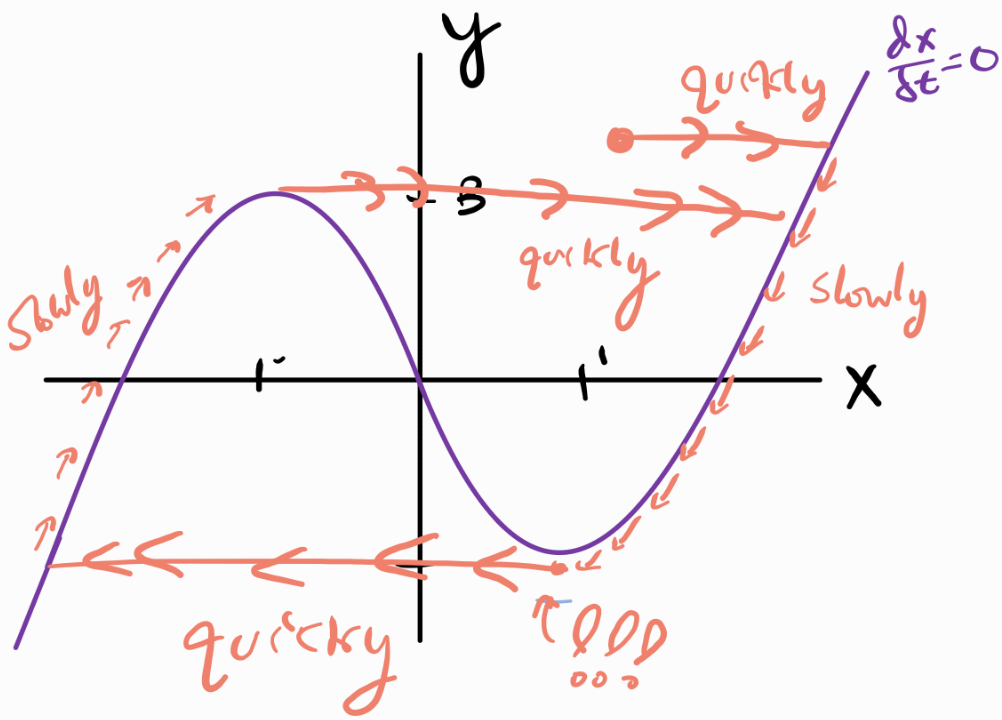
First x converges to the cubic nullcline $y = \frac{1}{3}x^3 - x$, direction depending on y_0

$\frac{dy}{dt} = -\frac{1}{\mu} x$ slowly with x s.t $\underbrace{\frac{x^3}{3} - x}_{F(x)} = y$

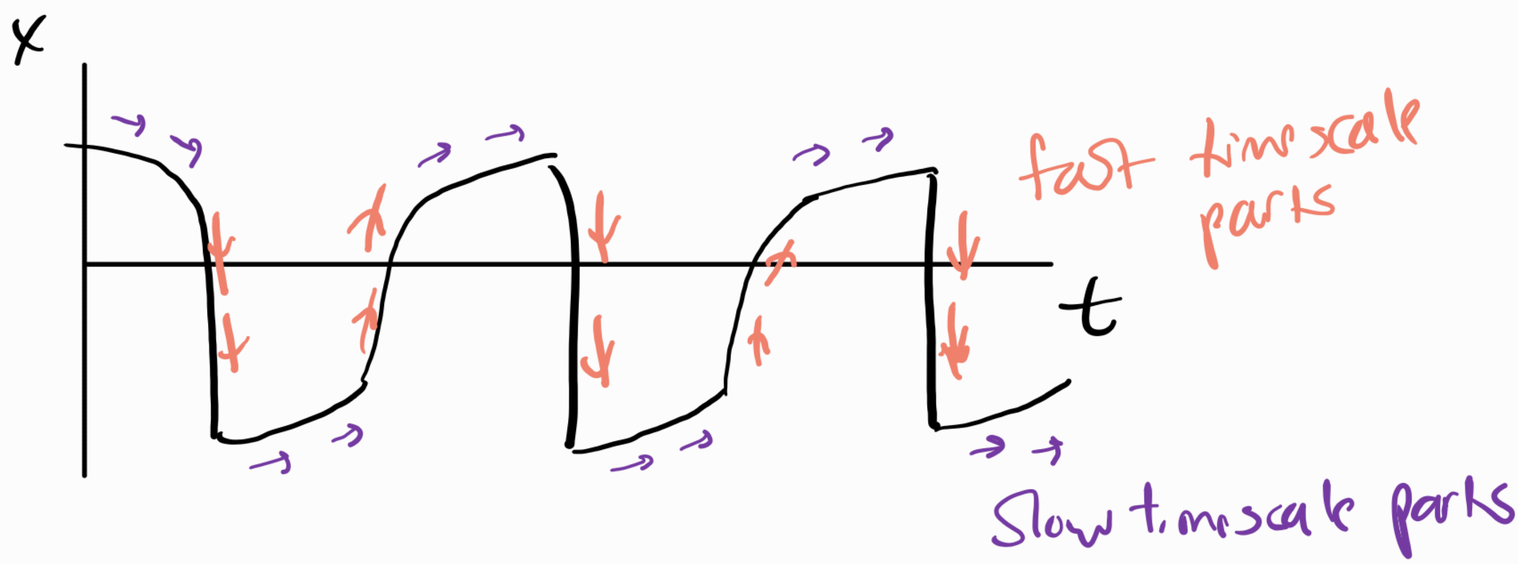
$\Downarrow$

if $x > 0$, $\frac{dy}{dt} < 0 \rightarrow$ crawl "down" nullcline

if $x < 0$, $\frac{dy}{dt} > 0 \rightarrow$ crawl "up" nullcline



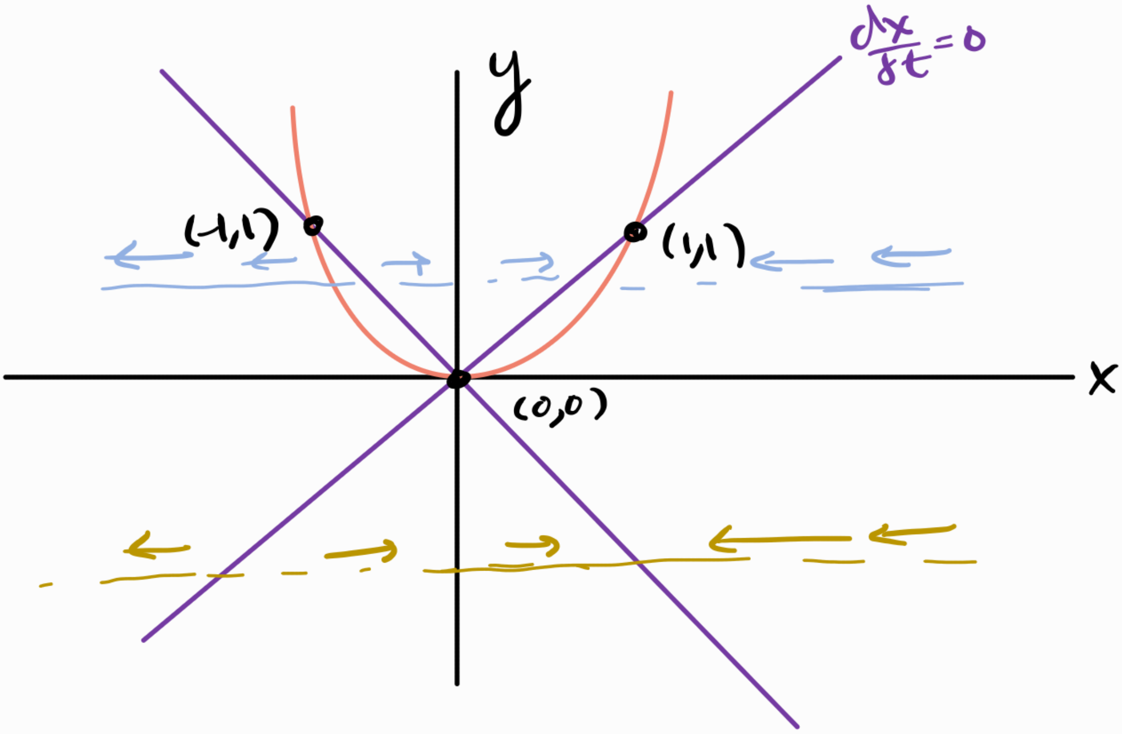
When looking at $x(t)$, you see **relaxation oscillations**



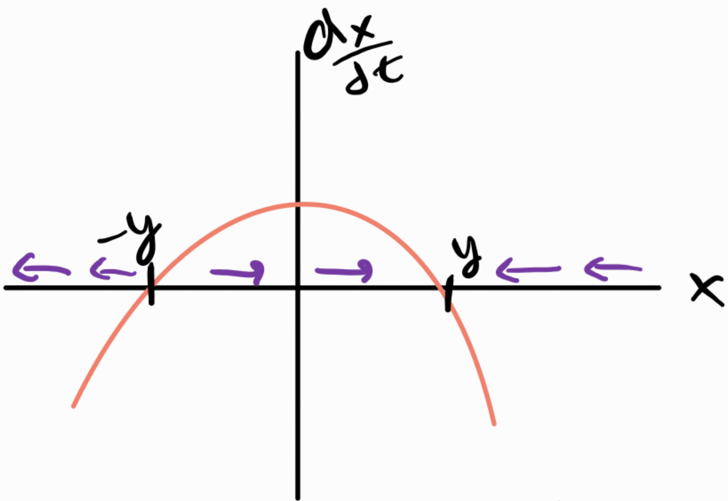
Example

$\mu >> 1$

$$\begin{cases} \dfrac{dx}{dt} = \overbrace{(y-x)(y+x)}^{y^2-x^2}\mu \\ \dfrac{dy}{dt} = y - x^2 \end{cases}$$



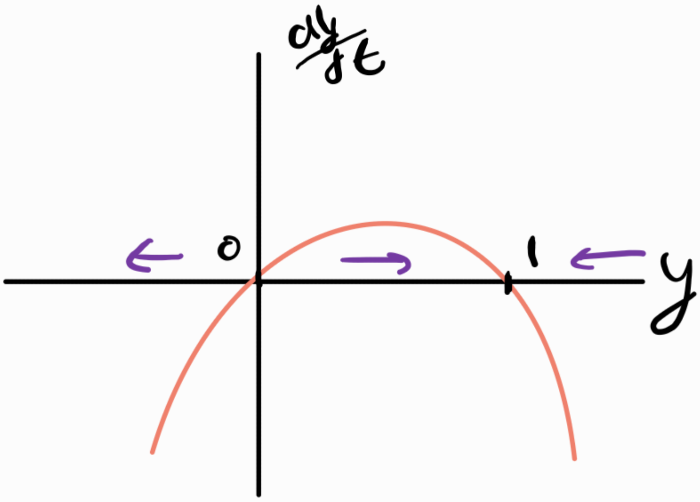
First look at x for fixed y:



If $x_0 > -y$, then $x \to y$ quickly.

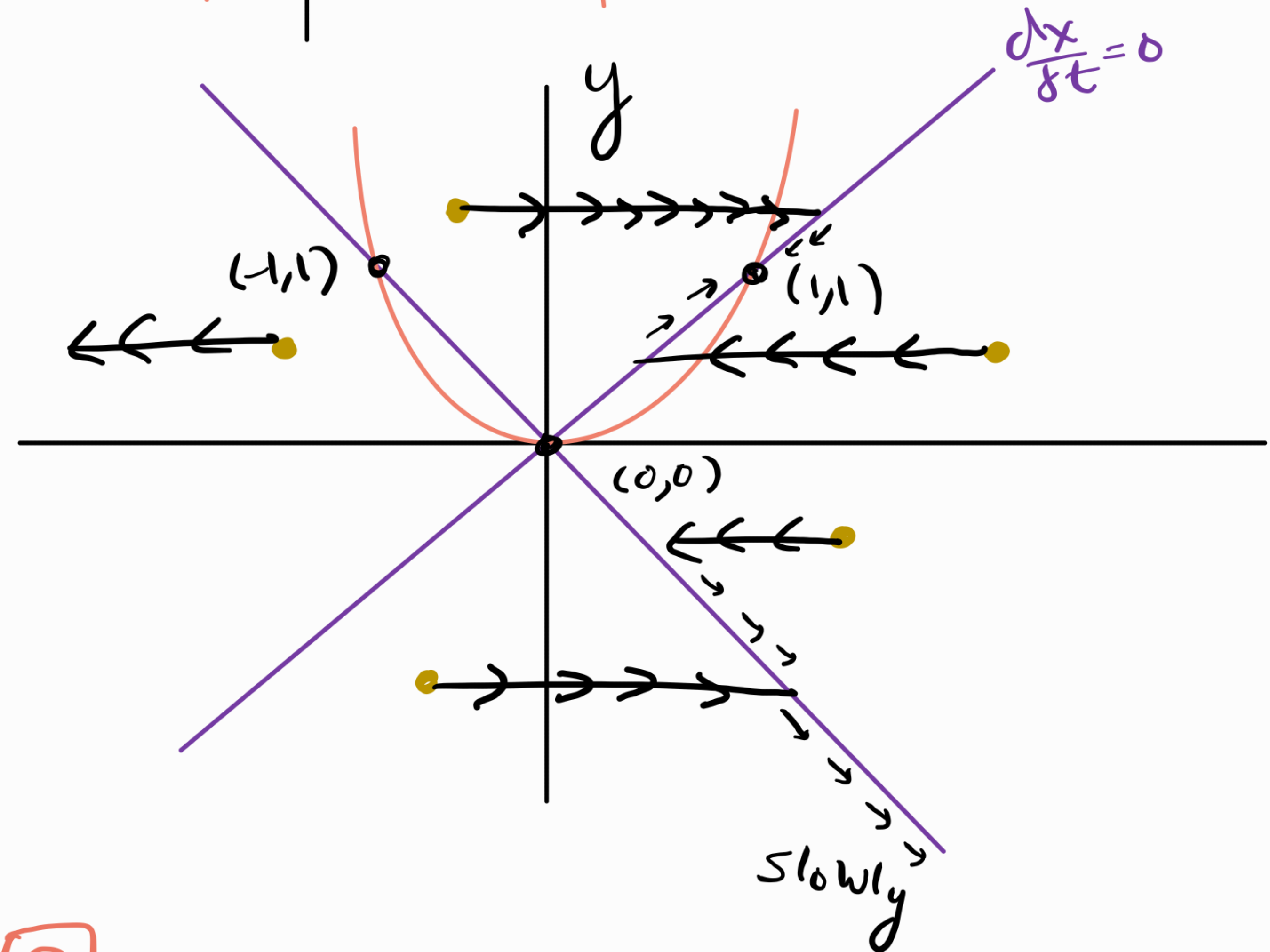
If $x_0 < -y$, then x diverges to $-\infty$

now, $x = y \Rightarrow \dfrac{dy}{dt} = y - y^2$

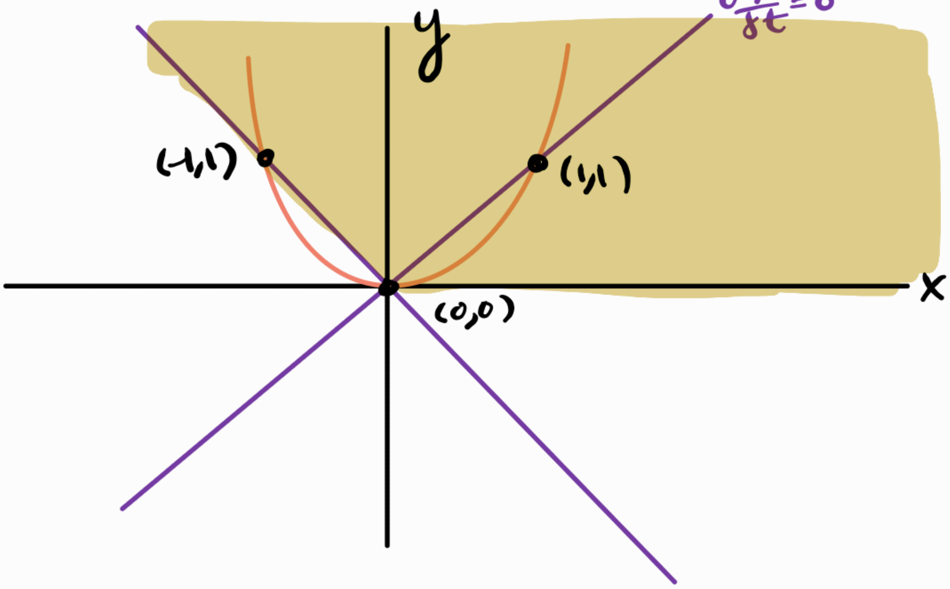


if $y_0 > 0$, converges slowly to 1

else, $y \to -\infty$



Q What is the basin of attraction for (1,1)?



all IC here converge to (1,1) as $t \to \infty$, albeit some faster than others

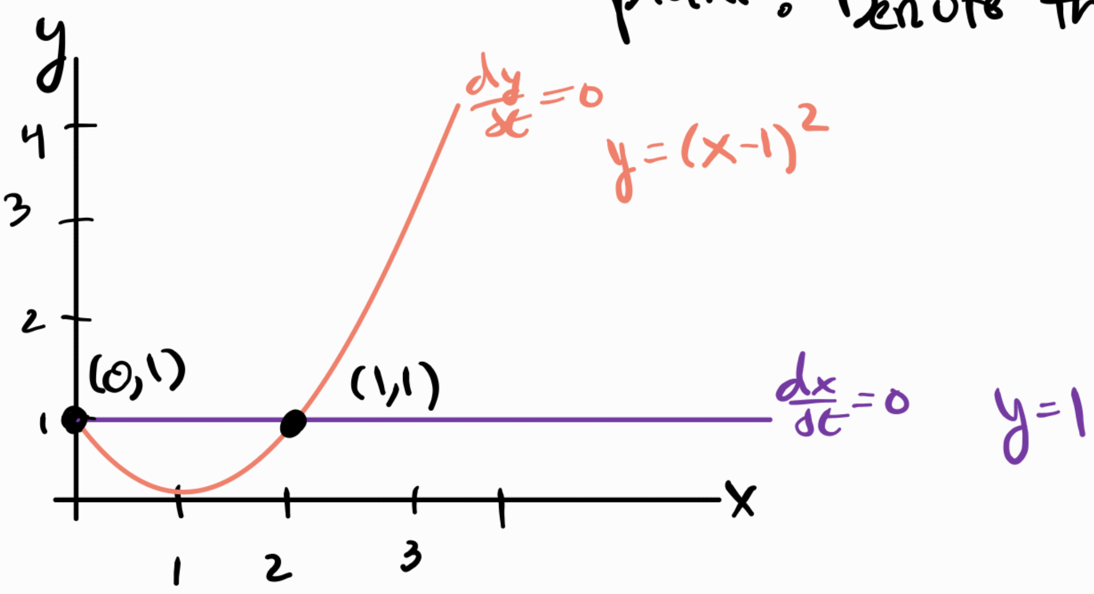
Quiz

Consider the non-dimensional dynamical system for chemical concentrations x, y:

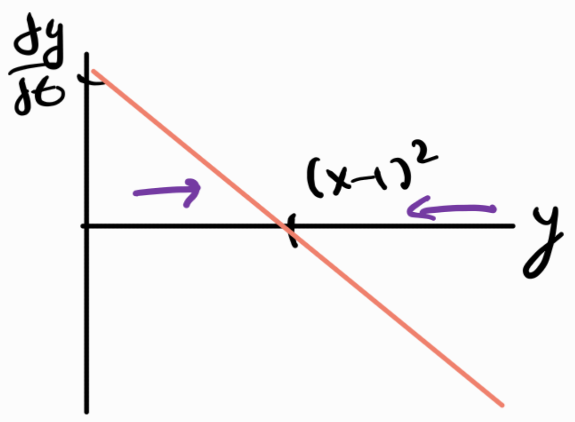
$$\begin{cases} \frac{dx}{dt} = \varepsilon(1-y) & (1) \\ \frac{dy}{dt} = (x-1)^2 - y & (2) \end{cases}$$

Taking $\varepsilon \ll 1$, x changes in time very slowly

a) Find and sketch the nullclines in the phase plane. Denote the steady states.



b) Assuming x is constant, consider (2) and conclude what will be the dynamics of y

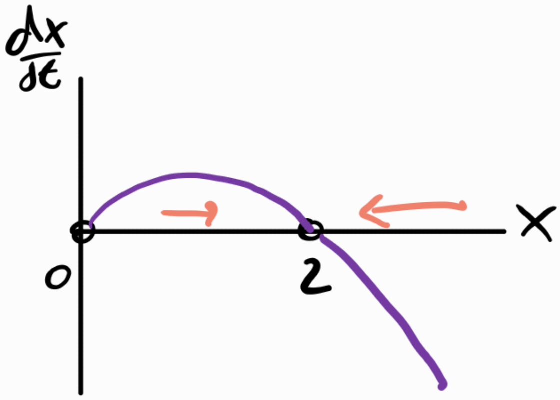


$\Rightarrow$ y quickly converges to $(x-1)^2$ $\forall y_0$

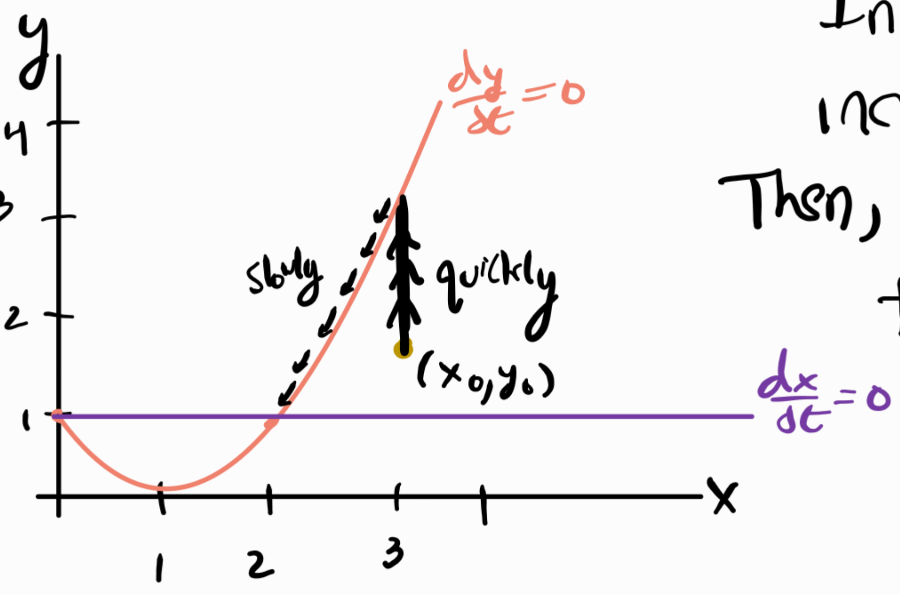
c) Now think about the slow dynamics of x. Explain qualitatively what happens if you start the system at $x_0=3, y_0=2$ (it may be helpful to draw in the phase plane).

if we take $y \to (x-1)^2$, $\frac{dx}{dt}$ becomes

$$\frac{dx}{dt} = \varepsilon(1-(x-1)^2) = \varepsilon(-x^2+2x) = -\varepsilon \cdot x \cdot (x-2)$$



$\forall x_0, x \to 2$ slowly



Initially, y rapidly increases to $(x_0-1)^2$. Then, x slowly decreases to 2, with y always rapidly adjusting to equal $(x-1)^2$

d) Make a conclusion about the behavior of the whole system based on your analysis

All initial conditions rapidly shoot to the y-nullcline, along which they slowly converge to $(2,1)$