

Probability Basics

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What is probability?

- ▶ Probability is a means to quantify uncertainty.
- ▶ If a variable can take on more than one value, probability can be used to describe the certainty that it will take each one of its possible values.
- ▶ Probabilities must lie between zero and one.
- ▶ The sum of the probabilities of all values of a variable must equal one.

What is probability? Notation

- ▶ If X is a variable with possible values $\{x_1, x_2 \dots x_k \dots x_K\}$, then

$$P(X = x_k) \tag{1}$$

gives the probability that X takes the value x_k .

- ▶ The rules of probability require that

$$0 \leq P(X = x_k) \leq 1 \quad \forall x_k \tag{2}$$

and

$$\sum_{k=1}^K P(X = x_k) = 1. \tag{3}$$

Joint probability

- If X is a variable with possible values $\{x_1, x_2 \dots x_k \dots x_K\}$, and Y is a variable with possible values $\{y_1, y_2 \dots y_k \dots y_L\}$, then

$$P(X = x_k, Y = y_l) \quad (4)$$

gives the probability that X takes the value x_k *and* Y takes the value of y_l .

- The rules of probability require that

$$0 \leq P(X = x_k, Y = y_l) \leq 1 \quad \forall x_k, y_l \quad (5)$$

and

$$\sum_{l=1}^L \sum_{k=1}^K P(X = x_k, Y = y_l) = 1. \quad (6)$$

Conditional, joint and marginal probability

- ▶ Conditional probability is the probability distribution of a variable when the value of another variable is known, i.e.

$$P(X = x_k | Y = y_l) \quad (7)$$

is read as the “the probability that X is x_k *given* Y is y_l .”

- ▶ Conditional probability can be derived from the joint probability as follows:

$$P(X = x_k | Y = y_l) = \frac{P(X = x_k, Y = y_l)}{P(Y = y_l)} \quad (8)$$

- ▶ Marginal probability can also be derived from the joint probability as follows:

$$P(X = x_k) = \sum_{l=1}^L P(X = x_k, Y = y_l). \quad (9)$$

Conditional, joint and marginal probability

- ▶ There at least three different types of probability distributions, referred to as *joint*, *marginal* or *conditional* probabilities.
- ▶ There are fundamental relationships between them.
- ▶ We can illustrate these concepts by looking at survival rates of males and females on *RMS Titanic*.

Frequency distribution of survival rates on Titanic

- ▶ The following table shows the number of men and women who died or survived on the *Titanic*:

	Men	Women
Perished	1352	109
Survived	338	316

- ▶ Altogether, there were 2115 onboard.

From frequencies to probabilities

- We can convert the frequencies

	Men	Women
Perished	1352	109
Survived	338	316

to the probabilities

	Men	Women
Perished	.64	.05
Survived	.16	.15

by dividing each frequency by the total number onboard, i.e. 2115.

Joint probability tables

- The table

	Men	Women
Perished	.64	.05
Survived	.16	.15

is a *joint probability* table.

- It provides the probability for every combination of the two variables.
- It is just another probability distribution, like what we have met already.
- Each element in the table lies between 0 and 1 and together they must sum to 1.

Marginal probabilities from joint probabilities

- From the table

	Men	Women
Perished	.64	.05
Survived	.16	.15

what is the overall probability of dying?

- Recall that each element in the table provides the probability for a combination of values of the two variables, e.g. the probability of dying and being male is .64.
- Following the rules of probability, we calculate the probability of dying as follows:

$$P(\text{Perished}) = P(\text{Male \& Perished}) + P(\text{Female \& Perished}), \quad (10)$$

$$= .64 + .05 = .69 \quad (11)$$

Conditional probabilities from joint probabilities

- From the table

	Men	Women
Perished	.64	.05
Survived	.16	.15

what is the probability of dying if the person is a man?

- First, the probability of being a man onboard *Titanic* is

$$P(\text{Male}) = P(\text{Male \& Perished}) + P(\text{Male \& Survived}), \quad (12)$$

$$= .64 + .16 = .8 . \quad (13)$$

Conditional probabilities from joint probabilities (cont'd)

- ▶ Then, what is the probability of being a man and dying? This is simply .64.
- ▶ Putting these together: 80% of the ship's total was male and 64% of the total were men who died. That means the fraction of men who died is 64/80, i.e.

$$P(\text{Perished}|\text{Male}) = \frac{P(\text{Male \& Perished})}{P(\text{Male})}, \quad (14)$$

$$= \frac{.64}{.8} = .8 \quad (15)$$

Conditional, joint and marginal: More Rules

- As we have

$$P(X = x_k | Y = y_l) = \frac{P(X = x_k, Y = y_l)}{P(Y = y_l)} \quad (16)$$

then we also have

$$P(X = x_k, Y = y_l) = P(X = x_k | Y = y_l)P(Y = y_l), \quad (17)$$

- Likewise, given that

$$P(Y = y_l | X = x_k) = \frac{P(Y = y_l, X = x_k)}{P(X = x_k)} \quad (18)$$

then

$$P(Y = y_l, X = x_k) = P(Y = y_l | X = x_k)P(X = x_k). \quad (19)$$

Chain rule



$$P(X, Y, Z) = P(X|Y, Z)P(Y|Z)P(Z), \quad (20)$$

$$= P(X|Z, Y)P(Z|Y)P(Y), \quad (21)$$

$$= P(Y|X, Z)P(X|Z)P(Z), \quad (22)$$

$$= \dots \quad (23)$$

Independence and conditional independence

- ▶ If $P(X, Y) = P(X)P(Y)$, then X and Y are independent.
- ▶ If X and Y are independent, their correlation is 0.
- ▶ If $P(X, Y|Z) = P(X|Z)P(Y|Z)$, then X and Y are conditionally independent, conditional on Z .
- ▶ If X and Y are conditionally independent, conditional of Z , the partial correlation of X and Y given Z is zero.