Tidal Equation

Equations

The linear shallow-water equations on the sphere are:

$$\begin{aligned}
\frac{\partial u'}{\partial t} + \frac{\partial\left(\overline{u}u'\right)}{a\cos\theta\partial\lambda} + v'\frac{\partial\overline{u}}{a\partial\theta} - \frac{\overline{u}v'\tan\theta}{a} &= 2\Omega v'\sin\theta - \frac{g\partial h'}{a\cos\theta\partial\lambda} \\
\frac{\partial v'}{\partial t} + \frac{\partial\left(\overline{u}v'\right)}{a\cos\theta\partial\lambda} + \frac{2\overline{u}u'\tan\theta}{a} &= -2\Omega u'\sin\theta - \frac{g\partial h'}{a\partial\theta} \\
\frac{\partial h'}{\partial t} + v'\frac{\partial\overline{h}}{a\partial\theta} + \overline{u}\frac{\partial h'}{a\cos\theta\partial\lambda} + \overline{h}\nabla_H\cdot\mathbf{v}' &= 0,
\end{aligned} \tag{1}$$

where h is the height of the layer, $\boldsymbol{v} = (u, v)$ is the velocity, θ is latitude, λ is longitude, t is time, a is radius, g is gravity, and Ω is angular velocity. Overbars denote zonal-mean quantities (the background flow and height $\overline{u}$ and $\overline{h}$). Dashes denote perturbations to this background state.

The background state is stationary and in gradient wind balance:

$$\frac{1}{a}\frac{\partial}{\partial\theta}\left(\overline{h} + h_g\right) = -\left(2\Omega\overline{u}\sin\theta + \frac{\overline{u}^2}{a}\tan\theta\right). \tag{2}$$

The perturbed variables are wavelike in longitude and are uniformly damped, so are proportional to $\exp[im\lambda + \alpha t)]$. All variables are made non-dimensional with velocity scale $2\Omega a$, height scale $(2\Omega a)^2/g$ and time scale $1/(2\Omega)$, and denoted as such by an asterisk. This gives the following non-dimensional shallow-water equations:

$$\begin{aligned}
\alpha^* u_m^* + im\frac{\overline{u}^* u_m^*}{\cos\theta} + v_m^*\frac{\partial\overline{u}^*}{\partial\theta} - \overline{u}^* v_m^*\tan\theta &= v_m^*\sin\theta - \frac{imh_m^*}{\cos\theta}, \\
\alpha^* v_m^* + im\frac{\overline{u}^* v_m^*}{\cos\theta} + 2\overline{u}^* u_m^*\tan\theta &= -u_m^*\sin\theta - \frac{\partial h_m^*}{\partial\theta}, \\
\alpha^* h_m^* + im\overline{u}^*\frac{h_m^*}{\cos\theta} &= -\frac{\epsilon^*}{\cos\theta}\left[imu_m^* + \frac{\partial}{\partial\theta}\left(\cos\theta v_m^*\right)\right],
\end{aligned} \tag{3}$$

where Lamb's parameter is $\epsilon \equiv (2\Omega a)^2/gH$.

These can be written as

$$
\begin{aligned}
-\hat{\sigma}^* u_m^* - \overline{\zeta}^* v_m^* + \frac{m h_m^*}{\cos\theta} &= 0, \\
\hat{\sigma}^* v_m^* + f_1^* u_m^* + \frac{d h_m^*}{d\theta} &= 0, \\
\hat{\sigma}^* \epsilon \alpha h_m^* + \frac{m u_m^*}{\cos\theta} + \frac{1}{\cos\theta}\frac{d}{d\theta}(v\cos\theta) &= 0,
\end{aligned}
\tag{4}
$$

where

$$
\overline{\zeta}^* = f^* - \frac{1}{\cos\theta}\frac{d}{d\theta}(\overline{u}\cos\theta) \tag{5}
$$

is the absolute vorticity of the background flow,

$$
f_1 = f + 2\overline{u}\tan\theta \tag{6}
$$

is an effective Coriolis parameter modified by the background flow, and

$$
\hat{\sigma}^* = \sigma^* - \frac{m\overline{u}}{\cos\theta} \tag{7}
$$

is the Doppler-shifted time-derivate of the variables (see Chapter X).

Solving the first two shallow-water equations gives the two velocity components in terms of the height field:

$$
\begin{aligned}
u_m^* &= \frac{-\hat{\sigma}^* h_m^* m/\cos\theta - \overline{\zeta}^* dh_m^*/dy}{\Delta} \\
v_m^* &= \frac{\hat{\sigma}^* dh_m^*/dy + f_1^* h_m^* m/\cos\theta}{\Delta}
\end{aligned}
\tag{8}
$$

where $\Delta = f_1^* \overline{\zeta}^* - \hat{\sigma}^{*2}$.

Then, substituting these into the third shallow-water equation, while changing variables to $\mu = \sin\theta$ and $\phi_m^* = (1-\mu^2)^{-m/2} h_m^*$ to avoid the polar singularities (**?**), gives:

$$
\frac{\partial^2 \phi_m^*}{\partial \mu^2} - B(\sigma^*, \mu)\frac{\partial \phi_m^*}{\partial \mu} - A(\sigma^*, \mu)\,\phi_m^* = \frac{F(\theta, x)}{i\sigma} \tag{9}
$$

where

$$
\begin{aligned}
A(\sigma^*, \mu) \equiv & \frac{1}{1-\mu^2}\Bigg[m(m+1) - m\mu \frac{1}{\Delta^*}\frac{\partial \Delta^*}{\partial \mu} + \epsilon \Delta^* \\
& + \frac{m}{\Delta^* \hat{\sigma}^*}\left(f_1^* \frac{\partial \Delta^*}{\partial \mu} - \Delta * \frac{\partial f_1^*}{\partial \mu} \right)] \\
B(\sigma^*, \mu) \equiv & \frac{1}{\Delta^*}\frac{\partial \Delta^*}{\partial \mu} + \frac{2\mu(m+1)}{(1-\mu^2)} \\
\Delta^* \equiv & f_1^* \overline{\zeta}^* - \hat{\sigma}^{*2}
\end{aligned}
\tag{10}
$$

This equation is solved with a pseudo-spectral collocation method.

Collocation

Solutions

The full solutions are reconstructed