GPS Nonlinear System of Equations Report

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1 Introduction to the problem

1.1 Problem to explore

Nonlinear system of equations:
$(x - a_1)^2 + (y - b_1)^2 + (z - c_i)^2 = [(C * (t_1 - D))]$
$(x - a_2)^2 + (y - b_2)^2 + (z - c_i)^2 = [(C * (t_2 - D))]$
$(x - a_3)^2 + (y - b_3)^2 + (z - c_i)^3 = [(C * (t_3 - D))]$
$(x - a_4)^2 + (y - b_4)^2 + (z - c_i)^2 = [(C * (t_4 - D))]$

Note that the above formulation is a simplified version of the GPS observation equation, In my exploration, i consider and define all errors that should be accounted for, in the real world problem. In the code, you shall find that i define all of the possible errors to be zero, which leads to the formulation above.

1.2 GPS - Background

We begin by introducing the components of the GPS mechanism, from here we shall arrive at our nonlinear system of equations. One of the most trivial questions to ask is "why do we need satellites?" this question is closely related to our mathematical problem and you should keep it in mind. I shall refer to it later. The GPS mechanism is composed of three components, Space Vehicles (Satellites), ground stations, and receivers. The only component that requires an explanation is the ground stations, there are 5 of them on Earth, 4 are unmanned and a master control station. Their purpose is to keep track of the Space Vehicles and make sure that they are on path and do not malfunction. When i say on track, i mean that the must keep to their paths in the Constellation. The GPS Constellation is a grid of Space Vehicles that allows for any point on Earth to have visibility of 5-8 Satellites, let us note that at least 4 are required to solve the nonlinear system of linear equations. The GPS mechanism's purpose is evident, it is to allow for the calculation of receivers on Earth, thereby alerting the user of his/her location. Therefore, we seek to solve for the users (x,y,z) coordinates. How do we do this?. Suppose that the user has visibility of 4 Satellites, these vehicles broad cast ephemeris data, which includes the Satellites (x,y,z) position, its time t^k and its time offset/err, say dt^k. Let the receivers time be t, whose err/offset is unknown. We then can solve for an approximate value of the distance between the receiver and the Satellite, this we call the pseudo range and denote it as $P^k = C(t - t^k)$, where C is the speed of light constant. We now note the crux of our problem the observation equation.

1.3 Observation equation

$P^k = p^k + Cdt - Cdt^k + T^k + {}^{\text{`}}I^k + e^k$

$p^k = ((x^k - x)^2 + (y^k - y)^2 + (z^k - z)^2)^{1/2}$ - geometrical distance between the satellite and receiver

T- Tropospheric error

I- Ionospheric error

e- Observation error

k $\Leftrightarrow$ the k-th satellite, k = 1,2,3,4 in our case

1.4 Observation equation Linearized

We now must linearize our observation equation, this we do through a Taylor expansion.

1.4.1 Linearization

We linearize p^k,

$p^k = ((x^k - x)^2 + (y^k - y)^2 + (z^k - z)^2)^{1/2} = f(x, y, z)$

$x_{n+1} = x_n + \Delta x_n$

$y_{n+1} = y_n + \Delta y_n$

$z_{n+1} = z_n + \Delta z_n$

$f(x_{n+1}, y_{n+1}, z_{n+1}) = f(x_n, y_n, z_n) + \frac{\delta f(x_n,y_n,z_n)}{\delta x_n}\Delta x_n + \frac{\delta f(x_n,y_n,z_n)}{\delta y_n}\Delta y_n + \frac{\delta f(x_n,y_n,z_n)}{\delta z_n}\Delta z_n$

$\frac{\delta f(x_n,y_n,z_n)}{\delta x_n}\Delta x_n = \frac{-x^k - x_n}{p_n^k}$

$\frac{\delta f(x_n,y_n,z_n)}{\delta y_n}\Delta y_n = \frac{-y^k - y_n}{p_n^k}$

$\frac{\delta f(x_n,y_n,z_n)}{\delta z_n}\Delta z_n = \frac{-z^k - z_n}{p_n^k}$

We now have, for the k-th satellite, the following linear equation.

$P_n^k = p_n^k - \frac{x^k - x_n}{p_n^k}\Delta x_n - \frac{y^k - y_n}{p_n^k}\Delta y_n - \frac{z^k - z_n}{p_n^k}\Delta z_n + c(dt_n - dt^k)) + T_n^k + {}^{\text{`}}I_n^k + e_n^k$

$\begin{bmatrix} -\frac{x^k - x_n}{p_n^k} & -\frac{y^k - y_n}{p_n^k} & -\frac{z^k - z_n}{p_n^k} \end{bmatrix} \begin{bmatrix} \Delta x_n \\ \Delta y_n \\ \Delta z_n \\ c(dt_n) \end{bmatrix} = P_n^k - p_n^k + Cdt^k - T^k - {}^{\text{`}}I^k - e^k$ We now simplify in the following way.

$b_n^k = P_n^k - p_n^k + Cdt^k - T^k - {}^{\text{`}}I^k - e^k$ Note that for $k = 1, 2, 3, 4$ where k represents the k-th Satellite, for each Satellite then we have a corresponding linear equation, we can now construct a system of linear equations, and say the following.

$min_n|| \begin{bmatrix} -\frac{x^1-x_n}{p_n^1} & -\frac{y^1-y_n}{p_n^1} & -\frac{z^1-z_n}{p_n^1} \\ -\frac{x^2-x_n}{p_n^2} & -\frac{y^2-y_n}{p_n^2} & -\frac{z^3-z_n}{p_n^3} \\ -\frac{x^3-x_n}{p_n^3} & -\frac{y^3-y_n}{p_n^3} & -\frac{z^3-z_n}{p_n^3} \end{bmatrix} \begin{bmatrix} \Delta x_n \\ \Delta y_n \\ \Delta z_n \\ c(dt_n) \end{bmatrix} - \begin{bmatrix} b_n^1 \\ b_n^2 \\ b_n^3 \\ b_n^4 \end{bmatrix} ||$ We then solve the Least Square problem through an iterative process, please take a look at the code included with this paper for an example of the iterative process.

NOTE: MOST ERRORS ARE APPROXIMATED TO BE ZERO, REVIEW CODE FOR MORE INFO ON THIS MATTER.

1.5 Answering the question

"why do we need satellites?", remember this question?. We need them because they serve are reference points and allow us to "triangulate" to the receivers position, this then means that we need the satellites locations to be sparse and not clustered, and the more Satellites used to calculate the position the better the approximation. Through the iterative procedure we fit data/points into the LS and as we fit, we gain a sequence of points that converges to the approximate position (x,y,z).

I explored the LS method of solving the nonlinear system of equations. Newtons method can also be used and code pertaining to said method is included with this paper. I shall not document its procedure, for it is far to similar to the 2D case and i have taken the liberty to document the mathematics in the code, as reference for the reader. I shall however note that both LS and Newtons method require linearization.

1.6 Ending comments

This paper is not about comparing methods of solving a nonlinear system of equations, but rather to explore the most popular method of solving the GPS nonlinear system of equation. Moreover, do to my lack of experimental data i could not go about creating an experiment that compared their speed, for more on this problem please take a look at the code provided along with this paper. Note that the experimental data for each experiment in the code provided is different. Please keep this in mind.

2 Notes on code included

2.1 LS method - code

Please click on the GpsLs file, within this folder you shall find the code you will need to analyze and execute. To run an example of LS in action, with experimental data, please go ahead and open and execute the GPS Expe.m procedure, it shall generate, a convergent sequence to the approximate solution and graph, the receivers position,and said sequence of approximate positions. To analyze the iterative procedure, please open nonLinSolver4.m.

2.2 Newton's method - code

Please go ahead and open the file named GpsNewt, within it you shall find the iterative procedure and an example of Newton's method in action with the use of experimental data. I have also documented Newton's method, within newTonsMethod.m, which is where you shall find the iterative procedure for Newton's method. For the experimental procedure please go ahead and open GPSnewtExamp.m, it shall accomplish the same thing as GPS Expe.m, with different experimental data.

3 Sources

3.1 Source 1

http://www.colorado.edu/geography/gcraft/notes/gps/gps.bak3

3.2 Source 2

http://mason.gmu.edu/ jtrichil/project2writeupmath447 =¿ used for exp. data

3.3 Source 3

http://www.imagingshop.com/linear-and-nonlinear-least-squares-with-math-net/

3.4 Source 4

http://www4.ncsu.edu/ mtchu/Teaching/Lectures/MA325/NLS$_G PS.pdf$

3.5 Source 5

http://www.adv-radio-sci.net/9/203/2011/ars-9-203-2011.pdf*

3.6 Source 6

http://www.mathworks.com/moler/leastsquares.pdf*

3.7 Source 7

http://www.uni-stuttgart.de/gi/research/paper/1996/207.pdf*

3.8 Source 8

Numerical Mathematics and Computing, Sixth edition Ward Cheney, David Kincaid