

BETWEEN WHITEHEAD GROUPS AND UNIFORMIZATION

SH486

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ABSTRACT. For a given stationary set S of countable ordinals we prove (in **ZFC**) that the assertion “every S -ladder system has \aleph_0 -uniformization” is equivalent to “every strongly \aleph_1 -free abelian group of cardinality \aleph_1 with non-freeness invariant $\subseteq S$ is \aleph_1 -coseparable, i.e. $\text{Ext}(G, \oplus_{i=0}^{\infty} \mathbb{Z}) = 0$ (in particular Whitehead, i.e. $\text{Ext}(G, \mathbb{Z}) = 0$)”. This solves problems B3 and B4 from the monograph of Eklof and Mekler [EM02].

§ 0. INTRODUCTION

On the subject and all background needed see Eklof-Mekler [EM02].

By [She74] there may be a non-free Whitehead group of cardinality \aleph_1 , by [She77] this may occur even under **CH**, in both proofs the main idea is that the group theoretic problem is similar to the combinatorial problem of uniformization. This intuition materialized as a theorem stating that the existence of a non-free Whitehead (\aleph_1 -coseparable, resp.) group of cardinality \aleph_1 is equivalent to the existence of a ladder system on a stationary subset of ω_1 that admits 2-uniformization (\aleph_0 -uniformization, resp.); see [She80] and [EM02, XIII.0, XIII.2.] where also a comprehensive list of problems is presented. More is done in Eklof-Mekler-Shelah [EMS92], [EMS94], Eklof-Shelah [ES94]; see the updated book Eklof-Mekler [EM02]. In fact if a group G of cardinality \aleph_1 is Whitehead, then it is \aleph_1 -free (i.e. each countable subgroup of G is free), and **MA** + \neg **CH** proves that for an abelian group G of cardinality \aleph_1 being a “Shelah” group (a strengthening of the notion of \aleph_1 -free groups, see Definition 0.2) is equivalent to being Whitehead [EM02, XII. 2.5, XIII. 3.6]. Therefore it is natural to ask whether uniformization principles themselves together with a strengthening of \aleph_1 -freeness imply the Whitehead property. P.C. Eklof, A.H. Mekler and S. Shelah explored the close connection of \aleph_0 -uniformization and the characterization of \aleph_1 -free abelian groups of cardinality \aleph_1 in terms of the Whitehead property and the following important problem remained open:

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References like [She, Th0.2=Ly5] means the label of Th.0.2 is y5. The reader should note that the version on the second author’s website is usually more updated than the one in the mathematical archive.

Problem. *Are the following two assertions equivalent?*

- (a) *Every strongly \aleph_1 -free abelian group of cardinality \aleph_1 is Whitehead,*
- (b) *every ladder system $\langle \eta_\delta : \delta < \omega_1 \text{ limit} \rangle$ has \aleph_0 -uniformization.*

This problem was first stated explicitly more than 30 years ago (in fact it arose implicitly more than 40 years ago), and is the crux of the problems listed in the Appendix of [EM02]. Our main theorem here settles it, and (the main case of, i.e. , the case of a strongly \aleph_1 -free group, of) **B3** and **B4**:

B3. *If we have, say, all uniformization results that can be deduced from **MA** + $\neg\mathbf{CH}$, then is every strongly \aleph_1 -free (every Shelah) group of cardinality \aleph_1 a Whitehead group?*

B4. *If every strongly \aleph_1 -free group is a Whitehead group, are they also all \aleph_1 -coseparable?*

Although the case of Shelah groups is certainly of interest, historically the stress has always been on strongly \aleph_1 -free groups. Model theoretic considerations suggest that the general case of Shelah groups might be more intricate than the more natural (and more familiar to algebraists) property of being strongly \aleph_1 -free [Ek174]. We also remark that by the already cited theorem [EM02, XII. 2.5, XIII. 3.6] we cannot replace strongly \aleph_1 -free by \aleph_1 -free.

In §1, in the proof of Theorem 1.3 the group theoretic problem is translated to a combinatorial one which is proved to be equivalent to the uniformization of ladder systems in §2 (Theorem 2.1). We have to remark that in some sense §1 is enough for answering **B3**, as (B_S) from Theorem 2.1 is a combinatorial principle which follows from \mathbf{MA}_{\aleph_1} by Claim 2.4. Still it is better to show that (A_S) from Theorem 2.1 (i.e. the ladder system uniformization with parameter S) can serve as a sufficient (and in fact necessary) condition, since that is a classical, old principle, and it is nicer to phrase.

Moreover, in Proposition 1.7 we prove an equivalence between uniformization of all ladder systems $\bar{\eta}^*$ that are “very similar” to a fixed ladder system $\bar{\eta}$ on a fixed stationary set S , and all strongly \aleph_1 -free groups of cardinality \aleph_1 whose structure in terms of non-freeness in some sense can be described by $\bar{\eta}^*$ (see $(B)_{S, \bar{\eta}^*}$ from the proposition) have the Whitehead property: We will argue that the Whitehead property of the groups in $K_{S, \bar{\eta}}$ (see Definition 1.35) implies \aleph_0 -uniformization of $\bar{\eta}$. This class is related to the more general (but less convenient to use) class where the equation is $k_n y_{\delta, n+1} = y_{\delta, n} + x_{\eta_\delta(n)}$ (related to the ones used in [She81]).

* * *

Notation 0.1.

- 1) Let $\alpha, \beta, \gamma, \delta$ be ordinals (here always $< \omega_1$), m, n, k, ℓ are natural numbers, Ω is the set of countable limit ordinals.
- 2) For sets u, v of ordinals, $\text{OP}_{u,v}$ is the following function: $\text{OP}_{u,v}(\alpha) = \beta$ iff $\alpha \in v$, $\beta \in u$ and $\text{otp}(\alpha \cap v) = \text{otp}(\beta \cap u)$. Here we let u, v, w vary on finite subsets of ω_1 .
- 3) By a sequence we mean a function on an ordinal, where for a sequence $\bar{s} = \langle s_\alpha : \alpha < \text{dom}(\bar{s}) \rangle$ the length of \bar{s} (in symbols $\ell g(\bar{s})$) denotes $\text{dom}(\bar{s})$. Moreover, for sequences \bar{s}, \bar{t} let $\bar{s} \frown \bar{t}$ denote the natural concatenation (of length $\ell g(\bar{s}) + \ell g(\bar{t})$).

4) Let $\forall^* n$ mean for every n large enough; let us define $x \subseteq^* y$, if $|x \setminus y| < \aleph_0$, so for the functions f_1, f_2 $f_1 \subseteq^* f_2$ iff $\{x \in \text{dom}(f_1) : x \notin \text{dom}(f_2) \text{ or } f_1(x) \neq f_2(x)\}$ is finite.

5) \mathbb{Z} is the additive group of integers, \mathbb{Z}_ω is the direct sum of \aleph_0 -many copies of \mathbb{Z} . The word group here always means abelian group.

Definition 0.2.

1) An abelian group G is \aleph_1 -free if all of its countable subgroups are free abelian groups, i.e. isomorphic to the direct sum $\oplus_{i < \mu} \mathbb{Z}$ for some cardinal μ (where of course for a countable free abelian group $\mu \leq \omega$). By classical results [Fuc73, Thm. 19.1.] it is equivalent to demanding that each finite rank subgroup is a free abelian group.

2) An \aleph_1 -free group G is a Shelah group, if for every countable subgroup $H \leq G$ there exists a countable subgroup $H' \leq G$ with $H \leq H'$, which moreover satisfies that for every countable group $F \leq G$

$$F \cap H' = H \Rightarrow F/H \text{ is free.}$$

3) An \aleph_1 -free group G is strongly \aleph_1 -free, if for each countable subgroup $H \leq G$ there exists a countable subgroup $H' \leq G$ with $H \leq H'$, and G/H' being \aleph_1 -free.

4) G is a W -group means G is a Whitehead group which means: if $\mathbb{Z} \subseteq H$, $H/\mathbb{Z} \cong G$ then \mathbb{Z} is a direct summand of H .

5) G is \aleph_1 -coseparable means: if $\mathbb{Z}_\omega \subseteq H$ and $H/\mathbb{Z}_\omega \cong G$ then \mathbb{Z}_ω is a direct summand of H where \mathbb{Z}_ω is the direct sum of ω -many copies of \mathbb{Z} . We call an \aleph_1 -coseparable group also a W_ω -group.

Definition 0.3.

1) For $S \subseteq \Omega$ we say that $\bar{\eta}$ is an S -ladder system if $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$, η_δ a strictly increasing ω -sequence of ordinals $< \delta$ with limit δ .

2) We say $\bar{\eta}$ has κ -uniformization when: for every $\bar{c} = \langle c_\delta : \delta \in S \rangle$, c_δ a function from $\text{ran}(\eta_\delta)$ to κ there is a function c from ω_1 to κ , uniformizing \bar{c} , i.e., for every $\delta \in S$, $c_\delta \subseteq^* c$, i.e. for every $n < \omega$ large enough $c(\eta_\delta(n)) = c_\delta(\eta_\delta(n))$.

3) We say that the ladder systems $\bar{\eta}^1, \bar{\eta}^2$ on the stationary set S are very similar when $\bar{\eta}^\ell = \langle \eta_\delta^\ell : \delta \in S \rangle$, and for each $\delta \in S$ for some $k_1, k_2 < \omega$ we have

$$\{\eta_\delta^1(k_1 + n) + \omega : n < \omega\} = \{\eta_\delta^2(k_2 + n) + \omega : n < \omega\}$$

(which is equivalent to stating that

$$\{\eta_\delta^1(n) + \omega : n < \omega\} =^* \{\eta_\delta^2(n) + \omega : n < \omega\}.$$

)

Definition 0.4.

1) We say S is stationary when for every closed unbounded $C \subseteq \Omega$, $S \cap C \neq \emptyset$.

2) Call S simple if $S \subseteq \{\alpha < \omega_1 : \omega^2 \text{ divides } \alpha\}$ (i.e. $S \cap \{\alpha + \omega : \alpha < \omega_1\} = \emptyset$).

§ 1. ABELIAN GROUPS

Before stating our main theorem we need an array of definitions.

Definition 1.1.

a) If G is an abelian group, $A \subseteq G$, we let $\text{cl}_G(A)$ be the subgroup of G that A generates (if G is fixed, we may suppress it).

- b) The subgroup $H \leq G$ is a pure subgroup if for every $g \in G$, $n \in \mathbb{Z}$, $ng \in H$ implies $g \in H$. As the groups in this paper are mostly torsion free (except for some quotient groups of the \aleph_1 -free groups, when we divide by a non-pure subgroup), for every such group G and set $A \subseteq G$ we have that the pure closure $\text{cl}_G^{\text{pr}}(A) := \cap \{K : A \subseteq K, K \text{ a pure subgroup of } G\}$ of A is equal to $\{x : \text{for some } n > 0, nx \in \text{cl}_G(A)\}$.
- c) For $K_1, K_2 \subseteq G$, let $K_1 + K_2 = \{x + y : x \in K_1, y \in K_2\}$. If $K_1 \cap K_2 = \{0\}$ we write $K_1 \oplus K_2$.
- d) For $K \leq G$ we say that $\{g_i : i \in I\} \subseteq G$ is independent over K if there exists no non-trivial \mathbb{Z} -linear combination of $\{g_i + K : i \in I\}$ being equal to 0 in the quotient group G/K .
- e) We call a set independent if it is independent over the trivial subgroup $\{0\}$.
- f) If G is an abelian group of cardinality \aleph_1 , then we say $\bar{G}' = \langle G'_i : i < \omega_1 \rangle$ is a filtration of G , when
 - f1) $\bar{G}' = \langle G'_i : i < \omega_1 \rangle$ is increasing, continuous,
 - f2) for each α G'_α is a countable pure subgroup of G ,
 - f3) $G = \bigcup_{\alpha < \omega_1} G'_\alpha$.

Definition 1.2. For a fixed stationary set S we will consider the following assertions.

- (A)_S \equiv every S -ladder system $\langle \eta_\delta : \delta \in S \rangle$ has \aleph_0 -uniformization,
- (B)_S \equiv if G, \bar{G}' satisfy $(*)_S(G, \bar{G}')$ below then G is a Whitehead group (W -group in short)
- $(*)_S(G, \bar{G}')$: $(\alpha)^*$ G is an abelian group of cardinality \aleph_1 ,
 $(\beta)^*$ G is \aleph_1 -free, i.e. every countable subgroup of it is free,
 $(\gamma)^*$ $\bar{G}' = \langle G'_i : i < \omega_1 \rangle$ is a filtration of G , and

$$\{\delta \in \Omega : G/G'_\delta \text{ is not } \aleph_1\text{-free}\} \subseteq_{\text{NS}} S$$
 (which means that the set $\{\delta \in \Omega : G/G'_\delta \text{ is not } \aleph_1\text{-free}\}$ is contained in S up to a non-stationary set),
 $(\delta)^*$ G is strongly \aleph_1 -free, i.e., for every countable $H \leq G$ there is a countable subgroup $H' \leq G$ extending H such that G/H' is \aleph_1 -free (follows from (γ) if $\omega_1 \setminus S$ is stationary),
- (C)_S \equiv if G, \bar{G}' satisfy $(*)_S(G, \bar{G}')$ above, then G is \aleph_1 -coseparable (one of the equivalent definitions is $\text{Ext}(G, \mathbb{Z}_\omega) = 0$ where \mathbb{Z}_ω is the direct sum of ω copies of \mathbb{Z} , i.e. G is a W_ω -group).

Theorem 1.3. *Let S be a stationary subset of ω_1 . Then*

$$(A)_S \iff (B)_S \iff (C)_S.$$

Before proving this we state its immediate corollary:

Corollary 1.4. The following are equivalent:

- (a) every ladder system $\langle \eta_\delta : \delta < \omega_1 \text{ limit} \rangle$ has \aleph_0 -uniformization.
- (b) Every strongly \aleph_1 -free abelian group of cardinality \aleph_1 is Whitehead,

However, unfortunately the following falls out of the scope of the present work:

Problem 1.5. Are the following two assertions equivalent?

- (a) Every ladder system $\langle \eta_\delta : \delta < \omega_1 \text{ limit} \rangle$ has \aleph_0 -uniformization.
- (b) Every Shelah group of cardinality \aleph_1 is Whitehead,

Proof. (Theorem 1.3) Note that $(C)_S \Rightarrow (B)_S$ is immediate. For the implication $(B)_S \Rightarrow (A)_S$ see [EM02, XIII. Proposition 2.9.]. However, this citation only proves that $(B)_S$ implies that every S -ladder system $\langle \eta_\delta : \delta \in S \rangle$ has 2-uniformization, and the statement that

$$\begin{aligned} & (\text{every } S\text{-ladder system } \langle \eta_\delta : \delta \in S \rangle \text{ has 2-uniformization}) \Rightarrow \\ & \Rightarrow (\text{every } S\text{-ladder system } \langle \eta_\delta : \delta \in S \rangle \text{ has } \aleph_0\text{-uniformization}) \end{aligned}$$

is only an exercise, so a possible reference for this is [She78, Lemma 1.4]. Because of some inaccuracies in this Lemma, here (in Lemma 1.6) we provide a cleaned-up proof for the sake of completeness.

When we will have finished Lemma 1.6 we turn to our main theorem, i.e. that $(A)_S$ implies $(C)_S$. Towards this Proposition 1.7 is the key, and we will need a sequel of lemmas to be able to reduce the task to the proposition. After having proved all the ingredients for the second clause of Proposition 1.7 and put them together (right before Definition 1.35), we will argue that $(A)_S$ implies $(C)_S$.

Lemma 1.6. [She78, Lemma 1.4] *Suppose that $S \subseteq \Omega$ is stationary. If every S -ladder system has 2-uniformization, then every S -ladder system has \aleph_0 -uniformization.*

Proof. Let $\langle \eta_\delta^0 : \delta \in S \rangle$ be an S -ladder system, $\langle c_\delta^0 : \delta \in S \rangle$ be a coloring with \aleph_0 -many colors, so c_δ^0 is defined on $\{\eta_\delta^0(i) : i \in \omega\}$. We define a ladder system $\bar{\eta}^1 = \langle \eta_\delta^1 : \delta \in S \rangle$ and a coloring $\langle c_\delta^1 : \delta \in S \rangle$ with two colors. Define the function $\varepsilon : \omega_1 \times \omega \rightarrow \omega_1$ so that it is an injection, and $\varepsilon(\alpha, n) + \omega = \alpha + \omega$, i.e. for each limit β , $\varepsilon(\beta + m, n)$ is of the form $\beta + l$ for some finite l , moreover $\varepsilon(\alpha, n) < \varepsilon(\alpha, n+1)$ for each α, n . Fix a $\delta \in S$, we are going to construct η_δ^1 . Define $k_0 = c_\delta^0(\eta_\delta^0(0)) + 1 \in \omega$, and let

$$\begin{aligned} \eta_\delta^1(0) &= \varepsilon(\eta_\delta^0(0), 0), \\ \eta_\delta^1(1) &= \varepsilon(\eta_\delta^0(0), 1), \\ &\vdots \\ \eta_\delta^1(k_0 - 1) &= \varepsilon(\eta_\delta^0(0), k_0 - 1), \end{aligned}$$

and $c_\delta^1(\eta_\delta^1(j)) = 0$ for $j < k_0 - 1$, $c_\delta^1(\eta_\delta^1(k_0 - 1)) = 1$. We define the sequence k_i by induction with the rule $k_{i+1} = k_i + c_\delta^0(\eta_\delta^0(i+1)) + 1$, and parallel to that the function $\eta_\delta^1 \upharpoonright [k_i, k_{i+1})$, and $c_\delta^1(\eta_\delta^1(\ell))$ for $k_i \leq \ell < k_{i+1}$ as follows. Let $\eta_\delta^1(k_i + j) = \varepsilon(\eta_\delta^0(i+1), j)$ for $0 \leq j \leq c_\delta^0(\eta_\delta^0(i+1))$. As for the coloring, set

$$(1.1) \quad \begin{aligned} c_\delta^1(\eta_\delta^1(k_i + j)) &= 0 && \text{for } j < c_\delta^0(\eta_\delta^0(i+1)), \\ c_\delta^1(\eta_\delta^1(k_i + j)) &= 1 && \text{for } j = k_{i+1} - k_i - 1 = c_\delta^0(\eta_\delta^0(i+1))). \end{aligned}$$

Suppose that the function $d^1 : \omega_1 \rightarrow \{0, 1\}$ is a uniformization for $\langle c_\delta^1 : \delta \in S \rangle$. We let $d^0(\alpha)$ be

$$(1.2) \quad d^0(\alpha) = \min\{n \in \omega : d^1(\varepsilon(\alpha, n)) = 1\}$$

(or $d^0(\alpha) = 0$, if the set in question is empty). Fix a $\delta \in S$, suppose that for all $\ell \geq m$ $d^1(\eta_\delta^1(\ell)) = c_\delta^1(\eta_\delta^1(\ell))$ holds, and let $i \in \omega$ be minimal such that $k_i \geq m$ (where k_i is the auxiliary sequence used for constructing η_δ^1 and c_δ^1). We claim that for $n > i$ the equality $c_\delta^0(\eta_\delta^0(n)) = d^0(\eta_\delta^0(n))$ holds true. So fix such an n , and let $\alpha = \eta_\delta^0(n)$, hence

$$(1.3) \quad \eta_\delta^1(k_{n-1} + j) = \varepsilon(\alpha, j) \text{ for } j \leq c^0(\alpha) = c^0(\eta_\delta^0(n)).$$

Note that $m \leq k_{n-1}$, so $d^1(\eta_\delta^1(\ell)) = c_\delta^1(\eta_\delta^1(\ell))$ for $\ell \in [k_{n-1}, k_n)$, which means that

$$(\forall j \leq k_n - k_{n-1} - 1 (= c_\delta^0(\eta_\delta^0(n)))) : d^1(\varepsilon(\alpha, j)) = c_\delta^1(\varepsilon(\alpha, j)).$$

Recalling (1.3) we obtain $d^1(\varepsilon(\eta_\delta^0(n), j)) = d^1(\eta_\delta^1(k_{n-1} + j)) = c_\delta^1(\eta_\delta^1(k_{n-1} + j))$ for $j \leq c_\delta^0(n)$, so by (1.1) this translates to $d^1(\varepsilon(\eta_\delta^0(n), j)) = d^1(\eta_\delta^1(k_{n-1} + j)) = 0$ if $j < c_\delta^0(\eta_\delta^0(n))$, and $d^1(\varepsilon(\eta_\delta^0(n), c_\delta^0(\eta_\delta^0(n)))) = d^1(\eta_\delta^1(k_{n-1} + c_\delta^0(\eta_\delta^0(n)))) = 1$. Therefore, by the way we defined d^0 (1.2), $d^0(\eta_\delta^0(n)) = c_\delta^0(\eta_\delta^0(n))$, as desired. $\square_{\text{Lemma 1.6}}$

Proposition 1.7. *Fix a stationary set $S \subseteq \omega_1$:*

- (i) *If $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ is a ladder system on S and $\bar{\eta}$ satisfies $\text{ran}(\eta_\delta) \subseteq \Omega$ for each $\delta \in S$ (so necessarily S is simple), $\bar{\eta}^*$ is very similar to $\bar{\eta}$, then*

$$(A)_{S, \bar{\eta}} \Rightarrow (A)_{S, \bar{\eta}^*},$$

- (ii) *if $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ is a ladder system on S and $\bar{\eta}$ satisfies $\text{ran}(\eta_\delta) \subseteq \Omega$ for each $\delta \in S$, and the ladder system $\bar{\eta}^*$ is very similar to $\bar{\eta}$, then*

$$(A)_{S, \bar{\eta}} \Rightarrow (C)_{S, \bar{\eta}^*},$$

- (iii) *for every ladder system $\bar{\eta}^*$ on S*

$$(B)_{S, \bar{\eta}^*} \Rightarrow (A)_{S, \bar{\eta}^*},$$

where under $(A)_{S, \bar{\eta}}$, $(B)_{S, \bar{\eta}^}$, $(C)_{S, \bar{\eta}^*}$ we mean the following:*

- (A) $_{S, \bar{\eta}}$ \equiv *the ladder system $\langle \eta_\delta : \delta \in S \rangle$ has \aleph_0 -uniformization,*
(B) $_{S, \bar{\eta}^*}$ \equiv *whenever G and the ladder system $\bar{\eta}^* = \langle \eta_\delta^* : \delta \in S \rangle$ satisfy $(*)_{S, \bar{\eta}^*}^+(G, \bar{G}')$ below for some \bar{G}' then G is a W -group,*
 $(*)_{S, \bar{\eta}^*}^+(G, \bar{G}')$:

- (α) $^+$ G is an abelian group of cardinality \aleph_1 ,
(β) $^+$ G is \aleph_1 -free, i.e. every countable subgroup of it is free,
(γ) $^+$ the sequence $\bar{G}' = \langle G'_\alpha : \alpha < \omega_1 \rangle$ is a filtration of G , where for each $\alpha < \omega_1$ $G'_{\alpha+1}/G'_\alpha$ is of rank 1, and
 $\{\delta \in \omega_1 : G/G'_\delta \text{ is not } \aleph_1\text{-free}\} \subseteq S$,
(δ) $^+$ G is strongly \aleph_1 -free, i.e., for every countable $H \leq G$ there is a countable $H' \leq G$ extending H such that G/H' is \aleph_1 -free (follows from (γ) $^+$ if $\omega_1 \setminus S$ is not bounded in ω_1),
(ε) $^+$ for each $\delta \in S$ for some sequence $\langle y_{\delta+\ell} : \ell < \omega \rangle \in {}^\omega G'_{\delta+\omega}$ which is a maximal independent sequence over G'_δ , we have for each $n \in \omega$:

$$\left\{ \alpha < \delta : \begin{array}{l} \text{cl}^{\text{Pr}}(G'_{\alpha+1} \cup \{y_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{Pr}}(G'_\alpha \cup \{y_{\delta+\ell} : \ell < n\}) + G'_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^*).$$

- (C) $_{S, \bar{\eta}^*}$ \equiv *if G satisfies $(*)_{S, \bar{\eta}^*}^+(G, \bar{G}')$ above for some \bar{G}' then G is a W_ω -group,*

Remark 1.8. Probably we cannot state the alternative version of Proposition 1.7 defining $(A)_{S, \bar{\eta}}$, $(B)_{S, \bar{\eta}}$ modulo nonstationary sets. However, $(A)_{S, \bar{\eta}} \iff (A)_{S', \bar{\eta}'}$ if $(S, \bar{\eta})$, $(S', \bar{\eta}')$ are club similar, see the definition below.

Definition 1.9. We call $(S^0, \bar{\eta}^0)$, $(S^1, \bar{\eta}^1)$ (where S^0, S^1 are stationary subsets of ω_1 , $\bar{\eta}^i$ is a ladder system on S^i) club similar, when there exists a club set C , and a bijection $\pi : \omega_1 \rightarrow \omega_1$ such that

- (1) $C \cap S^0 = C \cap S^1$,
- (2) $\pi \upharpoonright (C \cap S^0) = \text{id} \upharpoonright (C \cap S^0)$,

(3) for each $\delta \in \omega_1$ we have $\delta \in S^0 \iff \pi(\delta) \in S^1$, and if they hold, then

$$\{\pi(\eta_\delta^0(n)) : n \in \omega\} =^* \{\eta_{\pi(\delta)}^1(n) : n \in \omega\}.$$

Lemma 1.10 proves (i). Claim 1.11, Claim 1.13, Claim 1.25 together will prove clause (ii) in Proposition 1.7. (After finishing the proof of Claim 1.25, i.e. having proved Lemma 1.34, we will put the pieces together and formally verify (ii).)

After that we will turn back to the proof of (iii) at the end of the section (starting with Definition 1.35, concluding with Lemma 1.41).

Lemma 1.10. *If $S \subseteq \omega_1 \cap \Omega$ is a stationary set, and the ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ is such that for every $\delta \in S$ $\text{ran}(\eta_\delta) \subseteq \Omega$, and $\bar{\eta}$ has \aleph_0 -uniformization, then every ladder system $\bar{\eta}^*$ very similar to $\bar{\eta}$ has \aleph_0 -uniformization.*

Proof. First observe that since $\bar{\eta}$ and $\bar{\eta}^*$ are very similar, for each δ for large enough n (depending on δ) we have $\eta_\delta(n) + \omega = \eta_\delta^*(n^*) + \omega$ for some n^* (depending on n and δ of course), moreover, since $\eta_\delta(n)$ is limit this means that $\eta_\delta^*(n^*) = \eta_\delta(n) + l$ for some $l \in \omega$. In fact, for such δ , for some N_δ , there exists a strictly increasing sequence $\langle N_{\delta,n}^* : n \geq N_\delta \rangle$ such that

$$\eta_\delta(n) + \omega = \eta_\delta^*(n^*) + \omega \text{ whenever } n^* \in [N_{\delta,n}^*, N_{\delta,n+1}^*),$$

hence for all $(\forall n \geq N_\delta)$ for some $l_0 < l_1 < \dots < l_{N_{\delta,n+1}^* - N_{\delta,n}^* - 1}$

$$(1.4) \quad \left. \begin{aligned} \eta_\delta(n) + l_0 &= \eta_\delta^*(N_{\delta,n}^*), \\ \eta_\delta(n) + l_1 &= \eta_\delta^*(N_{\delta,n}^* + 1), \\ &\dots \\ \eta_\delta(n) + l_{N_{\delta,n+1}^* - N_{\delta,n}^* - 1} &= \eta_\delta^*(N_{\delta,n+1}^* - 1). \end{aligned} \right\} (\forall n \geq N_\delta)$$

Now define the coloring $\langle c_\delta : \delta \in S \rangle$ of the ladder system $\bar{\eta}$ by

$$c_\delta(\eta_\delta(n)) = \{l \in \omega : \eta_\delta(n) + l \in \text{ran}(\eta_\delta^*)\} \in [\omega]^{<\omega},$$

and fix $\langle H_\alpha : \alpha < \omega_1 \rangle$, $\langle m_\delta : \delta \in S \rangle$ such that for every $\delta \in S$, $\forall n \geq m_\delta$ $c_\delta(\eta_\delta(n)) = H_{\eta_\delta(n)}$. We may as well assume that $m_\delta \geq N_\delta$, so that

$$(1.5) \quad \emptyset \neq \{l \in \omega : \eta_\delta(n) + l \in \text{ran}(\eta_\delta^*)\} = H_{\eta_\delta(n)} \quad (\forall n \geq m_\delta)$$

Suppose that $\langle c'_\delta : \delta \in S \rangle$ is a coloring of $\bar{\eta}^*$ with \aleph_0 -many colors, define the coloring $c''_\delta : \text{ran}(\eta_\delta \upharpoonright (\omega \setminus m_\delta))$ on $\bar{\eta}$ by

$$(1.6) \quad \begin{aligned} c''_\delta(\eta_\delta(n)) &= \langle c'_\delta(\eta_\delta(n) + l_j) : j < |H_{\eta_\delta(n)}| \rangle, \\ \text{where } H_{\eta_\delta(n)} &= \{l_0 < l_1 < \dots < l_{|H_{\eta_\delta(n)}| - 1}\}. \end{aligned}$$

(Note that, since $n \geq m_\delta$, $\eta_\delta(n) + l \in \text{dom}(c'_\delta) = \text{ran}(\eta_\delta^*)$ for all $l \in H_{\eta_\delta(n)}$.) It remains to check that a uniformization of \bar{c}'' gives rise to a uniformization of \bar{c}' . Let the function c'' uniformize \bar{c}'' ; we define c' as follows. The domain of c' is defined for any limit β so that

$$\text{dom}(c') \cap [\beta, \beta + \omega) = \{\beta + l_0, \beta + l_1, \dots, \beta + l_{i-1}\},$$

where $H_\beta = \{l_0, l_1, \dots, l_{i-1}\}$ (possibly $\text{dom}(c') \cap [\beta, \beta + \omega) = \emptyset$). Set $c'(\beta + l)$'s so that

$$(1.7) \quad c''(\beta) = \langle c'(\beta + l_j) : j < |H_\beta| \rangle$$

hold (and so $c''(\beta)$ is a finite sequence).

Fix $\delta \in S$, (recalling that c'' is a uniformization for \bar{c}'') set $m''_\delta \geq m_\delta$ so that

$$(1.8) \quad \text{for } n \geq m''_\delta : c''(\eta_\delta(n)) = c''_\delta(\eta_\delta(n))$$

We claim that setting $m'_\delta = N_{\delta, m''_\delta}^*$, whenever m satisfies $m \geq m'_\delta$, then $c'(\eta_\delta^*(m)) = c'_\delta(\eta_\delta^*(m))$, which will finish the proof of the Lemma. (Note that, recalling (1.4), since $m''_\delta \geq m_\delta$, and $m_\delta \geq N_\delta$, we clearly have $m''_\delta \geq N_\delta$, so N_{δ, m''_δ}^* is defined.) Fix $m \geq m'_\delta$, let $n \geq m''_\delta$ be the unique natural number for which

$$m \in [N_{\delta, n}^*, N_{\delta, n+1}^*).$$

It suffices to show that

$$(1.9) \quad \begin{aligned} &\langle c'_\delta(\eta_\delta^*(N_{\delta, n}^*)), c'_\delta(\eta_\delta^*(N_{\delta, n}^* + 1)), \dots, c'_\delta(\eta_\delta^*(N_{\delta, n+1}^* - 1)) \rangle = \\ &\langle c'(\eta_\delta^*(N_{\delta, n}^*)), c'(\eta_\delta^*(N_{\delta, n}^* + 1)), \dots, c'(\eta_\delta^*(N_{\delta, n+1}^* - 1)) \rangle. \end{aligned}$$

Now, since $n \geq N_\delta$, recalling (1.4) we obtain that for some $l_0 < l_1 < \dots < l_{N_{\delta, n+1}^* - N_{\delta, n}^* - 1}$:

$$(1.10) \quad \begin{aligned} &\eta_\delta(n) + l_j = \eta_\delta^*(N_{\delta, n}^* + j) \quad (\forall j < N_{\delta, n+1}^* - N_{\delta, n}^*), \\ &\text{and} \\ &\{l \in \omega : \eta_\delta(n) + l \in \text{ran}(\eta_\delta^*)\} = \{l_0, l_1, \dots, l_{N_{\delta, n+1}^* - N_{\delta, n}^* - 1}\}. \end{aligned}$$

Now (1.10) together with (1.5) (and $n \geq m_\delta$) give us

$$(1.11) \quad H_{\eta_\delta(n)} = \{l_0, l_1, \dots, l_{N_{\delta, n+1}^* - N_{\delta, n}^* - 1}\}.$$

Then (1.9) (what we want to verify) is equivalent to

$$\langle c'_\delta(\eta_\delta(n) + l_j) : j < |H_{\eta_\delta(n)}| \rangle = \langle c'(\eta_\delta(n) + l_j) : j < |H_{\eta_\delta(n)}| \rangle,$$

which is equivalent to (reformulating both the left-hand side and the right-hand side by (1.6) and (1.7)):

$$(1.12) \quad c''_\delta(\eta_\delta(n)) = c''(\eta_\delta(n)).$$

Finally, remembering that c'' uniformizes \bar{c}'' , $n \geq m''_\delta$, (1.8) implies that (1.12) holds true (and so (1.9), too), as desired. $\square_{\text{Lemma 1.10}}$

The following will be useful for the proof of Theorem 1.3 (as Proposition 1.7 has stronger premises).

Claim 1.11. *Let G be a group, \bar{G}^* , S satisfy $(*)_S(G, \bar{G}^*)$ from Definition 1.2, and suppose that $C^0 \subseteq \omega_1$ is a club, then $(**)_{S \cap C^0 \setminus \{\alpha + \omega : \alpha < \omega_1\}}(G, \bar{G}')$ holds for some \bar{G}' , where the symbol $(**)_{S \cap C^0 \setminus \{\alpha + \omega : \alpha < \omega_1\}}(G, \bar{G}')$ denotes the following assertion:*

$(**)_{S \cap C^0 \setminus \{\alpha + \omega : \alpha < \omega_1\}}(G, \bar{G}')$:

- $(\alpha)^{**}$ G is an abelian group of cardinality \aleph_1 ,
- $(\beta)^{**}$ G is \aleph_1 -free, i.e. every countable subgroup of it is free,
- $(\gamma)^{**}$ the sequence $\bar{G}' = \langle G'_i : i < \omega_1 \rangle$ is a filtration of G for which

$$\{\delta \in \Omega : G/G'_\delta \text{ is not } \aleph_1\text{-free}\} \subseteq S,$$
 and for each $\alpha < \omega_1$ $G_{\alpha+1}/G_\alpha$ is of rank 1,
- $(\delta)^{**}$ G is strongly \aleph_1 -free.

Proof. We can choose the sequence $\langle G_\alpha^0 : \alpha \in \omega_1 \cap (\Omega \cup \{0\}) \rangle$ such that:

- \boxplus_1 (a) $G_0^0 = \{0\}$,
- (b) each G_α^0 is a pure subgroup of G ,
- (c) for each α the quotient group $G/G_{\alpha+\omega}^0$ is \aleph_1 -free (hence strongly \aleph_1 -free), and $G_{\alpha+\omega}^0/G_\alpha^0$ is of rank \aleph_0 ,

(d) the sequence $\langle G_\alpha^0 : \alpha \in \omega_1 \cap \Omega \rangle$ is continuous, increasing, $\bigcup_{\alpha \in \omega_1 \cap \Omega} G_\alpha^0 = G$.

\boxplus_2 So for a suitable club C of ω_1 with $0 \in C$, for each $i \in C$ we have $G_i^0 = G_i^*$, and

$$(1.13) \quad \forall \alpha \in C \setminus S : G/G_\alpha^0 = G/G_\alpha^* \text{ is } \aleph_1\text{-free}$$

(as $\langle G_i^* : i \in \omega_1 \rangle$ was chosen to be a witness of $(*)_S(G, \bar{G}^*)$). Moreover, we can assume that

$$(1.14) \quad \{\alpha + \omega : \alpha < \omega_1\} \cap C = \emptyset,$$

i.e. each element of C is divisible by ω^2 , and

$$C \subseteq C^0.$$

We are going to define the filtration $\bar{G}' = \langle G'_i : i \in \omega_1 \rangle$ witnessing $(**)_{S \cap C}(G, \bar{G}')$ with $G_\alpha^* = G_\alpha^0 = G'_\alpha$ for each $\alpha \in C$. (1.13) and (1.14) imply that this suffices.

For each $\alpha \in C$ let $\{y_{\alpha+n} : n \in \omega\} \subseteq G_{\alpha+\omega}^0$ be a maximal independent set over G_α^0 so that

\boxplus_3 if $G_{\alpha+\omega}^0/G_\alpha^0$ is free, then $\{y_{\alpha+n} + G_\alpha^0 : n \in \omega\}$ is a basis of $G_{\alpha+\omega}^0/G_\alpha^0$.

Now letting $\text{succ}_C(\alpha) = \min(C \setminus (\alpha + \omega))$

\boxplus_4 we choose (for each $\alpha \in C$) $\{y_\gamma : \gamma \in [\alpha + \omega, \text{succ}_C(\alpha))\}$ so that $\{y_\gamma + G_{\alpha+\omega}^0 : \gamma \in [\alpha + \omega, \text{succ}_C(\alpha))\}$ forms a basis of the free group $G_{\text{succ}_C(\alpha)}^0/G_{\alpha+\omega}^0$, moreover (as $\text{succ}_C(\alpha)$ is not of the form $\beta + \omega$ for any β by \boxplus_2),

\boxplus_{5a} for some strictly increasing sequence $\langle \delta_n : n \in \omega \rangle$ in $[\alpha + \omega, \text{succ}_C(\alpha))$ cofinal in $\text{succ}_C(\alpha)$ with $\delta_0 = \alpha$ (equivalently, $\langle \delta_n + \omega : n \in \omega \rangle \in {}^\omega[\alpha + \omega + \omega, \text{succ}_C(\alpha))$ is cofinal) we demand also that

$\{y_\gamma + G_{\delta_n+\omega}^0 : \gamma \in [\delta_n + \omega, \delta_{n+1} + \omega)\}$ is a base of $G_{\delta_{n+1}+\omega}^0/G_{\delta_n+\omega}^0$

Now it is easy to see by induction on $\alpha \in C$, that

\boxplus_5 for each $\alpha \in C$ the set $\{y_j : j < \alpha\}$ is a maximal independent set in G_α^0 , and as G_α^0 is a pure subgroup of G , necessarily $G_\alpha^0 = \text{cl}^{\text{pr}}(\{y_j : j < \alpha\})$, moreover from the choice of $\{y_j : \alpha \leq j < \alpha + \omega\}$, $\alpha \in C$ implies that $G_{\alpha+\omega}^0 = \text{cl}^{\text{pr}}(\{y_j : j < \alpha + \omega\})$,

\boxplus_6 letting $G'_i = \text{cl}^{\text{pr}}(\{y_j : j < i\})$ ($i < \omega_1$) it is easy to see that the sequences G'_i, y_i ($i < \omega_1$) satisfy the following.

(a) $G = \bigcup_{i < \omega_1} G'_i$, and $\langle G'_i : i < \omega \rangle$ is an increasing continuous chain of countable pure subgroups of G such that $G'_0 = \{0\}$, G'_{i+1}/G'_i is of rank 1,

(b) $\{y_i : i < \alpha\}$ is a maximal independent family in G'_α for every ordinal $\alpha < \omega_1$,

(c) for each $i \in C$ we have $G'_i = G_i^0 = G_i^*$, and $G'_{i+\omega} = G_{i+\omega}^0$,

(d) for each limit ordinal $\alpha \notin S \cap C$ the quotient G'_{ω_1}/G'_α is \aleph_1 -free (hence for every $\alpha = \beta + \omega$ as the elements of C are divisible by ω^2 \boxplus_2),

Proof. (\boxplus_6) (a) follows from the definition of the G'_i 's, while (c) follows from \boxplus_5 . Now for (d) fix $\alpha < \omega_1$ limit, $\alpha \notin S \cap C$. If $\alpha \in C$, then $G'_{\alpha+\omega} = G_{\alpha+\omega}^0$ by (c) (as well as $G'_\alpha = G_\alpha^0$), and using $\alpha \notin S$ and (1.13) we are done. If $\alpha \in [\beta + \omega, \text{succ}_C(\beta))$ for some $\beta \in C$ (so $\alpha + \omega \in [\beta + \omega, \text{succ}_C(\beta))$ by (1.14)), then recalling the construction of y_γ 's for $\gamma \in [\beta + \omega, \text{succ}_C(\beta))$ in \boxplus_4 there is some δ_n such that $\delta_n + \omega \geq \alpha$. Now

it clearly suffices to argue that for each countable group H with $G'_\alpha \subseteq H$ both $H/(H \cap G'_{\delta_n+\omega})$ and $(H \cap G'_{\delta_n+\omega})/G'_\delta$ are free, as extending a free group by a free group always yields a free group. As \boxplus_4 implies that

$$G'_\alpha = G'_{\beta+\omega} \oplus (\oplus_{\gamma \in [\beta+\omega, \alpha)} \mathbb{Z}y_\gamma), \text{ and}$$

$$G'_{\delta_n+\omega} = G'_{\beta+\omega} \oplus (\oplus_{\gamma \in [\beta+\omega, \delta_n+\omega)} \mathbb{Z}y_\gamma),$$

clearly $G'_{\delta_n+\omega}/G'_\alpha = \oplus_{\gamma \in [\alpha, \delta_n+\omega)} \mathbb{Z}(y_\gamma + G'_\alpha)$, so $G'_{\delta_n+\omega}/G'_\alpha$ is a free group, but then $(H \cap G'_{\delta_n+\omega})/G'_\alpha$ is free as well, as every subgroup of a free group is free.

Now recall the so called Second Isomorphism Theorem.

Lemma 1.12. *(Second Isomorphism Theorem) If H, N are subgroups of a (not necessarily abelian) group G , and N is normal, then $H/(H \cap N) \simeq HN/N$ through the mapping $h(H \cap N) \mapsto hN$.*

This implies that $H/(H \cap G'_{\delta_n+\omega}) \simeq (H + G'_{\delta_n+\omega})/G'_{\delta_n+\omega}$. But we have already seen that $G'_{\delta_n+\omega} = G_{\delta_n+\omega}^0$ in (c), and $G/G_{\delta_n+\omega}^0$ is \aleph_1 -free by \boxplus_1 / (c), so its countable subgroup $(H + G'_{\delta_n+\omega})/G_{\delta_n+\omega}^0 = (H + G'_{\delta_n+\omega})/G'_{\delta_n+\omega}$ must be free. \square

\square Claim 1.11

Lemma 1.12 immediately implies the following.

- \boxplus_7 If a filtration $\bar{G}'' = \langle G''_\alpha : \alpha < \omega_1 \rangle$ of G satisfies $(*)_S(G, \bar{G}'')$ for some $S \cap \{\alpha + \omega : \alpha < \omega_1\} = \emptyset$, then for each ordinal α , the groups $G''_\alpha \subseteq G''_{\alpha+\omega} \subseteq G$ satisfy: if $K \subseteq G$ is countable, then $K/(G''_{\alpha+\omega} \cap K) \simeq (K + G''_{\alpha+\omega})/G''_{\alpha+\omega}$ is free, in particular, if $G''_{\alpha+\omega} \cap K = G''_\alpha$, then K/G''_α is free.

We now prove the following, slightly more general claim than what is needed for Theorem 1.3 (but essential for Proposition 1.7/(ii)).

Claim 1.13. *If G is an abelian group, $S \subseteq \Omega \setminus \{\alpha + \omega : \alpha < \omega_1\}$ is stationary, and the filtration $\bar{G} = \langle G_\alpha : \alpha < \omega_1 \rangle$ satisfies $(**)_{S(G, \bar{G})}$ (from Claim 1.11), then there are*

- 1) a maximal independent sequence $\langle x_\alpha : \alpha < \omega_1 \rangle \in {}^{\omega_1}G$,
- 2) for each $\delta \in S$ a strictly increasing sequence $\langle \gamma_n^\delta : n \in \omega \rangle$ with limit δ ,
- 3) and a ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$

satisfying

- (\bullet)_i for each $\alpha < \omega_1$: $G_\alpha = \text{cl}^{\text{pr}}(\{x_i : i < \alpha\})$,
- (\bullet)_{ii} whenever G/G_δ is \aleph_1 -free (in particular, if $\delta \notin S$ is limit) then $G_{\delta+\omega} = G_\delta \oplus (\oplus_{n \in \omega} \mathbb{Z}x_{\delta+n})$, and
- (\bullet)_{iii} if $\delta \in S$, $n \in \omega$, then for the set

$$v_n^\delta = \left\{ \begin{array}{l} \alpha < \delta : \text{cl}^{\text{pr}}(G_{\alpha+1} \cup \{x_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{pr}}(G_\alpha \cup \{x_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\}.$$

we have

$$v_n^\delta \subseteq \text{ran}(\eta_\delta).$$

- (\bullet)_{iv} for each $\delta \in S$, $n \leq m \in \omega$ and $\gamma_* \leq \gamma_n$, we have

$$\text{cl}^{\text{pr}}(G_{\gamma_*} \cup \{x_{\delta+i} : i < m\}) = \text{cl}^{\text{pr}}(G_{\gamma_*} \cup \{x_{\delta+i} : i < n\}) \oplus (\oplus_{j=n}^{m-1} \mathbb{Z}x_{\delta+j}).$$

Moreover, if $\bar{\eta}^0 = \langle \eta_\delta^0 : \delta \in S \rangle$ is a ladder system on S , and $(**)_{S, \bar{\eta}^0}^+(G, \bar{G})$ holds, which is the conjunction of clauses $(\alpha)^{**} - (\delta)^{**}$ from $(**)_{S, \bar{\eta}^0}(G, \bar{G})$ (from Claim 1.11) and the additional clause

$(\varepsilon)^{**}$ for each $\delta \in S$ for some sequence $\langle y_{\delta+\ell} : \ell < \omega \rangle \in {}^\omega G_{\delta+\omega}$ which is a maximal independent sequence over G_δ , for each $n \in \omega$

$$\left\{ \begin{array}{l} \alpha < \delta : \quad \text{cl}^{\text{Pr}}(G_{\alpha+1} \cup \{y_{\delta+\ell} : \ell < n\}) \neq \\ \quad \text{cl}^{\text{Pr}}(G_\alpha \cup \{y_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^0),$$

holds, then we can assume that

$$(\bullet)_v \text{ran}(\eta_\delta) \subseteq \text{ran}(\eta_\delta^0).$$

Proof. First we construct the x_β 's. We proceed by induction on $\delta \in \Omega$ (i.e. δ is limit), and choose the $x_{\delta+\ell}$'s ($\ell \in \omega$). Fixing such a δ , if $\delta \notin S$ then using $(**)_{S, \bar{\eta}^0}(G, \bar{G})$ it can be easily seen from clause $(\gamma)^{**}$ that $G_{\delta+\omega}/G_\delta$ is a free abelian group of rank \aleph_0 so we can pick a system $x_\delta, x_{\delta+1}, x_{\delta+2}, \dots$ such that $\{x_{\delta+i} + G_\delta : i \in \omega\}$ is a basis of $G_{\delta+\omega}/G_\delta$ (e.g. apply induction and in each step invoke Fact 1.18 with $H_0 = G_{\delta+n}/G_{\delta+n}$, $H_1 = G_{\delta+n+1}/G_{\delta+n}$ to obtain x_{δ_n}).

So we can assume that $\delta \in S$. Without loss of generality:

$$\blacksquare_1 \quad \delta \in S \Rightarrow G_{\delta+\omega}/G_\delta \text{ is not free,}$$

as we can decrease S (and $(**)_{S, \bar{\eta}^0}(G, \bar{G})$ will still hold).

Definition 1.14. We fix a system $\{z_{\delta+i} : i \in \omega\} \subseteq G_{\delta+\omega}$ being a maximal independent family over G_δ so that if moreover $(\varepsilon)^{**}$ holds, then the system $y_{\delta+\ell} = z_{\delta+\ell}$ ($\ell < \omega$) witnesses $(\varepsilon)^{**}$.

So for every $n < \omega$ we can consider the set

$$(1.15) \quad w_n^\delta = \left\{ \begin{array}{l} \alpha < \delta : \quad \text{cl}^{\text{Pr}}(G_{\alpha+1} \cup \{z_{\delta+\ell} : \ell < n\}) \neq \\ \quad \text{cl}^{\text{Pr}}(G_\alpha \cup \{z_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\}.$$

Subclaim 1.15. If $\{z_{\delta+i} : i \in \omega\} \subseteq G_{\delta+\omega}$ is a maximal independent family over G_δ , and w_n^δ 's are defined as in (1.15), then for each $n < \omega$ and ordinal $\alpha_* < \delta$ the set $w_n^\delta \cap \alpha_*$ is finite.

Proof. Fixing $\alpha_* < \delta$, $n \in \omega$ we define for each $\alpha \leq \alpha_*$ the groups

$$(1.16) \quad \begin{aligned} H_\alpha &= \text{cl}^{\text{Pr}}(G_\alpha \cup \{z_{\delta+\ell} : \ell < n\}), \text{ and} \\ K_\alpha &= H_\alpha / G_\alpha. \end{aligned}$$

Observe that by Lemma 1.12 (the Second Isomorphism Theorem) if $\alpha \leq \beta$ we have that the canonical mapping

$$\varphi_{\beta, \alpha} : K_\alpha = H_\alpha / G_\alpha \rightarrow (H_\alpha + G_\beta) / G_\beta \leq H_\beta / G_\beta = K_\beta$$

(i.e. $\varphi_{\beta, \alpha}(h + G_\alpha) = h + G_\beta$) is an embedding of K_α into K_β , moreover $\varphi_{\gamma, \beta} \circ \varphi_{\beta, \alpha} = \varphi_{\gamma, \alpha}$ for each $\alpha \leq \beta \leq \gamma \leq \alpha_*$, so $\{K_\alpha, \varphi_{\beta, \alpha} : \alpha \leq \beta \leq \alpha_*\}$ forms a direct system, so we can assume that $K_\alpha \leq K_\beta$ if $\alpha \leq \beta$. It is easy to check that

- (\star)₁ $H_\alpha + G_\beta \neq H_\beta$ iff $\varphi_{\beta, \alpha}$ is not surjective, and
- (\star)₂ $\bigcup_{\alpha < \beta} K_\alpha = K_\beta$ if $\beta \leq \alpha_*$ is limit, i.e. the sequence is continuous.

Observe that $\alpha_* + \omega < \delta$ as $\alpha_* < \delta \in S$ (no $\gamma \in S$ is of the form $\xi + \omega$) but note that α_* is not necessarily limit. Recalling that $\{z_{\delta+\ell} : \ell < \omega\}$ is independent over G_δ , thus over $G_{\alpha_*+\omega}$ and G_β for any $\beta \leq \alpha_*$, too, so by the definition of H_β we have $H_\beta \cap G_{\alpha_*+\omega} = G_\beta$. Therefore by (the main clause of) \boxplus_7

(\star)₃ the quotient $K_\beta = H_\beta/G_\beta = H_\beta/(H_\beta \cap G_{\alpha_*+\omega}) \simeq (H_\beta + G_{\alpha_*+\omega})/G_{\alpha_*+\omega}$ is free whenever $\beta \leq \alpha_*$ (so free for each $\beta < \delta$).

In order to finish the proof assume on the contrary that for infinitely many $\alpha < \alpha_*$ we have $K_\alpha \subsetneq K_{\alpha+1}$, let $\beta \leq \alpha_* < \delta$ be a limit point of this set, i.e. β is a limit ordinal, and for cofinally many $\alpha < \beta$, $K_\alpha \subsetneq K_{\alpha+1}$ holds. But K_β is a free group of finite rank (of rank n), so finitely generated, and so for some $\alpha < \beta$ $K_\alpha = K_\beta$ must hold, which is a contradiction. $\square_{\text{Subclaim 1.15}}$

For future reference we remark the following fact that follows from the previous proof.

Lemma 1.16. *If $H \leq H' \leq G$ are pure subgroups in G , $g_0, g_1, \dots, g_{j-1} \notin H'$ are independent over H' , and G/H' is \aleph_1 -free, then $\text{cl}^{\text{pr}}(H \cup \{g_0, g_1, \dots, g_{j-1}\})/H$ is a free group (of finite rank).*

*In particular, if G is an abelian group, $S \subseteq \Omega \setminus \{\alpha + \omega : \alpha < \omega_1\}$ is stationary, and the filtration $\bar{G} = \langle G_\alpha : \alpha < \omega_1 \rangle$ satisfies $(**)_{\bar{S}}(G, \bar{G})$ (thus $G/G_{\alpha+\omega}$ is \aleph_1 -free), then for any $\delta \in S$, $\beta < \delta$, $g_0, g_1, \dots, g_{j-1} \notin G_\delta$ which are independent over G_δ we have*

$$\text{cl}^{\text{pr}}(G_\beta \cup \{g_0, g_1, \dots, g_{j-1}\})/G_\beta \text{ is free (of finite rank).}$$

Proof. Note that $\text{cl}^{\text{pr}}(H \cup \{g_0, g_1, \dots, g_{j-1}\}) \cap H' = H$ (as otherwise for some $h' - h \in H' \setminus H$ we would have $h' - h = k_0 g_0 + \dots + k_{j-1} g_{j-1}$ contradicting that the g_i 's form an independent system over H') and proceed as in the proof above. $\square_{\text{Lemma 1.16}}$

In addition to the claim observe that:

■₂ if $\delta \in S$ and $\alpha_* < \delta$, then

$$\text{cl}^{\text{pr}}(G_{\alpha_*} \cup \{z_{\delta+\ell} : \ell < n\})/\text{cl}(G_{\alpha_*} \cup \{z_{\delta+\ell} : \ell < n\}),$$

is a finite group (and a torsion group, of course), similarly, if $\alpha < \beta < \delta$, and $w_n^\alpha \subsetneq w_n^\beta$, then

$$\text{cl}^{\text{pr}}(G_\beta \cup \{z_{\delta+\ell} : \ell < n\})/(\text{cl}^{\text{pr}}(G_\alpha \cup \{z_{\delta+\ell} : \ell < n\}) + G_\beta)$$

is a finite group.

At this point recall that if we assume $(\varepsilon)^{**}$, then

$$\left\{ \begin{array}{l} \alpha < \delta : \text{cl}^{\text{pr}}(G_{\alpha+1} \cup \{y_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{pr}}(G_\alpha \cup \{y_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^0),$$

and since in that case $z_{\delta+n} = y_{\delta+n}$ ($n \in \omega$) by Definition 1.14, we have

$$\left\{ \begin{array}{l} \alpha < \delta : \text{cl}^{\text{pr}}(G_{\alpha+1} \cup \{z_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{pr}}(G_\alpha \cup \{z_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^0).$$

Note that if we choose $x_{\delta+\ell}$ to be any $x' \in G_{\delta+\ell+1} \setminus G_{\delta+\ell}$, then the sequence will satisfy $(\bullet)_i$. This justifies the following.

Definition 1.17.

by induction on $n \in \omega$ we define $\langle x_{\delta+n} : n < \omega \rangle$ so that for each ℓ , $x_{\delta+\ell} \in G_{\delta+\ell+1} \setminus G_{\delta+\ell}$ holds as follows.

1) Choose first

$$\gamma_0^\delta < \dots \gamma_n^\delta < \gamma_{n+1}^\delta < \dots < \delta = \bigcup_{n < \omega} \gamma_n^\delta,$$

such that if $(\varepsilon)^{**}$ holds, then moreover

$$(1.17) \quad \forall n : w_{n+1}^\delta \setminus \gamma_n^\delta \subseteq \text{ran}(\eta_\delta^0).$$

(recalling how we defined w_n^δ 's in (1.15), i.e.

$$w_n^\delta = \left\{ \alpha < \delta : \begin{array}{l} \text{cl}^{\text{Pr}}(G_{\alpha+1} \cup \{z_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{Pr}}(G_\alpha \cup \{z_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\}.$$

2) Second, by induction on $n < \omega$ choose $x_{\delta+n} \in \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\} \cup \{z_{\delta+n}\}) \setminus G_{\delta+n}$ as above such that:

$$\text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\} \cup \{z_{\delta+n}\}) = \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\}) \oplus \mathbb{Z}x_{\delta+n}.$$

Choosing such $x_{\delta+n}$'s is possible by the following known fact (applying to the finite rank group $H_1 = \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\} \cup \{z_{\delta+n}\})/G_{\gamma_n^\delta}$, which is free, since $\text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\} \cup \{z_{\delta+n}\}) \cap G_{\gamma_n^\delta+\omega} = G_{\gamma_n^\delta}$ (recall \boxplus_7), and letting $H_0 = \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\})/G_{\gamma_n^\delta}$.

Fact 1.18. *Let $H_0 \leq H_1$ be free abelian groups of finite rank, where $\text{cl}_{H_1}^{\text{Pr}}(H_0) = H_0$, i.e. H_0 is a pure subgroup, and for some g we have $\text{cl}^{\text{Pr}}(H_0 \cup \{g\}) = H_1$. Then for some $g^* \in H_1 \setminus H_0$ we have $H_1 = H_0 \oplus \mathbb{Z}g^*$.*

Proof. Since H_0 is a pure subgroup H_1/H_0 is torsion-free. As H_1 is a finite rank free group, it is finitely generated, so is H_1/H_0 , therefore w.l.o.g. H_1/H_0 is a finitely generated group of rank one. This means that H_1/H_0 is isomorphic to a subgroup H^* of the additive group \mathbb{Q} , with the generating set $\{n_1q^*, \dots, n_kq^*\}$ with q^* not necessarily belonging to H^* , but the greatest common divisor of $\{n_1, n_2, \dots, n_k\}$ is 1. Now by a standard elementary argument for some $\ell_1, \ell_2, \dots, \ell_k \in \mathbb{Z}$

$$\ell_1n_1 + \ell_2n_2 + \dots + \ell_kn_k = 1,$$

so $H^* = \mathbb{Z}q^*$, i.e. a free group. Choosing g_* so that $g_* + H_0$ generates H_1/H_0 clearly works. $\square_{\text{Fact 1.18}}$

Fact 1.19. *Let $H_0 \leq H_1$ be free abelian groups of finite rank, where $\text{cl}_{H_1}^{\text{Pr}}(H_0) = H_0$, i.e. H_0 is a pure subgroup, and for some H_0 -independent set $\{g_0, g_1, \dots, g_n\}$ we have $\text{cl}^{\text{Pr}}(H_0 \cup \{g_i : i < n\}) = H_1$. Then for some $g_i^* \in \text{cl}^{\text{Pr}}(H_0 \cup \{g_j : j \leq i\})$ ($i < n$) we have $H_1 = H_0 \oplus (\oplus_{i < n} \mathbb{Z}g_i^*)$.*

Proof. By induction, apply Fact 1.18 at each step for $\text{cl}^{\text{Pr}}(H_0 \cup \{g_j : j \leq i\})$ and $\text{cl}^{\text{Pr}}(H_0 \cup \{g_j : j \leq i+1\})$. $\square_{\text{Fact 1.19}}$

Note the following simple observation which is easy to check:

■₃ By induction on ℓ one can prove that $x_{\delta+\ell} \in \text{cl}^{\text{Pr}}(G_{\gamma_\ell^\delta} \cup \{z_{\delta+j} : j \leq \ell\})$, and so

$$\forall \gamma_* \geq \gamma_\ell^\delta \quad \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{z_{\delta+j} : j \leq \ell\}) = \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+j} : j \leq \ell\}).$$

Definition 1.20. For $\delta \in S$, $\gamma \leq \delta$, $n \in \omega$ we define the group

$$H_{\gamma,n}^\delta = \text{cl}^{\text{Pr}}(G_\gamma \cup \{x_{\delta+\ell} : \ell < n\}).$$

So using this, the choice of the $x_{\delta+n}$'s in 1.17 implies that

■₄ for each $\delta \in S$, $n \in \omega$:

$$H_{\gamma_n^\delta, n+1}^\delta = \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n+1\}) = \text{cl}^{\text{Pr}}(G_{\gamma_n^\delta} \cup \{x_{\delta+\ell} : \ell < n\}) \oplus \mathbb{Z}x_{\delta+n}.$$

Now we are ready to argue $(\bullet)_{iv}$. After proving Subclaim 1.22 we are going to finish the proof of Claim 1.13 by constructing $\eta_\delta : \omega \rightarrow \delta$, verifying $(\bullet)_{iii}$, and possibly $(\bullet)_v$.

We have to remark the following (easy) corollary of ■₄:

Subclaim 1.21. *For each $\delta \in S$, $n \in \omega$, $\gamma_* \leq \gamma_n^\delta$, it is the case that*

$$(H_{\gamma_*, n+1}^\delta =) \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+\ell} : \ell < n+1\}) = \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+\ell} : \ell < n\}) \oplus \mathbb{Z}x_{\delta+n}.$$

Proof. Since $H_{\gamma_*, n}^\delta$ is the pure subgroup generated by $\{x_\alpha : \alpha < \gamma_*\} \cup \{x_{\delta+k} : k < n\}$ and the set $\{x_\alpha : \alpha < \omega_1\}$ is independent we only have to verify that

$$H_{\gamma_*, n+1}^\delta = H_{\gamma_*, n}^\delta + \mathbb{Z}x_{\delta+n}.$$

Now suppose that

$$g \in H_{\gamma_*, n+1}^\delta \setminus (H_{\gamma_*, n}^\delta \oplus \mathbb{Z}x_{\delta+n}).$$

On the one hand $g \in H_{\gamma_*, n+1}^\delta = \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+\ell} : \ell < n+1\})$ implies that

$$(1.18) \quad g = \sum_{k=0}^{s-1} q_k x_{i_k} + \sum_{k=0}^n p_k x_{\delta+k},$$

$\{i_0, i_1, \dots, i_{s-1}\} \subseteq \gamma_*$, $\{q_0, q_1, \dots, p_0, p_1, \dots, p_n\} \subseteq \mathbb{Q}$. On the other hand $g \in H_{\gamma_*, n+1}^\delta \subseteq H_{\gamma_n^\delta, n+1}^\delta = H_{\gamma_n^\delta, n}^\delta \oplus \mathbb{Z}x_{\delta+n}$, so $g = g_0 + zx_{\delta+n}$, where $z \in \mathbb{Z}$, and g_0 is a \mathbb{Q} -linear combination of elements of the independent set $\{x_\alpha : \alpha < \gamma_n^\delta\} \cup \{x_{\delta+k} : k < n\}$. This means that $p_n = z \in \mathbb{Z}$, and $g_0 \in \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+\ell} : \ell < n\})$, as desired.

□_{Subclaim 1.21}

Subclaim 1.22. *For each $\delta \in S$, $n \leq m \in \omega$, $\gamma_* \leq \gamma_n^\delta$, we have*

$$H_{\gamma_*, m}^\delta = H_{\gamma_*, n}^\delta \oplus (\oplus_{j=n}^{m-1} \mathbb{Z}x_{\delta+j}).$$

Proof. Fix $\delta \in S$, $n \in \omega$, we proceed by induction on m . For $m = n$ the statement is void, for $m = n+1$ the statement is exactly Subclaim 1.21. Assume that we already know that $H_{\gamma_*, m}^\delta = H_{\gamma_*, n}^\delta \oplus (\oplus_{j=n}^{m-1} \mathbb{Z}x_{\delta+j})$. Now by Subclaim 1.21 (as $\gamma_n^\delta \leq \gamma_m^\delta$, so $\gamma_* \leq \gamma_m^\delta$)

$$H_{\gamma_*, m+1}^\delta = H_{\gamma_*, m}^\delta \oplus \mathbb{Z}x_{\delta+m} = H_{\gamma_*, n}^\delta \oplus (\oplus_{j=n}^{m-1} \mathbb{Z}x_{\delta+j}) \oplus \mathbb{Z}x_{\delta+m},$$

□_{Subclaim 1.22}

Recall that the v_n^δ 's were defined in $(\bullet)_{iii}$ as

$$v_n^\delta = \left\{ \begin{array}{l} \alpha < \delta : \quad \text{cl}^{\text{Pr}}(G_{\alpha+1} \cup \{x_{\delta+\ell} : \ell < n\}) \neq \\ \quad \text{cl}^{\text{Pr}}(G_\alpha \cup \{x_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\}.$$

and observe that Subclaim 1.22 clearly implies that

■₅ for each $\delta \in S$, $n \leq m < \omega$ we have

$$(\forall \delta \in S, \forall n \leq m \in \omega) \quad v_m^\delta \cap \gamma_n^\delta = v_n^\delta \cap \gamma_n^\delta,$$

since an easy calculation yields that for any fixed $\gamma_* < \gamma_n^\delta$ (so $\gamma_* + 1 \leq \gamma_n^\delta$) the conditions

$$H_{\gamma_*,n}^\delta + G_{\gamma_*+1} \neq H_{\gamma_*+1,n}^\delta,$$

and the assertion

$$H_{\gamma_*,m}^\delta + G_{\gamma_*+1} = H_{\gamma_*,n}^\delta + (\mathbb{Z}x_{\delta+n} + \mathbb{Z}x_{\delta+n+1} + \cdots + \mathbb{Z}x_{\delta m-1}) + G_{\gamma_*+1} \\ \neq$$

$$H_{\gamma_*+1,m}^\delta = H_{\gamma_*+1,n}^\delta + (\mathbb{Z}x_{\delta+n} + \mathbb{Z}x_{\delta+n+1} + \cdots + \mathbb{Z}x_{\delta m-1})$$

are equivalent. (In fact it can be shown that $v_n^\delta \subseteq v_{n+1}^\delta$ holds for each n .)

■₆ Define $v^\delta := \bigcup_{n < \omega} v_n^\delta$.

Subclaim 1.23. *For each $\delta \in S$, the set $v^\delta = \bigcup_{n < \omega} v_n^\delta$ is a subset of δ such that $\alpha < \delta \Rightarrow v^\delta \cap \alpha$ is finite.*

Proof. Fix $\alpha < \delta$, set n to be so that $\alpha < \gamma_n^\delta$, we need to verify that $\bigcup_{m < \omega} v_m^\delta \cap \alpha$ is finite. By ■₅ clearly

$$\bigcup_{m \geq n} v_m^\delta \cap \alpha = v_n^\delta \cap \alpha,$$

whereas for each $v_i^\delta \cap \alpha$ ($i < n$) (and for $i = n$) is finite by applying Subclaim 1.15 with the roles $z_{\delta+i} = x_{\delta+i}$ ($i < \omega$).

Therefore, we obtained that $v^\delta \cap \alpha = \bigcup_{i \leq n} (v_i^\delta \cap \alpha)$, which is finite, as desired.

□_{Subclaim1.23}

Moreover, for future reference we have to remark that

■₇ if $\beta < \beta'$ are two consecutive elements of v^δ , then for each n and $\theta \in (\beta, \beta']$

$$H_{\beta',n}^\delta = \text{cl}^{\text{pr}}(\{x_\alpha : \alpha < \beta'\} \cup \{x_{\delta+j} : j < n\}) = H_{\theta,n}^\delta + G_{\beta'}$$

(by an easy induction argument, using the definition of v_n^δ 1.15, since $v_n^\delta \cap (\beta, \beta') = \emptyset$).

If v^δ is bounded in δ (e.g. by some γ_n^δ) we get a contradiction to “ $G_{\delta+\omega}/G_\delta$ is not free”. (Why would this lead to a contradiction? If $v_\delta \subseteq \gamma_m^\delta$ for some $m < \omega$, then using Subclaim 1.22

$$H_{\gamma_m^\delta, \infty}^\delta = \text{cl}^{\text{pr}}(\{g_\alpha : \alpha < \gamma_m^\delta\} \cup \{x_{\delta+\ell} : \ell < \omega\}) = H_{\gamma_m^\delta}^\delta \oplus (\oplus_{\ell=m}^\infty \mathbb{Z}x_{\delta+\ell}),$$

and so the group $H_{\gamma_m^\delta, \infty}^\delta / G_{\gamma_m^\delta}$ is free ((★)₃), but from our indirect assumption and

■₇ $H_{\gamma_m^\delta, \infty}^\delta + G_\delta = G_{\delta+\omega}$, so by the isomorphism theorem $H_{\gamma_m^\delta, \infty}^\delta / G_{\gamma_m^\delta} \simeq G_{\delta+\omega} / G_\delta$ is free, contradicting ■₁.)

Therefore, by Subclaim 1.23, and $|v^\delta| = \aleph_0$:

■₈ if $\delta \in S$ then v^δ is an unbounded subset of δ of order type ω .

Definition 1.24.

□₁ We let $\eta_\delta : \omega \rightarrow \delta$ enumerate v^δ in increasing order.

Finally we have to argue that $(\varepsilon)^{**}$ implies $(\bullet)_v$, i.e. $v^\delta = \bigcup_{n \in \omega} v_n^\delta \subseteq \text{ran}(\eta_\delta^0)$. So suppose that $n \in \omega$ is the least natural number such that $v_{n+1}^\delta \not\subseteq \text{ran}(\eta_\delta^0)$, pick $\theta \in v_{n+1}^\delta \setminus (v_n^\delta \cup \text{ran}(\eta_\delta^0))$ (note that by $(\bullet)_{iii}$ $v_0^\delta = \emptyset$, thus $n \geq 0$ necessarily). Now \blacksquare_5 and $\theta \in v_{n+1}^\delta \setminus v_n^\delta$ imply that $\gamma_n^\delta \leq \theta$, so we obtain

$$(1.19) \quad \theta \in v_{n+1}^\delta \setminus (\text{ran}(\eta_\delta^0) \cup \gamma_n^\delta).$$

But by \blacksquare_3

$$\forall \gamma_* \geq \gamma_n^\delta : \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{y_{\delta+j} : j \leq n\}) = \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+j} : j \leq n\}),$$

hence recalling the definition of v_{n+1}^δ and w_{n+1}^δ in $(\bullet)_{iii}$, $(\varepsilon)^{**}$ clearly $v_{n+1}^\delta \setminus \gamma_n^\delta = w_{n+1}^\delta \setminus \gamma_n^\delta$. Now by the definition of the γ_m^δ 's (1.17) from 1.17 $w_{n+1}^\delta \setminus \gamma_n^\delta \subseteq \text{ran}(\eta_\delta^0)$, so

$$\theta \in v_{n+1}^\delta \setminus \gamma_n^\delta = w_{n+1}^\delta \setminus \gamma_n^\delta \subseteq \text{ran}(\eta_\delta^0)$$

contradicting (1.19), we are done.

□_{Claim 1.13}

Claim 1.25. *Suppose that G is an abelian group, $S \subseteq \omega_1 \setminus \{\alpha + \omega : \alpha < \omega_1\}$ is stationary, the filtration $\bar{G} = \langle G_i : i \in \omega_1 \rangle$ satisfies $(**)_{\bar{S}}(G, \bar{G})$ (from Claim 1.11), and $\langle x_\alpha : \alpha < \omega_1 \rangle$, $\bar{\eta}^1$, $\langle \gamma_n^\delta : \delta \in S, n \in \omega \rangle$ satisfy 1)-3), $(\bullet)_i - (\bullet)_{iv}$ of Claim 1.13, i.e.*

- (1) $\langle x_\alpha : \alpha < \omega_1 \rangle \in {}^{\omega_1}G$ is a maximal independent sequence,
- (2) for each $\delta \in S$ the sequence $\langle \gamma_n^\delta : n \in \omega \rangle$ is strictly increasing with limit δ ,
- (3) and $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ is a ladder system,

for which

- $(\bullet)_i$ for each $\alpha < \omega_1$: $G_\alpha = \text{cl}^{\text{Pr}}(\{x_i : i < \alpha\})$,
- $(\bullet)_{ii}$ whenever G/G_δ is \aleph_1 -free (in particular, if $\delta \notin S$ is limit) then $G_{\delta+\omega} = G_\delta \oplus (\oplus_{n \in \omega} \mathbb{Z}x_{\delta+n})$, and
- $(\bullet)_{iii}$ if $\delta \in S$, $n \in \omega$, then for the set

$$v_n^\delta = \left\{ \alpha < \delta : \begin{array}{l} \text{cl}^{\text{Pr}}(G_{\alpha+1} \cup \{x_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{Pr}}(G_\alpha \cup \{x_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\}.$$

we have

$$v_n^\delta \subseteq \text{ran}(\eta_\delta).$$

- $(\bullet)_{iv}$ for each $\delta \in S$, $n \leq m \in \omega$ and $\gamma_* \leq \gamma_n$, we have

$$\text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+i} : i < m\}) = \text{cl}^{\text{Pr}}(G_{\gamma_*} \cup \{x_{\delta+i} : i < n\}) \oplus (\oplus_{j=n}^{m-1} \mathbb{Z}x_{\delta+j}).$$

Then

(every ladder system $\bar{\eta}'$ very similar to $\bar{\eta}^1$ has \aleph_0 -uniformization) \Rightarrow

$\Rightarrow (G \text{ is a } W_\omega \text{ group}).$

Proof.

Definition 1.26. Fix $\delta \in S$. Using $\{x_\alpha : \alpha < \omega_1\}$, γ_n^δ ($n \in \omega$) and $\bar{\eta}^1$

- $(\tau)_1$ for each n we can find a $Y_{\delta,n}$ such that:

- (α) $Y_{\delta,n} \subseteq \{x_i : i \leq \eta_\delta^1(n)\}$ is finite, and
- (β) letting m be such that $\eta_\delta^1(n) \in [\gamma_{m-1}^\delta, \gamma_m^\delta)$ we demand

$$\text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < m\}) = G_{\eta_\delta^1(n)+1} + \text{cl}^{\text{Pr}}(\{x_{\delta+\ell} : \ell < m\} \cup Y_{\delta,n}),$$

(in other words, one can have a system

$$g_0^{n,m}, g_1^{n,m}, \dots, g_{m-1}^{n,m} \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < m\}),$$

each $g_j^{n,m}$ is a \mathbb{Q} -linear combination of elements of $\{x_{\delta+\ell} : \ell < m\} \cup Y_{\delta,n}$ so that $g_0^{n,m} + G_{\eta_\delta^1(n)+1}, g_1^{n,m} + G_{\eta_\delta^1(n)+1}, \dots, g_{m-1}^{n,m} + G_{\eta_\delta^1(n)+1}$ freely generates $H_{\eta_\delta^1(n)+1,m}^\delta / G_{\eta_\delta^1(n)+1}$, possible by e.g. using Lemma 1.16, and we can assume that $x_{\eta_\delta^1(n)} \in Y_{\delta,n}$ holds,

(T)₂ define $u_{\delta,n} \subseteq \eta_\delta^1(n)$ by the equality $Y_{\delta,n} = \{x_\alpha : \alpha \in u_{\delta,n}\} \cup \{x_{\eta_\delta^1(n)}\}$.

We claim that

\blacktriangle_1 if $\delta \in S$, $n \in \omega$ is fixed, and m is chosen such that $\eta_\delta^1(n) \in [\gamma_{m-1}^\delta, \gamma_m^\delta)$, then it follows from the demand above on $Y_{\delta,n}$ that for any $k \leq m$

$$\text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < k\}) = G_{\eta_\delta^1(n)+1} + \text{cl}^{\text{Pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n}),$$

and so we can write $g_0^{n,k}, g_1^{n,k}, \dots, g_{k-1}^{n,k}$ as a \mathbb{Q} -linear combination of elements of $\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n}$ so that $g_0^{n,k} + G_{\eta_\delta^1(n)+1}, g_1^{n,k} + G_{\eta_\delta^1(n)+1}, \dots, g_{k-1}^{n,k} + G_{\eta_\delta^1(n)+1}$ freely generates $H_{\eta_\delta^1(n)+1,k}^\delta / G_{\eta_\delta^1(n)+1}$.

(Why is \blacktriangle_1 true? If $g \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < k\})$, then $g \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < m\})$ obviously, and so

$$g = g' + \sum_{j < m} k_j \cdot g_j^{n,m}$$

with k_j 's being integers, $g_j^{n,m}$'s are from 1.26, $g' \in G_{\eta_\delta^1(n)+1}$. But since each $g_j^{n,m} \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < m\})$, and the set $\{x_{\delta+n} : n < m\}$ is independent over G_δ , hence over $G_{\eta_\delta^1(n)+1}$ too, so

$$\sum_{j < m} k_j \cdot g_j^{n,m} \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < k\}) = \text{cl}^{\text{Pr}}(\{x_\alpha : \alpha < \eta_\delta^1(n)+1\} \cup \{x_{\delta+\ell} : \ell < k\}).$$

But we know that

$$\sum_{j < m} k_j \cdot g_j^{n,m} \in \text{cl}^{\text{Pr}}(Y_n \cup \{x_{\delta+\ell} : \ell < m\}),$$

where $Y_n \subseteq \{x_\alpha : \alpha < \eta_\delta^1(n) + 1\}$. So recalling that $\{x_\beta : \beta < \omega_1\}$ is an independent system, we get that

$$\begin{aligned} \sum_{j < m} k_j \cdot g_j^{n,m} &\in \text{cl}^{\text{Pr}}(Y_n \cup \{x_{\delta+\ell} : \ell < m\}) \cap \text{cl}^{\text{Pr}}(\{x_\alpha : \alpha < \eta_\delta^1(n)+1\} \cup \{x_{\delta+\ell} : \ell < k\}) = \\ &= \text{cl}^{\text{Pr}}(Y_n \cup \{x_{\delta+\ell} : \ell < k\}), \end{aligned}$$

so $g \in \text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1}) + \text{cl}^{\text{Pr}}(Y_n \cup \{x_{\delta+\ell} : \ell < k\})$, indeed, as desired.)

Moreover, (using that $\eta_\delta^1(n) \in [\gamma_{m-1}^\delta, \gamma_m^\delta)$ implies $\eta_\delta^1(n) + 1 \leq \gamma_m^\delta$), clause $(\bullet)_{iv}$ in the premises says that if $k \geq m$ then $H_{\eta_\delta^1(n)+1,k}^\delta = H_{\eta_\delta^1(n)+1,m}^\delta \oplus (\oplus_{j=m}^{k-1} \mathbb{Z}x_{\delta+j})$, so

\blacktriangle_2 if $\delta \in S$, $n \in \omega$ are fixed, then for every $k \geq m$ (hence for every $k \in \omega$)

$$\text{cl}^{\text{Pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < k\}) = G_{\eta_\delta^1(n)+1} + \text{cl}^{\text{Pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n}).$$

Now we recall that if k is fixed, then $(\eta_\delta^1(n), \eta_\delta^1(n+1)) \cap v_k^\delta = \emptyset$ by condition $(\bullet)_{iii}$ we obtain:

$$\text{cl}^{\text{pr}}(G_{\eta_\delta^1(n+1)} \cup \{x_{\delta+\ell} : \ell < k\}) = \text{cl}^{\text{pr}}(G_{\eta_\delta^1(n)+1} \cup \{x_{\delta+\ell} : \ell < k\}) + G_{\eta_\delta^1(n+1)}.$$

this means that in \blacktriangle_2 we could as well write $G_{\eta_\delta^1(n+1)}$ in place of $G_{\eta_\delta^1(n)+1}$:

\blacktriangle_3 $(\forall \delta \in S, \forall n \in \omega)$ for every $k \in \omega$

$$\text{cl}^{\text{pr}}(G_{\eta_\delta^1(n+1)} \cup \{x_{\delta+\ell} : \ell < k\}) = G_{\eta_\delta^1(n+1)} + \text{cl}^{\text{pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n}).$$

Similarly,

\blacktriangle_4 for each $V \subseteq \eta_\delta^1(n+1)$ with $u_{\delta,n} \subseteq V$, letting $G_V := \text{cl}^{\text{pr}}(\{x_\alpha : \alpha \in V\})$, and $G_V^{+\delta,k} := \text{cl}^{\text{pr}}(\{x_\alpha : \alpha \in V \cup [\delta, \delta+k]\})$ an easy calculation yields that

$$G_V^{+\delta,k} = G_V + \text{cl}^{\text{pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n}),$$

equivalently, there exists some independent set $\{g_0^{V,k}, g_1^{V,k}, \dots, g_{k-1}^{V,k}\} \in \text{cl}^{\text{pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n})$, such that

$$G_V^{+\delta,k}/G_V \simeq \oplus_{j < k} \mathbb{Z}(g_j^{V,k} + G_V),$$

which equivalence is justified by the following: first let $H_1 = \text{cl}^{\text{pr}}(\{x_{\delta+\ell} : \ell < k\} \cup Y_{\delta,n})$, $H_0 = \text{cl}^{\text{pr}}(Y_{\delta,n})$, and apply Fact 1.19, so $H_1 = H_0 \oplus (\oplus_{i < k} \mathbb{Z}g_i)$ (where for each $i < k$ $g_i \in \text{cl}^{\text{pr}}(Y_{\delta,n} \cup \{x_{\delta+j} : j \leq i\})$ holds). Now observe, that $H_1 \cap G_V = H_0$, so by the second isomorphism theorem the mapping $g + H_0 \mapsto g + G_V$ is an isomorphism from $H_1/(H_1 \cap G_V)$ onto $H_1 + G_V/G_V$, as desired. But note that by \blacktriangle_2 $H_1 + G_V$ must be $G_V^{+\delta,k}$, otherwise the isomorphism between

$$H_1 + G_V/G_V$$

and

$$H_1 + G_V + G_{\eta_\delta^1(n+1)}/G_{\eta_\delta^1(n+1)} = \text{cl}^{\text{pr}}(G_{\eta_\delta^1(n+1)} \cup \{x_{\delta+\ell} : \ell < k\})/G_{\eta_\delta^1(n+1)}$$

(viewing both groups as subgroups of $\oplus_{j < k} \mathbb{Q}x_{\delta_j}$) cannot be surjective.

Now we can turn to proving that G is W_ω -group. In order to do this we fix

\blacktriangle_5 the groups $K \leq H$, and a homomorphism φ from H onto G , where K is the kernel of φ , and K is a countable free group, so w.l.o.g. $K = \mathbb{Z}_\omega = \oplus_{j < \omega} \mathbb{Z}$.

We reduce the task of finding a suitable homomorphism to a more general combinatorial problem. This will need some preparations: from Definition 1.27 to 1.31, in this section we prepare the ground for stating Theorem 2.1, as Section 2 is devoted exclusively to the proof of that theorem. At the end of the section Definition 1.33 and Lemma 1.34 put the present problem (i.e. that of Proposition 1.7 (ii)) in the frame of Theorem 2.1, and justify that.

Definition 1.27. We say that \mathfrak{p}^0 is a S -uniformization frame when $\mathfrak{p}^0 = (S, \bar{\eta}, \bar{u}) = (S^{\mathfrak{p}^0}, \bar{\eta}^{\mathfrak{p}^0}, \bar{u}^{\mathfrak{p}^0})$ satisfies

- (a) $\langle \eta_\delta : \delta \in S \rangle$ is an S -ladder system, i.e., $S \subseteq \omega_1$ is a stationary set of limit ordinals and for each $\delta \in S$, η_δ is an ω -sequence of ordinals $< \delta$, strictly increasing, with limit δ ;
- (b) $\bar{u} = \langle u_{\delta,n} : \delta \in S, n \in \omega \rangle$, with $u_{\delta,n}$ being a finite subset of $\eta_\delta(n)$,

Definition 1.28. Let \mathfrak{p}^0 be a S -uniformization frame.

1) We define the sets

$$b_{n,k}^\delta = (b_{n,k}^\delta)^{\mathfrak{p}^0} = \bigcup_{j \leq n} (u_{\delta,j} \cup \{\eta_\delta(j)\}) \cup [\delta, \delta + k] \quad (\delta \in S, n, k \in \omega),$$

we call $b_{n,k}^\delta$ a \mathfrak{p}^0 -basic set.

2) We let

$$\mathcal{B} = \mathcal{B}^{\mathfrak{p}^0} = \{b_{n,k}^\delta : \delta \in S, n, k \in \omega\},$$

and call \mathcal{B} the base of \mathfrak{p}^0 .

3) A subset x of ω_1 is called \mathfrak{p}^0 -closed (or closed if \mathfrak{p}^0 is clear from the context) when :

$$(\alpha) \quad \epsilon + 1 \in x \Rightarrow \epsilon \in x,$$

$$(\beta) \quad \text{if } \delta \in S \cap x, \text{ then for some } 0 < N = N(\delta) \leq \omega \text{ for every } n < \omega:$$

$$(i) \quad \eta_\delta(n) \in x \text{ iff } n < N,$$

$$(ii) \quad [\eta_\delta(n), \eta_\delta(n+1)) \cap x \neq \emptyset \rightarrow \eta_\delta(n) \in x,$$

$$(iii) \quad \text{if } \eta_\delta(n) \in x \text{ then } u_{\delta,n} \subseteq x.$$

Definition 1.29. We say $\mathfrak{p} = (S, \bar{\eta}, \bar{u}, \Psi) = (S^{\mathfrak{p}}, \bar{\eta}^{\mathfrak{p}}, \bar{u}^{\mathfrak{p}}, \Psi^{\mathfrak{p}})$ is a special S -uniformization problem when $\mathfrak{p}^0 = (S^{\mathfrak{p}}, \bar{\eta}^{\mathfrak{p}}, \bar{u}^{\mathfrak{p}})$ is an S -uniformization frame, and there exists a countable set C such that

- \square_a Ψ is a function with domain $\mathcal{B}^{\mathfrak{p}^0}$, and for each $b \in \mathcal{B}^{\mathfrak{p}^0}$ $\Psi(b)$ is a countable non-empty family of functions with domain b , range $\subseteq C$ satisfying that for any n there is a countable $\Upsilon_n = \Upsilon_n^{\mathfrak{p}}$ such that if $b \in \mathcal{B}^{\mathfrak{p}^0}$, $|b| = n$ then $\{f \circ \text{OP}_{b,n} : f \in \Psi(b)\} \in \Upsilon_n$.
- \square_b If we define the function Ψ^+ with domain $[\omega_1]^{<\aleph_0} = \{u : u \subseteq \omega_1 \text{ finite}\}$ by $\Psi^+(u) = \{f \in {}^u C : \forall b \subseteq u (b \in \mathcal{B}^{\mathfrak{p}^0} \Rightarrow f \restriction b \in \Psi(b))\}$, then for each $u \in [\omega_1]^{<\aleph_0}$ $\Psi^+(u)$ is a non-empty family (which must be countable, of course).
- \square_c If $b, b' \in \mathcal{B}^{\mathfrak{p}^0}$, $b \subseteq b'$ and $f \in \Psi(b')$ then $f \restriction b \in \Psi(b)$.
- \square_d If $u \subseteq v \in [\omega_1]^{<\aleph_0}$, $f \in \Psi^+(u)$ and u is finite and \mathfrak{p}^0 -closed then $(\exists f')[f \subseteq f' \in \Psi^+(v)]$.

Claim 1.30. If $\mathfrak{p} = (S, \bar{\eta}, \bar{u}, \Psi) = (S^{\mathfrak{p}}, \bar{\eta}^{\mathfrak{p}}, \bar{u}^{\mathfrak{p}}, \Psi^{\mathfrak{p}})$ is a special S -uniformization problem, then for each $n < \omega$ there exists a countable set Υ_n^+ such that:

$$\forall x \in [\omega_1]^n : \{f \circ \text{OP}_{x,n} : f \in \Psi^+(x)\} \in \Upsilon_n^+.$$

Proof. If $x \in [\omega_1]^n$, then clearly there are only finitely many $b \in \mathcal{B}^{\mathfrak{p}^0}$ with $b \subseteq x$. Now by the definition of Ψ^+ in \square_b the set

$$\{f \circ \text{OP}_{x,n} : f \in \Psi^+(x)\}$$

can be coded by $\langle \text{OP}_{x,n}^{-1}(b) : b \in \mathcal{B}^{\mathfrak{p}^0}, b \subseteq x \rangle$, i.e. the relative position of the finitely many basic sets that are included, and

$$\langle \{f \circ \text{OP}_{b,n} : f \in \Psi(b)\} : b \subseteq x, b \in \mathcal{B}^{\mathfrak{p}^0} \rangle.$$

But note that this latter sequence has its entries from $\bigcup_{k \in \omega} \Upsilon_k$ by \square_a , so there are only countably many ways to choose the parameters (which will define the set in question), we are done.

□Claim1.30

Definition 1.31.

- 1) We say f is a solution of \mathfrak{p} , a special S -uniformization problem when f is a function with domain ω_1 such that for every $u \in [\omega_1]^{<\aleph_0}$ we have $f \restriction u \in \Psi^+(u)$ (equivalently, for each $b \in \mathcal{B}$, $b \subseteq u$ $f \restriction b \in \Psi(b)$).
- 2) We say that f is a partial solution of \mathfrak{p} if f is a function with domain $\subseteq \omega_1$ such that for every finite $u \subseteq \text{Dom}(f)$ we have $f \restriction u \in \Psi^+(u)$.
- 3) We say \mathfrak{p} is simple if $\alpha < \omega_1 \Rightarrow \alpha + \omega \notin S^{\mathfrak{p}}$.

Convention 1.32. From now on if \mathfrak{p} is a special S -uniformization problem, then \mathfrak{p} -closed will mean \mathfrak{p}^0 -closed.

Here we state the main theorem of Section 2.

Theorem 2.6. *Let S be a stationary set of limit ordinals $< \omega_1$ [which is simple, i.e., $(\forall \alpha < \omega_1)(\alpha + \omega \notin S)$] and $\bar{\eta}^*$ be an S -ladder system. Then $(\mathbf{D})_{S, \bar{\eta}^*} \Leftrightarrow (\mathbf{E})_{S, \bar{\eta}^*}$ where*

- $(\mathbf{D})_{S, \bar{\eta}^*}$ *for any ladder system $\bar{\eta}$ on S if $\bar{\eta}$ is very similar to $\bar{\eta}^*$, then it has \aleph_0 -uniformization*
- $(\mathbf{E})_{S, \bar{\eta}^*}$ *for every ladder system $\bar{\eta}$ very similar to $\bar{\eta}^*$, for every simple special S -uniformization problem \mathfrak{p} with $\bar{\eta}^{\mathfrak{p}} = \bar{\eta}$, \mathfrak{p} has a solution.*

Now we only have to translate our lifting problem to a special S -uniformization problem to finish the proof of Theorem 1.3.

Recalling \blacktriangle_5 fix a map $F : \{x_\alpha : \alpha < \omega_1\} \rightarrow H$ such that $\varphi \circ F(x_\alpha) = x_\alpha$, (note that F does not necessarily induce a homomorphism from $G = \text{cl}^{\text{pr}}(\{x_\alpha : \alpha < \omega_1\})$, only from the generated free group $\text{cl}(\{x_\alpha : \alpha < \omega_1\})$). Note that in the case that F induces a homomorphism $\tilde{F} : \text{cl}^{\text{pr}}(\{x_\alpha : \alpha < \omega_1\}) \rightarrow H$, then clearly $\varphi \circ \tilde{F} = \text{id}_G$. Now observe that

\boxplus_8 the stationary set S together with $\bar{\eta}^1$, \bar{u} from Definition 1.26 form an S -uniformization frame \mathfrak{p}^0 .

So if we can form an S -uniformization problem, each solution of which yields a suitable homomorphism from the entire G , then we will be done.

Definition 1.33. Fix $\delta \in S$, and $b \in \mathcal{B}^{\mathfrak{p}}$ (from Definition 1.28, so $b = \bigcup_{j \leq n} (u_{\delta, j} \cup \{\eta_\delta^1(j)\}) \cup [\delta, \delta + k]$ for some n, k).

- (\odot) Define $f \in \Psi(b)$, iff $f : b \rightarrow \mathbb{Z}_\omega = \bigoplus_{k \in \omega} \mathbb{Z}$, and the function $F^{+f} : \{x_\alpha : \alpha \in b\} \rightarrow H$, $F^{+f}(x_\alpha) = F(x_\alpha) + f(\alpha)$ extends to a homomorphism $F^{+f} : \text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in b\}) \rightarrow H$.

We encapsulate the missing claims in the following assertion.

Lemma 1.34.

- $(\odot)_1$ Ψ (from Definition 1.33) together with the S -uniformization frame \mathfrak{p}^0 (from Definition 1.26) satisfies the demands in Definition 1.29, moreover,
- $(\odot)_2$ for every \mathfrak{p} -closed set U , and partial solution f of \mathfrak{p} with $\text{dom}(f) = U$ the defined map

$$\begin{aligned} F^{+f} : \{x_\alpha : \alpha \in U\} &\rightarrow H, \\ x_\alpha &\mapsto F(x_\alpha) + f(\alpha) \end{aligned}$$

induces a homomorphism $\widetilde{F^{+f}} : \text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in U\}) \rightarrow H$.

Proof. First recall that every countable subgroup of G is free, and every free group is W_ω -group (just pick a set freely generating the group and pick an arbitrary preimage for each generator), in particular:

- \Box_1 for every pure subgroup $G_* = \text{cl}_G^{\text{pr}}(\{g_i : i < N \leq \omega\}) \leq G$ of either finite or countable infinite rank there exists a homomorphism $\psi : G_* \rightarrow H$ with $\varphi \circ \psi = \text{id}_{G_*}$.

Using Fact 1.19 for every pure subgroup $G_* = \text{cl}_G^{\text{pr}}(\{g_i : i < N \leq \omega\}) \leq G$ of either finite or countable infinite rank, and pure subgroup G_{**} of G_* of finite rank G_{**} is a direct summand of G_* (i.e. $G_* = G_{**} \oplus G'$), (and of course G' is free) so

- \Box_2 for every pure subgroup $G_* = \text{cl}_G^{\text{pr}}(\{g_i : i < N \leq \omega\}) \leq G$ of either finite or countable infinite rank, pure subgroup G_{**} of G_* of finite rank, and homomorphism $\psi^0 : G_{**} \rightarrow H$ with $\varphi \circ \psi^0 = \text{id}_{G_{**}}$ there exists a homomorphism $\psi : G_* \rightarrow H$ extending ψ^0 with $\varphi \circ \psi = \text{id}_{G_*}$.

For the requirement \Box_a it is clear that for each finite $b \in \mathcal{B}$ the collection $\Psi(b)$ of mappings is non-empty. We claim that for any fixed $b \in \mathcal{B}$, $|b| = n$,

- \Box_3 there exists $\bar{k} = \langle k_j^\alpha : j < n, \alpha \in b \rangle$ with $(\forall j < n, \forall \alpha \in b) k_j^\alpha \in \mathbb{Z}$, such that for each $f_0 \in \Psi(b)$ we have

$$\Psi(b) = \{f_{\bar{k}, \bar{d}} : \bar{d} \in {}^n \mathbb{Z}_\omega\}$$

$$(\text{where } f_{\bar{k}, \bar{d}}(\alpha) = f_0(\alpha) + \sum_{\ell=0}^{n-1} k_\ell^\alpha \cdot d_\ell),$$

which would clearly complete the proof of \Box_a . So pick a basis $\{e_0, e_1, \dots, e_{n-1}\} \subseteq \text{cl}_G^{\text{pr}}(\{x_i : i \in b\})$, write $x_\alpha = \sum_{j=0}^{n-1} k_j^\alpha e_j$, and fix $f_0 \in \Psi(b)$. As for each $f : b \rightarrow \mathbb{Z}_\omega$ there is at most only one way $F^{+f} : \{x_\alpha : \alpha \in b\} \rightarrow H$ can extend to a homomorphism $\widetilde{F^{+f}}$ from $\text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in b\})$ there is obviously a bijection between the sets $\{\widetilde{F^{+f}} - \widetilde{F^{+f_0}} : f \in \Psi(b)\}$ and ${}^n \mathbb{Z}_\omega$ (i.e. the attained value on the base $\{e_0, e_1, \dots, e_{n-1}\}$). Changing back from the base $\{e_0, e_1, \dots, e_{n-1}\}$ to $\{x_\alpha : \alpha \in b\}$ it is easy to see \Box_3 . For \Box_b note that

- \Box_4 whenever $f : u \rightarrow \mathbb{Z}_\omega$ is such that the mapping $F^{+f} : \{x_\alpha : \alpha \in u\}$ extends to a homomorphism from $\text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in u\})$ it is obviously a homomorphism from $\text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in b\})$ for any \mathfrak{p} -closed $b \subseteq u$,

and similarly \Box_c is as well obvious. For \Box_d recall \Box_1 and \Box_2 so it is enough to prove that if U is \mathfrak{p} -closed, $f : U \rightarrow \mathbb{Z}_\omega$ satisfies that for each $b \in \mathcal{B}$ $f \upharpoonright b \in \Psi(b)$, then F^{+f} extends to a homomorphism $\widetilde{F^{+f}} : \text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in U\}) \rightarrow H$. (And so if U is finite, $U \subseteq V$, then \Box_2 means that we can further extend the homomorphism, so recalling Definition 1.33 and \Box_b the corresponding mapping from V will belong to $\Psi^+(V)$.)

So fix U and f with $[(b \subseteq U) \rightarrow (f \upharpoonright b \in \Psi(b))]$. Let $\delta + k < \omega_1$ (with δ limit) be minimal such that F^{+f} does not extend to $\text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in U \cap (\delta + k)\})$. Since this extension (if exists) is unique it is easy to see that only successor steps may be problematic, so $\delta + k - 1 \in U$. Moreover, $\delta \in S$ necessarily, since the x_α 's were given by Claim 1.13, and so otherwise $G_{\delta+k} = G_\delta \oplus (\oplus_{i < k} \mathbb{Z}x_{\delta+i})$, so if $F^{+f} \upharpoonright \{x_\alpha : \alpha \in U \cap \delta\}$ extends to $G_{U \cap \delta}$ (and $F^{+f} \upharpoonright \{x_\alpha : \alpha \in [\delta, \delta + k)\}$ clearly extends to the k -rank abelian free group), then it extends to their direct sums as well.

Now there exists $u = \{\alpha_0, \alpha_1, \dots, \alpha_{m-1}\} \subseteq U \cap (\delta + \omega)$ finite, and $k_0, k_1, \dots, k_{m-1} \in \mathbb{Z}$, $d \in \mathbb{Z} \setminus \{0\}$ such that

$$g^* = \frac{k_0 x_{\alpha_0} + k_1 x_{\alpha_1} + \dots + k_{m-1} x_{\alpha_{m-1}}}{d} \in \text{cl}^{\text{pr}}(\{x_\alpha : \alpha \in U \cap \delta + \omega\}) \leq G,$$

but

$$\sum_{j=0}^{m-1} k_j (F(x_{\alpha_j}) + f(\alpha_j)) \in H \text{ is not divisible by } d.$$

Now w.l.o.g. (since the \mathfrak{p} -closedness of U implies that $(\forall \beta < \omega_1) : (\beta + 1 \in U \rightarrow \beta \in U)$):

\Box_5 for some $k \leq m$ we have $\alpha_{m-k+j} = \delta + j$ for $j = 0, 1, \dots, k-1$, whereas $\{\alpha_0, \alpha_1, \dots, \alpha_{m-k-1}\} \subseteq \delta$.

By the minimality of δ the function F^{+f} extends to a homomorphism from $\text{cl}_G^{\text{pr}}(\{x_\alpha : \alpha \in U \cap \delta\})$. As U is \mathfrak{p} -closed, (recalling Definition 1.28/3)) either $\text{ran}(\eta_\delta^1) \subseteq U$, and then each $u_{\delta,n} \subseteq U$, or there exists a minimal n such that $U \cap \delta \subseteq \eta_\delta^1(n)$. No matter which case holds, we have

\Box_6 $\forall j \in \omega : (\eta_\delta^1(j) \in U) \rightarrow (u_{\delta,j} \subseteq U)$,

therefore as $u_{\delta,j} \subseteq \eta_\delta^1(j)$'s are finite $((\mathfrak{T})_1, (\mathfrak{T})_2$ in Definition 1.26) w.l.o.g.

\Box_7 if $\eta_\delta^1(j) \in \{\alpha_0, \alpha_1, \dots, \alpha_{m-k-1}\}$, then $\bigcup_{\ell \leq j} (u_{\delta,\ell} \cup \{\eta_\delta^1(\ell)\}) \subseteq \{\alpha_0, \alpha_1, \dots, \alpha_{m-k-1}\}$.

Let $j_0 \in \omega$ be such that

\Box_8 $\bigcup_{\ell \leq j_0} (u_{\delta,\ell} \cup \{\eta_\delta^1(\ell)\}) \subseteq \{\alpha_0, \alpha_1, \dots, \alpha_{m-k-1}\} \subseteq \eta_\delta^1(j_0 + 1)$.

Therefore by $(\mathfrak{T})_2$ $Y_{\delta,j_0} \subseteq \{x_{\alpha_i} : i \in m\}$, and by \blacktriangle_3 we obtain $g^* = g_0 + g_1$, where $g_1 \in \text{cl}^{\text{pr}}(Y_{\delta,j_0} \cup \{x_\alpha : \alpha \in [\delta, \delta + k]\})$, and $g_0 \in G_{\eta_\delta^1(j_0+1)}$. But as $g^* \in \text{cl}^{\text{pr}}(\{x_\alpha : \alpha \in (U \cap \eta_\delta^1(j_0 + 1)) \cup [\delta, \delta + k]\})$, $Y_{\delta,j_0} \subseteq \{x_\alpha : \alpha \in U \cap \eta_\delta^1(j_0 + 1)\}$, obviously $g_1 \in \text{cl}^{\text{pr}}(\{x_\alpha : \alpha \in U \cap \eta_\delta^1(j_0 + 1)\})$. But $F^{+f} \upharpoonright \{x_\alpha : \alpha \in U \cap \delta\}$ and $F^{+f} \upharpoonright (Y_{\delta,j_0} \cup \{x_{\delta+i} : i < k\}) = F^{+f} \upharpoonright \{x_\alpha : \alpha \in b_{j_0,k}^\delta\}$ both extend to the corresponding generated pure subgroups, so to their sums, so to g^* as well, which is a contradiction.

$\square_{\text{Lemma 1.34}}$

$\square_{\text{Claim 1.25}}$

Having finished the proof of the last ingredient, we can put all the necessary claims together to argue (ii) of Proposition 1.7.

Proof. (Clause (ii) of Proposition 1.7) Suppose that $S \subseteq \Omega$ is stationary, $\bar{\eta}$ is a ladder system for which $\text{ran}(\eta_\delta) \subseteq \Omega$ for each $\delta \in S$ (so necessarily S is simple, $\alpha + \omega \notin S$ for any α). Let G be a group of size \aleph_1 , $\bar{G}' = \langle G'_\alpha : \alpha < \omega_1 \rangle$ a filtration of G , and the ladder system $\bar{\eta}^*$ are such that $\bar{\eta}^*$ is very similar to $\bar{\eta}$, $(*)_{S, \bar{\eta}^*}^+(G, \bar{G}')$ holds from the proposition, i.e. clauses $(\alpha)^+(\varepsilon)^+$ are satisfied. This implies that $(**)_{S, \bar{\eta}^*}(G, \bar{G}')$ from Lemma 1.11 holds, and we can apply Claim 1.13. Let $\langle x_\alpha : \alpha < \omega_1 \rangle$, and the ladder system $\bar{\eta}^1$ (and γ_n^δ 's) be given by the claim. Moreover, since we are in the proof of Clause (ii) of Proposition 1.7, we assume $(\varepsilon)^+$:

$(\varepsilon)^+$ for each $\delta \in S$ for some sequence $\langle y_{\delta+\ell} : \ell < \omega \rangle \in {}^\omega G'_{\delta+\omega}$ which is a maximal independent sequence over G'_δ , we have for each $n \in \omega$:

$$\left\{ \begin{array}{l} \alpha < \delta : \text{cl}^{\text{pr}}(G'_{\alpha+1} \cup \{y_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{pr}}(G'_\alpha \cup \{y_{\delta+\ell} : \ell < n\}) + G'_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^*).$$

This means that clause $(\varepsilon)^{**}$ stated in Claim 1.13 with the role $\bar{\eta}^0 := \bar{\eta}^*$ i.e.

$(\varepsilon)^{**}$ for each $\delta \in S$ for some sequence $\langle y_{\delta+\ell} : \ell < \omega \rangle \in {}^\omega G_{\delta+\omega}$ which is a maximal independent sequence over G_δ , for each $n \in \omega$

$$\left\{ \begin{array}{l} \alpha < \delta : \text{cl}^{\text{pr}}(G_{\alpha+1} \cup \{y_{\delta+\ell} : \ell < n\}) \neq \\ \text{cl}^{\text{pr}}(G_\alpha \cup \{y_{\delta+\ell} : \ell < n\}) + G_{\alpha+1} \end{array} \right\} \subseteq^* \text{ran}(\eta_\delta^0),$$

is also satisfied, we also know that for each $\delta \in S$, for the output η_δ^1 of the claim: $\text{ran}(\eta_\delta^1) \subseteq^* \text{ran}(\eta_\delta^*)$.

At this point we claim that every ladder system $\bar{\eta}^2$ that is very similar to $\bar{\eta}^1$ has \aleph_0 -uniformization, using that every ladder system very similar to $\bar{\eta}$ has \aleph_0 -uniformization: we define η_δ^3 so that $\eta_\delta^3(n) \in \Omega$ for each n and $\bar{\eta}^3$ is a ladder system very similar to $\bar{\eta}^2$. Then, since for $\delta \in S$ we have $\text{ran}(\eta_\delta) \subseteq \Omega$, and

$$\{\eta_\delta(n) + \omega : n \in \omega\} = \{\eta_\delta^*(n) + \omega : n \in \omega\},$$

we obtain

$$\{\eta_\delta(n) + \omega : n \in \omega\} \supseteq^* \{\eta_\delta^1(n) + \omega : n \in \omega\} = \{\eta_\delta^3(n) + \omega : n \in \omega\}.$$

Since both $\eta_\delta(n)$'s and $\eta_\delta^3(m)$'s are limit ordinals, clearly

$$\{\eta_\delta(n) : n \in \omega\} \supseteq^* \{\eta_\delta^3(n) : n \in \omega\}.$$

Now the fact that $\bar{\eta}$ has \aleph_0 -uniformization implies that $\bar{\eta}^3$ has \aleph_0 -uniformization, and by Lemma 1.10, $\bar{\eta}^2$ has \aleph_0 -uniformization, too. This means that we can appeal to Lemma 1.34 (applying it to $\langle x_\alpha : \alpha < \omega_1 \rangle$, and the ladder system $\bar{\eta}^1$, and the γ_δ^n 's), which completes the proof of clause (ii) of Proposition 1.7.

□_{Clause (ii) of Proposition 1.7}

Having proved clause (ii) of Proposition 1.7, with all the necessary ingredients we can verify $(A)_S \Rightarrow (C)_S$ from Theorem 1.3. So suppose that $S \subseteq \omega_1$ is a stationary set of limit ordinals, and every S -ladder system has \aleph_0 -uniformization. Fix a strongly \aleph_1 -free abelian group G and filtration \bar{G}' which satisfy $(*)_S(G, \bar{G}')$ from Definition 1.2. Now applying Claim 1.11 we can replace S with $S \setminus \{\alpha + \omega : \alpha < \omega_1\}$ (with the fact that every S -ladder system has \aleph_0 -uniformization remaining true), get the filtration $\bar{G} = \langle G_\alpha : \alpha < \omega_1 \rangle$ and $(**)_{S, \bar{\eta}}(G, \bar{G})$ (from Claim 1.11). Moreover, we can apply Claim 1.13, which gives us $\bar{x} = \langle x_\alpha : \alpha < \omega_1 \rangle$ and the ladder system $\bar{\eta}$ on our new S , and that \bar{G} , \bar{x} and $\bar{\eta}$ satisfy $(\bullet)_i - (\bullet)_{iv}$. It remains to invoke (ii) from Proposition 1.7: $(A)_{S, \bar{\eta}}$ holds (with $\bar{\eta}$ given by the claim), so we obtain $(C)_{S, \bar{\eta}}$. Thus it suffices to check that $(*)_{S, \bar{\eta}^*}^+(G, \bar{G}')$ holds for some filtration \bar{G}' : set $\bar{G}' = \bar{G}$, $\bar{\eta}^* = \bar{\eta}$, $y_{\delta+n} = x_{\delta+n}$ ($\delta \in S$, $n \in \omega$) (recall $(\bullet)_i - (\bullet)_{iv}$). This finishes the proof of $(A)_S \Rightarrow (C)_S$ (assuming Theorem 2.6).

It is only left to prove clause (iii) from Proposition 1.7, so we fix the ladder system $\bar{\eta}^*$ on S .

⊙₁ We define the free abelian group F to be generated by

$$\{\dot{x}_\alpha : \alpha < \omega_1\} \cup \{\dot{y}_{\delta, i, j} : \delta \in S, i \leq j \in \omega\}.$$

First we will need some definitions and claims

Definition 1.35.

- (1) For a sequence $\bar{p} = \langle p_{\delta,i,j} : \delta \in S, i, j \in \omega \rangle \in {}^{S \times \Delta^+} \mathbb{Z}$ (where by Δ^+ we mean $\langle i, j : i \leq j \in \omega \rangle$) we define the abelian group $G_{\bar{p}}$ by the generators

$$\{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,i,j} : \delta \in S, i \leq j \in \omega\}$$

and by the relations

$$(1.20) \quad p_{\delta,i,j} y_{\delta,i,j} = y_{\delta,0,0} - (x_{\eta_\delta^*(j)} - x_{\eta_\delta^*(i)}) \quad (\delta \in S, i < j),$$

and

$$(1.21) \quad p_{\delta,i,i} y_{\delta,i+1,i+1} = y_{\delta,0,0} - x_{\eta_\delta^*(i)} \quad (\delta \in S, i \in \omega),$$

i.e. a group isomorphic to $F/K_{\bar{p}}$, where $K_{\bar{p}}$ is the subgroup of F generated by

$$\begin{aligned} &\{p_{\delta,i,j} \dot{y}_{\delta,i,j} - (\dot{y}_{\delta,0,0} - (\dot{x}_{\eta_\delta^*(j)} - \dot{x}_{\eta_\delta^*(i)})) : \delta \in S, i < j \in \omega\}, \\ &\{p_{\delta,i,i} \dot{y}_{\delta,i+1,i+1} - (\dot{y}_{\delta,0,0} - \dot{x}_{\eta_\delta^*(i)}) : \delta \in S, i \in \omega\}, \end{aligned}$$

- (2) Define $K_{S,\bar{\eta}^*} = \{G_{\bar{p}} : \bar{p} \in {}^{S \times \Delta^+} \mathbb{Z}, (\forall \delta \in S) \langle p_{\delta,i,j} : i, j \in \omega \rangle \text{ is a sequence of pairwise relatively prime positive integers}\}$.
- (3) With a slight abuse of notation we define the group $G_{\bar{p}}$ when $\bar{p} = \langle p_{\delta,i} : \delta \in S, i \in \omega \rangle \in {}^{S \times \Delta^+} \mathbb{Z}$ to be $G_{\bar{p}'}$, where $\bar{p}' = \langle p'_{\delta,i,j} : \delta \in S, i, j \in \omega \rangle$ is defined by the equalities

$$\begin{aligned} p'_{\delta,i,i} &= p_{\delta,i} \quad (i \in \omega), \\ p'_{\delta,i,j} &= 1 \quad (i < j). \end{aligned}$$

The following is standard, but for the sake of completeness we include the proof.

Lemma 1.36. *Suppose that we are given $\bar{p} = \langle p_{\delta,i,j} : \delta \in S, i, j \in \omega \rangle \in {}^{S \times \omega} \mathbb{Z}$. If for each δ the sequence $\langle p_{\delta,i,j} : i \leq j \in \omega \rangle$ consists of pairwise relatively prime positive integers (i.e. there are no common prime factors of $p_{\delta,i,j}$ and $p_{\delta,k,l}$ if $(i,j) \neq (k,l)$, possibly $p_{\delta,i,j} = 1$ for some (i,j)), then for the filtration $\bar{G}^{\bar{p}} = \langle G_\beta^{\bar{p}} : \beta \in \omega_1 \rangle$ defined as*

$$G_\beta^{\bar{p}} = \text{cl}^{\text{pr}}(\{x_\alpha, y_{\alpha,0,0} : \alpha < \beta\})$$

the assertion $()_{S,\bar{\eta}^*}^+(G_{\bar{p}}, \bar{G}^{\bar{p}})$ (from Proposition 1.7) holds, in particular the groups in $K_{S,\bar{\eta}^*}$ are strongly \aleph_0 -free.*

Proof. First we prove that the collection

$$I := \{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,0,0} : \delta \in S\}$$

forms a maximal independent family. If this were true, then it would imply that $\{x_\alpha + G_\beta^{\bar{p}} : \beta \leq \alpha < \omega_1\} \cup \{y_{\delta,0,0} + G_\beta^{\bar{p}} : \beta \leq \delta \in S\}$ is independent in $G_{\bar{p}}/G_\beta^{\bar{p}}$.

As $G_{\bar{p}} = \text{cl}^{\text{pr}}(\{x_\alpha : \alpha < \omega_1\} \cup \{y_{\delta,0,0} : \delta \in S\})$ we only have to argue that I is independent. Working with the representation of $G_{\bar{p}}$ as $F/K_{\bar{p}}$, it is enough to show that the set

$$\dot{I} := \{\dot{x}_\alpha : \alpha < \omega_1\} \cup \{\dot{y}_{\delta,0,0} : \delta \in S\} \subseteq F$$

is independent over $K_{\bar{p}}$. But the generator set

$$\begin{aligned} &\{p_{\delta,i,j} \dot{y}_{\delta,i,j} - (\dot{y}_{\delta,0,0} - (\dot{x}_{\eta_\delta^*(j)} - \dot{x}_{\eta_\delta^*(i)})) : \delta \in S, i < j \in \omega\} \\ &\cup \end{aligned}$$

$$\{p_{\delta,i,i}\dot{y}_{\delta,i+1,i+1} - (\dot{y}_{\delta,0} - \dot{x}_{\eta_{\delta}^*(i)}) : \delta \in S, i \in \omega\}$$

freely generates $K_{\bar{p}}$, so comparing the coefficients clearly

$$\text{cl}(\dot{I}) \cap K_{\bar{p}} = \{0\},$$

so

⊖₁ the system

$$I = \{x_{\alpha} : \alpha < \omega_1\} \cup \{y_{\delta,0,0} : \delta \in S\} \text{ is independent in } G_{\bar{p}},$$

indeed. In particular,

⊖₂ $G_{\bar{p}}$ is torsion-free.

Second, we can argue that for any fixed $\beta \notin S$, the quotient $G_{\bar{p}}/G_{\beta}^{\bar{p}}$ is \aleph_1 -free. By [Fuc73, Thm. 19.1.] it suffices to prove that every subgroup of finite rank is free. Since a subgroup of a free group is always free it is enough to show that a cofinal system of the finite rank subgroups (w.r.t. \leq) consists of free group. So fix $J = \{x_{\alpha_0}, x_{\alpha_1}, \dots, x_{\alpha_{j-1}}\}$, $L = \{y_{\delta_0,0,0}, y_{\delta_1,0,0}, \dots, y_{\delta_{\ell-1},0,0}\}$ for some $j, \ell < \omega$, where $\beta \leq \alpha_0 < \alpha_1 < \dots < \alpha_{j-1} < \gamma$ and $\beta \leq \delta_0 < \delta_1 < \dots < \delta_{\ell-1} < \gamma$. We would like to prove that $\text{cl}^{\text{pr}}((L \cup J) + G_{\beta}^{\bar{p}}) \leq G_{\bar{p}}/G_{\beta}^{\bar{p}}$ is free, and by the above remark we can increase J and L , and assume that

⊖₃ if $y_{\delta_{k_0},0,0} \neq y_{\delta_{k_1},0,0} \in L$, $\eta_{\delta_{k_0}}^*(n_0) = \eta_{\delta_{k_1}}^*(n_1)$ for some $0 < n_0, n_1$, then $x_{\eta_{\delta_{k_0}}^*(n_0)} = x_{\eta_{\delta_{k_1}}^*(n_1)} \in J$.

Now we claim the following. By the characterization theorem of finitely generated abelian groups [Fuc73, Thm 15.5.] the claim finishes the proof of Lemma 1.36.

Claim 1.37. *If L and J are as above, then $\text{cl}^{\text{pr}}((L \cup J) + G_{\beta}^{\bar{p}}) \leq G_{\bar{p}}/G_{\beta}^{\bar{p}}$ is generated by*

$$\begin{aligned} & \{z + G_{\beta}^{\bar{p}} : z \in L \cup K\} \cup \\ & \cup \left\{ \frac{y_{\delta,0,0} - x_{\eta_{\delta}^*(n)}}{p_{\delta,n,n}} + G_{\beta}^{\bar{p}} : (y_{\delta,0,0} \in L, n \in \omega) \wedge (x_{\eta_{\delta}^*(n)} \in J \vee \eta_{\delta}^*(n) < \beta) \right\}, \\ & \cup \left\{ \frac{y_{\delta,0,0} - (x_{\eta_{\delta}^*(m)} - x_{\eta_{\delta}^*(n)})}{p_{\delta,n,m}} + G_{\beta}^{\bar{p}} : (y_{\delta,0,0} \in L, n < m \in \omega) \wedge \right. \\ & \quad \left. (x_{\eta_{\delta}^*(n)} \in J \vee \eta_{\delta}^*(n) < \beta) \wedge (x_{\eta_{\delta}^*(m)} \in J \vee \eta_{\delta}^*(m) < \beta) \right\} \end{aligned}$$

in particular it is finitely generated.

Proof. (Claim 1.37) Recalling the definition of the group $G_{\bar{p}}$ we have that the set $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ generates $G_{\bar{p}}/G_{\beta}^{\bar{p}}$, where

$$\mathcal{G}_0 = \{x_{\alpha} + G_{\beta}^{\bar{p}} : \beta \leq \alpha < \omega_1\},$$

$$\mathcal{G}_1 = \{y_{\delta,0,0} + G_{\beta}^{\bar{p}} : \delta \in S \setminus \beta\},$$

$$\mathcal{G}_2 = \{(y_{\delta,0,0} - x_{\eta_{\delta}^*(n)})/p_{\delta,n,n} + G_{\beta}^{\bar{p}} : \delta \in S \setminus \beta, n \in \omega\},$$

$$\mathcal{G}_3 = \{(y_{\delta,0,0} - (x_{\eta_{\delta}^*(m)} - x_{\eta_{\delta}^*(n)}))/p_{\delta,n,m} + G_{\beta}^{\bar{p}} : \delta \in S \setminus \beta, n < m \in \omega\},$$

so it is enough to verify the following subclaim.

Subclaim 1.38. *Let $h \in G_{\bar{p}}/G_{\beta}^{\bar{p}}$ be given, suppose that*

$$\begin{aligned} h = & \sum_{\alpha < \omega_1} k_{\alpha}(x_{\alpha} + G_{\beta}^{\bar{p}}) + \\ & + \sum_{\delta \in S \setminus \beta} k_{\delta,0,0}(y_{\delta,0,0} + G_{\beta}^{\bar{p}}) + \\ & + \sum_{\delta \in S, n \in \omega} k_{\delta,n+1,n+1}((y_{\delta,0,0} - x_{\eta_{\delta}^*(n)})/p_{\delta,n,n} + G_{\beta}^{\bar{p}}) + \\ & + \sum_{\delta \in S, n < m \in \omega} k_{\delta,n,m}((y_{\delta,0,0} - (x_{\eta_{\delta}^*(m)} - x_{\eta_{\delta}^*(n)}))/p_{\delta,n,m} + G_{\beta}^{\bar{p}}). \end{aligned}$$

Then the following hold:

i) for every $\delta \in S \setminus \beta$, $n < m \in \omega$, if $y_{\delta,0,0} \notin L$, then

$$p_{\delta,n,m} \nmid k_{\delta,n,m} \rightarrow h \notin \text{cl}^{\text{Pr}}(L \cup J),$$

and

$$p_{\delta,n,n} \nmid k_{\delta,n+1,n+1} \rightarrow h \notin \text{cl}^{\text{Pr}}(L \cup J),$$

ii) for every $\delta \in S \setminus \beta$, $n < m \in \omega$, if $(y_{\delta,0,0} \in L)$, and either $(x_{\eta_\delta^*(n)} \notin J \ \& \ \eta_\delta^*(n) \geq \beta)$, or $(x_{\eta_\delta^*(m)} \notin J \ \& \ \eta_\delta^*(m) \geq \beta)$, then

$$p_{\delta,n,m} \nmid k_{\delta,n,m} \rightarrow h \notin \text{cl}^{\text{Pr}}(L \cup J),$$

(This would finish the proof of Claim 1.37, i.e. $h \in \text{cl}^{\text{Pr}}(L \cup J)$ implies that h is a \mathbb{Z} -linear combinations of the finite set in the claim and possibly adding $(x_\alpha + G_\beta^{\bar{p}})$'s ($\alpha \geq \beta$, $x_\alpha \notin J$), or $(y_{\delta,0,0} + G_\beta^{\bar{p}})$'s ($\delta \in S \setminus \beta$, $y_{\delta,0,0} \notin L$). But since $h \in \text{cl}^{\text{Pr}}(L \cup J)$, we have that for some $0 \neq \ell \in \mathbb{Z}$, ℓh is a \mathbb{Z} -linear combination of elements from $L \cup K$.)

Proof. (Subclaim 1.38) Fix $\delta \geq \beta$. We prove first i), so we fix $\delta \in S$, and we need to show that if $y_{\delta,0,0} \notin L$, $n < m \in \omega$, then

$$p_{\delta,n,m} \nmid k_{\delta,n,m} \rightarrow h \notin \text{cl}^{\text{Pr}}(L \cup J),$$

and

$$p_{\delta,n,n} \nmid k_{\delta,n+1,n+1} \rightarrow h \notin \text{cl}^{\text{Pr}}(L \cup J).$$

Suppose on the contrary, and consider a counterexample $h \in \text{cl}^{\text{Pr}}(L \cup J)$, and the values

$$k_{\delta,n,m} \quad (n, m \in \omega),$$

which are not all 0's. Clearly (recalling \ominus_1):

$$k_{\delta,0,0} + \sum_{n < m} \frac{k_{\delta,n,m}}{p_{\delta,n,m}} + \sum_{n=0}^{\infty} \frac{k_{\delta,n+1,n+1}}{p_{\delta,n,n}} = 0.$$

Now since the sequence $\langle p_{\delta,n,m} : n, m \in \omega \rangle$ consists of pairwise relatively prime integers, even the fact that this sum is an integer implies

$$\forall n < m \in \omega : p_{\delta,n,m} \mid k_{\delta,n,m},$$

and

$$\forall n \in \omega : p_{\delta,n,n} \mid k_{\delta,n+1,n+1}.$$

Now we can argue ii). Now for this fixed δ we can replace the relevant elements of $\mathcal{G}_2 \cup \mathcal{G}_3$ (i.e. those that have $y_{\delta,0,0}$ occurring in them) by the sum of elements from \mathcal{G}_0 and \mathcal{G}_1 , we arrive at another \mathbb{Z} -linear combination with the same sum h , which derived decomposition satisfies

$$[(\forall n < m) \ k_{\delta,n,m} = 0 \ \& \ k_{\delta,n+1,n+1} = 0]$$

(only $k_{\delta,0,0}$ may be nonzero).

By the above argument we can of course assume that

$$(1.22) \quad (\forall \delta \in S, \forall n \in \omega) : [y_{\delta,0,0} \notin L \rightarrow k_{\delta,n+1,n+1} = 0]$$

and

$$(1.23) \quad (\forall \delta \in S, \forall n < m \in \omega) : [y_{\delta,0,0} \notin L \rightarrow k_{\delta,n,m} = 0].$$

So let $\alpha \geq \beta$ such that

$$(1.24) \quad x_\alpha \notin J.$$

Again, $h \in \text{cl}^{\text{pr}}(L \cup J)$ implies

$$\begin{aligned} 0 = & k_\alpha - \\ & - \sum_{\delta \in S, y_{\delta,0,0} \in L, n \in \omega, \eta_\delta^*(n) = \alpha} \frac{k_{\delta,n+1,n+1}}{p_{\delta,n,n}} \\ & - \sum_{\delta \in S, y_{\delta,0,0} \in L, n < m \in \omega, \eta_\delta^*(n) = \alpha} \frac{k_{\delta,n,m}}{p_{\delta,n,m}} \\ & + \sum_{\delta \in S, y_{\delta,0,0} \in L, l < n \in \omega, \eta_\delta^*(n) = \alpha} \frac{k_{\delta,l,n}}{p_{\delta,l,n}}. \end{aligned}$$

But by \ominus_3 it follows from $x_\alpha \notin J$ that there is at most one $\delta_0 \in S$ for which $y_{\delta_0,0,0} \in L$, and $\alpha \in \eta_{\delta_0}^*(n)$ for some n . W.l.o.g. we can assume that there is such a δ_0 , (and $\eta_{\delta_0}^*(n_0) = \alpha$) otherwise we are done. So

$$\begin{aligned} 0 = & k_\alpha - \\ & - \frac{k_{\delta_0,n_0+1,n_0+1}}{p_{\delta_0,n_0,n_0}} \\ & - \sum_{m > n_0} \frac{k_{\delta_0,n_0,m}}{p_{\delta_0,n_0,m}} \\ & + \sum_{l < n_0} \frac{k_{\delta_0,l,n_0}}{p_{\delta_0,l,n_0}} \end{aligned}$$

Using that $\langle p_{\delta_0,n,m} : n \leq m \rangle$ consists of pairwise relatively prime numbers, we are done.

□Subclaim1.38

□Claim1.37

□Lemma1.36

In order to complete the proof of clause (iii) we have to verify that if for each \bar{p} satisfying the demands from Lemma 1.36 the group $G_{\bar{p}}$ is Whitehead, then $\bar{\eta}^*$ has \aleph_0 -uniformization.

Claim 1.39. *Suppose that the sequence $\bar{p} \in {}^{S \times \Delta^+} \mathbb{Z}$ satisfies that for each $\delta \in S$ $\langle p_{\delta,n,m} : n \leq m \in \omega \rangle$ is a sequence of pairwise relative prime integers, and the group $G_{\bar{p}} \in K_{S,\bar{\eta}^*}$ (from Definition 1.35) is Whitehead.*

Then for every sequence $\bar{a} = \langle a_{\delta,n,m} : \delta \in S, n \leq m \in \omega \rangle \in {}^{S \times \omega} \mathbb{Z}$ there is a solution $\langle b_\delta : \delta \in S \rangle \in {}^S \mathbb{Z}$, $\langle c_\alpha : \alpha < \omega_1 \rangle \in {}^{\omega_1} \mathbb{Z}$ of \bar{a} , which means that

$$(1.25) \quad \forall \delta \in S, n < m \in \omega : b_\delta + a_{\delta,n,m} \equiv c_{\eta_\delta^*(m)} - c_{\eta_\delta^*(n)} \pmod{p_{\delta,n,m}},$$

and

$$(1.26) \quad \forall \delta \in S, n \in \omega : b_\delta + a_{\delta,n,n} \equiv c_{\eta_\delta^*(n)} \pmod{p_{\delta,n,n}}.$$

Proof. Define the abelian group H generated by

$$\{x_\alpha^*, y_{\delta,n,m}^* : \alpha < \omega_1, \delta \in S, n \leq m \in \omega\} \cup \{\zeta\}$$

freely, except for the relations

$$(1.27) \quad p_{\delta,n,m} y_{\delta,n,m}^* = y_{\delta,0,0}^* - (x_{\eta_\delta^*(m)}^* - x_{\eta_\delta^*(n)}^*) - a_{\delta,n,m} \zeta \quad (\delta \in S, n < m \in \omega),$$

and

$$(1.28) \quad p_{\delta,n,n} y_{\delta,n+1,n+1}^* = y_{\delta,0,0}^* - (x_{\eta_\delta^*(n)}^* - x_{\eta_\delta^*(n)}^*) - a_{\delta,n,n} \zeta \quad (\delta \in S, n \in \omega).$$

Now similarly to calculations in the proof of Lemma 1.36 one can check that

⊙₁ the system

$$I^* = \{x_\alpha^* : \alpha < \omega_1\} \cup \{y_{\delta,0,0}^* : \delta \in S\} \cup \{\zeta\}$$

is a maximal independent family in H , and

⊙₂ $\text{cl}_H^{\text{pf}}(\{\zeta\}) = \mathbb{Z}\zeta$ (using the condition on \bar{p} , by the same argument as in Subclaim 1.37).

It is also straightforward to check that there is a homomorphism $\varphi : H \rightarrow G_{\bar{p}}$ sending x_α to x_α^* , $y_{\delta,0,0}^*$ to $y_{\delta,0,0}$, and ζ to 0, since it respects the relations defining the groups in (1.20) and (1.21). Now since each element in H is a \mathbb{Q} -linear combination of I^* , ⊙₂ implies that

$$\ominus_3 \text{Ker}(\varphi) = \mathbb{Z}\zeta \simeq \mathbb{Z},$$

so there is a homomorphism $f : G_{\bar{p}} \rightarrow H$ with $\varphi \circ f = \text{id}_{G_{\bar{p}}}$. Now clearly

$$\ominus_4 f(y_{\delta,n,m}) - y_{\delta,n,m}^* \in \mathbb{Z}\zeta \quad (\delta \in S, n \leq m),$$

$$\ominus_5 f(x_\alpha) - x_\alpha^* \in \mathbb{Z}\zeta \quad (\alpha < \omega_1),$$

so we can define the vectors \bar{b}, \bar{c} by the following equations:

$$\ominus_6 b_\delta \zeta = f(y_{\delta,0,0}) - y_{\delta,0,0}^* \quad (\delta \in S),$$

$$\ominus_7 c_\alpha \zeta = f(x_\alpha) - x_\alpha^* \quad (\alpha < \omega_1).$$

Then recalling (1.27), fixing $\delta \in S$, $n < m \in \omega$ and using $f(y_{\delta,n,m}) = y_{\delta,n,m}^* + e\zeta$ for some $e \in \mathbb{Z}$ (which holds by ⊙₃) on the one hand

$$y_{\delta,0,0}^* - (x_{\eta_\delta^*(m)}^* - x_{\eta_\delta^*(n)}^*) - a_{\delta,n,m}\zeta = p_{\delta,n,m}y_{\delta,n,m}^*,$$

and on the other hand,

$$\begin{aligned} p_{\delta,n,m}(y_{\delta,n,m}^* + e\zeta) &= p_{\delta,n,m}f(y_{\delta,n,m}) = f(p_{\delta,n,m}y_{\delta,n,m}) = f(y_{\delta,0,0} - (x_{\eta_\delta^*(m)} - x_{\eta_\delta^*(n)})) = \\ &= f(y_{\delta,0,0}) - (f(x_{\eta_\delta^*(m)}) - f(x_{\eta_\delta^*(n)})) = b_\delta \zeta + y_{\delta,0,0}^* - (c_{\eta_\delta^*(m)}\zeta + x_{\eta_\delta^*(m)}^* - c_{\eta_\delta^*(n)}\zeta - x_{\eta_\delta^*(n)}^*), \\ \text{so } p_{\delta,n,m} \cdot e \cdot \zeta &= p_{\delta,n,m}(y_{\delta,n,m}^* + e\zeta - y_{\delta,n,m}^*) \text{ can be written as:} \end{aligned}$$

$$\left(b_\delta \zeta + y_{\delta,0,0}^* - (c_{\eta_\delta^*(m)}\zeta + x_{\eta_\delta^*(m)}^* - c_{\eta_\delta^*(n)}\zeta - x_{\eta_\delta^*(n)}^*)\right) - \left(y_{\delta,0,0}^* - (x_{\eta_\delta^*(m)}^* - x_{\eta_\delta^*(n)}^*) - a_{\delta,n,m}\zeta\right),$$

hence

$$p_{\delta,n,m} \cdot e \cdot \zeta = b_\delta \zeta - (c_{\eta_\delta^*(m)}\zeta - c_{\eta_\delta^*(n)}\zeta) - (-a_{\delta,n,m}\zeta)$$

so by ⊙₁ the coefficients of ζ must be equal:

$$p_{\delta,n,m} \cdot e = b_\delta - (c_{\eta_\delta^*(m)} - c_{\eta_\delta^*(n)}) + a_{\delta,n,m}.$$

Finally,

$$b_\delta + a_{\delta,n,m} \equiv c_{\eta_\delta^*(m)} - c_{\eta_\delta^*(n)} \pmod{p_{\delta,n,m}},$$

as desired. (Checking that $b_\delta + a_{\delta,n,n} \equiv c_{\eta_\delta^*(n)} \pmod{p_{\delta,n,n}}$ is completely analogous.)

□_{Claim 1.39}

Claim 1.40. *Suppose that*

- *the groups in $K_{S,\bar{\eta}^*}$ (from Definition 1.35 are all Whitehead,*

moreover, assume that we are given the sequence $\langle d_{\delta,n} : \delta \in S, n \in \omega \rangle \in {}^{S \times \omega} \omega$.

Then there exists a sequence $\langle d_\alpha^ : \alpha < \omega_1 \rangle \in {}^{\omega_1} \omega$ with the property*

$$(\forall \delta \in S) \quad \forall^\infty n : d_{\eta_\delta^*(n)}^* \geq d_{\delta,n},$$

and

$$(\forall \delta \in S) \quad \forall^\infty n \quad \forall k > 0 : d_{\eta_\delta^*(n+k)}^* \neq d_{\eta_\delta^*(n)}^*,$$

i.e. for some N the sequence $\langle d_{\eta_\delta^(l)}^* : l \geq N \rangle$ is injective.*

Proof. For each $\delta \in S$ define the sequence $p_{\delta,n,m}$ ($n \leq m \in \omega$) so that each $p_{\delta,n,m}$ is a prime, and if $(n,m) \neq (k,l)$, then $p_{\delta,n,m} \neq p_{\delta,k,l}$, moreover,

$$p_{\delta,n,n} \geq 4d_{\delta,n},$$

Note that this necessarily implies that

$$(\forall M)(\exists N)(\forall m \geq n \geq N) : p_{\delta,n,m} \geq M.$$

Now by our assumptions the group $G_{\bar{p}}$ is Whitehead, and defining $\langle a_{\delta,n,m} : \delta \in S, n \leq m \in \omega \rangle$ as

$$(1.29) \quad a_{\delta,n,m} = \lfloor p_{\delta,n,m}/2 \rfloor,$$

then for some $\langle b_\delta : \delta \in S \rangle$, $\langle c_\alpha : \alpha < \omega_1 \rangle$ (1.25) holds. This means that for each $\delta \in S$, for every large enough n $|b_\delta| \leq p_{\delta,n,n}/4$, so for such n 's (1.29) and (1.26) clearly imply

$$|c_{\eta_\delta^*(n)}| \geq p_{\delta,n,n}/4 \geq d_{\delta,n}.$$

Moreover (for the same fixed δ) if N is so large that $N \leq n < m$ implies the inequality $|b_\delta| \leq p_{\delta,n,m}/4$, then one can easily get from (1.25) that

$$|c_{\eta_\delta^*(m)} - c_{\eta_\delta^*(n)}| \geq p_{\delta,n,n}/4 > 0,$$

so choosing $d_\alpha^* = |c_\alpha|$ works.

□ Claim 1.40

Claim 1.41. *Suppose that*

- *for every $G \in K_{S,\bar{\eta}^*}$ (from Definition 1.35 G is Whitehead.*

Then $\bar{\eta}^$ has \aleph_0 -uniformization.*

Proof. Fix a system $\langle f_\delta : \delta \in S \rangle$, where for each $\delta \in S$

$$f_\delta : \text{ran}(\eta_\delta^*) \rightarrow \omega \text{ is a coloring with countably many colors,}$$

and using Claim 1.40 fix $\langle d_\alpha^* : \alpha < \omega_1 \rangle$ with

$$(1.30) \quad \forall \delta \in S : \lim_{n \rightarrow \infty} d_{\eta_\delta^*(n)}^* = \infty,$$

w.l.o.g. $\forall \alpha \ d_\alpha^* \in 2\mathbb{Z} + 1$. Now apply Claim 1.40 again in order to obtain a system $\langle e_\alpha : \alpha < \omega_1 \rangle \in {}^{\omega_1}\omega$ satisfying

$$(1.31) \quad (\forall \delta \in S) \ \forall^\infty n : e_{\eta_\delta^*(n)} \geq d_{\eta_\delta^*(n)}^* \cdot (f_\delta(\eta_\delta^*(n)) + 1),$$

and

$$(1.32) \quad (\forall \delta \in S) \ (\exists N = N(\delta)) \ (\forall n, m \geq N) : [(n \neq m) \rightarrow (e_{\eta_\delta^*(n)} \neq e_{\eta_\delta^*(m)})]$$

☒₁ Choose q_α to be the e_α 'th prime (so in particular

$$(\forall \delta \in S) \ \forall^\infty n : q_{\eta_\delta^*(n)} \geq d_{\eta_\delta^*(n)}^* \cdot (f_\delta(\eta_\delta^*(n)) + 1)),$$

☒₂ let for each $\delta \in S, n \in \omega$

$$\begin{aligned} p_{\delta,n} &= q_{\eta_\delta^*(n)}, & \text{if } n \geq N(\delta), \\ &= 1 & \text{otherwise.} \end{aligned}$$

☒₃ and note that by (1.32) $\langle p_{\delta,n} : n \in \omega \rangle$ is a sequence of pairwise relatively prime integers, with

$$(\forall^\infty n) \ p_{\delta,n} = q_{\eta_\delta^*(n)}.$$

Now

⊠₄ define the system $\langle a_{\delta,n} : \delta \in S, n \in \omega \rangle$ as

$$a_{\delta,n} = d_{\eta_\delta(n)}^* \cdot f_\delta(\eta_\delta^*(n)),$$

and apply Claim 1.40 again, i.e. we obtain the solution $\langle b_\delta : \delta \in S \rangle \in {}^S\mathbb{Z}$, $\langle c_\alpha : \alpha < \omega_1 \rangle \in {}^{\omega_1}\mathbb{Z}$:

$$(1.33) \quad \forall \delta \in S, n \in \omega : b_\delta \equiv c_{\eta_\delta^*(n)} - a_{\delta,n} \pmod{p_{\delta,n}}.$$

Now

⊠₅ for each $\alpha < \omega_1$ we define $f^*(\alpha) \in \omega$ as follows. Let

$$c_\alpha^* \in \{kq_\alpha + \ell d_\alpha^* : k \in \mathbb{Z}, \ell \in \{0, 1, \dots, \lfloor q_\alpha/d_\alpha^* \rfloor - 1\}\}$$

be the element of that set of minimal distance from c_α (if that set is not empty), let $f^*(\alpha) = \ell_0 \in \{0, 1, \dots, \lfloor q_\alpha/d_\alpha^* \rfloor - 1\}$, where $c_\alpha^* = k_0 q_\alpha + \ell_0 d_\alpha^*$.

Fixing $\delta \in S$, using ⊠₂ (in order to prove that f^* uniformizes the f_δ 's) it clearly suffices to prove that

$$(1.34) \quad (\forall \delta \in S) \forall^\infty n \exists k \in \mathbb{Z} |kq_{\eta_\delta^*(n)} + f_\delta(\eta_\delta^*(n)) \cdot d_{\eta_\delta^*(n)}^* - c_{\eta_\delta^*(n)}^*| < d_{\eta_\delta^*(n)}^*/2$$

(recall that $0 \leq f_\delta(\eta_\delta^*(n)) \cdot d_{\eta_\delta^*(n)}^* < q_{\eta_\delta^*(n)}$ holds for each δ for large enough n). By (1.33),

$$b_\delta + a_{\delta,n} = b_\delta + d_{\eta_\delta(n)}^* \cdot f_\delta(\eta_\delta^*(n)) \equiv c_{\eta_\delta^*(n)}^* \pmod{p_{\delta,n}}.$$

First recall that $p_{\delta,n} = q_{\eta_\delta^*(n)}$ for large enough n (by ⊠₃), so for any large enough n , for some $k = k(n) \in \mathbb{Z}$

$$b_\delta + d_{\eta_\delta(n)}^* \cdot f_\delta(\eta_\delta^*(n)) + kq_{\eta_\delta^*(n)} = c_{\eta_\delta^*(n)}^*.$$

Rearranging this equation one obtains

$$d_{\eta_\delta(n)}^* \cdot f_\delta(\eta_\delta^*(n)) + kq_{\eta_\delta^*(n)} - c_{\eta_\delta^*(n)}^* = -b_\delta.$$

Therefore for large enough $n = n(\delta)$ we have $|b_\delta| < d_{\eta_\delta^*(n)}^*/2$ (by (1.30)), obtaining (1.34), we are done.

□_{Claim1.41}

□_{Theorem1.3 assuming Theorem2.1}

§ 2. SPECIAL UNIFORMIZATION PROBLEMS: A COMBINATORIAL EQUIVALENCE

The purpose of this section is to prove our main theorem:

Theorem 2.1. *Let S be a stationary set of limit ordinals $< \omega_1$, which is simple (i.e. $\forall \alpha: \alpha + \omega \notin S$). Then $(A)_S \Rightarrow (B)_S$, where:*

- (A)_S every S -ladder system has \aleph_0 -uniformization,
- (B)_S every special S -uniformization \mathfrak{p} has a solution, see below.

We shall prove more in 2.6 below.

Definition 2.2. Assume \mathfrak{p} is a special S -uniformization problem, $\Psi = \Psi^{\mathfrak{p}}$.

1) We say x_1, x_2 are Ψ -isomorphic when: $|x_1| = |x_2|$ and for any $b_\ell \subseteq x_\ell$, $b_\ell \in \mathcal{B}^{\mathfrak{p}}$, f_ℓ a function with domain b_ℓ (for $\ell = 1, 2$) satisfying

$$\text{OP}_{u_1, u_2}(b_2) = b_1,$$

$$f_1 \circ \text{OP}_{b_1, b_2} = f_2$$

we have $f_1 \in \Psi(b_1)$ iff $f_2 \in \Psi(b_2)$.

Observation 2.3.

- 1) Let \mathfrak{p} be a special S -uniformization problem. The number of Ψ -isomorphism types is $\leq \aleph_0$.
- 2) For every finite $u \subseteq \omega_1$ there is a finite \mathfrak{p} -closed v extending u .
- 3) The relation “ $\bar{\eta}', \bar{\eta}$ ” are very similar S -ladder systems” is an equivalence relation on the set of S -ladder systems (see Definition 1.31(4)).

Proof. his follows by clause (c) of Definition 1.27, and see Claim 2.7 below. \square_T

As a warm-up we show (but we rely on Claim 2.7 below).

Claim 2.4. *If \mathfrak{p} is a special S -uniformization problem, then the forcing notion $\mathbb{Q}_{\mathfrak{p}} = (\mathcal{F}_{\mathfrak{p}}, \subseteq)$ satisfies the c.c.c. and $\Vdash_{\mathbb{Q}_{\mathfrak{p}}} “\cup \{f : f \in G_{\mathbb{Q}_{\mathfrak{p}}}\} \text{ is a solution for } \mathfrak{p}”$ where $\mathcal{F}_{\mathfrak{p}} = \cup \{\Psi^+(u) : u \subseteq \omega_1 \text{ is finite, } \mathfrak{p}\text{-closed}\}$.*

Proof. For the second phrase just note that $\mathcal{F}_{\mathfrak{p}}$ is non-empty (by clause \square_b of Definition 1.29 and for every $\alpha < \omega_1$ the set $\mathcal{I}_{\alpha} = \{f \in \mathcal{F}_{\mathfrak{p}} : \alpha \in \text{Dom}(f)\}$ is dense open in $\mathbb{Q}_{\mathfrak{p}}$ by clause \square_d in Definition 1.29.

As for the first phrase, “ $\mathbb{Q}_{\mathfrak{p}}$ satisfies the c.c.c.”, let $f_\alpha \in \Psi(x_\alpha)$ for $\alpha < \omega_1$, where each x_α is \mathfrak{p} -closed. As each x_α is finite, for some stationary $S_0 \subseteq S$ the sequence $\langle x_\alpha : \alpha \in S_0 \rangle$ is a Δ -system with heart u^* , and for $\alpha \neq \beta$ in S_0 , $x_\alpha \cap \alpha = x_\beta \cap \beta = x_\alpha \cap x_\beta = u^*$. Without loss of generality $\alpha \in S_0 \Rightarrow f_\alpha \restriction u^* = f^*$ and $\beta < \alpha \in S_0 \Rightarrow x_\beta \subseteq \alpha$. Observe that this together with the closedness of the x_β 's imply that

$$(2.1) \quad \alpha < \beta \in S_0 \Rightarrow \sup(x_\alpha) + \omega \leq \min(x_\beta \setminus u_*).$$

Let $\beta(\alpha) = \min(S_0 \setminus (\alpha + 1))$ for $\alpha \in S_0$. We define $v_\alpha = u^*$ if the ordinal $\delta := \min(x_{\beta(\alpha)} \setminus \beta(\alpha)) \notin S$, and otherwise if this $\delta \in S$ let

$$v_\alpha = \text{cl}^{\mathfrak{p}} \left(\bigcup_{n < \omega, \eta_\delta(n) < \alpha} (\{\eta_\delta(n)\} \cup u_{\delta, n}) \cup u^* \right),$$

so clearly $v_\alpha \subseteq \alpha$, and it is finite by Claim 2.7 below.

Subclaim 2.5. *For each $\alpha \in S_0$ the finite set $y_\alpha = x_{\beta(\alpha)} \cup v_\alpha$ is \mathfrak{p} -closed.*

Proof. First we check clause (α) of Definition 1.28 3). Pick $\epsilon + 1 \in y_\alpha$. Since both $x_{\beta(\alpha)}$ and v_α satisfy (α) , the ordinal ϵ must be in either $x_{\beta(\alpha)}$, or v_α . Now let $\delta \in S \cap y_\alpha$. If $\delta < \alpha$, then necessarily $\delta \in v_\alpha$ and since v_α is \mathfrak{p} -closed we are done. At this point we also note that

$$(2.2) \quad (\forall \delta \in S \cap y_\alpha \cap \alpha) (\forall n, k) : [b_{n,k}^\delta \subseteq y_\alpha \rightarrow b_{n,k}^\delta \subseteq v_\alpha] \quad (\text{since } v_\alpha \supseteq u^*).$$

Second, suppose that $\delta \geq \alpha$. Then $\delta \in x_{\beta(\alpha)} \setminus \alpha = x_{\beta(\alpha)} \setminus \beta(\alpha)$ (since $x_{\beta(\alpha)} \cap \beta(\alpha) = u^*$ as $\beta(\alpha) \in S_0$). We will distinguish two cases depending on whether $\delta = \min(x_{\beta(\alpha)} \setminus \beta(\alpha))$, or $\delta > \min(x_{\beta(\alpha)} \setminus \beta(\alpha))$.

If $\delta > \min(x_{\beta(\alpha)} \setminus \beta(\alpha))$, then for each n we have

$$\eta_\delta(n) \in x_{\beta(\alpha)} \iff \eta_\delta(n) \leq \max(\delta \cap x_{\beta(\alpha)} \setminus \beta(\alpha)) \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq x_{\beta(\alpha)},$$

(since $x_{\beta(\alpha)}$ is \mathfrak{p} -closed), so

$$(2.3) \quad \eta_\delta(n) \in y_\alpha \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq y_\alpha \iff \eta_\delta(n) \leq \max(\delta \cap x_{\beta(\alpha)} \setminus \beta(\alpha)).$$

Now if $\delta = \min(x_{\beta(\alpha)} \setminus \beta(\alpha))$, then for each n

$$\eta_\delta(n) \in v_\alpha \iff \eta_\delta(n) < \alpha \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq v_\alpha,$$

so

$$(2.4) \quad \eta_\delta(n) \in y_\alpha \iff \eta_\delta(n) < \alpha \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq y_\alpha,$$

and we are done. For future reference we note that (2.2), (2.3), (2.4) together imply

$$(2.5) \quad (\forall \delta \in S \cap y_\alpha \setminus \alpha) (\forall n, k) : [b_{n,k}^\delta \subseteq y_\alpha \rightarrow (b_{n,k}^\delta \subseteq v_\alpha \vee b_{n,k}^\delta \subseteq x_{\beta(\alpha)})].$$

□_{Subclaim 2.5}

Now for some $v^* \in [\omega_1]^{<\aleph_0}$ and stationary subset S_1 of S_0 we have

$$\alpha \in S_1 \Rightarrow (v_\alpha = v^*).$$

For each $\alpha \in S_1$ we can find $g_\alpha \in \Psi^+(y_\alpha) = \Psi^+(x_{\beta(\alpha)})$ extending $f_{\beta(\alpha)}$, which exists by clause \Box_d in Definition 1.29. Now w.l.o.g. we can assume that for every $\alpha_1 < \alpha_2$ in S_1 we have $\beta(\alpha_1) < \alpha_2$ (intersecting S_1 with a club if necessary), and $g_{\alpha_1} \upharpoonright v^* = g_{\alpha_2} \upharpoonright v^*$ (by passing down to a smaller stationary set). Observe that if $\alpha_1 < \alpha_2$ in S_1 , then $y_{\alpha_1} \cup y_{\alpha_2} = v^* \cup x_{\beta(\alpha_1)} \cup x_{\beta(\alpha_2)}$ is \mathfrak{p} -closed, since by the proof of Subclaim 2.5 for any $\delta \in (S \cap y_{\alpha_2} \setminus \alpha_2)$ either for all $n \in \omega$:

$$\eta_\delta(n) \in y_{\alpha_2} \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq y_{\alpha_2} \iff \eta_\delta(n) \leq \max(\delta \cap x_{\beta(\alpha_2)} \setminus \beta(\alpha)),$$

(which is the case if $\delta > \min(x_{\beta(\alpha_2)} \setminus \beta(\alpha)) = \min(y_{\alpha_2} \setminus \beta(\alpha))$), or

$$\eta_\delta(n) \in y_{\alpha_2} \iff \eta_\delta(n) < \alpha_2 \iff \{\eta_\delta(n)\} \cup u_{\delta,n} \subseteq y_{\alpha_2}$$

(when $\delta = \min(x_{\beta(\alpha_2)} \setminus \beta(\alpha)) = \min(y_{\alpha_2} \setminus \beta(\alpha))$). This means that

$$(2.6) \quad \text{for } \delta \in (S \cap y_{\alpha_2} \setminus \alpha_2) : (\forall n, k) : [b_{n,k}^\delta \subseteq y_{\alpha_1} \cup y_{\alpha_2}] \rightarrow [b_{n,k}^\delta \subseteq y_{\alpha_2}].$$

Whereas if $\delta \in (S \cap (y_{\alpha_2} \cup y_{\delta_1}) \cap \alpha_2)$, then $\delta \in y_{\delta_1}$ (this holds for any $\epsilon \in (y_{\alpha_2} \cup y_{\delta_1}) \cap \alpha_2$), so

$$(2.7) \quad \text{if } \delta \in (S \cap (y_{\alpha_2} \cup y_{\delta_1}) \cap \alpha_2) : (\forall n, k) : [b_{n,k}^\delta \subseteq y_{\alpha_1} \cup y_{\alpha_2}] \rightarrow [b_{n,k}^\delta \subseteq y_{\alpha_1}].$$

Therefore, as $f_{\beta(\alpha_1)} \cup f_{\beta(\alpha_2)}$ is a function (since they agree on u^*), so (by (2.6), (2.7)) $f_{\beta(\alpha_1)} \cup f_{\beta(\alpha_2)} \in \Psi^+(y_{\alpha_1} \cup y_{\alpha_2})$, hence are compatible in $\mathbb{Q}_\mathfrak{p}$ as required. □_{Claim 2.4}

Our main theorem (Theorem 2.1) will follow from the following slightly more general one:

Theorem 2.6. *Let S be a stationary set of limit ordinals $< \omega_1$ which is simple, i.e., $(\forall \alpha < \omega_1)(\alpha + \omega \notin S)$, and $\bar{\eta}^*$ be an S -ladder system. Then $(\mathbf{D})_{S, \bar{\eta}^*} \Leftrightarrow (\mathbf{E})_{S, \bar{\eta}^*}$ where*

- $(\mathbf{D})_{S, \bar{\eta}^*}$ *for any ladder system $\bar{\eta}$ on S if η is very similar to $\bar{\eta}^*$ then it has \aleph_0 -uniformization*
- $(\mathbf{E})_{S, \bar{\eta}^*}$ *for every ladder system $\bar{\eta}$ very similar to $\bar{\eta}^*$, for every simple special S -uniformization problem \mathbf{p} with $\bar{\eta}^{\mathbf{p}} = \bar{\eta}$, then \mathbf{p} has a solution.*

We shall prove Claim 2.6 together with proving Theorem 2.1.

Proof. Proof of 2.6

$(\mathbf{E})_{S, \bar{\eta}^*} \Rightarrow (\mathbf{D})_{S, \bar{\eta}^*}$

$(\mathbf{D})_{S, \bar{\eta}^*}$ is essentially a special case of $(\mathbf{E})_{S, \bar{\eta}^*}$: let $\bar{\eta}$ be very similar to $\bar{\eta}^*$ we can replace $\eta_\delta(n)$ by $\eta_\delta(n) + 1$ and use the value of the prospective solution f at δ to say from which n onwards the uniformization demand for c_δ holds, that is, $c_\delta(\eta_\delta(n)) = f(\eta_\delta(n))$ if $n \geq f(\delta)$.

In detail, this will look as follows. Given $\langle \eta_\delta : \delta \in S \rangle$, $\langle c_\delta : \delta \in S \rangle$ as in $(\mathbf{D})_{S, \bar{\eta}^*}$ recalling Definition 1.29 we now define a special S -uniformization problem \mathbf{p} with

- (i) $\bar{\eta}^{\mathbf{p}} = \langle \eta'_\delta : \delta \in S \rangle$, where $\eta'_\delta(n) = \eta_\delta(n) + 1$ ($n < \omega$),
- (ii) $u_{\delta, n}^{\mathbf{p}} = \emptyset$ for every $\delta \in S$, $n < \omega$ (so $b_{n, k}^\delta = \{\eta'_\delta(i) : i \leq n\} \cup [\delta, \delta + k]$),
- (iii) $\Psi^{\mathbf{p}}(b_{n, k}^\delta)$ is the set of all functions f from $b_{n, k}^\delta$ to ω such that:
 - $(\forall i \in [f(\delta), n]) : f(\eta'_\delta(i)) = c_\delta(\eta_\delta(i))$,

It has to be checked that

- (α) (\mathbf{p}) is indeed a special S -uniformization problem (recall that $\{\text{ran}(\eta_\delta) : \delta \in S\}$ is an almost disjoint family), and
- (β) $\bar{\eta}', \bar{\eta}$ are very similar.

The second clause is immediate, let us deal with the first clause for which we need to check that $\Psi^{\mathbf{p}}$ satisfies clauses \square_a - \square_b from Definition 1.29. For a fixed $\delta \in S$, $n, k \in \omega$ we note that $\Psi^{\mathbf{p}}(b_{n, k}^\delta)$ is determined by $\langle c_\delta(\eta_\delta(j)) : j \leq n \rangle \in {}^{n+1}\omega$, so \square_a holds. It is easy to see that clauses \square_b and \square_c hold. Finally, fix a finite $u \subseteq \omega$ that is \mathbf{p} -closed, $f \in \Psi^+(u)$, $v \supseteq u$, $|v| < \aleph_0$. Now observe that if we could define $f(\alpha)$ for all the α 's that belong to $v \setminus u$, and α is of the form $\alpha = \eta'_\delta(\ell)$ for some $\ell \in \omega$, $\delta \in u$ (and so $\alpha = \eta_\delta(\ell) + 1$) so that $f(\alpha) = c_\delta(\eta_\delta(\ell))$, then we would be done.

We claim that whenever $\delta < \delta'$ both belong to $u \cap S$, and $\eta_\delta(k) = \eta_{\delta'}(l)$ for some $l, k \in \omega$, then $\eta'_\delta(k) = \eta'_{\delta'}(l) \in u$. This will finish the verification of clause (α). But since u is \mathbf{p} -closed, in fact for each l with $\eta'_{\delta'}(l) < \delta$ we have $\eta'_{\delta'}(l) \in u$, since $\delta, \delta' \in u$ (just recall 3) (β) (ii)).

Now let f' be a solution to the special uniformization problem $\mathbf{p} = (S, \bar{\eta}', \bar{u}, \Psi)$, as guaranteed in $(B)_{S, \bar{\eta}^*}$ (i.e., as defined in Definition 1.31(1)). We check the function $c : \omega_1 \rightarrow \omega$ defined by: $c(\alpha) =: f(\alpha + 1)$ is as required in $(\mathbf{D})_{S, \bar{\eta}^*}$ for $\bar{\eta}$:

Fix $\delta \in S$ and $n \in \omega$ for which $n \geq f(\delta)$. Consider the set $b_{n, 0}^\delta$, containing $\eta'_\delta(n) = \eta_\delta(n) + 1$, and δ . Since f is a solution, $f_0 = f \upharpoonright b_{n, 0}^\delta \in \Psi^{\mathbf{p}}(b_{n, k}^\delta)$, thus

(recalling the definition of $\Psi^{\mathbf{p}}(b_{n,k}^\delta)$ in clause (iii)):

$$f(\eta'_\delta(n)) = f_0(\eta'_\delta(n)) = c_\delta(\eta_\delta(n)),$$

as $n \geq f_0(\delta) = f(\delta)$. Therefore, (for $n \geq f(\delta)$):

$$c(\eta_\delta(n)) = f(\eta_\delta(n) + 1) = f(\eta'_\delta(n)) = c_\delta(\eta_\delta(n)),$$

as desired.

(D) $_{S, \bar{\eta}^*} \Rightarrow$ (E) $_{S, \bar{\eta}^*}$.

So from now on we are going to work under the assumption that for all $\bar{\eta}$ similar to $\bar{\eta}^*$ have \aleph_0 -uniformization. After the following preparation Lemma 2.9, Lemma 2.18, Lemma 2.23 together will conclude the statement.

Let \mathbf{p} be a special S -uniformization problem with $\bar{\eta}^{\mathbf{p}} = \bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$, also $\langle h_\delta : \delta \in S \rangle$, $\langle u_{\delta,n} : \delta \in S, n \in \omega \rangle$ and Ψ as in Definition 1.29 are given such that $\bar{\eta}$ is very similar to $\bar{\eta}^*$ and $S = S^{\mathbf{p}}$. So it is enough to prove that \mathbf{p} has a solution.

(We usually suppose \mathbf{p} is the index so closed means \mathbf{p} -closed, etc.) The proof is broken to stages (and definitions).

Claim 2.7. *Let $\mathbf{p}^0 = (S, \bar{\eta}, \bar{u})$ be an S -uniformization frame. For every finite $x \subseteq \omega_1$ there is a \mathbf{p}^0 -closed finite y , $x \subseteq y \subseteq \omega_1$ satisfying $\max(x) = \max(y)$, moreover, if we are given $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$ with for each α : $v_\alpha \in [\alpha]^{<\aleph_0}$, then we can prescribe y to be \bar{v} -closed, by which we mean*

$$\alpha \in y \rightarrow v_\alpha \subseteq y.$$

Proof. We prove this by induction on $\delta_x = \max\{\alpha \in \{0\} \cup \Omega : \alpha \leq \sup(v)\}$

Case 1, $\delta_x = 0$: trivial, let $u = [0, \sup(v)]$.

Case 2, $\delta_x > 0$ and $\delta_x \notin S$:

Let m be such that $\max(x) = \delta + m$, note that $\sup(x \cap \delta) < \delta$ and so by the induction hypothesis there is a \mathbf{p} -closed finite $y' \subseteq \delta_x$ containing $x \cap \delta_x$ and we let $y = y' \cup [\delta_x, \delta_x + m]$.

Case 3, $\delta_x > 0$ and $\delta_x \in S$: Choose m such that $\max(x) = \delta_x + m$ and choose n such that $\eta_{\delta_x}(n) > \sup(x \cap \delta_x)$. Let

$$x^+ = (x \cap \delta_x) \cup \{\eta_{\delta_x}(\ell) : \ell < n\} \cup \bigcup_{\ell < n} u_{\delta_x, \ell}.$$

so $\max(x^+) < \eta_{\delta_x}(n) < \delta_x$ (so $\delta_{x^+} < \delta_x$). Hence by the induction hypothesis there is a $\mathbf{p} - \bar{v}$ -closed y' such that for some $k < \omega$ we have

$$x^+ \subseteq y' \subseteq \max(x^+) + 1 \leq \eta_{\delta_x}(n) < \delta_x$$

and let $y = y' \cup [\delta_x, \delta_x + m]$, remembering that $\delta_x + m = \max(x)$ it is easy to check that we are done.

□_{Claim2.7}

Also note that

(T)₁ the intersection of a family of $\mathbf{p} - \bar{v}$ -closed subsets of ω_1 is closed,

hence

(T)₂ for any finite $x \subseteq \omega_1$ the closure of x , $\text{cl}^{\mathbf{p} - \bar{v}} = \cap \{y : y \text{ is finite } \mathbf{p} - \bar{v}\text{-closed and contains } x\}$ is finite, $\mathbf{p} - \bar{v}$ -closed, contains x and has the same maximum.

Note the following assertions

(T)₃ If c_δ is a function with domain $\text{ran}(\eta_\delta)$ for each $\delta \in S$ satisfying
 (*) for each $\alpha < \omega_1$, the set $\mathbf{c}^\alpha =: \{c_\delta(\alpha) : \alpha \in \text{ran}(\eta_\delta), \delta \in S\}$ is countable,

then we can uniformize $\langle c_\delta : \delta \in S \rangle$, i.e.

- find a function c satisfying $\text{Dom}(c) = \omega_1$ and $c(\alpha)$ belongs to $\{c_\delta(\alpha) : \alpha \in \text{ran}(\eta_\delta)\} \cup \{0\}$, for every $\alpha < \omega_1$ and $\bigwedge_{\delta \in S} (c_\delta \subseteq^* c)$;
- moreover, there are $\bar{m} = \langle m_\delta : \delta \in S \rangle$ and a function $\mathbf{m} : \omega_1 \rightarrow \omega$ such that for every $\delta \in S$ we have:

$$n \in [m_\delta, \omega) \Rightarrow f(\eta_\delta(n)) = c_\delta(\eta_\delta(n)),$$

$$\text{and } (\forall^* n)(\mathbf{m}(\eta_\delta(n)) = m_\delta),$$

[Why? Let $g_\alpha : \{c_\delta(\alpha) : \alpha \in \text{ran}(\eta_\delta)\} \rightarrow \omega$ be one to one, let $c'_\delta : \text{ran}(\eta_\delta) \rightarrow \omega$ be $c'_\delta(\eta_\delta(n)) = g_{\eta_\delta(n)}(c_\delta(\eta_\delta(n)))$. Now we can find a function f' which uniformize $\langle c'_\delta : \delta \in S \rangle$, see Definition 0.3(2). It exists as we are assuming $(\mathbf{D})_{S, \bar{\eta}^*}$. Let $m_\delta = \text{Min}\{m : 0 < m < \omega \text{ and } (\forall n)(m \leq n < \omega \rightarrow f'(\eta_\delta(n)) = c'_\delta(\eta_\delta(n)))\}$.

Then we apply $(\mathbf{D})_{S, \bar{\eta}^*}$ to $\langle c''_\delta : \delta \in S \rangle$ where c''_δ is defined by $c''_\delta(\eta_\delta(n)) = m_\delta$, getting f'' . Let $\mathbf{m} = f''$ and define c with domain ω_1 by $c(\alpha)$ satisfies $g_\alpha(c(\alpha)) = f'(\alpha)$ when $f'(\alpha) \in \text{ran}(g_\alpha)$ and $c(\alpha) = 0$ otherwise.]

For a given sequence $\langle c_\delta^i : \delta \in S \rangle$, we use c^i, \mathbf{m}^i to denote uniformizing functions like c and \mathbf{m} as above.

Definition 2.8. Suppose that $\mathbf{q} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem satisfying

- a) $u_{\delta, n} \cup \{\eta_\delta(n)\} \subseteq u_{\delta, n+1}$,
- b) $\delta = \bigcup_{n < \omega} u_{\delta, n}$.

We call \mathbf{q} *very special* if in addition to a), b) there is a system $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$ with for each $\delta \in S$, $n \in \omega$: $u_{\delta, n} = v_{\eta_\delta(n)}$.

Lemma 2.9. Let $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ be a special S -uniformization problem. If $\bar{\eta}$ has \aleph_0 -uniformization, then there is a very special S -uniformization problem \mathbf{p}' , each solution of which is a solution of \mathbf{p} .

Proof.

Definition 2.10. If $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem, and we are given $\bar{v} = \langle v_{\delta, n} : \delta \in S, n \in \omega \rangle$ with

$$(\forall \delta \in S, \forall n \in \omega) : u_{\delta, n} \subseteq v_{\delta, n} \subseteq \eta_\delta(n),$$

(so $\mathbf{p}_{+\bar{v}}^0 = (S, \bar{\eta}, \bar{v})$ is an S -uniformization frame), and for each $b = (b_{n, k}^\delta)^{\mathbf{p}_{+\bar{v}}^0} \in \mathcal{B}^{\mathbf{p}_{+\bar{v}}^0}$ let

$$f \in \Phi((b_{n, k}^\delta)^{\mathbf{p}_{+\bar{v}}^0}) \iff \forall b \in \mathcal{B}^{\mathbf{p}} : b \subseteq (b_{n, k}^\delta)^{\mathbf{p}_{+\bar{v}}^0} \rightarrow (f \upharpoonright b \in \Psi(b)),$$

then we can define the tuple

$$\mathbf{p}_{+\bar{v}} = (S, \bar{\eta}, \bar{v}, \Phi).$$

Claim 2.11. *If $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem, and we are given $\bar{v} = \langle v_{\delta,n} : \delta \in S, n \in \omega \rangle$ with*

$$(\forall \delta \in S, \forall n \in \omega) : u_{\delta,n} \subseteq v_{\delta,n} \subseteq \eta_\delta(n),$$

(so $\mathbf{p}_{+\bar{v}}^0 = (S, \bar{\eta}, \bar{v})$ is an S -uniformization frame), then $\mathbf{p}_{+\bar{v}} = (S, \bar{\eta}, \bar{v}, \Phi)$ from the definition above is a special S -uniformization problem, moreover, each solution g of $\mathbf{p}_{+\bar{v}}$ is a solution of \mathbf{p} .

Proof. First we check \square_a in Definition 1.28. Let $w \in [\omega_1]^{<\omega_1}$ be an element of $\mathcal{B}^{\mathbf{p}+\bar{v}}$ (recall Definition 1.28). Now as w is finite $w \cap S$ is also finite, and it is easy to see that $\{b \in \mathcal{B}^{\mathbf{p}} : b \subseteq w\}$ is finite. Now since $(b_{n,k}^\delta)^{\mathbf{p}^0+\bar{v}} \supseteq b_{n,k}^\delta \in \mathcal{B}^{\mathbf{p}^0}$, and the fact that

$$(\forall k \in \omega) \quad \Upsilon_k = \{\{f \circ \text{OP}_{b,k} : f \in \Psi(b)\} : b \in \mathcal{B}^{\mathbf{p}}, |b| = k\} \text{ is countable}$$

it can be easily seen that \square_b indeed holds.

For \square_c notice that if $b' \subseteq b''$ are in $\mathcal{B}^{\mathbf{p}+\bar{v}}$, then by a reformulation of the definition of Φ :

$$\begin{aligned} \Phi(b') &= \bigcap_{b \in \mathcal{B}^{\mathbf{p}}, b \subseteq b'} \{f : b' \rightarrow C : f \restriction b \in \Psi(b)\}, \\ \Phi(b'') &= \bigcap_{b \in \mathcal{B}^{\mathbf{p}}, b \subseteq b''} \{f : b'' \rightarrow C : f \restriction b \in \Psi(b)\}. \end{aligned}$$

Finally as every $\mathbf{p}_{+\bar{v}}$ -closed set is \mathbf{p} -closed as well, moreover, observe that if a $\mathbf{p}_{+\bar{v}}$ -closed set w contains the \mathbf{p} -basic set $b_{n,k}^\delta$ as a subset, then for $b' = (b_{n,k}^\delta)^{\mathbf{p}^0+\bar{v}}$ we have $w \supseteq b' \supseteq b_{n,k}^\delta$. This implies

$$w \text{ is } \mathbf{p}_{+\bar{v}}\text{-closed} \rightarrow \Phi^+(w) = \Psi^+(w),$$

so \square_d holds as well.

As for the moreover part again recall that for each $b \in \mathcal{B}^{\mathbf{p}}$ there exists $b' \in \mathcal{B}^{\mathbf{p}+\bar{v}}$ with $b' \supseteq b$, so if g is a function with $\text{dom}(g) = \omega_1$, $\forall b' \in \mathcal{B}^{\mathbf{p}+\bar{v}} \quad g \restriction b' \in \Phi(b')$, then $\forall b \in \mathcal{B}^{\mathbf{p}} \quad g \restriction b \in \Psi(b)$. $\square_{\text{Claim 2.11}}$

(\mathbf{T})₄ So using the claim, replacing $u_{\delta,n}$ by any finite $u'_{\delta,n}$ satisfying $u_{\delta,n} \subseteq u'_{\delta,n} \subseteq \eta_\delta(n)$, and redefining Ψ as there it suffices to solve the derived special uniformization problem.

(\mathbf{T})₅ we can define by induction on $\delta \in S$, $n \in \omega$ the finite sets $u'_{\delta,n}$ (and the S -uniformization frame $\mathbf{p}_{+\bar{u}'}^0 = (S, \bar{\eta}, \bar{u}')$) satisfying $u_{\delta,n} \subseteq u'_{\delta,n}$ so that

- (a) $u'_{\delta,n} \cup \{\eta_\delta(n)\} \subseteq u'_{\delta,n+1}$,
- (b) $\delta = \bigcup_{n < \omega} u'_{\delta,n}$,

and the special S -uniformization problem $\mathbf{p}_{+\bar{u}'} = (S, \bar{\eta}, \bar{u}', \Psi)$ from Definition 2.10 it suffices to solve the problem $\mathbf{p}_{+\bar{u}'}$.

Observe that

(\mathbf{T})₆ if $u \subseteq \omega_1$ is \mathbf{p} -closed and $\alpha < \omega_1$ are given, then $u \cap \alpha$ is \mathbf{p} -closed.

Definition 2.12. Suppose that $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem, $\bar{m} = \langle m_\delta : \delta \in S \rangle$ is a sequence of natural numbers, define $\mathbf{p} \setminus \bar{m} = \mathbf{p}'$ as follows. Let

- ₁ the ladder system $\bar{\eta} - \bar{m} = \bar{\eta}'$ on S be defined as

$$\eta'_\delta(n) = \eta_\delta(n + m_\delta) \quad (\delta \in S, n \in \omega),$$

- ₂ and the system \bar{u}' is defined as

$$u'_{\delta,n} = u_{\delta,n+m_\delta} (\subseteq \eta_\delta(n+m_\delta) = \eta'_\delta(n)),$$

- ₃ and for each \mathbf{p}' -basic set $(b')^\delta_{n,k} = \bigcup_{j \leq n} (u'_{\delta,j} \cup \{\eta'_\delta(j)\}) \cup [\delta, \delta+k)$ (so here $(b')^\delta_{n,k}$ just equals the \mathbf{p} -basic set $b^\delta_{n+m_\delta,k}$), let $\Psi'((b')^\delta_{n,k}) = \Psi(b^\delta_{n+m_\delta,k})$.

Claim 2.13. *Suppose that $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem, and furthermore \bar{u} satisfies clauses a) and b) from Definition 2.8. Assume that $\bar{m} = \langle m_\delta : \delta \in S \rangle$ is a sequence of natural numbers. Then $\mathbf{p}' = \mathbf{p} \setminus \bar{m} = (S, \bar{\eta}', \bar{u}', \Psi')$ is a special S -uniformization problem, moreover, for every solution f of \mathbf{p}' , f is a solution of \mathbf{p} too.*

Proof. It is easy to see that a) and b) from Definition 2.8 imply $\mathcal{B}^{\mathbf{p}'} \subseteq \mathcal{B}^{\mathbf{p}}$, and $\Psi(b) = \Psi'(b)$ if $b \in \mathcal{B}^{\mathbf{p}'}$, so \square_b, \square_c automatically hold. Also each \mathbf{p}' -closed set w is \mathbf{p} -closed, and so

$$\begin{aligned} \Psi^+(w) &= \bigcap_{b \in \mathcal{B}^{\mathbf{p}'}, b \subseteq w} \{f : w \rightarrow C : f \upharpoonright b \in \Psi'(b)\} = \\ &= \bigcap_{b \in \mathcal{B}^{\mathbf{p}'}, b \subseteq w} \{f : w \rightarrow C : f \upharpoonright b \in \Psi(b)\}, \end{aligned}$$

and if $b \in \mathcal{B}^{\mathbf{p}} \setminus \mathcal{B}^{\mathbf{p}'}$, then $b = b^\delta_{n,k}$ for some $n < m_\delta$, $b \subseteq (b^\delta_{0,k})^{\mathbf{p}'}$, and if $b \subseteq w$, w is \mathbf{p}' -closed, then $(b^\delta_{m_\delta,k})^{\mathbf{p}} = (b^\delta_{0,k})^{\mathbf{p}'} \subseteq w$. Finally, since for any $f \in \Psi'((b^\delta_{0,k})^{\mathbf{p}'})$ we have $f \in \Psi((b^\delta_{m_\delta,k})^{\mathbf{p}})$, and $f \upharpoonright b \in \Psi(b)$ (by \square_c for \mathbf{p}). This easily implies that for any \mathbf{p}' -closed w , we have $\Psi^+(w) = (\Psi^+(w))^{\mathbf{p}'}$, and so \mathbf{p}' is indeed a special S -uniformization problem (similarly to the proof of Claim 2.11). (Note that \mathbf{p} -closed sets are not necessarily \mathbf{p}' -closed.)

For the moreover part observe that each old basic set $b^\delta_{n,k} = \bigcup_{j \leq n} (u_{\delta,j} \cup \{\eta_\delta(j)\}) \cup [\delta, \delta+k)$ is contained in the new basic set

$$(b^\delta_{\max(0,n-m_\delta),k})^{\mathbf{p}'} = \bigcup_{j \leq \max(0,n-m_\delta)} (u'_{\delta,j} \cup \{\eta'_\delta(j)\}) \cup [\delta, \delta+k),$$

and for each g with $g \in \Psi'((b^\delta_{\max(0,n-m_\delta),k})^{\mathbf{p}'})$ (since we have

$$\Psi'(b) = \Psi(b)^{\mathbf{p}} \text{ for } b \in \mathcal{B}^{\mathbf{p}'}$$

by definition) we have $g \upharpoonright (b^\delta_{n,k})^{\mathbf{p}} \in \Psi((b^\delta_{n,k})^{\mathbf{p}})$ as \square_c holds for \mathbf{p} .

□_{Claim2.13}

Definition 2.14. If $\mathbf{q} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a very special S -uniformization problem witnessed by $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$, then we call a set $X \subseteq \omega_1$ \bar{v} -closed, if X is \mathbf{q} -closed, and for each $\alpha \in X$ the inclusion $v_\alpha \subseteq X$ also holds.

- (\mathbf{T})₇ if the special S -uniformization problem $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is very special witnessed by $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$, then with some slight abuse of notation we may refer to also the tuple $(S, \bar{\eta}, \bar{v}, \Psi)$ as \mathbf{p} .

We summarize some useful facts about very special uniformization problems for future reference.

Fact 2.15. *If $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a very special uniformization problem, which fact is witnessed by $\bar{v} = \langle v_\gamma : \gamma < \omega_1 \rangle$, then*

$$(T)_8 \quad (a) \quad v_{\eta_\delta(n)} \cup \{\eta_\delta(n)\} \subseteq v_{\delta, n+1},$$

$$(b) \quad \delta = \bigcup_{n < \omega} v_{\eta_\delta(n)},$$

therefore

$$(c) \quad \text{ran}(\eta_\delta \upharpoonright n+1) \subseteq v_{\delta, n+1}$$

Also, $b_{n,k}^\delta = v_{\eta_\delta(n)} \cup \{\eta_\delta(n)\} \cup [\delta, \delta+k]$ (so $\mathcal{B}^p = \{b_{n,k}^\delta : \delta \in S, n, k \in \omega\}$)

$$(T)_9 \quad \text{for } x \in [\omega_1]^{<\aleph_0}$$

$$g \in \Psi^+(x) \iff [\forall \delta \in S \forall n, k : (b_{n,k}^\delta \subseteq x) \rightarrow (g \upharpoonright b_{n,k}^\delta \in \Psi(b_{n,k}^\delta))].$$

$$(T)_{10} \quad \text{for } x \subseteq \omega_1 \text{ is } \mathbf{p}\text{-closed, if}$$

(a) for each $\alpha \in x$, if $\alpha \in \text{ran}(\eta_\delta)$ for some $\delta \in S \cap x$, then $v_\alpha \subseteq x$,

(b) for each $\delta \in x$ either for some n $[\eta_\delta(n) \in x \wedge x \cap \delta \subseteq \eta_\delta(n+1)]$, or $\text{ran}(\eta_\delta) \subseteq x$,

(c) for each $\alpha + 1 \in x$: $\alpha \in x$.

$$(T)_{11} \quad \text{If } \bar{m} \in {}^S\omega, \text{ then the special } S\text{-uniformization problem } \mathbf{p}' = \mathbf{p} \setminus \bar{m} \text{ is also very special witnessed by the same } \bar{v}, \text{ and then}$$

- if $x \subseteq \omega_1$ is $(\mathbf{p} \setminus \bar{m}) - \bar{v}$ -closed, then it is $\mathbf{p} - \bar{v}$ -closed as well,
- for the basic sets we have

$$\mathcal{B}^{\mathbf{p} \setminus \bar{m}} = \{b_{n,k}^\delta \in \mathcal{B}^p : n \geq m_\delta, k \in \omega\},$$

$$\Psi^{\mathbf{p} \setminus \bar{m}} = \Psi^p \upharpoonright \mathcal{B}^{\mathbf{p} \setminus \bar{m}},$$

- hence

$$\text{if } x \in [\omega_1]^{<\aleph_0}, \text{ then } (\Psi^+)^{\mathbf{p} \setminus \bar{m}}(x) = (\Psi^+)^{\mathbf{p}}(x).$$

$$(T)_{12} \quad \text{If } \bar{v}' = \langle v'_\alpha : \alpha < \omega_1 \rangle \text{ is such that for each } \alpha \text{ } v_\alpha \subseteq v'_\alpha \subseteq \alpha \text{ and } v'_\alpha \text{ is finite, then letting } u'_{\delta,n} = v'_{\eta_\delta(n)} \text{ the special uniformization problem } \mathbf{p}_{+\bar{v}'} \text{ (as in Definition 2.10) is very special witnessed by } \bar{v}'.$$

Observe that Claim 2.13 and the following finishes the proof of Lemma 2.9.

Claim 2.16. *If $\mathbf{p} = (S, \bar{\eta}, \bar{u}, \Psi)$ is a special S -uniformization problem, \bar{u} satisfies a)-b) from Definition 2.8 (where $\bar{\eta}$ has \aleph_0 -uniformization), then for some $\bar{m} = \langle m_\delta : \delta \in S \rangle \in {}^S\omega$ the special S -uniformization problem $\mathbf{p} \setminus \bar{m}$ is very special.*

Proof. We have to find a sequence \bar{m} , such that for any α if $\alpha = \eta_\delta(n_0 + m_\delta) = \eta'_\delta(n_1 + m_{\delta'})$, then $u_{\delta, n_0} = u_{\delta', n_1}$. So define the functions c_δ for $\delta \in S$ by $c_\delta(\eta_\delta(n)) = u_{\delta, n}$. Again, as for any fixed $\alpha < \omega_1$

$$\{c_\delta(\alpha) : \delta \in S \wedge \alpha \in \text{ran}(\eta_\delta)\} \subseteq [\alpha]^{<\aleph_0}$$

there exists some function c^* uniformizing $\langle c_\delta : \delta \in S \rangle$ and \bar{m} with for each δ , $n \geq m_\delta$ the equality $c^*(\eta_\delta(n)) = c_\delta(\eta_\delta(n))$ holds, which is exactly what we wanted. $\square_{\text{Claim 2.16}}$

$\square_{\text{Lemma 2.9}}$

Definition 2.17. We call the very special S -uniformization problem $\mathbf{q} = (S, \bar{\eta}, \bar{v}, \Psi)$ *nice* if it satisfies that for every $\delta \in S$ $\text{ran}(\eta_\delta) \subseteq \Omega$, and for the system $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$: $\alpha \notin \Omega \rightarrow v_\alpha = \emptyset$.

Lemma 2.18. *Let $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ be a very special S -uniformization problem. If each ladder system $\bar{\eta}^* = \langle \eta_\delta^* : \delta \in S \rangle$ very similar to $\bar{\eta}$ has \aleph_0 -uniformization then there exists a very special S -uniformization problem $\mathbf{p}' = (S, \bar{\eta}', \bar{v}', \Psi')$, and a sequence of natural numbers $\bar{m} = \langle m_\delta : \delta \in S \rangle$, such that $\mathbf{p}'' = \mathbf{p}' \setminus \bar{m}$ is nice, and if \mathbf{p}'' has a solution, then so does \mathbf{p} .*

Proof.

Definition 2.19. Assume that E is an equivalence relation on ω_1 , satisfying that each equivalence class is a finite convex set (and $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ is a very special S -uniformization problem). We define $\mathbf{p}' = \mathbf{p}/E$ as follows.

Let $\{e_i : i < \omega_1\}$ list the equivalence classes in increasing order (hence $\delta \in e_\delta$ for δ limit ordinal $< \omega_1$); assume further $\bar{\eta}' = \langle \eta'_\delta : \delta \in S \rangle, \bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$ are defined as follows:

(α) η'_δ enumerates in increasing order

$$\{\alpha : (\exists n) \eta_\delta(n) \in e_\alpha\}$$

(so $\bar{\eta}' = \langle \eta'_\delta : \delta \in S \rangle$ is an S -ladder system),

(β) if $\alpha < \omega_1$, then

•₁ $v_\alpha^* = \{\beta < \alpha : \text{for some } \gamma \in e_\alpha \ e_\beta \cap v_\gamma \neq \emptyset\}$,

•₂ then $v'_\alpha = \text{cl}^{\bar{v}^*}(v_\alpha^*)$ (where $\bar{v}^* = \langle v_\alpha^* : \alpha < \omega_1 \rangle$),

(γ) Ψ' with $\text{dom}(\Psi') = \mathcal{B}^{(\mathbf{p}')}^0 = \{(b_{n,k}^\delta)'\} : \delta \in S, n, k \in \omega\}$ is defined as follows

$$\Psi'((b_{n,k}^\delta)') = \left\{ \begin{array}{l} f_g : \left(\text{dom}(g) = \bigcup_{\alpha \in (b_{n,k}^\delta)'} e_\alpha \right) \& \\ \left(\forall b \in \mathcal{B}^{\mathbf{p}} : b \subseteq \bigcup_{\alpha \in (b_{n,k}^\delta)'} e_\alpha \rightarrow g \upharpoonright b \in \Psi(b) \right) \end{array} \right\},$$

where for $g \in \Psi^+(\bigcup_{\alpha \in b} e_\alpha)$ we let $\text{dom}(f_g) = b$ and $f_g(\alpha) = \langle g(\beta_0), g(\beta_1), \dots, g(\beta_{j-1}) \rangle$

where $e_\alpha = \{\beta_0 < \beta_1 < \dots < \beta_{j-1}\}$,

(δ) $\mathbf{p}' = \mathbf{p}/(E, \bar{m}) = \mathbf{p}/(\{e_\alpha : \alpha < \omega_1\})$ is $(S, \bar{\eta}', \bar{v}', \Psi')$.

Claim 2.20. *Suppose that $\mathbf{p} = (S, \bar{\eta}, \bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle, \Psi)$ is a very special S -uniformization problem. Assume that E is an equivalence relation on ω_1 , satisfying that each equivalence class is a finite convex set. Then*

- (*)₁ $\mathbf{p}' = (S, \bar{\eta}', \bar{v}', \Psi')$ forms a very special S -uniformization problem (defined above in Definition 2.19),
- (*)₂ if f' is a solution for \mathbf{p}' then there is a solution for \mathbf{p} (which is defined naturally satisfying the equality $\langle f(\beta) : \beta \in e_\alpha \rangle = f'(\alpha)$),
- (*)₃ new closed sets essentially are old closed sets, that is, if $u' \subseteq \omega_1$ is \mathbf{p}' -closed (i.e. for $\bar{\eta}', \bar{v}'$) then $\bigcup \{e_\alpha : \alpha \in u'\}$ is \mathbf{p} -closed (i.e. for $\bar{\eta}, \bar{v}$),
- (*)₄ $\bar{\eta}', \bar{\eta}, \bar{\eta}^*$ are very similar (see Definition 1.31(4)),
- (*)₅ $\mathbf{p}' = (S, \bar{\eta}', \bar{v}', \Psi')$ satisfies
 - (a) for each $\delta \in S$: $\bigcup_{n \in \omega} v'_{\eta'_\delta(n)} = \delta$,
 - (b) for each $\delta \in S, n \in \omega$: $v'_{\eta'_\delta(n)} \cup \{\eta'_\delta(n)\} \subseteq v'_{\eta'_\delta(n+1)}$.

Proof. We first verify $(*)_3$. If x is \mathbf{p}' -closed, letting $x^e = \bigcup \{e_\alpha : \alpha \in x\}$, then it is enough to show that if $\delta \in x$ (equivalently, $\delta \in x^e$), and $\eta_\delta(k) \notin x^e$, then $(\eta_\delta(k), \delta) \cap x^e = \emptyset$, or else, $\eta_\delta(k) \in x^e$ implies $v_{\eta_\delta(k)} \subseteq x^e$.

First, if $\eta_\delta(k) \notin x^e$, then for the natural number n defined by $\eta_\delta(k) \in e_{\eta'_\delta(n)}$ clearly $\eta'_\delta(n) \notin x$, so for each $\alpha \in (\eta'_\delta(n), \delta)$, $\alpha \notin x$, which in turn implies that $[\min(e_{\eta'_\delta(n)}, \delta) \cap x^e = \emptyset$, so $(\eta_\delta(k), \delta) \cap x^e = \emptyset$. On the other hand, if $\eta_\delta(k) \in x^e$, then $\eta'_\delta(n) \in x$, so $v'_{\eta'_\delta(n)} \subseteq x$, in particular $v^*_{\eta'_\delta(n)} \subseteq x$, and so recalling (β) again $v_{\eta_\delta(k)} \subseteq x^e$.

Now we can turn to $(*)_5$, here we use that v'_α 's are defined in two steps (instead of just letting $v'_\alpha = v^*_\alpha$). So fix $\delta \in S$, $n \in \omega$, it is clear from the construction that $\eta'_\delta(n) \in v^*_{\eta'_\delta(n+1)} \subseteq v'_{\eta'_\delta(n+1)}$ (since $\eta_\delta(\ell) \in v_{\eta_\delta(k)}$ always holds if $\ell < k$ by $(\tau)_8$). For $v'_{\eta'_\delta(n)} = \text{cl}^{\bar{v}^*}(v^*_{\eta'_\delta(n)}) \subseteq v'_{\eta'_\delta(n+1)}$ we only have to argue that $v^*_{\eta'_\delta(n)} \subseteq v'_{\eta'_\delta(n+1)}$. But as we have already seen that $\eta'_\delta(n) \in v^*_{\eta'_\delta(n+1)}$, the \bar{v}^* -closure of $v^*_{\eta'_\delta(n+1)}$ must contain $v^*_{\eta'_\delta(n)}$.

We are ready to argue $(*)_1$. For \Box_a it can be easily seen that $\Psi'((b^\delta_{n,k})')$ can be defined by a single member of $\bigcup_{\ell < \omega} \Upsilon_\ell$, and the position of the (finitely many) \mathbf{p} -basic sets in $\bigcup_{\alpha \in (b^\delta_{n,k})'} e_\alpha$. It is straightforward to check that if $y = x^e = \bigcup_{\alpha \in x} e_\alpha$ and $g \in (\Psi^+)'(y)$, then $f_g \in \Psi^+(x)$. Moreover, if x is \mathbf{p}' -closed, then $y = x^e$ is \mathbf{p} -closed, and similarly to the proof of Claim 2.11 one can easily check that $f_g \in (\Psi^+)(y)$ is equivalent to $g \in (\Psi^+)'(x)$ (check that if some \mathbf{p} -basic set $b^\delta_{n^*,k^*} \subseteq x^e$ contributes to the restrictions (where n^*, k^*, n, k as in (γ)) then by the \mathbf{p}' -closedness of x there is a corresponding \mathbf{p}' -basic set $(b^\delta_{n,k})' \subseteq x$).

Finally, let $f' = f_g$ for some function g with domain ω_1 . It is straightforward to check that as for each $\delta \in S$, $n, k \in \omega$ $f' = f_g \upharpoonright (b^\delta_{n,k})' \in \Psi'((b^\delta_{n,k})')$, by the definition of Ψ' we have $g \upharpoonright b^\delta_{n^*,n^*} \in \Psi(b^\delta_{n^*,k^*})$ (where n^*, k^* as in (γ)), and for each $b^\delta_{n^*,k^*}$ there exists some $(b^\delta_{n',k'})'$ with $b^\delta_{n^*,k^*} \subseteq \bigcup_{\alpha \in (b^\delta_{n',k'})'} e_\alpha$, so we are done.

□_{Claim 2.20}

Applying Claim 2.20 we have the following (thus finishing the proof of Lemma 2.18):

- $(\tau)_{13}$ (if the very special $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ is given) for some E and \bar{m} the very special S -uniformization $\mathbf{p}'' = (\mathbf{p}/E) \setminus \bar{m} = (S, \bar{\eta}'', \bar{v}'', \Psi'')$ given by Definition 2.19 satisfies $\text{ran}(\eta''_\delta) \subseteq \Omega$, and $\alpha \notin \Omega \rightarrow v''_\alpha = \emptyset$.

Proof. $((\tau)_{13})$ Define for $\delta \in S$ the function $\eta'_\delta : \omega \rightarrow \delta$ listing

$$\{\alpha \in \Omega : [\alpha, \alpha + \omega) \cap \text{ran}(\eta_\delta) \neq \emptyset\}.$$

As each $\delta \in S$ is divisible by ω^2 according to our assumptions (i.e. S is simple we are proving Theorem 2.6) $\bar{\eta}' = \langle \eta'_\delta : \delta \in S \rangle$ is an S -ladder system, which is also very similar to $\bar{\eta}$, and hence to $\bar{\eta}^*$. For $\delta \in S$, define c_δ by $c_\delta(\eta'_\delta(n)) = \eta_\delta \upharpoonright (k_{\delta,n} + 1)$, where $k_{\delta,n} < \omega$ is the maximal k such that $\eta_\delta(k) < \eta_\delta(n) + \omega$. Note that the assumptions of $(\tau)_3$ hold (concerning $(*)$, $c_\delta(\alpha)$ is from ${}^\omega(\alpha + \omega)$ which is countable). So the conclusion of (α) of $(\tau)_3$ holds, say, witnessed by c . Let $\alpha E \beta$ iff $\alpha = \beta$ or $\alpha = \delta + k$, $\beta = \delta + \ell$ for some $\delta \in S$ and $k, \ell \in \omega$, and $c(\delta)$ is a finite sequence with last element in $[\alpha, \alpha + \omega) \cap [\beta, \beta + \omega)$. Then use Claim 2.20 to obtain the very special S -uniformization problem $\mathbf{p}' = \mathbf{p}/E$. So if $\mathbf{p}' = (S, \bar{\eta}', \bar{v}', \Psi')$, then for each $\delta \in S$, $\text{ran}(\eta'_\delta)$ contains only finitely many ordinals outside Ω , say,

the largest is at most the $m_\delta - 1$ 'th. Now apply $(\tau)_{11}$, letting $\mathbf{p}'' = \mathbf{p}' - \bar{m}$, and if $\mathbf{p}'' = (S, \eta'', \bar{v}'', \Psi'')$, where each $\text{ran}(\eta''_\delta)$ consists of only limit ordinals, then we can clearly assume that $v''_{\alpha+1} = \emptyset$ for $\alpha < \omega_1$, this does not change the set $\mathcal{B}^{\mathbf{p}''}$ of \mathbf{p}'' -basic sets or \mathbf{p}'' -closed sets (Definitions 1.28, 1.29), since $\text{ran}(\eta_\delta) \subseteq \Omega$, so it doesn't change the set of solutions. \square

$\square_{\text{Lemma 2.18}}$

We summarize the steps taken so far in the following lemma.

Lemma 2.21. *Suppose that S is a stationary set of limit ordinals which is simple, every ladder system $\bar{\eta}$ very similar to $\bar{\eta}^*$ has \aleph_0 -uniformization. For every special S -uniformization problem \mathbf{p} there exists a very special S -uniformization problem $\mathbf{q} = (S, \bar{\eta}^q, \bar{v}^q, \Psi^q)$ which is nice, and every solution to \mathbf{q} is a solution to \mathbf{p} , too, and every ladder system very similar to $\bar{\eta}^q$ has \aleph_0 -uniformization.*

Proof. By Claim 2.11 and $(\tau)_5$ we reduce the problem to solving the special uniformization problem $\mathbf{p}_{+\bar{u}'} = (S, \bar{\eta}, \bar{u}', \Psi)$, where \bar{u}' satisfies a), b) from Definition 2.8. Then Claim 2.16 yields $\mathbf{p}'' = (S, \bar{\eta}'', \bar{v}'', \Psi'')$, a very special S -uniformization problem on the ladder system $\bar{\eta}'' = \bar{\eta} \setminus \bar{m}$ for some \bar{m} , which is very similar to $\bar{\eta}$.

Finally, applying Lemma 2.18 to \mathbf{p}'' , it reduces our problem to solving a very special S -uniformization problem $\mathbf{p}''' = (S, \bar{\eta}''', \bar{v}''', \Psi''')$, where $\bar{\eta}'''$ is very similar to $\bar{\eta}''$ (and satisfies that for each $\delta \in S$ $\text{ran}(\eta_\delta)$ contains finitely many ordinals outside Ω , as is seen from the following). Moreover, for some \bar{m} , $\mathbf{p}''' \setminus \bar{m}$ is a very special S -uniformization problem on $\bar{\eta}^{(4)} = \bar{\eta}''' \setminus \bar{m}$ which is nice, and $\bar{\eta}^{(4)}$ being a finite modification of $\bar{\eta}'''$, is very similar to $\bar{\eta}'''$ (so $\mathbf{q} = \mathbf{p}''' \setminus \bar{m}$ works). $\square_{\text{Lemma 2.21}}$

Fact 2.22. *If the nice S -uniformization problem $\mathbf{q} = (S, \bar{\eta}, \bar{v}, \Psi)$ and $\langle m_\delta : \delta \in S \rangle$ are given, then $\mathbf{q} \setminus \bar{m}$ is also nice, each solution f of the latter is solution for \mathbf{q} as well.*

Lemma 2.23. *Let $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ be a very special S -uniformization problem, which is nice, and the S -ladder system $\bar{\eta} = \langle \eta_\delta : \delta \in S \rangle$ has \aleph_0 -uniformization. Then \mathbf{p} has a solution.*

Proof. In this finishing lemma we will not introduce new types of uniformization problems, but we still need to remove (for each $\delta \in S$) proper initial segments of $\text{ran}(\eta_\delta)$ (i.e. appealing to Fact 2.22) finitely many times, obtaining nicer and nicer systems (Claim 2.26, Lemma 2.28, Lemma 2.34). The consecutive refinements (applications of Fact 2.22) may ruin earlier steps, for example the system \bar{v} in the following claim, or the reader may think of the difference between \mathbf{p} -closedness and $\mathbf{p} - \bar{m}$ -closedness, so we will have to do it carefully. However, finally after Lemma 2.34 we will not replace \mathbf{p} by $\mathbf{p} - \bar{m}$, instead, in Claim 2.37 we will construct a solution for \mathbf{p} using $\mathbf{p} - \bar{m}$ -“histories” for \bar{m} given by Lemma 2.34 (see below in Definition 2.31).

After Claim 2.37 we will put the pieces together to complete the proof of the present lemma (Lemma 2.23).

Claim 2.24. *Suppose that $\bar{\eta}$ is a ladder system on S with \aleph_0 -uniformization. Then there exists $\bar{v} = \langle v_\alpha : \alpha < \omega_1 \rangle$ and ladder system $\bar{\eta}'$ on S such that*

a) $\bar{\eta}' = \bar{\eta} - \bar{m}$ for some $\bar{m} \in {}^S\omega$, i.e. for each $\delta \in S$

$$\eta'_\delta(n) = \eta_\delta(n + m_\delta),$$

- b) for each $\alpha < \omega_1$, ν_α is a finite strictly increasing sequence of ordinals $< \alpha$,
 c) for $\delta \in S$, $n < \omega$ we have

$$\nu_{\eta_\delta(n)} = \eta'_\delta \upharpoonright n,$$

Proof. For each $\delta \in S$ define

$$c_\delta(\eta_\delta(n)) = \eta_\delta \upharpoonright n,$$

(note that for each α $\{c_\delta(\alpha) : \delta \in S\} \subseteq {}^{<\omega}\alpha$, so is countable), let $c : \omega_1 \rightarrow {}^{<\omega}\omega_1$ uniformize c_δ . Finally define $\nu_\alpha^0 = c(\alpha)$, so w.l.o.g. for each $\alpha < \omega_1$ $\nu_\alpha^0 = \langle \nu_\alpha^0(0), \nu_\alpha^0(1), \dots, \nu_\alpha^0(|\nu_\alpha^0| - 1) \rangle \in {}^{<\omega}\alpha$ and let

$$k_\alpha = \max \left(\{n < |\nu_\alpha^0| : \nu_{\nu_\alpha^0(n)}^0 \neq \nu_\alpha^0 \upharpoonright n\} \cup \{-1\} \right) + 1,$$

and let

$$m_\delta = \min \{m \in \omega : (\forall \ell \geq m) \eta_\delta \upharpoonright \ell = \nu_{\eta_\delta(\ell)}^0\}.$$

Now observe that for each $\delta \in S$, if $\ell \geq m_\delta$, then $k_{\eta_\delta(\ell)} = m_\delta$, so letting

$$\nu_\alpha = \langle \nu_\alpha^0(k_\alpha), \nu_\alpha^0(k_\alpha + 1), \dots, \nu_\alpha^0(|\nu_\alpha^0| - 1) \rangle \in {}^{|\nu_\alpha^0| - k_\alpha}\alpha,$$

$$\eta'_\delta(n) = \eta_\delta(n + m_\delta)$$

works. □_{Claim 2.24}

Definition 2.25. Assume that $\bar{\eta}$ is a ladder system on the stationary set S . We say that the system of sets $\langle A_n : n \in \omega \rangle$ is a $\bar{\eta}$ -rank, if the following hold:

- (\star)_a $\langle A_n : n \in \omega \rangle$ is pairwise disjoint, covers $S \cup (\cup \{\text{ran}(\eta_\delta) : \delta \in S\})$,
- (\star)_b if $\delta \in S \cap A_n$, then $\text{ran}(\eta_\delta) \subseteq A_{n+1}$,

Observe that

- (\top)₁₄ if \bar{A} is an $\bar{\eta}$ -rank for the ladder system $\bar{\eta}$, and $\langle m_\delta : \delta \in S \rangle \in {}^S\omega$, then \bar{A} is also an $\bar{\eta} - \bar{m}$ -rank.

Claim 2.26. Suppose that $\bar{\eta}$ is a ladder system on S with \aleph_0 -uniformization. Then there exists $\bar{m} = \langle m_\delta : \delta \in S \rangle$ and $\bar{A} = \langle A_n : n \in \omega \rangle$ such that letting $\bar{\eta}' = \bar{\eta} - \bar{m}$ (from Definition 2.12) \bar{A} is a $\bar{\eta}'$ -rank.

Proof. First apply Claim 2.24, so (after possibly replacing η with $\eta - \bar{m}^*$ for some \bar{m}^*) we can assume that there exists $\langle \nu_\alpha : \alpha < \omega_1 \rangle$ such that

$$\forall \delta \in S, n \in \omega : \nu_{\eta_\delta(n)} = \eta_\delta \upharpoonright n.$$

Define for $\delta \in S$ the function $c_\delta : \text{ran}(\eta_\delta) \rightarrow \omega$ as

$$c_\delta(\eta_\delta(n)) = |\nu_\delta|,$$

let $c^* : \omega_1 \rightarrow \omega$ uniformize $\langle c_\delta : \delta \in S \rangle$. Now define the function $g : \omega_1 \rightarrow \omega$ by $g(\alpha) = \min(c^*(\alpha) + 1, |\nu_\alpha|)$, (note that

$$(2.8) \quad g(\alpha) \leq |\nu_\alpha| \quad (\alpha \in \omega_1),$$

and choose d_δ so that

$$(2.9) \quad m \geq d_\delta \rightarrow [g(\eta_\delta(m)) = c^*(\eta_\delta(m)) + 1 = |\nu_\delta| + 1] \quad (> |\nu_\delta| \geq g(\delta)).$$

Let $B_k = \{\alpha < \omega_1 : g(\alpha) = k\}$. So (2.9) and (2.8) together imply that for each δ

- $\delta \in B_{g(k)}$,
- while $\text{ran}(\eta_\delta \upharpoonright [d_\delta, \infty)) \subseteq B_k$ for some $k > g(\delta)$,

- so we let $C_k = \{\delta \in S : \text{ran}(\eta_\delta \upharpoonright [d_\delta, \infty)) \subseteq B_k\}$, which is a partition of S , and

$$(\forall k \in \omega) C_k \subseteq \bigcup_{j < k} B_j.$$

Now we construct the A_i 's as follows in the form of $A_i = f^{-1}(i)$ for a suitable function f . We define $f \upharpoonright \bigcup_{j \leq i} B_j, \langle m_\delta : \delta \in (\bigcup_{j \leq i} C_j) \rangle$ by induction on i so that for each $i \in \omega$

(γ) for each $\delta \in \bigcup_{j \leq i} C_j$ we have

$$\forall m \geq m_\delta : f(\eta_\delta(m)) = f(\delta) + 1.$$

So suppose that $f \upharpoonright \bigcup_{j \leq i} B_j, \langle m_\delta : \delta \in (\bigcup_{j \leq i} C_j) \rangle$ are already constructed. For each $\delta \in S \cap (\bigcup_{j \leq i} B_j)$ with $\text{ran}(\eta_\delta \upharpoonright [d_\delta, \infty)) \subseteq B_{i+1}$ define the colouring $c'_\delta : \text{ran}(\eta_\delta) \rightarrow \omega$ as $c'_\delta(\eta_\delta(n)) = f(\delta)$. Now if c' uniformizes the c'_δ 's, then let $f \upharpoonright B_{i+1} = c' \upharpoonright B_{i+1}$, and choose for each $\delta \in S$ a suitable m_δ . $\square_{\text{Claim 2.26}}$

Definition 2.27. Let $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ be a very special S -uniformization problem. We call the system $\bar{w} = \langle w_\alpha : \alpha < \omega_1 \rangle$ \mathbf{p} -pressed down, if

- ₁ for each α , $w_\alpha \subseteq \alpha$ is \mathbf{p} -closed,
- ₂ for each $\delta \in S$, n we have $v_{\eta_\delta(n)} \subseteq w_{\eta_\delta(n)} \subseteq w_{\eta_\delta(n+1)}$, (in particular $\text{ran}(\eta_\delta \upharpoonright n) \subseteq w_{\eta_\delta(n)}$)
- ₃ for each $\delta \in S$, $n \in \omega$: $w_\delta \subseteq w_{\eta_\delta(n)}$.

Lemma 2.28. Let $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ very special S -uniformization problem with $\bar{\eta}$ admitting \aleph_0 -uniformization, the $\bar{\eta}$ -rank $\langle A_n : n \in \omega \rangle$ given by Lemma 2.28. Then there exists $\bar{m} = \langle m_\delta : \delta \in S \rangle$ and a $(\mathbf{p} \setminus \bar{m})$ -pressed down system $\bar{w} = \langle w_\alpha : \alpha < \omega_1 \rangle$. (And niceness is of course preserved if \mathbf{p} is nice.)

Proof. We can apply Claim 2.26 and so remove an initial segment from each $\text{ran}(\eta_\delta)$. First we construct $\bar{m} = \langle m_\delta : \delta \in S \rangle, \bar{w}^{(0)} = \langle w_\alpha^{(0)} : \alpha < \omega_1 \rangle$ with

- ₁ for each $\delta \in S$, $n \geq m_\delta$: $v_{\eta_\delta(n)} \subseteq w_{\eta_\delta(n)}^{(0)}$,
- ₂ for each $\delta \in S$, $n \geq m_\delta$: $w_{\eta_\delta(n)}^{(0)} = w_\delta^{(0)} \cup v_{\eta_\delta(n)}$.

We are going to construct $\langle w_\delta^{(0)} : \delta \in A_i \rangle$ by induction on i . If $\gamma \in A_0$, then $\gamma \notin \text{ran}(\eta_\delta)$ for any $\delta \in S$, we can let $w_\gamma^{(0)} = v_\gamma \subseteq \gamma$.

Assuming that w_γ is constructed for each $\gamma \in \bigcup_{j \leq i} A_j$, and m_δ for $\delta \in \bigcup_{j < i} A_j$ note that if $\gamma \in A_{i+1} \cap \text{ran}(\eta_\delta)$, then $\delta \in A_i$ by our assumptions on the A_n 's $(\star)_b$ in Claim 2.26). For $\delta \in A_i \cap S$ define $c_\delta : \text{ran}(\eta_\delta) \rightarrow [\omega_1]^{<\omega}$,

$$c_\delta(\eta_\delta(n)) = \begin{cases} w_\delta^{(0)} \cup v_{\eta_\delta(n)} & \text{if } w_\delta^{(0)} \subseteq \eta_\delta(n), \\ v_{\eta_\delta(n)}, & \text{otherwise,} \end{cases}$$

note that $v_\alpha \subseteq \alpha$ by the definition of very special uniformization problems (Definition 2.8), so $w_\delta \subseteq \eta_\delta(n)$ implies that $c_\delta(\eta_\delta(n)) \in [\eta_\delta(n)]^{<\aleph_0}$ (since $v_{\eta_\delta(n)} \subseteq \eta_\delta(n)$), thus we can find a function c^* with $\text{dom}(c^*) = A_{i+1}$ uniformizing the c_δ 's. W.l.o.g. we can assume that $c^*(\gamma) \supseteq v_\gamma$, set $w_\gamma^{(0)} = c^*(\gamma)$. Choose m_δ so that $n \geq m_\delta$ implies $c_\delta(\eta_\delta(n)) = c^*(\eta_\delta(n))$ and $w_\delta^{(0)} \cup v_{\eta_\delta(n)} = c^*(\eta_\delta(n))$ (thus $w_{\eta_\delta(n)}^{(0)} = c^*(\eta_\delta(n)) = w_\delta^{(0)} \cup v_{\eta_\delta(n)}$ if $n \geq m_\delta$).

Once we defined the entire sequences \bar{m} , $\bar{w}^{(0)}$ it is straightforward to check that defining $w_\alpha = \text{cl}^{\mathbf{p}} \setminus \bar{m}(w_\alpha^{(0)})$ works (since $v_\gamma \subseteq w_\gamma^{(0)}$ for $\gamma \in \omega_1$, and for each δ , $n \geq m_\delta$, we have $w_{\eta_\delta(n)}^{(0)} = w_\delta^{(0)} \cup v_{\eta_\delta(n)}$, so $w_{\eta_\delta(n)}^{(0)} \supseteq w_\delta^{(0)}$).

□_{Lemma 2.28}

Observe that if we have a very special S -uniformization problem $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$, and thus obtain the $\bar{\eta}$ -rank $\langle A_i : i \in \omega \rangle$, and the \mathbf{p} -pressed down system \bar{w} then it can be easily seen by induction on k that

(T)₁₅ for any $\delta_0, \delta_1, \dots, \delta_k$, and n_1, n_2, \dots, n_k with $\delta_{i+1} = \eta_{\delta_i}(n_{i+1})$ we have

$$w_{\delta_0} \subseteq w_{\delta_k} \text{ \& } v_{\delta_0} \subseteq w_{\delta_k}.$$

Recall that if $\eta_\delta(n) = \delta^*$, then $\text{ran}(\eta_\delta \upharpoonright n) \subseteq v_{\delta^*}$ by (c) from (T)₈, so

(T)₁₆ if we fix any $\delta_0, \delta_1, \dots, \delta_\ell$, and n_1, n_2, \dots, n_ℓ with $\delta_{i+1} = \eta_{\delta_i}(n_{i+1})$, then for any $j < \ell$ $w_{\delta_\ell} \supseteq \text{ran}(\eta_{\delta_j} \upharpoonright n_{j+1})$, in particular $\delta_\ell > \eta_{\delta_j}(n_{j+1} - 1)$, so

$$(\forall j < \ell) \quad \delta_\ell \in (\eta_{\delta_j}(n_{j+1} - 1), \eta_{\delta_j}(n_{j+1})).$$

Definition 2.29. If $\bar{\eta}$ is a ladder system on a stationary set S , then the sequence $\langle h_\alpha : \alpha \in \omega_1 \rangle \in {}^\omega \omega$ is permitted (for $\bar{\eta}$), if for each $\delta \in S$ the sequence $h_{\eta_\delta(n)}$ ($n < \omega$) is strictly increasing.

Claim 2.30. If $\bar{\eta}$ is a ladder system on a stationary set S admitting \aleph_0 -uniformization, then there exist sequences $\bar{m} = \langle m_\delta : \delta \in S \rangle \in {}^S \omega$, $\bar{h} = \langle h_\alpha : \alpha \in \omega_1 \rangle \in {}^\omega \omega$, such that \bar{h} is permitted for $\bar{\eta}' = \bar{\eta} - \bar{m}$.

Proof. The proof is an easy application of \aleph_0 -uniformization, and it is left to the reader. □

Definition 2.31. Let $\bar{\eta}$ be a ladder system on the stationary set S .

- 1) We say that $\bar{\delta} = (\delta_i, \dots, \delta_{i+\ell})$ is a $\bar{\eta}$ -history, if $i \in \mathbb{Z}$, $\ell > 0$, and for each $0 \leq k < \ell$: $\delta_{i+k+1} = \eta_{\delta_{i+k}}(n_{i+k+1})$ for some $n_{i+k+1} \in \omega$.
- 2) We let $\bar{n}^{\bar{\delta}} = \langle n_{i+k+1} : k < \ell \rangle$ denote this sequence associated to $\bar{\delta}$.
- 3) We call the $\bar{\eta}$ -history $\bar{\delta} = (\delta_i, \dots, \delta_{i+\ell})$ maximal, if there is no $\bar{\eta}$ -history $\bar{\delta}^*$ properly extending $\bar{\delta}$ (i.e. no suitable δ_{i-1} , nor $\delta_{i+\ell+1}$ does exist),
- 4) We call the $\bar{\eta}$ -history $\bar{\delta} = (\delta_i, \dots, \delta_{i+\ell})$ long, if there is no δ_{i-1} such that $(\delta_{i-1}, \delta_i, \dots, \delta_{i+\ell})$ is a $\bar{\eta}$ -history.

Observe that

(T)₁₇ if $\bar{\eta}$ is a ladder system on S , $\langle A_n : n \in \omega \rangle$ is a $\bar{\eta}$ -rank (Definition 2.25), then for each $\bar{\eta}$ -history $\bar{\delta} = (\delta_i, \dots, \delta_{i+\ell})$ necessarily $\delta_{i+k} \in A_{j+k}$ for some $j \in \omega$ for each $k \leq \ell$.

Moreover, the following also holds.

Claim 2.32. Assume that

- (A) $\bar{\eta}$ is a ladder system on S , $\langle A_n : n \in \omega \rangle$ is a $\bar{\eta}$ -rank (Definition 2.25),
- (B) $\mathbf{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ is a very special S -uniformization problem that is nice,
- (C) $\bar{w} = \langle w_\alpha : \alpha < \omega_1 \rangle$ is a \mathbf{p} -pressed down system,

then

- (1) for any $\delta < \omega_1$ the set w_δ is \mathbf{p} -closed,

- (2) and for any maximal $\bar{\eta}$ -history $\bar{\delta} = \langle \delta_i, \delta_{i+1}, \dots, \delta_{i+\ell} \rangle$ and for any choice of $t_{i+k} \in \omega$ ($k < \ell$) the set

$$w^* = w_{\delta_\ell} \cup \cup \{[\delta_k, t_k] : k \in [i, i + \ell]\}$$

is \mathfrak{p} -closed.

Proof. The fact that w_δ is \mathfrak{p} -closed is just part of the definition of being \mathfrak{p} -pressed down. Second, use that $\text{ran}(\eta_\delta) \subseteq \Omega$ (by $(\mathfrak{T})_{13}$ as \mathfrak{p} is nice), while for a fixed history $\bar{\delta} = \langle \delta_i, \delta_{i+1}, \dots, \delta_{i+\ell} \rangle$, $v_{\delta_{i+k}} \subseteq w_{\delta_{i+k}} \subseteq w_{\delta_{i+\ell}}$, so

$$v_{\delta_{i+k+1}} \cup \{\delta_{i+k+1}\} \subseteq w^* \cap \delta_{i+k} \subseteq \eta_{\delta_{i+k}}(n_{i+k+1} + 1) \\ \text{(where } \delta_{i+k+1} = \eta_{\delta_{i+k}}(n_{i+k+1}))$$

implies that $w^* = w_{\delta_\ell} \cup \cup \{[\delta_k, t_k] : k \in [i, i + \ell]\}$ is indeed \mathfrak{p} -closed. $\square_{\text{Claim 2.32}}$

Definition 2.33. Let $\bar{\eta}$ be a ladder system on the stationary set S , assume that $\bar{\delta} = \langle \delta_i, \dots, \delta_{i+\ell} \rangle$ is an $\bar{\eta}$ -history, and $\langle h_\alpha : \alpha < \omega_1 \rangle$ is permitted for $\bar{\eta}$, assume that $\langle w_\alpha : \alpha < \omega_1 \rangle$ is given. Then we let

$$w_{\bar{\delta}}^* = w_{\bar{\delta}}^*(\bar{\eta}, \bar{h}, \bar{w}) = w_{\delta_{i+\ell}} \cup (\cup \{[\delta_k, \delta_k + h_{\delta_{k+1}}] : k \in [i, i + \ell]\}).$$

Lemma 2.34. Assume that

- (A) $\bar{\eta}$ is a ladder system on the stationary set S (with \aleph_0 -uniformization on $\bar{\eta}$), admitting the rank \bar{A} , and $\langle h_\alpha : \alpha < \omega_1 \rangle$ is permitted.
- (B) $\mathfrak{p} = (S, \bar{\eta}, \bar{v}, \Psi)$ is a very special S -uniformization problem that is nice, $\bar{w} = \langle w_\alpha : \alpha < \omega_1 \rangle$ forms a \mathfrak{p} -pressed down system.

Then there exists $\langle \psi_\alpha : \alpha < \omega_1 \rangle$, $\langle \nu_\alpha^* : \alpha < \omega_1 \rangle$, $\langle h_\alpha^* : \alpha < \omega_1 \rangle$, $\langle p_\alpha : \alpha < \omega_1 \rangle$, $\bar{m} = \langle m_\alpha : \alpha \in S \rangle$, $\langle \nu_\alpha^{**} : \alpha < \omega_1 \rangle$ such that letting $\bar{\eta}' = \bar{\eta} - \bar{m}$ the following holds for every $\gamma < \omega_1$:

- (\blacktriangle)₁ if $\gamma = \eta'_\delta(n)$ for some $\delta \in S$, then $\psi_\gamma \in \bigcup_{k \in \omega} (\Upsilon^+)_k^{\mathfrak{p}}$ (from Claim 1.30),
- (\blacktriangle)₂ for every long $\bar{\eta}' = \bar{\eta} - \bar{m}$ -history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_\ell = \gamma \rangle$, the sequence $h_\delta^* \in {}^{\omega > \omega}$ satisfies

$$h_\delta^* = \langle h_{\delta_1}, \dots, h_{\delta_\ell} = h_\gamma \rangle,$$

and if $|w_\delta^*| = k$, then

$$\psi_\gamma = \{g \circ OP_{w_\delta^*, k} : g \in (\Psi^+)^{\mathfrak{p}}(w_\delta^*)\} \in (\Upsilon_k^+)^{\mathfrak{p}}$$

(here we insist on $(\Psi^+)^{\mathfrak{p}}$ for technical reasons, and do not write $(\Psi^+)^{\mathfrak{p} \setminus \bar{m}}$ instead, which would not be the same, as w_δ^* may not be $\mathfrak{p} - \bar{m}$ -closed),

- (\blacktriangle)₃ for each $\delta \in S$ with $\eta'_\delta(n) = \gamma$ for some n : $\nu_\gamma^* = \eta'_\delta \upharpoonright n \in {}^n(\eta'_\delta(n))$.
- (\blacktriangle)₄ $\nu_\gamma^{**} = \langle \nu_{\gamma, k}^{**} : k < \ell \rangle \in {}^{<\omega}({}^{<\omega}\gamma)$, where for every long $\bar{\eta}' = \bar{\eta} - \bar{m}$ -history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_j = \gamma \rangle$ we have $j = \ell$ (i.e. $\ell g(\bar{\delta}) = \ell + 1 = \ell g(\nu_\gamma^{**}) + 1$), and

$$(\forall k < \ell) : \nu_{\gamma, k}^{**} = \nu_{\delta_k}^*,$$

(so if $\delta_k = \eta_{\delta_{k-1}}(n)$, $k < \ell$ then $\nu_{\gamma, k}^{**} = \eta'_{\delta_{k-1}} \upharpoonright n$),

- (\blacktriangle)₅ $p_\gamma \in {}^\ell\{0, 1\}$, where for every long $\bar{\eta}'$ -history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_j = \gamma \rangle$ we have $j = \ell$ (i.e. $p_\gamma = \langle p_\gamma(0), p_\gamma(1), \dots, p_\gamma(j-1) \rangle$, so $\ell g(\bar{\delta}) = \ell g(p_\gamma) + 1$), moreover for $0 < k < \ell$

$$p_\gamma(k) = 1 \iff (\exists \delta' \neq \delta'') : \delta_k \in \text{ran}(\eta'_{\delta'}) \cap \text{ran}(\eta'_{\delta''}).$$

Proof. We will proceed by induction on $i < \omega$, and define

- (α) $\langle m_\delta : \delta \in (S \cap \bigcup_{j < i} A_j) \rangle$,
 (β) $\langle \psi_\alpha : \alpha \in \bigcup_{j \leq i} A_j \rangle, \langle \nu_\alpha^* : \alpha \in \bigcup_{j \leq i} A_j \rangle, \langle h_\alpha^* : \alpha \in \bigcup_{j \leq i} A_j \rangle, \langle p_\alpha : \alpha \in \bigcup_{j \leq i} A_j \rangle$.

Recall that for each $\varepsilon \in A_i$ if $\varepsilon = \eta_\delta(n)$ then $\delta \in A_{i-1}$ by Definition 2.25, therefore if $\varepsilon \in A_0$, then $\langle \varepsilon \rangle$ is a long history, let $h_\varepsilon^* = \nu_\varepsilon^* = \nu^{**} = \langle \rangle$ (the empty sequence) and if $k = |w_{\langle \varepsilon \rangle}^*|$, then simply let

$$\psi_\varepsilon = \{g \circ \text{OP}_{w_{\langle \varepsilon \rangle}^*, k} : g \in (\Psi^+)^{\mathbb{P}}(w_{\langle \varepsilon \rangle}^*)\}.$$

Suppose that $i \geq 0$ is fixed, and the objects in (α), (β) are already defined, so if $\gamma \in \bigcup_{j \leq i} A_j$, then $h_\gamma^*, \psi_\gamma, \nu_\gamma^*, \nu_\gamma^{**}, p_\gamma$ satisfy clauses $(\blacktriangle)_1$ – $(\blacktriangle)_5$ (recall that under $\eta'_\delta(n)$ we mean $\eta_\delta(n + m_\delta)$, note that m_δ is defined if $\delta \in \bigcup_{j < i} A_j$).

First we choose $\langle \nu_\gamma^{**}, \psi_\gamma, h_\gamma^*, p_\gamma : \gamma \in A_{i+1} \cap (\bigcup_{\delta \in (S \cap A_i)} \text{ran}(\eta_\delta)) \rangle$, then we are able to define $\langle \nu_\gamma^* : \gamma \in A_{i+1} \rangle, \langle m_\delta : \delta \in A_i \rangle$, which determines $\bar{\eta}' \upharpoonright A_i = (\bar{\eta} - \bar{m}) \upharpoonright A_i$ (note that there may be some

$$\gamma \in A_{i+1} \cap \left(\bigcup_{\delta \in (S \cap A_i)} \text{ran}(\eta_\delta) \right) \setminus \left(\bigcup_{\delta \in (S \cap A_i)} \text{ran}(\eta'_\delta) \right).$$

Finally we can choose $\langle \nu_\gamma^{**}, \psi_\gamma, h_\gamma^*, p_\gamma : \gamma \in A_{i+1} \setminus (\bigcup_{\delta \in (S \cap A_i)} \text{ran}(\eta'_\delta)) \rangle$, similarly to the case of $\gamma \in A_0$.

Before the induction step we have to observe the following corollaries of the induction hypothesis $(\blacktriangle)_2$:

- (\star_1) if $\varepsilon \in \bigcup_{j \leq i} A_j$, then for the long $\bar{\eta}'$ -histories $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_\ell = \varepsilon \rangle, \bar{\delta}' = \langle \delta'_0, \delta'_1, \dots, \delta'_{\ell'} = \varepsilon \rangle$ we always have

$$h_\varepsilon^* = \langle h_{\delta_0}, h_{\delta_1}, \dots, h_{\delta_\ell} (= h_\varepsilon) \rangle = \langle h_{\delta'_0}, h_{\delta'_1}, \dots, h_{\delta'_{\ell'}} (= h_\varepsilon) \rangle,$$

in particular

$$\ell = \ell',$$

moreover,

- (\star_2) if $\varepsilon \in \bigcup_{j \leq i} A_j$, $\bar{\delta}, \bar{\delta}'$ are as above in (\star_1), $n \in \omega$, then

$$w_{\bar{\delta} \frown \langle \eta_\varepsilon(n) \rangle}^* \cap \varepsilon = w_{\bar{\delta}' \frown \langle \eta_\varepsilon(n) \rangle}^* \cap \varepsilon$$

as $w_{\delta_i} \subseteq w_{\eta_{\delta_i}(n)}$ by Definition 2.27 (even

$$(2.10) \quad w_{\bar{\delta} \frown \langle \eta_\varepsilon(n) \rangle}^* \cap (\varepsilon + \omega) = w_{\bar{\delta}' \frown \langle \eta_\varepsilon(n) \rangle}^* \cap (\varepsilon + \omega)$$

is true). Moreover,

$$w_{\bar{\delta} \frown \langle \eta_\varepsilon(n) \rangle}^* \setminus w_{\bar{\delta}}^* = w_{\bar{\delta}' \frown \langle \eta_\varepsilon(n) \rangle}^* \setminus w_{\bar{\delta}'}^* =$$

$$= [\varepsilon + 1, \varepsilon + h_{\eta_\varepsilon(n)}] \cup \{\eta_\varepsilon(n)\} \cup (w_{\eta_\varepsilon(n)} \setminus w_\varepsilon)$$

(since $\delta_0 > \delta_1 > \dots > \delta_\ell = \varepsilon > \eta_\varepsilon(n)$, and $\delta'_0 > \delta'_1 > \dots > \delta'_{\ell'} = \varepsilon > \eta_\varepsilon(n)$, and $w_\alpha \subseteq \alpha$ by \bullet_1 in Definition 2.27), in particular,

$$(w_{\bar{\delta} \frown \langle \eta_\varepsilon(n) \rangle}^* \setminus w_{\bar{\delta}}^*) \cap \eta_\varepsilon(n) = (w_{\bar{\delta}' \frown \langle \eta_\varepsilon(n) \rangle}^* \setminus w_{\bar{\delta}'}^*) \cap \eta_\varepsilon(n) = w_{\eta_\varepsilon(n)} \setminus w_\varepsilon,$$

which easily implies that

(★₃) if $\varepsilon \in \bigcup_{j \leq i} A_j$, $\bar{\delta}$, $\bar{\delta}'$ are as above in (★₁), $n \in \omega$, then for any $k \leq \ell$

$$(\text{OP}_{w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*, w_{\bar{\delta}' \cap \langle \eta_\varepsilon(n) \rangle}})[\delta'_k, \delta'_k + h_{\delta'_{k+1}}] = [\delta_k, \delta_k + h_{\delta_{k+1}}],$$

and of course

$$\text{OP}_{w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*, w_{\bar{\delta}' \cap \langle \eta_\varepsilon(n) \rangle}}(\alpha) = \alpha, \quad \text{if } \alpha \leq \varepsilon,$$

hence

$$(2.11) \quad \text{OP}_{w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*, w_{\bar{\delta}' \cap \langle \eta_\varepsilon(n) \rangle}} \upharpoonright w_{\bar{\delta}'}^* = \text{OP}_{w_{\bar{\delta}}^*, w_{\bar{\delta}'}^*}.$$

Now fix $\varepsilon \in A_i$, define the functions $c_\varepsilon^{h^*}, c_\varepsilon^\psi, c_\varepsilon^p$ with domain $\text{ran}(\eta_\varepsilon)$ as follows.

- _a $c_\varepsilon^{h^*}(\eta_\varepsilon(n)) = h_\varepsilon^* \cap \langle h_{\eta_\varepsilon(n)} \rangle \in {}^{<\omega}\omega$,
- _b $c_\varepsilon^{\psi^*}(\eta_\varepsilon(n)) = \nu_\varepsilon^{**} \cap \langle \nu_\varepsilon^* \rangle$,
- _c for $c_\varepsilon^\psi : \text{ran}(\eta_\varepsilon) \rightarrow \bigcup_{k \leq \omega} (\Upsilon_k^+)^p$ fix a long $\bar{\eta}' \upharpoonright (S \cap \bigcup_{j < i} A_j) = (\bar{\eta} - \bar{m}) \upharpoonright (S \cap \bigcup_{j < i} A_j)$ -history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_\ell = \varepsilon \rangle$, and consider

$$w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^* = w_{\eta_\varepsilon(n)} \cup (\cup \{[\delta_k, \delta_k + h_{\delta_{k+1}}] : k \in [0, \ell)\} \cup [\varepsilon, \varepsilon + h_{\eta_\varepsilon(n)}]).$$

First note that by our inductive hypothesis

$$h_\varepsilon^* = \langle h_{\delta_1}, h_{\delta_2}, \dots, h_{\delta_\ell} (= \varepsilon) \rangle,$$

so (since $w_\varepsilon \subseteq w_{\eta_\varepsilon(n)} \subseteq \eta_\varepsilon(n)$) $k = |w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*|$ does not depend on the particular choice of $\bar{\delta}$, but we also shall argue later for

$$(2.12) \quad c_\varepsilon^\psi(\eta_\varepsilon(n)) := \{g \circ \text{OP}_{w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*, k} : g \in (\Psi^+)^p(w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*)\} \in \Upsilon_k^p$$

(where $k = |w_{\bar{\delta} \cap \langle \eta_\varepsilon(n) \rangle}^*|$) being independent of $\bar{\delta}$ (and only depending on ε).

- _d Finally let $c_\varepsilon^p(\eta_\varepsilon(n)) = p_\varepsilon \cap \langle j \rangle$, where $j = 1$, if there are $\delta' \neq \delta''$ with $\varepsilon \in \text{ran}(\eta'_{\delta'}) \cap \text{ran}(\eta'_{\delta''})$ (i.e. $\varepsilon = \eta_{\delta'}(n') = \eta_{\delta''}(n'')$ for some $n' \geq m_{\delta'}$, $n'' \geq m_{\delta''}$), otherwise $j = 0$.

Claim 2.35. *The value $c_\varepsilon^\psi(\eta_\varepsilon(n))$ (defined in (2.12) using the long history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_\ell = \varepsilon \rangle$) does not depend on the particular $\bar{\delta}$.*

Proof. Fix $\varepsilon \in S \cap A_i$, $n \in \omega$, and long η' -histories $\bar{\xi} = \langle \xi_0, \xi_1, \dots, \xi_\ell = \varepsilon \rangle$, $\bar{\zeta} = \langle \zeta_0, \zeta_1, \dots, \zeta_\ell = \varepsilon \rangle$.

We have to use the induction hypothesis on $\langle \psi_\delta : \delta \in \bigcup_{j \leq i} A_j \rangle$ and recall how $(\Psi^+)^p$ is defined from $(\Psi)^p$ in Claim 1.30. Let

$$x_\xi := w_{\bar{\xi} \cap \langle \eta_\varepsilon(n) \rangle}^* = w_{\eta_\varepsilon(n)} \cup (\cup \{[\xi_k, \xi_k + h_{\xi_{k+1}}] : k \in [0, \ell)\} \cup [\varepsilon, \varepsilon + h_{\eta_\varepsilon(n)}]),$$

and

$$x_\zeta := w_{\bar{\zeta} \cap \langle \eta_\varepsilon(n) \rangle}^* = w_{\eta_\varepsilon(n)} \cup (\cup \{[\zeta_k, \zeta_k + h_{\zeta_{k+1}}] : k \in [0, \ell)\} \cup [\varepsilon, \varepsilon + h_{\eta_\varepsilon(n)}]).$$

It is clear from the way we defined $w_{\bar{\delta}}^*$'s (Definition 2.33) and the inductive assumption on h_ε^* (or just on ψ_ε) that

$$|w_{\bar{\zeta} \cap \langle \eta_\varepsilon(n) \rangle}^*| = |w_{\bar{\xi} \cap \langle \eta_\varepsilon(n) \rangle}^*|,$$

let k denote this number. So suppose that the function f_0 satisfies

$$(2.13) \quad f_0 = g_0 \circ \text{OP}_{x_\xi, k} \quad \text{for some } g_0 \in (\Psi^+)^p(x_\xi),$$

but

$$(2.14) \quad g_1 = f_0 \circ \text{OP}_{k, x_\zeta} \notin (\Psi^+)^{\mathfrak{p}}(x_\zeta)$$

(so

$$(2.15) \quad g_0 = g_1 \circ \text{OP}_{x_\zeta, x_\xi}).$$

This means (recalling $(\mathfrak{T})_9$ from Fact 2.15) that for some $b \in \mathcal{B}^{\mathfrak{p}}$ and $b \subseteq x_\zeta$:

$$g_1 \upharpoonright b \notin \Psi(b).$$

Now again by Fact 2.15 this b is of the form $b_{n,s}^\delta = \bigcup_{j \leq n} (v_{\eta_\delta}(j) \cup \{\eta_\delta(j)\}) \cup [\delta, \delta + s]$ for some $\delta \in S$, $n, s \in \omega$.

Now assume first that $\delta \leq \zeta_\ell = \varepsilon$, but then (2.10) in (\star_1) implies that $b \subseteq x_\xi \cap x_\zeta$, and $\text{OP}_{x_\zeta, x_\xi} \upharpoonright b = \text{id}_b$, so $g_0 \upharpoonright b = g_1 \upharpoonright b \notin \Psi^{\mathfrak{p}}(b)$, hence $g_0 \notin (\Psi^+)^{\mathfrak{p}}(x_\xi)$ contradicting (2.13).

Therefore we can assume that $\delta > \varepsilon$ (so $\delta = \zeta_j$ for some $j < \ell$), then w.l.o.g. by \square_c from Definition 1.29 we can assume that n, s are maximal such that $b = b_{n,s}^\delta \subseteq x_\zeta$, so

$$b = v_{\zeta_{j+1}} \cup \{\zeta_{j+1}\} \cup [\zeta_j, \zeta_j + h_{\zeta_{j+1}}].$$

But note that then

$$b \subseteq w_\zeta^*$$

(since $\{\zeta_{j+1}\} \cup [\zeta_j, \zeta_j + h_{\zeta_{j+1}}] \cup w_{\zeta_\ell} \subseteq w_\zeta^*$ by Definition 2.33 and $w_{\zeta_\ell} \supseteq w_{\zeta_{j+1}} \supseteq v_{\zeta_{j+1}}$ by our demands on $\langle w_\alpha : \alpha < \omega_1 \rangle$), so

$$g_1 \upharpoonright w_\zeta^* \notin (\Psi^{\mathfrak{p}})^+(w_\zeta^*).$$

Now by the inductive assumption on ψ_ε in $(\blacktriangle)_2$ necessarily

$$g^* := (g_1 \upharpoonright w_\zeta^*) \circ \text{OP}_{w_\zeta^*, w_\xi^*} \notin (\Psi^{\mathfrak{p}})^+(w_\xi^*),$$

but as $\text{OP}_{x_\zeta, x_\xi} \upharpoonright w_\xi^* = \text{OP}_{w_\zeta^*, w_\xi^*}$ by (2.11) in (\star_3) ,

$$g^* = (g_1 \circ \text{OP}_{x_\zeta, x_\xi}) \upharpoonright w_\xi^* \notin (\Psi^{\mathfrak{p}})^+(w_\xi^*),$$

and by (2.15)

$$g_0 \upharpoonright w_\xi^* \notin (\Psi^{\mathfrak{p}})^+(w_\xi^*)$$

implying $g_0 \notin (\Psi^{\mathfrak{p}})^+(x_\xi)$, contradicting (2.13).

□_{Claim2.35}

Now having the claim proven we already saw that $c_\varepsilon(\eta_\varepsilon(n))$ does not depend on the long η' -history ending with ε , once we manage to uniformize the system it will only depend on $\eta_\varepsilon(n)$, not on ε .

It is easy to see that for each fixed α the sets $\{c_\varepsilon^{h^*}(\alpha) : \alpha \in \text{dom}(c_\varepsilon^{h^*})\}$, $\{c_\varepsilon^\psi(\alpha) : \alpha \in \text{dom}(c_\varepsilon^\psi)\}$ and $\{c_\varepsilon^{\mathfrak{p}}(\alpha) : \alpha \in \text{dom}(c_\varepsilon^{\mathfrak{p}})\}$ are all countable, while

$$(2.16) \quad \{c_\varepsilon^{\nu^{**}}(\alpha) : \alpha \in \text{ran}(\text{dom}(\eta_\varepsilon))\} \subseteq {}^{<\omega}({}^{<\omega}\alpha),$$

which can be seen by the following: $c_\varepsilon^{\nu^{**}}(\alpha) = \nu_\varepsilon^{**} \frown \langle \nu_\varepsilon^* \rangle$ by its very definition in \bullet_b , and by our inductive assumptions $(\blacktriangle)_2$ for each long η' -history $\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_\ell = \varepsilon \rangle$ the sequence $c_\varepsilon^{\nu^{**}}(\alpha)$ is equal to $\langle \nu_{\delta_0}^*, \nu_{\delta_1}^*, \dots, \nu_{\delta_{\ell-1}}^* \rangle \frown \langle \nu_{\delta_\ell}^* \rangle$. By $(\blacktriangle)_4$ this latter is equal to

$$\langle \langle \rangle, (\eta'_{\delta_0} \upharpoonright n_1), (\eta'_{\delta_1} \upharpoonright n_2), \dots, (\eta'_{\delta_{\ell-1}} \upharpoonright n_\ell) \rangle$$

(where $\eta'_{\delta_k}(n_{k+1}) = \delta_{k+1}$, and $\nu_{\delta_0}^* = \langle \rangle$), so

$$c_\varepsilon^{\nu^{**}}(\alpha) \in {}^{<\omega}\delta_1 \times {}^{<\omega}\delta_2 \times \cdots \times {}^{<\omega}\delta_\ell.$$

But as $\bar{\delta} \cap \langle \delta_{\ell+1} := \alpha \rangle$ is an η -history, so recalling $(\mathbf{T})_{16}$ if $k \leq \ell$ and $\beta \in \text{ran}(\eta_{\delta_k}) \cap \delta_{k+1}$, then $\beta \in \alpha = \delta_{\ell+1}$, in particular,

$$\text{for each } k < \ell : \eta'_{\delta_k} \upharpoonright n_{k+1} \text{ has range } \subseteq \alpha,$$

we are done with proving (2.16).

Thus, we can uniformize all these functions, and pick some suitable d^{h^*} , d^ψ , $d^{\nu^{**}}$ and d^p that uniformize the respective mappings, so fix $m_\delta^0 : \delta \in A_i$ so that $(n \geq m_\delta^0) \rightarrow d^t(\eta_\delta(n)) = c_\delta^t(\eta_\delta(n))$ for any $t \in \{h^*, \psi, \nu^{**}, p\}$.

Now for obtaining ν^* apply Claim 2.24 to the ladder system $\langle \eta_\delta^0 : \delta \in A_i \cap S \rangle$, $\eta_\delta^0(n) = \eta_\delta(n + m_\delta^0)$ giving us

$$\odot \langle m_\delta^1 : \delta \in A_i \cap S \rangle, \text{ and } \langle \nu_\alpha : \alpha < \omega_1 \rangle \text{ with}$$

$$\begin{aligned} \nu_{\eta_\delta^0(n+m_\delta^1)} &= \nu_{\eta_\delta(n+m_\delta^0+m_\delta^1)} = \\ &= \eta^0 \upharpoonright [m_\delta^1, m_\delta^1 + n) = \eta \upharpoonright [m_\delta^0 + m_\delta^1, m_\delta^0 + m_\delta^1 + n). \end{aligned}$$

For each $\delta \in A_i \cap S$ we let $m_\delta = m_\delta^0 + m_\delta^1$, so we defined the ladder system

$$\bar{\eta}' \upharpoonright (A_i \cap S), \quad \eta'_\delta(n) = \eta_\delta(m_\delta^0 + m_\delta^1 + n) \quad (n \in \omega).$$

Finally, for $\alpha \in A_{i+1}$ let

$$\begin{aligned} \nu_\alpha^* &= \begin{cases} \nu_\alpha, & \text{if } \alpha \in \text{ran}(\eta'_\delta) \text{ for some } \delta \in A_i \\ \langle \rangle, & \text{otherwise,} \end{cases} \\ \nu_\alpha^{**} &= \begin{cases} d^{\nu^{**}}(\alpha), & \text{if } \alpha \in \text{ran}(\eta'_\delta) \text{ for some } \delta \in A_i \\ \langle \rangle, & \text{otherwise,} \end{cases} \\ \psi_\alpha &= \begin{cases} d^\psi(\alpha), & \text{if } \alpha \in \text{ran}(\eta'_\delta) \text{ for some } \delta \in A_i \\ \langle \rangle, & \text{otherwise,} \end{cases} \\ p_\alpha &= \begin{cases} d^p(\alpha), & \text{if } \alpha \in \text{ran}(\eta'_\delta) \text{ for some } \delta \in A_i \\ \langle \rangle, & \text{otherwise,} \end{cases} \\ h_\alpha^* &= \begin{cases} d^{h^*}(\alpha), & \text{if } \alpha \in \text{ran}(\eta'_\delta) \text{ for some } \delta \in A_i \\ \langle \rangle, & \text{otherwise.} \end{cases} \end{aligned}$$

It is straightforward to check all our demands in $(\blacktriangle)_1$ - $(\blacktriangle)_5$ for each $\delta \in S \cap A_i$, $\gamma \in A_{i+1}$.

□_{Lemma 2.34}

Lemma 2.36. *Assume that*

- (A) $\bar{\eta}$ is a ladder system on the stationary set S (with \aleph_0 -uniformization on $\bar{\eta}$), admitting the rank \bar{A} , and $\langle h_\alpha : \alpha < \omega_1 \rangle$ is permitted.
- (B) $\mathbf{q} = (S, \bar{\eta}, \bar{v}, \Psi)$ is a nice very special S -uniformization problem, $\bar{w} = \langle w_\alpha : \alpha < \omega_1 \rangle$ is a \mathbf{q} -pressed down system.

Moreover, suppose that there exists $\bar{\psi}$, \bar{v}^* , \bar{p} , \bar{v}^{**} , \bar{h}^* and \bar{m} as in Lemma 2.34 (and a derived ladder system $\bar{\eta}' = \bar{\eta} - \bar{m}$)

Fix $\varepsilon < \omega_1$, $\bar{\delta} = \langle \delta_0, \dots, \delta_\ell = \varepsilon \rangle$, $\bar{\delta}' = \langle \delta'_0, \dots, \delta'_\ell = \varepsilon \rangle$ are long $\bar{\eta}'$ -histories, and suppose that $\theta < \varepsilon$, $\xi = \langle \xi_0, \dots, \xi_k = \theta \rangle$ is a long $\bar{\eta}'$ -history. Then

- (Y)₁ $\{\delta_i : i \leq \ell\} \cap \{\delta'_i : i \leq \ell\} = \{\delta_i : i \geq j^\bullet\}$ for some $j^\bullet \leq \ell$,
 (Y)₂ $\{\delta_i : i \leq \ell\} \cap \{\xi_i : i \leq k\}$ is either empty, or is of the form $\{\delta_i : j_0 \leq i \leq j_1\}$ for some $0 \leq j_0 \leq j_1 \leq \ell$, moreover,

$$\{\delta_i : j_0 \leq i \leq j_1\} = \{\xi_i : j_0 \leq i \leq j_1\},$$

- (Y)₃ if $j_0 \leq j_1$ are as in (Y)₂ (so $\bar{\delta}$ and $\bar{\xi}$ intersect each other), then there exists a long $\bar{\eta}'$ -history $\bar{\xi}' = \langle \xi'_0, \xi'_1, \dots, \xi'_k \rangle$ with

- (a) $\xi'_k = \theta$, and
 (b) $\{\delta'_i : i \leq \ell\} \cap \{\xi'_i : i \leq k\} = \{\delta'_i : j_0 \leq i \leq j_1\}$,
 (c) moreover,

$$(OP_{w_{\bar{\delta}'}, w_{\bar{\delta}}^*})^{w_{\bar{\delta}}^* \cap w_{\bar{\xi}}^*} = w_{\bar{\delta}}^* \cap w_{\bar{\xi}'}^*,$$

and

$$(OP_{w_{\bar{\xi}'}, w_{\bar{\xi}}^*})^{w_{\bar{\delta}}^* \cap w_{\bar{\xi}}^*} = w_{\bar{\delta}}^* \cap w_{\bar{\xi}'}^*,$$

- (Y)₄ there exists a long $\bar{\eta}'$ -history $\bar{\xi} = \langle \xi_0, \xi_1, \dots, \xi_{k^*} \rangle$ such that $\xi_{k^*} < \varepsilon$, $\varepsilon \notin \{\xi_j : j \leq k^*\}$, and whenever $\bar{\vartheta} = \langle \vartheta_0, \vartheta_1, \dots, \vartheta_m \rangle$ is a long $\bar{\eta}'$ -history with $\vartheta_m < \varepsilon$, $\varepsilon = \delta_\ell \notin \{\vartheta_j : j \leq m\}$, then for all $i < \ell$ $\delta_i \in \{\vartheta_j : j \leq m\}$ implies $\delta_i \in \{\xi_0, \xi_1, \dots, \xi_{k^*}\}$ and

$$w_{\bar{\vartheta}}^* \cap [\delta_i, \delta_i + \omega) \subseteq w_{\bar{\xi}}^* \cap [\delta_i, \delta_i + \omega).$$

Proof. First, assume on the contrary, that for the long $\bar{\eta}'$ -histories $\bar{\delta}$, $\bar{\delta}'$ ($\delta_\ell = \delta'_\ell$) for some $i < \ell$ $\delta_i = \delta'_i$, but $\delta_{i+1} \neq \delta'_{i+1}$ (so necessarily $i+1 < \ell$). Then on the one hand $\delta_{i+1}, \delta'_{i+1} \in \text{ran}(\eta'_{\delta'_i})$, so by $(\blacktriangle)_3$ $\nu_{\delta_{i+1}}^* \neq \nu_{\delta'_{i+1}}^*$ as one is a proper initial segment of the other. On the other hand by $(\blacktriangle)_4$

$$\nu_{\delta_{i+1}}^* = \nu_{\delta_\ell, i+1}^{**} = \nu_{\delta'_\ell, i+1}^{**} = \nu_{\delta'_{i+1}}^*,$$

a contradiction, this proves (Y)₁.

For (Y)₂ let $\delta_i = \xi_j$, but $\delta_{i-1} \neq \xi_{j-1}$ (so $\delta_{i-1} \notin \text{ran}(\bar{\xi})$ by $(\top)_{17}$). Then apply (Y)₁ to $\langle \delta_0, \delta_1, \dots, \delta_i \rangle$, $\langle \xi_0, \xi_1, \dots, \xi_j \rangle$. Finally, follows from $(\blacktriangle)_2$ or $(\blacktriangle)_4$ that $i = j$.

To prove (Y)₃ let j^\bullet be given by (Y)₁ let j_0, j_1 be given by (Y)₂ for $\bar{\delta}$, $\bar{\xi}$, so that

$$(2.17) \quad \{\delta_i : j_0 \leq i \leq j_1\} = \{\xi_i : j_0 \leq i \leq j_1\}.$$

Define $\xi'_i = \delta'_i$ for $i \in [j_0, j_1]$, and $\xi'_i = \xi_i$ for $i > j_1$ (and $i \leq k$). Before completing the definition of $\bar{\xi}'$ we check that this segment is a η' -history, which is clear once we have showed

$$(2.18) \quad \xi_{j_1+1} \in \text{ran}(\eta'_{\delta'_{j_1}}).$$

Of course we can assume that $\delta'_{j_1} \neq \delta_{j_1}$ (as $\delta_{j_1} = \xi_{j_1}$ and so (2.18) would follow), from which necessarily $j_1 < \ell$.

We will need the following two observations.

- (\square)₁ $\nu_{\delta'_{j_1+1}}^* = \nu_{\delta'_{j_1+1}}^*$, either because $j_1 + 1 = \ell$ (and then $\delta_{j_1+1} = \delta'_{j_1+1} = \varepsilon$), or else if $j_1 + 1 < \ell$, then $\nu_{\varepsilon, j_1+1}^{**} = \nu_{\delta'_{j_1+1}}^* = \nu_{\delta'_{j_1+1}}^*$ by $(\blacktriangle)_4$, so

$$\nu_{\delta'_{j_1+1}}^* = \eta'_{\delta'_{j_1}} \restriction n = \eta'_{\delta_{j_1}} \restriction n \text{ for some } n,$$

where

$$(2.19) \quad \eta'_{\delta'_{j_1}}(n) = \delta'_{j_1+1},$$

and

$$(2.20) \quad \eta'_{\delta_{j_1}}(n) = \delta_{j_1+1}.$$

Moreover, as $\delta_{j_1} = \xi_{j_1}$ we have

$$(\Box)_2 \quad \nu_{\xi_{j_1+1}}^* = \eta'_{\xi_{j_1}} \upharpoonright n^\bullet = \eta'_{\delta_{j_1}} \upharpoonright n^\bullet \text{ for some } n^\bullet \text{ by } (\blacktriangle)_2, \text{ and } \eta'_{\delta_{j_1}}(n^\bullet) = \eta'_{\xi_{j_1}}(n^\bullet) = \xi_{j_1+1}.$$

So (\Box_1) and (\Box_2) together mean that it suffices to argue that

$$(2.21) \quad n^\bullet < n,$$

because then

$$(2.22) \quad \eta'_{\delta'_{j_1}}(n^\bullet) = \eta'_{\delta_{j_1}}(n^\bullet) = \xi_{j_1+1},$$

i.e. 2.18 holds. As $\eta'_{\delta_{j_1}}(n) = \delta_{j_1+1}$, and $\eta'_{\xi_{j_1}}(n^\bullet) = \xi_{j_1+1}$ and $\delta_{j_1+1} \neq \xi_{j_1+1}$ by our assumptions, so if $n^\bullet > n$, then it follows from (\Box_2) that $\eta'_{\xi_{j_1}}(n) = \eta'_{\delta_{j_1}}(n) = \delta_{j_1+1}$ by (2.20), and necessarily $\delta_{j_1+1} < \xi_{j_1+1} = \eta'_{\xi_{j_1}}(n^\bullet)$. So applying $(\top)_{16}$ to $\bar{\xi}$ we obtain that for each $j \leq k$ $\xi_j > \delta_{j+1} \geq \delta_\ell = \varepsilon$ holds, contradicting $\theta < \varepsilon$.

Now recalling $(\Upsilon)_2$ we only have to extend $\langle \xi'_{j_0}, \xi'_{j_0+1}, \dots, \xi'_k \rangle$ to a long η' -history $\langle \xi'_0, \xi'_1, \dots, \xi'_k \rangle$, such that $\xi'_{j_0-1} \neq \delta'_{j_0-1}$. So we can assume that $0 < j_0$. Note that we may assume that $\delta_{j_0} \neq \delta_\ell = \varepsilon$, i.e. $j_0 < \ell$, as otherwise letting $\bar{\xi}' = \bar{\xi}$ would work. Recall $(\blacktriangle)_5$, so $p_\varepsilon(j_0)$ is defined, and $\delta_{j_0} \in \text{ran}(\eta'_{\delta_{j_0-1}}) \cap \text{ran}(\eta'_{\xi_{j_0-1}})$ implies that $p_\varepsilon(j_0) = 1$. Applying this fact to $\bar{\delta}'$ we obtain that

$$\delta'_{j_0} \in \text{ran}(\eta'_{\delta'_{j_0-1}}) \cap \text{ran}(\eta'_\gamma) \text{ for some } \gamma \neq \delta'_{j_0-1},$$

let $\xi'_{j_0-1} = \gamma$, and extending it to a long history, we are done. (It follows from $(\Upsilon)_1$, that two long η' -histories $\bar{\xi}$ and $\bar{\xi}'$ with the same ending must have the same length.)

For the moreover part recall how we defined the w^* 's (Definition 2.33):

$$w_\delta^* = w_{\delta_\ell} \cup (\cup\{[\delta_i, \delta_i + h_{\delta_{i+1}}] : i < \ell\}),$$

and note that $\delta_\ell = \delta'_\ell$, so by $(\blacktriangle)_2$ $h_{\delta_i} = h_{\delta'_i}$ for each $i \leq \ell$, and

$$w_{\bar{\xi}}^* = w_{\xi_k} \cup (\cup\{[\xi_i, \xi_i + h_{\xi_{i+1}}] : i < k\}),$$

similarly $\xi_k = \xi'_k$, so by $(\blacktriangle)_2$ $h_{\xi_i} = h_{\xi'_i}$ for each $i \leq k$. We recall that

$$\begin{aligned} \max(w_{\xi_k}) &< \xi_k < \xi_{k-1} < \dots < \xi_0, \\ \max(w_{\xi'_k}) &< \xi'_k < \xi'_{k-1} < \dots < \xi'_0, \\ \max(w_{\delta_\ell}) &< \delta_\ell < \delta_{\ell-1} < \dots < \delta_0, \\ \max(w_{\delta'_\ell}) &< \delta'_\ell < \delta'_{\ell-1} < \dots < \delta'_0, \end{aligned}$$

(and $w_{\delta_\ell} = w_{\delta'_\ell}$), so one can easily check that the moreover part of $(\Upsilon)_3$ would follow if we could argue that

$$(2.23) \quad (\forall i \leq k) : [\xi_i, \xi_i + h_{\xi_{i+1}}] \cap w_{\delta_\ell} = [\xi'_i, \xi'_i + h_{\xi'_{i+1}}] \cap w_{\delta_\ell}.$$

Note that (2.23) would follow from the assertion

$$(2.24) \quad (\forall i \leq k) : \xi_i \in w_{\delta_\ell} \iff \xi'_i \in w_{\delta_\ell}.$$

For (2.24) note that by definition $\xi_j = \xi'_j$ for $j > j_1$, so we only have to care about ξ_j 's, ξ'_j 's for $j \leq j_1$. Therefore, as $\eta'_{\delta_{j_1}}(n) = \delta_{j_1+1}$, $\eta'_{\delta'_{j_1}}(n) = \delta'_{j_1+1}$, $n > n^\bullet$ ((2.19), (2.20), (2.21)), and $\eta'_{\delta'_{j_1}}(n^\bullet) = \eta'_{\delta_{j_1}}(n^\bullet) = \xi_{j_1+1} = \xi'_{j_1+1}$, (by (2.22)), and clearly

$$\begin{aligned}\eta'_{\delta_{j_1}}(n) &= \delta_{j_1} = \xi_{j_1} > \delta_{j_1+1} > \dots > \delta_\ell > \eta'_{\delta_{j_1}}(n^\bullet) = \xi_{j_1+1}, \\ \eta'_{\delta'_{j_1}}(n) &= \delta'_{j_1} = \xi'_{j_1} > \delta'_{j_1+1} > \dots > \delta'_\ell > \eta'_{\delta'_{j_1}}(n^\bullet) = \xi'_{j_1+1}.\end{aligned}$$

This means that $\xi_{j_1}, \xi'_{j_1} > \max(w_{\delta_\ell})$, and we are done.

For $(\Upsilon)_4$ first let $k^\bullet = \ell$, if $\nu_\varepsilon^* \neq \langle \rangle$, otherwise letting $k^\bullet < \ell$ to be the largest $j < \ell$ such that $\nu_{\varepsilon, j}^* \neq \langle \rangle$ (equivalently, $\nu_{\delta_j}^* \neq \langle \rangle$) if such a j exists, and let $k^\bullet = 0$ if there is no such j . We will argue that choosing

$$(2.25) \quad \bar{\xi} = \langle \xi_0, \xi_1, \dots \rangle = \langle \delta_0, \delta_1, \dots, \delta_{k^\bullet-1}, \nu_{\delta_{k^\bullet}}^* (|\nu_{\delta_{k^\bullet}}^*| - 1) \rangle$$

works (or else, if $\nu_\varepsilon^* = \langle \rangle$ and so there is no $j < \ell$ with $\nu_{\delta_j}^* \neq \langle \rangle$, then for any $\bar{\eta}'$ -history $\bar{\vartheta}$ with last entry smaller than ε , the set of entries of $\bar{\vartheta}$ does not meet that of $\bar{\delta}$). No matter which case we are in we can assume that

$$(2.26) \quad \forall t > 0 : \nu_{\delta_{k^\bullet+t}}^* = \langle \rangle.$$

In the first case note that if $k^\bullet > 0$, then $\nu_{\delta_{k^\bullet}}^* = \eta'_{\delta_{k^\bullet-1}} \upharpoonright |\nu_{\delta_{k^\bullet}}^*|$ by $(\blacktriangle)_3$, so $\bar{\xi}$ is indeed a long $\bar{\eta}'$ -history. Also it is easy to see by $(\blacktriangle)_4$ (or $(\blacktriangle)_3$ if $k^\bullet = \ell$) that for the last entry of $\bar{\xi}$:

$$\xi_{k^\bullet} = \nu_{\delta_{k^\bullet}}^* (|\nu_{\delta_{k^\bullet}}^*| - 1) < \varepsilon.$$

Clearly it is enough to verify that picking an arbitrary $\bar{\eta}'$ -history $\bar{\vartheta}$ with $\vartheta_{|\bar{\vartheta}|-1} < \varepsilon$ and $\varepsilon \notin \{\vartheta_0, \vartheta_1, \dots, \vartheta_{|\bar{\vartheta}|-1}\}$ necessarily

$$(\forall t \geq 0) \delta_{k^\bullet+t} \notin \{\vartheta_0, \vartheta_1, \dots, \vartheta_{|\bar{\vartheta}|-1}\}.$$

But observe that if a long $\bar{\eta}'$ -history $\bar{\xi}$ goes through $\delta_{k^\bullet+t}$, then it can go through $\delta_{k^\bullet+t-1}, \delta_{k^\bullet+t-2}, \dots, \delta_{k^\bullet}$ with the same ending point $\xi_{|\bar{\xi}|-1}$.

So it is enough to argue that if $\bar{\vartheta}$ is a long $\bar{\eta}'$ -history with $\vartheta_j = \delta_{k^\bullet}$ for some j then necessarily $\vartheta_{|\bar{\vartheta}|-1} \geq \varepsilon = \delta_\ell$. Recall $(\Upsilon)_2$, pick the largest $t \geq 0$ such that $\vartheta_{j+t} = \delta_{k^\bullet+t}$ (so $k^\bullet + t < \ell$, $\delta_{k^\bullet+t} \neq \varepsilon$ as $\varepsilon \notin \{\vartheta_i : i < |\bar{\vartheta}|\}$ by our assumptions). Then $\vartheta_{j+t+1} \neq \delta_{k^\bullet+t+1}$ are of the form $\eta'_{\delta_{k^\bullet+t}}(n_1), \eta'_{\delta_{k^\bullet+t}}(n_2)$ for some $n_1, n_2 < \omega$, respectively, and as $\nu_{\delta_{k^\bullet+t+1}}^* = \langle \rangle = \eta'_{\delta_{k^\bullet+t}} \upharpoonright n_2$ by $(\blacktriangle)_3$ we obtain that $n_2 < n_1$, so $\vartheta_{j+t+1} > \delta_{k^\bullet+t+1}$. But then it follows from $(\Upsilon)_{16}$, that all entries of ϑ are bigger than $\delta_{k^\bullet+t+1}$, so bigger than ε .

Since for every $\varrho \in S$ $h_{\eta'_\varrho(n)}$ is strictly increasing (by our assumptions) recalling the definition of $w_\vartheta^* = w_{\vartheta_{|\bar{\vartheta}|-1}} \cup (\cup \{[\delta_j, \delta_j + h_{\vartheta_{j+1}}] : j \in [0, |\bar{\vartheta}|)\})$ in Definition 2.33 (and a similar reasoning applying $(\Upsilon)_{16}$) it is straightforward to check the moreover part.

□_{Lemma 2.36}

Now using Lemmas 2.34 and 2.36 we will solve the nice S -uniformization problem \mathfrak{p} with the aid of \bar{m} and all the values resulting from Lemma 2.36.

Claim 2.37. *Assume that $\bar{\eta}$ is a ladder system on the stationary set S , $\langle A_i : i \in \omega \rangle$ is a $\bar{\eta}$ -rank, \bar{w} is a \mathfrak{p} -pressed down system, and Lemma 2.34 holds with $\bar{m} = \langle m_\delta : \delta \in S \rangle$.*

Then there exists a function F with $\text{dom}(F) = \omega_1$, which is a solution of \mathfrak{p} .

Proof. We define

- (*)₁ $\Xi_\varepsilon = \{\bar{\delta} = \langle \delta_0, \delta_1, \dots, \delta_{|\bar{\delta}|-1} \rangle : \bar{\delta} \text{ is a maximal } \bar{\eta}' = \bar{\eta} - \bar{m}\text{-history ending with } \varepsilon, \text{ i.e. } \delta_{|\bar{\delta}|-1} = \varepsilon\},$

let

- (*)₂ $\Gamma = \{\alpha < \omega_1 : \Xi_\alpha \neq \emptyset\}$ (so $\Gamma = \bigcup_{\delta \in S} (\text{ran}(\eta_\delta \upharpoonright [m_\delta, \infty)) \setminus S$).
 (*)₃ We choose for each $\zeta \in \Gamma$ a maximal history $\bar{\delta}^\zeta \in \Xi_\zeta$.
 (*)₄ By induction on $\zeta \in \Gamma$ we choose a function f_ζ such that
 (a) $\text{dom}(f_\zeta) = w_{\bar{\delta}^\zeta}^*$, and
 (b) $f_\zeta \in (\Psi^+)^{\mathfrak{p}}(w_{\bar{\delta}^\zeta}^*)$, moreover,
 (c) introducing the object

$$G_\zeta = \bigcup_{\alpha < \zeta} \bigcup_{\bar{\vartheta} \in \Xi_\alpha} g_{\bar{\vartheta}},$$

which we require to be a function, where by $g_{\bar{\vartheta}}$ we mean the function with domain $w_{\bar{\vartheta}}^*$ defined as follows.

- (*)₅ For $\alpha \in \zeta \cap \Gamma$ and $\bar{\vartheta} \in \Xi_\alpha$ we let $g_{\bar{\vartheta}} = f_\alpha \circ \text{OP}_{w_{\bar{\delta}^\alpha}^*, w_{\bar{\vartheta}}^*}$ where $\text{dom}(f_\alpha) = w_{\bar{\delta}^\alpha}^*$ (so $g_{\bar{\vartheta}}$ is a function with domain $w_{\bar{\vartheta}}^*$), and
 (*)₆ we let $B_\zeta = \bigcup_{\alpha < \zeta} \bigcup_{\bar{\vartheta} \in \Xi_\alpha} w_{\bar{\vartheta}}^*$.

Note that by (*)₅

$$(2.27) \quad (\text{for } \alpha < \zeta) : \forall \bar{\delta}, \bar{\delta}' \in \Xi_\alpha : G_\zeta \upharpoonright w_{\bar{\delta}'}^* \circ (\text{OP}_{w_{\bar{\delta}'}^*, w_{\bar{\delta}}^*}) = G_\zeta \upharpoonright w_{\bar{\delta}}^*,$$

moreover, by our demands on \bar{m} using (\blacktriangle)₂ and (*)₄b clearly

$$(2.28) \quad \forall \bar{\delta}' \in \Xi_\alpha : G_\zeta \upharpoonright w_{\bar{\delta}'}^* \in (\Psi^{\mathfrak{p}})^+(w_{\bar{\delta}'}^*),$$

First we argue that (provided we can carry out the induction) the resulted function $F = \bigcup_{\alpha < \omega_1} G_\alpha$ is a solution for \mathfrak{p} . Note that (by (Υ)₈/ (a) in Fact 2.15, and recalling $u_\alpha \subseteq w_\alpha \subseteq \alpha$ in Definition 2.27)

- (*)₇ if $\delta \in S$, then $\bigcup_{n \in \omega} w_{\eta_\delta(m_\delta+n)} = \delta$,

so by choosing for each n a maximal η' -history through $\eta'_\delta(m_\delta + n)$ and δ clearly $\delta \subseteq B_\delta$. As S is stationary F is an entire function, defined on each ordinal $\alpha < \omega_1$. We only have to check that for each δ , and each $n, k < \omega$

$$F \upharpoonright b_{n,k}^\delta \in \Psi^{\mathfrak{p}}(b_{n,k}^\delta).$$

For this having δ, n, k fixed choose n^\bullet large enough so that

$$\eta'_\delta(n^\bullet) = \eta_\delta(m_\delta + n^\bullet) \geq \eta_\delta(n),$$

and

$$h_{\eta'_\delta(n^\bullet)} \geq k.$$

Now if $\zeta \in \Gamma$ is such that there exists a long $\bar{\eta}'$ -history $\bar{\delta} \in \Xi_\zeta$ through $\eta'_\delta(n^\bullet)$ and δ , then by Definition 2.33 we have

$$w_{\bar{\delta}}^* \supseteq b_{m_\delta+n^\bullet, h_{\eta'_\delta(n^\bullet)}}^\delta \supseteq b_{n,k}^\delta,$$

and by (2.28)

$$F \upharpoonright w_{\bar{\delta}}^* = G_{\zeta+1} \upharpoonright w_{\bar{\delta}}^* \in (\Psi^{\mathfrak{p}})^+(w_{\bar{\delta}}^*),$$

and by \square_b $F \upharpoonright b_{n,k}^\delta \in \Psi^{\mathfrak{p}}(b_{n,k}^\delta)$, we are done.

This means that the next claim completes the proof of Claim 2.37, so that of our main theorem.

Subclaim 2.38. *We can construct the sequence $\langle f_\zeta : \zeta \in \Gamma \rangle$ satisfying $(\otimes)_4$.*

Proof. We proceed by induction, obviously only the successor step is interesting. So assume that f_β is constructed for each $\zeta \cap \Gamma$, we need to construct a suitable f_ζ .

We have to characterize the set $B_\zeta \cap w_{\delta_\zeta}^*$ (and $B_\zeta \cap w_{\bar{\delta}}^*$ for $\bar{\delta} \in \Xi_\zeta$), the set where our function is already defined, and we have to argue that the induction hypothesis ensures that the required compatibility in $(\otimes)_5$ can be achieved.

Appealing to $(\Upsilon)_4$ of Lemma 2.36 with $\bar{\delta}^\zeta = \langle \delta_0^\zeta, \delta_1^\zeta, \dots, \delta_\ell^\zeta = \zeta \rangle$ there exists a long $\bar{\eta}'$ -history $\xi = \langle \xi_0, \xi_1, \dots, \xi_k \rangle$ with $\xi_k < \zeta$, and with the property that whenever for a maximal $\bar{\eta}'$ -history $\bar{\vartheta}$ we have $\vartheta_j = \delta_i^\zeta$ for some $j < \ell g(\bar{\vartheta})$, $i < \ell$, but $\vartheta_{|\bar{\vartheta}|-1} < \zeta$, then $w_{\bar{\vartheta}}^* \cap [\delta_i^\zeta, \delta_i^\zeta + \omega] \subseteq w_\xi^*$, in particular, $\delta_i^\zeta = \xi_n$ for some $n < k$. This also implies that

$$(2.29) \quad B_\zeta \cap w_\xi^* \subseteq \zeta \cup w_\xi^*.$$

Recalling the definition of $w_{\bar{\delta}}^*$'s ($w_{\bar{\delta}}^* = w_{\delta_{\ell g(\bar{\delta})-1}}^* \cup \bigcup_{k < \ell g(\bar{\delta})-1} [\delta_k, \delta_k + h_{\delta_{k+1}}]$ Definition 2.33, where in fact the sequence of h_{δ_k} 's only depend on $\delta_{\ell g(\bar{\delta})-1}$) this clearly implies that

$$(2.30) \quad B_\zeta \cap w_{\bar{\xi}}^* \subseteq \zeta \cup w_{\bar{\xi}}^*$$

(as $\bar{\xi}$ works for any $\bar{\vartheta}$ and i).

By extending $\bar{\xi}$ to be a maximal $\bar{\eta}'$ -history if necessary we can assume that $\rho := \xi_k \in \Gamma$. Now $(\Upsilon)_3$ from Lemma 2.36 yields that for every $\bar{\delta}' \in \Xi_\zeta$ there exists $\bar{\xi}' \in \Xi_\rho$, such that

$$(\text{OP}_{w_{\bar{\xi}'}^*, w_{\bar{\xi}}^*}) "w_{\bar{\delta}'}^* \cap w_{\bar{\xi}}^* = w_{\bar{\delta}'}^* \cap w_{\bar{\xi}'}^*,$$

and

$$(\text{OP}_{w_{\bar{\delta}'}^*, w_{\bar{\delta}^\zeta}^*}) "w_{\bar{\delta}^\zeta}^* \cap w_{\bar{\xi}}^* = w_{\bar{\delta}'}^* \cap w_{\bar{\xi}}^*,$$

so

$$(2.31) \quad \text{OP}_{w_{\bar{\xi}'}^*, w_{\bar{\xi}}^*} \upharpoonright w_{\bar{\delta}^\zeta}^* \cap w_{\bar{\xi}}^* = \text{OP}_{w_{\bar{\delta}'}^*, w_{\bar{\delta}^\zeta}^*} \upharpoonright w_{\bar{\delta}^\zeta}^* \cap w_{\bar{\xi}}^*,$$

as both functions are order preserving. So (2.31) together with (2.29) (and (2.27)) imply that

$$(2.32) \quad G_\zeta \circ \text{OP}_{w_{\bar{\delta}^\zeta}^*, w_{\bar{\delta}'}^*} \upharpoonright (w_{\bar{\delta}^\zeta}^* \cap B_\zeta) = G_\zeta \upharpoonright (w_{\bar{\delta}'}^* \cap B_\zeta)$$

(since $\text{OP}_{w_{\bar{\delta}^\zeta}^*, w_{\bar{\delta}'}^*}$ fixes each ordinal smaller or equal to $\zeta = \delta_\ell^\zeta = \delta_\ell'$).

We claim that it is enough to extend $G_\zeta \upharpoonright B_\zeta \cap w_{\bar{\delta}^\zeta}^*$ to a function f_ζ defined on $w_{\bar{\delta}^\zeta}^*$ with $f_\zeta \in (\Psi^p)^+(w_{\bar{\delta}^\zeta}^*)$. So suppose that such f_ζ exists, we need that $G_{\zeta+1} = G_\zeta \upharpoonright \bigcup \{g_{\bar{\delta}'} : \bar{\delta}' \in \Xi_\zeta\}$ is a function, which in turn would imply that

$$G_{\zeta+1} \circ \text{OP}_{w_{\bar{\delta}^\zeta}^*, w_{\bar{\delta}'}^*} = G_{\zeta+1} \upharpoonright w_{\bar{\delta}'}^*.$$

Suppose that $\beta \in w_{\bar{\delta}'}^* \cap w_{\bar{\delta}''}^*$ for some $\bar{\delta}', \bar{\delta}'' \in \Xi_\zeta$. Then recalling that $w_{\bar{\delta}'}^* = w_{\delta_{|\bar{\delta}'|-1}}^* \cup (\cup \{[\delta_i, \delta_i + h_{\delta_{i+1}}] : i < |\bar{\delta}'| - 1\})$, $\text{OP}_{w_{\bar{\delta}'}^*, w_{\bar{\delta}''}^*} = \text{OP}_{w_{\bar{\delta}'}^*, w_{\bar{\delta}^\zeta}^*} \circ \text{OP}_{w_{\bar{\delta}^\zeta}^*, w_{\bar{\delta}''}^*}$ fixes β , so $\text{OP}_{\bar{\delta}^\zeta, \bar{\delta}''}(\beta) = \text{OP}_{\bar{\delta}^\zeta, \bar{\delta}'}(\beta)$, and we are done (remembering $(\otimes)_5$).

For this (i.e. for a suitable f_ζ) we only have to prove (recalling \square_d from Definition 1.29) that $B_\zeta \cap w_{\bar{\delta}^\zeta}^*$ is \mathfrak{p} -closed, and check that $G_\zeta \upharpoonright B_\zeta \cap w_{\bar{\delta}^\zeta}^* \in (\Psi^p)^+(B_\zeta \cap w_{\bar{\delta}^\zeta}^*)$, i.e. for \mathfrak{p} -basic sets $b \subseteq w_{\bar{\delta}^\zeta}^*$ $G \upharpoonright b \in \Psi^p(b)$. So fix $\delta^\bullet \in S$, $\delta^\bullet \in B_\zeta \cap w_{\bar{\delta}^\zeta}^*$. First

assume that $\delta^\bullet < \zeta$, so $\delta^\bullet \in w_\zeta$ by the definition of $w_{\delta^\zeta}^*$. But then as w_ζ is \mathfrak{p} -closed by our present assumptions on \bar{w} and Definition 2.27, so for some n and k

$$\eta_{\delta^\bullet}(n) \cup v_{\eta_{\delta^\bullet}(n)} \subseteq w_\zeta \cap \delta^\bullet \subseteq \eta_{\delta^\bullet}(n+1),$$

and

$$w_\zeta \cap [\delta^\bullet, \delta^\bullet + \omega] = [\delta^\bullet, \delta^\bullet + k].$$

(note that here we use the original ladder system η). Then for some n' $\eta'_{\delta^\bullet}(n') = \eta_{\delta^\bullet}(m_{\delta^\bullet} + n') \geq \eta_{\delta^\bullet}(n)$, and $h_{\eta'_{\delta^\bullet}(n')} \geq k$ (as \bar{h} is permitted for $\bar{\eta}$). Then choosing a maximal $\bar{\eta}'$ -history $\bar{\xi}$ going through $\eta'_{\delta^\bullet}(n')$ and δ^\bullet , the following necessarily holds:

$$b_{n,k}^{\delta^\bullet} \subseteq b_{m_{\delta^\bullet} + n', h_{\eta(m_{\delta^\bullet} + n')}}^{\delta^\bullet} = \{\eta_{\delta^\bullet}(m_{\delta^\bullet} + n')\} \cup v_{\eta_{\delta^\bullet}(m_{\delta^\bullet} + n')} \cup [\delta^\bullet, \delta^\bullet + h_{\eta(m_{\delta^\bullet} + n')}] \subseteq w_{\bar{\xi}}^*$$

(note that $b_{m_{\delta^\bullet} + n', h_{\eta(m_{\delta^\bullet} + n')}}^{\delta^\bullet}$ is a \mathfrak{p} -basic set as well). As $G_\zeta \upharpoonright w_{\bar{\xi}}^* \in (\Psi^{\mathfrak{p}})^+(w_{\bar{\xi}}^*)$ by (2.28), clearly $G_\zeta \upharpoonright b_{n,k}^{\delta^\bullet} \in \Psi^{\mathfrak{p}}(b_{n,k}^{\delta^\bullet})$ (by \square_c), we are done.

Now since $\zeta \in \Gamma$, so $\zeta \notin S$, the other possibility is that $\delta^\bullet > \zeta$. Then $\delta^\bullet \in w_{\delta^\zeta}^*$ means that $\delta^\bullet = \delta_i^\zeta$ for some $i < \ell$ ($\delta^\bullet \in S$), and so $\delta_{i+1}^\zeta = \eta'_{\delta^\bullet}(n)$ for some n . Note that by $(\tau)_{16}$

$$(2.33) \quad \eta'_{\delta^\bullet}(n-1) < \zeta < \eta'_{\delta^\bullet}(n) = \delta_{i+1}^\zeta,$$

and then

$$(2.34) \quad w_{\delta_{i+1}^\zeta}^* \cup \{\eta'_{\delta^\bullet}(n)\} \subseteq w_\zeta \cup \{\eta'_{\delta^\bullet}(n)\} \subseteq w_{\delta^\zeta}^* \cap \delta^\bullet \subseteq \delta_{i+1}^\zeta + 1 = \eta'_{\delta^\bullet}(n) + 1$$

by Definition 2.27 and the \mathfrak{p} -closedness of $w_{\delta^\zeta}^*$, and

$$(2.35) \quad [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n)}] = w_{\delta^\zeta}^* \cap [\delta^\bullet, \delta^\bullet + \omega].$$

Now $\delta^\bullet \in B_\zeta$ implies that

- (\bullet)₁ there is a maximal η' -history $\bar{\xi} = \langle \xi_0, \xi_1, \dots, \xi_k \rangle$ with $\xi_k < \zeta$ going through δ^\bullet , so $\xi_j = \delta^\bullet$, and $\xi_{j+1} = \eta'_{\delta^\bullet}(n^\bullet)$ for some j and n^\bullet (where $m_{\delta^\bullet} \geq n^\bullet$, since $\bar{\xi}$ is an $\bar{\eta}$ -history).

At this point let n^\bullet be maximal such that (\bullet)₁ holds (i.e. for some $\bar{\eta}'$ -history $\bar{\xi} = \langle \xi_0, \dots, \xi_k \rangle$, $\bar{\xi}$ has δ^\bullet as an entry, and $\xi_k < \zeta$), then by finding a maximal history through $\eta'_{\delta^\bullet}(n^\bullet)$ we can assume that $\xi_{j+1} = \eta'_{\delta^\bullet}(n^\bullet)$, and n^\bullet is the maximal t such that $\eta'_{\delta^\bullet}(t) \in B_\zeta$. Such maximal n^\bullet exists by the following. We are going to argue that

- (\bullet)₂ either $n^\bullet = n - 1$, or $n^\bullet = n$ where n is from (2.33).

First recall (2.33), so either $\eta'_{\delta^\bullet}(n-1) \in \Gamma$, or it belongs to S , and there is a maximal η' -history through it and δ , so $n^\bullet \geq n - 1$. On the other hand, $\zeta < \eta'_{\delta^\bullet}(n) < \eta'_{\delta^\bullet}(n+1)$, and again by $(\tau)_{16}$ any history through $\eta'_{\delta^\bullet}(n+t)$ for $t > 0$ must have a last entry bigger than $\eta'_{\delta^\bullet}(n+t-1)$, so bigger than $\eta'_{\delta^\bullet}(n)$, contradicting the fact that so far we only considered maximal η' -histories with last entry smaller than ζ .

Now in order to finish the proof of the subclaim it suffices to show that

- (1) $(w_{\delta^\zeta}^* \cap B_\zeta) \cap \delta^\bullet \subseteq \eta'_{\delta^\bullet}(n^\bullet + 1)$,
- (2) $(w_{\delta^\zeta}^* \cap B_\zeta) \cap [\delta^\bullet, \delta^\bullet + \omega] = [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n^\bullet)}]$,
- (3) so $b_{m_{\delta^\bullet} + l, k}^{\delta^\bullet} \subseteq B_\zeta \cap w_{\delta^\zeta}^*$ iff $l \leq n^\bullet$ and $k \leq h_{\eta'_{\delta^\bullet}(n^\bullet)}$,
- (4) and $G_\zeta \upharpoonright b_{m_{\delta^\bullet} + l, h_{\eta'_{\delta^\bullet}(n^\bullet)}}^{\delta^\bullet} \in \Psi^{\mathfrak{p}}(b_{m_{\delta^\bullet} + l, h_{\eta'_{\delta^\bullet}(n^\bullet)}}^{\delta^\bullet})$,

this will imply that $w_{\delta_\zeta}^* \cap B_\zeta$ is \mathfrak{p} -closed and $G_\zeta \upharpoonright w_{\delta_\zeta}^* \cap B_\zeta \in (\Psi^+)^{\mathfrak{p}}$, as δ^\bullet was an arbitrary element of $(w_{\delta_\zeta}^* \cap B_\zeta) \cap S$.

It follows from the maximality of n^\bullet (and the definition of w_ξ^* , and \bar{h} being permitted) that

$$(2.36) \quad B_\zeta \cap [\delta^\bullet, \delta^\bullet + \omega] = [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n^\bullet)}] = w_\xi^* \cap [\delta^\bullet, \delta^\bullet + \omega].$$

Then similarly to (2.34), (2.35)

$$(2.37) \quad w_{\eta'_{\delta^\bullet}(n^\bullet)} \cup \{\eta'_{\delta^\bullet}(n^\bullet)\} \cup [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n^\bullet)}] \subseteq w_\xi^* \subseteq B_\zeta.$$

If $n^\bullet = n - 1$, then by the maximality of n^\bullet we would have $\eta'_{\delta^\bullet}(n) \notin B_\zeta$ so putting together (2.34), (2.35) and (2.37) we have

$$(2.38) \quad w_{\eta'_{\delta^\bullet}(n^\bullet)} \cup \{\eta'_{\delta^\bullet}(n^\bullet)\} \subseteq (B_\zeta \cap w_{\delta_\zeta}^*) \cap \delta^\bullet \subseteq \eta'_{\delta^\bullet}(n^\bullet + 1),$$

and

$$(2.39) \quad (B_\zeta \cap w_{\delta_\zeta}^*) \cap [\delta^\bullet, \delta^\bullet + \omega] = [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n^\bullet)}].$$

On the other hand, if $n^\bullet = n$, then clearly $w_{\delta_\zeta}^* \cap [\delta^\bullet, \delta^\bullet + \omega] = [\delta^\bullet, \delta^\bullet + h_{\eta'_{\delta^\bullet}(n^\bullet)}]$, and (2.38), (2.39) also hold.

In each case $b^\bullet := b_{m_{\delta^\bullet} + n^\bullet, h_{\eta'_{\delta^\bullet}(n^\bullet)}}^{\delta^\bullet}$ is a maximal \mathfrak{p} -basic set at δ^\bullet that is a subset of $B_\zeta \cap w_{\delta_\zeta}^*$ (since $v_{\eta'_{\delta^\bullet}(n^\bullet)} \subseteq w_{\eta'_{\delta^\bullet}(n^\bullet)}^*$ by Definition 2.27). It is also clear that $B_\zeta \cap w_{\delta_\zeta}^*$ is \mathfrak{p} -closed. Moreover, $b^\bullet \subseteq w_\xi^*$ for the long $\bar{\eta}'$ -history $\bar{\xi}$ going through δ^\bullet and $\eta'_{\delta^\bullet}(n^\bullet)$, so $G_\zeta \upharpoonright w_\xi^* \in (\Psi^{\mathfrak{p}})^+(w_\xi^*)$ by (2.28), so $G_\zeta \upharpoonright b^\bullet \in \Psi^{\mathfrak{p}}(b^\bullet)$ by \square_b .

$\square_{\text{Subclaim 2.38}}$

$\square_{\text{Claim 2.37}}$

In order to finish the proof of Lemma 2.23, suppose that \mathfrak{p} is a very special S -uniformization system which is nice. By Claim 2.26 there exists some \bar{m} and $\bar{A} = \langle a_n : n \in \omega \rangle$ such that \bar{A} is an $\bar{\eta}' = \bar{\eta} - \bar{m}$ -rank. By Lemma 2.28 for some \bar{m}' the system \bar{w} is a $\bar{\eta}'' = \bar{\eta} - \bar{m}'$ -pressed down system. Therefore, Lemma 2.34 and Claim 2.37 apply to $\mathfrak{p}'' = (\mathfrak{p} - \bar{m}) - \bar{m}'$, and we have a solution to that (i.e. to \mathfrak{p}'').

$\square_{\text{Lemma 2.23}}$

$\square_{\text{Theorem 2.1}}$

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