Directivity & Beam-Patterns

ECE 551 Project

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Agenda

- Dipole/Donut
  - Derive Directivity
  - Calculate Directivity
  - Plot Beam Pattern
- Continuous Line Source
  - Derive Directivity Factor
  - Plot Beam Pattern
  - Derive Directivity
- Circular Baffled Piston
  - Derive Directivity Factor
  - Plot Beam Pattern
  - Derive Directivity
- Simple Source Line Array
  - Derive Directivity Factor
  - Plot Beam Pattern
  - Define Trade Off for different Windows
  - Derive Directivity
  - Steerable Beampattern

Dipole/Donut

Dipole: Directional Factor

The pressure produced at a field point $\vec{r}$ due to two sources

$$P_1 = \frac{A}{(r+\Delta r_1)} e^{j(\omega t - k(r+\Delta r_1))}$$

$$P_2 = \frac{-A}{(r-\Delta r_2)} e^{j(\omega t - k(r-\Delta r_2))}$$

$\Delta r_1 = \Delta r_2 = \frac{d}{2}\sin\theta$ in the far field limit

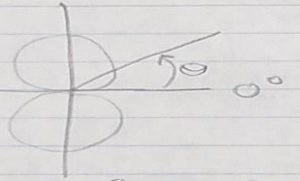
The ratio of $\Delta r / r \ll 1$. Therefore pressure is expressed as followed

$$P = \frac{A}{r}\left(e^{-jk\frac{d}{2}\sin\theta} - e^{jk\frac{d}{2}\sin\theta}\right)e^{j(\omega t - kr)}$$

$$P = \frac{A(-2j)}{r}\left[\sin\left(\tfrac{1}{2}kd\sin\theta\right)\right]e^{j(\omega t - kr)}$$

At low frequencies the expression for pressure is reduced to

$$P = j\frac{A}{r}kd\sin\theta$$

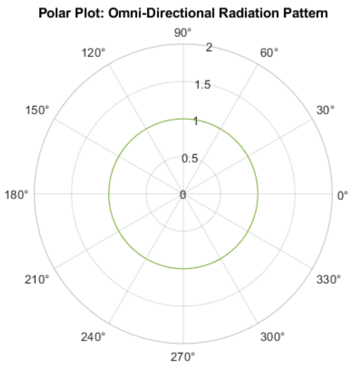


The Directivity Factor is the radiation that depends on the angle

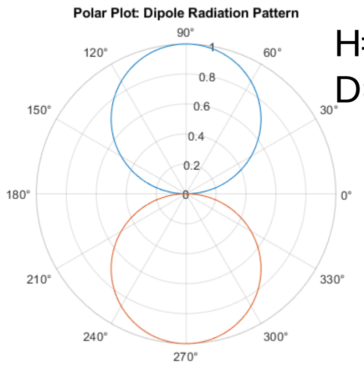
$$H(\theta) = \sin\theta$$

Beam Pattern = $b = 20\log H(\theta)$

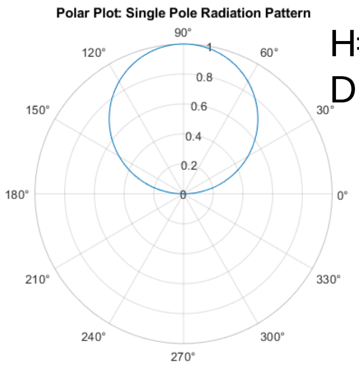
Dipole Beam Pattern in MATLAB



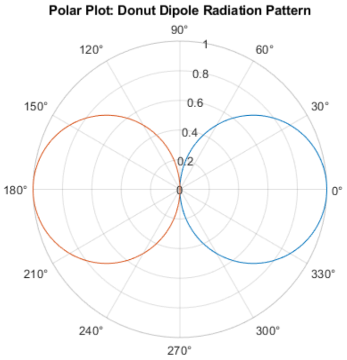
H= 1
D=1



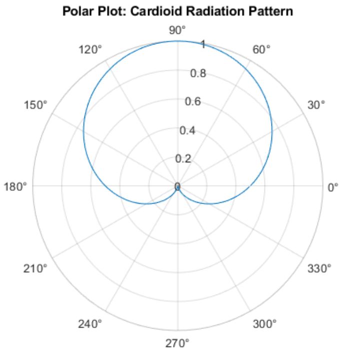
H= sin(theta)
D=3



H= (½)sin(theta)
D=6



H= cos(theta)
D=1.5



H=(1+sin(theta))/2
D=3

Directivity Derived General Form

Directivity (D) is a calculation of the power concentration indicating how directional a source (or receiver) is relative to a non-directional (aka omni-directional) source.

$$H(\theta, \varphi) \qquad P(r,\theta,\varphi) = P_{ax}(r) \cdot H(\theta,\varphi).$$

$$D = \frac{I_{\mathrm{ax}}(r)[\mathrm{directional}]}{I_{\mathrm{s}}(r)[\mathrm{spherical}]} = \frac{P_{\mathrm{ax}}^2(r)}{P_{\mathrm{s}}^2(r)}; \qquad D = \frac{4\pi}{\int_{4\pi} H^2(\theta,\varphi)\,d\Omega}. \qquad D = \frac{4\pi}{\int_0^{2\pi} \int_0^{\pi} H^2(\theta,\varphi)\sin\theta\,d\theta\,d\varphi}$$

Directivity Calculation of Dipole:

Directional factor, $H(\theta) = \cos\theta$

$$D = \frac{4\pi(1)^2}{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi}$$

$$\int_0^{2\pi} d\phi = 2\pi - 0 = 2\pi$$

$$= \frac{4\pi}{2\pi \int_0^{\pi} \cos^2\theta \sin\theta \, d\theta}$$

$$u = \cos\theta$$

$$du = -\sin\theta \, d\theta$$

$$= \frac{4\pi}{-2\pi \int_1^{-1} u^2 du}$$

$$= \frac{4\pi}{-2\pi \left[\frac{u^3}{3}\right]_1^{-1}}$$

$$= \frac{-2}{-\frac{2}{3}}$$

$$= 3$$

Directivity Calculation of Donut Dipole:

Directional factor, $H(\theta) = \sin\theta$

$$D = \frac{4\pi(1)^2}{\int_0^{2\pi}\int_0^{\pi} \sin^2\theta \sin\theta \, d\theta \, d\phi}$$

$$\int_0^{2\pi} d\phi = 2\pi - 0 = 2\pi$$

$$= \frac{4\pi}{2\pi \int_0^{\pi} (1-\cos^2\theta) \sin\theta d\theta}$$

$$= \frac{4\pi}{2\pi \int_0^{\pi} (\sin\theta d\theta - \cos^2\theta \sin\theta d\theta)}$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \frac{2}{[-\cos\theta]_0^{\pi} + \int_1^{-1} u^2 \, du}$$

$$= \frac{2}{(1+1) - \frac{2}{3}}$$

$$= 1.5$$

Directivity Calculation of Cardiod:

Directional factor, $H(\theta) = \dfrac{1+\cos\theta}{2}$

$$D = \frac{4\pi(1)^2}{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \, \sin\theta \, d\theta \, d\phi}$$

$$\int_0^{2\pi} d\phi = 2\pi - 0 = 2\pi$$

$$= \frac{4\pi(1)^2}{2\pi \int_0^{\pi} \left(\frac{1+\cos\theta}{2}\right)^2 \sin\theta \, d\theta}$$

$$= \frac{2}{\frac{1}{4}\int_0^{\pi} (\sin\theta \, d\theta + 2\cos\theta\sin\theta \, d\theta + \cos^2\theta \sin\theta \, d\theta)}$$

$u = \cos\theta$

$du = -\sin\theta \, d\theta$

$$= \frac{8}{[-\cos\theta]_0^{\pi} + \int_0^{\pi} \sin 2\theta \, d\theta - \int_1^{-1} u^2 \, du}$$

$$= \frac{8}{2 + \frac{1}{2}[-\cos 2\theta]_0^{\pi} - \left[\frac{u^3}{3}\right]_1^{-1}}$$

$$= \frac{8}{2 + \frac{1}{2}(-1+1) + \frac{2}{3}}$$

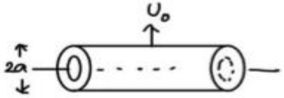
$$= \frac{8}{2 + \frac{2}{3}}$$

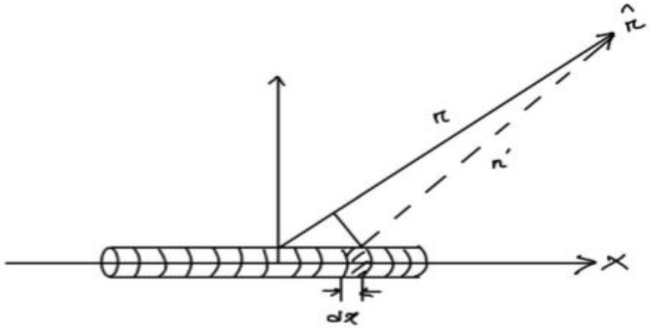
$$= 3$$

Continuous line source

Derivation of Directional factor of CLS:

Consider a long cylindrical source of finite length made of individual cylindrical elements each of length dx having source strength $dQ = U_0 2\pi a \, dx$ as illustrated below -





Each simple source element generates a pressure at the field point (r, θ) given by -

$$dp = j \frac{\rho_0 c}{4\pi r'} k \underbrace{U_0 2\pi a \cdot dx}_{dQ = \int U \cdot \hat{n} \, ds} \cdot e^{j(\omega t - kr')}$$

where geometry reveals that

$r' = r - x \sin\theta$

The total pressure is due to all differential elements and is

$$p = j \frac{\rho_0 c}{4\pi} k a^2 2\pi U_0 \int_{-L/2}^{+L/2} \frac{1}{r'} e^{j(\omega t - kr')} dx$$

In the far field where $r >> L$; the denominator may be replaced with $r' = r$, however this simplification is not permitted in the exponent and the integral reduces to -

$$p = j \frac{\rho_0 c U k a}{2r} e^{-jkr} \int_{-L/2}^{+L/2} e^{-j(kx\sin\theta)} . e^{j\omega t} dx$$

Now,

$$\int_{-L/2}^{+L/2} e^{-j(kx\sin\theta)} = \frac{1}{-jk\sin\theta} \left[e^{-jk\sin\theta \frac{L}{2}} - e^{jk\sin\theta\frac{L}{2}} \right]$$

$$= \frac{2j}{jk\sin\theta} \left[\frac{e^{-jk\sin\theta \frac{L}{2}} - e^{jk\sin\theta\frac{L}{2}}}{2j} \right]$$

$$= L \frac{1}{(\frac{1}{2}kL\sin\theta)} \left[\sin(\frac{1}{2}kL\sin\theta) \right]$$

which evaluated yields

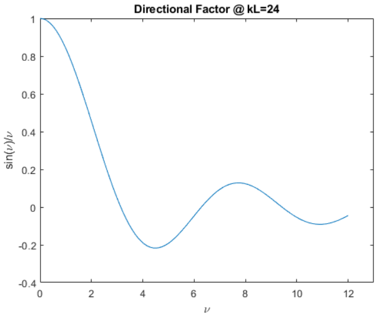
$$p = j \frac{\rho_0 c U k a \cdot L}{2r} \left[\frac{\sin(\frac{1}{2}kL\sin\theta)}{(\frac{1}{2}kL\sin\theta)} \right] \cdot e^{j(\omega t - kr)} ,$$

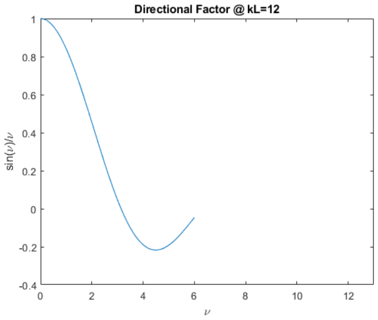
Seperating variables, we can express the acoustic pressure as

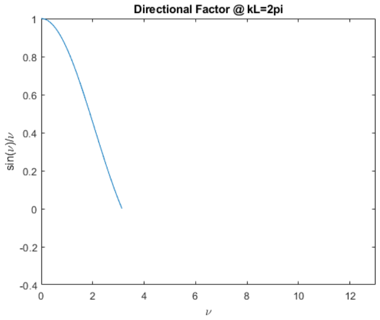
$$p = P_{ax} \cdot H(\theta)$$

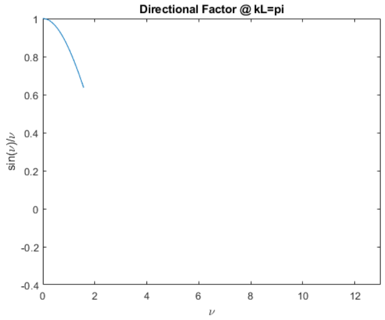
where, $H(\theta) = \frac{\sin v}{v}$; where $v = \frac{1}{2} kL \sin\theta$

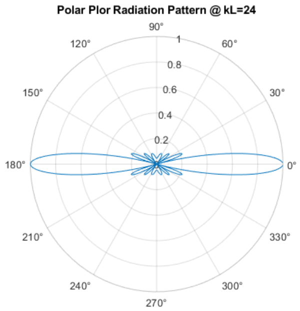
CLS: Beam Pattern MATLAB Plots

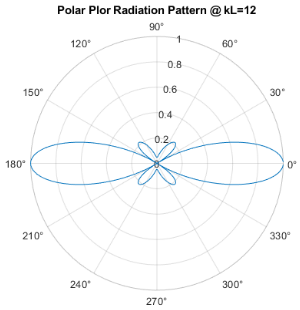


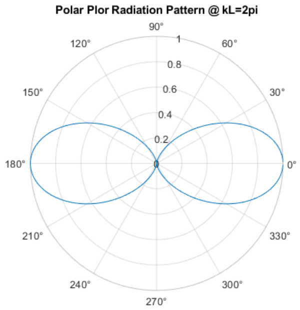


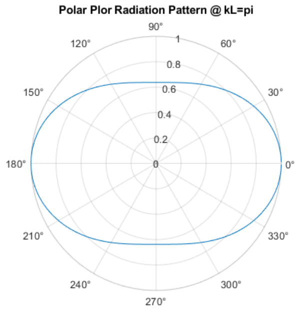












Circular Baffled Piston

Continuous Line Source – Directivity Derived

$$D = 4\pi \Big/ 2\int_0^{\pi/2} H^2(\theta) 2\pi \cos\theta \, d\theta$$

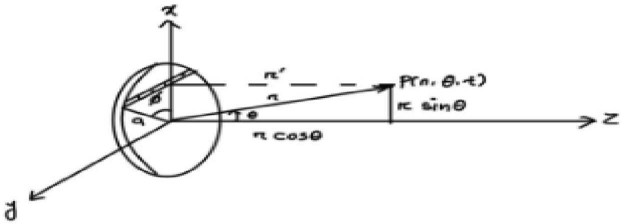
$$\int_0^{\infty} \left(\frac{\sin v}{v}\right)^2 dv = \frac{\pi}{2}$$

$$D = \frac{kL}{2} \Big/ \int_0^{kL/2} \left(\frac{\sin v}{v}\right)^2 dv$$

$$D \approx kL/\pi = 2L/\lambda$$

Derivation of Directional factor of a Baffled Plane Circular Piston.

Consider a circular flat piston mounted on an infinite baffle.



where ϕ is the angle in the plane of the piston

The approach to calculate the radiated pressure is the same as before. We consider each differential element acting as a simple source. Each differential surface element of length $2a\sin\phi$ contributes to the radiation with a source strength

$$dQ = 2U_0 \sin\phi \, dx$$

Thus the pressure due to this linear element is

$$dp = j\rho_0 c \frac{U_0}{\pi r'} ka \sin\phi \, e^{j(\omega t - kr')} dx$$

For $r >> a$, r' has the approximate form

$$r' \cong r\left(1 - \frac{a}{r}\sin\theta\cos\phi\right) = r + \Delta r$$

and the total acoustic pressure is then +a

$$P = j\,\rho_0 c \frac{v_0}{\pi r} ka\, e^{j(\omega t - kr)} \int_{-a}^{+a} e^{jka\sin\theta\cos\phi} \sin\phi \, dz$$

where we may approximate $r = r'$ in the denominator but we keep r' in the phase term.

Noting $x = a\cos\phi$, it is convenient to integrate over ϕ,

$$P = j\,\rho_0 c \frac{v_0}{\pi} ka^2 e^{j(\omega t - kr)} \int_0^{\pi} e^{jka\sin\theta\cos\phi} \sin^2\phi \, dz$$

Noting that the imaginary part of the integral of sin vanishes due to symmetry from 0 to π, and that the Real Part may be evaluated as

$$\int_0^{\pi} \cos(z\cos\phi) \sin^2\phi \, d\phi = \pi \frac{J_1(z)}{z}$$

We arrive the result that

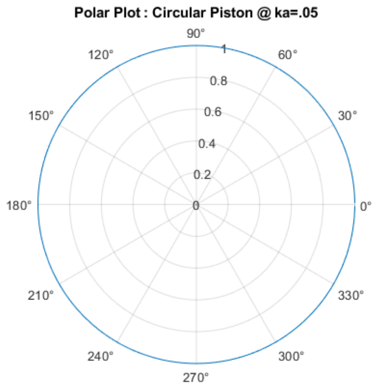
$$P = j \frac{\rho_0 c}{2} V_0 \frac{a}{r} ka\, e^{j(\omega t - kr)} \Bigg[\underbrace{\frac{2J_1(ka\sin\theta)}{ka\sin\theta}}_{H(\theta)} \Bigg]$$

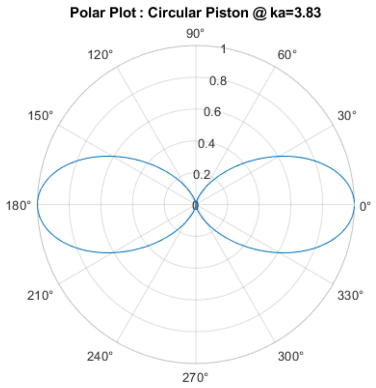
The directivity factor is

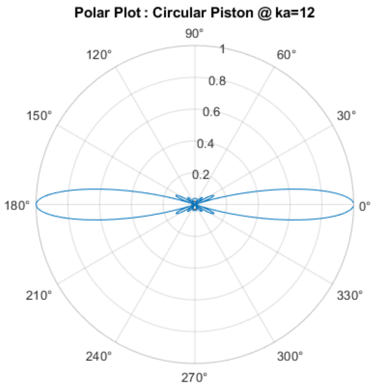
$$H(\theta) = \left| \frac{2J_1(x)}{x} \right|$$

$$x = ka\sin\theta$$

Circular Piston: Beam Pattern MATLAB







A larger ka=2pi*a/lambda leads to larger directivity.

Circular Piston: Directivity Derived

$$D = 4\pi \Big/ \int_0^{\pi/2} \left[\frac{2J_1(ka \sin\theta)}{ka \sin\theta} \right]^2 2\pi \sin\theta \, d\theta$$

$$D = \frac{(ka)^2}{1 - J_1(2ka)/ka}$$

$$D \approx (ka)^2 \qquad ka \gg 1$$

Simple Source Line Array

Simple Source Line Array Design

Array Parameters

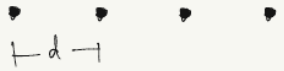
- Sound Speed
  - Water: 1500 m/s
  - Need this in order to get our lambda and wave number
- Operating Frequency
- fc= 1kHz
  - We need this in order to determine lambda and also our element spacing
- Element Spacing
  - Determined by lambda/2
  - The space between each element in our array
  - This will affect our radiation pattern,
- Wave Number
- Number of Elements
  - Increasing the number of elements leads to a narrower main lobe and more side lobes

Line Array: Directional Factor

Line Array

Directional factor

$$H(\theta) = \frac{1}{N}\,\frac{\sin(Nx)}{\sin(x)}$$

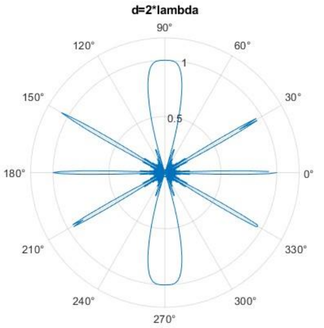


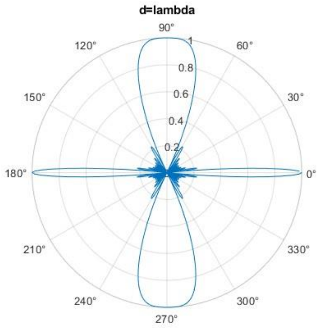
$$x = \frac{1}{2} k d \sin\theta$$

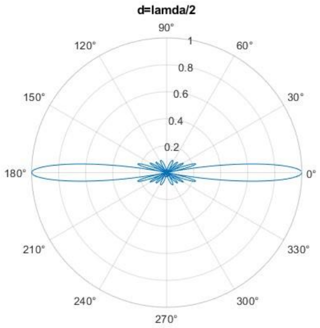
this is similiar to the Directional factor of a CLS: $H(\theta) = \dfrac{\sin\left(\frac{1}{2}kL\sin\theta\right)}{\frac{1}{2}kL\sin\theta}$

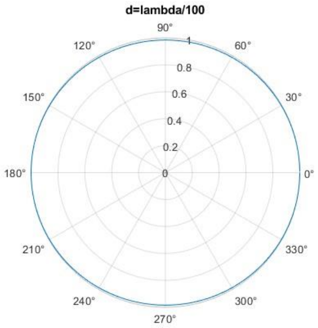
the only difference is that the N number of elements

Line Array MATLAB Element Spacing



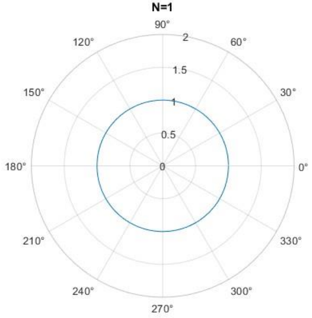


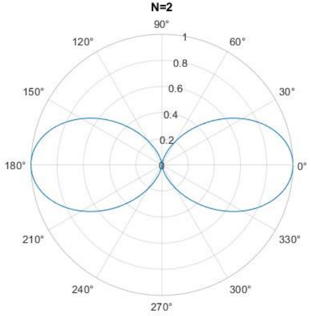


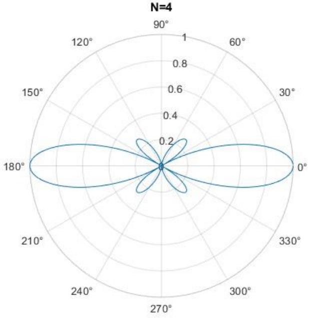


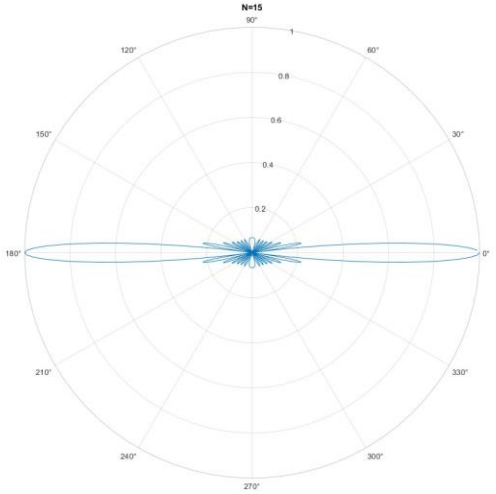
The ideal element spacing is lambda/2
Large element spacing leads to extra main lobes that degrades directivity
Too small of element spacing leads to an omni directional radiation pattern

Line Array in MATLAB with Different Number of Elements









An increase in element in our array leads to better directivity.
The array is made of uniform element.
Each radiating omni-direction. The combined interference of each element lead to a constructive interference in the acoustic phase center and side lobes.
The points where the interference is destructive creates notches.

Directivity and Radiation Pattern

Line Array : Directivity

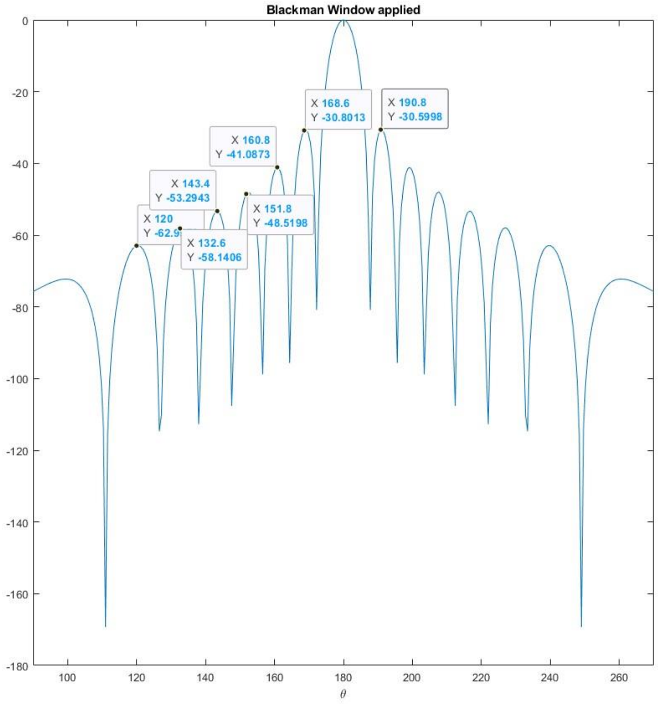
$$D \simeq 2L_x/\lambda = 2 \cdot d \cdot (N-1)/\lambda$$

L_x = lenght of full array

Which would make sense b/c we get better directivity when we add more elements

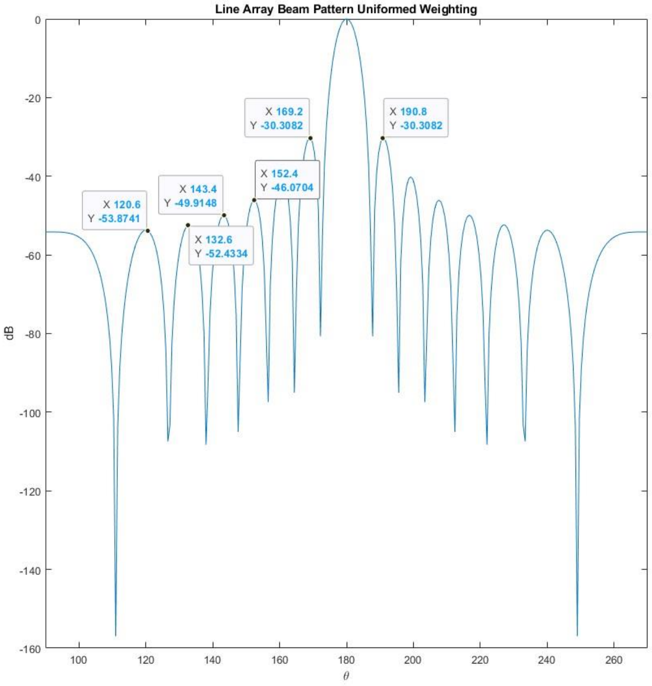
Windows

Blackman



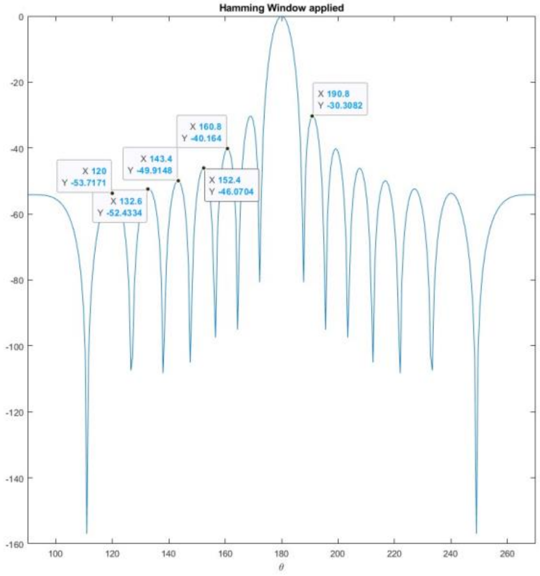
- First side lobe is at –30.5998dB which is lower
- The other side lobe are lowered
- Beam width 4.8 degrees

Uniformed Weighted



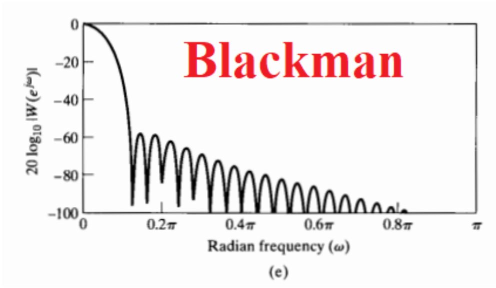
- All elements are weighted equally
- The first side lobe is at –30.3082 dB
- Beam width 4.8 degrees

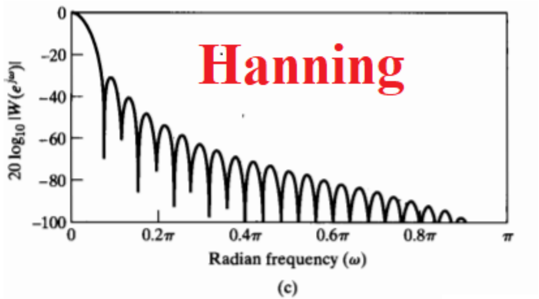
Hamming

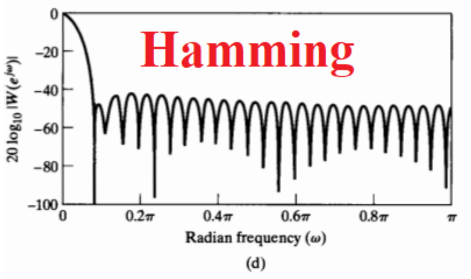


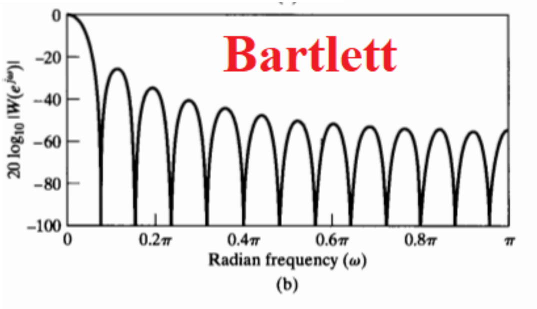
- Hamming weights
- First side lobe at –30.3082dB
- We didn't notice a difference between the uniform weights and the Hamming weighted beam pattern
- Beam width 4.8 degrees

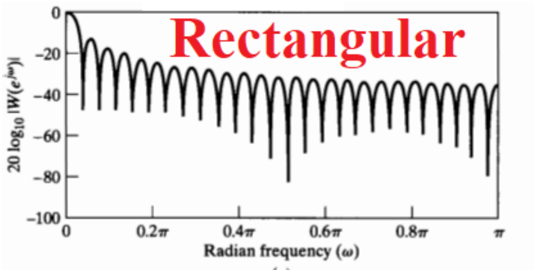
Different Windows





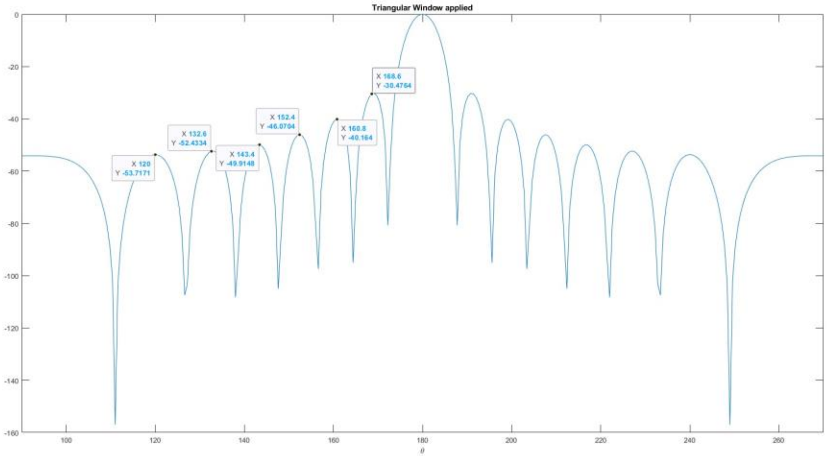






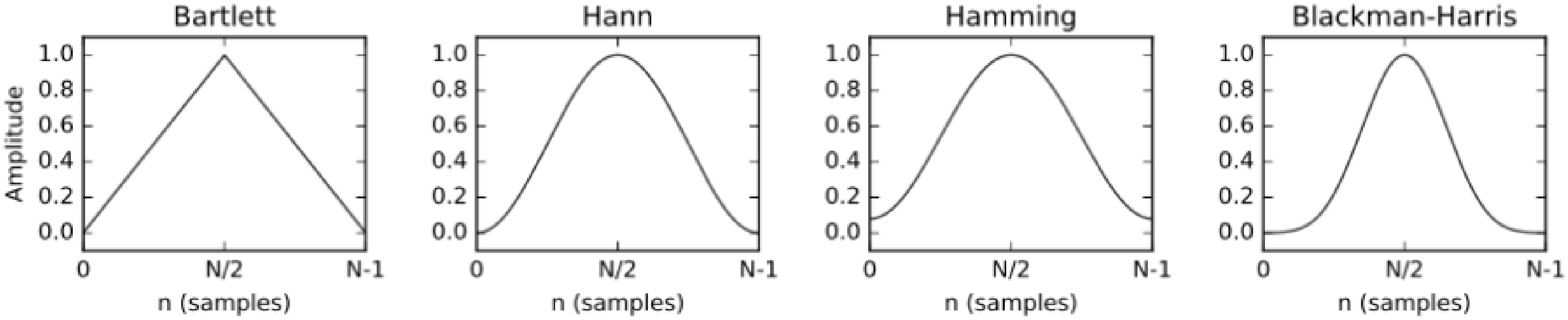
Discrete Time Signal Processing
Second Edition
Alan V. Oppenheim, Ronald W. Schafer

Triangular



- First side lobe at –30.4764 dB
- Compared to the Blackman the side lobe for the Triangular window are lowered as much
- Beam width 4.8 degrees

Different Windows



Images from this webpage- https://wvurail.org/dspira/labs/05/

Different Windows

TABLE 7.1 COMPARISON OF COMMONLY USED WINDOWS

Type of Window	Peak Side-Lobe Amplitude (Relative)	Approximate Width of Main Lobe	Peak Approximation Error, $20\log_{10}\delta$ (dB)	Equivalent Kaiser Window, β	Transition Width of Equivalent Kaiser Window
Rectangular	−13	$4\pi/(M+1)$	−21	0	$1.81\pi/M$
Bartlett	−25	$8\pi/M$	−25	1.33	$2.37\pi/M$
Hanning	−31	$8\pi/M$	−44	3.86	$5.01\pi/M$
Hamming	−41	$8\pi/M$	−53	4.86	$6.27\pi/M$
Blackman	−57	$12\pi/M$	−74	7.04	$9.19\pi/M$

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Steering

Line Array Beam Steering

τ $\quad 2\tau$ $\quad 3\tau$ $\quad 4\tau$

$$H(\theta) = \frac{1}{N} \frac{\sin\left[\frac{N}{2} kd\left(\sin - \frac{c\tau}{d}\right)\right]}{\sin\left(\frac{1}{2} kd\left(\sin - \frac{c\tau}{d}\right)\right)}$$

this term changes the phase which change the main lobe direction