LATENT VARIABLE MODELS

Session 1 – Introduction

Course prerequisites

- Knowledge of statistics
  - Statistical testing (e.g., chi-square & normal distributions)
  - Regression (GLMs)
  - Var, cov, cor, mean
- Knowledge of psychometrics
  - Validity
  - PCA, EFA, CFA
  - Reliability
  - IRT
- Programming in R

Course materials

Book(s):

- Beaujean, A. A. (2014). *Latent variable modeling using R: A step-by-step guide*.
  - Good as a starting guide, not an authorative standard
- Kaplan, D. (2009). *Structural Equation Modeling: Foundations and Extensions.*
  - Authorative standard. But more technical and not focused on specific software, not practical for current course.

Brightspace materials:

- Lecture slides
- Markdown files with examples, exercises and answers

Book examples

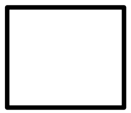
- You are strongly advised to copy and run R code from examples in Beaujean book and from Brightspace:
  - They give you a step-by-step guide on how to perform analyses
  - They give you a starter for making the exercises
  - If you make a mistake, you will get an error or warning message, from which you can learn alot! (But only if you try to decipher it, <u>red = good!)</u>

Structural Equation Modeling

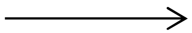
- **Structural**: We aim to *explain associations between* (that is, the structure of) *observed variables*
- **Equations**: Associations between variables are described using mathematical formulae
- **Modeling**: we construct models (hypotheses, theories) of reality. The models can be statistically tested. That is: rejected by the data (or not), but never proven 'true' or 'right'.
  - All models are wrong, but some are useful.

Structural Equation Modeling

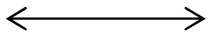
SEMS are graphically represented using these building blocks:



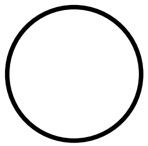
Observed (manifest) variable



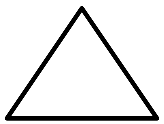
Directional relationship (regression relationship)



Non-directional relationship (correlation/(co)variance)



Latent variable



Constant term (i.e., not a variable, e.g., intercept)

Structural Equation Modeling

- Arrows in SEM denote regression relationships
- All generalized linear models (GLMs) can be formulated as a SEM:
  - t-test
  - ANOVA
  - Multiple linear regression
  - Multiple logistic regression
  - …..
- Also, SEM can be used to models for multilevel or longitudinal data (i.e., GLMMs)

Example dataset

Variables in the model:

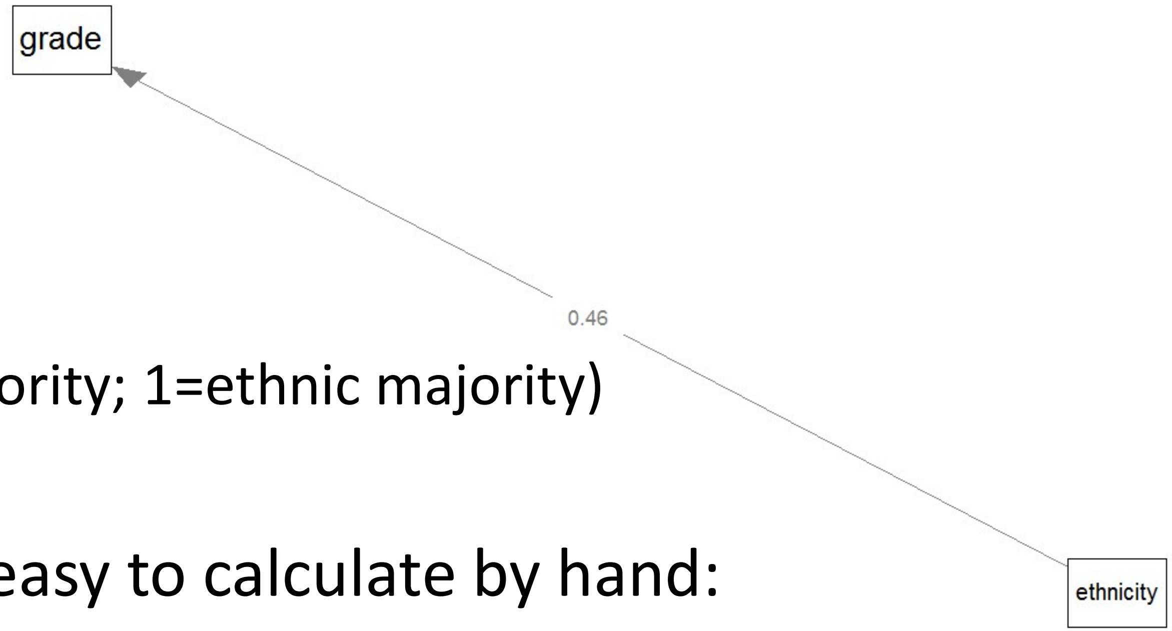
grade | ethnicity | homework | SES | Prev_ach

- GPA in 10th grade
- Ethnicity
- Homework (8th grade)
- Socio-economic status
- Previous achievement (8th grade)

- Sample covariance matrix **S**:

	grade	homwrk	prv_ch	ethnct	SES
grade	2.185				
homework	0.335	0.649			
prev_ach	6.429	2.067	79.092		
ethnicity	0.081	0.028	1.201	0.175	
SES	0.338	0.176	3.541	0.106	0.690

Model: Univariate regression

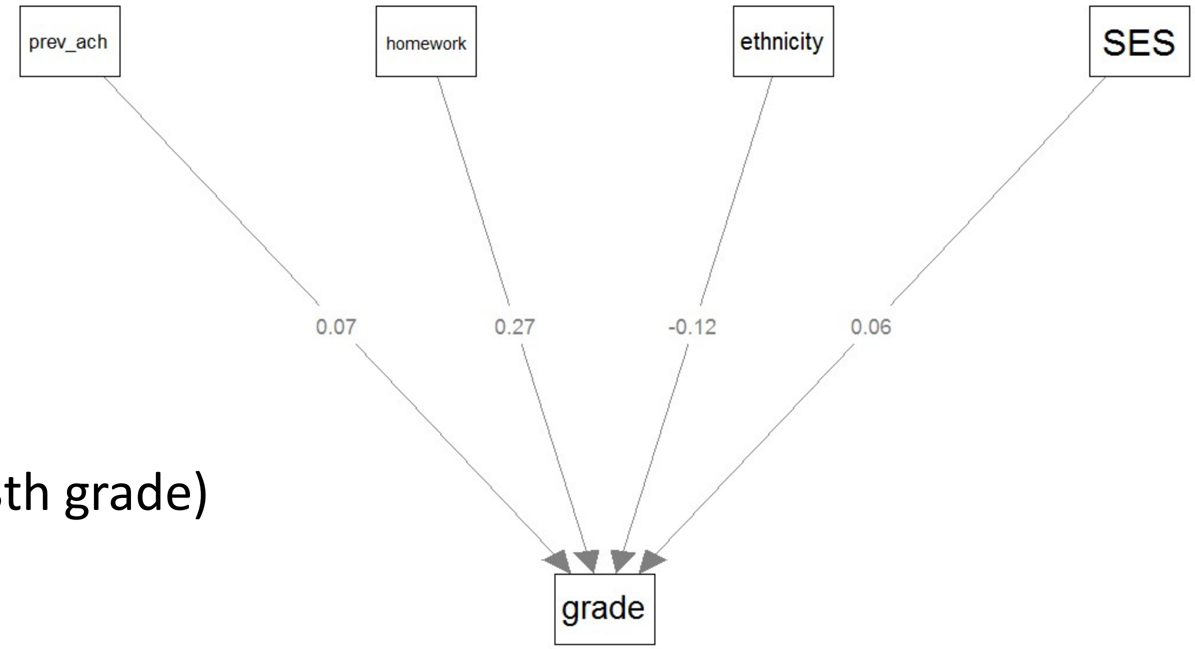


- Dependent:
  - GPA in 10th grade
- Independent:
  - ethnicity (0=ethnic minority; 1=ethnic majority)

- Regression coefficient easy to calculate by hand:
  - $\hat{b}_{xy} = \frac{cov_{x,y}}{var_x} = \frac{0.0814}{0.1752}$ = 0.4646
  - standardized $\hat{b}_{xy} = \hat{\rho}_{xy} = \frac{cov_{x,y}}{s_x s_y} = s_x \frac{\hat{b}_{xy}}{s_y} = 0.132$
- Measure of fit or (strength of) association: ${\hat{\rho}_{xy}}^2$

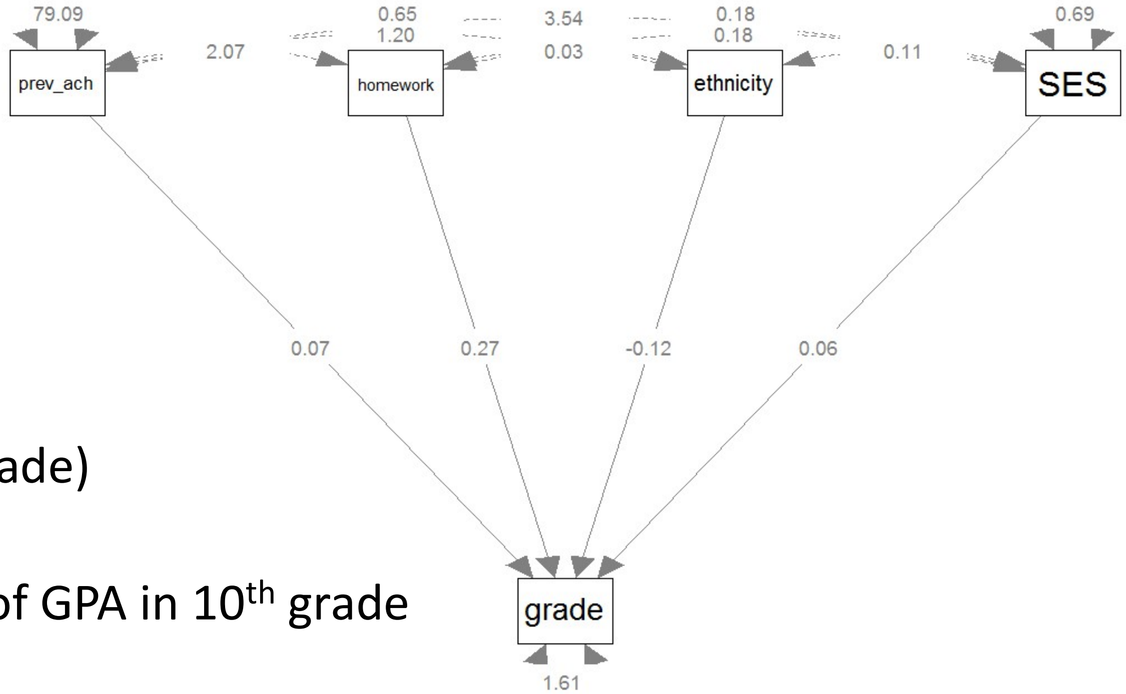
Model: Multiple regression

- Dependent:
  - GPA in 10th grade
- Independent:
  - Ethnicity
  - Homework (8th grade)
  - Previous achievement (8th grade)
  - Socio-economic status



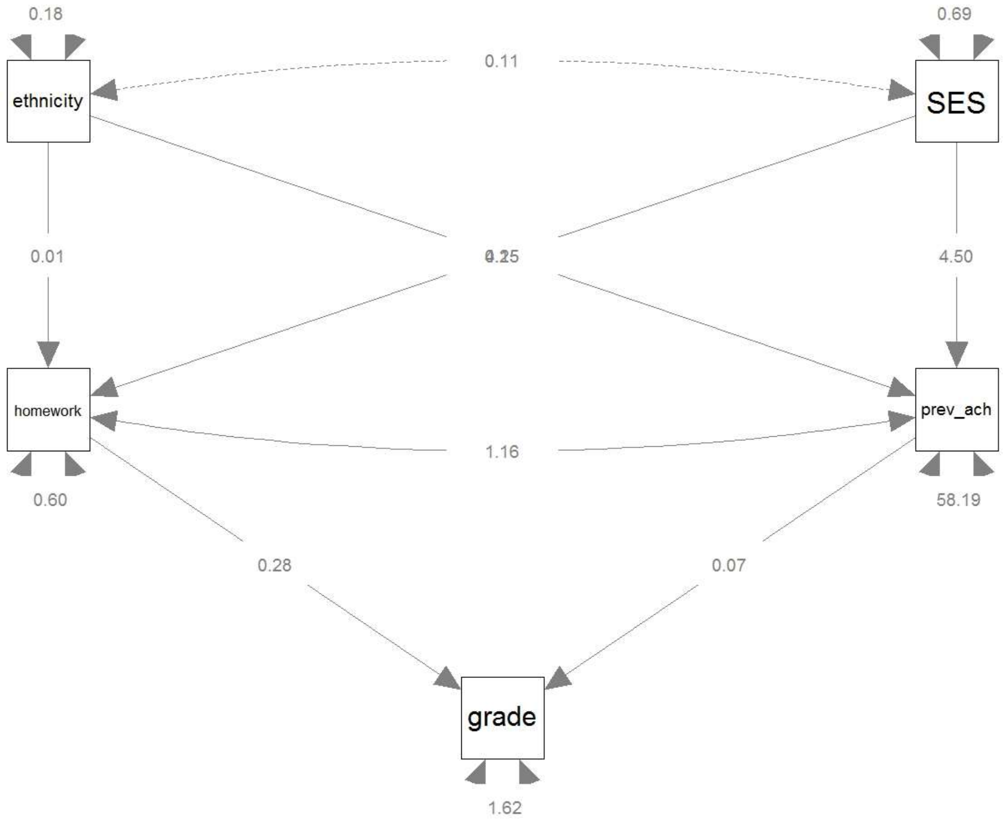
- Regression estimates are now a vector of partial regression coefficients, need matrix algebra to compute: $\boldsymbol{\hat{\beta}} = (\boldsymbol{X^T X})^{-1}\boldsymbol{X^T y}$
- Measure of fit: multiple correlation (R=.512), or variance explained (R^2=.262)
- Measure of (strength of) association: $\hat{b}_{xy}$ or standardized $\hat{b}^*_{xy} = s_x \frac{\hat{b}_{xy}}{s_y}$ (where $\hat{b}_{xy}$ is a partial regression coefficient)

Model: SEM



- *Endogenous* variables:
  - GPA in 10th grade
- *Exogenous* variables:
  - Ethnicity
  - Homework (8th grade)
  - Previous achievement (8th grade)
  - Socio-economic status
  - (disturbance/error/residual) of GPA in 10th grade
- Regression estimates are still a vector of partial regression coefficients, need matrix algebra and optimization to compute
- Measure of (strength of) associations: Partial regression coefficients
- Overall model fit: How well are the observed variables' (co)variances reproduced by the model?
  - Quantified by a χ^2value and model fit indices

Model: SEM



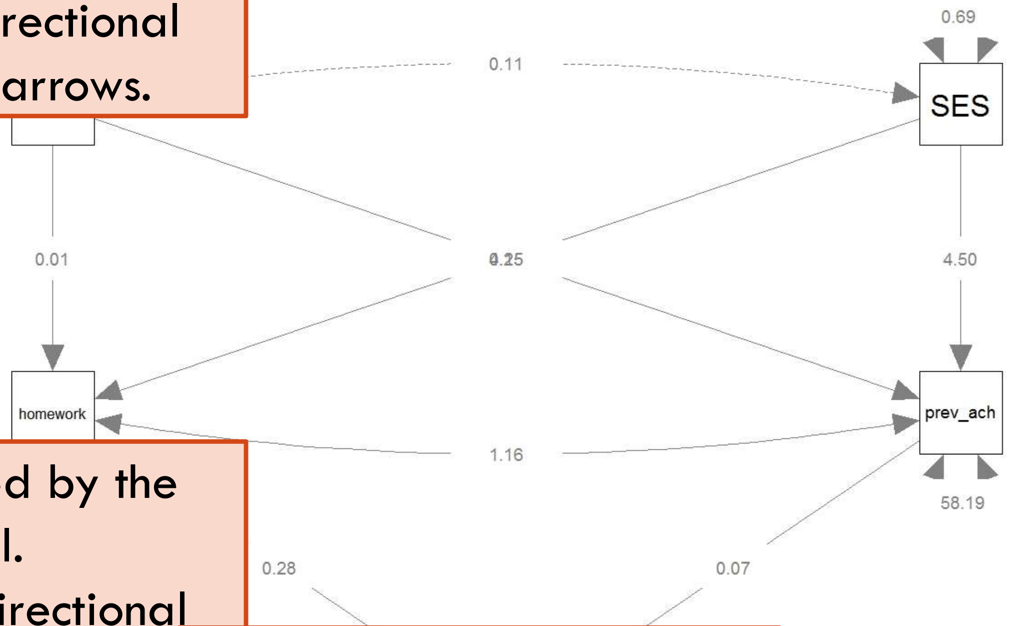
- *Endogenous* variables:
  - Homework (8th grade)
  - Previous achievement (8th grade)
  - GPA in 10th grade
- *Exogenous* variables:
  - Ethnicity
  - Socio-economic status
  - (disturbances/errors/residuals of
    - Homework (8th grade)
    - Previous achievement (8th grade)
    - GPA in 10th grade
- Measure of (strength of) associations: partial regression coefficients
- Overall model fit: How well are the observed variables' (co)variances reproduced by the model?
  - Quantified by a χ^2value and model fit indices

Model: SEM

Explained by the model.
Have unidirectional incoming arrows.

From Greek *endo*, meaning 'inside', and *gignomai*

- ***Endogenous* variables:**
  - Homework (8th grade)
  - Previous achievement (8th grade)
  - GPA in 10th grade
- ***Exogenous* variables:**
  - Ethnicity
  - Socio-economic sta
  - (disturbances/error
    - Homework (8th gr
    - Previous achievement (8th grade)
    - GPA in 10th grade



Not explained by the model.
Have no unidirectional incoming arrow(s).

From Greek exo, meaning 'outside', and *gignomai*, meaning 'to produce'

- Measure of (strength of) associations: partial regression coefficients
- Overall model fit: How well are the observed variables' (co)variances reproduced by the model?
  - Quantified by a χ^2value and model fit indices

SEM using lavaan

To fit a SEM in R with lavaan, we need two things:

1. Data, which can be:
   - Raw data, which is often an external file (e.g., .sav, .xls) which needs to be loaded into R (most common case in practice)
   - Covariance or correlation matrix, which can be an external file, or can be entered manually (most often the case in book's examples and exercises)
2. Model specification:
   - A long character string that specifies whether population parameters are fixed (e.g., to a constant like 1 or 0) restricted (e.g., to be equal to another parameter) or should be freely estimated, using lavaan model syntax

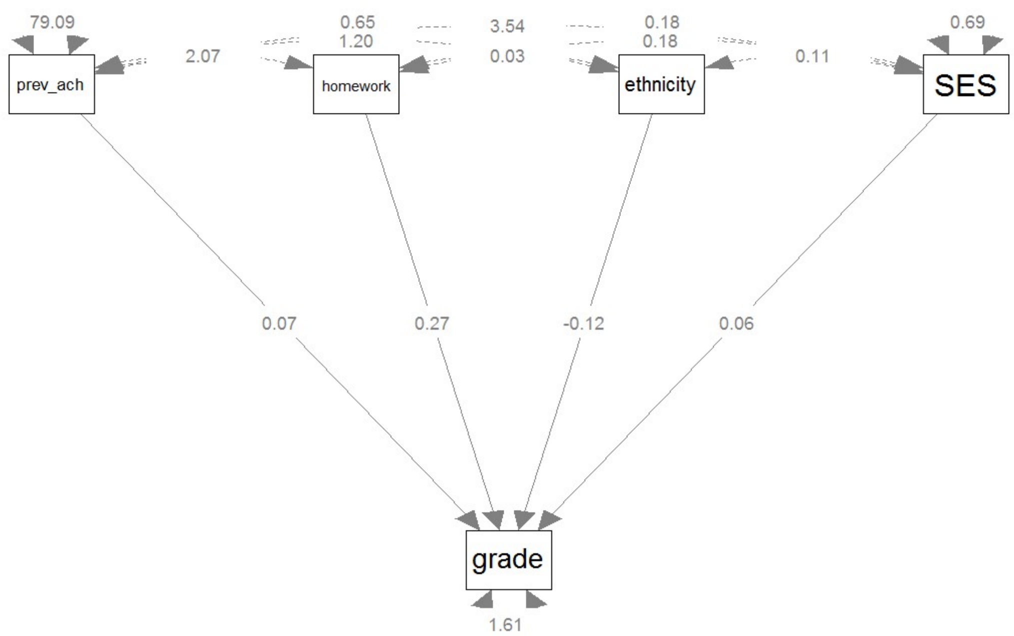
Lavaan model syntax

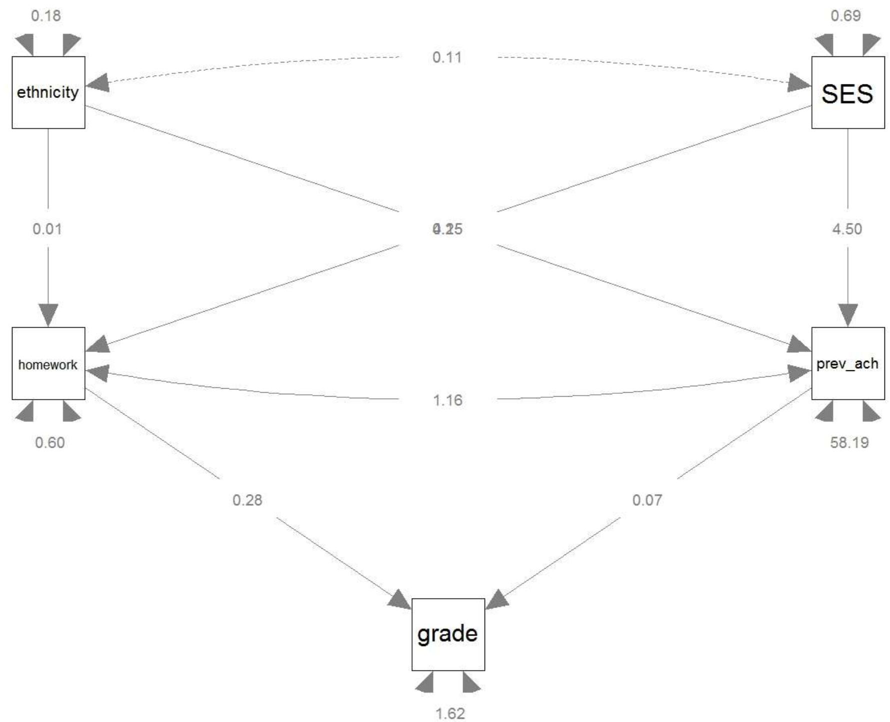
<table>
<tr><th>Syntax</th><th>Command</th><th>Example</th></tr>
<tr><td>~</td><td>Regress onto</td><td>Regress B onto A: B ~ A</td></tr>
<tr><td>~~</td><td>(Co)variance</td><td>Variance of A: A ~~ A
Covariance of A and B: A ~~ B</td></tr>
<tr><td>~1</td><td>Constant/mean/intercept</td><td>Regress B onto A, and and include the intercept in the model: B ~ 1 + A or
B ~ A
B ~ 1</td></tr>
<tr><td>=~</td><td>Define reflective latent variable</td><td>Define Factor 1 by A-D:
F1 =~A+B+C+D</td></tr>
<tr><td><~</td><td>Define formative latent variable</td><td>Define Factor 1 by A-D:
F1 <~ 1*A+B+C+D</td></tr>
<tr><td>:=</td><td>Define non-model parameter</td><td>Define parameter u2 to be twice the square of u:
u2 := 2*(u^2)</td></tr>
<tr><td>*</td><td>Label parameters (the label has to be pre-multiplied)</td><td>Label the regression of Z onto X as b:
Z ~b*X</td></tr>
<tr><td>|</td><td>Define the number of thresholds (for categorical endogenous variables)</td><td>Variable u has three thresholds:
u | t1 + t2 + t3</td></tr>
</table>

grade | ethnicity | homework | SES | Prev_ach

Syntax	Command	Example
~	Regress onto	Regress B onto A: `B ~ A`
~~	(Co)variance	Variance of A: `A ~~ A`
		Covariance of A and B: `A ~~ B`

Q: How do we specify these models in lavaan syntax?





Computation time!

Example 2.4.1

- get PDF from Github

Make Exercise 2.1:

- Get Exercises_week_1.pdf from Github (adapted version of the exercises in the Beaujean book)

Structural Equation Modeling

- Fitted model is used to explain associations between observed variables
- I.e., to explain sample covariances between observed variables:

$$\mathrm{cov}_{xy} = \left(\frac{1}{N-1}\right)\sum_i (X_i - \bar{X})(Y_i - \bar{Y})$$

$$\mathrm{cov}_{xy} = r_{xy} SD_x SD_y$$

  - Note: means, and skewness & kurtosis can be also be involved in SEM (discussed later in course)

Structural Equation Modeling

- With SEM, we obtain a fitted model that minimizes the difference between
  - sample matrix of observed covariances $\mathbf{S}$ and
  - population matrix of model-implied covariances $\widehat{\mathbf{\Sigma}}$
    - In addition, we try to keep the model parsimoneous through applying restrictions (i.e., specifying the model) so that not all possible paths are estimated
- These covariance matrices contain all (co)variances of the observed variables in the model. Note that:
  - Covariance matrices are always symmetric, because cov(x,y)=cov(y,x)
  - Covariance matrices have the variance of the observed variables on the diagonal. I.e., cov(x,x) = var(x)

Model-implied (co)variances

- Variables in the model:

grade	ethnicity	homework	SES	Prev_ach

- Observed covariance matrix **S**:

	grade	homwrk	prv_ch	ethnct	SES
grade	2.185				
homework	0.335	0.649			
prev_ach	6.429	2.067	79.092		
ethnicity	0.081	0.028	1.201	0.175	
SES	0.338	0.176	3.541	0.106	0.690

- Once the model is estimated, the model-implied covariance matrix $\hat{\Sigma}$ can be calculated using path analysis, or equivalently, matrix algebra

Model-implied (co)variances

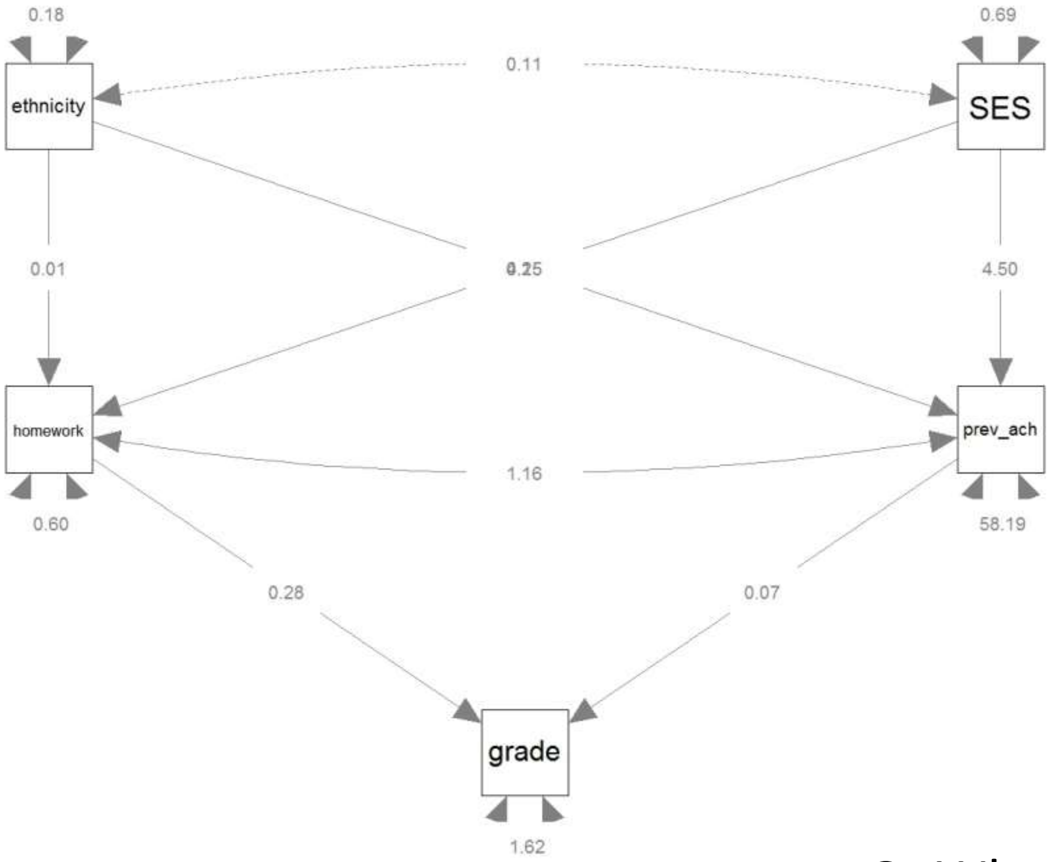
Path analysis:

- Model-implied covariance between variables X and Y can be computed as follows:
  - Find all paths leading from X to Y
  - Multiply all parameter values along a given path from X to Y, but:
    - No loops: may not go through same variable more than once
    - May switch forward/backward direction only once within a path
    - May go through double-headed arrow only once within a path
  - Summing all values thus obtained

- Variances of variables are calculated as follows:
  - For exogenous variables, model-implied variances are equal to sample variances, so are given (not computed)
  - For endogenous variables, variances are computed like covariances (rules above)

Model-implied (co)variances

Model:

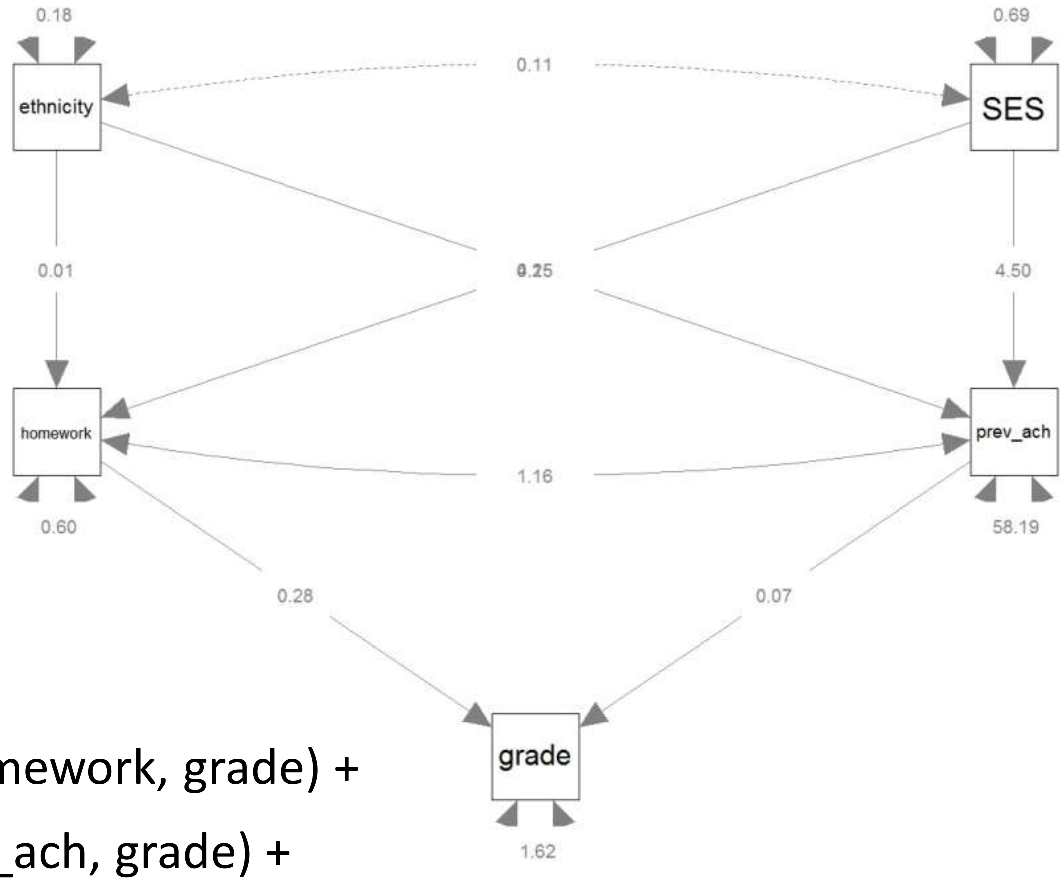


Parameter estimates:

```
      lhs op       rhs    est
    grade  ~  prev_ach  0.074
    grade  ~  homework  0.281
 homework  ~ ethnicity  0.007
 homework  ~       SES  0.254
 prev_ach  ~ ethnicity  4.147
 prev_ach  ~       SES  4.496
 homework ~~  prev_ach  1.158
    grade ~~     grade  1.616
 homework ~~  homework  0.604
 prev_ach ~~  prev_ach 58.190
ethnicity ~~ ethnicity  0.175
ethnicity ~~       SES  0.106
      SES ~~       SES  0.690
```

Q: What is the model-implied cov(SES, grade)?

Model-implied (co)variances



model-implied cov(SES, grade)=

var(SES) * b(SES, homework) * b(homework, grade) +

var(SES) * b(SES, prev_ach) * b(prev_ach, grade) +

cov(SES, ethnicity) * b(ethnicity, homework) * b(homework, grade) +

cov(SES, ethnicity) * b(ethnicity, prev_ach) * b(prev_ach, grade)

Model-implied (

Note that Beaujean's examples in section 2.1.3 seem more simple, because he uses the standardized solution. Then all variances of exogenous variables equal 1 and can be omitted, which simplifies calculations a lot.

model-implied cov(SES,grade)=

var(SES) * b(SES, homework) * b(homework, grade) +

var(SES) * b(SES, prev_ach) * b(prev_ach, grade) +

cov(SES, ethnicity) * b(ethnicity, homework) * b(homework, grade) +

cov(SES, ethnicity) * b(ethnicity, prev_ach) * b(prev_ach, grade) =

.690 * .254 * .281 +

.690 * 4.496 * .074 +

.106 * .007 * .281 +

.106 * 4.147 * .074 =

0.3115514

lhs	op	rhs	est
grade	~	prev_ach	0.074
grade	~	homework	0.281
homework	~	ethnicity	0.007
homework	~	SES	0.254
prev_ach	~	ethnicity	4.147
prev_ach	~	SES	4.496
homework	~~	prev_ach	1.158
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prev_ach	~~	prev_ach	58.190
ethnicity	~~	ethnicity	0.175
ethnicity	~~	SES	0.106
SES	~~	SES	0.690

Model-implied (co)variances

- A SEM is a system of linear equations, which we can represent by matrices
  - Although non-linear SEM also exists, but outside the scope of this course
- The path tracing rules represent matrix algebra but more tedious/confusing/error prone
- Beaujean’s book hardly involves formulas, and no matrix notation. To get a good understanding of SEM, you need to know about underlying matrices and vectors

Model-implied (co)variances

- In lavaan, the (co)variance stucture of a fitted model is given by four parameter matrices
- Matrix algebra gives us the model-implied covariance matrix:

$$\widehat{\boldsymbol{\Sigma}} = \boldsymbol{\Lambda}(\mathbf{I} - \boldsymbol{\beta})^{-1} \boldsymbol{\psi} \left[(\mathbf{I} - \boldsymbol{\beta})^{-1} \right]^{\mathbf{T}} \boldsymbol{\Lambda}^{\mathbf{T}} + \boldsymbol{\Theta}$$

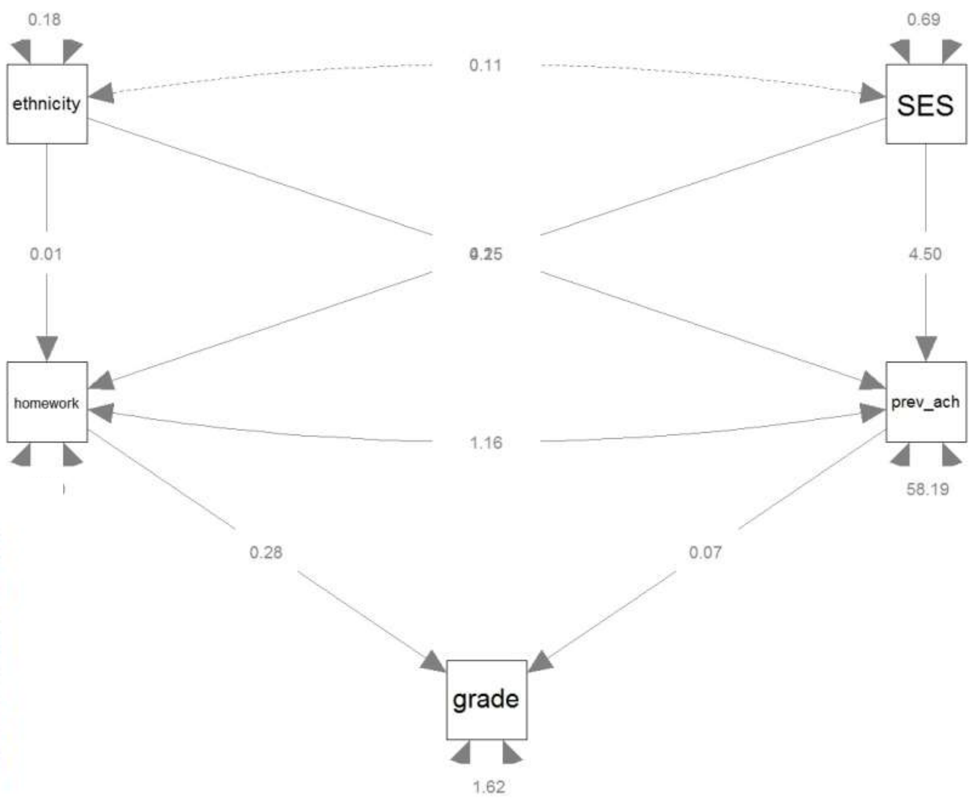
- Today, our models assume no measurement error, so $\boldsymbol{\Lambda}$ is an identity matrix and $\boldsymbol{\Theta}$ all zeros. Thus, the above formula simplifies to:

$$\widehat{\boldsymbol{\Sigma}} = (\mathbf{I} - \boldsymbol{\beta})^{-1} \boldsymbol{\psi} \left[(\mathbf{I} - \boldsymbol{\beta})^{-1} \right]^{\mathbf{T}}$$

Model-implied (co)variances

- Let p be the number of observed variables in the model
- If we have observed variables only:
  - $\boldsymbol{\beta}$ is a p x p matrix of regression coefficients, relating predictor to criterion variables
    - ‘Contains’ single-headed (directed) arrows, therefore non-symmetric
    - The columns reflect the variables as predictors, the rows reflect the variables as responses
  - $\boldsymbol{\psi}$ is a p x p matrix of (co)variances not explained by the regression equations
    - ‘Contains’ double headed (undirected) arrows, therefore symmetric
- $\boldsymbol{\psi}$ and $\boldsymbol{\beta}$ describe the **structural** model
- Often, SEM models also involve a **measurement** model (described by $\boldsymbol{\Lambda}$ and $\boldsymbol{\Theta}$, which will be introduced next session)

Model-implied (co)variances




```
> beta
          grade homwrk prv_ch ethnct   SES
grade         0  0.281  0.074  0.000 0.000
homework      0  0.000  0.000  0.007 0.254
prev_ach      0  0.000  0.000  4.147 4.496
ethnicity     0  0.000  0.000  0.000 0.000
SES           0  0.000  0.000  0.000 0.000
> psi
          grade  homwrk prv_ch ethnct SES
grade      1.616
homework   0.000  0.604
prev_ach   0.000  1.158 58.190
ethnicity  0.000  0.000  0.000  0.175
SES        0.000  0.000  0.000  0.106  0.690
```

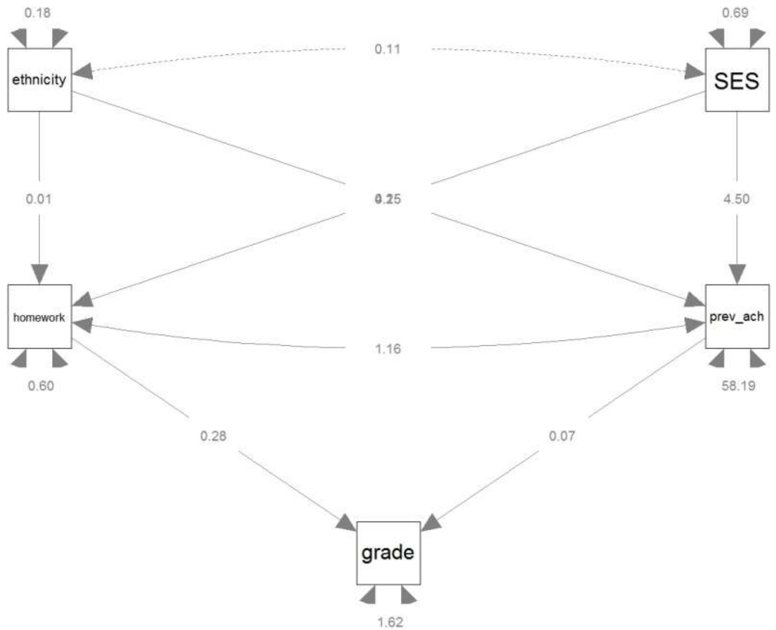
Structural and measurement model

Two main components of SEMs:

- the ***structural model*** contains *causal* regression relationships between endogenous and exogenous variables
  - path models (without measurement errors) can be viewed as SEMs that contain only the structural model
- the ***measurement model*** contains the associations between latent variables and their indicators
  - confirmatory factor analysis models contain only the measurement part

Model-implied (co)variances

Model:



Parameter estimates:

lhs	op	rhs	est
grade	~	prev_ach	0.074
grade	~	homework	0.281
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prev_ach	~	SES	4.496
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homework	~~	homework	0.604
prev_ach	~~	prev_ach	58.190
ethnicity	~~	ethnicity	0.175
ethnicity	~~	SES	0.106
SES	~~	SES	0.690

Model-implied covariance matrix $\widehat{\boldsymbol{\Sigma}}$:

	grade	homwrk	prv_ch	ethnct	SES
grade	2.185				
homework	0.335	0.649			
prev_ach	6.429	2.067	79.092		
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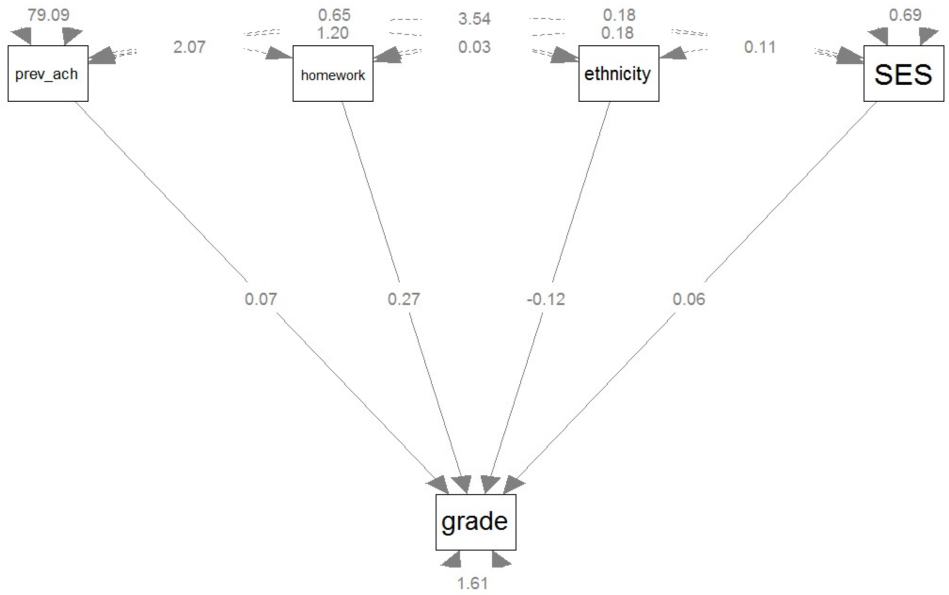
Model-implied (co)variances

Model:



Parameter estimates:

lhs	op	rhs	est
grade	~	prev_ach	0.073
grade	~	homework	0.271
grade	~	ethnicity	-0.119
grade	~	SES	0.063
grade	~~	grade	1.612
prev_ach	~~	prev_ach	79.092
prev_ach	~~	homework	2.067
prev_ach	~~	ethnicity	1.201
prev_ach	~~	SES	3.541
homework	~~	homework	0.649
homework	~~	ethnicity	0.028
homework	~~	SES	0.176
ethnicity	~~	ethnicity	0.175
ethnicity	~~	SES	0.106
SES	~~	SES	0.690

Model-implied covariance matrix $\widehat{\boldsymbol{\Sigma}}$:

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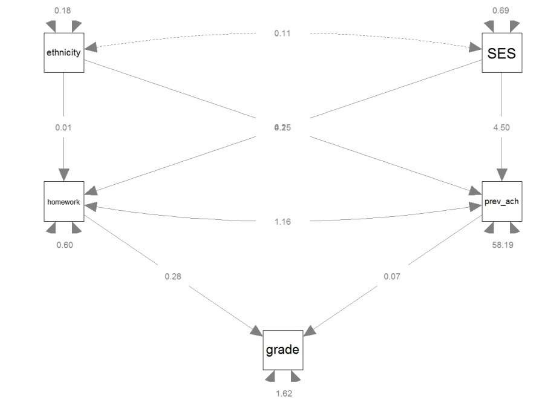
Model-implied (co)variances



$\hat{\Sigma} =$

	grade	homwrk	prv_ch	ethnct	SES
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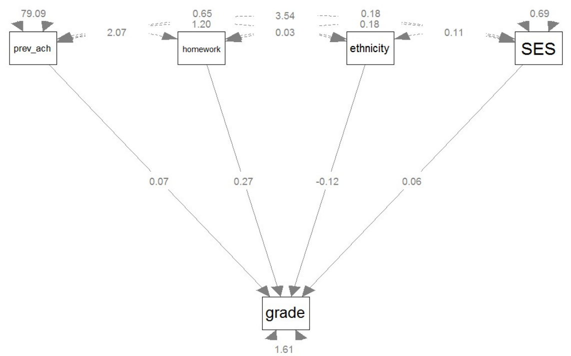

$\hat{\Sigma} =$

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SES	0.338	3.541	0.176	0.106	0.690

Which model fits data best (i.e., approximates sample covariances best)?

Which is most parsimonious (i.e., estimates lowest number of population parameters)?

S =

	grade	homwrk	prv_ch	ethnct	SES
grade	2.185				
homework	0.335	0.649			
prev_ach	6.429	2.067	79.092		
ethnicity	0.081	0.028	1.201	0.175	
SES	0.338	0.176	3.541	0.106	0.690

Variances of exogenous variables often not explicitly depicted

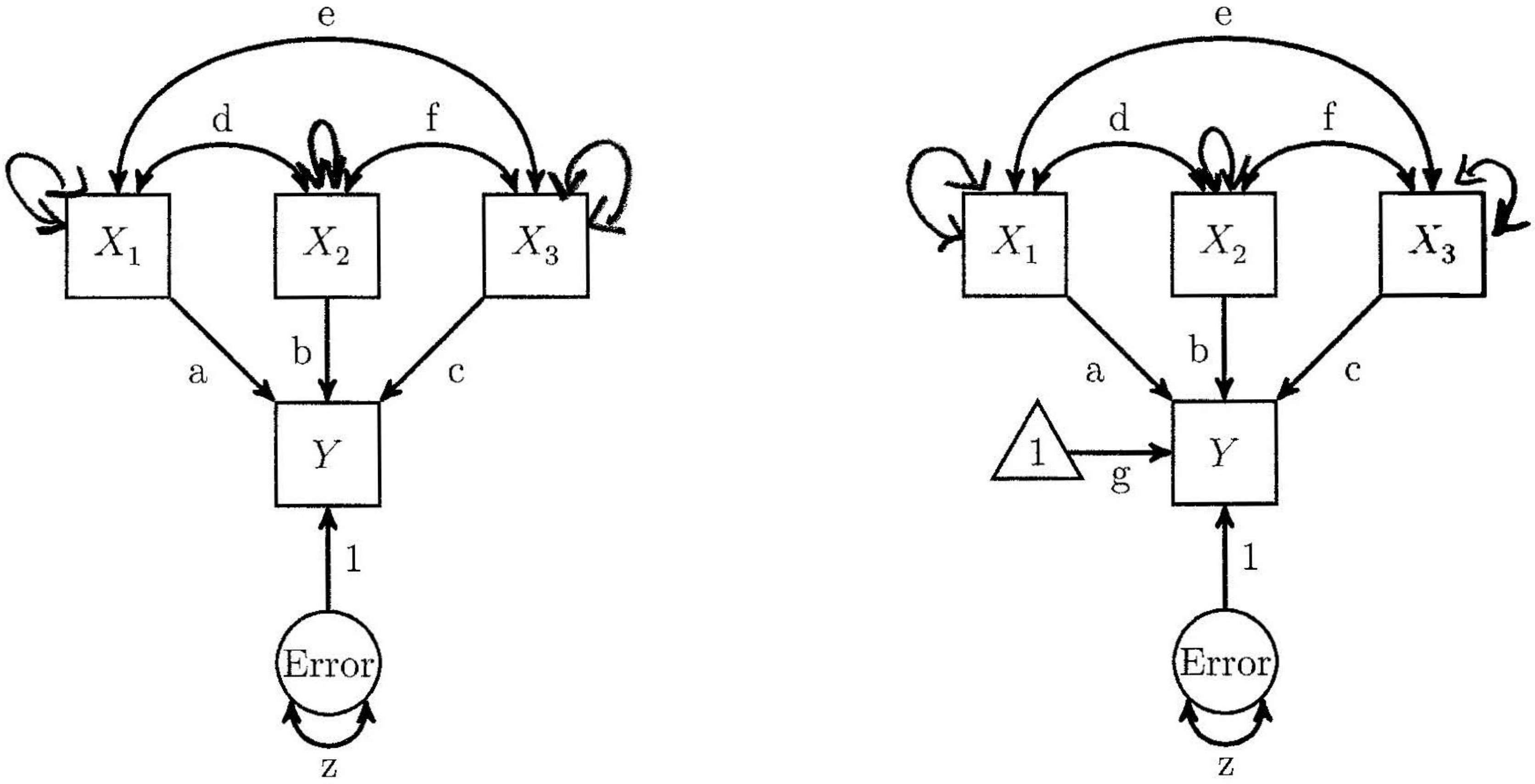
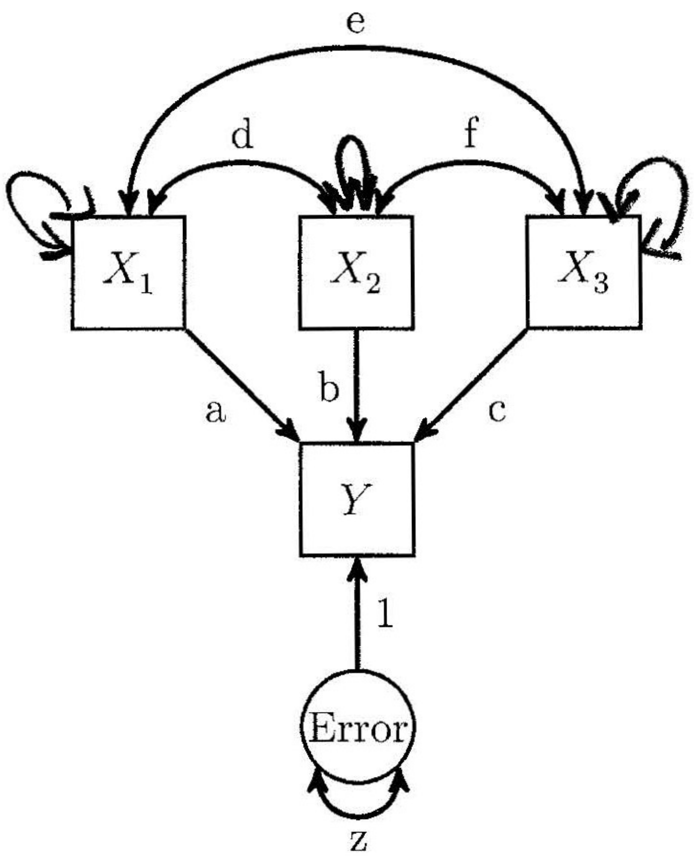
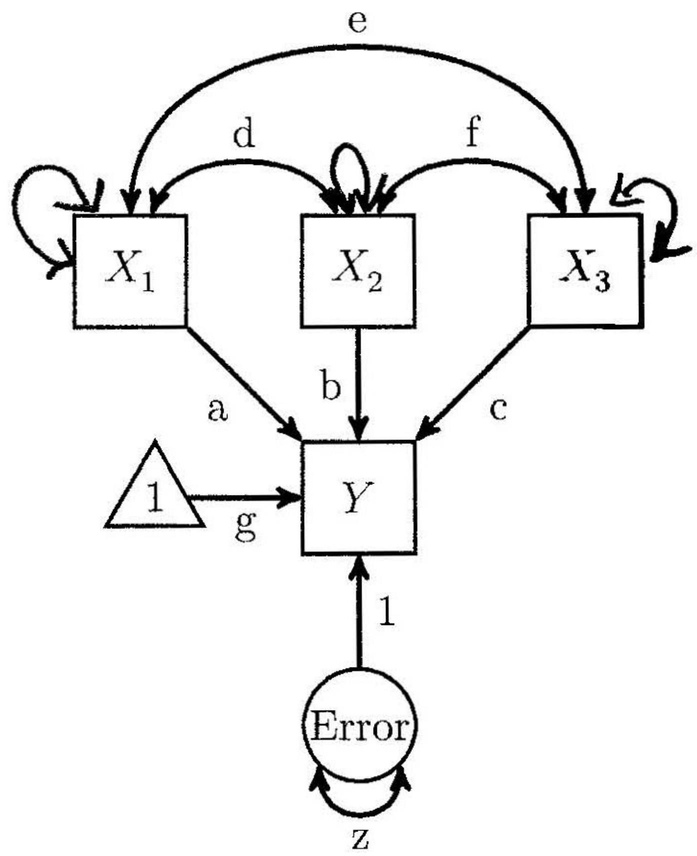


Figure 2.2 Path model of a multiple regression with three predictor (exogenous) variables.

Mean structure often omitted



(co)variance structure only
all means omitted (i.e., assumed zero)
$Y = aX_1 + bX_2 + cX_3 + error$



(co)variance and mean structure
means freely estimated
$Y = g + aX_1 + bX_2 + cX_3 + error$

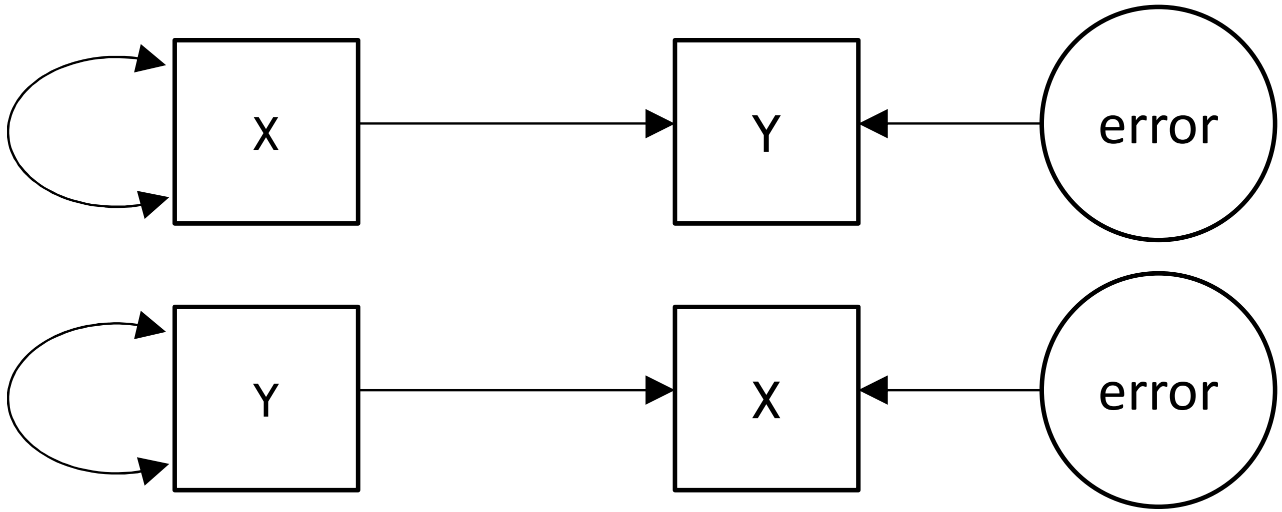
Figure 2.2 Path model of a multiple regression with three predictor (exogenous) variables.

Error terms

- Errors are also latent variables: they are hypothetical, not directly observed
- Error is defined as the difference between observed (sample) variance and variance explained by other variables in the model
  - Therefore, a variable that has an error/disturbance term is an endogenous variable (vice versa)
  - Errors/disturbance terms are always exogenous (i.e., no incoming directional arrows)

Causation

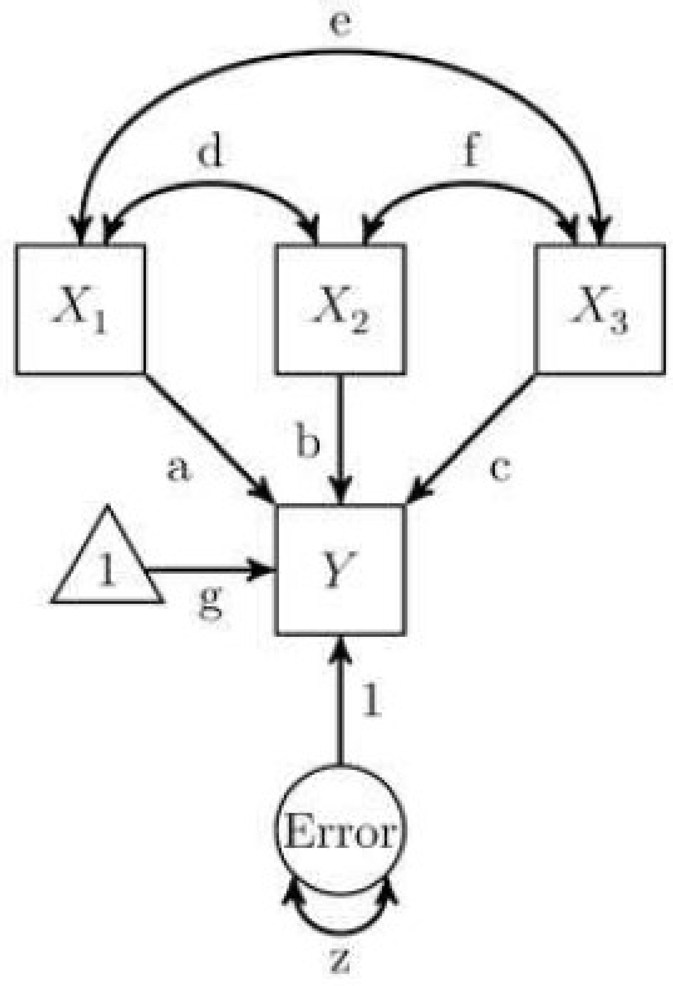
- Causality needs to be ensured through the research design, cannot be statistically proven



- Both models above will fit the observed data equally well, it is up to the researcher to decide on the direction of the arrows!
  - In the SEM model, it is merely a matter of scaling: $b_x = \frac{cov_{xy}}{var_x}$ and $b_y = \frac{cov_{xy}}{var_y}$

Path & partial regression coefficients

- Path coefficients (a, b, c, g and 1) are partial regression coefficients
- That is, the expected increase in the response variable, when the predictor variable increases by 1, controlling for (= keeping constant) all the other predictor variables
  - Note that the intercept is always 1, so cannot in- or decrease



Standardized coefficients

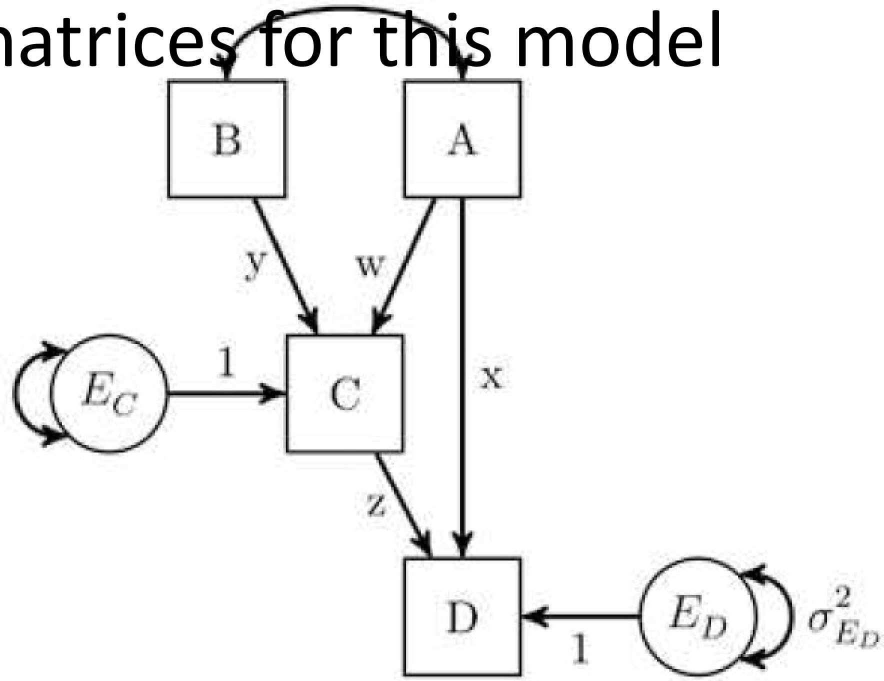
- Parameter estimates (path coefficients) can be standardized and unstandardized
  - Unstandardized: Interpret like regression coefficients
    - Expected increase in Y if X increases by 1
  - Standardized: Interpret like correlation coefficients
    - Expected increase in SDs of Y if X increases by 1 SD
    - 0: no linear association; -1: perfect negative association; 1: perfect positive association
    - squared standardized coefficient = prop. of variance in Y explained by X (vice versa)

Lavaan model syntax

Syntax	Command	Example
`~`	Regress onto	Regress B onto A: `B ~ A`
`~~`	(Co)variance	Variance of A: `A ~~ A`
		Covariance of A and B: `A ~~ B`
`~1`	Constant/mean/intercept	Regress B onto A, and and include the intercept in the model: `B ~ 1 + A` or
		`B ~ A`
		`B ~ 1`
`=~`	Define reflective latent variable	Define Factor 1 by A-D:
		`F1 =~A+B+C+D`
`<~`	Define formative latent variable	Define Factor 1 by A-D:
		`F1 <~ 1*A+B+C+D`
`:=`	Define non-model parameter	Define parameter u2 to be twice the square of u:
		`u2 := 2*(u^2)`
`*`	Label parameters (the label has to be pre-multiplied)	Label the regression of Z onto X as b:
		`Z ~b*X`
`|`	Define the number of thresholds (for categorical endogenous variables)	Variable u has three thresholds:
		`u | t1 + t2 + t3`

Lavaan syntax exercise

1) How do we write the model below in lavaan syntax?
2) How can we label and refer to the indirect effect from A on D via C in lavaan syntax?
3) What do the beta and psi matrices for this model look like?



Note that Beaujean often labels paths in lavaan syntax, but that is not required - I never do it, unless there are indirect effects that I want to explicitly define in the model. Labeling or not does not make a difference for the estimated parameters and model fit.

Homework

- Exercises 2.2 and 2.3 (see PDF on Brightspace)
- See Example-2.4.1.pdf on Brightspace for instructions on extracting beta and psi matrices