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Operations Research

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Introduction in Operations Research

Science of Operations Research determines the optimal solutions of the real life problems. Operations research started in 1938 in England since the Second World War, they were locking for manufacturing Radars and identifying the optimal positions of Radars against German attacks.

Also, in 1940, in America, a team of scientists was interested in this science and established Operations Research Department in different Military Colleges. This was an important reason to the end of the Second World War.

Different countries requested from the scientists, the Engineers and the Military experts to study and develop the air, the sea and the land defense facilities in order to find the best (the optimal) military facilities.

Now, Operations Research different branches can be used in all our real life problems in Industry, Economy and many other areas.

Now, some definitions of operations research are considered as follows:

Definition of Operations Research

Science of operations research is the new accurate scientific technique to form mathematical models of the complicated real life problems in order to find it's optimal solution to help the decision maker to take the good and suitable decision, where, the decision maker DM knows the problem completely, the decision variables (unknowns), a definite objective function and a set of definite constraints.

To study (solve) the problems in the real life, use the following scientific technique:

1- Define and identify the problem completely.

2- Collect and update the data of the problem. Be sure that, the data is correct and save.

3- Formulate the suitable mathematical model of the problem.

4- Using suitable scientific methods and computer to find the optimal solution (or optima) of the mathematical model of the problem.

5- Introduce recommendations to the decision maker to decide the optimal decision to be applied in the real life.

Due to the great process of science in many countries in the 20th century, some Arabic countries started to use operations research in marine transportation and in military forces.

Also some professionals of planning, management, economy, commerce, engineering, statistics and others started to apply operations research models and techniques in many areas such as education, industry, agriculture, health, environment and others.

Fields of Operations Research

OR is one of the most important branches in new applied mathematics. Some OR fields are:

1- Mathematical Programming

This field includes the following:

Linear Programming

Nonlinear Programming

Parametric Programming

Engineering Programming

Dynamic Programming

Stochastic Programming

Integer Programming

Multi-Objective Programming

Game Theory

2- Probability Applications

Includes the following:

Markov and Poisson Preprocesses

Queuing (waiting lines) Theory

Inventory Theory

Simulation and Others

3- Statistical Methods

Includes the following:

Analysis of Correlation, Regression (Prediction) and Variance

Design and Analysis of Experiments.

Note that:

Methods of Mathematical Programming are useful for obtaining the maximum or the minimum value of a function of severable variables under a set of conditions (constraints) in equations or inequalities forms.

In 1947, G. B. Dantiz and his students started to use mathematical programming in American Air Weapons and started the simplex method, then they used this method in Transportation, Assignment, marketing, feeding , Production and Others.

In this book, the study is concentrated on the linear programming models and the solution graphically and algebraically. The sensitivity analysis also is considered in this book. The transportation models and the assignment models are considered in the study.

Chapter 1

Formulation of Linear Programming Models and Graphical Solutions

1-1 The general form of a L. P. P.

Assume that the decision variables of the problem are $x_1, x_2, \ldots, x_n$ then the general form of a LPP is

objective function

$$\max imize\ (\min imize)\ z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

subject to a set of linear constraints .

in The form $a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \ (\leq or = or \geq)\ b_1$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \ (\leq or = or \geq)\ b_2$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \ (\leq or = or \geq)\ b_m$$

The non negative condition

$$x_1 \geq 0,\ x_2 \geq 0, \ldots, x_n \geq 0$$

where the coefficients given

$$c_1, c_2, \ldots, c_n, a_{11}, \ldots, a_{mn}$$

$a_{ij}, b_{ij}\ \forall i = 1 \ldots m$ → are constants

$\forall j = 1, 2 \ldots n$

and the R.H.S. values $b_1, b_2, \quad ..., b_m$ are given

the condition

$$x_1 \geq 0, \quad x_2 \geq 0, \quad ..., \quad x_n \geq 0$$

Is called the Non-negativity condition.

1-2 Some L. P. Models

1-2-1 Diet Problem (Example)

وجبات

A factory produces children meals contain milk, meat and eggs. Each child needs every day foods give at least 45 unit of vitamin A, 55 units vitamin B. If the prices are

Minimum

10 Pounds per kilo of milk, it contains 3 units of vitamin A and 12 units of vitamin B,

100 Pounds per kilo of meat, it contains 5 units of vitamin A and 10 units of vitamin B,

12 Pounds per dozen (=12) of eggs, it contains 9 units of vitamin A and 8 units of vitamin B.

Formulate the mathematical model of this problem to determine the optimum different quantities of milk, meat and eggs satisfying the required vitamins of each meal and satisfy the minimum possible price.

Solution (Formulation of LP Model)

Assume that x_1, x_2, x_3 are the required quantities, where

x_1 is the quantity in kilos of milk per one meal,

x_2 is the quantity in kilos of meat per one meal,

x_2 is the quantity in dozens of eggs per one meal.

This data can be summarized in the table:

Vitamin	Quantity of milk in kilos	Quantity of meat in kilos	Quantity of eggs in dozens	The minimum number of vitamins
A	3	5	9	45
B	12	10	8	55
Price in Pounds	10	100	30	

Then, the linear programming model is as follows:

$$\min imize \quad z = 10\ x_1 + 100\ x_2 + 30\ x_3$$

$$subject\ to$$

$$3x_1 + 5x_2 + 9x_3 \geq 45 \quad (vit.\ A)$$

$$12\ x_1 + 10x_2 + 8\ x_3 \geq 55\ (vit.\ B)$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\ .$$

Now, after studying the solution methods, we try to find the optimal solution x_1^*, x_2^*, x_3^* which satisfy the set of constraints and gives the minimum value of the objective function $z^* = 10x_1^* + 100x_2^* + 30x_3^*$

1-2-2 Production Problem (Example)

السلع

A factory produces three types of commodities A, B, C where, the production process can be done in three stages I, II, III as follows:

Each unit of commodity A takes 2 min. in stage I, 3 min. in stage II and one min. in stage III

Each unit of commodity B takes 2 min. in stage I, 5 min. in stage II and 3 min. in stage III

Each unit of commodity C takes 3 min. in stage I, 2 min. in stage II and 4 min. in stage III

If the maximum number of minutes every day in the three stages I, II, III are 520, 540, 560 respectively,

ربح

If the profit per unit of the three types A, B, C are \$4, \$3, \$5 respectively,

Formulate the mathematical model of this problem to determine the optimum number of units to be produced of the three types A, B, C and satisfy the maximum possible profit.

Solution (Formulation of LP Model)

Assume that x_1, x_2, x_3 are the decision variables, where

x_1 is the number of units of commodity of type A to be produced per day

x_2 is the number of units of commodity of type B to be produced per day

x_3 is the number of units of commodity of type C to be produced per day

this data can be summarized in the table:

Commodity type	Time in minutes per unit in stage I	Time in minutes per unit in stage II	Time in minutes per unit in stage III	Profit in $ per commodity unit
A	2	3	1	4
B	2	5	3	3
C	3	2	4	5
Max. no. of min./ day	520	540	560	

Then, the linear programming model is as follows:

$$\max imize \qquad z = 4\,x_1 + 3\,x_2 + 5\,x_3$$

subject to

$$2x_1 + 2x_2 + 3x_3 \le 520 \quad (stage\, I)$$

$$3\,x_1 + 5x_2 + 2\,x_3 \le 540 \quad (stage\ II)$$

$$x_1 + 3\,x_2 + 4\,x_3 \le 560 \quad (stage\ III)$$

$$x_1 \ge 0,\ \ x_2 \ge 0\,,\ \ x_3 \ge 0$$

Now, after studying the solution methods, we try to find the optimal solution $x_1^*,\ x_2^*\ ,\ x_3^*$ which satisfy the set of constraints and gives the maximum value of the objective function $z^* = 4\,x_1^* + 3\,x_2^* + 5\,x_3^*$.

1-2-3 Paints Production Problem (Example)

A factory produces three types of paints, Exterior and Interior, where, the production needs two types of raw materials $I,\ II$ as follows:

Each ton of Exterior paint consumes one ton of raw material I and two tons of raw material II

Each ton of Interior paint consumes two tons of raw material I and one ton of raw material II

The maximum availability of raw material *I* and *II*

Per day are 6 tons and 8 tons , respectively.

The sale price per ton of External and Internal paint are $3000 and $2000 , respectively.

this data can be summarized in the table:

Paint type	Raw material I	Raw material II	Sale price per ton
External	1	2	$ 3000
Internal	2	1	$ 2000
Tons available /day	6	8	

The survey show that, the daily produced quantity of the internal paint cannot exceed the daily produced quantity of the External paint by more than one ton. Also, the daily produced quantity of the internal paint is a maximum of 2 tons. Formulate the mathematical model of this problem in order to determine the optimal daily production of each type to maximize the sales income (so, maximize the profit).

<u>Solution (Formulation of LP Model)</u>

Assume that x_1, x_2 are the decision variables,

where

x_1 is the number of tons of External paint to be produced per day

x_2 is the number of tons of Internal paint to be produced per day

the linear programming model is as follows

$$\begin{aligned}
&\text{maximize} && z = 3x_1 + 2x_2 \\
&\text{subject to} && \\
& && x_1 + 2x_2 \le 6 && (\text{resource } 1) \\
& && 2x_1 + x_2 \le 8 && (\text{resource } 2) \\
& && -x_1 + x_2 \le 1 && (\text{resource } 3) \\
& && x_2 \le 2 && (\text{resource } 4) \\
& && x_1 \ge 0, \quad x_2 \ge 0 &&
\end{aligned}$$

Now, we try to find the optimal solution $x_1^*, \ x_2^*$ which satisfy the set of constraints and gives the maximum value of the objective function

$$z^* = 3x_1^* + 2x_2^*$$

<u>Note that</u> the money unit is considered in $1000

1-3 Graphical Solution

If the L. P. P. has only two variables, it can be solved graphically. Sometimes, if it has three variables.

In general, if no. of variables is more than 2, the problem can be solved by one of the algebraic methods. Before solving the problem graphically, we need to know the following concepts:

1-3-1 The Feasible Domain

It is the region (area) satisfies the set of all constraints, this region can be bounded or unbounded.

1-3-2 Extrema (Max. or Min. Points)

The set of all corner points of the feasible region obtained from the intersection of the straight lines representing the constraints, is called the set of extreme points, from which we determine the maximum the maximum and the minimum points.

1-3-3 Optimal Solution

It is one of the extreme points that at which the objective function is maximized or minimized according to the problem.

1-3-4 Theory of Existence of Solutions of L. P.P

1- If the solution space is bounded, then both the Maximum and the minimum values are exist.

2- If the solution space is unbounded from above, the coefficients of the objective function are positive, then the minimum is exist but the maximum is not exist.

3- If the solution space is empty, then the minimum and the maximum does not exist.

1-3-5 Obtaining the Optimal Solution Graphically

1- Sketch the horizontal x_1 axis and the vertical axis x_2

2- If the non-negativity condition is satisfied then, the solution space is in the first quarter in the plane. If the solution space is not satisfied, then the solution space is in any area of the plane.

3- Sketch the straight lines representing the constraints, then identify the half plane satisfying the constraint.

4- Determine the solution space, which is the area (the region) bounded by the set of all constraints, this region can be bounded or unbounded.

5- The optimal solution is one of the extreme points satisfies the maximum or the minimum value of the objective function. This can be obtained by two methods:

5-a Substitute by the extreme point in the objective function, make comparison between the O.F. values to determine the maximum or the minimum.

5-b Sketch the objective function line at a suitable value of Z such that, the line is cutting the solution space, then move this line in the direction of max. or min. according to the problem or (sketch parallel lines) until we find a line is a tangent to one of the extreme points, this point is the optimal solution.

<u>Example 1-3-1</u>

Find, graphically, the optimal solution of the following L.P.P:

$$\max imize \quad z = 3x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 \le 6 \quad \rightarrow (1)$$ (0,3) (6,0)

$$2x_1 + x_2 \le 8 \quad \rightarrow (2)$$ (0,8) (4,0)

$$-x_1 + x_2 \le 1 \quad \rightarrow (3)$$ (0,1) (-1,0)

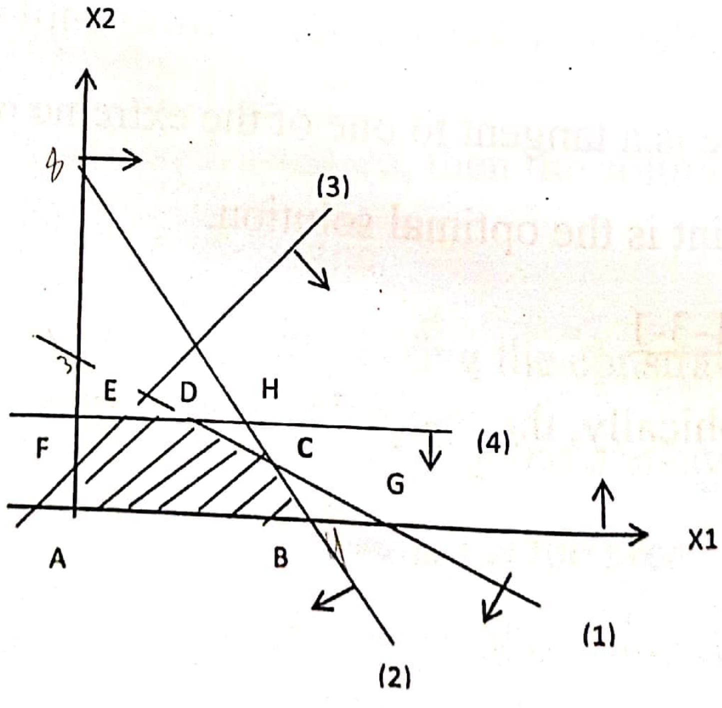
$$x_2 \le 2 \quad \rightarrow (4)$$

$$x_1 \ge 0, \; x_2 \ge 0$$

<u>Solution</u>

The solution space is the region ABCDEF, where

$A(0, 0)$, $B(4, 0)$, $C\left(\frac{10}{3}, \frac{4}{3}\right)$, $D(2, 2)$, $E(1, 2)$, $F(0, 1)$



The optimal solution is at the point $C\left(\frac{10}{3}, \frac{4}{3}\right)$ obtained from the intersection of the two lines

$$x_1 + 2x_2 = 6, \qquad 2x_1 + x_2 = 8$$

Then, the optimal solution is

$$x_1^* = \frac{10}{3}, \quad x_2^* = \frac{4}{3}, \quad z^* = 3x_1^* + 2x_2^* = \frac{38}{3}$$

<u>Example 1-3-2</u>

Find, graphically, the optimal solution of the following L.P.P:

$$\max imize \qquad z = 4x_1 + 5x_2$$

$$subject\ to$$

12 45 21

(3,0) (0,9) (4,1)

(3,3) 27

$$2x_1 + x_2 \le 9 \qquad \to (1)$$

16

(4,0)

(0,3) 15

$$x_1 \le 4 \qquad \to (2)$$

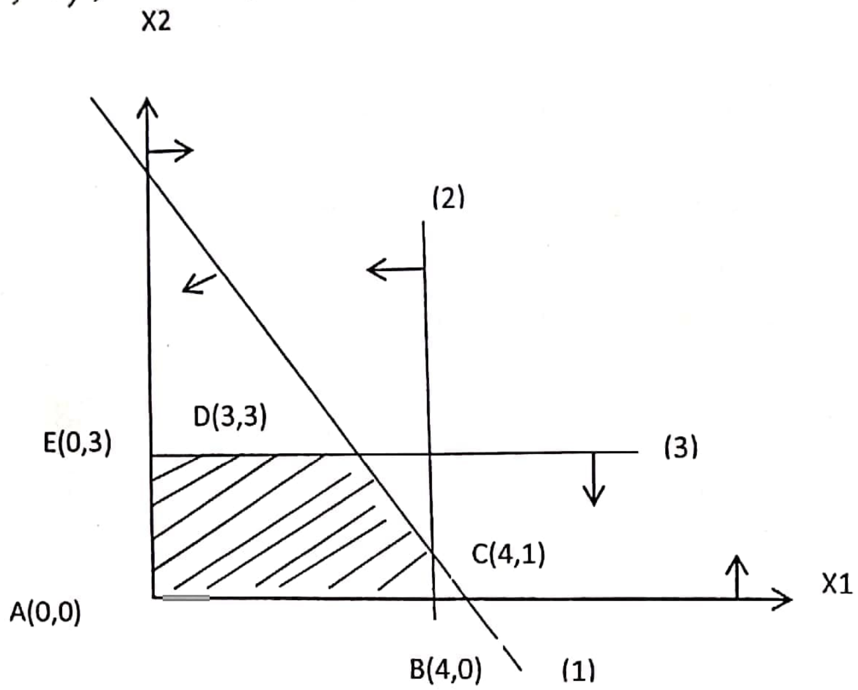
$$x_2 \le 3 \qquad \to (3)$$

$$x_1 \ge 0, \quad x_2 \ge 0$$

<u>Solution</u>

The solution space is the region ABCDEF, where

$$A(0,\ 0),\ B(4,0),\ C(4\ ,\ 1),\ D(3,3)\ ,\ \ E(0,3)$$



The optimal solution is at the point $D(3,\ 3)$ obtained from the intersection of the two lines

$$2x_1 + x_2 = 9\ ,\ \ x_2 = 3$$

Then, the optimal solution is

$$x_1^* = 3,\ \ x_2^* = 3\ ,\ \ z^* = 4x_1^* + 5x_2^* = 27$$

<u>Example 1-3-3</u>

Find, graphically, the optimal solution of the following L.P.P:

$$\begin{aligned} &\min imize && z = x_1 + x_2 \\ &subject\ to && \end{aligned}$$

$$2x_1 + x_2 \geq 12 \quad \rightarrow (1)$$

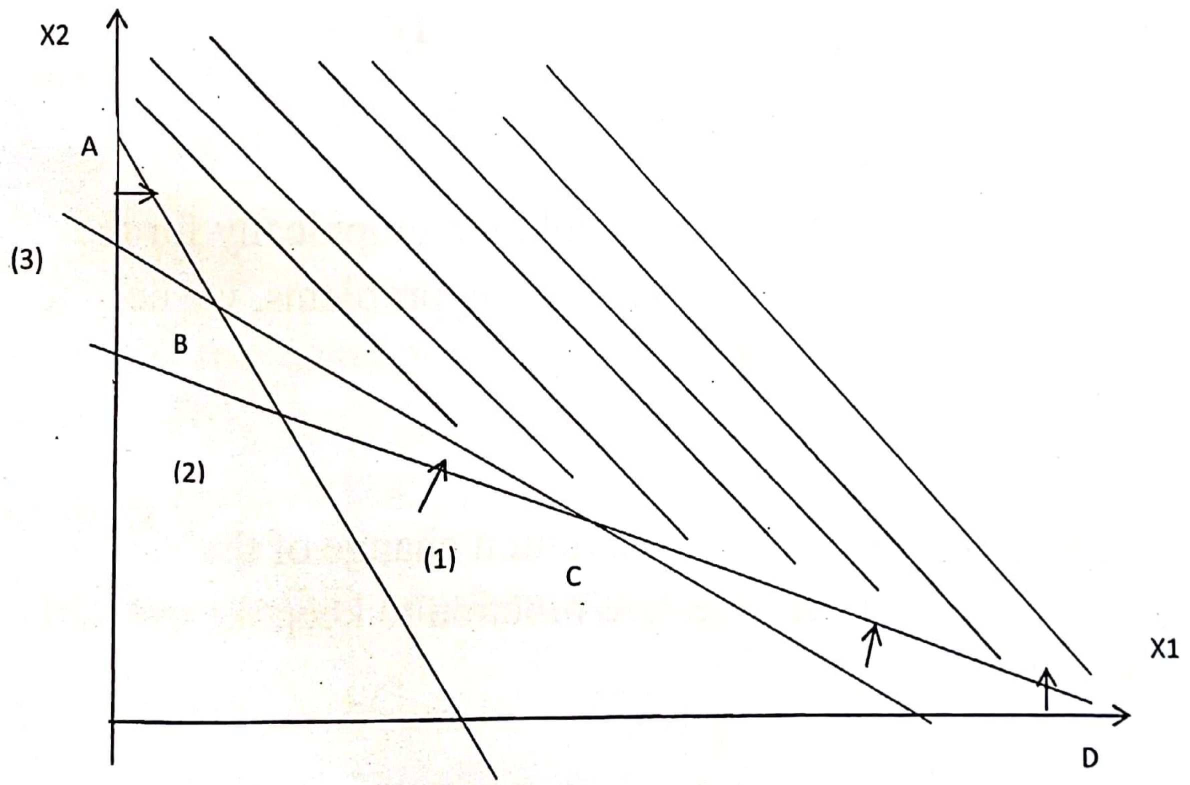
$$x_1 + 6x_2 \geq 28 \quad \rightarrow (2)$$

$$5x_1 + 8x_2 \geq 74 \quad \rightarrow (3)$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

Solution

The solution space is unbounded from Right and from above, where



The extreme points are

$$A(0,\ 12),\ B(2, 8),\ C(10\ ,\ 3),\ D(28, 0)$$

The optimal solution is at the point $B(2,8)$ obtained from the intersection of the two lines

$$2x_1 + x_2 = 12, \qquad 5x_1 + 8x_2 = 74$$

Then, the optimal solution is

$$x_1^* = 2, \quad x_2^* = 8, \quad z^*_{min} = x_1^* + x_2^* = 10$$

<u>Note that</u>

For the same solution space, the maximum value of the objective function approaches to infinity, so the solution in case of maximization <u>Does Not Exist.</u>

1-4 Graphical Sensitivity Analysis

(Post Optimal Analysis Graphically)

After obtaining the optimal solution graphically for the maximizations linear programming problems, we need to answer two questions:

<u>Question 1:</u> what is the maximum change of the coefficients of the objective function to keep the optimal solution?

<u>Question 2:</u> what is the unit worth of each resource?

<u>Note that:</u>

1- The resource no. i is the value in the R.H.S. of the constraint no. i of Type ($\leq$).

2- The Binding constraints are that constraints satisfied as equations at the optimal solution. The straight lines of these constraints intersects at the optimal solution.

3- The Nonbinding constraints are that constraints satisfied as inequalities ($<$ *or* $>$) at the optimal solution.

4- The steps of sensitivity analysis graphically can be applied directly for the above examples as follows:

Example 1-4-1

Answer the two questions of sensitivity analysis of example 1-3-1.

Answer of Q1

Assume that the O.F. equation and it's slope are

$$z = C_1 x_1 + C_2 x_2 \,, \qquad m_1 = -\frac{C_1}{C_2}$$

The Binding constraints equations and their slopes are

$$x_1 + 2x_2 = 6 \quad , \qquad m_2 = -\frac{1}{2}$$

$$2x_1 + x_2 = 8 \quad , \qquad m_2 = -\frac{2}{1}$$

To keep the optimal solution at the point $C\left(\frac{10}{3}, \frac{4}{3}\right)$, the slope of the O.F. line must be between the two slopes of the two binding constraints as follows

$$m_2 \le m \le m_1$$

then

$$-2 \le -\frac{C_1}{C_2} \le -\frac{1}{2}$$

at $C_1 = 3$ we have

$$\frac{3}{2} \le C_2 \le 6$$

at $C_2 = 2$ we have

$$1 \le C_1 \le 4$$

/

<u>Answer of Q2</u>

Resource 1 value = R.H.S. of constraint (1)

$= 6$ tons/day (raw material I)

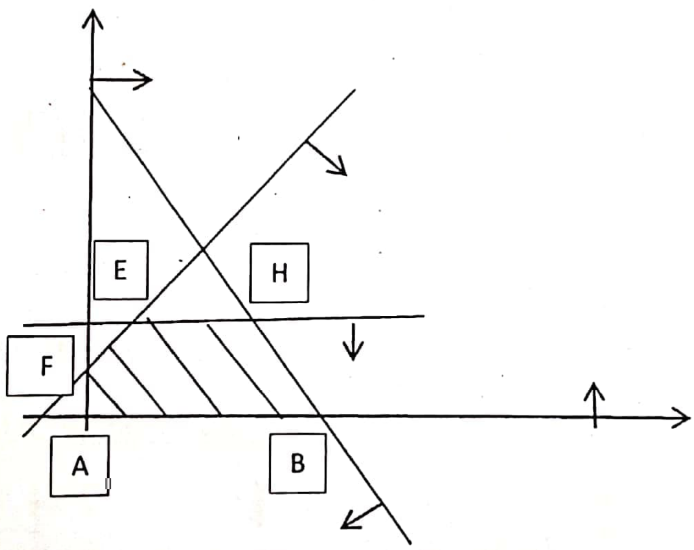
Resource 2 value = R.H.S. of constraint (2)

$= 6$ tons/day (raw material II)

<u>Resource 1:</u>

The value 6 tons can be increased until we find the new solution space ABHEF as follows:

$$A(0,\ 0),\ B(4,0),\ H(3,2),\ E(1,2),\ F(0,1)$$



The two extreme points of the second binding constraint (2) in the new solution space are B and H, at these points; find the two values of <u>resource 1</u> and the two values of the <u>objective function z</u>, as follows:

Extreme point	Resource (1) value	O.F. value
B(4,0)	4	12
H(3,2)	7	13

The unit worth of resource (1) is

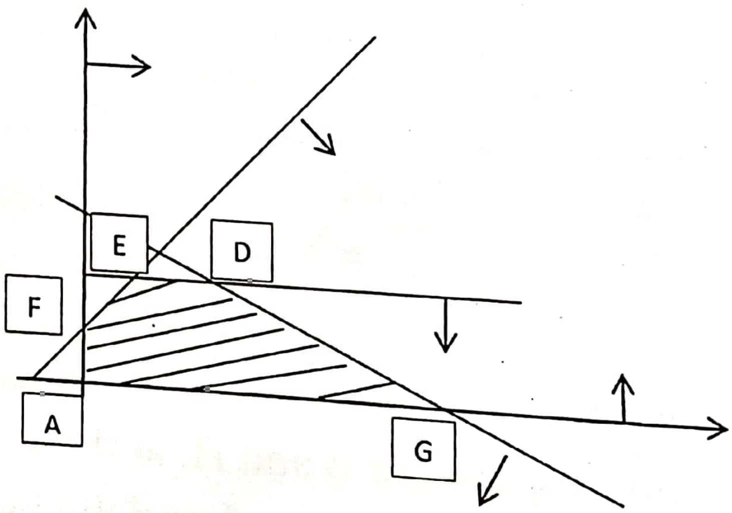
$$y_1 = \frac{Z_H - Z_B}{(resource\ 1)_H - (resource\ 1)_B}$$

$$y_1 = \frac{13-12}{7-4} = \frac{1}{3} \qquad th.\ \$$$

Resource 2:

The value 8 tons can be increased until we find the new solution space AGDEF as follows:

$$A(0,\ 0),\ G(6,\ 0),\ D(2,2)\ ,\ E(1,\ 2),\ F(0,\ 1)$$



The two extreme points of the first binding constraint (1) in the new solution space are G and D, at these points; find the two values of resource 2 and the two values of the objective function z, as follows:

Extreme point	Resource (2) value	O.F. value
G(6,0)	12	18
D(2,2)	6	10

The unit worth of resource (2) is

$$y_2 = \frac{Z_G - Z_D}{(resource\ 1)_G - (resource\ 1)_D}$$

$$y_2 = \frac{18-10}{12-6} = \frac{4}{3} \qquad th.\$$$

<u>Note that:</u>

The priority of increasing resources to increase the profit is that for <u>resource (2)</u> because

$$y_1 = \frac{1}{3}, \quad y_2 = \frac{4}{3} \quad \Rightarrow y_2 > y_1$$

Exercise 1

For the following linear programming problems, find the optimal solution (if it is exist).

(1)

$$\begin{aligned} &\text{minimize} \quad z = 5x_1 + 2x_2 \\ &\text{subject to} \\ &\quad 2x_1 + 5x_2 \geq 10 \\ &\quad 4x_1 - x_2 \geq 12 \\ &\quad x_1 + x_2 \geq 4 \\ &\quad x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

(2)

$$\begin{aligned} &\text{minimize} \quad z = 3x_1 + 2x_2 \\ &\text{subject to} \\ &\quad x_1 + 2x_2 \geq 6 \\ &\quad 2x_1 + x_2 \geq 8 \\ &\quad x_0 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

(3)

$$\begin{aligned} \text{minimize} \quad & z = 20x_1 + 30x_2 \\ \text{subject to} \quad & \\ & x_1 + x_2 \ge 50 \\ & x_1 \le 30 \\ & x_2 \le 40 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$

(4)

$$\begin{aligned} \text{minimize} \quad & z = 30x_1 + 40x_2 \\ \text{subject to} \quad & \\ & 2x_1 + x_2 \ge 12 \\ & x_1 + x_2 \ge 9 \\ & x_1 + 3x_2 \ge 15 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{aligned}$$

(5)

$$\begin{aligned} \min imize \quad & z = -2\,x_1 + 3x_2 \\ subject\ to \quad & \\ & -x_1 + x_2 \leq 1 \\ & x_1 + 2x_2 \geq 4 \\ & 4x_1 + 5\,x_2 \leq 20 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

Use the same solution space to

$$Maximize \quad z = 5x_1 - 2x_2$$

(6)

$$\begin{aligned} \min imize \quad & z = 12\,x_1 - 5\,x_2 \\ subject\ to \quad & \\ & 3x_1 - x_2 \geq 4 \\ & 2x_1 + 3x_2 \leq 6 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

(7) Determine graphically, the solution space of the following set of linear constraints:

$$x_1 + x_2 \le 4$$
$$4x_1 + 3x_2 \le 12$$
$$-x_1 + x_2 \ge 1$$
$$x_1 + x_2 \le 6$$
$$x_1 \,,\; x_2 \ge 0$$

what is the redundant equations?

Reduce the constraints to the minimum number of constraints that have the same solution space.

(8) Find the optimal solution graphically, of the model

$\max imize$

$$z = 4x_1 + 4x_2$$

subjct to

$$2x_1 + 7x_2 \le 21$$
$$7x_1 + 2x_2 \le 49$$
$$x_1 \,,\; x_2 \ge 0$$

What is the maximum change of the coefficients of the objective function to keep the optimal solution ?

(9) A company produces two types of commodities, A and B. The production time per unit of type A is twice as the time per unit of type B. If the company decided to produce ONLY type B, it can produce a total of 500 unit per day. The market does not need more than 2oo unit of type A and 150 unit of type B, per day. If the profit per unit of type A is \$5 and \$8 of type B. Formulate the problem as a L. P. P. the solve it graphically to find the optimal no. of units of each type produced per day in order to maximize the total profit.

(10) Find the optimal solution graphically, of the model

max *imize*

$$z = 5x_1 + 6x_2$$

subjct to

$$x_1 - 2x_2 \geq 2$$
$$-2x_1 + 3x_2 \geq 2$$
$$x_1 \ , \ x_2 \qquad \textit{unrestricted in sign}$$

(11) Show graphically that, the following model has no solution.

max *imize*

$$z = 3x_1 + 2x_2$$

subjct to

$$2x_1 + x_2 \le 2$$

$$3x_1 + 4x_2 \ge 12$$

$$x_1 \; , \; x_2 \ge 0$$