Binary logistic regression: Part II

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Overview of logistic regression topic

Here are the things about logistic regression that you need to know or do:

1. The assumed statistical model (i.e. what model of the data does logistic regression assume).
2. How we infer the unknown coefficients.
3. What do the inferred coefficients mean.
4. How to do prediction in logistic regression.
5. How to do model comparison.

Logistic regression's assumed model (simple case)

- For all $i \in 1 \dots n$,

$$\begin{aligned} y_i &\sim \text{Bernoulli}(\theta_i), \\ \text{logit}(\theta_i) &= a + bx_i. \end{aligned}$$

or equivalently

$$\begin{aligned} y_i &\sim \text{Bernoulli}(\theta_i), \\ \theta_i &= \text{ilogit}(a + bx_i), \end{aligned}$$

where

$$\text{logit}(\theta_i) \triangleq \log\left(\frac{\theta}{1-\theta}\right),$$

and

$$\text{ilogit}(a + bx_i) \triangleq \frac{1}{1 + e^{-(a+bx_i)}}$$

Logistic regression's assumed model (multiple regression case)

- For all $i \in 1 \ldots n$,

$$
\begin{aligned}
y_i &\sim \mathrm{Bernoulli}(\theta_i), \\
\mathrm{logit}(\theta_i) &= \beta_0 + \sum_{k=1}^{K} \beta_k x_{ki}
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
y_i &\sim \mathrm{Bernoulli}(\theta_i), \\
\theta_i &= \mathrm{ilogit}(\beta_0 + \sum_{k=1}^{K} \beta_k x_{ki}).
\end{aligned}
$$

Inference of unknown coefficients

- With the assumed model just described, the values of the parameters $\beta_0, \beta_1 \ldots \beta_K$ are unknown.
- We infer these by *maximum likelihood estimation*.

The likelihood function: Definition

- If $\mathcal{D}$ is a set of observed data whose probability distribution is parameterized by θ, i.e. its probability distribution is $P(\mathcal{D}|\theta)$, then the likelihood function, $L(\theta|\mathcal{D})$ gives the probability of $\mathcal{D}$ as a function of θ.
- More precisely, the likelihood is any function proportional to $P(\mathcal{D}|\theta)$ (treated as a function of θ), i.e.

$$L(\theta|\mathcal{D}) \propto P(\mathcal{D}|\theta).$$

- For demo,

  `https://lawsofthought.shinyapps.io/binomial_likelihood/`

Maximum likelihood estimation

- The likelihood can be used for estimating the values of an unknown parameter.
- For example, given some observed number m of gold coins in a sample of n coins, the true value of p can be estimated by choosing the parameter that maximizes the likelihood function.
- This is the maximum-likelihood estimate.
- More precisely, $\hat{\theta}$ is the maximum likelihood estimate of the unknown parameter of a probability distribution if

$$\hat{\theta} = \underset{\theta}{\operatorname{argmax}} L(\theta|\mathcal{D}).$$

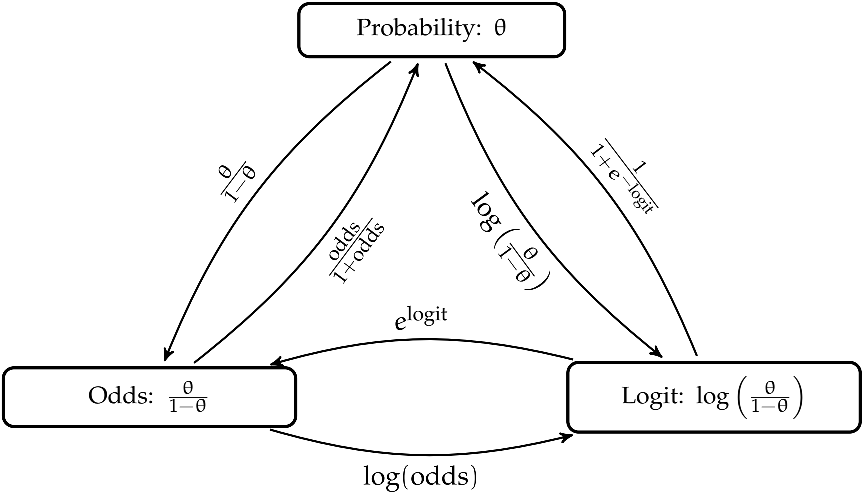
Prediction

- Given inferred values for $\beta_0, \beta_1 \ldots \beta_K$, the predicted log odds of the outcome variable taking the value of 1 if the predictor variables's values are $x_1, x_2 \ldots x_K$ is

$$\beta_0 + \sum_{k=1}^{K} \beta_k x_k$$

- Knowing the predicted log odds, the predicted probability or predicted odds is easily calculated.

From probabilities to odds to logits, and back



Understanding β coefficients

- In linear models, a coefficient for a predictor variable has a straightforward interpretation: 1 unit change for a predictor variable corresponds to β change in the outcome variable.
- As logistic regression curves are nonlinear, the change in the outcome variable is not a constant function of change in the predictor.
- This makes interpretation more challenging.
- The most common means to interpret β coefficients is in terms of odds ratios.

Odds ratios

- We have seen that an odds in favour of an event are $\frac{p}{1-p}$.
- We can compare two odds with an odds ratio.
- For example, the odds of getting a certain job for someone with a MBA might be $\frac{p}{1-p}$, while the odds of getting the same job for someone without an MBA might be $\frac{q}{1-q}$.
- The ratio of the odds for the MBA to those of the non-MBA are

$$\frac{p}{1-p} \Big/ \frac{q}{1-q}$$

- This gives the factor by which odds for the job change for someone who gains an MBA.

β *coefficients as (log) odds ratios*

- Consider a logistic regression model with a single dichotomous predictor, i.e.

$$\log\left(\frac{P(y_i = 1)}{1 - P(y_i = 1)}\right) = \alpha + \beta x_i,$$

  where $x_i \in \{0, 1\}$.
- The log odds that $y_i = 1$ when $x_i = 1$ is $\alpha + \beta$.
- The log odds that $y_i = 1$ when $x_i = 0$ is α.
- The log odds that $y_i = 1$ when $x_i = 1$ minus the log odds that $y_i = 1$ when $x_i = 0$ is

$$(\alpha + \beta) - \alpha = \beta.$$

β *coefficients as (log) odds ratios*

- Let's denote the probability that $y_i = 1$ when $x_i = 1$ by p, and denote the probability that $y_i = 1$ when $x_i = 0$ by q.
- Subtracting the log odds is the log of the odds ratio, i.e.

$$\log\left(\frac{p}{1-p}\right) - \log\left(\frac{q}{1-q}\right) = \log\left(\frac{p}{1-p} \Big/ \frac{q}{1-q}\right) = \beta$$

- As such,

$$e^{\beta} = \frac{p}{1-p} \Big/ \frac{q}{1-q}.$$

- This provides a general interpretation for the β coefficients.

Model Fit with Deviance

- Once we have the maximum likelihood estimate for the parameters, we can calculate *goodness of fit*.
- The *deviance* of a model is defined

$$-2 \log L(\hat{\alpha}, \hat{\beta} | \mathcal{D}),$$

  where $\hat{\alpha}$, $\hat{\beta}$ are the mle estimates.
- This is counterpart to R^2 for generalized linear models.

Model Fit with Deviance: Model testing

- In a model with one predictor, a null model would be that $P(y_i = 1)$ is not a function of x_i.
- The difference in the deviance of the null model minus the deviance of the full model is

$$\Delta_D = D_0 - D_1 = -2 \log \frac{L(\hat{\alpha}|\mathcal{D})}{L(\hat{\alpha}, \hat{\beta}|\mathcal{D})}.$$

- Under the null hypothesis, Δ_D is distributed as χ^2 with 1df.

Deviance based model testing

- In general, we can compare any two *nested* models using χ^2 test applied to differences in deviance.
- The deviance of the subset model minus that of the full model will always be (approximately) distributed a χ^2 with df equalling the difference in the number of parameters between the two models.
- In other words, under the null hypothesis that subset and full models are identical, the difference in the deviances will be distributed as a χ^2 with df equal to the difference in the number of parameters between the two models.