Chapter 3

Penalty and Two-Phase Methods (Using Artificial Variables)

3-1 M- Technique (Method of Penalty)

This method is used to find the optimal solution of linear programming models in which some constraints are of types $(=)$, $(\geq)$ where, we need to add artificial variables R_1 , R_2, ... to the L. H. S. of the constraints after subtracting surplus variables to form the standard form. These variables R_1 , R_2, ... are the starting basic variables of the standard form.

This method depends on adding the variables

$$R_1 \;,\; R_2, \;\ldots$$

to the O. F. with coefficients

$-M$ for each variable in case of maximization,

$+M$ for each variable in case of minimization.

The value of $+M$ is very large approaches to infinity.

Then apply the steps of the simplex method until obtaining the optimal solution.

Note that:

Before constructing the starting tableau, the O. F. must be in terms of ONLY starting non-basic variables using the constraints. It means that, the O. F. coefficients of the artificial variables R_1 , R_2, ... in the starting tableau are zeros.

Example 3-1

Use penalty method (M- technique) to solve the following L. P. model

$$\min imize \quad z = 4x_1 + x_2$$
$$subject\ to$$

$$3x_1 + x_2 = 3$$

$$4x_1 + 3x_2 \geq 6$$

$$x_1 + 2x_2 \leq 4$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Solution

After adding slack variable to the L. S. of constraint no. 3, subtracting a surplus variable from the L. S. of constraint no. 2 then adding artificial variables to L. S. of constraints 1 and 2, the standard form of the model is as follows:

$$\begin{aligned} \min imize \quad & z = 4x_1 + x_2 + M R_1 + M R_2 \\ subject\ to \quad & \\ & 3x_1 + x_2 + R_1 = 3 \\ & 4x_1 + 3x_2 - s_1 + R_2 = 6 \\ & x_1 + 2x_2 + s_2 = 4 \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad R_1 \geq 0, \quad R_2 \geq 0$$

Substitute from constraints 1 and 2 in the O. F. , then

$$z = 4x_1 + x_2 + M R_1 + M R_2$$

$$z = 4x_1 + x_2 + M(3 - 3x_1 - x_2) + M(6 - 4x_1 - 3x_2 + s_1)$$

$$z - (4 - 7M)x_1 - (1 - 4M)x_2 - M s_1 = 9M$$

<u>Starting tableau (1)</u>

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
z	$-4+7M$	$-1+4M$	$-M$	**0**	**0**	**0**	$9M$
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
s_2	1	2	0	0	0	1	4

Introduce x_1 as an entering variable and drop R_1 as the leaving variable, so the pivot element is 3 , then obtain tableau 2 as follows:

Tableau (2)

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
z	0	$\frac{1}{3}+\frac{5}{3}M$	$-M$	$\frac{4}{3}-\frac{7}{3}M$	**0**	**0**	$4+2M$
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
s_2	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

Introduce x_2 as an entering variable and drop R_2 as the leaving variable, so the pivot element is $\frac{5}{3}$, then obtain tableau 3 as follows:

ناقص M لو كان Max
موجب M لو كان Min

Tableau (3)

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
z	0	0	$\frac{1}{5}$	$\frac{8}{5}-M$	$-\frac{1}{5}-M$	**0**	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
s_2	0	0	1	1	-1	1	1

Introduce s_1 as an entering variable and drop s_2 as the leaving variable, so the pivot element is 1 , then obtain tableau 4 as follows:

Tableau (4) (optimal)

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
z	0	0	0	$\frac{7}{5}-M$	$-M$	$-\frac{1}{5}$	$\frac{17}{5}$
x_1	1	0	0	$\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$
x_2	0	1	0	$-\frac{1}{5}$	0	$\frac{3}{5}$	$\frac{9}{5}$
s_1	0	0	1	1	-1	1	1

Since all coefficients of z row are non-positive, the solution in tableau 4 is the optimal as follows:

$$x_1^* = \frac{2}{5}, \quad x_2^* = \frac{9}{5},$$

$$z^* = 4x_1^* + x_2^* = \frac{17}{5}$$

Exercise 3-1

Use the graphical method and the M- technique (penalty method) to solve the following linear programming models:

1-

$$\begin{aligned} \text{minimize} \quad & z = 5x_1 + 2x_2 \\ \text{subject to} \quad & \\ & 2x_1 + 5x_2 \geq 10 \\ & 4x_1 - x_2 \geq 12 \\ & x_1 + x_2 \geq 4 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

2-

$$\begin{aligned} \text{minimize} \quad & z = 3x_1 + 2x_2 \\ \text{subject to} \quad & x_1 + 2x_2 \geq 6 \\ & 2x_1 + x_2 \geq 8 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

3-

$$\begin{aligned} \min imize \quad & z = 20\,x_1 + 30\,x_2 \\ subject\ \ to \quad & \\ & x_1 + x_2 \geq 50 \\ & x_1 \leq 30 \\ & x_2 \leq 40 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

4-

$$\begin{aligned} \min imize \quad & z = 30\,x_1 + 40\,x_2 \\ subject\ \ to \quad & \\ & 2x_1 + x_2 \geq 12 \\ & x_1 + x_2 \geq 9 \\ & x_1 + 3\,x_2 \geq 15 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

5-

$$\begin{aligned} &\min imize \quad z = -2x_1 + 3x_2 \\ &subject\ \ to \\ &\qquad -x_1 + x_2 \le 1 \\ &\qquad x_1 + 2x_2 \ge 4 \\ &\qquad 4x_1 + 5x_2 \le 20 \\ &\qquad x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$

6-

$$\begin{aligned} &\min imize \quad z = 12x_1 - 5x_2 \\ &subject\ \ to \\ &\qquad 3x_1 - x_2 \ge 4 \\ &\qquad 2x_1 + 3x_2 \le 6 \\ &\qquad x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$

3-2 Two - Phase Method

This method is used to find the optimal solution of linear programming models in which some constraints are of types (=), (≥) where, we need to add artificial variables R_1 , R_2, ... to the L. H. S. of the constraints after subtracting surplus variables to form the standard form. These variables R_1 , R_2, ... are the starting basic variables of the standard form.

This method depends on <u>two phases:</u>

<u>Phase I</u>

<u>Step 1</u>: Solve the L. P. P.

$$\begin{aligned} &\text{minimize} \quad r = R_1 + R_2 + \dots \\ &\text{subject to} \\ &\quad \textit{the set of constraints in the} \\ &\quad \textit{standard form containing} \\ &\quad \textit{artificial variables} \end{aligned}$$

<u>Note that:</u>

Before starting the solution, the coefficients of R_1 , R_2, ... in the O. F. must be zeros using the su constraints.

Step 2

If the optimal minimum value of r is zero, that is if $\min imum \quad r = 0$, the original L.P.P. has a solution, then go to phase 2. Otherwise, if the minimum value of r is positive, then the problem has no solution and stop.

Phase II

Step 1

The optimal solution of phase I is a starting solution of phase II. In the optimal tableau of phase I, cross out the objective function row (r- row) and the columns of artificial variables.

Step 2

Formulate the original objective function z in terms of the non-basic variables in the optimal tableau of phase I.

Step3

Insert the new form of z in the optimal tableau in phase I and apply the simplex algorithm for maximization or (minimization) until obtaining the optimal solution of the original problem.

Example 3-2-1

Use the two-phase method to solve the following L.P.P.:

$$\begin{aligned} &\text{minimize} \quad z = 4x_1 + x_2 \\ &\text{subject to} \\ &3x_1 + x_2 = 3 \\ &4x_1 + 3x_2 \geq 6 \\ &x_1 + 2x_2 \leq 4 \\ &x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

Solution

Phase I

Solve the following L. P.P.

$$\begin{aligned} &\text{minimize} \quad r = R_1 + R_2 \\ &\text{subject to} \\ &3x_1 + x_2 + R_1 = 3 \\ &4x_1 + 3x_2 - s_1 + R_2 = 6 \\ &x_1 + 2x_2 + s_2 = 4 \\ &x_1 \geq 0, \quad x_2 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0, \quad R_1 \geq 0, \quad R_2 \geq 0 \end{aligned}$$

Substitute from the constraints in the O. F. we have

$$z = R_1 + R_2$$

$$z = (3 - 3x_1 - x_2) + (6 - 4x_1 - 3x_2 + s_1)$$

$$z + 7x_1 + 4x_2 - s_1 = 9$$

Starting tableau (1)

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
r	7	4	−1	**0**	**0**	**0**	9
R_1	3	1	0	1	0	0	3
R_2	4	3	-1	0	1	0	6
s_2	1	2	0	0	0	1	4

Introduce x_1 as an entering variable and drop R_1 as the leaving variable, so the pivot element is 3 , then obtain tableau 2 as follows:

Tableau (2)

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
r	0	$\frac{5}{3}$	−1	$-\frac{7}{3}$	**0**	**0**	2
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	0	1
R_2	0	$\frac{5}{3}$	-1	$-\frac{4}{3}$	1	0	2
s_2	0	$\frac{5}{3}$	0	$-\frac{1}{3}$	0	1	3

Introduce x_2 as an entering variable and drop R_2 as the leaving variable, so the pivot element is $\frac{5}{3}$, then obtain tableau 3 as follows:

Tableau (3) optimal

Basic	x_1	x_2	s_1	R_1	R_2	s_2	solution
r	0	0	0	-1	-1	0	0
x_1	1	0	$\frac{1}{5}$	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	$-\frac{4}{5}$	$\frac{3}{5}$	0	$\frac{6}{5}$
s_2	0	0	1	1	-1	1	1

Since minimum r equal zero, go to phase II.

Phase II

Step 1

Cross out the columns of artificial variables and the O. F. row , we find that the basic variables are x_1 , x_2 the non basic variable is s_2. Obtain tableau (4) as follows:

Tableau (4)

Basic	x_1	x_2	s_1	s_2	solution
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
s_2	0	0	1	1	1

Substitute in z by $x_2 = \frac{6}{5} + \frac{3}{5} s_1$ $\quad x_1 = \frac{3}{5} - \frac{1}{5} s_1$,

then $z = 4x_1 + x_2 \quad = \quad -\frac{1}{5} s_1 + \frac{18}{5}$

Step 2

Insert the coefficients of z in the new form, in tableau (4) obtain the following:

Tableau (5)

Basic	x_1	x_2	s_1	s_2	solution
z	0	0	$\frac{1}{5}$	0	$\frac{18}{5}$
x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$
x_2	0	1	$-\frac{3}{5}$	0	$\frac{6}{5}$
s_2	0	0	1	1	1

Introduce s_1 as an entering variable and drop s_2 as the leaving variable, so the pivot element is 1 , then obtain tableau 6 as follows:

Tableau (6) optimal

Basic	x_1	x_2	s_1	s_2	solution
z	0	0	0	$-\frac{1}{5}$	$\frac{17}{5}$
x_1	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$
x_2	0	1	0	$\frac{3}{5}$	$\frac{9}{5}$
s_1	0	0	1	1	1

non Positive

The optimal solution is

$$x_1^* = \frac{2}{5}, \quad x_2^* = \frac{9}{5}, \quad z^* = 4x_1^* + x_2^* = \frac{17}{5}$$

Which is the same solution obtained by M- technique.

Exercise 3-2

Use the two-phase method to find the optimal solution of the following L. P. Models:

1-

$$\begin{aligned} &\text{minimize} && z = 5x_1 + 2x_2 \\ &\text{subject to} && \\ &&& 2x_1 + 5x_2 \geq 10 \\ &&& 4x_1 - x_2 \geq 12 \\ &&& x_1 + x_2 \geq 4 \\ &&& x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

2-

$$\begin{aligned} &\text{minimize} && z = 3x_1 + 2x_2 \\ &\text{subject to} && x_1 + 2x_2 \geq 6 \\ &&& 2x_1 + x_2 \geq 8 \\ &&& x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

3-

$$\text{minimize} \quad z = 20\, x_1 + 30\, x_2$$
$$\text{subject to}$$
$$x_1 + x_2 \geq 50$$
$$x_1 \leq 30$$
$$x_2 \leq 40$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

4-

$$\text{minimize} \quad z = 30\, x_1 + 40\, x_2$$
$$\text{subject to}$$
$$2x_1 + x_2 \geq 12$$
$$x_1 + x_2 \geq 9$$
$$x_1 + 3\, x_2 \geq 15$$
$$x_1 \geq 0, \quad x_2 \geq 0$$

5-

$$\begin{aligned} \min imize \quad & z = -2x_1 + 3x_2 \\ subject\ \ to \quad & \\ & -x_1 + x_2 \le 1 \\ & x_1 + 2x_2 \ge 4 \\ & 4x_1 + 5x_2 \le 20 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$

6-

$$\begin{aligned} \min imize \quad & z = 12x_1 - 5x_2 \\ subject\ \ to \quad & \\ & 3x_1 - x_2 \ge 4 \\ & 2x_1 + 3x_2 \le 6 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$