LATENT VARIABLE MODELS

Session 2: Basic CFA models

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Contents this afternoon:

- Reflective and formative factor models
- Model estimation, evaluation, modification
  - Parameter estimation: Maximum likelihood (ML)
  - Assessing model fit
  - Model modification
  - Robust ML (perhaps cover in session 4, ordered-categorical items)

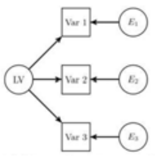
Latent variables

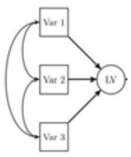
- Latent variables (LVs) are variables that are not directly observed, but are inferred from other variables that are directly observed (OVs)
- LVs represent a construct or concept that researchers are interested in, but cannot directly measure:
  - E.g., depression, anxiety, aggressiveness, socio-economic status, wellbeing, quality of life, social skills, intelligence, mathematical abilities, …
  - In this workshop: focus on continuous LVs
    - LVs can also be categorical (latent classes), but outside scope of this course

Factor analysis

- Confirmatory factor analysis (CFA)
  - We have a (relatively) clear idea about:
    - number of factors underlying observed variables
    - with which observed variables they are related
    - what they represent
- Exploratory factor analysis (EFA)
  - When we have no clear idea about that
  - Not in this course
- Both assume arrows to go from factor to indicator (i.e., reflective model)

Reflective and formative LVs





- Reflective LV:
  - OVs reflect the LV
  - LV ('underlying' factor) 'causes' the OVs
  - LV is exogenous
  - OVs are endogenous
  - OVs must be correlated
  - e.g., depression, intelligence, …

- Formative LV:
  - OVs form the LV
  - OVs 'cause' the LV
  - LV is endogenous
  - OVs are exogenous
  - OVs not necessarily correlated
  - E.g, SES, physical health (balanced diet, regular exercise, sufficient sleep)

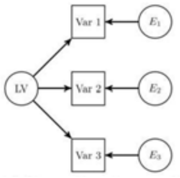
Errors are also LVs, but do not represent a construct, and are always exogenous

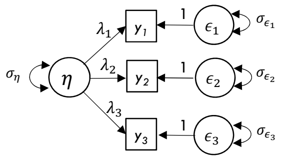
Reflective measurement model

The score on item *i* of person *j* is given by:

$$y_{ij} = \tau_i + \lambda_i \, \eta_j + \epsilon_{ij}$$

Often, we assume a centered model (i.e., all means are zero), so τ_i (item intercepts) are zero and can be omitted:



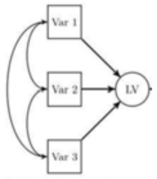


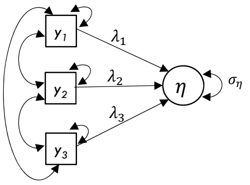
Formative measurement model

The score on the latent factor of person *j* is given by:

$$\eta_j = \tau + \lambda_i \, y_{ij}$$

Often, we assume a centered model (i.e., all means are zero), so τ (intercepts) is zero and can be omitted:





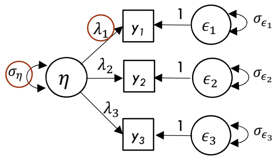
Coefficients

- A factor loading is a regression coefficient:
  - **Unstandardized** factor loading:
    - expected increase in OV, when LV increases by 1 (reflective model)
    - Expected increase in LV, when OV increase by 1 (formative model)
  - **Standardized** factor loading:
    - bivariate correlation between OV and LV
    - expected increase in SDs of OV, when LV increases by 1 SD (reflective model)

Identification

- E.g., we have 1 reflective LV, with 3 indicator variables
- There are 6 pieces of information about the scales and associations of the variables in the sample data
- In the (population) model, there are 7 unknowns (parameters) to estimate
  - We assign a constant value to ('fix') one of the parameters
  - Then values of other 6 parameters can be freely estimated, using the sample information
    - In other words: this yields a unique solution -> the model is identified

	X_1	X_2	X_3
X_1	$Var(X_1)$		
X_2	$Cov(X_1, X_2)$	$Var(X_2)$	
X_3	$Cov(X_1, X_3)$	$Cov(X_2, X_3)$	$Var(X_3)$



Identification: Reflective LVs

- Minimum requirements for identification of reflective LVs – rules of thumb:
  - > 3 indicator variables per LV (preferred)
    - Scale of LV has to be set by fixing a single parameter
    - Some errors are allowed to correlate
  - 3 indicator variables per LV (not preferred)
    - Scale of LV has to be set by fixing a parameter
    - No error covariances
  - 2 indicator variables per LV (not preferred)
    - Scale of LV has to be set by fixing a parameter
    - No error covariances
    - Both loadings set to equality
  - 1 indicator variable per LV (should be avoided)
    - Better use observed variable, without underlying LV

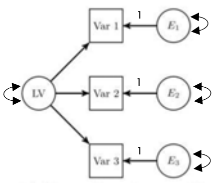
Identification: Reflective LVs

3 ways to identify scale of an LV:

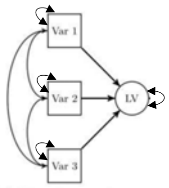
1. Standardize LV: fix LV's variance to 1
   - In lavaan: use model syntax, or set 'std.lv = TRUE' in cfa() function
2. Marker variable: set factor loading of an item to 1
   - Best practice: use the item most strongly correlated with the factor
   - Most common practice: use first item (not a major sin but always check if marker item is substantially correlated with factor)
     - Default in lavaan's cfa() function
3. Effects coding: set sum of loadings equal to the number of indicator variables
   - See example 3.3.1 in Beaujean book
   - Very rarely used, so skipped in this course

Yield same *standardized* solution, but different *unstandardized* solutions.

Identification: Reflective vs. Formative LVs



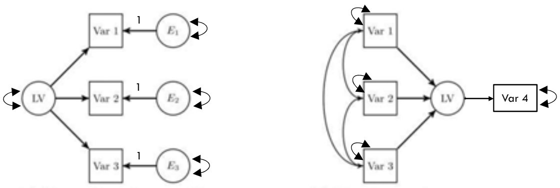
(a) *LV* is a reflective latent variable.



(b) *LV* is a formative latent v

Reflective model can be identified through a single restriction: then # of parameters to be estimated = # of sample statistics	Formative model is not identified here. Formative measurement models require at least 1 additional variable, caused by the formative LV, to be identified.

Identification



(a) *LV* is a reflective latent variable.

(b) *LV* is a formative latent va

Reflective model can be identified through a single restriction: If we fix a factor loading or variance to 1, there are 6 sample stats and 6 params (arrows) to be estimated	Formative model with additional variable is identifiable through a single restriction: If we fix a factor loading or variance to 1, there are 10 sample stats and 10 params (arrows) to be estimated

Identification

- If the model involves (co)variances only (i.e., no mean structure)
  - If # OVs in the model = *P*, then # of sample statistics = *P*(*P*+1)/2
  - So max. # of (model, population) parameters that can be freely estimated with *P* observed variables is *P*(*P*+1)/2
- SEM models can be:
  - **Just identified**
    - No. of free parameters = *P*(*P*+1)/2
    - Model always fits data perfectly
  - **Underidentied**
    - No. of free parameters > *P*(*P*+1)/2
    - Free parameters cannot be estimated, because there is no unique solution
  - **Overidentified**
    - No. of free parameters < *P*(*P*+1)/2
    - All free parameters can be estimated. Generally, model fits data imperfectly -> degree of model fit can be quantified and compared between models

Identification

Two basic conditions for model identification:

1) The number of free(ly estimated) parameters in the model ≤ the number of non-redundant (unique) elements in the sample variance-covariance matrix
2) Each latent variable needs to be assigned a scale (i.e., mean and variance)

Thus:

- In SEMs with OVs only, the model is always (just- or over-) identified
- In models with LVs, some parameter values have to be fixed to a constant by the user for the model to be identified
- Further assumptions:
  - Normality: all latent variables (latent factors and residuals/errors) are normally distributed (thus note: OVs need not be normally distributed)
  - Linearity (associations between variables in the model are linear)

Identification

- With overidentified models, we can select a 'best' model by comparing the models' trade-offs between
  - Models' misfit to the data
    - Closer fit (less misfit) to data is better
    - Quantified by chi-square value
  - Parsimonity
    - More parsimoneous is better (Occam's razor)
    - Quantified by df

- *df* = # knowns - # unknowns
  - # of sample stats (knowns) - # of free model parameters (unknowns)
- Just identified models have *df* = 0
- Overidentified models have *df* > 0
- Under identified models have *df* < 0 (cannot be estimated)

Question

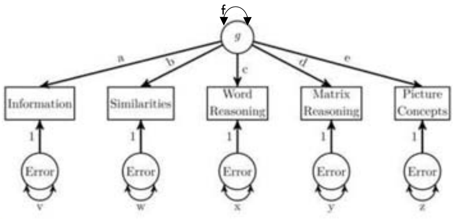


Figure 3.3 Single-factor model of five Wechsler Intelligence Scale for Children-Fourth Edition subtests.

How many sample (co)variances are there?
How many population parameters to be freely estimated?
How many degrees of freedom?

Examples and exercises

- Example 3.3 and 3.4 – Part I

SEM parameter matrices

- This mornings examples involved a **structural model** only:
  - **β**: a matrix of regression coefficients (single-headed arrows)
  - **Ψ**: a matrix of (co)variances not explained by the regression equations (double headed arrows)
- SEMs with LVs also involve a **measurement** model:
  - **Λ**: a matrix of factor loadings, relating observed variables to reflective latent variables
  - **Θ**: a matrix of measurement error variances

SEM parameter matrices

- When SEMs involve both a **measurement** and **structural** model, model-implied covariance matrix is given by: $\hat{\mathbf{\Sigma}} = \mathbf{\Lambda}(\mathbf{I}-\mathbf{\beta})^{-1}\mathbf{\Psi}\left[(\mathbf{I}-\mathbf{\beta})^{-1}\right]^{\mathrm{T}}\mathbf{\Lambda}^{\mathrm{T}} + \mathbf{\Theta}$
- If model involves **measurement** model only, this simplifies to: $\hat{\mathbf{\Sigma}} = \mathbf{\Lambda}\mathbf{\Psi}\mathbf{\Lambda}^{\mathrm{T}} + \mathbf{\Theta}$
- If model involves **structural** model only, this simplifies to: $\hat{\mathbf{\Sigma}} = (\mathbf{I}-\mathbf{\beta})^{-1}\mathbf{\Psi}\left[(\mathbf{I}-\mathbf{\beta})^{-1}\right]^{\mathrm{T}}$

SEM parameter matrices

If *P* is the number of observed variables and Q the number of latent variables in the model*, then:

- **β** (beta) is a QxQ matrix
  - Regression coefficients between latent vars
- **Ψ** (psi) is a QxQ matrix
  - (Co)variances of latent vars
- **Λ** (lambda) is a *P*xQ matrix
  - Factor loadings, relating observed to latent vars
- **Θ** (theta) is a *PxP* matrix
  - Measurement error (co)variances of observed vars

* and there are no formative latent variables and all regression relationships specified are between latent variables only

Examples and exercises

- Example 3.3 and 3.4 - part II
- Additional Exercise 1a

Parameter estimation: ML

- Most often, parameter estimation in a SEM is performed by maximum likelihood (ML)
- Sometimes, ML estimates have closed form solutions, and can be calculated directly using a fomula
  - e.g., ML estimates for the population mean and variance:

$$\hat{\mu}_X = \bar{X} = \frac{1}{N}\sum_N X \qquad \hat{\sigma}_X^2 = \frac{1}{N}\sum_N \left(X - \bar{X}\right)^2$$

- SEMs generally have a large number of parameters to be estimated, and an iterative procedure is needed (or much faster) to estimate the parameters
  - Therefore, output reports 'lavaan converged normally after … iterations'

Parameter estimation and model fit

The outcome of the optimization process provides:

1. The ML estimates of the parameter values
2. The standard errors of the ML parameter estimates
   - Based on the 2nd order derivative of the likelihood function
   - With large sample sizes, the ratio of each estimated parameter to its standard error is approximately *z*-distributed
     - Gives a z- and p-value for each parameter in the output
3. The value of the likelihood function F_{ML}
   - Under the null hypothesis (i.e., the model-implied cov matrix is the true cov matrix in the population), -2 times the log-likelihood value at the final parameter estimates follows a chi-square distribution with *df* degrees of freedom
     - Allows for a statistical test of overall model fit when *df* > 0
     - When *df* = 0, the model always fits perfectly: likelihood = 1 and log(likelihood) = 0

Assessing model fit

- Model fit should be evaluated in several ways:
  1. Overal model fit: assessed with model fit indices
  2. Individual parameter estimates
     - Parameter estimates substantial and statistically (in)significant where expected?
     - Are estimated parameter values plausible? E.g., expected sign of regression coefficients? Values as large or small as expected? E.g., |standardized factor loadings| > .30 ?
  3. Possible sources of misfit
     - Strikingly large residuals (co)variances or means?
     - Strikingly large modification index values?

Assessing overal model fit

- Statistical test of model fit: χ^2 (df)
  - Tests whether difference between the population and model-implied covariance matrix is zero
- In a SEM model, χ^2 value quantifies difference between:
  - observed (sample) covariance matrix $\boldsymbol{S}$ and
  - model-implied (population) covariance matrix $\hat{\boldsymbol{\Sigma}}$
    - $\chi^2 = 0$ if model fits perfectly, when $\hat{\boldsymbol{\Sigma}} - \boldsymbol{S} = \boldsymbol{0}$
    - In all other cases, $\chi^2 > 0$
      - The larger the difference between $\hat{\boldsymbol{\Sigma}}$ and $\boldsymbol{S}$, the larger the χ^2 value

Assessing overal model fit

- The larger the difference between $\hat{\boldsymbol{\Sigma}}$ and $\boldsymbol{S}$, the larger the χ^2 value, but:
- χ^2 value is also affected by other factors, affecting type I and II error rates of the χ^2 test:
  - Sample size
    - χ^2 value almost always significant with sample sizes > 75
    - χ^2 assesses statistical significance, but what about substantial significance?
    - One remedy: fit indices, are less dependent on sample size
  - Model complexity
    - More observed variables in model -> larger χ^2 value
    - Remedy: Evaluate individual parameter estimates and residual (co)variances to assess model fit
  - Departures from multivariate normality
    - Increasing non-normality -> in- or deflated χ^2 value
    - Remedy: use robust ML estimation

Assessing overal model fit

- In addition to χ^2(df), many other model fit indices
  - Lavaan provides > 40 of them for a single model
  - Have to make a selection:
    - Incremental fit indices (e.g., CFI)
    - Parsimony-based indices (e.g., RMSEA, AIC, BIC)
    - Absolute fit indices (e.g., SRMR)

Incremental fit indices

- Higher values indicate better fitting model (range: 0-1; rarely, values > 1 occur)
- Compare the fit of the proposed model with that of a null model
  - The null model has:
    - Zero correlation between variables in the model (so no latent variables)
    - Variances of observed variables equal to sample variances
- Value depends on the average size of the correlations in the data
  - If average correlation between variables is not very high, then incremental fit indices not very high.

Incremental fit indices

- Comparative fit index
  - Let d = $\chi^2 - df$
  - $$\text{CFI} = \frac{\text{d(Null Model) - d(Proposed Model)}}{\text{d(Null Model)}}$$
- Bentler-Bonett Index or Normed Fit Index (NFI)
  - $$\frac{\chi^2\text{(Null Model)} - \chi^2\text{(Proposed Model)}}{\chi^2\text{(Null Model)}}$$
  - Not so often used, due to no penalty for model complexity
- Tucker Lewis Index or Non-normed Fit Index (NNFI):
  - $$\frac{\chi^2\text{/df(Null Model)} - \chi^2\text{/df(Proposed Model)}}{\chi^2\text{/df(Null Model) - 1}}$$

Parsimony-based indices

- Information-theoretic criteria:
    - Model with lowest value has best fit
    - Note that there are various ways to calculate AIC, so never compare between software packages!
  - AIC: Akaike's Information Criterion
    - Penalty for every additional, freely estimated parameter is 2
  - BIC: Bayesian Information Criterion
    - Penalty for every additional, freely estimated parameter is nat.log(N), where N is the total sample size
  - SSABIC: Sample-Size Adjusted BIC
    - Penalty for every additional, freely estimated parameter is ln([N+2]/24)

Parsimony-based indices

- RMSEA: Root Mean Square Error of Approximation

$$RMSEA = \sqrt{\frac{\chi^2 - df}{df \cdot (N-1)}}$$

  - Lower values indicate better fitting model
    - Also, confidence interval can be calculated
    - And the p-value for RMSEA <= 0.05 (if p-value > .05, hypothesis of close fit is retained)
- χ^2/df ratio
  - Smaller values indicate better fit
  - Various rules of thumb have been proposed, ranging from 2 to 6 (what is good depends also on sample size)

Absolute fit indices

- SRMR: Standardized Root Mean Squared Residual

$$RMR = \sqrt{\frac{\sum_{i=1}^{p}\sum_{j=1}^{i}(s_{ij} - \hat{\sigma}_{ij})^2}{p(p+1)/2}}$$

s_{ij} is an element of the empirical covariance matrix S,

$\hat{\sigma}_{ij}$ is an element of the model-implied matrix covariance $\Sigma(\hat{\theta})$, and

p is the number of observed variables.

  - Average difference between the observed and model-implied correlations
  - Has no penalty for model complexity
  - SRMR = 0 indicates perfect fit

Overall model fit – cut-off values

- Based on simulations, Hu & Bentler (1999) derived the following cut-off values for good model fit:
  - CFI/TLI $\geq$ .95
  - SRMR $\leq$ .08
  - RMSEA $\leq$ .06
- Other authors suggest more lenient criteria
  - Sometimes, CFI $\geq$ .90 and/or RMSEA $\leq$ .08 called 'adequate' or 'acceptable'
- Model fit is not an all-or-nothing question, rules-of-thumb above offer a good starting point

Examples and exercises

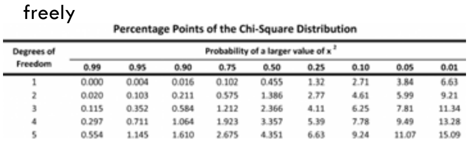
- Example 3.3 and 3.4 – part III
- Exercise 3.1
- Exercise 3.2

Improving model fit

- **Residual (co)variances**
  - Observed sample (co)variances minus model-implied covariances
  - Can be obtained in lavaan with the residuals() function
  - Using this information, the model may be improved

Improving model fit

- **Modification indices**
- Give an estimate of how much the χ^2-value of model fit will decrease when a parameter is freely estimated
- It can be interpreted as a χ^2-value with 1 df
  - Rule of thumb: if MI > 5, consider estimating parameter freely



Percentage Points of the Chi-Square Distribution

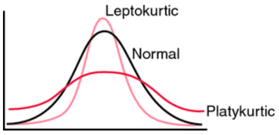
Degrees of Freedom	Probability of a larger value of x ²								
	0.99	0.95	0.90	0.75	0.50	0.25	0.10	0.05	0.01
1	0.000	0.004	0.016	0.102	0.455	1.32	2.71	3.84	6.63
2	0.020	0.103	0.211	0.575	1.386	2.77	4.61	5.99	9.21
3	0.115	0.352	0.584	1.212	2.366	4.11	6.25	7.81	11.34
4	0.297	0.711	1.064	1.923	3.357	5.39	7.78	9.49	13.28
5	0.554	1.145	1.610	2.675	4.351	6.63	9.24	11.07	15.09

Examples and exercises

- Example 3.3 and 3.4 - parts IV and V
- Additional exercise 1c
- Additional exercise 2

Robust ML estimation

- Robust ML estimation, like ML, assumes the data follow a multivariate normal distribution, but that the data have more or less kurtosis than a normal distribution
  - Thus does not correct for skewness!
- Kurtosis: measure of the shape of the distribution
  - From Greek word for bulging
- The degree of kurtosis in a data set is related to how incorrect the log-likelihood value will be
  - Leptokurtic data:
    χ^2 too large, SEs too small
  - Platykurtic data:
    χ^2 too small, SEs too large



Robust ML estimation

- **Parameter estimates** under MLR are just ML estimates
- SEs and model χ^2 value are adjusted under MLR, depending on kurtosis of data:
  - Model χ^2 value and associated fit statistics are adjusted
    - smaller χ^2 when data are leptokurtic
    - larger χ^2 when data are platykurtic
  - Model SEs are adjusted
    - smaller SEs when data are leptokurtic
    - larger SEs when data show platykurtosis
- If data have normal kurtosis, no adjustment is made (so safe to always use MLR)

Robust ML estimation

- Invoked by adding argument 'estimator = "MLR" ' in model-fitting function (e.g., lavaan(), sem(), cfa(), growth functions)
- Works only when raw data is supplied
  - When only covariance matrix (and/or means) are supplied, there is no info about the kurtosis of the data, so adjusted the standard errors and test statistic is not possible

Examples and exercises

- Additional exercise 3