Introducing Bayesian Inference

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Inference in a Bernoulli Distribution

e.g., inferring a coin's bias

> *. . . Polish mathematicians Tomasz Gliszczynski and Waclaw Zawadowski. . . spun a Belgian one euro coin 250 times, and found it landed heads up 140 times . . . When tossed 250 times, the one euro coin came up heads 139 times and tails 111. . . .*
>
> *The Guardian*, January 4, 2002[1]

- A sample of $n = 250$ coin tosses can be modelled as n independent and identically distributed (iid) Bernoulli random variables with parameter θ, which represents the coin's bias.
- In other words, our *probabilistic generative model* of the data is

$$x_i \sim \text{Bernoulli}(\theta), \quad \text{for } i \in \{1, 2 \ldots n\}.$$

  where $x_i \in \{0, 1\}$ represents the outcome of the coin flip i.
- Our aim is to infer the probable values of θ given an observation of $m = 139$ (or $m = 140$, etc.).

[1] See `http://bit.ly/1BOKu9b` for original story and `http://bit.ly/1BOKx4Q` for discussion.

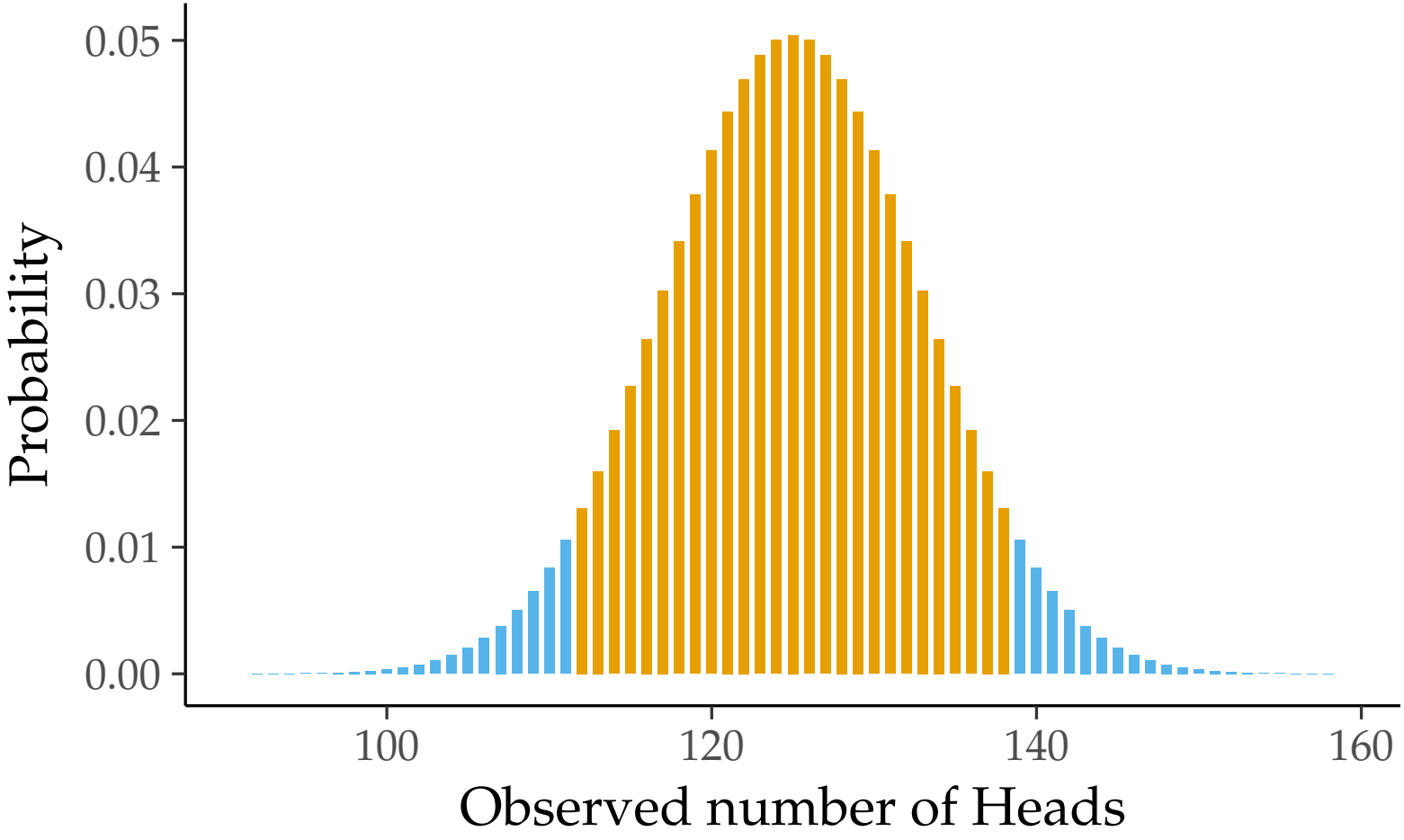
Sampling theory based inference

In a nutshell

- The traditional sampling theory based approach to inference begins by considering the *sampling distribution* of outcomes according to some hypothetical value of θ, which we signify with θ_0.
- In our case, the sampling distribution is a binomial distribution with parameters $n = 250$ and $\theta = \theta_0$.
- We then calculate the probability of observing data *as or more extreme* that the result we obtained, i.e. $m = 139$ Heads.
- This probability is exactly the p-value, and effectively tells us how typical or unsurprising an outcome of $m = 139$ Heads would be if the true value of θ was θ_0.
- Furthermore, the set of values of θ_0 that we would not reject at the α level of significance is exactly the $1 - \alpha$ *confidence interval*.

Sampling theory based inference

A binomial distribution with $n = 250$ *and* $\theta = 0.5$.



- Areas shaded in blue are values *as or more extreme* than $m = 139$.
- For the interactive version:
  https://lawsofthought.shinyapps.io/binomial_test/.

Inference using the likelihood function

- The likelihood function is a function over the parameter space, i.e. over θ.
- It gives the probability[2] of the observed data, i.e. $m = 139$ Heads in $n = 250$ coin flips, for all possible values of θ.
- As such, it tells us how well any possible value of θ predicts the observed data.
- We usually denote the likelihood function as follows:

$$L(\theta|\mathcal{D}) \propto P(\mathcal{D}|\theta),$$

where $\mathcal{D}$ signifies the observed data, i.e. $\mathcal{D} = \{n = 250, m = 130\}$.

[2] Strictly speaking, it gives the probability times an arbitrary positive constant c.

The likelihood function versus sampling distribution

- The sampling distribution is the binomial distribution:

$$P(m|\theta, n) = \binom{n}{m} \theta^m (1-\theta)^{n-m}.$$

  This is always a function of m, with θ and n fixed.
- The corresponding likelihood function treats the same function

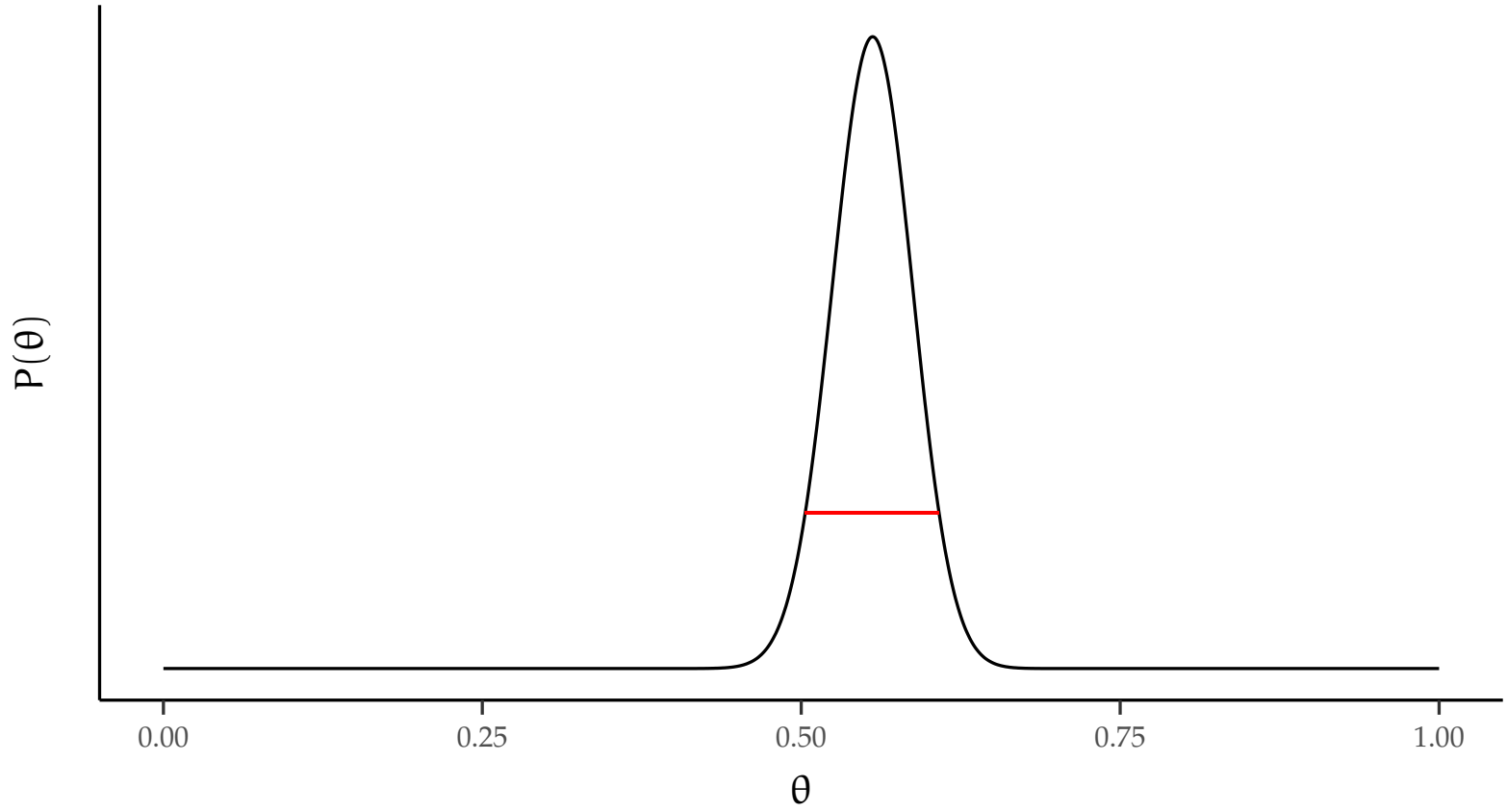
$$P(m|\theta, n) = \binom{n}{m} \theta^m (1-\theta)^{n-m},$$

  as a function of θ, but now with m and n fixed.
- As such, the binomial likelihood is

$$L(\theta|\mathcal{D}) \propto \theta^m (1-\theta)^{n-m}.$$

Binomial likelihood

Given $\mathfrak{n} = 250$ *and* $\mathfrak{m} = 139$



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- The red interval gives the values of θ that are more than $\frac{1}{8}$ the value of the maximum likelihood estimator.
- For the interactive version: https://lawsofthought.shinyapps.io/binomial_likelihood/.

The likelihood principle

> *Within the framework of a statistical model, all the information which the data provide concerning the relative merits of two hypotheses is contained in the likelihood ratio of those hypotheses on the data.*
>
> Edwards *Likelihood* (1972, p.30)

- In any statistical model, all information provided by the observed data that are relevant to the parameters is given by the likelihood function.
- In our case, all information in the observed data concerning the true value of θ is given by $\theta^m(1-\theta)^{n-m}$.

The law of the likelihood

> *Within the framework of a statistical model, a particular set of data supports one statistical hypothesis better than another if the likelihood of the first hypothesis, on the data, exceeds the likelihood of the second hypothesis.*
>
> Edwards *Likelihood* (1972, p.30)

- In our case, how well the observed data support the hypothesis that $\theta = \theta_1$ versus $\theta = \theta_0$ is given by

$$\frac{L(\theta_1|\mathcal{D})}{L(\theta_0|\mathcal{D})},$$

and the data support the hypothesis $\theta = \theta_1$ more than $\theta = \theta_0$ if

$$\frac{L(\theta_1|\mathcal{D})}{L(\theta_0|\mathcal{D})} > 1.0.$$

The likelihood interval

- The likelihood interval gives the set of possible values of θ for which the data provide non-trivial evidence.
- The $\gamma \in (0, 1)$ likelihood interval is

$$\left\{\theta\colon \frac{L(\theta|\mathcal{D})}{L(\theta_{MLE}|\mathcal{D})} > \gamma\right\},$$

  where θ_{MLE} is the maximum likelihood estimator.
- All values within the γ likelihood interval have evidential support that is greater than γ times the support of θ_{MLE}.
- Typical values of γ are $\frac{1}{8}$, $\frac{1}{32}$, etc.

Bayesian inference of a coin's bias

- The probabilistic generative model of the Euro coin toss data is as follows:
  - The coin's bias corresponds to the fixed but unknown value of the parameter θ of a Bernoulli random variable.
  - The observed outcomes $x_1, x_2 \cdots x_n$ are n iid samples from Bernoulli(θ).
- This generative model can be extended by assuming that θ is itself drawn from a prior distribution $P(\theta)$:

$$\begin{aligned} \theta &\sim P(\theta), \\ x_i &\sim \text{Bernoulli}(\theta), \quad \text{for } i \in \{1, 2 \ldots n\}. \end{aligned}$$

- In other words, we assume that a value for θ was randomly drawn from $P(\theta)$ and then n binary variables were sampled from Bernoulli(θ).
- We refer to $P(\theta)$ as the *prior* distribution on θ.

Conjugate Priors

- For a given likelihood function, a *conjugate prior* distribution is a prior probability distribution that leads to a posterior distribution of the same parametric family.
- Using conjugate priors allows Bayesian inference and other probabilistic calculations to be performed analytically.
- Only a small subset of probabilistic models have conjugate priors.
- However, conjugate priors play a vital role in Monte Carlo methods like Gibbs sampling even in complex models.

The beta distribution

- For the binomial likelihood function

$$\theta^m(1-\theta)^{n-m}$$

a conjugate prior is the beta distribution

$$\text{Beta}(\theta|\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

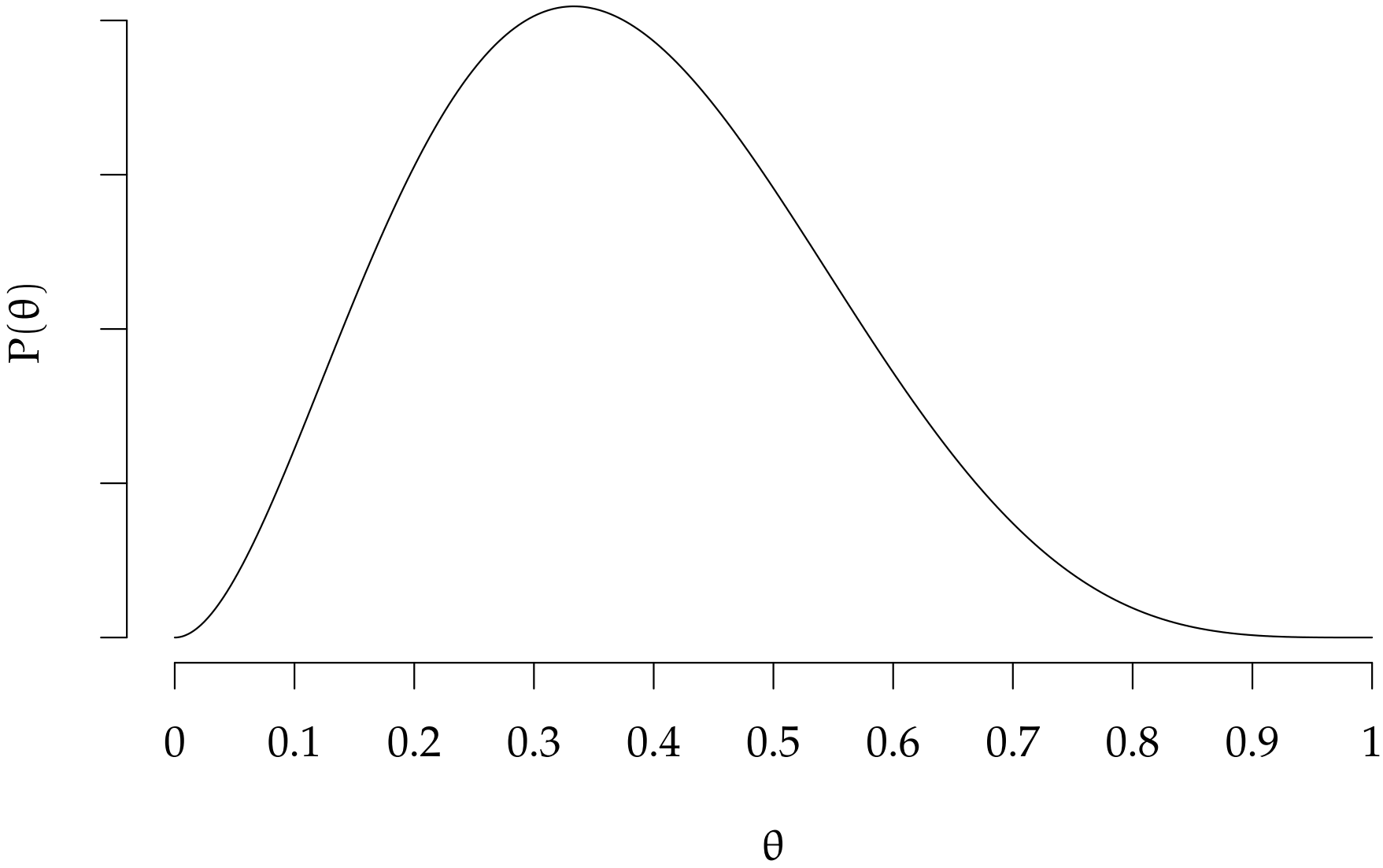
- The *normalizing constant* term is

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} = \frac{1}{B(\alpha,\beta)},$$

where $B(\alpha,\beta)$ is the beta function:

$$B(\alpha,\beta) = \int \theta^{\alpha}(1-\theta)^{\beta-1}\,d\theta.$$

The beta distribution



The beta distribution with $\alpha = 3$ and $\beta = 5$.

Posterior distribution

- Denoting the observed data by $D = (n, m)$, with the beta prior, the posterior distribution is

$$\begin{aligned} P(\theta|D, \alpha, \beta) &= \frac{P(D|\theta)P(\theta|\alpha, \beta)}{\int P(D|\theta)P(\theta|\alpha, \beta)\, d\theta}, \\ &\propto \overbrace{\theta^m(1-\theta)^{n-m}}^{\text{likelihood}} \times \overbrace{\theta^{\alpha-1}(1-\theta)^{\beta-1}}^{\text{prior}}, \\ &\propto \theta^{m+\alpha-1}(1-\theta)^{n-m+\beta-1}, \\ &= \text{Beta}(m+\alpha, n-m+\beta). \end{aligned}$$

where the normalizing constant is the reciprocal of the beta function

$$\begin{aligned} \frac{\Gamma(m+\alpha)\Gamma(n-m+\beta)}{\Gamma(n+\alpha+\beta)} &= \int \theta^{\alpha+m-1}(1-\theta)^{\beta+n-m-1}\, d\theta. \\ &= B(\alpha+m, \beta+n-m). \end{aligned}$$

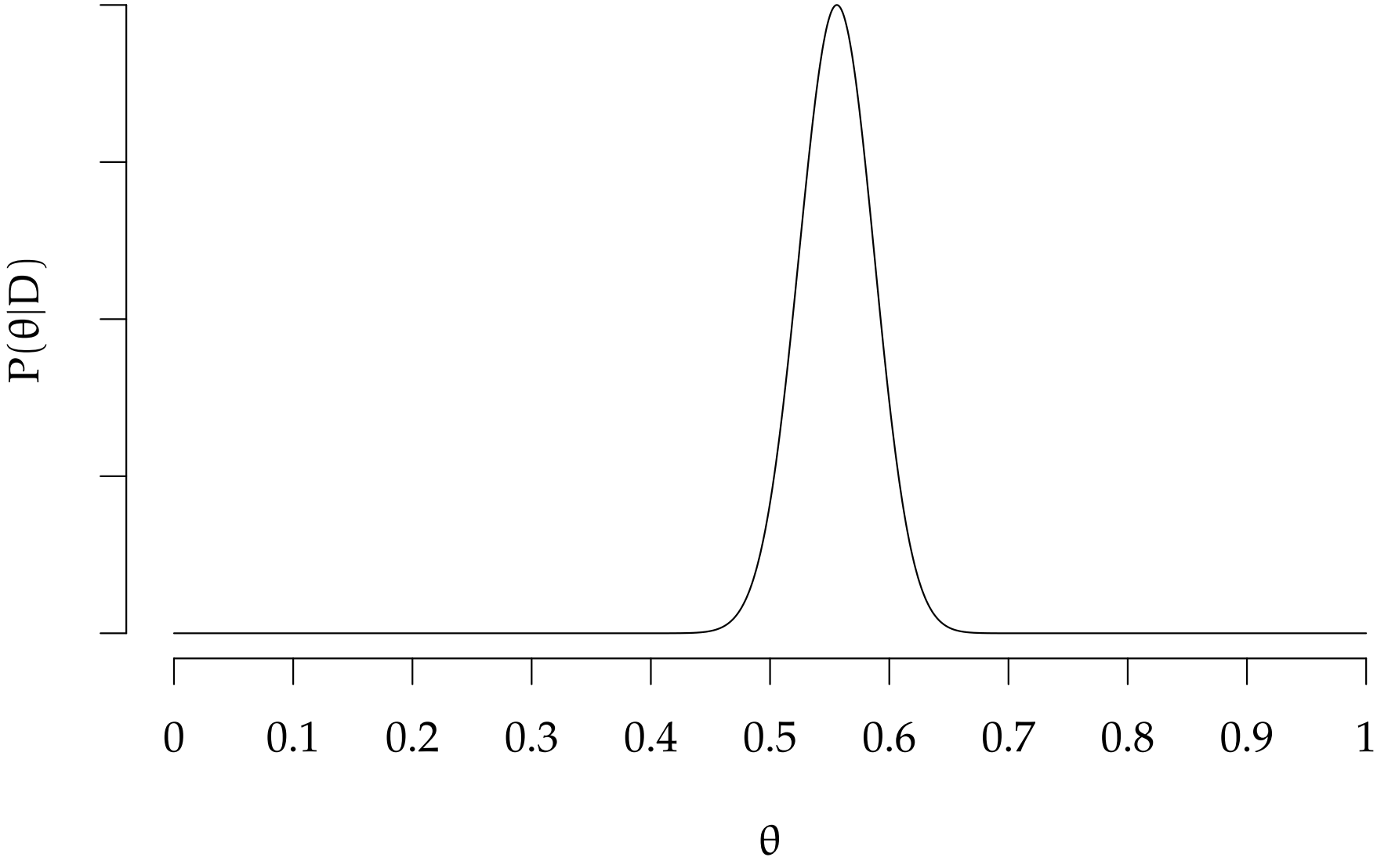
Posterior distribution

- For our Euro coin example, our observed data are $n = 250$ and $m = 139$.
- A noninformative uniform prior on θ is $\text{Beta}(\alpha = 1, \beta = 1)$.
- With this prior, the posterior distribution is

$$\begin{aligned}\text{Beta}(m + \alpha, n - m + \beta) &= \text{Beta}(139 + 1, 250 - 139 + 1), \\ &= \text{Beta}(140, 112)\end{aligned}$$

Posterior distribution

when $n = 250$, $m = 139$, $\alpha = 1$ *and* $\beta = 1$



For an interactive version:

https://lawsofthought.shinyapps.io/bayesian_coin_inference/.

Summarizing the posterior distribution

- The mean, variance and modes of any beta distribution are as follows:

$$\langle\theta\rangle = \frac{\alpha}{\alpha+\beta},$$
$$V(\theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)},$$
$$\text{mode}(\theta) = \frac{\alpha-1}{\alpha+\beta-2}.$$

- Thus, in our case of $\text{Beta}(140, 112)$, we have

$$\langle\theta\rangle = 0.5556,$$
$$V(\theta) = 0.001, \quad \text{sd}(\theta) = 0.0312,$$
$$\text{mode}(\theta) = 0.556.$$

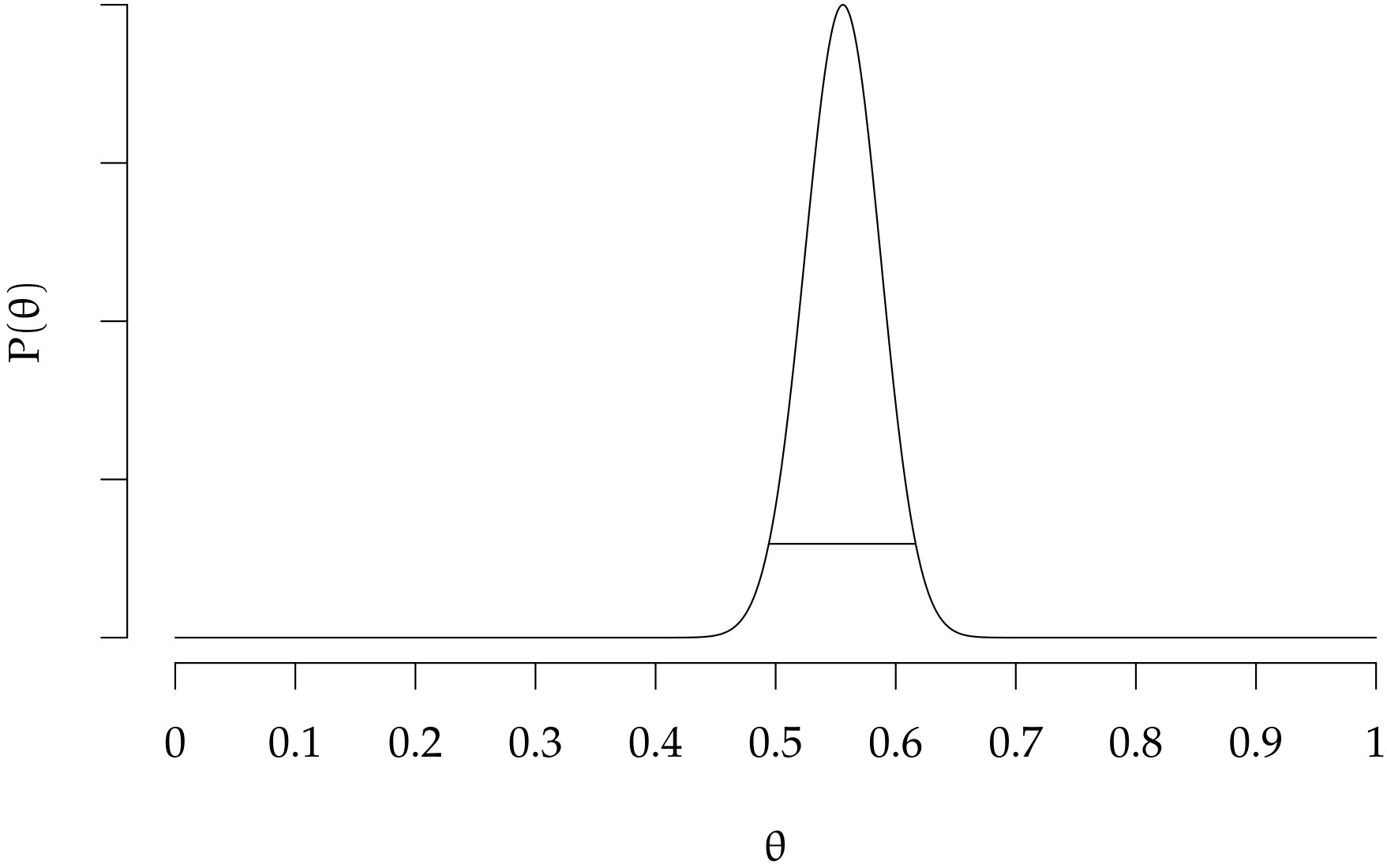
High posterior density (HPD) intervals

- HPD intervals provide ranges that contain specified probability mass. For example, the 0.95 HPD interval is the range of values that contain 0.95 of the probability mass of the distribution.
- The φ HPD interval for the probability density function $P(x)$ is computed by finding a probability density value p^* such that

$$P(\{x\colon P(x) \geqslant p^*\}) = \varphi.$$

- In other words, we find the value p^* such that the probability mass of the set of points whose density is greater than than p^* is exactly φ.
- In general, the HPD is not trivial to compute but in the case of symmetric distributions, it can be easily computed from the cumulative density function.

The 0.95 HPD *interval*



The posterior distribution, with its 0.95 HPD, when $n = 250$, $m = 139$, $\alpha = 1$ and $\beta = 1$. In this case, the HPD interval is $(0.494, 0.617)$.

Posterior predictive distribution

- Given that we have observed m heads in n coin tosses, what is the probability that the *next* coin toss is heads.
- This is given by the *posterior predictive* probability that $x = 1$:

$$\begin{aligned} P(x = 1|D, \alpha, \beta) &= \int P(x = 1|\theta) \overbrace{P(\theta|D, \alpha, \beta)}^{\text{Posterior}} \, d\theta, \\ &= \int \theta \times P(\theta|D, \alpha, \beta) \, d\theta, \\ &= \langle \theta \rangle, \\ &= \frac{\alpha + m}{\alpha + \beta + n}. \end{aligned}$$

- Thus, given 139 heads in 250 tosses, the predicted probability that the next coin will also be heads is ≈ 0.5556.

Marginal likelihood

- The posterior distribution is

$$P(\theta|D,\alpha,\beta) = \frac{\overbrace{P(D|\theta)}^{\text{Likelihood}}\ \overbrace{P(\theta|\alpha,\beta)}^{\text{Prior}}}{\underbrace{\int P(D|\theta)P(\theta|\alpha,\beta)\ d\theta}_{\text{Marginal likelihood}}}.$$

  where the *marginal likelihood* gives the likelihood of the model given the observed data:

$$\int P(D|\theta)P(\theta|\alpha,\beta)\ d\theta \stackrel{\text{def}}{=} P(D|\alpha,\beta).$$

- In this example, it has a simple analytical form:

$$P(D|\alpha,\beta) = B(\alpha+m,\beta+n-m) = \frac{\Gamma(m+\alpha)\Gamma(n-m+\beta)}{\Gamma(n+\alpha+\beta)}.$$

Model comparison with Bayes factors

- Given D, we can compare the probability of model M_1 relative to model M_0 as follows:

$$\frac{P(M_1|D)}{P(M_0|D)} = \underbrace{\frac{P(D|M_1)}{P(D|M_0)}}_{\text{Bayes factor}} \times \underbrace{\frac{P(M_1)}{P(M_0)}}_{\text{Priors odds}}.$$

- When both models are equally probable a priori, then the relative posterior probabilities is determined by the Bayes factor.
- We can compare our model M_1, i.e. with $\alpha = \beta = 1$, with the M_0 model that $\theta = \frac{1}{2}$.

$$\frac{P(D|M_1)}{P(D|M_0)} = \frac{\int P(D|\theta)P(\theta|\alpha = 1, \beta = 1)\, d\theta}{\int P(D|\theta)\delta(\theta - \frac{1}{2})\, d\theta.}$$

Model comparison with Bayes factors

- We can compare our model M_1, i.e. with $\alpha = \beta = 1$, with the M_0 model that $\theta = \frac{1}{2}$.

$$\begin{aligned}\frac{P(D|M_1)}{P(D|M_0)} &= \frac{\int P(D|\theta)P(\theta|\alpha = 1, \beta = 1)\, d\theta}{\int P(D|\theta)\delta(\theta - \frac{1}{2})\, d\theta.}, \\ &= \frac{\Gamma(\alpha + m)\Gamma(\beta + n - m)}{\Gamma(\alpha + \beta + n)} \Big/ \tfrac{1}{2}^m (1 - \tfrac{1}{2})^{n-m}, \\ &= \frac{m!(n-m)!}{(n+1)!} \Big/ \frac{1}{2^n}.\end{aligned}$$

If $n = 250$, $m = 139$, then

$$= \frac{139!111!}{251!} \Big/ \frac{1}{2^{250}} = 0.38.$$

- This is a factor of 2.65 in favour of the unbiased coin hypothesis.
- Note that the classical statistics null hypothesis test gives a p-value of $p = 0.0875$.

If n is a positive integer $\Gamma(n) = (n-1)!$.