Probabilistic Mixture models

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Parametric model fitting

- So far, we have encountered four distinct probability distributions used to model the conditional distribution of outcome variables in regression model:
  - Normal distributions. These are used to model the outcome variable in standard linear regression models.
  - Bernoulli distributions. These are models of binary outcome variables, and are used in binary logistic regression as elsewhere.
  - Poisson distributions. These are used to model count variables, and are used in Poisson regression.
  - Negative binomial distributions. The negative binomial distribution can be seen as similar to the Poisson distribution, but with a additional width parameter.

In any cases, we can fit these models to data by modifying their parameters to achieve the best fit, often done by maximum likelihood estimation.

Fitting parametric models

- Assume our data is n observations $y_1, y_2 \ldots y_n$.
- If we assume that

$$y_i \sim N(\mu, \sigma^2), \quad \text{for } i \in 1 \ldots n,$$

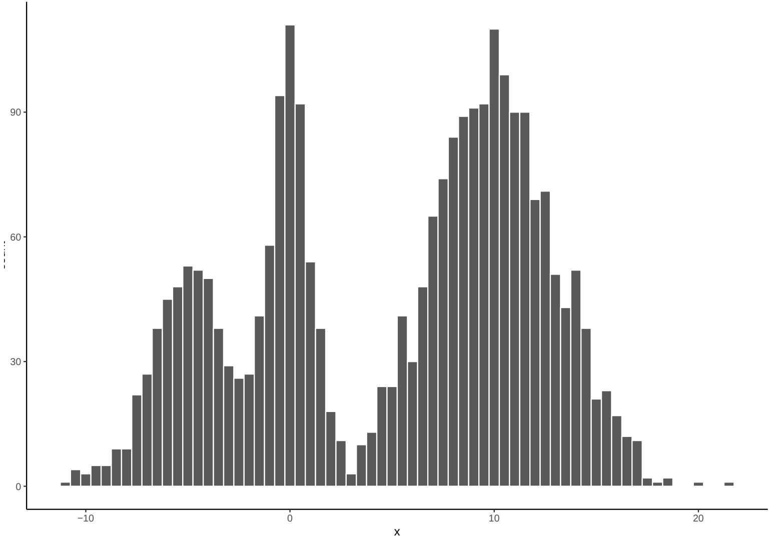
then we can calculate the likelihood function for μ and σ^2, i.e.

$$L(\mu, \sigma^2 | y_1 \ldots y_n) \propto \prod_{i=1}^{n} P(y_i | \mu, \sigma^2),$$

and maximize this function for μ and σ^2.

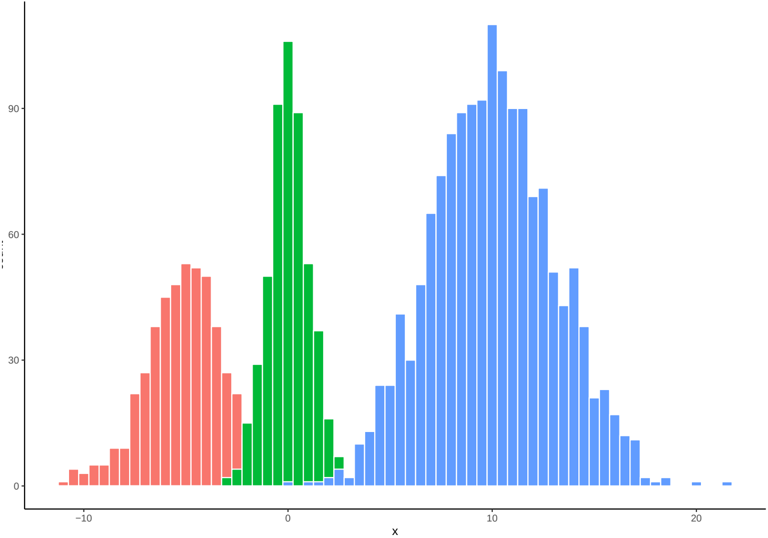
Irregular distributions

- What should we do when encounter data of the following form?



Mixture model

- A mixture model assumes that the data is sampled from independent component distributions, each of which can be modelled by parametric distributions.



Latent variables

- With irregular data, even if assume it is derived from a mixture of independent distributions, we do not know which data point came from which distributions.
- In other words, we have a set of data $y_1, y_2 \dots y_n$, but we don't know which distribution each data point came from or even how many distributions there are.
- In this situation, we assume that for each y_i data point, there is an z_i that tells us which distribution y_i came from.
- This z_i is a *latent* variable. It has some value, but we don't or can't observe it directly.
- Another name for a model of this kind is a *latent class model*. We assume each y_i belongs to some class, but we just don't or can't observe what that class is.

Mixture models: The probabilistic generative model

- We start by assuming that there are K distinct hidden classes, e.g. $K = 3$.
- So each $z_i \in \{1, 2, 3\}$.
- Then, our model is

$$y_i \sim \begin{cases} N(\mu_1, \sigma_1^2), & \text{if } z_i = 1 \\ N(\mu_2, \sigma_2^2), & \text{if } z_i = 2 \\ N(\mu_3, \sigma_3^2), & \text{if } z_i = 3 \end{cases},$$

$$z_i \sim P(\pi),$$

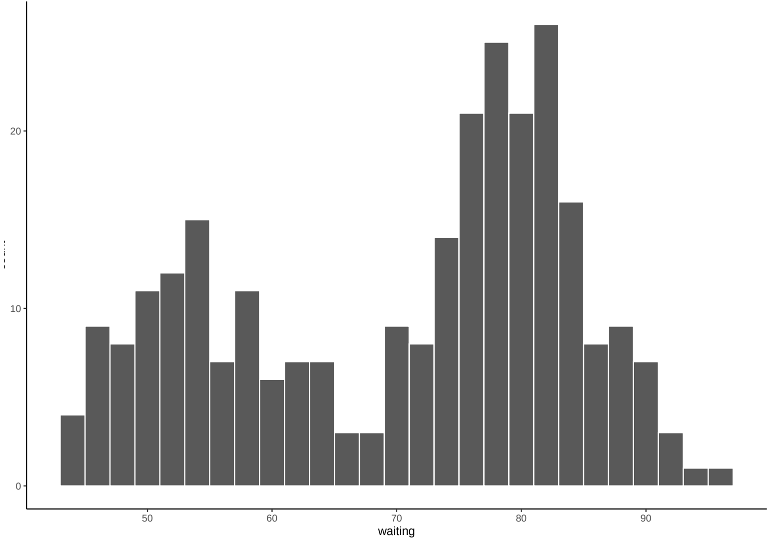
where $\pi = [\pi_1, \pi_2, \pi_3]$ is a probability distribution of $\{1, 2, 3\}$, i.e. π_1 gives the probability that the latent's class's value is class 1, π_2 gives the probability that the latent's class's value is class 2, π_3 gives the probability that the latent's class's value is class 3.

Mixture models: Inference

- In a normal mixture model with $K = 3$ components, we have 9 parameters:
  - μ_1, σ_1^2: The parameters of component distribution 1.
  - μ_2, σ_2^2: The parameters of component distribution 2.
  - μ_3, σ_3^2: The parameters of component distribution 2.
  - π_1, π_2, π_3: The relative probabilities of each component.
- In addition, we have the probability distribution over each value $x_1, x_2 \ldots x_n$.
- Inferring these values by maximum likelihood estimation is usually done by the *expectation-maximization* algorithm.

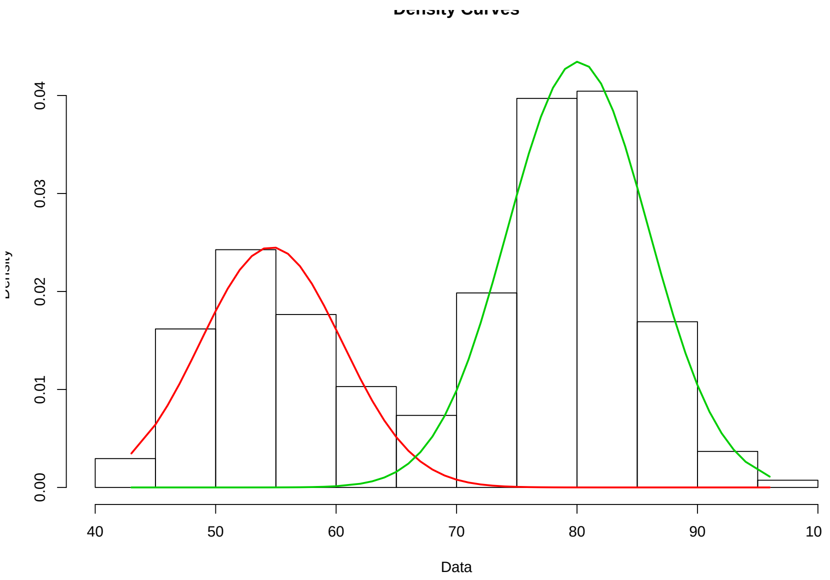
Example: Old faithful

- The distribution of waiting times.



Example: Old faithful

```r
M <- normalmixEM(faithful$waiting, k=2)
```



Example: Old faithful

- The inferred means are

```
## [1] 54.61489 80.09109
```

- The inferred standard deviations

```
## [1] 5.871243 5.867717
```

- The relative probabilities of the two components

```
## [1] 0.3608869 0.6391131
```

Example: Old faithful

- The probabilities for each z_i (for first 10 values)

comp.1	comp.2
0	1
1	0
0.004	0.996
0.967	0.033
0	1
1	0
0	1
0	1
1	0
0	1

Mixture regression models

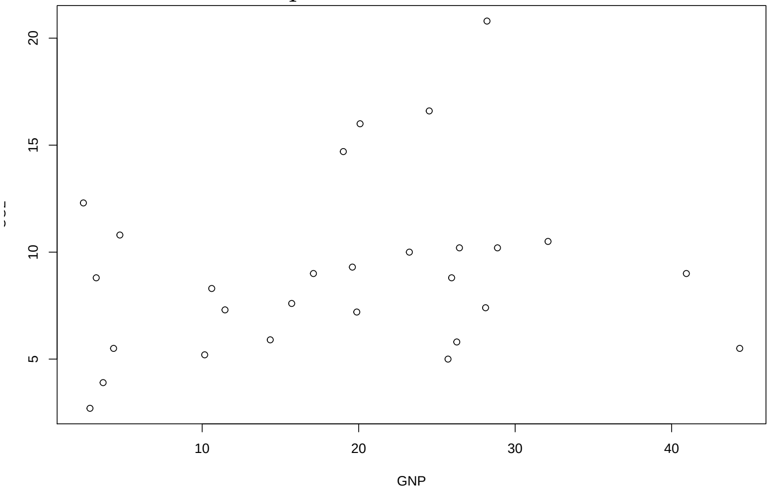
- In a mixture of regressions, we assume that there are K regression models.
- Each data point being associated with one of them.
- Again, we don't know which component it came from. This is given by a latent variable.

$$y_i \sim \begin{cases} N(\alpha_1 + \beta_1 x_i, \sigma_1^2), & \text{if } z_i = 1 \\ N(\alpha_2 + \beta_2 x_i, \sigma_2^2), & \text{if } z_i = 2 \end{cases},$$

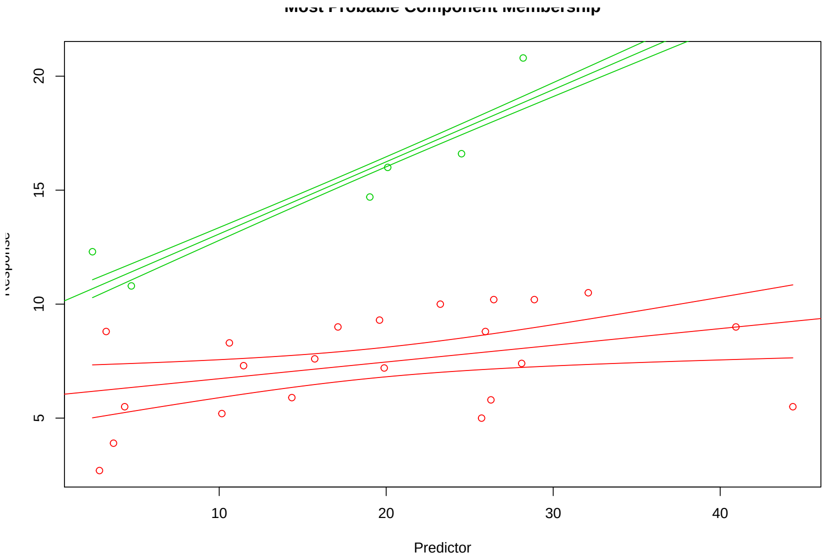
$$z_i \sim P(\pi),$$

Mixture regression models

- A mixture of two scatterplots?



Mixture regression models



Mixture regression models

- The inferred coefficients are

	comp.1	comp.2
beta.0	5.998	9.914
beta.1	0.07324	0.3166

- The inferred standard deviations

```
## [1] 2.025023 1.316106
```

- The relative probabilities of the two models

```
## [1] 0.7875039 0.2124961
```

Mixture of regressions

- The probabilities for each z_i (for first 10 values)

comp.1	comp.2
0.006	0.994
1	0
0	1
1	0
1	0
0	1
1	0
1	0
1	0
0.193	0.807